Pre-model structure on the universe in a two level type theory

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INRIA, Nantes

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MLTT_2 and a pre-model structure on $\mathcal{UF}_i$

$\mathrm{MLTT}_2^{\mathcal{F}}$ and a pre-model structure on $\mathcal{U}_i^{\mathrm{S}}$

Model of $\mathrm{MLTT}_2^{\mathcal{F}}$

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Model of $\mathrm{MLTT}_2^{\mathcal{F}}$

The framework: MLTT_2

Fibrant fragment

$\Pi, \Sigma, =$ (path equality)

Cyl_f (cylinders, a HIT)

$\mathcal{UF}_0, \mathcal{UF}_1, \ldots$ hierarchy of fibrant types

univalence not needed

Strict fragment

$\Pi, \Sigma, \equiv$ (strict equality)

$\mathcal{U}^{\mathrm{S}}_0, \mathcal{U}^{\mathrm{S}}_1, \ldots$ hierarchy of pre-types

UIP and funext for $\equiv$

THE FRAMEWORK: MLTT_2

Differences with the previous talk:

$$\mathcal{U}_i \quad \rightsquigarrow \quad \mathcal{UF}_i \qquad \text{(fibrant types)}$$

$$\overset{\mathrm{s}}{=} \quad \rightsquigarrow \quad \equiv \qquad \text{(strict equality)}$$

$$\equiv \quad \rightsquigarrow \quad \simeq_{\beta\eta} \qquad \text{(conversion)}$$

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$$\equiv \rightsquigarrow \simeq_{\beta\eta} \qquad \text{(conversion)}$$

A judgment for fibrancy:

$$\Gamma \vdash A \ \mathsf{Fib}$$

For instance:

$$\frac{\Gamma \vdash A : \mathcal{U}_i^{\mathrm{S}} \qquad \Gamma \vdash A \ \mathsf{Fib}}{\Gamma \vdash A : \mathcal{UF}_i}$$

A PRE-MODEL STRUCTURE ON $\mathcal{UF}_i$

In MLTT_2, $\mathcal{UF}_i$ and $\mathcal{U}_i^{\mathrm{S}}$ are categories (for $\equiv$).

GOAL: Equip them with a pre-model structure.

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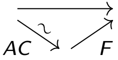
DEFINITION

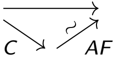
*A **pre-model structure** is given by :*

- *3 classes of arrows W, F and C*

  ($AF := F \cap W$ and $AC := C \cap W$)

such that:

- *an arrow can be factorized as* 

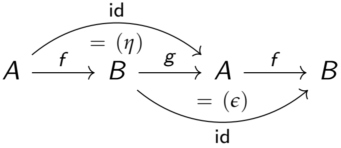
- *an arrow can be factorized as* 

- *various lifting problems are satisfied ...*

A PRE-MODEL STRUCTURE ON $\mathcal{UF}_i$

Weak equivalences are given by type equivalences:

$$f \in W \quad \text{iff} \quad \mathsf{IsEquiv}\, f$$



$$+ \; f(\eta_x) = \epsilon_{f(x)}$$

F-AC factorization (2008)

THE IDENTITY TYPE WEAK FACTORISATION SYSTEM

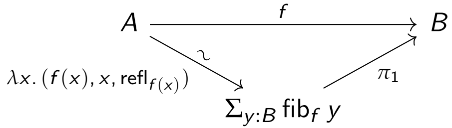
NICOLA GAMBINO AND RICHARD GARNER

Abstract. We show that the classifying category $\mathcal{C}(\mathbb{T})$ of a dependent type theory $\mathbb{T}$ with axioms for identity types admits a non-trivial weak factorisation system. We provide an explicit characterisation of the elements of both the left class and the right class of the weak factorisation system. This characterisation is applied to relate identity types and the homotopy theory of groupoids.

