Proof of the slicing result

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$$A \xrightarrow{\text{kp}} \mathbf{KP}(f_0) \xrightarrow{\text{kp}} \mathbf{KP}(f_1) \xrightarrow{\text{kp}} \dots \ \mathbf{KP}(f_n) \longrightarrow \dots$$

where $f_0 := f : A \to B$ et $f_{n+1} := \mathbf{lift}_{f_n} : \mathbf{KP}(f_n) \to B$.

Theorem 1. *The colimit of this diagram is* $\Sigma_{y:B} \| \mathbf{fib}_f\, y \|$ *, the* image *of* f*.*

We will show that the diagram considered is equivalent to another diagram, of which the colimit is clearly the image of f.
More precisely, we go through an intermediate diagram. So, our aim is to prove that the first line is equivalent to the second, and the second to the third in the following diagram:

$$\begin{array}{ccccccccc}
A & \xrightarrow{\text{kp}} & \mathbf{KP}(f_0) & \xrightarrow{\text{kp}} & \mathbf{KP}(f_1) & \xrightarrow{\text{kp}} & \dots & \mathbf{KP}(f_n) & \dots \\
\wr\downarrow e_0 & & \wr\downarrow e_1 & & \wr\downarrow e_2 & & & \wr\downarrow e_{n+1} & \\
\Sigma_y \mathbf{fib}_{f_0}\, y & \xrightarrow{\Sigma\overline{\text{kp}}} & \Sigma_y \mathbf{fib}_{f_1}\, y & \xrightarrow{\Sigma\overline{\text{kp}}} & \Sigma_y \mathbf{fib}_{f_2}\, y & \xrightarrow{\Sigma\overline{\text{kp}}} & \dots & \Sigma_y \mathbf{fib}_{f_{n+1}}\, y & \dots \\
\wr\downarrow \Sigma h_0 & & \wr\downarrow \Sigma h_1 & & \wr\downarrow \Sigma h_2 & & & \wr\downarrow \Sigma h_{n+1} & \\
\Sigma_y \mathbf{fib}_{f_0}\, y & \xrightarrow{\Sigma\alpha} & \Sigma_y \{\mathbf{fib}_{f_0}\, y\} & \xrightarrow{\Sigma\alpha} & \Sigma_y \{\{\mathbf{fib}_{f_0}\, y\}\} & \xrightarrow{\Sigma\alpha} & \dots & \Sigma_y \{_\}^{n+1}(\mathbf{fib}_{f_0}\, y) & \dots
\end{array}$$

where $\overline{\text{kp}} : \mathbf{fib}_{f_n} \to \mathbf{fib}_{f_{n+1}}$ is defined by $\lambda(x;\, p).\, (\text{kp}x;\, p)$ and where $\Sigma\overline{\text{kp}}$ and $\Sigma\alpha$ are the functions induced by $\overline{\text{kp}}$ and α on the second components of sigma types.

Let's go.
1. First, the image of f is well a colimit of the third line.
Indeed, from the Floris' result, $\| \mathbf{fib}_f\, y \|$ is a colimit of the diagram

$$\mathbf{fib}_{f_0}\, y \xrightarrow{\alpha} \{\mathbf{fib}_{f_0}\, y\} \xrightarrow{\alpha} \{\{\mathbf{fib}_{f_0}\, y\}\} \xrightarrow{\alpha} \dots \ \{\_\}^{n+1}(\mathbf{fib}_{f_0}\, y) \ \dots$$

for all $y : B$. Hence the result by commutation of colimits with sigmas.

2. Now, let's show that the two first lines are equivalent.
This equivalence is rather easy, in fact, it is true for every diagram of this shape. We just have to apply the equivalence between a type and the sum of the fibers ($A \simeq \Sigma_{b:B} \mathbf{fib}_g\, b$ for $g : A \to B$) to all types of the diagram. As a consequence, the e_n are defined by $e_n := \lambda x.\, (f_n(x);\, (x;\, \mathbf{refl}))$. The n-th square commutes because $e_{n+1} \circ \text{kp}$ and $\Sigma\overline{\text{kp}} \circ e_n$ are equal on the second component of $\Sigma_y \mathbf{fib}_{f_{n+1}}\, y$ (which is sufficient to prove that they are extensionnaly equal).

