Dynamics of vortex cap solutions on the rotating unit sphere

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Abstract

In this work, we analytically study the existence of periodic vortex cap solutions for the homogeneous and incompressible Euler equations on the rotating unit 2-sphere, which was numerically conjectured in [28, 29, 60, 61]. Such solutions are piecewise constant vorticity distributions, subject to the Gauss constraint and rotating uniformly around the vertical axis. The proof is based on the bifurcation from zonal solutions given by spherical caps. For the one–interface case, the bifurcation eigenvalues correspond to Burbea's frequencies obtained in the planar case but shifted by the rotation speed of the sphere. The two–interfaces case (also called band type or strip type solutions) is more delicate. Though, for any fixed large enough symmetry, and under some non-degeneracy conditions to avoid spectral collisions, we achieve the existence of at most two branches of bifurcation.

Contents

1 Introduction

The purpose of this paper is to give an analytical proof of the emergence of uniformly rotating (around the z-axis) vortex caps with $\mathbf{m}$-fold symmetries ($\mathbf{m} \in \mathbb{N}^*$) for the incompressible Euler equations set on the rotating unit sphere $\mathbb{S}^2$ defined by

$$\mathbb{S}^2 \triangleq \Big\{(x,y,z) \in \mathbb{R}^3 \quad \text{s.t.} \quad x^2+y^2+z^2=1\Big\}.$$

In particular, we shall implement bifurcation techniques in order to find non-trivial vortex cap solutions close to the trivial stationary flat (spherical) ones, which was numerically conjectured in [28, 29, 61, 60]. In this introduction we present the model of interest, discuss some historical background, derive the contour dynamics equations, expose our results and give the organization of this work.

1.1 Euler equations on the rotating unit sphere

This study deals with the homogeneous incompressible Euler equations on the two dimensional unit sphere in rotation around the vertical axis. Such a model, sometimes called *barotropic*, is commonly used in geophysical fluid dynamics for meteorological predictions or to study the motion of planets' atmosphere. We may refer the reader to [43, Sec. 13.4.1] and [66] for a rather complete introduction to these equations. In order to present the model, we shall first recall some basic notions in differential calculus/geometry. The set $\mathbb{S}^2$ is endowed with a smooth manifold structure described by the following two charts

$$\begin{array}{rcl} C_1 : (0,\pi)\times(0,2\pi) & \to & \mathbb{R}^3 \\ (\theta,\varphi) & \mapsto & \big(\sin(\theta)\cos(\varphi)\,,\,\sin(\theta)\sin(\varphi)\,,\,\cos(\theta)\big), \\ C_2 : (0,\pi)\times(0,2\pi) & \to & \mathbb{R}^3 \\ (\vartheta,\phi) & \mapsto & \big(-\sin(\vartheta)\cos(\phi)\,,\,-\cos(\vartheta)\,,\,-\sin(\vartheta)\sin(\phi)\big). \end{array}$$

For our purpose, we shall mainly work in the chart C_1 where the variables θ and φ are called *colatitude* and *longitude*, respectively. Notice that the physical literature mentioned above rather considers the latitude/longitude convention, but we found more convenient to work with the other one. In the colatitude/longitude chart C_1, the Riemannian metric of $\mathbb{S}^2$ is given by

$$\mathtt{g}_{\mathbb{S}^2}(\theta,\varphi) \triangleq d\theta^2 + \sin^2(\theta)d\varphi^2. \tag{1.1}$$

Therefore, denoting N and S the north and south poles, we have that for any $p\in\mathbb{S}^2\setminus\{N,S\}$, an orthonormal basis of the tangent space $T_p\mathbb{S}^2$ is given by

$$\mathbf{e}_\theta \triangleq \partial_\theta, \qquad \mathbf{e}_\varphi \triangleq \frac{1}{\sin(\theta)}\partial_\varphi.$$

We have used the classical identification between tangent vectors and directional differentiations. In these coordinates, the Riemannian volume is given by

$$d\sigma = \sin(\theta)d\theta d\varphi.$$

Therefore, for any function $\mathtt{f}:\mathbb{S}^2\to\mathbb{R}$, we define

$$f(\theta,\varphi)\triangleq \mathtt{f}\big(C_1(\theta,\varphi)\big), \qquad \int_{\mathbb{S}^2}\mathtt{f}(\xi)d\sigma(\xi)\triangleq\int_0^{2\pi}\int_0^{\pi} f(\theta,\varphi)\sin(\theta)d\theta d\varphi. \tag{1.2}$$

In the sequel, with a small abuse of notation, we shall denote f for both $\mathtt{f}$ or f with no possible confusion according to the context, since one is in the cartesian variables ξ and the other one is in the spherical coordinates (θ,φ). The gradient of f is defined as follows

$$\nabla f(\theta,\varphi)\triangleq \partial_\theta f(\theta,\varphi)\mathbf{e}_\theta + \frac{\partial_\varphi f(\theta,\varphi)}{\sin(\theta)}\mathbf{e}_\varphi.$$

Similarly, we define its orthogonal as

$$\nabla^\perp f(\theta,\varphi)\triangleq J\nabla f(\theta,\varphi), \qquad \underset{(\mathbf{e}_\theta,\mathbf{e}_\varphi)}{\mathrm{Mat}}(J)=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

The Laplace-Beltrami operator applied to f is defined by

$$\Delta f(\theta,\varphi)\triangleq\frac{1}{\sin(\theta)}\partial_\theta\big[\sin(\theta)\partial_\theta f(\theta,\varphi)\big]+\frac{1}{\sin^2(\theta)}\partial_\varphi^2 f(\theta,\varphi).$$

For a vector field $U(\theta,\varphi)=U_\theta(\theta,\varphi)\mathbf{e}_\theta+U_\varphi(\theta,\varphi)\mathbf{e}_\varphi$, the divergence expresses as

$$(\nabla\cdot U)(\theta,\varphi)\triangleq\frac{1}{\sin(\theta)}\partial_\theta\big[\sin(\theta)U_\theta(\theta,\varphi)\big]+\frac{1}{\sin(\theta)}\partial_\varphi U_\varphi(\theta,\varphi).$$

The incompressible Euler equations on the rotating sphere $\mathbb{S}^2$ with angular rotation speed $\widetilde{\gamma}$ are given by

$$(E_{\widetilde{\gamma}})\begin{cases}\partial_t\Omega(t,\theta,\varphi)+U(t,\theta,\varphi)\cdot\nabla\big(\Omega(t,\theta,\varphi)-2\widetilde{\gamma}\cos(\theta)\big)=0,\\ U(t,\theta,\varphi)=\nabla^\perp\Psi(t,\theta,\varphi),\\ \Delta\Psi(t,\theta,\varphi)=\Omega(t,\theta,\varphi).\end{cases} \tag{1.3}$$

We mention that the term $-2\widetilde{\gamma}\, U(t,\theta,\varphi)\cdot\nabla\cos(\theta)$ corresponds to the Coriolis force coming from the rotation of the sphere. In the sequel, we shall work with the following quantity called *absolute vorticity*

$$\overline{\Omega}(t,\theta,\varphi)\triangleq\Omega(t,\theta,\varphi)-2\widetilde{\gamma}\cos(\theta). \tag{1.4}$$

The second equation in (1.3) states that the velocity field U is divergence-free. Then, the divergence theorem implies the following so-called *Gauss constraint*

$$\forall t\geqslant 0,\qquad \int_{\mathbb{S}^2}\overline{\Omega}(t,\xi)d\sigma(\xi)=\int_{\mathbb{S}^2}\Omega(t,\xi)d\sigma(\xi)=0. \tag{1.5}$$

Notice that the first equality above is justified by

$$\int_{\mathbb{S}^2}\big[\overline{\Omega}(t,\xi)-\Omega(t,\xi)\big]d\sigma(\xi)=-4\pi\widetilde{\gamma}\int_0^\pi\cos(\theta)\sin(\theta)d\theta=\big[\pi\widetilde{\gamma}\cos(2\theta)\big]_0^\pi=0.$$

According to [4], the stream function Ψ can be computed from the vorticity Ω through the following integral representation

$$\Psi(t,\xi)=\int_{\mathbb{S}^2}G(\xi,\xi')\Omega(t,\xi')d\sigma(\xi'),\qquad G(\xi,\xi')\triangleq\frac{1}{2\pi}\log\left(\frac{|\xi-\xi'|_{\mathbb{R}^3}}{2}\right), \tag{1.6}$$

where $|\cdot|_{\mathbb{R}^3}$ is the usual Euclidean norm in $\mathbb{R}^3$. In the colatitude/longitude coordinates, we have

$$G(\theta,\varphi,\theta',\varphi')=\frac{1}{4\pi}\log\Big(1-\cos(\theta)\cos(\theta')-\sin(\theta)\sin(\theta')\cos(\varphi-\varphi')\Big)-\frac{\log(2)}{4\pi}\cdot \tag{1.7}$$

In what follows, we shall denote for simplicity

$$D(\theta,\theta',\varphi,\varphi')\triangleq 1-\cos(\theta)\cos(\theta')-\sin(\theta)\sin(\theta')\cos(\varphi-\varphi'). \tag{1.8}$$

Observe that we can write

$$\begin{aligned}D(\theta,\theta',\varphi,\varphi')&=1-\cos(\theta-\theta')+\sin(\theta)\sin(\theta')\big(1-\cos(\varphi-\varphi')\big)\\&=2\Big[\sin^2\left(\tfrac{\theta-\theta'}{2}\right)+\sin(\theta)\sin(\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big].\end{aligned} \tag{1.9}$$

The above function D will play an important role since it describes the singularity of the integral operator defining the stream function. Indeed, with this last expression, we recover that

$$D(\theta,\theta',\varphi,\varphi')\geqslant 0\qquad\text{and}\qquad D(\theta,\theta',\varphi,\varphi')=0\quad\Leftrightarrow\quad\theta=\theta'\text{ and }\big(\varphi=\varphi'\text{ or }\theta\in\{0,\pi\}\text{ or }\theta'\in\{0,\pi\}\big).$$

Notice that once one works with the colatitude/longitude coordinates instead of the physical ones, the Euclidean norm is deformed implying that the Green kernel is anisotropic and the north and south poles are degenerating points. This will generate extra complexity later when dealing with the regularity of the stream function in the new coordinates.

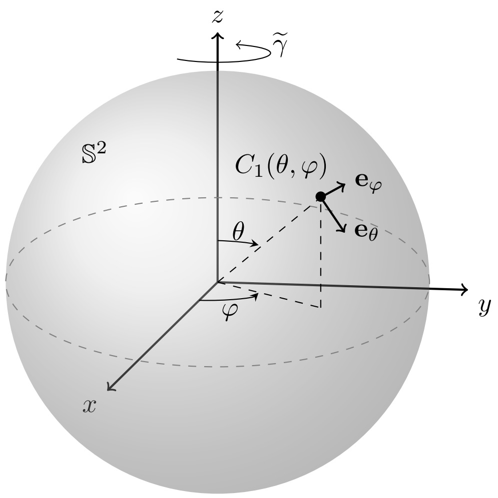


Figure 1: Convention colatitude/longitude for spherical coordinates.

1.2 Historical discussion

We shall expose here some relevant results linked to our work.

Vortex patches in the plane

Recall that the planar homogeneous incompressible Euler equations write

$$\begin{cases}\partial_t\boldsymbol{\omega}(t,x_1,x_2)+\mathbf{v}(t,x_1,x_2)\cdot\nabla\boldsymbol{\omega}(t,x_1,x_2)=0,\\ \mathbf{v}(t,x_1,x_2)=\nabla^{\perp}_{\mathbb{R}^2}\boldsymbol{\Psi}(t,x_1,x_2),\\ \Delta_{\mathbb{R}^2}\boldsymbol{\Psi}(t,x_1,x_2)=\boldsymbol{\omega}(t,x_1,x_2),\end{cases}\qquad \nabla^{\perp}_{\mathbb{R}^2}\triangleq\begin{pmatrix}-\partial_{x_2}\\ \partial_{x_1}\end{pmatrix},\qquad \Delta_{\mathbb{R}^2}\triangleq\partial^2_{x_1}+\partial^2_{x_2}. \tag{1.10}$$

Vortex patches are weak solutions to (1.10) in the Yudovich class and taking the form $t\mapsto\mathbf{1}_{D_t}$ where D_t is a bounded planar domain. The vorticity jump here is normalized to 1. The dynamics is described by the evolution of the boundary ∂D_t. More precisely, according to [55, p. 174], if $z(t,\cdot):\mathbb{T}\to\partial D_t$ is a parametrization of (one connected component of) the boundary at time t, then it must solve the following contour dynamics equation

$$\mathrm{Im}\Big(\partial_t z(t,x)\overline{\partial_x z(t,x)}\Big)=\partial_x\Big(\boldsymbol{\Psi}\big(t,z(t,x)\big)\Big). \tag{1.11}$$

The solutions that we construct in this study are the analogous to the V-states obtained in the planar case. These later are particular vortex patches where the dynamics is given by a uniform rotation of the initial domain around its center of mass (fixed to be the origin), namely $D_t=e^{\mathrm{i}\Omega t}D_0$ with $\Omega\in\mathbb{R}$. As remarked by Rankine in 1858, any radial domain is a V-state rotating with any angular velocity $\Omega\in\mathbb{R}$. Later, Kirchhoff [62] discovered non-trivial explicit examples of V-states with elliptic shapes. Close to the unit disc, such solutions were first obtained numerically by Deem and Zabusky [26] and then analytically by Burbea [5] through bifurcation techniques. He constructed a countable family of local curves of V-states with $\mathbf{m}$-fold symmetries (i.e. invariance by $\frac{2\pi}{\mathbf{m}}$ angular rotation) bifurcating from the disc at the angular velocities

$$\Omega_{\mathbf{m}}=\frac{\mathbf{m}-1}{2\mathbf{m}}\cdot \tag{1.12}$$

The global continuation of the Burbea's bifurcation branches was found in [48] through global bifurcation arguments. Some rigidity results giving necessary conditions on the angular velocity to obtain non-trivial V-states can be found in [30, 51, 41]. In the last decade, there have been intensive rigorous studies on the subject, giving more properties of the V-states and exploring their existence around different topological structures. In particular, the bifurcation from the annulus

$$A_b\triangleq\big\{z\in\mathbb{C}\quad \text{s.t.}\quad b<|z|<1\big\},\quad b\in(0,1),$$

has been studied by Hmidi-de la Hoz-Mateu-Verdera [25]. They proved that under the condition

$$\mathtt{p}(b,\mathbf{m})\triangleq 1+b^{\mathbf{m}}-\frac{1-b^2}{2}\mathbf{m}<0,$$

there are exactly two branches of periodic vortex patches, with two interfaces, emerging at the angular velocities

$$\Omega^{\pm}_{\mathbf{m}}(b)\triangleq\frac{1-b^2}{4}\pm\frac{1}{2\mathbf{m}}\sqrt{\left(\frac{1-b^2}{2}\mathbf{m}-1\right)^2-b^{2\mathbf{m}}}. \tag{1.13}$$

The degenerate case $\mathtt{p}(b,\mathbf{m})=0$ has been discussed in [53, 68]. Periodic multiple patches were found via a desingularization of an appropriate distribution of point vortices. The first work studying the desingularization of two point vortices using contour dynamics equations is due to Hmidi-Mateu [54]. Other extensions, in the same spirit, have been achieved in [31, 32, 45, 50]. Similar results were obtained using variational arguments in [6, 7, 10, 67] or via gluing techniques in [21, 22]. Recently, the authors in [33] studied the global continuation of the desingularization of the vortex pairs. The existence of a non-uniform rotating vorticity distribution has been treated in [16, 37, 36]. In [40], Gómez-Serrano, Park and Shi have constructed stationary configurations of multi-layered patches with finite kinetic energy using Nash-Moser techniques. Some additional references can be found in [15, 24, 52, 55, 56].

Many of the results, mentioned above, apply not only to the planar Euler equations but also to other active scalar equations such as the generalized surface quasi-geostrophic equation or the quasi-geostrophic shallow-water equations, see [1, 8, 9, 11, 14, 16, 17, 23, 27, 31, 32, 33, 34, 35, 37, 38, 44, 45, 48, 50, 52, 53, 54, 56, 58, 64, 65].

We end this discussion by very recent new perspectives concerning the existence of quasi-periodic patches using the Nash-Moser scheme together with KAM theory, see [3, 39, 46, 47, 49, 57, 64].

Around stationary solutions on the rotating unit sphere

Looking for stationary solutions to (1.3) is equivalent to solving the following equation on the stream function

$$\nabla^{\perp}\Psi(\theta,\varphi)\cdot\nabla\Big(\Delta\Psi(\theta,\varphi)-2\widetilde{\gamma}\cos(\theta)\Big)=0. \tag{1.14}$$

If $\Psi(\theta,\varphi)=\Psi(\theta)$ is longitude independant, then it automatically solves (1.14) because in this case, $\nabla^{\perp}\Psi(\theta)$ is colinear to $\mathbf{e}_{\varphi}$ and $\nabla\Big(\Delta\Psi(\theta)-2\widetilde{\gamma}\cos(\theta)\Big)$ is colinear to $\mathbf{e}_{\theta}$. Such type of solutions are called *zonal solutions.* Also, one can easily check that any solution of the semilinear elliptic problem

$$\Delta\Psi(\theta,\varphi)-2\widetilde{\gamma}\cos(\theta)=F\big(\Psi(\theta,\varphi)\big), \tag{1.15}$$

with $F\in C^1(\mathbb{R},\mathbb{R})$, solves (1.14), but the converse is not true in general. Constantin and Germain [19] showed that any solution to (1.15) with $F'>-6$ must be zonal (modulo rotation) and stable in $H^2(\mathbb{S}^2)$ provided the additional constraint $F'<0$. Notice that the -6 corresponds to the second eigenvalues of the Laplace-Beltrami operator. The zonal Rossby-Haurwitz stream functions of degree $n\in\mathbb{N}$ are special stationary solutions in the form

$$\Psi_n(\theta)=\beta Y_n^0(\theta)+\frac{2\widetilde{\gamma}}{n(n+1)-2}\cos(\theta),\qquad \beta\in\mathbb{R}^*,$$

where Y_n^0 is the spherical harmonic. We refer the reader to [2] for an introduction to spherical harmonics. In [19], the authors also discussed the local and global bifurcation of non-zonal solutions to (1.15) from Rossby-Haurwitz waves. They also proved the stability in $H^2(\mathbb{S}^2)$ of Rossby-Haurwitz zonal solutions of degree 2 as well as the unstability in $H^2(\mathbb{S}^2)$ of more general non-zonal Rossby-Haurwitz type solutions. Very recently, the stability of the degree 2 Rossby-Haurwitz waves in $L^p(\mathbb{S}^2)$ spaces with $p\in(1,\infty)$ has been obtained by Cao-Wang-Zuo [12]. Recently, Nualart [63] proved the existence of non-zonal stationary solutions Gevrey-close to the zonal Rossby-Haurwitz stream functions of degree 2. The Lyapunov stability of zonal monotone vorticities in $L^p(\mathbb{S}^2)$ for $p\in(2,\infty)$ was discussed by Caprino-Marchioro [13]. We mention that stationary solutions to (1.3) correspond to traveling solutions of the non-rotating case (E_0). More generally, we have the following result, see [19, 63].

Lemma 1.1. *We consider two vorticities $\Omega,\widetilde{\Omega}$ related through*

$$\Omega(t,\theta,\varphi)=-2c\cos(\theta)+\widetilde{\Omega}(t,\theta,\varphi-ct),$$

with $c\in\mathbb{R}$, and associated to stream functions $\Psi,\widetilde{\Psi}$ and velocity fields $U,\widetilde{U}$:

$$\Psi(t,\theta,\varphi)=c\cos(\theta)+\widetilde{\Psi}(t,\theta,\varphi-ct),\qquad U(t,\theta,\varphi)=c\sin(\theta)\mathbf{e}_{\varphi}+\widetilde{U}(t,\theta,\varphi-ct).$$

Then, the following are equivalent

(i) (Ω,Ψ,U) *is a solution to* $(E_{\widetilde{\gamma}})$.

(ii) $(\widetilde{\Omega},\widetilde{\Psi},\widetilde{U})$ *is a solution to* $(E_{\widetilde{\gamma}+c})$.

For our later purposes, we shall prove that a longitude independent (absolute) vorticity generates a zonal (so stationary) flow. This is given by the following lemma, whose proof is postponed to Appendix A.4.

Lemma 1.2. *For any $\alpha\in\mathbb{R}$, we introduce the rotation of angle α around the z axis*

$$\mathcal{R}(\alpha)\triangleq\begin{pmatrix}\cos(\alpha) & -\sin(\alpha) & 0\\ \sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1\end{pmatrix}\in SO_3(\mathbb{R}).$$

Assume that

$$\forall\alpha\in\mathbb{R},\quad\forall\xi\in\mathbb{S}^2,\quad\Omega\big(\mathcal{R}(\alpha)\xi\big)=\Omega(\xi),$$

or equivalently

$$\forall\alpha\in\mathbb{R},\quad\forall\xi\in\mathbb{S}^2,\quad\overline{\Omega}\big(\mathcal{R}(\alpha)\xi\big)=\overline{\Omega}(\xi),$$

then

$$\forall\alpha\in\mathbb{R},\quad\forall\xi\in\mathbb{S}^2,\quad\Psi\big(\mathcal{R}(\alpha)\xi\big)=\Psi(\xi).$$

This means that the flow is zonal (and thus stationary).

Patch type solutions on the rotating 2-sphere

Our analysis is strongly motivated by previous numerical works concerning the existence of patch type solutions for the rotating 2-sphere. The pioneering results are due to Dritschel-Polvani [28, 29] where they considered the sphere at rest $(\widetilde{\gamma}=0)$ and found, numerically, vortex cap solutions with one and two interfaces. They also studied the numerical nonlinear stability. Later Kim [59] described the free boundary problem for patch type solutions in the rotating 2-sphere $(\widetilde{\gamma}\neq0)$ by using the stereographic projection. In [61], Kim, Sakajo and Sohn numerically observed the existence of vortex caps (only one interface) and vortex bands (two interfaces) [60]. They also showed the linear stability of those solutions.

1.3 Dynamics of vortex cap solutions

Here, we introduce the notion of vortex cap solutions on the sphere, which are the analogous to vortex patches in the plane. Then, we derive the fundamental contour dynamics equations, introduced in this work to track the evolution of the caps interfaces. Our formulation (1.21) is derived by following the ideas of [55, p. 174], implemented in the context of vortex patches, but adapted to fit with the non Euclidean geometry of the sphere.

