# Growth condition on the generator of BSDE with singular terminal value ensuring continuity up to terminal time

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**Abstract**

We study the limit behavior of the solution of a backward stochastic differential equation when the terminal condition is singular, that is it can be equal to infinity with a positive probability. In the Markovian setting, Malliavin's calculus enables us to prove continuity if a balance condition between the growth w.r.t. $y$ and the growth w.r.t. $z$ of the generator is satisfied. As far as we know, this condition is new. We apply our result to liquidity problem in finance and to the solution of some semi-linear partial differential equation ; the imposed assumption is also new in the literature on PDE.

**Keywords.** Backward stochastic differential equation, singular terminal condition, Malliavin's calculus

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# 1 Introduction

In this paper we are interested in the limit behavior of the solution of a backward stochastic differential equation (BSDE for short) with singular terminal condition. BSDEs' theory has been widely developed since more than 30 years because they are a very useful tool in two domains: stochastic optimal control and partial differential equations (PDEs for short). In the first topic, BSDEs naturally appear as the adjoint equation in the Pontryagin's maximum principle (see for example [31, Chapters 3 and 7]). The dynamics of this equation is given on a time interval $[0, T]$ in a backward way: a terminal condition is given and the equation should be solved from $T$ to 0. Concerning their application in PDEs theory, BSDEs provide an extension of the Feynman-Kac formula to non-linear PDE and the couple forward SDE and BSDE is a method of characteristics to solve a second-order PDE (see [21, Sections 5.4 and 5.7] or [32, Chapter5]).

To obtain a solution with suitable integrability condition, it is usually assumed that the terminal condition of the BSDE, denoted $\xi$, is integrable, as in [21, Section 5.3]. Then a priori estimates show that the solution of the BSDE is also integrable. But a large class of PDEs doesn't satisfy such constraint. Indeed for forward reaction-diffusion PDE of the form

$$\frac{\partial u}{\partial t} - \Delta u + u^q = 0,$$

with $q > 1$, it is known that the initial value of $u$ can be equal to $+\infty$. This property has been proved by analytic methods in [16] or by probabilistic[1] arguments in [8, 9]. Roughly speaking, the solution can blow up on a non-empty set: $\lim_{t\to 0} u(t, x) = +\infty$. Furthermore in the context of stochastic control, if a target is imposed on the final value of the state process $\Xi$, then a singularity appears in the related adjoint equation. The optimal liquidation problem in finance is a typical example where the mandatory liquidation constraint can be written as: $\Xi_T = 0$ and the terminal value for the BSDE is $+\infty$ almost surely (see [3, 27]).

Let us now fix some notations. In this paper, we consider BSDE of the following form

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,$$

where $W$ is a $d$-dimensional Brownian motion, the function $F$ is called the generator or the driver and $\xi$ is the terminal condition. Since we impose that the solution is adapted to the underlying filtration generated by $W$, the solution is the couple $(Y, Z)$. We classically assume that $F$ is continuous and monotone w.r.t. $y$ and Lipschitz continuous w.r.t. $z$. When $\xi$ is integrable, these assumptions are sufficient to obtain a solution (see [21, Section 5.3]). However to ensure the existence of a solution without integrability condition on $\xi$, [13, 23] show that it is sufficient to impose that there exist a positive process $\eta$ and $q > 1$ such that

$$\forall y \in \mathbb{R}_+, \quad \forall (s, z) \in [0, T] \times \mathbb{R}^d, \quad F(s, y, z) - F(s, 0, z) \leq -\eta_s |y|^q.$$

Then there exists a minimal solution $(Y, Z)$ such that a.s.

$$\liminf_{t\to T} Y_t \geq \xi = Y_T. \tag{1}$$

This behavior is sufficient to solve the related control problem with constraint, see [13, Section 2]. The possible lack of continuity at time $T$ is due to the singularity of $\xi$ ; in other words if $\xi$ is in $L^p(\Omega)$ for some $p > 1$, then a.s.

$$\lim_{t\to T} Y_t = \xi = Y_T. \tag{2}$$

A natural question is: if $\xi$ is not in some $L^p$ space, under which conditions on $F$ or on $\xi$, does (2) hold ? This question is called the continuity problem and has been studied in [22, 23, 28, 17, 1]. This question is important and we refer to [1, Section 1.1] for the implications of this problem. The known results can be summarized as follows:

- The existence of the left limit at time $T$ only depends on $F$, see [23, Theorem 3.1].
- Equality in (1) holds in the Markovian case and if the growth of $F$ is sufficient fast (when $y$ tends to $\infty$), see [23, Theorem 4.5]. Roughly speaking, $q > 3$ is assumed in this result.