3. Last, let's show the equivalence between the second and the third line.
We reason pointwise. So, we fix $y : B$ and we show the equivalence:

$$\begin{array}{ccccccccc}
\mathbf{fib}_{f_0}\, y & \xrightarrow{\overline{\mathrm{kp}}} & \mathbf{fib}_{f_1}\, y & \xrightarrow{\overline{\mathrm{kp}}} & \mathbf{fib}_{f_2}\, y & \xrightarrow{\overline{\mathrm{kp}}} & \dots & \mathbf{fib}_{f_{n+1}}\, y & \dots \\
\wr\big\downarrow h_0 & & \wr\big\downarrow h_1 & & \wr\big\downarrow h_2 & & & \wr\big\downarrow h_{n+1} & \\
\mathbf{fib}_{f_0}\, y & \xrightarrow{\alpha} & \{\mathbf{fib}_{f_0}\, y\} & \xrightarrow{\alpha} & \{\{\mathbf{fib}_{f_0}\, y\}\} & \xrightarrow{\alpha} & \dots & \{_\}^{n+1}(\mathbf{fib}_{f_0}\, y) & \dots
\end{array}$$

Let's pose $h_0 := \mathbf{id}$. To define recursively the other equivalences we need the two following lemmas.

Lemma 1. *For $f : A \to B$ we have:*

$$\mathbf{KP}\ f \ \simeq\ \Sigma_y\{\mathbf{fib}_f\, y\}.$$

The prove of this lemma is especially simple and elegant: We know that $\mathbf{KP}\, f$ is a colimit of the diagram $A \times_B A \rightrightarrows A$ and that $\Sigma_y\{\mathbf{fib}_f\, y\}$ is a colimit of $\Sigma_y(\mathbf{fib}_f\, y) \times (\mathbf{fib}_f\, y) \rightrightarrows \Sigma_y\, \mathbf{fib}_f\, y$ (always by commutation with sigmas). But those two diagrams are equivalent! Hence the equivalence of their colimits.

This second lemma show how to construct h_{n+1} for h_n:

Lemma 2. *Let $f : A \to B$ be a function, A a type, $y : B$ and $\epsilon : (\mathbf{fib}_f\, y) \simeq A$ an equivalence. Then, there exists an equivalence $\epsilon' : (\mathbf{fib}_{\mathbf{lift}_f}\, y) \simeq \{A\}$ such that*

$$\begin{array}{ccc}
\mathbf{fib}_f\, y & \xrightarrow{\overline{\mathrm{kp}}} & \mathbf{fib}_{\mathbf{lift}_f}\, y \\
\epsilon\big\downarrow\wr & & \wr\big\downarrow\epsilon' \\
A & \xrightarrow{\alpha} & \{A\}
\end{array}$$

commute.

If we note s the equivalence of the previous lemma, then ϵ' is given by:

$$\begin{aligned}
\mathbf{fib}_{\mathbf{lift}_f}\, y \ &\equiv\ \Sigma_{x:\mathbf{KP}\, f}\, \mathbf{lift}_f(x) = y && (1)\\
&\simeq\ \Sigma_{x:\, \Sigma_y\{\mathbf{fib}_f\, y\}}\ \mathbf{lift}_f \circ s^{-1}(x) \ =\ y && (2)\\
&\simeq\ \Sigma_{x:\, \Sigma_y\{\mathbf{fib}_f\, y\}}\ \pi_1(x) \ =\ y && (3)\\
&\simeq\ \{\mathbf{fib}_f\, y\} && (4)\\
&\simeq\ \{A\} && (5)
\end{aligned}$$

(1) $\simeq$ (2) : by s (lemma 1).
(2) $\simeq$ (3) : because we can show that $\pi_1 \circ s = \mathbf{lift}_f$
(4) $\simeq$ (5) : by functoriality of colimits (if $A \simeq B$ then $\{A\} \simeq \{B\}$).

The formalization of this can be found in the files `KernelPair_v1.v` and `CechNerve.v` of the hott-colimit library.