Fix $M\in\mathbb{N}\setminus\{0,1\}$ and $(\omega_k)_{1\leqslant k\leqslant M}\in\mathbb{R}^M$ such that

$$\forall k\in[\![1,M-1]\!],\quad \omega_k\neq\omega_{k+1}. \tag{1.16}$$

Consider a partition of the unit sphere in the form

$$\mathbb{S}^2=\bigsqcup_{k=1}^{M}\mathscr{C}_k(0),$$

where for any $k\in[\![1,M-1]\!]$, the boundary $\Gamma_k(0)\triangleq\partial\mathscr{C}_k(0)\cap\partial\mathscr{C}_{k+1}(0)$ is diffeomorphic to a circle. Take an initial condition in the form

$$\overline{\Omega}(0,\cdot)=\sum_{k=1}^{M}\omega_k\mathbf{1}_{\mathscr{C}_k(0)}. \tag{1.17}$$

The Gauss constraint (1.5) requires the following additional condition

$$\sum_{k=1}^{M}\omega_k\sigma\big(\mathscr{C}_k(0)\big)=0. \tag{1.18}$$

Observe that, by virtue of (1.3)-(1.4), the absolute vorticity $\overline{\Omega}$ solves the nonlinear transport equation

$$\partial_t\overline{\Omega}+U\cdot\nabla\overline{\Omega}=0.$$

Since the singularity of the Green function in (1.6) is logarithmic, then, similarly to Yudovich's theory [69] in the planar case, one can expect to obtain existence and uniqueness of a global in time weak solution which is Lagrangian, namely

$$\forall t\geqslant 0,\quad \forall\xi\in\mathbb{S}^2,\quad \overline{\Omega}(t,\xi)=\overline{\Omega}\big(0,\Phi_t^{-1}(\xi)\big),$$

where

$$\forall\xi\in\mathbb{S}^2,\quad \Phi_t(\xi)=\xi+\int_0^t U\big(s,\Phi_s(\xi)\big)ds.$$

Applying this remark with the initial condition (1.17) gives the following structure of the solution at any later time $t\geqslant 0$

$$\overline{\Omega}(t,\cdot)=\sum_{k=1}^{M}\omega_k\mathbf{1}_{\mathscr{C}_k(t)},\qquad\text{with}\qquad \forall k\in[\![1,M]\!],\quad \mathscr{C}_k(t)\triangleq\Phi_t\big(\mathscr{C}_k(0)\big). \tag{1.19}$$

Since U is solenoidal, then the flow $t\mapsto\Phi_t$ is measure preserving

$$\forall k\in[\![1,M]\!],\quad \sigma\big(\mathscr{C}_k(t)\big)=\sigma\big(\mathscr{C}_k(0)\big).$$

Any solution in the form (1.19) satisfying the conditions (1.16) and (1.18) is called a *vortex cap solution.* From now on, we fix a $k\in[\![1,M-1]\!]$. Assume that the initial boundary $\Gamma_k(0)$ can be described as the zero level set of a certain C^1 regular function $\mathtt{h}_k:\mathbb{S}^2\to\mathbb{R}$, namely

$$\Gamma_k(0)=\big\{\xi\in\mathbb{S}^2\quad\text{s.t.}\quad \mathtt{h}_k(\xi)=0\big\}.$$

Let us consider the following quantity

$$\forall t\geqslant 0,\quad \forall\xi\in\mathbb{S}^2,\quad F_k(t,\xi)\triangleq\mathtt{h}_k\big(\Phi_t^{-1}(\xi)\big). \tag{1.20}$$

Then, by construction $F_k(t,\cdot)$ describes the boundary $\Gamma_k(t)\triangleq\partial\mathscr{C}_k(t)\cap\partial\mathscr{C}_{k+1}(t)$. More precisely,

$$\Gamma_k(t)=\big\{\xi\in\mathbb{S}^2\quad\text{s.t.}\quad F_k(t,\xi)=0\big\}.$$

Differentiating the relation (1.20) with respect to time yields

$$\forall t\geqslant 0,\quad \forall\xi\in\mathbb{S}^2,\quad \partial_tF_k\big(t,\Phi_t(\xi)\big)+U\big(t,\Phi_t(\xi)\big)\cdot\nabla F\big(t,\Phi_t(\xi)\big)=0.$$

Now, take a parametrization $z_k(t,\cdot):\mathbb{T}\to\Gamma_k(t)$. We have

$$\forall t\geqslant 0,\quad \forall x\in\mathbb{T},\quad F_k\big(t,z_k(t,x)\big)=0.$$

Differentiating the previous relation with respect to time implies

$$\forall t\geqslant 0,\quad \forall x\in\mathbb{T},\quad \partial_t F_k\big(t,z_k(t,x)\big)+\partial_t z_k(t,x)\cdot\nabla F_k\big(t,z_k(t,x)\big)=0.$$

Putting together the foregoing calculations, we deduce that

$$\Big[\partial_t z_k(t,x)-U\big(t,z_k(t,x)\big)\Big]\cdot\nabla F_k\big(t,z_k(t,x)\big)=0.$$

The above scalar products can be understood either as taken in the tangent space $T_{z_k(t,x)}\mathbb{S}^2\cong\mathbb{R}^2$ or in the classical Euclidean space $\mathbb{R}^3$, both are equivalent. Indeed, in the spherical coordinates the Euclidean metric writes

$$\mathbf{g}_{\mathbb{R}^3}(r,\theta,\varphi)\triangleq dr^2+r^2\big(d\theta^2+\sin^2(\theta)d\varphi^2\big).$$

Since the sphere is described by the equation $r=1$, then the induced metric of $\mathbf{g}_{\mathbb{R}^3}$ on $\mathbb{S}^2$ is indeed $\mathbf{g}_{\mathbb{S}^2}$ as defined in (1.1). On $T_{z_k(t,x)}\mathbb{S}^2$, the operator J acts as a rotation of $-\frac{\pi}{2}$. Consequently, since $\partial_x z_k(t,x)$ is tangential to $\Gamma_k(t)$ and contained in $T_{z_k(t,x)}\mathbb{S}^2$, then $J\partial_x z_k(t,x)$ is orthogonal to $\Gamma_k(t)$ and contained in $T_{z_k(t,x)}\mathbb{S}^2$. In addition, since $\Gamma_k(t)$ is a level set of $F_k(t,\cdot)$, then $\nabla F_k\big(t,z_k(t,x)\big)$ is also orthogonal to $\Gamma_k(t)$ and contained in $T_{z_k(t,x)}\mathbb{S}^2$. We deduce that $J\partial_x z_k(t,x)$ and $\nabla F_k\big(t,z_k(t,x)\big)$ are proportional, which leads to

$$\forall t\geqslant 0,\quad \forall x\in\mathbb{T},\quad \Big[\partial_t z_k(t,x)-U\big(t,z_k(t,x)\big)\Big]\cdot\big(J\partial_x z_k(t,x)\big)=0.$$

But, using that $J^T=J^{-1}=-J$ and $\nabla^\perp=J\nabla$, we obtain

$$\begin{aligned}U\big(t,z_k(t,x)\big)\cdot\big(J\partial_x z_k(t,x)\big)&=\nabla^\perp\Psi\big(t,z_k(t,x)\big)\cdot\big(J\partial_x z_k(t,x)\big)\\&=-\Big(J\nabla^\perp\Psi\big(t,z_k(t,x)\big)\Big)\cdot\partial_x z_k(t,x)\\&=\nabla\Psi\big(t,z_k(t,x)\big)\cdot\partial_x z_k(t,x)\\&=\partial_x\Big(\Psi\big(t,z_k(t,x)\big)\Big).\end{aligned}$$

Hence the contour dynamics equations for the vortex cap solutions are

$$\forall k\in[\![1,M-1]\!],\quad \forall t\geqslant 0,\quad \forall x\in\mathbb{T},\quad \partial_t z_k(t,x)\cdot\big(J\partial_x z_k(t,x)\big)=\partial_x\Big(\Psi\big(t,z_k(t,x)\big)\Big),\tag{1.21}$$

which are comparable to (1.11).

1.4 Main results

Let us now present our main results. First observe that Lemma 1.2 implies that any spherical vortex cap in the form

$$\overline{\Omega}(\theta)=\omega_1\mathbf{1}_{0<\theta<\theta_1}+\omega_2\mathbf{1}_{\theta_1\leqslant\theta<\theta_2}+\ldots+\omega_{M-1}\mathbf{1}_{\theta_{M-2}\leqslant\theta<\theta_{M-1}}+\omega_M\mathbf{1}_{\theta_{M-1}\leqslant\theta<\pi},$$

with

$$M\in\mathbb{N}\setminus\{0,1\},\qquad \theta_0\triangleq 0<\theta_1<\ldots<\theta_{M-1}<\pi\triangleq\theta_M,\qquad \forall k\in[\![1,M-1]\!],\quad \omega_k\neq\omega_{k+1},$$

and supplemented by the Gauss condition

$$\sum_{k=1}^{M}\omega_k\big(\cos(\theta_{k-1})-\cos(\theta_k)\big)=0,$$

generates a stationary vortex cap solution to (1.3). In the sequel, we shall focus on the cases $M=2$ and $M=3$ (the latter will also be referred to as *vortex bands* or *vortex strips*). More precisely, we study the existence non-trivial periodic solutions living close to these structures. Due to the symmetry of the stationary vortex caps, we will look for rotating solutions around the vertical axis at uniform velocity c, that is

$$\forall t\geqslant 0,\quad \forall\xi\in\mathbb{S}^2,\quad \overline{\Omega}(t,\xi)=\overline{\Omega}\big(0,\mathcal{R}(ct)\xi\big),$$

and satisfy for some fixed $\mathbf{m}\in\mathbb{N}^*$ the $\mathbf{m}$-fold property

$$\forall\xi\in\mathbb{S}^2,\quad \overline{\Omega}\big(0,\mathcal{R}(\tfrac{2\pi}{\mathbf{m}})\xi\big)=\overline{\Omega}(0,\xi).$$

Our first result concerns the one-interface case $M=2$ and reads as follows.

Theorem 1.1. *Let $\widetilde{\gamma} \in \mathbb{R}$, $\mathbf{m} \in \mathbb{N}^*$ and $\theta_0 \in (0, \pi)$. Consider $\omega_N, \omega_S \in \mathbb{R}$ such that*

$$\frac{\omega_N + \omega_S}{\omega_N - \omega_S} = \cos(\theta_0).$$

There exists a branch of $\mathbf{m}$-fold uniformly rotating vortex cap solutions to (1.3) *with one interface bifurcating from*

$$\overline{\Omega}_{\mathrm{FC}}(\theta) \triangleq \omega_N \mathbf{1}_{0<\theta<\theta_0} + \omega_S \mathbf{1}_{\theta_0 \leqslant \theta < \pi},$$

at the velocity

$$c_{\mathbf{m}}(\widetilde{\gamma}) \triangleq \widetilde{\gamma} - (\omega_N - \omega_S)\frac{\mathbf{m} - 1}{2\mathbf{m}}.$$

Remark 1.1. *Let us make the following remarks.*

1. *The bifurcation points $c_{\mathbf{m}}(\widetilde{\gamma})$ correspond to a shift by the rotation speed $\widetilde{\gamma}$ of Burbea's frequencies* (1.12) *with a vorticity jump $[\![\overline{\Omega}]\!] \triangleq \omega_N - \omega_S = -1$. Hence, in the rotation frame of the sphere, the solutions bifurcate at Burbea's frequencies.*
2. *The local curve is parametrized in particular with $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto c^{\varepsilon}_{\mathbf{m}}(\widetilde{\gamma}) \in \mathbb{R}$ for some small $\varepsilon_0 > 0$. Denoting $\widetilde{\gamma}_{\mathbf{m}} \triangleq (\omega_N - \omega_S)\frac{\mathbf{m}-1}{2\mathbf{m}}$, we have $c^0_{\mathbf{m}}(\widetilde{\gamma}_{\mathbf{m}}) = 0$. Since the dependence of $c_{\mathbf{m}}(\widetilde{\gamma})$ in $\widetilde{\gamma}$ is affine, an application of the implicit function theorem allows to construct a curve $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \widetilde{\gamma}^{\varepsilon}_{\mathbf{m}} \in \mathbb{R}$ such that (up to reducing the size of ε_0)*
$$\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0), \quad c^{\varepsilon}_{\mathbf{m}}(\widetilde{\gamma}^{\varepsilon}_{\mathbf{m}}) = 0.$$
This means that we can construct a branch of non-trivial $\mathbf{m}$-fold solutions which are stationary in the geocentric frame. One could also obtain these solutions by implementing bifurcation theory with the parameter $\widetilde{\gamma}$.
3. *The bifurcation analysis is performed in Hölder spaces, but similarly to the planar case, we expect our solutions to be analytic.*

Next, we shall present our second result dealing with two interfaces ($M = 3$). In this case, the computations in the spectral study are much more involved. Our result is the following.

Theorem 1.2. *Let $\widetilde{\gamma} \in \mathbb{R}$ and $0 < \theta_1 < \theta_2 < \pi$. Fix $\omega_N, \omega_C, \omega_S \in \mathbb{R}$ such that*

$$\omega_N + \omega_S = (\omega_N - \omega_C)\cos(\theta_1) + (\omega_C - \omega_S)\cos(\theta_2). \tag{1.22}$$

Consider the following non-degeneracy conditions

1. $\omega_S \cos^2\left(\frac{\theta_1}{2}\right) + \omega_N \sin^2\left(\frac{\theta_2}{2}\right) \neq 0$;
2. $\omega_S \cos^2\left(\frac{\theta_1}{2}\right) + \omega_N \sin^2\left(\frac{\theta_2}{2}\right) = 0$ *supplemented by one of the following properties*

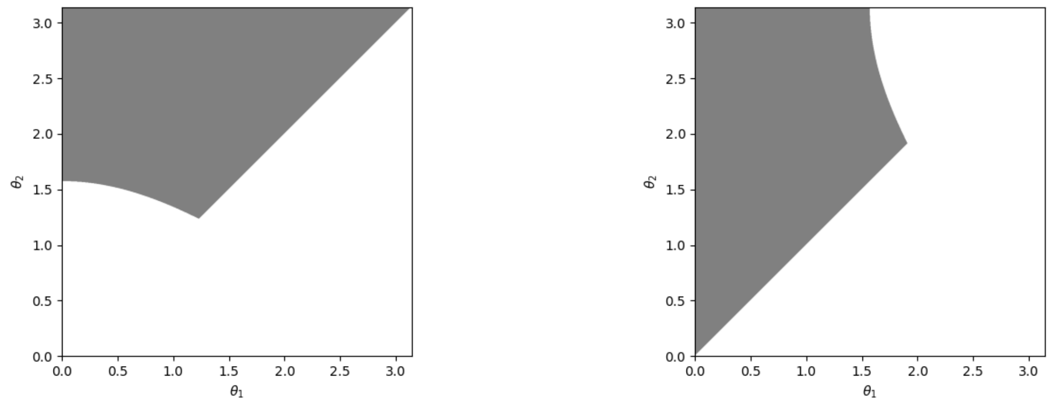
(H1+) $\omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0,$
(H2+) $\omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0$ and $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right),$
(H3+) $\omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0,$
(H4+) $\omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0$ and $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$
(H1−) $\omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0,$
(H2−) $\omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0$ and $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$
(H3−) $\omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0,$
(H4−) $\omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0$ and $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right).$

Let $\kappa \in \{+, -\}$. Assume that either the condition 1 holds or the condition 2 supplemented with $(\mathbf{Hk}\kappa)$ for some $k \in [\![1, 4]\!]$ hold. There exists $N(\theta_1, \theta_2) \triangleq N(\theta_1, \theta_2, \omega_N, \omega_S, \omega_C) \in \mathbb{N}^*$ such that for any $\mathbf{m} \in \mathbb{N}^*$ with $\mathbf{m} \geqslant N(\theta_1, \theta_2)$, there exists a branch of $\mathbf{m}$-fold uniformly rotating vortex strips for (1.3) *bifurcating from*

$$\overline{\Omega}_{\mathrm{FC2}}(\theta) \triangleq \omega_N \mathbf{1}_{0<\theta<\theta_1} + \omega_C \mathbf{1}_{\theta_1 \leqslant \theta < \theta_2} + \omega_S \mathbf{1}_{\theta_2 \leqslant \theta < \pi},$$

at the velocity

$$c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma}, \theta_1, \theta_2) \triangleq \widetilde{\gamma} + \frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N - \omega_S}{4\mathbf{m}}$$
$$+ \frac{\kappa}{4}\sqrt{\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_N + \omega_S - 2\omega_C}{\mathbf{m}}\right)^2 + \frac{1}{\mathbf{m}^2}(\omega_N - \omega_C)(\omega_C - \omega_S)\tan^{2\mathbf{m}}\left(\tfrac{\theta_1}{2}\right)\cot^{2\mathbf{m}}\left(\tfrac{\theta_2}{2}\right)}.$$



(a) $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right)$.

(b) $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right)$.

Figure 2: Representation of admissible couples $(\theta_1, \theta_2) \in (0, \pi)^2$ with $\theta_1 < \theta_2$.

Remark 1.2. *Let us make the following remarks.*

1. *The non-degeneracy conditions are required to avoid spectral collisions between both spectral components* $(c_n^+)_{n \geqslant N(\theta_1,\theta_2)}$ *and* $(c_n^-)_{n \geqslant N(\theta_1,\theta_2)}$ *when applying bifurcation techniques. This situation does not appear in the planar case [25], where both part of the spectrum are well-separated.*

2. *A priori, in this case the bifurcation points of Theorem 1.2 are independent from the planar case* (1.13) *even if they globally have a similar structure.*

3. *Similarly to Remark 1.1-2, we can construct, by implicit function theorem, at most two curves of non-trivial* $\mathbf{m}$*-fold solutions which are stationary in the geocentric frame.*

4. *The cases with more interfaces* ($M \geqslant 4$) *seem much more involved to study.*

1.5 Organization of the paper

This work is organized as follows. In Section 2 we provide the proof of Theorem 1.1 showing the analytical existence of non-trivial vortex caps with one interface bifurcating from the trivial one. First, we characterize the existence of such solutions with the non-trivial roots of a nonlinear and nonlocal functional. Later, we give the spectral properties of such functional and conlude the proof of the first main result. In Section 3 we analyze the two-interfaces problem (Theorem 1.2), which is more delicate since one has to study a coupled nonlinear system. The spectral study is more complex in this case, and we can show the existence of non-trivial vortex strips with large $\mathbf{m}$-fold symmetries under non-degeneracy conditions. Finally, the Appendix gives the proof of some technical lemmas and states the Crandall-Rabinowitz theorem.

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2 The one–interface case

This section is devoted to the proof of Theorem 1.1 dealing with the case of one interface ($M = 2$). We first discuss the stationary flat cap solution. Then, by choosing a suitable ansatz, we can rewrite the vortex cap equation (1.21) which leads to reformulate our problem in looking for the zeros of a nonlinear and nonlocal functional, see (2.4). Finally, we implement bifurcation arguments in order to show the existence of non-trivial roots of such functional. Hence, the proof relies on checking all the hypothesis of Crandall-Rabinowitz Theorem, see Appendix A.3.

2.1 Equation of interest

First, we shall discuss some properties of the spherical stationary solutions with one interface.

Lemma 2.1. *Let $\theta_0 \in (0,\pi)$. For any $\omega_N, \omega_S \in \mathbb{R}$ such that*

$$\frac{\omega_N + \omega_S}{\omega_N - \omega_S} = \cos(\theta_0), \tag{2.1}$$

the following function describing the flat cap (FC)

$$\overline{\Omega}_{\text{FC}}(\theta) \triangleq \omega_N \mathbf{1}_{0<\theta<\theta_0} + \omega_S \mathbf{1}_{\theta_0 \leqslant \theta < \pi},$$

is a stationary solution to Euler equations. In addition,

$$\partial_\theta \Psi_{\text{FC}}(\theta_0) = \left(\frac{\omega_N - \omega_S}{2} - \widetilde{\gamma}\right)\sin(\theta_0). \tag{2.2}$$

Proof. ► Observe that

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \overline{\Omega}_{\text{FC}}\big(\mathcal{R}(\alpha)\xi\big) = \overline{\Omega}_{\text{FC}}(\xi).$$

Hence, Lemma 1.2 applies and proves that this is a stationary solution.
► Notice that the constraint (2.1) is required since (1.5) and (1.2) imply

$$\begin{aligned} 0 = \int_{\mathbb{S}^2} \Omega_{\text{FC}}(t,\xi)d\sigma(\xi) &= \int_0^{2\pi}\int_0^{\pi} \Omega_{\text{FC}}(t,\theta,\varphi)\sin(\theta)d\theta d\varphi \\ &= 2\pi\left(\omega_N \int_0^{\theta_0}\sin(\theta)d\theta + \omega_S\int_{\theta_0}^{\pi}\sin(\theta)d\theta\right) \\ &= 2\pi\Big[\omega_N\big(1-\cos(\theta_0)\big) + \omega_S\big(1+\cos(\theta_0)\big)\Big]. \end{aligned}$$

► The potential velocity solves the elliptic equation

$$\Delta\Psi_{\text{FC}} = \overline{\Omega}_{\text{FC}} + 2\widetilde{\gamma}\cos(\theta), \qquad \text{i.e.} \qquad \partial_\theta\big[\sin(\theta)\partial_\theta\Psi_{\text{FC}}(\theta)\big] = \sin(\theta)\Big(\omega_N\mathbf{1}_{0<\theta<\theta_0} + \omega_S\mathbf{1}_{\theta_0\leqslant\theta<\pi}\Big) + \widetilde{\gamma}\sin(2\theta).$$

Integrating the previous relation gives

$$\partial_\theta\Psi_{\text{FC}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)}\big(1-\cos(\theta)\big) - \frac{\widetilde{\gamma}\cos(2\theta)}{2\sin(\theta)} + \frac{c}{\sin(\theta)}, & \text{if } \theta \in (0,\theta_0), \\ \frac{\omega_N}{\sin(\theta)}\big(1-\cos(\theta_0)\big) + \frac{\omega_S}{\sin(\theta)}\big(\cos(\theta_0)-\cos(\theta)\big) - \frac{\widetilde{\gamma}\cos(2\theta)}{2\sin(\theta)} + \frac{c}{\sin(\theta)}, & \text{if } \theta \in [\theta_0,\pi). \end{cases}$$

Since the flow is zonal, there is no velocity at the pole. Which implies that

$$\lim_{\theta\to 0^+}\partial_\theta\Psi_{\text{FC}}(\theta) = 0.$$

As a consequence, we must take the constant of integration c as follows

$$c \triangleq \frac{\widetilde{\gamma}}{2}.$$

Finally, using (2.1), we can write

$$\partial_\theta\Psi_{\text{FC}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)}\big(1-\cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in (0,\theta_0), \\ -\frac{\omega_S}{\sin(\theta)}\big(1+\cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in [\theta_0,\pi). \end{cases}$$

At $\theta = \theta_0$, using again (2.1), we find

$$\partial_\theta\Psi(\theta_0) = \frac{1}{2\sin(\theta_0)}\Big[\omega_N\big(1-\cos(\theta_0)\big) - \omega_S\big(1+\cos(\theta_0)\big)\Big] - \widetilde{\gamma}\sin(\theta_0) = \left(\frac{\omega_N-\omega_S}{2} - \widetilde{\gamma}\right)\sin(\theta_0).$$

The proof of Lemma 2.1 is now complete. □

Now, fix $\theta_0 \in (0,\pi)$ and let us consider a vortex cap solution close to $\overline{\Omega}_{\text{FC}}$ in the form

$$\overline{\Omega}(t,\theta,\varphi) = \omega_N\mathbf{1}_{0<\theta<\theta_0+f(t,\varphi)} + \omega_S\mathbf{1}_{\theta_0+f(t,\varphi)\leqslant\theta<\pi}, \qquad |f(t,\varphi)| \ll 1,$$