[1] The notion of superprocesses is used, which is completely different from our method.

- Going beyond the Markovian setting has been done in [17], but again under a strong growth condition on $F$, or in [28, 1], but for specific terminal conditions $\xi$.

The quadratic case ($q = 2$) nor the financial data-driven case of [2], where $q$ is estimated around 1.6, are not included in the existing literature.

The aim of the paper is to obtain the $\mathbb{P}$-a.s. equality

$$\liminf_{t\to T} Y_t = \xi = Y_T, \tag{3}$$

in the Markovian framework, without restriction on the growth of $F$, that is for any $q > 1$. Hence in the rest of the paper, we consider the system

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{4}$$

$$Y_t = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s, dW_s, \tag{5}$$

with unknown stochastic processes $(X, Y, Z)$ with values in $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$ and with measurable parameters: $b : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $\sigma : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^{m\times d}$, $g : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $F : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

In [22], the specific case $F(s, x, y, z) = -y|y|^{q-1}$, $q > 1$, is studied and it is proved that (3) holds. The arguments of the proof are not the same if $q > 3$ or not. When $q$ is sufficiently large, an a priori estimate on $Z$ and Hölder's inequality are used, whereas for $q \leq 3$, the Malliavin calculus is used to control the process $Z$, with the equality $Z_t = D_t Y_t$ and the Malliavin by-parts integration.

Here we are going to generalize this result, with two novelties. The first one of this paper concerns the drift $b$. To obtain the Malliavin derivative $DY$ of $Y$, it is usually assumed that $\sigma$ and $b$ have bounded derivatives w.r.t. $x$, which ensures that the Malliavin derivative $DX$ of $X$ solves a linear SDE with bounded coefficients, and thus $DX$ verifies some strong integrability properties. These properties are then used to derive the existence of $DY$ (see [10] or [18]). This setting is kept in [22, 23]. With the representation $Z_t = D_t Y_t$ and a by-parts integration, $\mathbb{E}(Z_t \phi(X_t))$ is transformed into $\mathbb{E}(Y_t \psi(X_t))$, where $\psi$ depends on $\phi$ and on the probability density function $p$ of $X$. To ensure the existence of $p$, a uniform ellipticity condition on $\sigma$, together with the boundedness of $b$ and $\sigma$, are also assumed in [22, 23].

In our paper, we do not still suppose that $b$ is Lipschitz conitnuous in $x$. In contrast to the other papers, we take advantage of the uniform ellipticity condition to remove the drift with a Girsanov transformation on the SDE (4). Therefore we only assume that $b$ is bounded, of class $C^1$, with derivatives of polynomial growth. We are aware that some works prove the existence of the Malliavin derivative for $X$, without the Lipschitz condition on $b$ (see for example [19]). However the existence of $DX$ (in a weak sense), without good integrability properties, is not sufficient to obtain the existence of $DY$. This is the reason why we modify the SDE in order to keep a Malliavin derivative with suitable integrability conditions. The key point is that if (3) holds under $\mathbb{P}$, it holds under any other probability measure equivalent to $\mathbb{P}$.

The second main novelty of this paper concerns the generator $F$. The used framework is standard for BSDE, that is $F$ is Lipschitz continuous w.r.t. $z$. But, to be able to manage this dependence on $z$, if $q \leq 3$, we suppose that:

$$|F(s, x, 0, z) - F(s, x, 0, 0)| \leq C(1 + |x|^\ell)|z|^\alpha, \quad 0 \leq \alpha < \frac{2(q-1)}{q+1} \leq 1. \tag{6}$$

Under this condition, we prove that (3) holds. If $q > 3$, we only suppose that $F$ is Lipschitz continuous in $z$ (as in [23]). Therefore there is an interplay between the growth of $F$ w.r.t. $y$, controlled by $q$, and the growth w.r.t. $z$, controlled by $\alpha$. In particular when $q$ is close to one, $f$ should be almost bounded in $z$. As far as we know, this property is new in the BSDEs' literature but also in the PDE theory.