1. Introduction

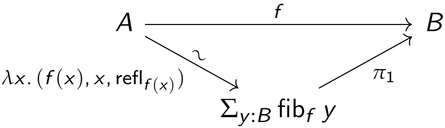
From the point of view of mathematical logic and theoretical computer science, Martin-Löf's axioms for identity types [25] admit a conceptually clear explanation in terms of the propositions-as-types correspondence [14, 22, 28]. The fundamental idea behind this explanation is that, for any two elements a, b of a type A, we have a new type $\mathrm{Id}_A(a, b)$, whose elements are to be thought of

F-AC factorization



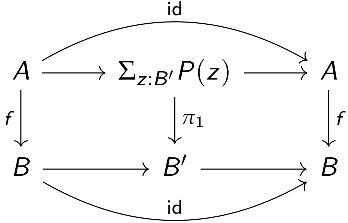
where $\mathsf{fib}_f\ y :\equiv \Sigma_{x:A}\, f\ x = y$

F-AC factorization



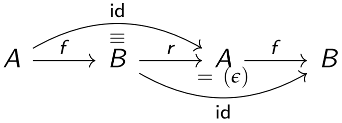
where $\mathsf{fib}_f\ y := \Sigma_{x:A}\, f\ x = y$

Fibrations



with $P : B' \to \mathcal{UF}_i$

Acyclic cofibrations (Injective equivalences)



$+\ \epsilon_{f(x)} \equiv \mathsf{refl}_{f(x)}$

AF-C factorization (2011)

MODEL STRUCTURES FROM HIGHER INDUCTIVE TYPES

PETER LEFANU LUMSDAINE

ABSTRACT. We show that for any dependent type theory with Martin-Löf identity types and *mapping cylinders* (defined as certain higher-dimensional inductive types), the category of contexts carries a *pre-model-structure*, i.e. a model structure minus the completeness conditions. The (trivial cofibrations,fibrations) are the Gambino-Garner weak factorisation system of [GG08], while the weak equivalences are equivalences in the sense of Voevodsky [Voe].

It follows that any categorical model of this type theory carries a pre-model-structure, and so, if it is additionally complete and co-complete, is a model category.

CONTENTS

1. Type-theoretic background 1
2. Type-theoretic mapping cylinders 3
3. A pre-model structure from mapping cylinders 4

AF-C factorization

```
Cyl {f : A → B} : B → Type :=
| top  : ∀ x, Cyl (f x)
| base : ∀ y, Cyl y
| eq   : ∀ x, base (f x) = top x.
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AF-C FACTORIZATION

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```
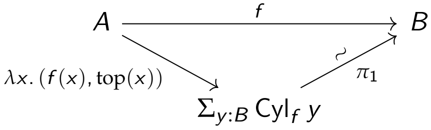
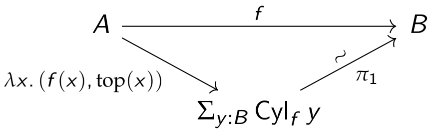

For all $y : B$, $\mathsf{Cyl}_f\ y$ is contractible.

And thus $\Sigma_{y:B}\, \mathsf{Cyl}_f\ y \simeq B$.

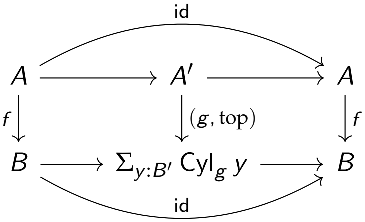
AF-C FACTORIZATION



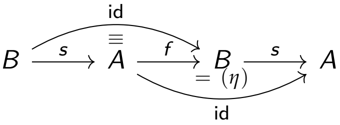
AF-C factorization



Cofibrations



Acyclic Fibrations
(Surjective Equivalences)



$$+\ \mathsf{ap}\ \ f\ \ \eta_x\ \equiv\ \mathsf{refl}_{f(x)}$$

A pre-model structure on $\mathcal{UF}_i$

Theorem

In $MLTT_2$, the (F, AC) and (AF, C) factorization systems give rise to a pre-model structure on $\mathcal{UF}_i$.

A PRE-MODEL STRUCTURE ON $\mathcal{UF}_i$

THEOREM

In MLTT$_2$, the (F, AC) and (AF, C) factorization systems give rise to a pre-model structure on $\mathcal{UF}_i$.

Formalized in Coq:

```
Record IsFibration {A B} (f : A → B) :=
  { fib_A : Type ;
    fib_P : fib_A → Type ;
    fib_Fib : FibrantF fib_P;
    fib_H : f RetractOf (π1 fib_P) }.
```

```
Theorem type_model_structure : model_structure TYPE_F.
Proof.
  rapply Build_model_structure.
  - exact (λ A B f, IsEquiv (repl_f f)).
  - exact Fib.
  - exact (@LLP TYPE AFib).
  - apply two_out_of_three_weak_equiv.
  - eapply wfs_iff_R. apply @AFib_ok.
```

Implementation in Coq

```
Axiom Fibrant : Type → Type.
Existing Class Fibrant.

Private Inductive paths {A : Type} (x : A) : A → Type :=
| idpath : paths x x.

Definition paths_ind {A} (FibA: Fibrant A) (x : A)
(P : ∀ y : A, paths x y → Type) (FibP : ∀ y p, Fibrant (P y p))
(u : P x idpath) (y : A) (p : paths x y) : P y p
 := match p with idpath ⇒ u end.
```

MLTT$_2$ AND A PRE-MODEL STRUCTURE ON $\mathcal{UF}_i$

MLTT$_2^{\mathcal{F}}$ AND A PRE-MODEL STRUCTURE ON $\mathcal{U}_i^{\mathrm{S}}$

MODEL OF MLTT$_2^{\mathcal{F}}$

Extending to $\mathcal{U}_i^{\mathrm{S}}$

We want to extend the result to $\mathcal{U}_i^{\mathrm{S}}$.

But the lifting properties are not satisfied anymore.

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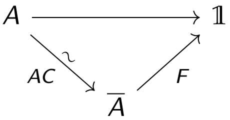
In a model category, the factorization



gives rise to a fibrant replacement.

Fibrant replacement

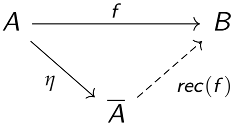
What could be a fibrant replacement in MLTT_2?

Fibrant replacement

What could be a fibrant replacement in MLTT_2?

A modality $\overline{A}$ such that:

- $\mathrm{Fib}\,\overline{A}$
- $\eta : A \to \overline{A}$
- if $\mathrm{Fib}\,B$:



$(rec(f) \circ \eta \equiv f)$

Fibrant replacement

Unfortunately, such a fibrant replacement is **inconsistent** in MLTT_2.

It was noticed by:

- Shulman et al. on the nLab
- also in Capriotti's thesis

This relies on the fact that $x = y \to \overline{x \equiv y}$

Fibrant replacement

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This relies on the fact that $x = y \to \overline{x \equiv y}$

$\Rightarrow$ we don't want $\overline{x \equiv y}$ to be fibrant.

In the model: the fibrant replacement is not stable under substitution.

A NEW TYPE THEORY: $\mathrm{MLTT}_2^{\mathcal{F}}$

$$\boldsymbol{\Gamma\,;\ \Delta \vdash A\ \ \mathbf{Fib}}$$

In the context Γ, the type family $\Delta \vdash A$ is *regularly fibrant*.

A NEW TYPE THEORY: $\mathrm{MLTT}_2^{\mathcal{F}}$

$$\boldsymbol{\Gamma\,;\ \Delta \vdash A\ \ \mathsf{Fib}}$$

In the context Γ, the type family $\Delta \vdash A$ is *regularly fibrant*.