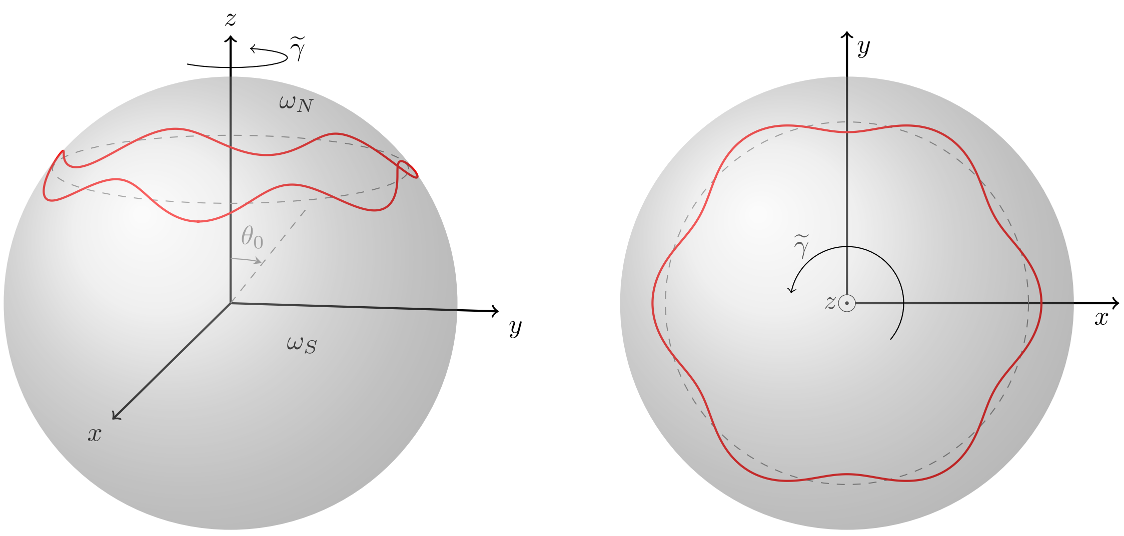

Figure 3: Representation of one interface (in red) vortex cap solutions with 6-fold symmetry.

with $\omega_N, \omega_S \in \mathbb{R}$ satisfying (2.1). The evolution is given by the dynamics of the interface which can be described (in the colatitude/longitude chart) through the following parametrization

$$z(t,\varphi) = C_1\big(\theta_0 + f(t,\varphi), \varphi\big) = \begin{pmatrix} \sin\big(\theta_0 + f(t,\varphi)\big)\cos(\varphi) \\ \sin\big(\theta_0 + f(t,\varphi)\big)\sin(\varphi) \\ \cos\big(\theta_0 + f(t,\varphi)\big) \end{pmatrix}.$$

Differentiating in time amounts to

$$\partial_t z(t,\varphi) = \partial_t f(t,\varphi) \begin{pmatrix} \cos\big(\theta_0 + f(t,\varphi)\big)\cos(\varphi) \\ \cos\big(\theta_0 + f(t,\varphi)\big)\sin(\varphi) \\ -\sin\big(\theta_0 + f(t,\varphi)\big) \end{pmatrix},$$

and the derivation in the longitude variable gives

$$\partial_\varphi z(t,\varphi) = \partial_\varphi f(t,\varphi) \begin{pmatrix} \cos\big(\theta_0 + f(t,\varphi)\big)\cos(\varphi) \\ \cos\big(\theta_0 + f(t,\varphi)\big)\sin(\varphi) \\ -\sin\big(\theta_0 + f(t,\varphi)\big) \end{pmatrix} + \begin{pmatrix} -\sin\big(\theta_0 + f(t,\varphi)\big)\sin(\varphi) \\ \sin\big(\theta_0 + f(t,\varphi)\big)\cos(\varphi) \\ 0 \end{pmatrix}.$$

The vector $J\partial_\varphi z(t,\varphi)$ can be obtained using the cross product

$$\begin{aligned} J\partial_\varphi z(t,\varphi) &= \partial_\varphi z(t,\varphi) \times z(t,\varphi) \\ &= \partial_\varphi f(t,\varphi) \begin{pmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\big(\theta_0 + f(t,\varphi)\big)\sin\big(\theta_0 + f(t,\varphi)\big)\cos(\varphi) \\ \cos\big(\theta_0 + f(t,\varphi)\big)\sin\big(\theta_0 + f(t,\varphi)\big)\sin(\varphi) \\ -\sin^2\big(\theta_0 + f(t,\varphi)\big) \end{pmatrix}. \end{aligned}$$

Therefore,

$$\partial_t z(t,\varphi) \cdot \big(J\partial_\varphi z(t,\varphi)\big) = \sin\big(\theta_0 + f(t,\varphi)\big)\partial_t f(t,\varphi).$$

Our ansatz corresponds to a vorticity in the form

$$\Omega(t,\theta,\varphi) = \omega_N \mathbf{1}_{0<\theta<\theta_0+f(t,\varphi)} + \omega_S \mathbf{1}_{\theta_0+f(t,\varphi)\leqslant\theta<\pi} + 2\widetilde{\gamma}\cos(\theta).$$

Consequently, according to (1.6), (1.7), (1.8) and (1.5), we have

$$\begin{aligned} \Psi\big(t, z(t,\varphi)\big) &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \log\Big(D\big(\theta_0 + f(t,\varphi), \theta', \varphi, \varphi'\big)\Big)\Omega(t,\theta',\varphi')\sin(\theta')d\theta' d\varphi' \\ &= \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(t,\varphi')} \log\Big(D\big(\theta_0 + f(t,\varphi), \theta', \varphi, \varphi'\big)\Big)\sin(\theta')d\theta' d\varphi' \\ &\quad + \frac{\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_0+f(t,\varphi')}^{\pi} \log\Big(D\big(\theta_0 + f(t,\varphi), \theta', \varphi, \varphi'\big)\Big)\sin(\theta')d\theta' d\varphi' \\ &\quad + \frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi} \log\Big(D\big(\theta_0 + f(t,\varphi), \theta', \varphi, \varphi'\big)\Big)\sin(2\theta')d\theta' d\varphi'. \end{aligned}$$

Remark that the unperturbed stream function can be written as follows

$$\begin{aligned}\Psi_{\rm FC}(\theta)&=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\log\Big(D\big(\theta,\theta',0,\varphi'\big)\Big)\Omega_{\rm FC}(\theta')\sin(\theta')d\theta' d\varphi'\\&=\frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0}\log\Big(D\big(\theta,\theta',0,\varphi'\big)\Big)\sin(\theta')d\theta' d\varphi'\\&\quad+\frac{\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_0}^{\pi}\log\Big(D\big(\theta,\theta',0,\varphi'\big)\Big)\sin(\theta')d\theta' d\varphi'\\&\quad+\frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi}\log\Big(D\big(\theta,\theta',0,\varphi'\big)\Big)\sin(2\theta')d\theta' d\varphi'.\end{aligned}$$

Thus, making appeal to Chasles' relation, we can write

$$\begin{aligned}\Psi\big(t,z(t,\varphi)\big)&=\Psi_{\rm FC}\big(\theta_0+f(t,\varphi)\big)+\Psi_p\{f\}\big(\theta_0+f(t,\varphi),\varphi\big)\triangleq\Psi\{f\}\big(\theta_0+f(t,\varphi),\varphi\big),\\\Psi_p\{f\}(\theta,\varphi)&\triangleq\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_0+f(t,\varphi')}\log\Big(D\big(\theta,\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta' d\varphi'.\end{aligned}$$

Therefore, the vortex cap equation (1.21) becomes

$$\partial_t f(t,\varphi)=\frac{\partial_\varphi\Big(\Psi_{\rm FC}\big(\theta_0+f(t,\varphi)\big)+\Psi_p\{f\}\big(\theta_0+f(t,\varphi),\varphi\big)\Big)}{\sin\big(\theta_0+f(t,\varphi)\big)}. \tag{2.3}$$

Looking for traveling solutions at speed $c\in\mathbb{R}$ leads to consider

$$f(t,\varphi)=f(\varphi-ct).$$

Inserting this into (2.3) gives

$$\mathscr{F}(c,f)(\varphi)\triangleq c\,\partial_\varphi f(\varphi)+\frac{\partial_\varphi\Big(\Psi_{\rm FC}\big(\theta_0+f(\varphi)\big)+\Psi_p\{f\}\big(\theta_0+f(\varphi),\varphi\big)\Big)}{\sin\big(\theta_0+f(\varphi)\big)}=0. \tag{2.4}$$

Observe that

$$\forall c\in\mathbb{R},\quad \mathscr{F}(c,0)=0.$$

That corresponds to a trivial line of roots of $\mathscr{F}$, corresponding to the flat cap stationary solution associated with the angle θ_0. In order to look for non-trivial roots, we shall use bifurcation arguments in terms of the Crandall-Rabinowitz Theorem. For that goal, we shall study the regularity of $\mathscr{F}$ and the spectral properties of its linearized operator.

2.2 Bifurcation study

In this section, we shall check the hypothesis of the Crandall-Rabinowitz Theorem for the functional $\mathscr{F}$ introduced in (2.4). Firstly, let us introduce the function spaces, in terms of Hölder regularity, that will be used in the bifurcation argument. Fix $\alpha\in(0,1)$, then the Hölder space $C^\alpha(\mathbb{T})$ consists in 2π-periodic functions $f:\mathbb{T}\to\mathbb{R}$ such that the following norm is finite:

$$\|f\|_{C^\alpha(\mathbb{T})}\triangleq\|f\|_{L^\infty(\mathbb{T})}+\sup_{\substack{(\varphi,\varphi')\in\mathbb{T}^2\\\varphi\neq\varphi'}}\frac{|f(\varphi)-f(\varphi')|}{|\varphi-\varphi'|^\alpha}.$$

The subspace $C^{1+\alpha}(\mathbb{T})$ of regular functions is associated with the following norm

$$\|f\|_{C^{1+\alpha}(\mathbb{T})}\triangleq\|f\|_{L^\infty(\mathbb{T})}+\|\partial_\varphi f\|_{C^\alpha(\mathbb{T})}.$$

Define also the following subspaces taking into account parity and symmetries

$$\begin{aligned}X_{\mathbf{m}}^{1+\alpha}&\triangleq\left\{f\in C^{1+\alpha}(\mathbb{T})\quad\text{s.t.}\quad\forall\varphi\in\mathbb{T},\,f(\varphi)=\sum_{n=1}^{\infty}f_n\cos(\mathbf{m}n\varphi),\quad f_n\in\mathbb{R}\right\},\\Y_{\mathbf{m}}^{\alpha}&\triangleq\left\{g\in C^{\alpha}(\mathbb{T})\quad\text{s.t.}\quad\forall\varphi\in\mathbb{T},\,g(\varphi)=\sum_{n=1}^{\infty}g_n\sin(\mathbf{m}n\varphi),\quad g_n\in\mathbb{R}\right\},\\B_{r,\mathbf{m}}^{1+\alpha}&\triangleq\Big\{f\in X_{\mathbf{m}}^{1+\alpha}\quad\text{s.t.}\quad\|f\|_{C^{1+\alpha}(\mathbb{T})}<r\Big\},\qquad r>0.\end{aligned}$$

The next proposition gathers the regularity properties for the functional $\mathscr{F}$ and gives the structure of its linearized operator at the flat cap.

Proposition 2.1. *Let $\alpha \in (0,1)$ and $\mathbf{m} \in \mathbb{N}^*$. There exists $r>0$ such that*

(i) The function $\mathscr{F} : \mathbb{R} \times B^{1+\alpha}_{r,\mathbf{m}} \to Y^\alpha_{\mathbf{m}}$ is well-defined and of class C^1.

(ii) The partial derivative $\partial_c d_f \mathscr{F} : \mathbb{R} \times B^{1+\alpha}_{r,\mathbf{m}} \to \mathcal{L}(X^{1+\alpha}_{\mathbf{m}}, Y^\alpha_{\mathbf{m}})$ exists and is continuous.

(iii) At the equilibrium $f=0$, the linearized operator admits the following Fourier representation

$$d_f\mathscr{F}(c,0)\left[\sum_{n=1}^{\infty} h_n \cos(\mathbf{m}n\varphi)\right] = \sum_{n=1}^{\infty} \mathbf{m}n\left[-c-(\omega_N-\omega_S)\frac{\mathbf{m}n-1}{2\mathbf{m}n}+\widetilde{\gamma}\right] h_n \sin(\mathbf{m}n\varphi). \tag{2.5}$$

In addition, if $c \neq \frac{\omega_N-\omega_S}{2} - \widetilde{\gamma}$, then the operator $d_f\mathscr{F}(c,0) : X^{1+\alpha}_{\mathbf{m}} \to Y^\alpha_{\mathbf{m}}$ is of Fredholm type with index zero.

Proof. **(i)** First, notice that the oddness and **m**-fold properties follow from the evenness and **m**-fold properties of f and changes of variables in the non-local part. Now, we need to check that $\mathscr{F}(c,f)$ belongs to $C^\alpha(\mathbb{T})$ provided that $f \in C^{1+\alpha}(\mathbb{T})$. Then, let us write $\mathscr{F}$ as

$$\mathscr{F}(c,f)(\varphi) = cf'(\varphi) + f'(\varphi)\frac{\big(\partial_\theta\Psi\{f\}\big)\big(\theta_0+f(\varphi),\varphi\big)}{\sin\big(\theta_0+f(\varphi)\big)} + \frac{\big(\partial_\varphi\Psi\{f\}\big)\big(\theta_0+f(\varphi),\varphi\big)}{\sin\big(\theta_0+f(\varphi)\big)}.$$

Notice that since $\theta_0 \notin \{0,\pi\}$ and $\|f\|_{C^{1+\alpha}(\mathbb{T})} < r$, hence by considering r small enough we find

$$\inf_{\varphi\in\mathbb{T}} \big|\sin\big(\theta_0+f(\varphi)\big)\big| \geqslant \delta_0 > 0, \qquad \delta_0 \triangleq \inf_{x\in[\theta_0-r,\theta_0+r]} |\sin(x)| \in (0,1). \tag{2.6}$$

Thus, in order to check that $\mathscr{F}$ is well-defined, it is enough to prove that

$$\big(\partial_\theta\Psi\{f\}\big)\big(\theta_0+f(\cdot),\cdot\big) \in C^\alpha(\mathbb{T}) \quad \text{and} \quad \big(\partial_\varphi\Psi\{f\}\big)\big(\theta_0+f(\cdot),\cdot\big) \in C^\alpha(\mathbb{T}). \tag{2.7}$$

Note that proving (2.7) uses the same techniques as the one used to show (2.9). Thus, we shall skip the details and only check that $f \mapsto (d_f\mathscr{F})$ is continuous using the expression in (2.4). Indeed, we can compute the Gateaux derivative of $\mathscr{F}$ and obtain

$$\begin{aligned} d_f\mathscr{F}(c,f)[h](\varphi) = {} & c\,h'(\varphi) - h(\varphi)\frac{\cos\big(\theta_0+f(\varphi)\big)}{\sin^2\big(\theta_0+f(\varphi)\big)}\partial_\varphi\Big(\Psi\{f\}\big(\theta_0+f(\varphi),\varphi\big)\Big) \\ & + \frac{1}{\sin\big(\theta_0+f(\varphi)\big)}\partial_\varphi\Big(h(\varphi)(\partial_\theta\Psi\{f\})\big(\theta_0+f(\varphi),\varphi\big)\Big) \\ & + \frac{1}{\sin\big(\theta_0+f(\varphi)\big)}\partial_\varphi\Big(\big(d_f\Psi\{f\}[h]\big)\big(\theta_0+f(\varphi),\varphi\big)\Big), \end{aligned}$$

with

$$\begin{aligned} \big(d_f\Psi\{f\}[h]\big)\big(\theta_0+f(\varphi),\varphi\big) &= \big(d_f\Psi_p\{f\}[h]\big)\big(\theta_0+f(\varphi),\varphi\big) \\ &= \frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi} h(\varphi')\log\Big(D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\Big)\sin\big(\theta_0+f(\varphi')\big)d\varphi', \end{aligned}$$

and

$$\begin{aligned} (\partial_\theta\Psi\{f\})(\theta,\varphi) = {} & \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\partial_\theta D(\theta,\theta',\varphi,\varphi')\sin(\theta')}{D(\theta,\theta',\varphi,\varphi')}d\theta' d\varphi' \\ & + \frac{\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_0+f(\varphi')}^{\pi} \frac{\partial_\theta D(\theta,\theta',\varphi,\varphi')\sin(\theta')}{D(\theta,\theta',\varphi,\varphi')}d\theta' d\varphi' \\ & + \frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial_\theta D(\theta,\theta',\varphi,\varphi')\sin(2\theta')}{D(\theta,\theta',\varphi,\varphi')}d\theta' d\varphi'. \end{aligned} \tag{2.8}$$

Notice that $d_f\mathscr{F}(c,f)$ is continuous in f provided that the functions

$$f \mapsto \partial_\varphi\Big((\partial_\theta\Psi)\{f\}\big(\theta_0+f(\varphi),\varphi\big)\Big) \quad \text{and} \quad f \mapsto \partial_\varphi\Big(d_f\Psi_p\{f\}[h]\big(\theta_0+f(\varphi),\varphi\big)\Big) \tag{2.9}$$

are continuous. Let us start with the first condition in (2.9) and, since the analysis is similar, we shall only give details for just one of the terms. Note that from (2.8), we can write

$$
\begin{aligned}
(\partial_\theta \Psi\{f\})\big(\theta_0+f(\varphi),\varphi\big) =& \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\partial_\theta D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\sin(\theta')}{D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)} d\theta' d\varphi' \\
&+ \frac{\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_0+f(\varphi')}^{\pi} \frac{\partial_\theta D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\sin(\theta')}{D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)} d\theta' d\varphi' \\
&+ \frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial_\theta D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\sin(2\theta')}{D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)} d\theta' d\varphi' \\
&\triangleq J_1\{f\}(\varphi)+J_2\{f\}(\varphi)+J_3\{f\}(\varphi).
\end{aligned}
$$

We focus on the first term J_1. Note that to check that the first condition in (2.9) is satisfied, we need to compute $\partial_\varphi J_1\{f\}$. However, let us simplify J_1 before differentiating. Observe that taking the derivative of (1.9) leads to

$$
\partial_\theta D(\theta,\theta',\varphi,\varphi') = 2\Big[\sin\left(\tfrac{\theta-\theta'}{2}\right)\cos\left(\tfrac{\theta-\theta'}{2}\right)+\cos(\theta)\sin(\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big], \tag{2.10}
$$

$$
\partial_{\theta'} D(\theta,\theta',\varphi,\varphi') = 2\Big[-\sin\left(\tfrac{\theta-\theta'}{2}\right)\cos\left(\tfrac{\theta-\theta'}{2}\right)+\sin(\theta)\cos(\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big]. \tag{2.11}
$$

Hence,

$$
\begin{aligned}
(\partial_\theta D+\partial_{\theta'}D)(\theta,\theta',\varphi,\varphi') &= 2\big[\cos(\theta)\sin(\theta')+\sin(\theta)\cos(\theta')\big]\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right) \\
&= 2\sin(\theta+\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right).
\end{aligned}
$$

Thus, adding and substracting $\partial_{\theta'}D$ appropriately and integrating by parts, we find

$$
\begin{aligned}
J_1\{f\}(\varphi) =& \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\sin^2\left(\frac{\varphi-\varphi'}{2}\right)\sin\big(\theta'+\theta_0+f(\varphi)\big)}{D(\theta_0+f(\varphi),\theta',\varphi,\varphi')}\sin(\theta')d\theta' d\varphi' \\
&- \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\partial_{\theta'} D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)}{D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)}\sin(\theta')d\theta' d\varphi' \\
=& \frac{\omega_N}{2\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\sin^2\left(\frac{\varphi-\varphi'}{2}\right)\sin\big(\theta'+\theta_0+f(\varphi)\big)}{D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)}\sin(\theta')d\theta' d\varphi' \\
&- \frac{\omega_N}{4\pi}\int_0^{2\pi}\log\Big(D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\Big)\sin\big(\theta_0+f(\varphi')\big)d\varphi' \\
&+ \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')}\log\Big(D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\Big)\cos(\theta')d\theta' d\varphi' \\
\triangleq& J_{1,1}\{f\}(\varphi)+J_{1,2}\{f\}(\varphi)+J_{1,3}\{f\}(\varphi).
\end{aligned}
$$

Let us work with $J_{1,1}\{f\}$, which is the most singular term. Now, we differentiate in φ and obtain

$$
\begin{aligned}
\partial_\varphi J_{1,1}\{f\}(\varphi) =& \frac{\omega_N}{2\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{f'(\varphi)\cos\big(\theta'+\theta_0+f(\varphi)\big)\sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D(\theta_0+f(\varphi),\theta',\varphi,\varphi')}\sin(\theta')d\theta' d\varphi' \\
&+ \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\sin\big(\theta'+\theta_0+f(\varphi)\big)\sin(\varphi-\varphi')}{D(\theta_0+f(\varphi),\theta',\varphi,\varphi')}\sin(\theta')d\theta' d\varphi' \\
&- \frac{\omega_N}{2\pi}\int_0^{2\pi}\int_0^{\theta_0+f(\varphi')} \frac{\sin\big(\theta'+\theta_0+f(\varphi)\big)\sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)}\partial_\varphi\Big(D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta' d\varphi' \\
\triangleq& J_{1,1,1}\{f\}(\varphi)+J_{1,1,2}\{f\}(\varphi)+J_{1,1,3}\{f\}(\varphi).
\end{aligned}
$$

Notice that the most singular integral is $J_{1,1,3}$. Thus, we only deal with that term. Remark that

$$
\begin{aligned}
\partial_\varphi\Big(D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)\Big) &= f'(\varphi)\partial_\theta D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big)+\partial_\varphi D\big(\theta_0+f(\varphi),\theta',\varphi,\varphi'\big) \\
&= 2f'(\varphi)\Big[\tfrac{1}{2}\sin\big(\theta_0+f(\varphi)-\theta'\big)+\cos\big(\theta_0+f(\varphi)\big)\sin(\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big] \\
&\quad+\sin\big(\theta_0+f(\varphi)\big)\sin(\theta')\sin(\varphi-\varphi').
\end{aligned}
$$

We can make the change of variables $\theta' = t(\theta_0 + f(\varphi'))$ to simplify the integral obtaining

$$J_{1,1,3}\{f\}(\varphi) = -\frac{\omega_N}{2\pi}\int_0^{2\pi}\int_0^1 \mathbb{K}\{f\}(t,\varphi,\varphi')dtd\varphi',$$

where

$$\begin{aligned}\mathbb{K}\{f\}(t,\varphi,\varphi') \triangleq{}& \frac{\sin\big((1+t)\theta_0 + f(\varphi) + tf(\varphi')\big)\sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2\big(\theta_0+f(\varphi), t(\theta_0+f(\varphi')),\varphi,\varphi'\big)}\sin\big(t(\theta_0+f(\varphi'))\big)\big(\theta_0+f(\varphi')\big)\\ &\times\Big\{2f'(\varphi)\Big[\tfrac{1}{2}\sin\big((1-t)\theta_0+f(\varphi)-tf(\varphi')\big)+\cos\big(\theta_0+f(\varphi)\big)\sin\big(t(\theta_0+f(\varphi'))\big)\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big]\\ &+\sin\big(\theta_0+f(\varphi)\big)\sin\big(t(\theta_0+f(\varphi))\big)\sin(\varphi-\varphi')\Big\}.\end{aligned}$$

Our goal is to check that $f\mapsto J_{1,1,3}\{f\}$ is continuous. For this aim, we take $f_1, f_2 \in B^{1+\alpha}_{r,m}$ and we estimate the difference at those points

$$J_{1,1,3}\{f_2\}(\varphi) - J_{1,1,3}\{f_1\}(\varphi).$$

In order to simplify, let us illustrate one of the terms since the order of singularity is the same at every term. For that, define

$$\tilde{J}_{1,1,3}\{f_1,f_2\}(\varphi) \triangleq \int_0^{2\pi}\int_0^1 \mathbb{K}\{f_1,f_2\}(t,\varphi,\varphi')dtd\varphi',$$

where

$$\begin{aligned}\mathbb{K}\{f_1,f_2\}(t,\varphi,\varphi') \triangleq{}& \frac{\sin\big((1+t)\theta_0+f_2(\varphi)+tf_2(\varphi')\big)\sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)}\sin^2\big(t(\theta_0+f_2(\varphi'))\big)\big(\theta_0+f_2(\varphi')\big)\sin(\varphi-\varphi')\\ &\times\Big[\sin\big(\theta_0+f_2(\varphi)\big)-\sin\big(\theta_0+f_1(\varphi)\big)\Big].\end{aligned}$$