If $F$ is linear w.r.t. $z$, Condition (6) cannot be satisfied. But the Girsanov transformation of the SDE, used if $b$ is not Lipschitz continuous, adds a new linear term in the generator $F$ of the form $\langle a(t, X_t), Z_t\rangle$. Therefore this condition is incompatible with this modification. Therefore we split $F$ into two parts: $F(t, x, y, z) = f(t, x, y, z) + \langle a(t, x), z\rangle$ and we show that it is sufficient that only the function $f$ satisfies Condition (6). In other words if $F$ is linear w.r.t. $z$, we don't need (6) to have (3).

Notice that our result (3) is not satisfied if we consider a BSDE with jumps

$$Y_t = \xi + \int_t^T F(s, X_s, Y_s, Z_s, U_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}} U_s(e) \tilde{N}(ds, de), \quad 0 \le t \le T,$$

where $\tilde{N}$ is a compensated Poisson random measure. Indeed a counter-example is developped in [6] with a simple Poisson process $N$, a simple driver $F(s, x, y, z, u) = -y|y|^{q-1}, q > 1$ and a simple SDE $X = N$. In this case, there is discontinuity in $T$: $\lim_{t \to T} Y_t = +\infty \neq \xi$.

**Breakdown of the paper** In the next section, we present the known results on BSDE with singular terminal condition.

The main result of this paper (Theorem 1) is stated at the beginning of Section 3. The rest of this section contains the proof of this result.

Essential points of our reasonning are based on the fact the we are in a Markovian framework, that is the randomness of the driver $F$ at the moment $t$ is given only through the random variable $X_t$ and $\xi$ is a function of $X_T$. First to obtain (3) on the non-singular set $\{\xi < +\infty\}$, we use test-processes $\varphi(X)$, where $\varphi$ is a test-function with compact support in the non-singular set. Instead of $Y$, we study $\varphi(X)Y$. At time $T$, $\varphi(X_T)Y_T = \varphi(X_T)\xi$ is integrable. Moreover in the Itô's formula, the cross variation term is of the form $\nabla\varphi(X)\sigma(X)Z$. To deal with this term, especially when $q$ is small, the Markovian setting is again crucial. Indeed we can represent the process $Z_t$ as the Malliavin derivative of $Y$: $Z_t = D_t Y_t$, and we use a Malliavin integration by parts (see Proposition 6 and Corollaries 2 and 3). Let us note here that the use of test functions can be generalized to smooth Itô functionals, as in [17]. However, outside the Markovian setting, we do not know how to control the cross-variation term when $q$ is small (see Condition (H) in [17], which requires $q > 3$).

To obtain these results, $X$ should have a Malliavin derivative with suitable integrability conditions. Under our setting, we use the Girsanov transformation to remove the drift part of the SDE. Note that if $b$ is Lischitz continuous, we don't need this transformation.

Since we are able to control the cross variation term, we can also deal to a linear term w.r.t. $z$: $\langle a, Z\rangle$. For large values of $q$, this linear term is superfluous and could be contained in $F$. But for $q < 3$, Condition (6) excludes linear growth and then adding this linear term $\langle a(t, x), z\rangle$ makes sense. This is the first reason why we assume the particular structure (13) for $F$. Another reason comes from the Girsanov transform. Indeed changing the probability measure to manage the drift $b$ in the SDE, adds a linear term in the BSDE. Let us note that the same conditions are imposed on $b$ (drift of the SDE) and $a$ (linear part of the BSDE).

In Section 4, we first state a comparison result for the minimal solutions of BSDEs with singular terminal condition. A direct consequence of this comparison principle for BSDE shows that (3) holds if the generator is bounded from above by a generator satisfying the assumptions of our main result.