$$\boldsymbol{\Gamma,\ \Delta\,;\ \cdot \vdash A\ \ \mathsf{Fib}}$$

$\Delta \vdash A$ is *degenerately fibrant* (weaker).

Some fibrancy rules of $\mathrm{MLTT}_2^{\mathcal{F}}$

$$\frac{\Gamma\,;\ \Delta \vdash A\ \ \mathsf{Fib} \qquad \Gamma\,;\ \Delta, x : A \vdash B\ \ \mathsf{Fib}}{\Gamma\,;\ \Delta \vdash \Pi x : A.\ B\ \ \mathsf{Fib}}$$

SOME FIBRANCY RULES OF $\mathrm{MLTT}_2^{\mathcal{F}}$

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$$\frac{\Gamma\,;\ \Delta \vdash A \ \ \mathsf{Fib} \qquad \Gamma \vdash \sigma : \Delta' \to \Delta}{\Gamma\,;\ \Delta' \vdash A\sigma \ \ \mathsf{Fib}}$$

E.g. if $\lambda\, n.\ P(n)$ is regularly fibrant, so is $\lambda\, n.\ P(n+2)$.

J-RULE

$$\frac{\Gamma \vdash A \ \mathsf{Fib} \qquad \Gamma \vdash t,\ t' : A \qquad \Gamma \vdash p : t =_A t' \qquad \boldsymbol{\Gamma ;\ y : A,\ q : t =_A y \vdash P\ \ \mathsf{Fib}} \qquad \Gamma \vdash u : P\{y := t, q := \mathsf{refl}_t\}}{\Gamma \vdash J_{=}(A,\ y.q.P,\ t,\ t',\ p,\ u) : P\left\{y := t', q := p\right\}}$$

$x = y \not\to \overline{x \equiv y}$ because $\lambda\, y.\ \overline{x \equiv y}$ only degenerately fibrant

(Degenerate) fibrant replacement

$$\frac{\Gamma \vdash A : \mathcal{U}_i^{\mathrm{S}}}{\Gamma \vdash \overline{A} : \mathcal{U}_i^{\mathrm{S}}} \qquad \frac{\Gamma \vdash A : \mathcal{U}_i^{\mathrm{S}}}{\Gamma ;\ \cdot \vdash \overline{A}\ \mathsf{Fib}} \qquad \frac{\Gamma \vdash A : \mathcal{U}_i^{\mathrm{S}}}{\Gamma \vdash \eta_A : A \to \overline{A}}$$

$$\frac{\Gamma ;\ z : \overline{A} \vdash P(z)\ \mathsf{Fib} \qquad \Gamma \vdash t : \Pi\, x : A.\ P(\eta_A\, x)}{\Gamma \vdash \mathsf{repl\_ind}_P\ t : \Pi\, z : \overline{A}.\ P(z)}$$

$$\mathsf{repl\_ind}_P\ t\, (\eta_A\, x) \simeq_{\beta\eta}\ t\, x$$

(Degenerate) fibrant replacement

We need a few more rules:

- Fibrant replacement of a function:

$$\overline{\mathrm{id}_A} \equiv \mathrm{id}_{\overline{A}} \qquad\qquad \overline{g \circ f} \equiv \overline{g} \circ \overline{f}$$

  where $\overline{f} : \overline{A} \to \overline{B}$.

(DEGENERATE) FIBRANT REPLACEMENT

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  where $\overline{f} : \overline{A} \to \overline{B}$.

- Extension of $P : A \to \mathcal{U}_i^{\mathrm{S}}$ to $\overline{A} \to \mathcal{U}_i^{\mathrm{S}}$:

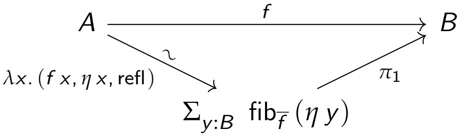
$$\frac{\Gamma \vdash P : A \to \mathcal{U}_i^{\mathrm{S}} \qquad \Gamma\,;\; x : A \vdash P\,x\ \ \mathsf{Fib}}{\Gamma\,;\; z : \overline{A} \vdash \mathsf{repl\_rec}_{A,\mathcal{U}_i^{\mathrm{S}}}\ P\,z\ \ \mathsf{Fib}}$$