To estimate the previous term in the Hölder space $C^\alpha(\mathbb{T})$, we use Proposition A.1 with kernel

$$K(\varphi,\varphi') \triangleq \int_0^1 \mathbb{K}\{f_1,f_2\}(t,\varphi,\varphi')dt.$$

Remark that the 1-Lipschitz property of the function sin implies

$$\Big|\sin\big(\theta_0+f_2(\varphi)\big)-\sin\big(\theta_0+f_1(\varphi)\big)\Big| \leqslant \|f_1-f_2\|_{L^\infty(\mathbb{T})}.$$

Now, we choose

$$\begin{cases} r<\frac{1}{2}|\theta_0-\frac{\pi}{2}|, & \text{if } \theta_0\neq\frac{\pi}{2},\\ r<\frac{\pi}{4}, & \text{if } \theta_0=\frac{\pi}{2}\end{cases}$$

and denote

$$\mathtt{m}_{\theta_0}(r) \triangleq \begin{cases}\theta_0-r, & \text{if } \theta_0\leqslant\frac{\pi}{2},\\ \theta_0+r, & \text{if } \theta_0>\frac{\pi}{2},\end{cases}\qquad \mathtt{M}_{\theta_0}(r) \triangleq \begin{cases}\theta_0-r, & \text{if } \theta_0>\frac{\pi}{2},\\ 1, & \text{if } \theta_0=\frac{\pi}{2},\\ \theta_0+r, & \text{if } \theta_0<\frac{\pi}{2}.\end{cases}$$

Notice that $\mathtt{m}_{\theta_0}(r)\in(0,\pi)$ and $\mathtt{M}_{\theta_0}(r)\in(0,\pi)$, thus, a convexity argument ensures that

$$\exists C_1,C_2>0,\quad \forall t\in[0,1],\quad C_1t\leqslant\sin\big(t\,\mathtt{m}_{\theta_0}(r)\big)\leqslant C_2t,\qquad C_1t\leqslant\sin\big(t\,\mathtt{M}_{\theta_0}(r)\big)\leqslant C_2t.$$

A direct estimation gives

$$\begin{aligned}\big|K(\varphi,\varphi')\big| &\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\frac{|\sin(\varphi-\varphi')|\sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)}\sin^2\big(t\,\mathtt{M}_{\theta_0}(r)\big)dt\\ &\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\frac{|\sin(\varphi-\varphi')|\sin^2\left(\frac{\varphi-\varphi'}{2}\right)t^2}{D^2\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)}dt.\end{aligned}$$

We need to control the denominator. For any $t\in[0,1]$, we have

$$\begin{aligned}D\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)&\geqslant 2\delta_0\Big[\sin^2\Big(\tfrac{(1-t)\theta_0+f_2(\varphi)-tf_2(\varphi')}{2}\Big)+\sin\big(t\,\mathtt{m}_{\theta_0}(r)\big)\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big]\\&\geqslant C\Big[\sin^2\Big(\tfrac{(1-t)\theta_0+f_2(\varphi)-tf_2(\varphi')}{2}\Big)+t\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big].\end{aligned}$$

For r small enough, we have

$$\big|(1-t)\theta_0+f_2(\varphi)-tf_2(\varphi')\big|\leqslant\theta_0+2\|f_2\|_{L^\infty(\mathbb{T})}\leqslant\theta_0+2r<\pi.$$

Hence, by concavity of the function sin on $(0,\frac{\pi}{2})$, we infer

$$\begin{aligned}\sin^2\Big(\tfrac{(1-t)\theta_0+f_2(\varphi)-tf_2(\varphi')}{2}\Big)&\geqslant C\big|\theta_0+f_2(\varphi)-t\big(\theta_0+f_2(\varphi')\big)\big|^2\\&\geqslant C\big|(1-t)\big(\theta_0+f_2(\varphi)\big)+t\big(f_2(\varphi)-f_2(\varphi')\big)\big|^2\\&\geqslant C\Big[(1-t)-t\|f_2\|_{\mathrm{Lip}}\Big|\sin\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big|\Big]^2\\&\geqslant C\Big[\tfrac{(1-t)^2}{2}-t^2\|f_2\|_{\mathrm{Lip}}^2\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big].\end{aligned}$$

To obtain the last estimate, we have used the following classical inequality

$$\forall(a,b)\in(\mathbb{R}_+)^2,\quad (a-b)^2\geqslant\tfrac{a^2}{2}-b^2.$$

Consequently,

$$\begin{aligned}D\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)&\geqslant C\Big[\tfrac{(1-t)^2}{2}-t^2\|f_2\|_{\mathrm{Lip}}^2\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)+t\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big]\\&\geqslant C\Big[\tfrac{(1-t)^2}{2}+\big(1-tr^2\big)t\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big].\end{aligned}$$

Therefore, for r small enough, we get for any $t\in[0,1]$,

$$D\big(\theta_0+f_2(\varphi),t(\theta_0+f_2(\varphi')),\varphi,\varphi'\big)\geqslant C\Big[(1-t)^2+t\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big].$$

Putting together the foregoing calculations yields

$$\big|K(\varphi,\varphi')\big|\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\frac{|\sin(\varphi-\varphi')|\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)t^2}{\Big[(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)\Big]^2}dt.$$

Now, observe that

$$\frac{t^2\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)}{(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)}\leqslant Ct$$

and then

$$\big|K(\varphi,\varphi')\big|\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\frac{t|\sin(\varphi-\varphi')|}{(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)}dt.$$

Next, use that for any $t\in[0,1]$,

$$\begin{aligned}\frac{t|\sin(\varphi-\varphi')|}{(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)}&\leqslant\frac{t|\sin(\varphi-\varphi')|}{\Big[(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)\Big]^{\frac12}\sqrt{t}\Big|\sin\Big(\frac{\varphi-\varphi'}{2}\Big)\Big|}\\&\leqslant\Big[(1-t)^2+t\sin^2\Big(\frac{\varphi-\varphi'}{2}\Big)\Big]^{-\frac12}\\&\leqslant C\Big[|1-t|+\sqrt{t}\Big|\sin\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big|\Big]^{-1}.\end{aligned}$$

The last inequality follows from the following classical estimate

$$\forall(a,b)\in(\mathbb{R}_+)^2,\quad \sqrt{a^2+b^2}\geqslant\tfrac{1}{\sqrt{2}}(a+b).$$

This implies in turn

$$
\begin{aligned}
|K(\varphi,\varphi')| &\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\Big[|1-t|+\sqrt{t}\,\Big|\sin\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big|\Big]^{-1}dt\\
&\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\int_0^1\Big|\sin\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big|^{-(1-\alpha)}t^{-\frac{1-\alpha}{2}}|1-t|^{-\alpha}dt\\
&\leqslant C\|f_1-f_2\|_{L^\infty(\mathbb{T})}\Big|\sin\Big(\tfrac{\varphi-\varphi'}{2}\Big)\Big|^{-(1-\alpha)},
\end{aligned}
$$

where we have used the classical interpolation estimate

$$
\forall\alpha\in(0,1),\quad\forall(a,b)\in(\mathbb{R}_+)^2,\quad(a+b)^{-1}\leqslant a^{-\alpha}b^{-(1-\alpha)}.
$$

The above computations allow to conclude that the hypothesis (A.1) is checked for the kernel K. Similarly, we can check that (A.2) is satisfied and hence Proposition A.1 can be applied obtaining the continuity in f. Let us continue with the second condition in (2.9). We can write

$$
\partial_\varphi\Big(d_f\Psi\{f\}[h]\big(\theta_0+f(t,\varphi),\varphi\big)\Big)=\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\frac{\partial_\varphi\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi')\big)\big]}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi')\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'.
$$

By adding and subtracting $\partial_{\varphi'}\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi')\big)\big]$ appropriately and integrating by parts, we find

$$
\begin{aligned}
&\partial_\varphi\Big(d_f\Psi\{f\}[h]\big(\theta_0+f(\varphi),\varphi\big)\Big)\\
&=\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\frac{\partial_\varphi\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]+\partial_{\varphi'}\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&\quad-\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\frac{\partial_{\varphi'}\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&=\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\frac{\partial_\varphi\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]+\partial_{\varphi'}\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&\quad+\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\log\Big(D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\Big)\partial_{\varphi'}\Big(\sin\big(\theta_0+f(\varphi')\big)h(\varphi')\Big)d\varphi'.
\end{aligned}
$$

Using the definition of D in (1.8), we infer

$$
\partial_\varphi D(\theta,\theta',\varphi,\varphi')=\sin(\theta)\sin(\theta')\sin(\varphi-\varphi')=-\partial_{\varphi'}D(\theta,\theta',\varphi,\varphi').
$$

Combined with (2.10)-(2.11), we get

$$
\begin{aligned}
&\partial_\varphi\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]+\partial_{\varphi'}\big[D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\big]\\
&=f'(\varphi)\partial_\theta D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)+f'(\varphi')\partial_{\theta'}D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\\
&=\sin\big(f(\varphi)-f(\varphi')\big)\big(f'(\varphi)-f'(\varphi')\big)+2f'(\varphi)\cos\big(\theta_0+f(\varphi)\big)\sin\big(\theta_0+f(\varphi')\big)\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)\\
&\quad+2f'(\varphi')\cos\big(\theta_0+f(\varphi')\big)\sin\big(\theta_0+f(\varphi)\big)\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big).
\end{aligned}
$$

As a consequence, we obtain

$$
\begin{aligned}
&\partial_\varphi\Big(d_f\Psi_p\{f\}[h]\big(\theta_0+f(t,\varphi),\varphi\big)\Big)\\
=&\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\frac{\sin\big(f(\varphi)-f(\varphi')\big)\big(f'(\varphi)-f'(\varphi')\big)}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&+\frac{\omega_N-\omega_S}{2\pi}\int_0^{2\pi}\frac{\cos\big(\theta_0+f(\varphi)\big)f'(\varphi)\sin\big(\theta_0+f(\varphi')\big)\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&+\frac{\omega_N-\omega_S}{2\pi}\int_0^{2\pi}\frac{\cos(\theta_0+f(\varphi'))f'(\varphi')\sin(\theta_0+f(\varphi))\sin^2\Big(\tfrac{\varphi-\varphi'}{2}\Big)}{D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f(\varphi')\big)h(\varphi')d\varphi'\\
&+\frac{\omega_N-\omega_S}{4\pi}\int_0^{2\pi}\log\Big(D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\Big)\partial_{\varphi'}\left(\sin\big(\theta_0+f(\varphi')\big)h(\varphi')\right)d\varphi'\\
\triangleq&\frac{\omega_N-\omega_S}{4\pi}\big[I_1\{f\}h(\varphi)+I_2\{f\}h(\varphi)+I_3\{f\}h(\varphi)+I_4\{f\}h(\varphi)\big].
\end{aligned}
$$

In the following, we show that $f\mapsto I_i\{f\}$ is continuous by showing that it has a modulus of continuity. Let us just give the details for I_1 and the others follow similarly. For that, take $f_1, f_2\in B^{1+\alpha}_{r,m}$, and estimate

$$\begin{aligned}I_1\{f_1\}h(\varphi)-I_1\{f_2\}h(\varphi)=&\int_0^{2\pi}K_1\{f_1,f_2\}(\varphi,\varphi')\big(f_1'(\varphi)-f_1'(\varphi')\big)h(\varphi')d\varphi'\\&+\int_0^{2\pi}K_2\{f_1,f_2\}(\varphi,\varphi')\big((f_1-f_2)'(\varphi)-(f_1-f_2)'(\varphi')\big)h(\varphi')d\varphi'\\&+\int_0^{2\pi}K_3\{f_1,f_2\}(\varphi,\varphi')\big(f_2'(\varphi)-f_2'(\varphi')\big)h(\varphi')d\varphi',\end{aligned}$$

where

$$\begin{aligned}K_1\{f_1,f_2\}(\varphi,\varphi')\triangleq&\frac{\sin\big(f_1(\varphi)-f_1(\varphi')\big)-\sin\big(f_2(\varphi)-f_2(\varphi')\big)}{D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f_1(\varphi')\big),\\K_2\{f_1,f_2\}(\varphi,\varphi')\triangleq&\frac{\sin\big(f_2(\varphi)-f_2(\varphi')\big)}{D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)}\sin\big(\theta_0+f_1(\varphi')\big),\\K_3\{f_1,f_2\}(\varphi,\varphi')\triangleq&\frac{\sin\big(f_2(\varphi)-f_2(\varphi')\big)}{D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)}\Big[\sin\big(\theta_0+f_1(\varphi')\big)-\sin\big(\theta_0+f_2(\varphi')\big)\Big]\\&+\frac{\sin\big(f_2(\varphi)-f_2(\varphi')\big)}{D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)D\big(\theta_0+f_2(\varphi),\theta_0+f_2(\varphi'),\varphi,\varphi'\big)}\\&\times\Big[D\big(\theta_0+f_2(\varphi),\theta_0+f_2(\varphi'),\varphi,\varphi'\big)-D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)\Big]\sin\big(\theta_0+f_2(\varphi')\big).\end{aligned}$$

Since the kernel of the integral operator has a non differentiable term, our purpose is to use Proposition A.2. For that, let us estimate each kernel K_i. First note that using (1.8) and (2.6), we get

$$\begin{aligned}D\big(\theta_0+f(\varphi),\theta_0+f(\varphi'),\varphi,\varphi'\big)\geqslant&\,2\sin\big(\theta_0+f(\varphi)\big)\sin\big(\theta_0+f(\varphi')\big)\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\\\geqslant&\,2\delta_0^2\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right).\end{aligned}\tag{2.12}$$

Using (2.12), it is easy to check that for K_1 we get

$$\begin{aligned}|K_1\{f_1,f_2\}(\varphi,\varphi')|\leqslant&\,C\big|(f_1-f_2)(\varphi)-(f_1-f_2)(\varphi')\big|\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-2}\\\leqslant&\,C\|f_1-f_2\|_{C^{1+\alpha}(\mathbb{T})}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1}\end{aligned}$$

and

$$|\partial_\varphi K_1\{f_1,f_2\}(\varphi,\varphi')|\leqslant C\|f_1-f_2\|_{C^{1+\alpha}(\mathbb{T})}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-2}.$$

Similarly for K_2 we obtain

$$\begin{aligned}|K_2\{f_1,f_2\}(\varphi,\varphi')|\leqslant&\,C\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1},\\|\partial_\varphi K_2\{f_1,f_2\}(\varphi,\varphi')|\leqslant&\,C\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-2}.\end{aligned}$$

Finally, we shall work with the last kernel K_3. Note that

$$\begin{aligned}&D\big(\theta_0+f_2(\varphi),\theta_0+f_2(\varphi'),\varphi,\varphi'\big)-D\big(\theta_0+f_1(\varphi),\theta_0+f_1(\varphi'),\varphi,\varphi'\big)\\&=2\sin^2\left(\tfrac{f_2(\varphi)-f_2(\varphi')}{2}\right)-2\sin^2\left(\tfrac{f_1(\varphi)-f_1(\varphi')}{2}\right)\\&\quad+2\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\Big\{\sin\big(\theta_0+f_2(\varphi)\big)\sin\big(\theta_0+f_2(\varphi')\big)-\sin\big(\theta_0+f_1(\varphi)\big)\sin\big(\theta_0+f_1(\varphi')\big)\Big\}\\&\leqslant2\left|\sin\left(\tfrac{f_2(\varphi)-f_2(\varphi')}{2}\right)\right|\left|\sin\left(\tfrac{f_2(\varphi)-f_2(\varphi')}{2}\right)-\sin\left(\tfrac{f_1(\varphi)-f_1(\varphi')}{2}\right)\right|\\&\quad+2\left|\sin\left(\tfrac{f_1(\varphi)-f_1(\varphi')}{2}\right)\right|\left|\sin\left(\tfrac{f_2(\varphi)-f_2(\varphi')}{2}\right)-\sin\left(\tfrac{f_1(\varphi)-f_1(\varphi')}{2}\right)\right|\\&\quad+2\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\big|\sin\big(\theta_0+f_2(\varphi)\big)\big|\big|\sin\big(\theta_0+f_2(\varphi')\big)-\sin\big(\theta_0+f_1(\varphi')\big)\big|\\&\quad+2\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)\big|\sin\big(\theta_0+f_1(\varphi')\big)\big|\big|\sin\big(\theta_0+f_2(\varphi)\big)-\sin\big(\theta_0+f_1(\varphi)\big)\big|\\&\leqslant C\|f_2-f_1\|_{C^{1+\alpha}(\mathbb{T})}\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right).\end{aligned}$$

To get the last estimate, we have used the 1-Lipschitz property of the function sin together with (A.5) and

$$\begin{aligned}\forall k\in\{1,2\},\quad \left|\sin\left(\tfrac{f_k(\varphi)-f_k(\varphi')}{2}\right)\right| &\leqslant |f_k(\varphi)-f_k(\varphi')|\\ &\leqslant \|f_k\|_{C^{1+\alpha}(\mathbb{T})}|\varphi-\varphi'|\\ &\leqslant Cr\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|.\end{aligned}$$

Hence

$$|K_3\{f_1,f_2\}(\varphi,\varphi')|\leqslant C\|f_2-f_1\|_{C^{1+\alpha}(\mathbb{T})}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1}.$$

Then, differentiating we find

$$|\partial_\varphi K_3\{f_1,f_2\}(\varphi,\varphi')|\leqslant C\|f_2-f_1\|_{C^{1+\alpha}(\mathbb{T})}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-2}.$$

Hence Proposition A.2 implies

$$\left\|\big(I_1\{f_1\}-I_2\{f_2\}\big)[h]\right\|_{C^\alpha(\mathbb{T})}\leqslant C\|f_1-f_2\|_{C^{1+\alpha}(\mathbb{T})}\|h\|_{L^\infty(\mathbb{T})},$$

concluding that $f\mapsto I_1\{f\}$ is continuous.

(ii) It follows from

$$\partial_c d_f\mathscr{F}(c,f)[h]=\partial_\varphi h. \tag{2.13}$$

(iii) We assume now that $f=0$. Since $\Psi_p\{0\}=0$, we have

$$d_f\mathscr{F}(c,0)[h](\varphi)=c\,\partial_\varphi h(\varphi)+\frac{1}{\sin(\theta_0)}\partial_\varphi\Big(\partial_\theta\Psi_{\text{FC}}(\theta_0)h(\varphi)+\big(d_f\Psi_p\{0\}[h]\big)(\theta_0,\varphi)\Big).$$

From (2.2), we deduce

$$\frac{1}{\sin(\theta_0)}\partial_\varphi\Big(\partial_\theta\Psi_{\text{FC}}(\theta_0)h\Big)=\left(\tfrac{\omega_N-\omega_S}{2}-\widetilde{\gamma}\right)\partial_\varphi h=-\left(\tfrac{\omega_N-\omega_S}{2}-\widetilde{\gamma}\right)\sum_{n=1}^{\infty}\mathbf{m}nh_n\sin(\mathbf{m}n\varphi). \tag{2.14}$$

After straightforward simplifications and using Lemma A.1, we get

$$\begin{aligned}\big(d_f\Psi_p\{0\}[h]\big)(\theta_0,\varphi)&=\frac{\omega_N-\omega_S}{4\pi}\sin(\theta_0)\int_0^{2\pi}h(\varphi')\log\Big(1-\cos^2(\theta_0)-\sin^2(\theta_0)\cos(\varphi-\varphi')\Big)d\varphi'\\ &=\frac{\omega_N-\omega_S}{4\pi}\sin(\theta_0)\sum_{n=1}^{\infty}h_n\int_0^{2\pi}\log\Big(1-\cos^2(\theta_0)-\sin^2(\theta_0)\cos(\varphi')\Big)\cos\big(\mathbf{m}n(\varphi-\varphi')\big)d\varphi'\\ &=\frac{\omega_N-\omega_S}{2}\sin(\theta_0)\sum_{n=1}^{\infty}h_nI_{\mathbf{m}n}(\theta_0,\theta_0)\cos(\mathbf{m}n\varphi)\\ &=-\frac{\omega_N-\omega_S}{2}\sin(\theta_0)\sum_{n=1}^{\infty}\frac{h_n}{\mathbf{m}n}\cos(\mathbf{m}n\varphi).\end{aligned}$$

Therefore,

$$\frac{1}{\sin(\theta_0)}\partial_\varphi\Big(\big(d_f\Psi_p\{0\}[h]\big)(\theta_0,\varphi)\Big)=\frac{\omega_N-\omega_S}{2}\sum_{n=1}^{\infty}h_n\sin(\mathbf{m}n\varphi). \tag{2.15}$$

Introducing the classical 2π-periodic Hilbert transform $\mathcal{H}$ defined by

$$\mathcal{H}h(\varphi)\triangleq\frac{1}{2\pi}\int_0^{2\pi}h(\varphi')\cot\left(\tfrac{\varphi-\varphi'}{2}\right)d\varphi',$$

which acts on the cosine basis as

$$\forall n\in\mathbb{N}^*,\quad \mathcal{H}\cos(n\varphi)=\sin(n\varphi),$$

we have

$$\frac{1}{\sin(\theta_0)}\partial_\varphi\Big(\big(d_f\Psi_p\{0\}[h]\big)(\theta_0,\varphi)\Big)=\frac{\omega_N-\omega_S}{2}\mathcal{H}h(\varphi). \tag{2.16}$$

Combining (2.14), (2.15) and (2.16), we obtain the Fourier representation (2.5) or equivalently, the following structure for the linearized operator

$$d_f\mathscr{F}(c,0)=\Big(c+\frac{\omega_N-\omega_S}{2}-\widetilde{\gamma}\Big)\partial_\varphi+\frac{\omega_N-\omega_S}{2}\mathcal{H}. \tag{2.17}$$

Clearly, if $c\neq\widetilde{\gamma}-\frac{\omega_N-\omega_S}{2}$, the operator $\Big(c+\frac{\omega_N-\omega_S}{2}-\widetilde{\gamma}\Big)\partial_\varphi : X_{\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}$ is an isomorphism. We shall prove the compactness of the Hilbert transform in the Hölder spaces. For that, we come back to the integral expression which can be rewritten as

$$\mathcal{H}h(\varphi)=\frac{1}{2\pi}\int_0^{2\pi}K(\varphi,\varphi')\partial_{\varphi'}h(\varphi')d\varphi',\qquad K(\varphi,\varphi')\triangleq\log\Big(\Big|\sin\big(\tfrac{\varphi-\varphi'}{2}\big)\Big|\Big).$$

For any $\delta\in(0,1)$, we have

$$|K(\varphi,\varphi')|\lesssim\Big|\sin\big(\tfrac{\varphi-\varphi'}{2}\big)\Big|^{-\delta},\qquad|\partial_\varphi K(\varphi,\varphi')|\lesssim\Big|\sin\big(\tfrac{\varphi-\varphi'}{2}\big)\Big|^{-(1+\delta)}.$$

Thus, we can apply Lemma A.1 (with $\delta=1-\beta$) and get

$$\forall\beta\in(\alpha,1),\quad\|\mathcal{H}h\|_{C^\beta(\mathbb{T})}\lesssim\|\partial_\varphi h\|_{L^\infty(\mathbb{T})}\lesssim\|h\|_{C^{1+\alpha}(\mathbb{T})}.$$

Since, for $\beta\in(\alpha,1)$, the injection $C^\beta(\mathbb{T})\hookrightarrow C^\alpha(\mathbb{T})$ is compact, we deduce that the operator $\mathcal{H}:C^{1+\alpha}(\mathbb{T})\to C^\alpha(\mathbb{T})$ is compact. Thus, (2.17) together with [18, Cor. 5.9] implies, for $c\neq\widetilde{\gamma}-\frac{\omega_N-\omega_S}{2}$, the desired Fredholmness property. This proves Proposition 2.1. ☐

According to (2.5), the candidates for bifurcation points define the following singular set

$$\mathcal{S}_c\triangleq\Big\{c_{\mathbf{m}}(\widetilde{\gamma})\triangleq\widetilde{\gamma}-(\omega_N-\omega_S)\tfrac{\mathbf{m}-1}{2\mathbf{m}},\quad\mathbf{m}\in\mathbb{N}^*\Big\}. \tag{2.18}$$

Finally, in the following proposition, we gather all the remaining conditions required to apply the Crandall-Rabinowitz Theorem. Then, Theorem 1.1 follows immediately from this proposition.