In Part 4.2, we apply this continuity result to the optimal liquidation problem studied in [13]. The goal is to minimize

$$J(t, \alpha) = \mathbb{E}\left( \int_t^T (\eta_s |\alpha_s|^p + \gamma_s |\Xi_s|^p) \, ds + \xi |\Xi_T|^p \middle| \mathcal{F}_t \right),$$

among all $\alpha$ such that the state process $\Xi$ is given by $d\Xi_t = \alpha_t dt$, with fixed $\Xi_0$. On the set $\{\xi = +\infty\}$, to have a finite cost, the terminal value $\Xi_T$ should be equal to zero. Hence the full liquidation studied in [3] corresponds to the case $\xi = +\infty$ a.s. The value function of this control problem, together with an optimal control, are given by the solution $(Y, Z)$ of a BSDE with terminal value $\xi$ (see BSDE (28)) with the limit behavior (1). The lack of continuity at time $T$ is interpreted as an extra cost due to the liquidation constraint. In the Markovian setting, we prove that there is no additional cost to minimize the control problem.

In Section 4.3, we also apply our result to the related PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) + \mathcal{L}(u)(t, x) + F(t, x, u(t, x), (\nabla u \sigma)(t, x)) = 0, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^m \\ u(T, x) = g(x), \quad \forall x \in \mathbb{R}^m, \end{cases} \tag{7}$$

where $\mathcal{L}$ is the infinitesimal generator of the SDE (4):

$$\mathcal{L}(u) = \langle b, \nabla u\rangle + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^* \nabla^2 u\right). \tag{8}$$

The singular case $\{g = +\infty\} \neq \varnothing$ has been studied in [9, 14, 16] when $F(s, x, y, z) = -y|y|^{q-1}$, but not when $F$ depends on $z$. In [24], $F$ could depend on $z$ but only for large values of $q$ (see [24, Theorem 2]). This paper fills the gap if (6) holds. In other words the minimal viscosity solution $u$ satisfies the continuity condition

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0).$$

In the appendix 5, we set out all technical inequalities required in the proof of our main result.

**Notations** In this paper we consider a deterministic time horizon $T \in \mathbb{R}_+^*$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a $d$-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ defined on the probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ the augmented filtration generated by $W$. For all $p \in [2, +\infty[$, we note:

- $\mathbb{D}^{1,p}$ is the domain of the Malliavin derivative operator in $L^p(\Omega)$. Furthermore we note $\mathbb{D}^{1,\infty} = \bigcap_{p\geq 2} \mathbb{D}^{1,p}$. For $A \in \mathbb{D}^{1,p}$ we note $(D_\theta A)_{0\leq\theta\leq T}$ its Malliavin derivative and for $X$ a $\mathbb{D}^{1,p}$-process we note $(D_\theta X_t)_{0\leq\theta,t\leq T}$.
- $S^p(0,T)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that
$$\mathbb{E}\left(\sup_{0\leq t\leq T} |A_t|^p\right) < +\infty$$
and $S^\infty(0,T) = \bigcap_{p\geq 1} S^p(0,T)$.
- $H^p(0,T)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that
$$\mathbb{E}\left(\int_0^T |A_t|^p dt\right) < +\infty$$
and $H^\infty(0,T) = \bigcap_{p\geq 1} H^p(0,T)$.
- Whenever the notation $T-$ appears in the definition of a process space, we mean the set of all processes whose restrictions satisfy the respective property when $T-$ is replaced by any $T - \varepsilon$, $\varepsilon > 0$. For example, $S^p(0,T-) = \bigcap_{\varepsilon>0} S^p(0,T-\varepsilon)$. Morevore we say that a sequence $(F_n)_{n\in\mathbb{N}}$ converges in $S^p(0,T-)$ to $F \in S^p(0,T-)$ if for any $\varepsilon > 0$, the sequence $(F_n)_{n\in\mathbb{N}}$ converges to $F$ in $S^p(0,T-\varepsilon)$.

In the rest of the paper, $C$ denotes a generic constant, which can depend on other coefficients, and may change from line to line.

# 2 Setting and known results

## 2.1 SDE with control of the density

To ensure existence and uniqueness of the solution $X$ of the SDE (4), we suppose that the parameters $b$ and $\sigma$ satisfy the next conditions.