  Consequence:

  If $\Gamma\,;\; x : A \vdash P(x)$ Fib then $\eta\, t = \eta\, t' \to P(t) \to P(t')$.

PRE-MODEL STRUCTURE ON $\mathcal{U}_i^{\mathrm{S}}$

- $f : A \to B$ is a weak-equivalence iff $\overline{f} : \overline{A} \to \overline{B}$ is a type equivalence
- (AC,F) factorization:



- idem for the (C, AF) factorization

A pre-model structure on $\mathcal{U}_i^{\mathrm{S}}$

Theorem

In $MLTT_2^{\mathcal{F}}$, there is a pre-model structure on the category $\mathcal{U}_i^{\mathrm{S}}$.

Formalized in Coq.

MLTT_2 and a pre-model structure on $\mathcal{UF}_i$

$\mathrm{MLTT}_2^{\mathcal{F}}$ and a pre-model structure on $\mathcal{U}_i^{\mathrm{S}}$

Model of $\mathrm{MLTT}_2^{\mathcal{F}}$

Model of $\mathrm{MLTT}_2^{\mathcal{F}}$

Interpretation of $\mathsf{MLTT}_2^{\mathcal{F}}$ in the Bezem-Coquand-Huber cubical model (without connections).

Our trick could probably replayed in other cubical models.

Interpretation of types (MLTT_2 and $\mathrm{MLTT}_2^{\mathcal{F}}$)

$\Gamma \vdash$

A cubical set is a presheaf on the cube category $\Gamma : \square^{\mathsf{op}} \to \mathsf{Set}$

$\Gamma \vdash A$

A cubical family $\Gamma \vdash A$ is given by:

- a set $A(I, \rho)$ for each $I \in \square$ and $\rho \in \Gamma(I)$
- a restriction $A(I, \rho) \to A(J, \rho f)$ for each $f : J \to I$ and $\rho \in \Gamma(I)$
- respecting identity and composition

Interpretation of fibrancy (MLTT_2)

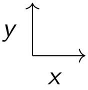
$\Gamma \vdash \boldsymbol{A}$ **Fib** is interpreted as:

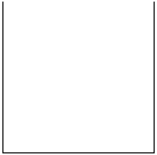
- for all $I \in \square$, S shape on I, $\rho \in \Gamma(I)$

  and $\vec{u}$ open-box of shape S in $A(\rho \quad)$, a there is *filler*

  $$\mathrm{fill}^{S}_{A(\rho \quad)}(\vec{u}) \in A(\rho \quad)$$

- such that . . .





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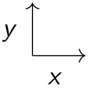

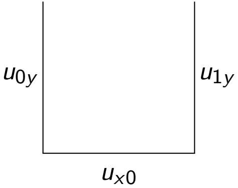

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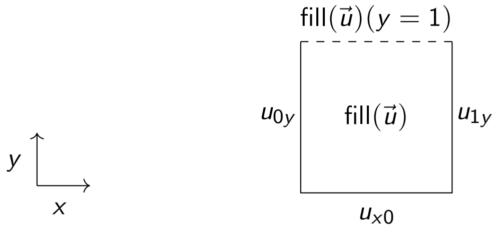


INTERPRETATION OF REGULAR FIBRANCY ($\mathrm{MLTT}_2^{\mathcal{F}}$)

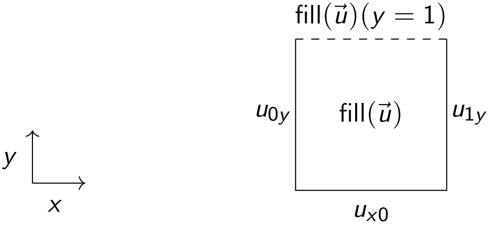
$.; \Gamma, \Delta \vdash A$ **Fib** is interpreted as:

- for all $I \in \square$, S shape on I, $\rho \in \Gamma(I)$

  $\delta \in \Delta(\rho)$ and $\vec{u}$ open-box of shape S in $A(\rho, \delta)$, a there is *filler*

$$\mathrm{fill}^{S}_{A(\rho,\delta)}(\vec{u}) \in A(\rho, \delta)$$

- such that ...



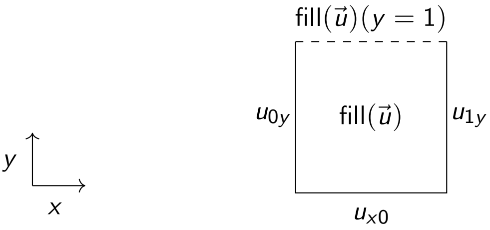