Proposition 2.2. *Let $\alpha\in(0,1)$, $\widetilde{\gamma}\in\mathbb{R}$ and $\mathbf{m}\in\mathbb{N}^*$.*

(i) The linear operator $d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big):X_{\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}$ is of Fredholm type with index zero.

(ii) The kernel of $d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big)$ is one dimensional. More precisely,

$$\ker\Big(d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big)\Big)=\mathtt{span}\big(\varphi\mapsto\cos(\mathbf{m}\varphi)\big). \tag{2.19}$$

(iii) The transversality condition is satisfied, namely

$$\partial_c d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big)[\varphi\mapsto\cos(\mathbf{m}\varphi)]\notin\mathrm{Im}\Big(d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),\widetilde{\gamma},0\big)\Big). \tag{2.20}$$

Proof. **(i)** By construction (2.18), our bifurcation points satisfy

$$c_{\mathbf{m}}(\widetilde{\gamma})=\widetilde{\gamma}-(\omega_N-\omega_S)\frac{\mathbf{m}-1}{2\mathbf{m}}\neq\widetilde{\gamma}-\frac{\omega_N-\omega_S}{2}\cdot$$

Hence, Proposition 2.1-(iii) gives the desired Fredholmness property.

(ii) The sequence $\big(c_{\mathbf{m}n}(\widetilde{\gamma})\big)_{n\in\mathbb{N}^*}$ being strictly monotone, then (2.5) and (2.18) give that the kernel is one dimensional and generated by $\varphi\mapsto\cos(\mathbf{m}\varphi)$.

(iii) To prove the transversality condition, we first need to describe the range. For this aim, we introduce on $Y_{\mathbf{m}}^{\alpha}$ the scalar product

$$\Big(\sum_{n=1}^{\infty}a_n\sin(\mathbf{m}n\varphi)\Big|\sum_{n=1}^{\infty}b_n\sin(\mathbf{m}n\varphi)\Big)\triangleq\sum_{n=1}^{\infty}a_nb_n.$$

Now we claim that

$$\mathrm{Im}\Big(d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big)\Big)=\mathtt{span}^{\perp_{(\cdot|\cdot)}}\big(\varphi\mapsto\sin(\mathbf{m}\varphi)\big). \tag{2.21}$$

Indeed, the first inclusion is obvious from (2.5)-(2.18). The converse inclusion is obtained because the range is closed and of codimension 1, which results from the Fredholmness property with zero index and the one dimensional kernel condition. Now, it remains to check the transversality condition. In view of (2.13), we infer

$$\partial_c d_f\mathscr{F}\big(c_{\mathbf{m}}(\widetilde{\gamma}),0\big)[\cos(\mathbf{m}\varphi)]=-\mathbf{m}\sin(\mathbf{m}\varphi)\in\mathtt{span}\big(\varphi\mapsto\sin(\mathbf{m}\varphi)\big). \tag{2.22}$$

Combining (2.21) and (2.22), the condition (2.20) follows. This achieves the proof of Proposition 2.2. ☐

3 The two–interfaces case: vorticity bands

This section is devoted to the proof of Theorem 1.2 dealing with the case of two interfaces ($M = 3$). As before, we shall reformulate the problem with a suitable functional and implement bifurcation techniques. The computations are more involved due to the interactions between the boundaries which leads to a vectorial analysis.

3.1 Equations of interest

We start again by some remarks on the flat solution.

Lemma 3.1. *Let $0 < \theta_1 < \theta_2 < \pi$. For any $\omega_N, \omega_C, \omega_S \in \mathbb{R}$ such that*

$$\omega_N + \omega_S = (\omega_N - \omega_C)\cos(\theta_1) + (\omega_C - \omega_S)\cos(\theta_2), \tag{3.1}$$

the following function describing the flat cap (FC2)

$$\overline{\Omega}_{\text{FC2}}(\theta) \triangleq \omega_N \mathbf{1}_{0<\theta<\theta_1} + \omega_C \mathbf{1}_{\theta_1 \leqslant \theta < \theta_2} + \omega_S \mathbf{1}_{\theta_2 \leqslant \theta < \pi}$$

is a stationary solution to Euler equations. In addition,

$$\partial_\theta \Psi_{\text{FC2}}(\theta_1) = \omega_N \tan\left(\tfrac{\theta_1}{2}\right) - \widetilde{\gamma}\sin(\theta_1), \qquad \Psi_{\text{FC2}}(\theta_2) = -\omega_S \cot\left(\tfrac{\theta_2}{2}\right) - \widetilde{\gamma}\sin(\theta_2). \tag{3.2}$$

Proof. ▶ Observe that

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \overline{\Omega}_{\text{FC2}}\big(\mathcal{R}(\alpha)\xi\big) = \overline{\Omega}_{\text{FC2}}(\xi).$$

Hence, Lemma 1.2 applies and proves that this is a stationary solution.
▶ The constraint (3.1) follows again from (1.5) and (1.2), namely

$$\begin{aligned} 0 = \int_{\mathbb{S}^2} \Omega_{\text{FC2}}(t,\xi) d\sigma(\xi) &= \int_0^{2\pi}\int_0^{\pi} \Omega_{\text{FC2}}(t,\theta,\varphi)\sin(\theta) d\theta d\varphi \\ &= 2\pi \left(\omega_N \int_0^{\theta_1} \sin(\theta) d\theta + \omega_C \int_{\theta_1}^{\theta_2} \sin(\theta) d\theta + \omega_S \int_{\theta_2}^{\pi} \sin(\theta) d\theta \right) \\ &= 2\pi \Big[\omega_N \big(1 - \cos(\theta_1)\big) + \omega_C \big(\cos(\theta_1) - \cos(\theta_2)\big) + \omega_S \big(1 + \cos(\theta_2)\big) \Big]. \end{aligned}$$

▶ The potential velocity solves the elliptic equation

$$\Delta \Psi_{\text{FC2}} = \Omega_{\text{FC2}}, \qquad \text{i.e.} \qquad \partial_\theta \big[\sin(\theta)\partial_\theta \Psi_{\text{FC2}}(\theta)\big] = \sin(\theta)\Big(\omega_N \mathbf{1}_{0<\theta<\theta_1} + \omega_C \mathbf{1}_{\theta_1 \leqslant \theta < \theta_2} + \mathbf{1}_{\theta_2 \leqslant \theta < \pi}\Big) + \widetilde{\gamma}\sin(2\theta).$$

Integrating the previous relation and chosing the constant of integration as in Lemma 2.1 gives

$$\partial_\theta \Psi_{\text{FC2}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)}\big(1 - \cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in (0,\theta_1), \\ \frac{\omega_N}{\sin(\theta)}\big(1 - \cos(\theta_1)\big) + \frac{\omega_C}{\sin(\theta)}\big(\cos(\theta_1) - \cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in [\theta_1,\theta_2), \\ \frac{\omega_N}{\sin(\theta)}\big(1 - \cos(\theta_1)\big) + \frac{\omega_C}{\sin(\theta)}\big(\cos(\theta_1) - \cos(\theta_2)\big) + \frac{\omega_S}{\sin(\theta)}\big(\cos(\theta_2) - \cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in [\theta_2,\pi). \end{cases}$$

Finally, using (3.1), we can write

$$\partial_\theta \Psi_{\text{FC2}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)}\big(1 - \cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in (0,\theta_1), \\ \frac{\omega_C}{\sin(\theta)}\big(\cos(\theta_2) - \cos(\theta)\big) - \frac{\omega_S}{\sin(\theta)}\big(1 + \cos(\theta_2)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in [\theta_1,\theta_2), \\ -\frac{\omega_S}{\sin(\theta)}\big(1 + \cos(\theta)\big) - \widetilde{\gamma}\sin(\theta), & \text{if } \theta \in [\theta_2,\pi). \end{cases}$$

At $\theta = \theta_1$, we find

$$\begin{aligned} \partial_\theta \Psi_{\text{FC2}}(\theta_1) &= \frac{\omega_N}{\sin(\theta_1)}\big(1 - \cos(\theta_1)\big) - \widetilde{\gamma}\sin(\theta_1) \\ &= \omega_N \tan\left(\tfrac{\theta_1}{2}\right) - \widetilde{\gamma}\sin(\theta_1). \end{aligned}$$

At $\theta = \theta_2$, we find

$$\begin{aligned} \partial_\theta \Psi_{\text{FC2}}(\theta_2) &= -\frac{\omega_S}{\sin(\theta_2)}\big(1 + \cos(\theta_2)\big) - \widetilde{\gamma}\sin(\theta_2) \\ &= -\omega_S \cot\left(\tfrac{\theta_2}{2}\right) - \widetilde{\gamma}\sin(\theta_2). \end{aligned}$$

□

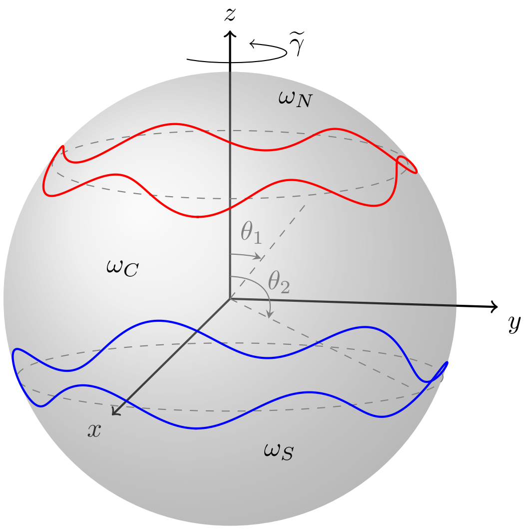

Figure 4: Representation of two interfaces (in red and blue) vortex cap solutions with 6-fold symmetry.

From now on, we fix

$$0<\theta_1<\theta_2<\pi, \tag{3.3}$$

and consider a vortex cap solution close to $\overline{\Omega}_{\mathrm{FC2}}$ in the form

$$\overline{\Omega}(t,\theta,\varphi)=\omega_N\mathbf{1}_{0<\theta<\theta_1+f_1(t,\varphi)}+\omega_C\mathbf{1}_{\theta_1+f_1(t,\varphi)\leqslant\theta<\theta_2+f_2(t,\varphi)}+\omega_S\mathbf{1}_{\theta_2+f_2(t,\varphi)\leqslant\theta<\pi},$$

with $\omega_N,\omega_C,\omega_S\in\mathbb{R}$ satisfying (3.1) and

$$\forall k\in\{1,2\},\quad |f_k(t,\varphi)|\ll 1.$$

For $k\in\{1,2\}$, the interface oscillating around $\theta=\theta_k$ can be parametrized by

$$z_k(t,\varphi)\triangleq\begin{pmatrix}\sin\big(\theta_k+f_k(t,\varphi)\big)\cos(\varphi)\\ \sin\big(\theta_k+f_k(t,\varphi)\big)\sin(\varphi)\\ \cos\big(\theta_k+f_k(t,\varphi)\big)\end{pmatrix}.$$

In view of (1.21), the parametrizations z_1 and z_2 must satisfy the following equations

$$\forall k\in\{1,2\},\quad \partial_t z_k(t,\varphi)\cdot\big(J\partial_\varphi z_k(t,\varphi)\big)=\partial_\varphi\Big(\Psi\big(t,z_k(t,\varphi)\big)\Big).$$

Proceeding as in Section 2.1, we obtain

$$\partial_t z_k(t,\varphi)\cdot\big(J\partial_\varphi z_k(t,\varphi)\big)=\sin\big(\theta_k+f_k(t,\varphi)\big)\partial_t f_k(t,\varphi).$$

Consequently, the unknowns f_1 and f_2 have to solve the following (coupled) system

$$\forall k\in\{1,2\},\quad \partial_t f_k(t,\varphi)=\frac{\partial_\varphi\Big(\Psi\big(t,z_k(t,\theta)\big)\Big)}{\sin\big(\theta_0+f_k(t,\varphi)\big)}\cdot$$

Now, the stream function writes

$$\begin{aligned}\Psi\big(t,z_k(t,\varphi)\big)=&\frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_1+f_1(t,\varphi')}\log\Big(D\big(\theta_k+f_k(t,\varphi),\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta'd\varphi'\\&+\frac{\omega_C}{4\pi}\int_0^{2\pi}\int_{\theta_1+f_1(t,\varphi')}^{\theta_2+f_2(t,\varphi')}\log\Big(D\big(\theta_k+f_k(t,\varphi),\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta'd\varphi'\\&+\frac{\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_2+f_2(t,\varphi')}^{\pi}\log\Big(D\big(\theta_k+f_k(t,\varphi),\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta'd\varphi'\\&+\frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi}\log\Big(D\big(\theta_k+f_k(t,\varphi),\theta',\varphi,\varphi'\big)\Big)\sin(2\theta')d\theta'd\varphi'.\end{aligned}$$

Remark that the unperturbed stream function can be written as follows

$$\begin{aligned}\Psi_{\text{FC2}}(\theta) = & \frac{\omega_N}{4\pi}\int_0^{2\pi}\int_0^{\theta_1}\log\Big(D(\theta,\theta',0,\varphi')\Big)\sin(\theta')d\theta' d\varphi'\\ & + \frac{\omega_C}{4\pi}\int_{\theta_1}^{\theta_2}\int_{\theta_0}^{\pi}\log\Big(D(\theta,\theta',0,\varphi')\Big)\sin(\theta')d\theta' d\varphi'\\ & + \frac{\omega_S}{4\pi}\int_{\theta_2}^{\pi}\int_{\theta_0}^{\pi}\log\Big(D(\theta,\theta',0,\varphi')\Big)\sin(\theta')d\theta' d\varphi'\\ & + \frac{\widetilde{\gamma}}{4\pi}\int_0^{2\pi}\int_0^{\pi}\log\Big(D(\theta,\theta',0,\varphi')\Big)\sin(2\theta')d\theta' d\varphi'.\end{aligned}$$

Making appeal to Chasles' relation, we can write

$$\begin{aligned}\Psi\big(t,z_k(t,\varphi)\big) & = \Psi_{\text{FC2}}\big(\theta_k+f_k(t,\varphi)\big)+\Psi_{p,2}\{f_1,f_2\}\big(\theta_k+f_k(t,\varphi),\varphi\big),\\ \Psi_{p,2}\{f_1,f_2\}(\theta,\varphi) & \triangleq \frac{\omega_N-\omega_C}{4\pi}\int_0^{2\pi}\int_{\theta_1}^{\theta_1+f_1(t,\varphi')}\log\Big(D(\theta,\theta',\varphi,\varphi')\Big)\sin(\theta')d\theta' d\varphi'\\ & \quad + \frac{\omega_C-\omega_S}{4\pi}\int_0^{2\pi}\int_{\theta_2}^{\theta_2+f_2(t,\varphi')}\log\Big(D(\theta,\theta',\varphi,\varphi')\Big)\sin(\theta')d\theta' d\varphi'.\end{aligned}$$

Therefore, the vortex cap equation (1.21) becomes

$$\forall k\in\{1,2\},\quad \partial_t f_k(t,\varphi) = \frac{\partial_\varphi\Big(\Psi_{\text{FC2}}\big(\theta_k+f_k(t,\varphi)\big)+\Psi_{p,2}\{f_1,f_2\}\big(\theta_k+f_k(t,\varphi),\varphi\big)\Big)}{\sin\big(\theta_k+f_k(t,\varphi)\big)}. \tag{3.4}$$

We look for traveling solutions at speed $c\in\mathbb{R}$

$$\forall k\in\{1,2\},\qquad f_k(t,\varphi)=f_k(\varphi-ct).$$

Thus, we shall solve

$$\mathscr{G}(c,f_1,f_2)=0,\qquad \mathscr{G}\triangleq(\mathscr{G}_1,\mathscr{G}_2),$$

where

$$\mathscr{G}_k(c,f_1,f_2)(\varphi)\triangleq c\,\partial_\varphi f_k(\varphi)+\frac{\partial_\varphi\Big(\Psi_{\text{FC2}}\big(\theta_k+f_k(\varphi)\big)+\Psi_{p,2}\{f_1,f_2\}\big(\theta_k+f_k(\varphi),\varphi\big)\Big)}{\sin\big(\theta_k+f_k(\varphi)\big)}.$$

Observe that

$$\forall c\in\mathbb{R},\quad \mathscr{G}(c,0,0)=0.$$

This leads again to implement bifurcation theory.

3.2 Spectral properties and proof of the main result

We check here the hypothesis of Crandall-Rabinowitz Theorem.

Proposition 3.1. *Let $\alpha\in(0,1)$ and $\mathbf{m}\in\mathbb{N}^*$. There exists $r>0$ such that*

(i) The function $\mathscr{G}:\mathbb{R}\times B_{r,\mathbf{m}}^{1+\alpha}\times B_{r,\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}\times Y_{\mathbf{m}}^{\alpha}$ is well-defined and of class C^1.

(ii) The partial derivative $\partial_c d_{(f_1,f_2)}\mathscr{G}:\mathbb{R}\times B_{r,\mathbf{m}}^{1+\alpha}\times B_{r,\mathbf{m}}^{1+\alpha}\to\mathcal{L}(X_{\mathbf{m}}^{1+\alpha}\times X_{\mathbf{m}}^{1+\alpha},Y_{\mathbf{m}}^{\alpha}\times Y_{\mathbf{m}}^{\alpha})$ exists and is continuous.

(iii) At the equilibrium $(f_1,f_2)=(0,0)$, the linearized operator admits the following Fourier representation

$$\begin{aligned} & d_{(f_1,f_2)}\mathscr{G}(c,0,0)\left[\sum_{n=1}^{\infty}h_n^{(1)}\cos(\mathbf{m}n\varphi),\sum_{n=1}^{\infty}h_n^{(2)}\cos(\mathbf{m}n\varphi)\right]\\ & = \sum_{n=1}^{\infty}\mathbf{m}nM_{\mathbf{m}n}(c,\theta_1,\theta_2)\begin{pmatrix}h_n^{(1)}\\ h_n^{(2)}\end{pmatrix}\sin(\mathbf{m}n\varphi),\end{aligned} \tag{3.5}$$

with

$$M_n(c,\theta_1,\theta_2)\triangleq\begin{pmatrix} -c+\frac{\omega_N-\omega_C}{2n}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}+\widetilde{\gamma} & \frac{\omega_C-\omega_S}{2n}\frac{\sin(\theta_2)}{\sin(\theta_1)}\tan^n\left(\frac{\theta_1}{2}\right)\cot^n\left(\frac{\theta_2}{2}\right)\\ \frac{\omega_N-\omega_C}{2n}\frac{\sin(\theta_1)}{\sin(\theta_2)}\tan^n\left(\frac{\theta_1}{2}\right)\cot^n\left(\frac{\theta_2}{2}\right) & -c+\frac{\omega_C-\omega_S}{2n}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\end{pmatrix}. \tag{3.6}$$

In addition, if $c\notin\left\{\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)},\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}\right\}$, then the operator $d_{(f_1,f_2)}\mathscr{G}(c,0,0):X_{\mathbf{m}}^{1+\alpha}\times X_{\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}\times Y_{\mathbf{m}}^{\alpha}$ is of Fredholm type with index zero.

Proof. **(i)** The proof is very close to Proposition 2.1-(i). Indeed, the functional involves terms corresponding to the self-interaction of each boundary with itself, which correspond to the one-interface analysis. It also involves new terms corresponding to the interaction between both boundaries, but which are non-singular (smooth kernels). Therefore, we omit the proof and just give the expression of the linearized operator. For $k \in \{1,2\}$,

$$\begin{aligned}
&d_{f_k}\mathscr{G}_k(c,f_1,f_2)[h_k](\varphi)\\
&= c\,\partial_\varphi h_k(\varphi) + h_k(\varphi)\frac{\cos\big(\theta_k+f_k(\varphi)\big)}{\sin^2\big(\theta_k+f_k(\varphi)\big)}\partial_\varphi\Big(\Psi_{\text{FC}}\big(\theta_k+f_k(\varphi)\big)+\Psi_{p,2}\{f_1,f_2\}\big(\theta_k+f_k(\varphi),\varphi\big)\Big)\\
&\quad+\frac{1}{\sin\big(\theta_k+f_k(\varphi)\big)}\partial_\varphi\Big(h_k(\varphi)\Big[\partial_\theta\Psi_{\text{FC}}\big(\theta_k+f_k(\varphi)\big)+\partial_\theta\Psi_{p,2}\{f_1,f_2\}\big(\theta_k+f_k(\varphi),\varphi\big)\Big]\Big)\\
&\quad+\frac{1}{\sin\big(\theta_k+f_k(\varphi)\big)}\partial_\varphi\Big(\big(d_{f_k}\Psi_{p,2}\{f_1,f_2\}[h]\big)\big(\theta_k+f_k(\varphi),\varphi\big)\Big)
\end{aligned}$$

and

$$d_{f_{3-k}}\mathscr{G}_k(c,f_1,f_2)[h_{3-k}](\varphi) = \frac{1}{\sin\big(\theta_k+f_k(\varphi)\big)}\partial_\varphi\Big(\big(d_{f_{3-k}}\Psi_{p,2}\{f_1,f_2\}[h_{3-k}]\big)\big(\theta_k+f_k(\varphi),\varphi\big)\Big).$$

If we denote $(\omega_N,\omega_C,\omega_S)=(\omega_1,\omega_2,\omega_3)$, then for $k\in\{1,2\}$, we have,

$$\big(d_{f_k}\Psi_{p,2}\{f_1,f_2\}[h_k]\big)(\theta,\varphi) = \frac{\omega_k-\omega_{k+1}}{4\pi}\int_0^{2\pi}h_k(\varphi')\log\Big(D\big(\theta,\theta_k+f_k(\varphi'),\varphi,\varphi'\big)\Big)\sin\big(\theta_k+f_k(\varphi')\big)d\varphi'.$$

(ii) Immediate since

$$\partial_c f_{(f_1,f_2)}\mathscr{G}(c,f_1,f_2)[h_1,h_2] = \begin{pmatrix}\partial_\varphi h_1 & 0\\ 0 & \partial_\varphi h_2\end{pmatrix}. \tag{3.7}$$

(iii) We assume now that $(f_1,f_2)=(0,0)$. Since $\Psi_{p,2}\{0,0\}=0$, we have

$$\begin{aligned}
d_{f_k}\mathscr{G}_k(c,0,0)[h_k](\varphi) &= c\,\partial_\varphi h_k(\varphi)+\frac{1}{\sin(\theta_k)}\partial_\varphi\Big(\partial_\theta\Psi_{\text{FC}}(\theta_k)h_k(\varphi)+\big(d_{f_k}\Psi_{p,2}\{0,0\}[h_k]\big)(\theta_k,\varphi)\Big),\\
d_{f_{3-k}}\mathscr{G}_k(c,0,0)[h_{3-k}](\varphi) &= \frac{1}{\sin(\theta_k)}\partial_\varphi\Big(\big(d_{f_{3-k}}\Psi_{p,2}\{0,0\}[h_{3-k}]\big)(\theta_k,\varphi)\Big).
\end{aligned}$$