**Assumption 1.**

1. *$\sigma$ is bounded and continuous on $[0,T]\times\mathbb{R}^m$ and of class $C^2$ with respect to $x$ with bounded first derivatives and bounded second derivatives of $\sigma\sigma^*$*
$$\forall i,j \in \{1,...,m\}, \quad \frac{\partial\sigma}{\partial x_i}, \frac{\partial^2(\sigma\sigma^*)}{\partial x_i\partial x_j} \in L^\infty([0,T]\times\mathbb{R}^m).$$
2. *$\sigma\sigma^*$ is uniformly $\lambda$-elliptic: there exists $\lambda > 0$ such that*
$$\forall s \in [0,T],\ \forall (x,y) \in \mathbb{R}^m\times\mathbb{R}^m, \quad \langle\sigma(s,x)\sigma^*(s,x)y,y\rangle \geq \lambda|y|^2.$$
3. *$b$ is bounded and continuous on $[0,T]\times\mathbb{R}^m$ and of class $C^1$ with respect to $x$ with polynomial growth derivatives: there exist $\ell \in [1,+\infty)$ and $C \geq 0$ such that*
$$\forall s \in [0,T], \forall x \in \mathbb{R}^m, \quad \left|\frac{\partial b}{\partial x_i}(s,x)\right| \leq C(1+|x|^\ell).$$

Under these conditions[2], according to [30, Theorem 6], the SDE (4) has a unique solution $X$ in $S^\infty(0,T)$. Furthermore we have according to [7]:

**Proposition 1.** *The process $X$ admits a probability density $p$ such that:*

1. *On $[\varepsilon, T] \times \mathbb{R}^m$, the density $p$ is continuous with respect to $(t,x)$ with a continuous derivative with respect to $x$.*
2. *There exists $c \in \mathbb{R}_+^*$ such that, for all $s \in ]0,T]$ and $x \in \mathbb{R}^m$,*

$$\frac{1}{cs^{\frac{1}{2}}} \exp\left(-c\frac{|x-x_0|^2}{s}\right) \leq p(s,x) \leq \frac{c}{s^{\frac{1}{2}}} \exp\left(-\frac{|x-x_0|^2}{cs}\right).$$

**Remark 1.** *The second property is the well known Aronson's estimate (see [4, 12, 25, 29]). If the function $b$ linearly grows with a second derivative with polynomial growth w.r.t. $x$, then the result is still true but we have to add a term $\exp(cs|x|^2)$ in the upper bound and $\exp(-s|x|^2/c)$ in lower bound (see [11, Proposition 1.2]). In general, if the function $\sigma$ is also with linear growth, then the Aronson estimate is not verified. However we only need a positive lower bound for the density to obtain our result. For example if $X$ is a geometric Brownian motion, the probability density is log-normal and thus our result also holds in this case.*

## 2.2 BSDE with singular terminal condition

For the BSDE (5), we use classical hypotheses on the generator $F$.

**Assumption 2.**

1. *The function $F$ is continuous and monotone with respect to $y$: there exists $\chi \in \mathbb{R}$ s.t.*

$$\forall (t,x,y,y',z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d, \quad (y-y')(F(t,x,y,z) - F(t,x,y',z)) \leq \chi |y-y'|^2.$$

2. *There exists a constant $\ell \in [1, +\infty)$ such that for all $\rho \in \mathbb{N}$, there exists $\widetilde{K}_\rho \geq 0$ such that*

$$\forall t \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d, \quad \sup_{|y| \leq \rho} |F(t,x,y,z)| \leq \widetilde{K}_\rho (1 + |x|^\ell + |z|).$$

3. *The function $F$ is of class $C^1$ and uniformly Lipschitz continuous in $z$: there exists $K \geq 0$ such that*

$$\forall s \in [0,T], x \in \mathbb{R}^m, y \in \mathbb{R}, z, z' \in \mathbb{R}^d, \quad |F(s,x,y,z) - F(s,x,y,z')| \leq K|z-z'|.$$

Under this framework, if $\xi = g(X_T) \in L^p(\Omega)$ for some $p > 1$, then there exists a unique solution $(Y,Z) \in S^p(0,T) \times H^p(0,T)$ to the BSDE (5) (see [5, Theorem 4.2]).

Our goal is to deal with singular terminal condition. We consider the Markovian framework with a terminal condition

$$\xi = g(X_T)$$

where $g : \mathbb{R}^m \to \overline{\mathbb{R}}_+$ is a deterministic measurable function. We assume the next setting for $g$.