INTERPRETATION OF DEGENERATE FIBRANCY

$\Gamma\,;\ \Delta \vdash A$ **Fib** is interpreted as:

- for all $I \in \square$, S shape on I, $\rho \in \Gamma(I)$ **degenerate** along the direction of S, $\delta \in \Delta(\rho)$ and $\vec{u}$ open-box of shape S in $A(\rho, \delta)$, a there is *filler*

$$\mathsf{fill}^{S}_{A(\rho,\,\delta)}(\vec{u}) \ \in\ A(\rho,\,\delta)$$

- such that ...



Interpretation of degenerate fibrancy

Why does it works?

- Proofs of fibrancy rules lift.
- Sufficient to interpret transport.

  From Huber thesis:

First we define the *transport* along a path. Let $\Gamma \vdash A$ be a type and $\Gamma.A \vdash C$ be a Kan type. Furthermore let $\Gamma \vdash a : A$, $\Gamma \vdash b : A$, $\Gamma \vdash e : C[a]$, and $\Gamma \vdash d : \mathtt{Id}_A(a,b)$. (Recall that $[a]$ is the substitution $(1,a)\colon \Gamma \to \Gamma.A$.) We define a term $\Gamma \vdash \mathtt{subst}_C(d,e) : C[b]$ as follows. For $\rho \in \Gamma(I)$ and a fresh $x = x_I$ we have that $d\rho \mathbin{@} x \in A\rho s_x$ with $(d\rho \mathbin{@} x)(x=0) = a\rho$ and $(d\rho \mathbin{@} x)(x=1) = b\rho$. Thus $(\rho s_x, d\rho \mathbin{@} x)$ connects $[a]\rho$ to $[b]\rho$ along x. We define

$$\mathtt{subst}_C(d,e)\rho = C(\rho s_x, d\rho \mathbin{@} x)^+_x(e) \in C[b]\rho \qquad (3.15)$$

Interpretation of the fibrant replacement

The fibrant replacement is interpreted by an inductive-recursive set (construction from Huber thesis).

Proposition

The degenerate fibrant replacement commutes with substitutions. For all $\sigma : \Gamma' \to \Gamma$, $\overline{A\sigma} = \overline{A}\sigma$.

Conclusion

1. In MLTT_2: pre-model structure on $\mathcal{UF}_i$
2. $\mathrm{MLTT}_2^{\mathcal{F}}$: a type theory with a fibrant replacement
3. In $\mathrm{MLTT}_2^{\mathcal{F}}$: pre-model structure on $\mathcal{U}_i^{\mathrm{S}}$
4. Interpretation of $\mathrm{MLTT}_2^{\mathcal{F}}$ in the cubical model

   (Cylinders remain to be done)
5. Implementation in Coq of both systems,

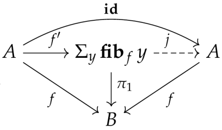
   1. and 3. are formalized.

Article: `https://hal.archives-ouvertes.fr/hal-01579822`

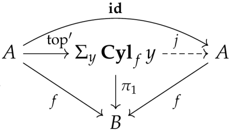
Formalization: `https://github.com/CoqHott/model-structures-Coq`

CHARACTERIZATION OF CLASSES IN MLTT_2

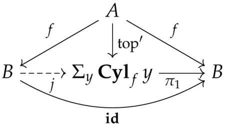
$f \in F$ iff.



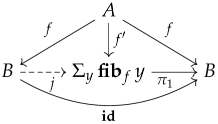
$f \in AF$ iff.



$f \in C$ iff.



$f \in AC$ iff.



where f' is $\lambda\, x.\ (f\ x,\ x,\ \mathbf{refl}_{f\ x})$ and top′ is $\lambda\, x.\ (x,\ \mathbf{top}_f\ x)$.