From (3.2), we deduce

$$\begin{aligned}
\frac{1}{\sin(\theta_1)}\partial_\varphi\Big(\partial_\theta\Psi_{\text{FC}}(\theta_1)h_1\Big) &= \left(\frac{\omega_N\tan\left(\frac{\theta_1}{2}\right)}{\sin(\theta_1)}-\widetilde{\gamma}\right)\partial_\varphi h_1\\
&= \left(\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}-\widetilde{\gamma}\right)\partial_\varphi h_1\\
&= \left(\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\right)\sum_{n=1}^{\infty}\mathbf{m}nh_n^{(1)}\sin(\mathbf{m}n\varphi)
\end{aligned}\tag{3.8}$$

and

$$\begin{aligned}
\frac{1}{\sin(\theta_2)}\partial_\varphi\Big(\partial_\theta\Psi_{\text{FC}}(\theta_2)h_2\Big) &= -\left(\frac{\omega_S\cot\left(\frac{\theta_2}{2}\right)}{\sin(\theta_2)}+\widetilde{\gamma}\right)\partial_\varphi h_2\\
&= -\left(\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\right)\partial_\varphi h_2\\
&= \left(\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\right)\sum_{n=1}^{\infty}\mathbf{m}nh_n^{(2)}\sin(\mathbf{m}n\varphi).
\end{aligned}\tag{3.9}$$

After straightforward simplifications and using Lemma A.1, we get for $k,\ell\in\{1,2\}$,

$$\begin{aligned}
\big(d_{f_\ell}\Psi_{p,2}\{0,0\}[h_\ell]\big)(\theta_k,\varphi) &= \frac{\omega_\ell-\omega_{\ell+1}}{4\pi}\sin(\theta_\ell)\int_0^{2\pi}h_\ell(\varphi')\log\Big(D(\theta_k,\theta_\ell,\varphi,\varphi')\Big)d\varphi'\\
&= \frac{\omega_\ell-\omega_{\ell+1}}{2}\sin(\theta_\ell)\sum_{n=1}^{\infty}h_n^{(\ell)}I_{\mathbf{m}n}(\theta_k,\theta_\ell)\cos(\mathbf{m}n\varphi)\\
&= -\frac{\omega_\ell-\omega_{\ell+1}}{2}\sin(\theta_\ell)\sum_{n=1}^{\infty}\frac{h_n^{(\ell)}}{\mathbf{m}n}\tan^{\mathbf{m}n}\left(\tfrac{\min(\theta_k,\theta_\ell)}{2}\right)\cot^{\mathbf{m}n}\left(\tfrac{\max(\theta_k,\theta_\ell)}{2}\right)\cos(\mathbf{m}n\varphi).
\end{aligned}$$

Therefore,

$$\frac{\partial_\varphi\Big(\big(d_{f_\ell}\Psi_{p,2}\{0,0\}[h_\ell]\big)(\theta_k,\varphi)\Big)}{\sin(\theta_k)}=\frac{\omega_\ell-\omega_{\ell+1}}{2}\frac{\sin(\theta_\ell)}{\sin(\theta_k)}\sum_{n=1}^{\infty}h_n^{(\ell)}\tan^{\mathbf{m}n}\left(\tfrac{\min(\theta_k,\theta_\ell)}{2}\right)\cot^{\mathbf{m}n}\left(\tfrac{\max(\theta_k,\theta_\ell)}{2}\right)\sin(\mathbf{m}n\varphi).$$

Putting together the foregoing calculations, we get (3.5)-(3.6). Now, denoting

$$\mathcal{Q}(\varphi)\triangleq\log\big(1-\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\cos(\varphi)\big),$$

we have

$$\begin{aligned}
&d_{(f_1,f_2)}\mathscr{G}(c,0,0)=I+K,\\
&I\triangleq\begin{pmatrix}\left(c+\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}-\widetilde{\gamma}\right)\partial_\varphi & 0\\ 0 & \left(c-\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}-\widetilde{\gamma}\right)\partial_\varphi\end{pmatrix},\\
&K\triangleq\begin{pmatrix}\frac{\omega_N-\omega_C}{2}\mathcal{H} & \frac{\omega_C-\omega_S}{2}\frac{\sin(\theta_2)}{\sin(\theta_1)}\partial_\varphi\mathcal{Q}*\cdot\\ \frac{\omega_N-\omega_C}{2}\frac{\sin(\theta_1)}{\sin(\theta_2)}\partial_\varphi\mathcal{Q}*\cdot & \frac{\omega_C-\omega_S}{2}\mathcal{H}\end{pmatrix}.
\end{aligned}\tag{3.10}$$

If $c\notin\left\{\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)},\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}\right\}$, then $I:X_{\mathbf{m}}^{1+\alpha}\times X_{\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}\times Y_{\mathbf{m}}^{\alpha}$ is an isomorphism. We have already studied the compact property of the Hilbert transform in the proof of Proposition 2.2, so we are left with the anti-diagonal terms. Actually, the corresponding symbol decays exponentially fast in n, which implies that $\partial_\varphi\mathcal{Q}*\cdot$ is smoothing at every order (so a fortiori compact in the considered functional framework). Thus, the operator $K:X_{\mathbf{m}}^{1+\alpha}\times X_{\mathbf{m}}^{1+\alpha}\to Y_{\mathbf{m}}^{\alpha}\times Y_{\mathbf{m}}^{\alpha}$ is compact. We deduce the desired Fredholmness property. ☐

We shall now study the spectrum.

Lemma 3.2. *Let $\widetilde{\gamma}\in\mathbb{R}$. There exists $N(\theta_1,\theta_2)\triangleq N(\theta_1,\theta_2,\omega_N,\omega_S,\omega_C)\in\mathbb{N}^*$ such that for any $n\in\mathbb{N}^*$ with $n\geqslant N(\theta_1,\theta_2)$, there exist two velocities*

$$\begin{aligned}
c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)\triangleq{}&\widetilde{\gamma}+\frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_N-\omega_S}{4n}\\
&\pm\frac{1}{4}\sqrt{\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right)^2+\frac{1}{n^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)}
\end{aligned}\tag{3.11}$$

for which the matrix $M_n\big(c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)$ is singular. The sequences $\big(c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)\big)_{n\geqslant N(\theta_1,\theta_2)}$ are strictly monotone and

$$\mathbb{L}\triangleq\left\{\lim_{n\to\infty}c_n^{+}(\widetilde{\gamma},\theta_1,\theta_2)\,,\,\lim_{n\to\infty}c_n^{-}(\widetilde{\gamma},\theta_1,\theta_2)\right\}=\left\{\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\,,\,\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}\right\}.\tag{3.12}$$

Moreover,

1. *If*
$$\omega_S\cos^2\left(\tfrac{\theta_1}{2}\right)+\omega_N\sin^2\left(\tfrac{\theta_2}{2}\right)\neq0.\tag{3.13}$$
then $|\mathbb{L}|=2$ and the following equations have no solution
$$c_p^{+}(\widetilde{\gamma},\theta_1,\theta_2)=c_q^{-}(\widetilde{\gamma},\theta_1,\theta_2),\qquad p,q\geqslant N(\theta_1,\theta_2).\tag{3.14}$$

2. *If*
$$\omega_S\cos^2\left(\tfrac{\theta_1}{2}\right)+\omega_N\sin^2\left(\tfrac{\theta_2}{2}\right)=0.\tag{3.15}$$
then
$$|\mathbb{L}|=1,\qquad\omega_N+\omega_S=\omega_C\neq0,\qquad\omega_N\omega_S<0.$$
In particular, (3.11) *simplifies into*
$$c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_N-\omega_S}{4n}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\pm\frac{1}{4n}\sqrt{\omega_C^2-\omega_N\omega_S\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)}.$$
In addition, for any $\mathbf{m}\in\mathbb{N}$ with $\mathbf{m}\geqslant N(\theta_1,\theta_2)$, under one of the additional constraints

(**H1**+) $\omega_C > 0, \qquad \omega_N > 0, \qquad \omega_S < 0,$

(**H2**+) $\omega_C > 0, \qquad \omega_N < 0, \qquad \omega_S > 0 \qquad \text{and} \qquad 2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right),$

(**H3**+) $\omega_C < 0, \qquad \omega_N > 0, \qquad \omega_S < 0,$

(**H4**+) $\omega_C < 0, \qquad \omega_N < 0, \qquad \omega_S > 0 \qquad \text{and} \qquad 2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$

the following equations have no solution

$$c^+_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2) = c^-_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2), \qquad k \in \mathbb{N}^*.$$

And under one of the additional constraints

(**H1**−) $\omega_C > 0, \qquad \omega_N < 0, \qquad \omega_S > 0,$

(**H2**−) $\omega_C > 0, \qquad \omega_N > 0, \qquad \omega_S < 0 \qquad \text{and} \qquad 2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$

(**H3**−) $\omega_C < 0, \qquad \omega_N < 0, \qquad \omega_S > 0,$

(**H4**−) $\omega_C < 0, \qquad \omega_N > 0, \qquad \omega_S < 0 \qquad \text{and} \qquad 2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right),$

the following equations have no solution

$$c^+_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2) = c^-_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2), \qquad k \in \mathbb{N}^*.$$

This means that there is no $\mathbf{m}$*-fold spectral collision, i.e. the* $\mathbf{m}$*-fold spectrum is simple.*

Proof. From (3.6), we have that the determinant of $M_n(c,\theta_1,\theta_2)$ is

$$\begin{aligned}
&\det\big(M_n(c,\theta_1,\theta_2)\big) \triangleq c^2 - \beta_n(\widetilde{\gamma},\theta_1,\theta_2)c + \gamma_n(\widetilde{\gamma},\theta_1,\theta_2) \in \mathbb{R}_2[c],\\
&\beta_n(\widetilde{\gamma},\theta_1,\theta_2) \triangleq \frac{\omega_N-\omega_S}{2n} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} + 2\widetilde{\gamma}, \qquad (3.16)\\
&\gamma_n(\widetilde{\gamma},\theta_1,\theta_2) \triangleq \left(\frac{\omega_C-\omega_S}{2n} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} + \widetilde{\gamma}\right)\left(\frac{\omega_N-\omega_C}{2n} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} + \widetilde{\gamma}\right)\\
&\qquad\qquad - \frac{1}{4n^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right). \qquad (3.17)
\end{aligned}$$

The discriminant of the previous polynomial is

$$\begin{aligned}
\Delta_n(\theta_1,\theta_2) &\triangleq \beta_n^2(\widetilde{\gamma},\theta_1,\theta_2) - 4\gamma_n(\widetilde{\gamma},\theta_1,\theta_2)\\
&= \frac{1}{4}\left[\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_N+\omega_S-2\omega_C}{n}\right)^2 + \frac{1}{n^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)\right].
\end{aligned}$$

Notice that $\Delta_n(\theta_1,\theta_2)$ is independent of $\widetilde{\gamma}$. We shall now prove that

$$\exists\, N(\theta_1,\theta_2) \in \mathbb{N}^*, \quad \forall n \in \mathbb{N}, \quad n \geqslant N(\theta_1,\theta_2) \quad \Rightarrow \quad \Delta_n(\theta_1,\theta_2) > 0. \qquad (3.18)$$

Assuming that (3.18) is true, then we conclude that, for n large enough, we have two distinct real roots

$$\begin{aligned}
c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2) &\triangleq \frac{1}{2}\beta_n(\widetilde{\gamma},\theta_1,\theta_2) \pm \frac{1}{2}\sqrt{\Delta_n(\theta_1,\theta_2)}\\
&= \widetilde{\gamma} + \frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N-\omega_S}{4n}\\
&\pm \frac{1}{4}\sqrt{\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_N+\omega_S-2\omega_C}{n}\right)^2 + \frac{1}{n^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)}.
\end{aligned}$$

1. First assume that (3.13) holds. From the proof of Lemma A.1, we know that $\tan\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right) < 1$, then

$$\forall k \in \mathbb{N}, \quad \frac{1}{n^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right) \underset{n\to\infty}{=} O_{\theta_1,\theta_2}\left(\frac{1}{n^k}\right). \qquad (3.19)$$

Then,

$$\Delta_\infty(\theta_1,\theta_2) = \frac{1}{4}\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}\right)^2 > 0, \qquad (3.20)$$

and (3.18) is true. Factorizing, we can write for any n sufficiently large

$$c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_N-\omega_S}{4n}\pm\frac{1}{4}\left|\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right|\pm\mathtt{r}_n(\theta_1,\theta_2),$$

with

$$\mathtt{r}_n(\theta_1,\theta_2)\triangleq\frac{1}{4}\left|\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right|\left[\sqrt{1+\frac{(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2n}\left(\frac{\theta_1}{2}\right)\cot^{2n}\left(\frac{\theta_2}{2}\right)}{\left(\left[\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}\right]n-(\omega_N+\omega_S-2\omega_C)\right)^2}}-1\right].$$

Notice that (3.19) implies

$$\forall k\in\mathbb{N},\quad \mathtt{r}_n(\theta_1,\theta_2)\underset{n\to\infty}{=}O_{\theta_1,\theta_2}\left(\frac{1}{n^k}\right).\tag{3.21}$$

We have the following dichotomy.

- If $\omega_S\cos^2\left(\frac{\theta_1}{2}\right)+\omega_N\sin^2\left(\frac{\theta_2}{2}\right)>0$, then for n large enough we have

$$\left|\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right|=\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n},$$

  and therefore

$$\begin{cases}c_n^{+}(\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_C-\omega_S}{2n}+\mathtt{r}_n(\theta_1,\theta_2),\\ c_n^{-}(\theta_1,\theta_2)=\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_N-\omega_C}{2n}-\mathtt{r}_n(\theta_1,\theta_2).\end{cases}$$

  As a consequence,

$$\begin{cases}c_{n+1}^{+}(\widetilde{\gamma},\theta_1,\theta_2)-c_n^{+}(\widetilde{\gamma},\theta_1,\theta_2)\underset{n\to\infty}{=}\frac{\omega_S-\omega_C}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right),\\ c_{n+1}^{-}(\widetilde{\gamma},\theta_1,\theta_2)-c_n^{-}(\widetilde{\gamma},\theta_1,\theta_2)\underset{n\to\infty}{=}\frac{\omega_C-\omega_N}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right).\end{cases}$$

  Since $\omega_N\neq\omega_C$ and $\omega_C\neq\omega_S$, then we conclude the asymptotic (strict) monotonicity of $n\mapsto c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)$. In addition,

$$\lim_{n\to\infty}c_n^{+}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)},\qquad\text{and}\qquad\lim_{n\to\infty}c_n^{-}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}.$$

- If $\omega_S\cos^2\left(\frac{\theta_1}{2}\right)+\omega_N\sin^2\left(\frac{\theta_2}{2}\right)<0$, then for n large enough we have

$$\left|\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right|=\frac{\omega_N+\omega_S-2\omega_C}{n}-\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}$$

  and therefore

$$\begin{cases}c_n^{+}(\theta_1,\theta_2)=\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_N-\omega_C}{2n}-\mathtt{r}_n(\theta_1,\theta_2),\\ c_n^{-}(\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_C-\omega_S}{2n}+\mathtt{r}_n(\theta_1,\theta_2).\end{cases}$$

  As a consequence,

$$\begin{cases}c_{n+1}^{+}(\widetilde{\gamma},\theta_1,\theta_2)-c_n^{+}(\widetilde{\gamma},\theta_1,\theta_2)\underset{n\to\infty}{=}\frac{\omega_C-\omega_N}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right),\\ c_{n+1}^{-}(\widetilde{\gamma},\theta_1,\theta_2)-c_n^{-}(\widetilde{\gamma},\theta_1,\theta_2)\underset{n\to\infty}{=}\frac{\omega_S-\omega_C}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right).\end{cases}$$

  The monotonicity conclusion is still valid. In addition, in this case, the limits are exchanged

$$\lim_{n\to\infty}c_n^{+}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\qquad\text{and}\qquad\lim_{n\to\infty}c_n^{-}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}.$$

The condition (3.13) implies that the above limits are well-separated. Together with the strict monotonicity property, we conclude (3.14).

2. Conversely, we assume that (3.15) holds. This condition can also be written

$$\omega_S = -\omega_N \frac{1-\cos(\theta_2)}{1+\cos(\theta_1)}\cdot \tag{3.22}$$

Notice that the Gauss constraint (3.1) can be written as follows

$$\omega_C = \frac{\omega_N\big(1-\cos(\theta_1)\big)+\omega_S\big(1+\cos(\theta_2)\big)}{\cos(\theta_2)-\cos(\theta_1)}\cdot \tag{3.23}$$

Plugging (3.22) into (3.23) gives

$$\begin{aligned}\omega_C &= \omega_N \frac{\big(1-\cos(\theta_1)\big)\big(1+\cos(\theta_1)\big)-\big(1-\cos(\theta_2)\big)\big(1+\cos(\theta_2)\big)}{\big(\cos(\theta_2)-\cos(\theta_1)\big)\big(1+\cos(\theta_1)\big)}\\ &= \omega_N \frac{\cos^2(\theta_2)-\cos^2(\theta_1)}{\big(\cos(\theta_2)-\cos(\theta_1)\big)\big(1+\cos(\theta_1)\big)}\\ &= \omega_N \frac{\cos(\theta_1)+\cos(\theta_2)}{1+\cos(\theta_1)}\cdot \end{aligned}\tag{3.24}$$

From (3.22) and (3.24), we get

$$\omega_N+\omega_S = \frac{\omega_N}{1+\cos(\theta_1)}\Big[1+\cos(\theta_1)-\big(1-\cos(\theta_2)\big)\Big] = \omega_N\frac{\cos(\theta_1)+\cos(\theta_2)}{1+\cos(\theta_1)} = \omega_C.$$

This last expression implies that $\omega_N \neq 0$ (resp. $\omega_S \neq 0$) otherwise $\omega_S = \omega_C$ (resp. $\omega_N = \omega_C$) which is excluded by construction. We also deduce

$$\omega_N - \omega_C = -\omega_S, \qquad \omega_C - \omega_S = \omega_N.$$

Now, using (3.22), we infer

$$\omega_N\omega_S = -\omega_N^2 \frac{1-\cos(\theta_2)}{1+\cos(\theta_1)} < 0. \tag{3.25}$$

In particular $\omega_N \neq \omega_S$ and have opposite sign. Now, assume for the sake of contradiction that $\omega_C = 0$, i.e. $\omega_N = -\omega_S$. Combined with (3.15) and the fact that $\frac{\theta_1}{2}, \frac{\theta_2}{2} \in (0, \frac{\pi}{2})$, we deduce

$$\cos^2\left(\tfrac{\theta_1}{2}\right) = \sin^2\left(\tfrac{\theta_2}{2}\right), \qquad \text{i.e.} \qquad \theta_1 = \theta_2.$$

This enters in contradiction with (3.3). Thus,

$$\omega_C \neq 0 \qquad \text{and} \qquad \omega_N + \omega_S - 2\omega_C = -\omega_C \neq 0. \tag{3.26}$$

In this case, the discriminant becomes

$$\forall n \in \mathbb{N}^*, \quad \Delta_n(\theta_1,\theta_2) = \frac{1}{4n^2}\left[\omega_C^2 - \omega_N\omega_S \tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)\right] > 0.$$

This implies in particular (3.18). Factorizing, we can write

$$\begin{aligned}c_n^{\pm}(\widetilde{\gamma},\theta_1,\theta_2) &= \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N-\omega_S}{4n} \pm \frac{|\omega_C|}{4n} \pm \mathtt{r}_n(\theta_1,\theta_2)\\ &= \widetilde{\gamma} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N-\omega_S}{4n} \pm \frac{|\omega_C|}{4n} \pm \mathtt{r}_n(\theta_1,\theta_2),\end{aligned}$$

with

$$\mathtt{r}_n(\theta_1,\theta_2) \triangleq \frac{|\omega_C|}{4n}\left[\sqrt{1-\frac{\omega_N\omega_S}{\omega_C^2}\tan^{2n}\left(\tfrac{\theta_1}{2}\right)\cot^{2n}\left(\tfrac{\theta_2}{2}\right)}-1\right].$$

We have the following dichotomy.

- If $\omega_C > 0$, then $|\omega_C| = \omega_C = \omega_N + \omega_S$ and therefore

$$\begin{cases} c_n^{+}(\theta_1,\theta_2) = \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N}{2n} + \mathtt{r}_n(\theta_1,\theta_2) = \widetilde{\gamma} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{2n} + \mathtt{r}_n(\theta_1,\theta_2),\\ c_n^{-}(\theta_1,\theta_2) = \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_S}{2n} - \mathtt{r}_n(\theta_1,\theta_2) = \widetilde{\gamma} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_S}{2n} - \mathtt{r}_n(\theta_1,\theta_2).\end{cases}\tag{3.27}$$

As a consequence,

$$\begin{cases} c^+_{n+1}(\widetilde{\gamma},\theta_1,\theta_2) - c^+_n(\widetilde{\gamma},\theta_1,\theta_2) \underset{n\to\infty}{=} -\frac{\omega_N}{2n(n+1)} + O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right), \\ c^-_{n+1}(\widetilde{\gamma},\theta_1,\theta_2) - c^-_n(\widetilde{\gamma},\theta_1,\theta_2) \underset{n\to\infty}{=} \frac{\omega_S}{2n(n+1)} + O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right). \end{cases}$$

This is sufficient to conclude the asymptotic strict monotonicity. In addition, since $\omega_N\omega_S < 0$, then both sequences have the same monotonicity asymptotically. Nevertheless, in this case, both part of the spectrum accumulate at the same point

$$\lim_{n\to\infty} c^+_n(\widetilde{\gamma},\theta_1,\theta_2) = \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} = \widetilde{\gamma} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} = \lim_{n\to\infty} c^-_n(\widetilde{\gamma},\theta_1,\theta_2).$$

Therefore, one must avoid the spectral collisions by a more careful analysis.
▶ Let us first study the equation

$$c^+_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2) = c^-_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2), \qquad k \in \mathbb{N}^*.$$

According to (3.27), this equation is equivalent to

$$-k\big(\omega_N + \widetilde{\mathtt{r}}_{\mathbf{m}}\big) = \omega_S + \widetilde{\mathtt{r}}_{k\mathbf{m}}, \qquad \widetilde{\mathtt{r}}_n \triangleq 2n\mathtt{r}_n(\theta_1,\theta_2) > 0. \tag{3.28}$$

Observe that $n \mapsto \widetilde{\mathtt{r}}_n$ is asymptotically decreasing and satisfies (3.21).
➢ If $\omega_N > 0$, then the equation (3.28) can be written

$$0 = \omega_C + 2(k-1)\omega_N + k\widetilde{\mathtt{r}}_{\mathbf{m}} + \widetilde{\mathtt{r}}_{k\mathbf{m}}.$$

Each term in the right hand-side is non-negative and $\omega_C > 0$, then this equation has no solution.
➢ Assume now that $\omega_N < 0$. By virtue of (3.25), we have $\omega_S > 0$. According to (3.21) and (3.28), we can select $\mathbf{m}$ large enough to ensure

$$\widetilde{\mathtt{r}}_{\mathbf{m}} < |\omega_N|.$$

Added to the asymptotic decay property of $n \mapsto \widetilde{\mathtt{r}}_n$ we get

$$\forall k \in \mathbb{N}\setminus\{0,1\}, \quad -k\big(\omega_N + \widetilde{\mathtt{r}}_{\mathbf{m}}\big) \geqslant 2\big(|\omega_N| - \widetilde{\mathtt{r}}_{\mathbf{m}}\big) \qquad \text{and} \qquad \omega_S + \widetilde{\mathtt{r}}_{2\mathbf{m}} \geqslant \omega_S + \widetilde{\mathtt{r}}_{k\mathbf{m}}.$$