**Assumption 3.**

1. *The function $g : \mathbb{R}^m \longrightarrow \overline{\mathbb{R}}_+$ is measurable.*
2. *The set of singularities $\mathcal{S} = \{x \in \mathbb{R}^m,\ g(x) = +\infty\}$ is closed in $\mathbb{R}^m$.*
3. *$g$ is continuously differentiable and Lipschitzian on each $\mathcal{O}_n = \{x \in \mathbb{R}^m,\ g(x) \leq n\}$ for every $n \in \mathbb{N}$.*
4. *The singular terminal condition $\xi = g(X(T))$ satisfies a local integrability condition: for all compact set $\mathcal{K}$ of $\mathbb{R}^m \backslash \mathcal{S}$,*

$$g(X_T)\mathbb{1}_{\mathcal{K}}(X_T) \in L^2(\Omega, \mathcal{F}_T).$$

[2] Note that boundedness of $b$ is sufficient to get existence and uniqueness.

To construct a solution when $\xi$ is not integrable, we proceed by truncation and consider the following BSDE: for any $n \in \mathbb{N}$

$$Y_t^n = \xi^n + \int_t^T F^n(s, X_s, Y_s^n, Z_s^n) ds - \int_t^T Z_s^n dW_s, \quad t \in [0, T] \tag{9}$$

with

$$\xi^n = \varphi_n(\xi), \quad F^n(s, x, y, z) = F(s, x, y, z) - F(s, x, 0, 0) + \varphi_n(F(s, x, 0, 0)),$$

where $(\varphi_n)_{n \in \mathbb{N}}$ a non-decreasing sequence of smooth non-decreasing functions such that

$$\forall n \in \mathbb{N}, \quad \forall u \in \mathbb{R}, \quad \varphi_n(u) = \begin{cases} u & \text{si} \quad u \leq n-1 \\ n & \text{si} \quad u \geq n+1 \end{cases}, \quad u \wedge (n-1) \leq \varphi_n(u) \leq u \wedge n. \tag{10}$$

**Proposition 2.** *Under Conditions* 1, 2 *and* 3*, the truncated BSDE (9) admits a unique solution* $(Y^n, Z^n)$ *in* $S^p(0, T) \times H^p(0, T)$ *for all* $p \in ]1, +\infty[$. *Moreover the sequence* $Y_n$ *is non-decreasing and the process* $Y^n$ *is bounded from above: there exists a constant* $C$ *such that for* $m \leq n$

$$\forall t \in [0, T], \quad Y_t^m \leq Y_t^n \leq C(T + 1)n.$$

*Proof.* Existence and uniqueness directly follows from [5, Theorem 4.2]. Now standard a priori estimate on the solution of a BSDE (see [21, Theorem 5.30]) and the comparison theorem (see [21, Theorem 5.33]) imply that there exists $\tilde{C} \geq 0$ such that if $m \leq n$, then a.s. for any $t \in [0, T]$

$$Y_t^m \leq Y_t^n \leq \tilde{C} e^{(\chi + 2\tilde{C}^2)T} n(T + 1),$$

where $\tilde{C}$ depends on the Lipschitz constant of $F$ w.r.t. $z$. This achieves the proof of this proposition. □

Since $Y_t^n$ is a non-decreasing sequence, its limit $Y_t$ exists. However the upper estimate on $Y^n$ is not sufficient to ensure that $Y_t$ is finite. But let us emphasize that for any $n$, $Y^0 \leq Y^n \leq Y$ with $Y^0 \in S^\infty(0, T)$.

Finally to obtain a suitable a priori estimate on $Y^n$, we add extra assumptions on $F$.

**Assumption 4.**

1. *The functions* $F$ *satisfies:*

$$\forall s \in [0, T], x \in \mathbb{R}^m, \quad F(s, x, 0, 0) \geq 0.$$

2. *There exists a constant* $q > 1$*, a positive process* $\eta$ *whose inverse is of polynomial growth: there exists* $C \in \mathbb{R}_+^*$ *and* $\ell \in [1, +\infty)$ *such that*

$$\forall s \in [0, T], x \in \mathbb{R}^m, \quad \frac{1}{\eta(s, x)} \leq C(1 + |x|^\ell)$$

*and*

$$\forall y \in \mathbb{R}_+, t \in [0, T], x \in \mathbb{R}^m, z \in \mathbb{R}^d, \quad F(t, x, y, z) - F(t, x, 0, z) \leq -\eta(t, x)|y|^q.$$

Then we have, according to [23] applied with $\varphi_n$ (controlled between $\cdot \wedge (n-1)$ and $\cdot \wedge n$) instead of with $\cdot \wedge n$, the following result.