Hence, it suffices to impose

$$2\big(|\omega_N| - \widetilde{\mathtt{r}}_{\mathbf{m}}\big) > \omega_S + \widetilde{\mathtt{r}}_{2\mathbf{m}}, \qquad \text{i.e.} \qquad 2|\omega_N| > \omega_S + 2\widetilde{\mathtt{r}}_{\mathbf{m}} + \widetilde{\mathtt{r}}_{2\mathbf{m}},$$

so that the equations (3.28) admit no solution for any $k \in \mathbb{N}^*$ (recall that $c^+_{\mathbf{m}} \neq c^-_{\mathbf{m}}$). Using (3.21), we deduce that, up to taking $\mathbf{m}$ large enough, the following condition is sufficient

$$2|\omega_N| > \omega_S. \tag{3.29}$$

But, according to (3.15), the constraint (3.29) is equivalent to

$$2\cos^2\left(\tfrac{\theta_1}{2}\right) > \sin^2\left(\tfrac{\theta_2}{2}\right).$$

▶ Now, we turn to the study of the equation

$$c^+_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2) = c^-_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2), \qquad k \in \mathbb{N}^*.$$

Using again (3.27), this equation is equivalent to

$$\omega_N + \widetilde{\mathtt{r}}_{k\mathbf{m}} = -k\big(\omega_S + \widetilde{\mathtt{r}}_{\mathbf{m}}\big).$$

This is basically the same equation as (3.28) where ω_S and ω_N have been exchanged. So either $\omega_S > 0$ and there is no solution, or $\omega_S < 0$ and there is no solution provided that $\mathbf{m}$ is large enough and

$$2|\omega_S| > \omega_N, \qquad \text{i.e.} \qquad 2\sin^2\left(\tfrac{\theta_2}{2}\right) > \cos^2\left(\tfrac{\theta_1}{2}\right).$$

- If $\omega_C < 0$, then $|\omega_C| = -\omega_C = -\omega_N - \omega_S$ and therefore

$$\begin{cases} c^+_n(\theta_1,\theta_2) = \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_S}{2n} + \mathtt{r}_n(\theta_1,\theta_2) = \widetilde{\gamma} + \frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_S}{2n} + \mathtt{r}_n(\theta_1,\theta_2), \\ c^-_n(\theta_1,\theta_2) = \widetilde{\gamma} - \frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N}{2n} - \mathtt{r}_n(\theta_1,\theta_2) = \widetilde{\gamma} + \frac{\omega_N}{2\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{2n} - \mathtt{r}_n(\theta_1,\theta_2). \end{cases} \tag{3.30}$$

As a consequence,

$$\begin{cases} c^{+}_{n+1}(\widetilde{\gamma},\theta_1,\theta_2)-c^{+}_{n}(\widetilde{\gamma},\theta_1,\theta_2) \underset{n\to\infty}{=} \frac{\omega_S}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right), \\ c^{-}_{n+1}(\widetilde{\gamma},\theta_1,\theta_2)-c^{-}_{n}(\widetilde{\gamma},\theta_1,\theta_2) \underset{n\to\infty}{=} -\frac{\omega_N}{2n(n+1)}+O_{\theta_1,\theta_2}\left(\frac{1}{n^3}\right). \end{cases}$$

As before, we can conclude the asymptotic strict monotonicity with the same limit.
► According to (3.30), the equation

$$c^{+}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2)=c^{-}_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),\qquad k\in\mathbb{N}^*$$

is equivalent to

$$-k\big(-\omega_S+\widetilde{\mathtt{r}}_{\mathbf{m}}\big)=-\omega_N+\widetilde{\mathtt{r}}_{k\mathbf{m}}$$

which is (3.28) with (ω_N,ω_S) replaced by $(-\omega_S,-\omega_N)$. So there is no solution for either $\omega_S<0$ or $\omega_S>0$ and $2\sin^2\left(\frac{\theta_2}{2}\right)>\cos^2\left(\frac{\theta_1}{2}\right)$.
► The equation

$$c^{+}_{k\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2)=c^{-}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),\qquad k\in\mathbb{N}^*$$

is equivalent to

$$-\omega_S+\widetilde{\mathtt{r}}_{k\mathbf{m}}=-k\big(-\omega_N+\widetilde{\mathtt{r}}_{\mathbf{m}}\big)$$

which is (3.28) with (ω_N,ω_S) replaced by $(-\omega_N,-\omega_S)$. So there is no solution for either $\omega_N<0$ or $\omega_N>0$ and $2\cos^2\left(\frac{\theta_1}{2}\right)>\sin^2\left(\frac{\theta_2}{2}\right)$.

□

In the following proposition, we gather all the remaining conditions required to apply the Crandall-Rabinowitz Theorem. Then, Theorem 1.2 follows immediatly.

Proposition 3.2. *Let $\alpha\in(0,1)$, $\kappa\in\{+,-\}$ and $\mathbf{m}\in\mathbb{N}^*$ with $\mathbf{m}\geqslant N(\theta_1,\theta_2)$. Assume that (3.13) holds or assume that (3.15) with $(\mathbf{Hk}\kappa)$ for some $k\in[\![1,4]\!]$ holds.*

(i) The linear operator $d_{(f_1,f_2)}\mathscr{G}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)$ is of Fredholm type with index zero.

(ii) The kernel of $d_{(f_1,f_2)}\mathscr{G}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)$ is one dimensional. More precisely,

$$\ker\Big(d_{(f_1,f_2)}\mathscr{G}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)\Big)=\mathtt{span}\big(u_0\big),\tag{3.31}$$

with

$$u_0:\varphi\mapsto\begin{pmatrix}-c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\\ \frac{\omega_N-\omega_C}{2\mathbf{m}}\frac{\sin(\theta_1)}{\sin(\theta_2)}\tan^{\mathbf{m}}\left(\frac{\theta_1}{2}\right)\cot^{\mathbf{m}}\left(\frac{\theta_2}{2}\right)\end{pmatrix}\cos(\mathbf{m}\varphi).$$

(iii) The transversality condition is satisfied, namely

$$\partial_c d_{(f_1,f_2)}\mathscr{G}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)[u_0]\not\in\mathrm{Im}\Big(d_{(f_1,f_2)}\mathscr{G}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)\Big).\tag{3.32}$$

Proof. **(i)** The strict monotonicity of $n\mapsto c^{\kappa}_{n}(\widetilde{\gamma},\theta_1,\theta_2)$ gives $c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2)\not\in\mathbb{L}$, where $\mathbb{L}$ is defined in (3.12). Together with Proposition 3.1-(iii) this implies the desired Fredholmness property.

(ii) The non-degeneracy conditions imply that

$$\forall n\in\mathbb{N}\setminus\{0,1\},\quad\det\Big(M_{\mathbf{m}n}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\Big)\neq0.$$

Together with the fact that the matrix $M_{\mathbf{m}}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)$ is singular and non-zero, we obtain from (3.5) the desired result because

$$\ker\Big(M_{\mathbf{m}}\big(c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\Big)=\mathtt{span}\begin{pmatrix}-c^{\kappa}_{\mathbf{m}}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\\ \frac{\omega_C-\omega_N}{2\mathbf{m}}\frac{\sin(\theta_1)}{\sin(\theta_2)}\tan^{\mathbf{m}}\left(\frac{\theta_1}{2}\right)\cot^{\mathbf{m}}\left(\frac{\theta_2}{2}\right)\end{pmatrix}.$$

(iii) The next step is to describe the range. For this aim, we introduce on $Y^{\alpha}_{\mathbf{m}}\times Y^{\alpha}_{\mathbf{m}}$ the scalar product

$$\left(\left(\sum_{n=1}^{\infty}a_n\sin(\mathbf{m}n\varphi),\sum_{n=1}^{\infty}c_n\sin(\mathbf{m}n\varphi)\right)\Big|\left(\sum_{n=1}^{\infty}b_n\sin(\mathbf{m}n\varphi),\sum_{n=1}^{\infty}d_n\sin(\mathbf{m}n\varphi)\right)\right)_2\triangleq\sum_{n=1}^{\infty}a_nb_n+c_nd_n.$$

Now we claim that

$$\mathrm{Im}\Big(d_{(f_1,f_2)}\mathscr{G}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)\Big)=\mathtt{span}^{\perp_{(\cdot|\cdot)_2}}\left(g_0:\varphi\mapsto\begin{pmatrix}-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_S-\omega_C}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\\ \frac{\omega_S-\omega_C}{2\mathbf{m}}\frac{\sin(\theta_2)}{\sin(\theta_1)}\tan^{\mathbf{m}}\left(\frac{\theta_1}{2}\right)\cot^{\mathbf{m}}\left(\frac{\theta_2}{2}\right)\end{pmatrix}\sin(\mathbf{m}\varphi)\right).\tag{3.33}$$

Indeed, as in the proof of Proposition 2.2-(iii), we shall prove the first inclusion and the second one is obtained by the Fredholmness property and the previous point. First observe that

$$v_{\mathbf{m}}^{\kappa}(\theta_1,\theta_2)\triangleq\begin{pmatrix}-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\\ \frac{\omega_S-\omega_C}{2\mathbf{m}}\frac{\sin(\theta_2)}{\sin(\theta_1)}\tan^{\mathbf{m}}\left(\frac{\theta_1}{2}\right)\cot^{\mathbf{m}}\left(\frac{\theta_2}{2}\right)\end{pmatrix}\in\ker\Big(M_{\mathbf{m}}^{\top}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\Big).$$

Now, consider

$$g:\varphi\mapsto\sum_{n=1}^{\infty}\mathbf{m}nM_{\mathbf{m}n}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\begin{pmatrix}h_n^{(1)}\\ h_n^{(2)}\end{pmatrix}\sin(\mathbf{m}n\varphi)\in\mathrm{Im}\Big(d_{(f_1,f_2)}\mathscr{G}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)\Big).$$

Then, denoting $\cdot$ the usual scalar product on $\mathbb{R}^2$, we have

$$\begin{aligned}\big(g\,|\,g_0\big)_2&=\left(\mathbf{m}M_{\mathbf{m}}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\begin{pmatrix}h_1^{(1)}\\ h_1^{(2)}\end{pmatrix}\right)\cdot v_{\mathbf{m}}^{\kappa}(\theta_1,\theta_2)\\&=\mathbf{m}\begin{pmatrix}h_1^{(1)}\\ h_1^{(2)}\end{pmatrix}\cdot\Big(M_{\mathbf{m}}^{\top}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)v_{\mathbf{m}}^{\kappa}(\theta_1,\theta_2)\Big)\\&=0.\end{aligned}$$

This proves the claim. Now, we turn to the transversality condition. We shall prove that the following quantity does not vanish

$$\begin{aligned}&\Big(\partial_c d_{(f_1,f_2)}\mathscr{G}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)[u_0]\,|\,g_0\Big)_2\\&=\mathbf{m}\left[\left(-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\right)^2-\frac{1}{4\mathbf{m}^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2\mathbf{m}}\left(\tfrac{\theta_1}{2}\right)\cot^{2\mathbf{m}}\left(\tfrac{\theta_2}{2}\right)\right].\end{aligned}$$

Using the fact that $\det\Big(M_{\mathbf{m}}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),\theta_1,\theta_2\big)\Big)=0$, we obtain

$$\begin{aligned}&\left(-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\right)^2-\frac{1}{4\mathbf{m}^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2\mathbf{m}}\left(\frac{\theta_1}{2}\right)\cot^{2\mathbf{m}}\left(\frac{\theta_2}{2}\right)\\&=\left(-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\right)\left(\frac{2\omega_C-\omega_N-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\right).\end{aligned}$$

According to (3.13) or (3.15)-(3.26), up to taking $\mathbf{m}$ large enough, we can ensure

$$\frac{2\omega_C-\omega_N-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}\neq0.$$

Besides, we can write

$$-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}=\frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{4\mathbf{m}}+\kappa\frac{1}{2}\sqrt{\Delta_{\mathbf{m}}(\theta_1,\theta_2)}.$$

Assume for the sake of contradiction that

$$-c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2)+\frac{\omega_C-\omega_S}{2\mathbf{m}}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}=0.$$

This equation is equivalent to

$$\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{2\mathbf{m}}=-\kappa\sqrt{\Delta_{\mathbf{m}}(\theta_1,\theta_2)}.$$

Taking the square, we end up with

$$\frac{1}{\mathbf{m}^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\tan^{2\mathbf{m}}\left(\tfrac{\theta_1}{2}\right)\cot^{2\mathbf{m}}\left(\tfrac{\theta_2}{2}\right)=0.$$

But, by construction $\omega_N\neq\omega_C$ and $\omega_C\neq\omega_S$. In addition $\tan\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right)\in(0,1)$. Contradiction. Consequently,

$$\Big(\partial_c d_{(f_1,f_2)}\mathscr{G}\big(c_{\mathbf{m}}^{\kappa}(\widetilde{\gamma},\theta_1,\theta_2),0,0\big)[u_0]\,|\,g_0\Big)_2\neq 0.$$

This ends the proof of Proposition 3.2. □

A Appendix

A.1 An integral

In this appendix, we give an explicit value for an integral with parameters oftenly used in this work.

Lemma A.1. *Let $n\in\mathbb{N}^*$ and $a,b\in(0,\pi)$. We define*

$$I_n(a,b)\triangleq\frac{1}{2\pi}\int_0^{2\pi}\cos(nx)\log\big(1-\cos(a)\cos(b)-\sin(a)\sin(b)\cos(x)\big)dx.$$

Then

$$I_n(a,b)=I_n(b,a)=-\frac{1}{n}\tan^n\left(\frac{\min(a,b)}{2}\right)\cot^n\left(\frac{\max(a,b)}{2}\right).$$

Proof. ► Let us first begin with the case $a=b$. Observe that

$$\begin{aligned}\log\big(1-\cos^2(a)-\sin^2(a)\cos(x)\big)&=\log\big(1-\cos(x)\big)+\log\big(\sin^2(a)\big)\\&=\log\Big(\sin^2\left(\tfrac{x}{2}\right)\Big)+\log(2)+\log\big(\sin^2(a)\big).\end{aligned}$$

Hence, using [15, Lem. A.3], we get

$$I_n(a,a)=\frac{1}{2\pi}\int_0^{2\pi}\log\Big(\sin^2\left(\tfrac{x}{2}\right)\Big)\cos(nx)dx=-\frac{1}{n}\cdot$$

► Now, we assume $a\neq b$ with, without loss of generality, $a<b$. We can write

$$I_n(a,b)=\frac{1}{\pi}\int_0^{\pi}\cos(nx)\log\big(1-\mu_{a,b}\cos(x)\big)dx,\qquad\mu_{a,b}\triangleq\frac{\sin(a)\sin(b)}{1-\cos(a)\cos(b)}>0.$$

Notice that

$$\begin{aligned}a\neq b\Leftrightarrow\cos(a-b)<1&\Leftrightarrow\cos(a)\cos(b)+\sin(a)\sin(b)<1\\&\Leftrightarrow\sin(a)\sin(b)<1-\cos(a)\cos(b)\\&\Leftrightarrow\mu_{a,b}<1.\end{aligned}$$

Performing an integration by parts yields

$$I_n(a,b)=-\frac{\mu_{a,b}}{n\pi}\int_0^{\pi}\frac{\sin(nx)\sin(x)}{1-\mu_{a,b}\cos(x)}dx.$$

Now we shall use the following result which can be found in [42, p. 391]

$$\frac{1}{1+\alpha^2}\int_0^{\pi}\frac{\sin(nx)\sin(x)}{1-\frac{2\alpha}{1+\alpha^2}\cos(x)}dx=\int_0^{\pi}\frac{\sin(nx)\sin(x)}{1-2\alpha\cos(x)+\alpha^2}dx=\begin{cases}\frac{\pi}{2}\alpha^{n-1}, & \text{if }\alpha^2<1,\\ \frac{\pi}{2\alpha^{n+1}}, & \text{if }\alpha^2>1.\end{cases}$$

We apply it with

$$\frac{2\alpha}{1+\alpha^2}=\mu_{a,b},\qquad\text{i.e.}\qquad\mu_{a,b}\alpha^2-2\alpha+\mu_{a,b}=0.$$

The discriminant of the previous second order polynomial equation is $\Delta=4(1-\mu_{a,b}^2)>0$, so we can take

$$\alpha=\alpha_{a,b}\triangleq\frac{1-\sqrt{1-\mu_{a,b}^2}}{\mu_{a,b}}\in(0,1).$$

Observe that

$$\begin{aligned}\alpha_{a,b} < 1 &\Leftrightarrow 1-\mu_{a,b} < \sqrt{1-\mu_{a,b}^2}\\ &\Leftrightarrow (1-\mu_{a,b})^2 < 1-\mu_{a,b}^2\\ &\Leftrightarrow \mu_{a,b} < 1.\end{aligned}$$

Hence,

$$I_n(a,b) = -\frac{\mu_{a,b}(1+\alpha_{a,b}^2)\alpha_{a,b}^{n-1}}{2n} = -\frac{\alpha_{a,b}^n}{n}.$$

We can also write

$$\begin{aligned}\alpha_{a,b} &= \frac{1}{\mu_{a,b}} - \sqrt{\frac{1}{\mu_{a,b}^2}-1}\\ &= \frac{1-\cos(a)\cos(b)-\sqrt{\big(1-\cos(a)\cos(b)\big)^2-\sin^2(a)\sin^2(b)}}{\sin(a)\sin(b)}.\end{aligned}$$

But

$$\begin{aligned}\big(1-\cos(a)\cos(b)\big)^2-\sin^2(a)\sin^2(b) &= 1+\cos^2(a)\cos^2(b)-2\cos(a)\cos(b)-\big(1-\cos^2(a)\big)\big(1-\cos^2(b)\big)\\ &= \cos^2(a)+\cos^2(b)-2\cos(a)\cos(b)\\ &= \big(\cos(a)-\cos(b)\big)^2.\end{aligned}$$

Since $a,b\in(0,\pi)$ with $a<b$, then $\cos(a)>\cos(b)$. Consequently,

$$\begin{aligned}\alpha_{a,b} &= \frac{1-\cos(a)\cos(b)-\big(\cos(a)-\cos(b)\big)}{\sin(a)\sin(b)}\\ &= \frac{\big(1-\cos(a)\big)\big(1+\cos(b)\big)}{\sin(a)\sin(b)}\\ &= \frac{2\sin^2\left(\frac{a}{2}\right)2\cos^2\left(\frac{b}{2}\right)}{2\sin\left(\frac{a}{2}\right)\cos\left(\frac{a}{2}\right)2\sin\left(\frac{b}{2}\right)\cos\left(\frac{b}{2}\right)}\\ &= \tan\left(\tfrac{a}{2}\right)\cot\left(\tfrac{b}{2}\right).\end{aligned}$$

Finally, we have

$$\forall n\in\mathbb{N}^*,\quad \forall\, 0<a<b<\pi,\quad I_n(a,b) = -\frac{\tan^n\left(\frac{a}{2}\right)\cot^n\left(\frac{b}{2}\right)}{n}.$$

□

A.2 Potential theory

This section is devoted to some results on the continuity of specific operators with singular kernels. The proof of the next result can be found in [58, Lem. 2.6].

Proposition A.1. *Let $\alpha\in(0,1)$. Consider a kernel $K:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ smooth out of the diagonal and satisfying, for some $C_0>0$,*

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad |K(\varphi,\varphi')|\leqslant C_0\left|\sin\left(\frac{\varphi-\varphi'}{2}\right)\right|^{-(1-\alpha)},\tag{A.1}$$

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad |\partial_\varphi K(\varphi,\varphi')|\leqslant C_0\left|\sin\left(\frac{\varphi-\varphi'}{2}\right)\right|^{-(2-\alpha)}.\tag{A.2}$$

Then, the integral operator $\mathcal{K}$ defined by

$$\forall\varphi\in\mathbb{T},\quad \mathcal{K}(f)(\varphi)\triangleq\int_0^{2\pi}K(\varphi,\varphi')f(\varphi')d\varphi'$$

is bounded from $L^\infty(\mathbb{T})$ into $C^\alpha(\mathbb{T})$. More precisely, we have the following estimate

$$\forall f\in L^\infty(\mathbb{T}),\quad \|\mathcal{K}(f)\|_{C^\alpha(\mathbb{T})}\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})},$$

with $C>0$ an absolute constant.

In some cases, the above proposition cannot be applied directly because the kernel K has a non differentiable term, and thus the condition (A.2) does not make sense. In those cases, let us give the alternative result.

Proposition A.2. *Let $\alpha\in(0,1)$ and $g\in C^{\alpha}(\mathbb{T})$, consider a kernel $K:\mathbb{T}\times\mathbb{T}\to\mathbb{R}$ smooth out of the diagonal and satisfying*

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad |K(\varphi,\varphi')|\leqslant C_0\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1},\tag{A.3}$$

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad |\partial_{\varphi}K(\varphi,\varphi')|\leqslant C_0\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-2}.\tag{A.4}$$

Then, the integral operator $\widetilde{\mathcal{K}}_g$ defined by

$$\forall\varphi\in\mathbb{T},\quad \widetilde{\mathcal{K}}_g(f)(\varphi)\triangleq\int_0^{2\pi}K(\varphi,\varphi')\big(g(\varphi)-g(\varphi')\big)f(\varphi')d\varphi'$$

is bounded from $L^{\infty}(\mathbb{T})$ into $C^{\alpha}(\mathbb{T})$. More precisely, we have the following estimate

$$\forall f\in L^{\infty}(\mathbb{T}),\quad \|\widetilde{\mathcal{K}}_g(f)\|_{C^{\alpha}(\mathbb{T})}\leqslant CC_0\|g\|_{C^{\alpha}(\mathbb{T})}\|f\|_{L^{\infty}(\mathbb{T})},$$

with $C>0$ an absolute constant.

Proof. The L^{∞} norm of $\widetilde{\mathcal{K}}_g(f)$ can be estimated as

$$\begin{aligned}\left|\widetilde{\mathcal{K}}_g(f)(\varphi)\right|&\leqslant C\|f\|_{L^{\infty}(\mathbb{T})}\|g\|_{C^{\alpha}(\mathbb{T})}\int_0^{2\pi}|K(\varphi,\varphi')||\varphi-\varphi'|^{\alpha}d\varphi'\\&\leqslant CC_0\|f\|_{L^{\infty}(\mathbb{T})}\|g\|_{C^{\alpha}(\mathbb{T})}\sup_{\varphi\in\mathbb{T}}\int_0^{2\pi}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1}|\varphi-\varphi'|^{\alpha}d\varphi'.\end{aligned}$$

Since we work on the torus, we can always assume that

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad 0<|\varphi-\varphi'|\leqslant\pi.$$

As a consequence, we have the following classical convexity estimate

$$\forall\varphi\neq\varphi'\in\mathbb{T},\quad \tfrac{2}{\pi}|\varphi-\varphi'|\leqslant 2\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|\leqslant|\varphi-\varphi'|.\tag{A.5}$$

Therefore, by using (A.5), a change of variable and the fact that $\alpha\in(0,1)$, we get

$$\forall\varphi\in\mathbb{T},\quad\int_0^{2\pi}\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|^{-1}|\varphi-\varphi'|^{\alpha}d\varphi'\leqslant C\int_0^{2\pi}\left|\sin\left(\tfrac{\varphi'}{2}\right)\right|^{-(1-\alpha)}d\varphi'<\infty.$$

Hence, we obtain

$$\|\widetilde{\mathcal{K}}_g(f)\|_{L^{\infty}(\mathbb{T})}\leqslant CC_0\|f\|_{L^{\infty}(\mathbb{T})}\|g\|_{C^{\alpha}(\mathbb{T})}.$$

For the Hölder regularity, take $\varphi_1\neq\varphi_2\in\mathbb{T}$. Define

$$d\triangleq 2\left|\sin\left(\tfrac{\varphi_1-\varphi_2}{2}\right)\right|=\left|e^{\mathrm{i}\varphi_1}-e^{\mathrm{i}\varphi_2}\right|$$

and for $\varphi\in\mathbb{T}$ and $r>0$,

$$B_{\varphi}(r)\triangleq\left\{\varphi'\in\mathbb{T}\quad\text{s.t.}\quad 2\left|\sin\left(\tfrac{\varphi-\varphi'}{2}\right)\right|<r\right\},\qquad B_{\varphi}^{c}(r)\triangleq\mathbb{T}\setminus B_{\varphi}(r).$$

Hence

$$\begin{aligned}\widetilde{\mathcal{K}}_g(f)(\varphi_1)-\widetilde{\mathcal{K}}_g(f)(\varphi_2)&=\int_0^{2\pi}K(\varphi_1,\varphi')\big(g(\varphi_1)-g(\varphi')\big)f(\varphi')d\varphi'-\int_0^{2\pi}K(\varphi_2,\varphi')\big(g(\varphi_2)-g(\varphi')\big)f(\varphi')d\varphi'\\&=\int_{B_{\varphi_1}(3d)}K(\varphi_1,\varphi')\big(g(\varphi_1)-g(\varphi')\big)f(\varphi')d\varphi'\\&\quad-\int_{B_{\varphi_1}(3d)}K(\varphi_2,\varphi')\big(g(\varphi_2)-g(\varphi')\big)f(\varphi')d\varphi'\\&\quad+\int_{B_{\varphi_1}^{c}(3d)}(K(\varphi_1,\varphi')-K(\varphi_2,\varphi'))\big(g(\varphi_1)-g(\varphi')\big)f(\varphi')d\varphi'\\&\quad+\int_{B_{\varphi_1}^{c}(3d)}K(\varphi_2,\varphi')\big(g(\varphi_1)-g(\varphi_2)\big)f(\varphi')d\varphi'\\&\triangleq I_1+I_2+I_3+I_4.\end{aligned}$$

Using (A.1), (A.5) and a change of variables, we arrive at

$$\begin{aligned}
|I_1| &\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|^{-1}|\varphi_1-\varphi'|^\alpha d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|^{-(1-\alpha)}d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_0^{\frac{3}{2}d}\frac{dw}{|w|^{1-\alpha}\sqrt{1-w^2}}dw\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}d^\alpha\\
&= CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|^\alpha.
\end{aligned}$$

In order to work with I_2, note that $B_{\varphi_1}(3d)\subset B_{\varphi_2}(4d)$. Thus, proceding as before, we infer

$$\begin{aligned}
|I_2| &\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_2-\varphi'}{2}\right)\right|^{-1}|\varphi_2-\varphi'|^\alpha d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B_{\varphi_2}(4d)}\left|\sin\left(\tfrac{\varphi_2-\varphi'}{2}\right)\right|^{-1}|\varphi_2-\varphi'|^\alpha d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B_{\varphi_2}(4d)}\left|\sin\left(\tfrac{\varphi_2-\varphi'}{2}\right)\right|^{-(1-\alpha)}d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|^\alpha.
\end{aligned}$$

For the third term I_3 we use the mean value theorem and (A.2) achieving

$$\begin{aligned}
|I_3| &\leqslant C\left|(\varphi_1-\varphi_2)\int_0^1\int_{B^c_{\varphi_1}(3d)}(\partial_x K)\big(\varphi_1+(1-s)(\varphi_2-\varphi_1),\varphi'\big)\big(g(\varphi_1)-g(\varphi')\big)f(\varphi')d\varphi' ds\right|\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|\int_0^1\int_{B^c_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1+(1-s)(\varphi_2-\varphi_1)-\varphi'}{2}\right)\right|^{-2}|\varphi_1-\varphi'|^\alpha d\varphi' ds.
\end{aligned}$$

Note that if $\varphi'\in B^c_{\varphi_1}(3d)$ and $s\in[0,1]$, then

$$\begin{aligned}
2\left|\sin\left(\tfrac{\varphi_1+(1-s)(\varphi_2-\varphi_1)-\varphi'}{2}\right)\right| &= \left|e^{\mathrm{i}(\varphi_1-\varphi')}-e^{\mathrm{i}(1-s)(\varphi_1-\varphi_2)}\right|\\
&\geqslant \left|e^{\mathrm{i}(\varphi_1-\varphi')}-1\right|-\left|e^{\mathrm{i}(1-s)(\varphi_1-\varphi_2)}-1\right|\\
&\geqslant \left|e^{\mathrm{i}(\varphi_1-\varphi')}-1\right|-\left|e^{\mathrm{i}(\varphi_1-\varphi_2)}-1\right| = \left|e^{\mathrm{i}\varphi_1}-e^{\mathrm{i}\varphi'}\right|-\left|e^{\mathrm{i}\varphi_1}-e^{\mathrm{i}\varphi_2}\right|\\
&\geqslant \tfrac{2}{3}\left|e^{\mathrm{i}\varphi_1}-e^{\mathrm{i}\varphi'}\right| = \tfrac{4}{3}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right| \qquad \text{(A.6)}
\end{aligned}$$

which implies, through (A.5) and a change of variables,

$$\begin{aligned}
|I_3| &\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|\int_{B^c_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|^{-2}|\varphi_1-\varphi'|^\alpha d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|\int_{B^c_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|^{-(2-\alpha)}d\varphi'\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|\int_d^1\frac{dw}{|w|^{2-\alpha}\sqrt{1-w^2}}\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|\frac{1}{|\varphi_1-\varphi_2|^{1-\alpha}}\\
&\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|^\alpha.
\end{aligned}$$

Let us finish with I_4, using that $g\in C^\alpha(\mathbb{T})$,

$$|I_4|\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B^c_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_2-\varphi'}{2}\right)\right|^{-1}|\varphi_1-\varphi_2|^\alpha d\varphi'.$$

Applying (A.6) with $s=0$, we get

$$2\left|\sin\left(\tfrac{\varphi_2-\varphi'}{2}\right)\right|\geqslant\tfrac{4}{3}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|.$$

Besides, for $\varphi' \in B^c_{\varphi_1}(3d)$, we have

$$\left|\sin\left(\tfrac{\varphi_1-\varphi_2}{2}\right)\right| \leqslant \tfrac{1}{3}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|.$$

Combining the foregoing facts, we end up with

$$\begin{aligned}|I_4| &\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}\int_{B^c_{\varphi_1}(3d)}\left|\sin\left(\tfrac{\varphi_1-\varphi'}{2}\right)\right|^{-(1-\alpha)}d\varphi'\\ &\leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|^\alpha.\end{aligned}$$

Putting together the preceding estimates yields

$$\left|\widetilde{\mathcal{K}}_g(f)(\varphi_1)-\widetilde{\mathcal{K}}_g(f)(\varphi_2)\right| \leqslant CC_0\|f\|_{L^\infty(\mathbb{T})}\|g\|_{C^\alpha(\mathbb{T})}|\varphi_1-\varphi_2|^\alpha,$$

concluding the proof. □

A.3 Crandall-Rabinowitz theorem

In this last appendix, we recall the classical Crandall-Rabinowitz Theorem whose proof can be found in [20].

Theorem A.1 (Crandall-Rabinowitz Theorem)**.** *Let $\lambda_0 \in \mathbb{R}$, X, Y be two Banach spaces, V be a neighborhood of 0 in X and $\mathscr{F} : \mathbb{R}\times V \to Y$ be a function with the properties,*

1. *$\mathscr{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.*
2. *The partial derivatives $\partial_\lambda\mathscr{F}$, $d_f\mathscr{F}$ and $\partial_\lambda d_f\mathscr{F}$ exist and are continuous.*
3. *The operator $d_f\mathscr{F}(\lambda_0, 0)$ is Fredholm of zero index and $\ker\big(d_f\mathscr{F}(\lambda_0, 0)\big) = \mathtt{span}(f_0)$ is one-dimensional.*
4. *Transversality assumption: $\partial_\lambda d_f\mathscr{F}(\lambda_0, 0)[f_0] \notin \mathrm{Im}\big(d_f\mathscr{F}(\lambda_0, 0)\big)$.*

If Z is any complement of $\ker\big(d_f\mathscr{F}(\lambda_0, 0)\big)$ in X, then there is a neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R}\times X$, an interval $(-a, a)$ with $a > 0$, and two continuous functions $\Phi : (-a, a) \to \mathbb{R}$, $\beta : (-a, a) \to Z$ such that $\Phi(0) = \lambda_0$ and $\beta(0) = 0$ and

$$\mathscr{F}^{-1}(0)\cap U = \big\{\big(\Phi(s), sf_0 + s\beta(s)\big) : |s| < a\big\} \cup \big\{(\lambda, 0) : (\lambda, 0) \in U\big\}.$$

A.4 Proof of Lemma 1.2

Fix $\alpha \in \mathbb{R}$. Since $\mathcal{R}(\alpha) \in SO_3(\mathbb{R})$, then it preserves the Euclidean norm $|\cdot|_{\mathbb{R}^3}$, i.e.

$$\forall \xi \in \mathbb{R}^3, \quad \big|\mathcal{R}(\alpha)\xi\big|_{\mathbb{R}^3} = |\xi|_{\mathbb{R}^3}.$$

As a consequence,

$$\begin{aligned}\forall(\xi,\xi') \in (\mathbb{S}^2)^2, \quad \forall\alpha\in\mathbb{R}, \quad G\big(\mathcal{R}(\alpha)\xi, \mathcal{R}(\alpha)\xi'\big) &= \frac{1}{2\pi}\log\left(\frac{\big|\mathcal{R}(\alpha)\xi - \mathcal{R}(\alpha)\xi'\big|_{\mathbb{R}^3}}{2}\right)\\ &= \frac{1}{2\pi}\log\left(\frac{\big|\mathcal{R}(\alpha)(\xi-\xi')\big|_{\mathbb{R}^3}}{2}\right)\\ &= \frac{1}{2\pi}\log\left(\frac{|\xi-\xi'|_{\mathbb{R}^3}}{2}\right)\\ &= G(\xi,\xi').\end{aligned}$$

Hence, using the change of variables $\xi' \mapsto \mathcal{R}(\alpha)\xi' \in SO(\mathbb{R}^3)$ (which preserves $\mathbb{S}^2$), we get for any $\xi \in \mathbb{S}^2$,

$$\begin{aligned}\Psi\big(\mathcal{R}(\alpha)\xi\big) &= \int_{\mathbb{S}^2} G\big(\mathcal{R}(\alpha)\xi, \xi'\big)\Omega(\xi')d\xi'\\ &= \int_{\mathbb{S}^2} G\big(\mathcal{R}(\alpha)\xi, \mathcal{R}(\alpha)\xi'\big)\Omega\big(\mathcal{R}(\alpha)\xi'\big)d\xi'\\ &= \int_{\mathbb{S}^2} G(\xi,\xi')\Omega(\xi')d\xi'\\ &= \Psi(\xi).\end{aligned}\tag{A.7}$$

This achieves the proof of Lemma 1.2.

References

[1] Weiwei Ao, Juan Dávila, Manuel del Pino, Monica Musso, and Juncheng Wei. Travelling and rotating solutions to the generalized inviscid surface quasi-geostrophic equation. *Transactions of the American Mathematical Society*, 374(9):6665–6689, 2021.

[2] Kendall Atkinson and Weimin Han. *Spherical harmonics and approximations on the unit sphere: an introduction*, volume 2044 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2012.

[3] Massimiliano Berti, Zineb Hassainia, and Nader Masmoudi. Time quasi-periodic vortex patches. *arXiv preprint arXiv:2202.06215*, 2022.

[4] Stefanella Boatto and David G. Dritschel. The motion of point vortices on closed surfaces. *Proceedings of the Royal Society A - Mathematical, Physical and Engineering Sciences*, 471(2176):20140890, 25, 2015.

[5] Jacob Burbea. Motions of vortex patches. *Letters in Mathematical Physics*, 6(1):1–16, 1982.

[6] Daomin Cao, Shanfa Lai, and Weicheng Zhan. Traveling vortex pairs for 2D incompressible Euler equations. *Calculus of Variations and Partial Differential Equations*, 60(5):Paper No. 190, 16, 2021.

[7] Daomin Cao, Zhongyuan Liu, and Juncheng Wei. Regularization of point vortices pairs for the Euler equation in dimension two. *Archive for Rational Mechanics and Analysis*, 212(1):179–217, 2014.

[8] Daomin Cao, Guolin Qin, Weicheng Zhan, and Changjun Zou. Existence and regularity of co-rotating and traveling-wave vortex solutions for the generalized SQG equation. *Journal of Differential Equations*, 299:429–462, 2021.

[9] Daomin Cao, Guolin Qin, Weicheng Zhan, and Changjun Zou. On the global classical solutions for the generalized SQG equation. *Journal of Functional Analysis*, 283(2):Paper No. 109503, 37, 2022.

[10] Daomin Cao and Jie Wan. Multiscale steady vortex patches for 2D incompressible Euler equations. *SIAM Journal on Mathematical Analysis*, 54(2):1488–1514, 2022.

[11] Daomin Cao, Jie Wan, Guodong Wang, and Weicheng Zhan. Rotating vortex patches for the planar Euler equations in a disk. *Journal of Differential Equations*, 275:509–532, 2021.

[12] Daomin Cao, Guodong Wang, and Bijun Zuo. Stability of degree-2 rossby-haurwitz waves. *arXiv preprint arXiv:2305.03279*, 2023.

[13] Silvia Caprino and Carlo Marchioro. On nonlinear stability of stationary Euler flows on a rotating sphere. *Journal of Mathematical Analysis and Applications*, 129(1):24–36, 1988.

[14] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Existence and regularity of rotating global solutions for the generalized surface quasi-geostrophic equations. *Duke Mathematical Journal*, 165(5):935–984, 2016.

[15] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Uniformly rotating analytic global patch solutions for active scalars. *Annals of PDE*, 2(1):Art. 1, 34, 2016.

[16] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Uniformly rotating smooth solutions for the incompressible 2D Euler equations. *Archive for Rational Mechanics and Analysis*, 231(2):719–785, 2019.

[17] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Global smooth solutions for the inviscid SQG equation. *Memoirs of the American Mathematical Society*, 266(1292):v+89, 2020.

[18] Christophe Cheverry and Nicolas Raymond. *A guide to spectral theory Applications and exercises.* Birkhäuser/Springer, Cham, 2021.

[19] Adrian Constantin and Pierre Germain. Stratospheric planetary flows from the perspective of the Euler equation on a rotating sphere. *Archive for Rational Mechanics and Analysis*, 245(1):587–644, 2022.

[20] Michael G. Crandall and Paul H. Rabinowitz. Bifurcation from simple eigenvalues. *Journal of Functional Analysis*, 8:321–340, 1971.

[21] Juan Davila, Manuel Del Pino, Monica Musso, and Juncheng Wei. Gluing methods for vortex dynamics in Euler flows. *Archive for Rational Mechanics and Analysis*, 235(3):1467–1530, 2020.

[22] Juan Davila, Manuel del Pino, Monica Musso, and Juncheng Wei. Leapfrogging vortex rings for the 3-dimensional incompressible euler equations. *arXiv preprint arXiv:2207.03263*, 2022.

[23] Francisco de la Hoz, Zineb Hassainia, and Taoufik Hmidi. Doubly connected V-states for the generalized surface quasi-geostrophic equations. *Archive for Rational Mechanics and Analysis*, 220(3):1209–1281, 2016.

[24] Francisco de la Hoz, Zineb Hassainia, Taoufik Hmidi, and Joan Mateu. An analytical and numerical study of steady patches in the disc. *Analysis & PDE*, 9(7):1609–1670, 2016.

[25] Francisco de la Hoz, Taoufik Hmidi, Joan Mateu, and Joan Verdera. Doubly connected V-states for the planar Euler equations. *SIAM Journal on Mathematical Analysis*, 48(3):1892–1928, 2016.

[26] Gary S. Deem and Norman J. Zabusky. Vortex waves: Stationary "V states," interactions, recurrence, and breaking. *Physical Review Letters*, 40:859–862, Mar 1978.

[27] David G. Dritschel, Taoufik Hmidi, and Coralie Renault. Imperfect bifurcation for the quasi-geostrophic shallow-water equations. *Archive for Rational Mechanics and Analysis*, 231(3):1853–1915, 2019.

[28] David G. Dritschel and Lorenzo M. Polvani. The roll-up of vorticity strips on the surface of a sphere. *Journal of Fluid Mechanics*, 234:47–69, 1992.

[29] David G. Dritschel and Lorenzo M. Polvani. Wave and vortex dynamics on the surface of a sphere. *Journal of Fluid Mechanics*, 255:35–64, 1993.

[30] Ludwig E. Fraenkel. *An introduction to maximum principles and symmetry in elliptic problems*, volume 128 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2000.

[31] Claudia García. Kármán vortex street in incompressible fluid models. *Nonlinearity*, 33(4):1625–1676, 2020.

[32] Claudia García. Vortex patches choreography for active scalar equations. *Journal of Nonlinear Science*, 31(5):Paper No. 75, 31, 2021.

[33] Claudia García and Susanna V. Haziot. Global bifurcation for corotating and counter-rotating vortex pairs. *arXiv preprint arXiv:2204.11327*, 2022.

[34] Claudia García, Taoufik Hmidi, and Joan Mateu. Time periodic doubly connected solutions for the 3D quasi-geostrophic model. *arXiv preprint arXiv:2206.10197*, 2022.

[35] Claudia García, Taoufik Hmidi, and Joan Mateu. Time Periodic Solutions for 3D Quasi-Geostrophic Model. *Communications in Mathematical Physics*, 390(2):617–756, 2022.

[36] Claudia García, Taoufik Hmidi, and Joan Mateu. Time periodic solutions close to localized radial monotone profiles for the 2D euler equations,. *arXiv preprint arXiv:2302.01279*, 2023.

[37] Claudia García, Taoufik Hmidi, and Juan Soler. Non uniform rotating vortices and periodic orbits for the two-dimensional Euler equations. *Archive for Rational Mechanics and Analysis*, 238(2):929–1085, 2020.

[38] Ludovic Godard-Cadillac, Philippe Gravejat, and Didier Smets. Co-rotating vortices with n fold symmetry for the inviscid surface quasi-geostrophic equation. *Indiana University Mathematics Journal*, 2020.

[39] Javier Gómez Serrano, Alexandru D. Ionescu, and Jaemin Park. Quasiperiodic solutions of the generalized SQG equation. *arXiv preprint arXiv:2303.03992.*, 2023.

[40] Javier Gómez-Serrano, Jaemin Park, and Jia Shi. Existence of non-trivial non-concentrated compactly supported stationary solutions of the 2d euler equation with finite energy. *arXiv preprint arXiv:2112.03821*, 2021.

[41] Javier Gómez-Serrano, Jaemin Park, Jia Shi, and Yao Yao. Symmetry in stationary and uniformly rotating solutions of active scalar equations. *Duke Mathematical Journal*, 170(13):2957–3038, 2021.

[42] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

[43] Gregory J. Hakim and James R. Holton. *An introduction to dynamic meteorology*. International Geophysics Series. Elsevier Academic Press, Burlington, MA, 4 edition, 2004.

[44] Zineb Hassainia and Taoufik Hmidi. On the V-states for the generalized quasi-geostrophic equations. *Communications in Mathematical Physics*, 337(1):321–377, 2015.

[45] Zineb Hassainia and Taoufik Hmidi. Steady asymmetric vortex pairs for Euler equations. *Discrete and Continuous Dynamical Systems. Series A*, 41(4):1939–1969, 2021.

[46] Zineb Hassainia, Taoufik Hmidi, and Nader Masmoudi. Kam theory for active scalar equations. *arXiv preprint arXiv:2110.08615*, 2021.

[47] Zineb Hassainia, Taoufik Hmidi, and Emeric Roulley. Invariant kam tori around annular vortex patches for 2D Euler equations. *arXiv preprint arXiv:2302.01311*, 2023.

[48] Zineb Hassainia, Nader Masmoudi, and Miles H. Wheeler. Global bifurcation of rotating vortex patches. *Communications on Pure and Applied Mathematics*, 73(9):1933–1980, 2020.

[49] Zineb Hassainia and Emeric Roulley. Boundary effects on the emergence of quasi-periodic solutions for euler equations. *arXiv preprint arXiv:2202.10053*, 2022.

[50] Zineb Hassainia and Miles H. Wheeler. Multipole vortex patch equilibria for active scalar equations. *SIAM Journal on Mathematical Analysis*, 54(6):6054–6095, 2022.

[51] Taoufik Hmidi. On the trivial solutions for the rotating patch model. *Journal of Evolution Equations*, 15(4):801–816, 2015.

[52] Taoufik Hmidi and Joan Mateu. Bifurcation of rotating patches from Kirchhoff vortices. *Discrete and Continuous Dynamical Systems. Series A*, 36(10):5401–5422, 2016.

[53] Taoufik Hmidi and Joan Mateu. Degenerate bifurcation of the rotating patches. *Advances in Mathematics*, 302:799–850, 2016.

[54] Taoufik Hmidi and Joan Mateu. Existence of corotating and counter-rotating vortex pairs for active scalar equations. *Communications in Mathematical Physics*, 350(2):699–747, 2017.

[55] Taoufik Hmidi, Joan Mateu, and Joan Verdera. Boundary regularity of rotating vortex patches. *Archive for Rational Mechanics and Analysis*, 209(1):171–208, 2013.

[56] Taoufik Hmidi, Joan Mateu, and Joan Verdera. On rotating doubly connected vortices. *Journal of Differential Equations*, 258(4):1395–1429, 2015.

[57] Taoufik Hmidi and Emeric Roulley. Time quasi-periodic vortex patches for quasi-geostrophic shallow-water equations. *arXiv preprint arXiv:2110.13751*, 2021.

[58] Taoufik Hmidi, Liutang Xue, and Zhi Xue. Emergence of time periodic solutions for the generalized surface quasi-geostrophic equation in the disc. *arXiv preprint arXiv:2210.08760*, 2022.

[59] Sun-Chul Kim. A free-boundary problem for Euler flows with constant vorticity on the sphere. *Journal of Mathematical Analysis and Applications*, 465(1):703–711, 2018.

[60] Sun-Chul Kim, Takashi Sakajo, and Sung-Ik Sohn. Stability of barotropic vortex strip on a rotating sphere. *Proceedings A*, 474(2210):20170883, 25, 2018.

[61] Sun-Chul Kim and Sung-Ik Sohn. Linear stability and nonlinear evolution of a polar vortex cap on a rotating sphere. *European Journal of Mechanics. B. Fluids*, 85:102–109, 2021.

[62] Gustav R. Kirchhoff. *Vorlesungen uber mathematische Physik. Mechanik.* Teubner, Leipzig, 1876.

[63] Marc Nualart. On zonal steady solutions to the 2D Euler equations on the rotating unit sphere. *arXiv preprint arXiv:2201.05522*, 2022.

[64] Emeric Roulley. Periodic and quasi-periodic Euler-α flows close to Rankine vortices. *arXiv preprint arXiv:2208.13109*, 2022.

[65] Emeric Roulley. Vortex rigid motion in quasi-geostrophic shallow-water equations. *Asymptotic Analysis*, pages 1–50, 2022.

[66] Yuri N. Skiba. *Mathematical problems of the dynamics of incompressible fluid on a rotating sphere*. Springer, Cham, 2017.

[67] Bruce Turkington. Corotating steady vortex flows with n-fold symmetry. *Nonlinear Analysis. Theory, Methods & Applications*, 9:351–369, 1985.

[68] Yuchen Wang, Xin Xu, and Maolin Zhou. Degenerate bifurcation of two-fold doubly-connected vortex patches. *arXiv preprint arXiv:2212.01869*, 2022.

[69] Victor I Yudovich. Non-stationary flow of an ideal incompressible liquid. *USSR Computational Mathematics and Mathematical Physics*, 3(6):1407–1456, 1963.

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