**Proposition 3.** *Under Conditions* 1, 2, 3 *and* 4*, the sequence* $(Y^n, Z^n)$ *converges to* $(Y, Z)$ *in* $S^\infty(0, T-) \times H^\infty(0, T-)$. *The limit* $(Y, Z)$ *is the minimal supersolution to the BSDE (5) on* $[0, T[$ *in the sense that:*

1. *The couple* $(Y, Z)$ *belongs to* $S^\infty(0, T-) \times H^\infty(0, T-)$.
2. *The process* $Y$ *is non negative.*
3. *For all* $0 \leq s \leq t < T$,

$$Y_s = Y_t + \int_s^t F(r, X_r, Y_r, Z_r) dr - \int_s^t Z_r dW_r.$$

4. *The process* $Y$ *satisfies the supercondition* (3) *on the left at* $t = T$*: a.s.* $\liminf_{t \to T} Y_t \geq \xi$.

5. *The process* $(Y,Z)$ *is minimal: if* $(\tilde{Y},\tilde{Z})$ *satisfies the four previous points, then a.s. for any* $t$, $Y_t \leq \tilde{Y}_t$.

A key step to obtain this result is the existence of a suitable a priori estimate on $Y^n$.

**Proposition 4.** *Under Conditions* 1 *to* 4*, for any* $r>1$*, there exists a constant* $K_r$ *depending on* $r>1$ *(and the constants in our assumptions) such that a.s. for any* $t\in[0,T]$ *and* $n\geq 1$*:*

$$Y_t^n \leq \frac{1}{(T-t+\frac{1}{n^{q-1}})^{q_*}}\left\{\frac{1}{n^{q-1}} + K_r\left[\mathbb{E}\left(\int_t^T \left(\left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*}F(s,X_s,0,0)\right)^r ds\middle|\mathcal{F}_t\right)\right]^{\frac{1}{r}}\right\} \tag{11}$$

*where* $q_*$ *is the Hölder conjugate of* $q$*:* $\dfrac{1}{q}+\dfrac{1}{q_*}=1$.

The proof of this proposition is set out in the appendix. As a consequence, the process $Y$ satisfies on $[0,T)$:

$$0 \leq Y_t^n \leq Y_t \leq \frac{K_r}{(T-t)^{q_*}}\left[\mathbb{E}\left(\int_t^T \left(\left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*}F(s,X_s,0,0)\right)^r ds\middle|\mathcal{F}_t\right)\right]^{\frac{1}{r}}. \tag{12}$$

**Remark 2.** *The non-negativity condition in* 4 *on* $F(s,x,0,0)$ *can be relaxed. Then the minimal supersolution* $Y$ *is only bounded from below by* $Y^0\in S^\infty(0,T)$*. We give details in Remark 7.*

As mentioned in the introduction, our aim is to prove that (3) holds: a.s. $\liminf_{t\to T} Y_t = \xi$. Note that on the event $\{\xi=+\infty\}$, we directly have $\lim_{t\to T} Y_t = \liminf_{t\to T} Y_t = \xi = +\infty$.

# 3 Main result and its proof

Taking into account:

- the fact that (6) rules out the linear case,
- the transformation of the SDE (4) when $b$ is not Lipschitz continuous (see Girsanov transformation 3.1.1),

we suppose that the driver of the BSDE (5) can be written:

$$F(t,X_t,Y_t,Z_t) = f(t,X_t,Y_t,Z_t) + \langle a(t,X_t),Z_t\rangle \tag{13}$$

where $f:[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ and $a:[0,T]\times\mathbb{R}^m\to\mathbb{R}^d$ are measurable deterministic functions. We assume that $a$ is bounded and $f$ satisfies Condition 2 and 4. Hence these two assumptions are also verified by $F$. Therefore one can easily check that Propositions 2, 3, 4 are still valid. Moreover to obtain the Malliavin differentiability of $(Y^n,Z^n)$, we impose that $F$ is of class $C^1$ w.r.t. $(x,y,z)$ with derivatives of polynomial growth. In details:

**Assumption 5.**

1. *The function* $f$ *is of class* $C^1$ *with respect to* $y$ *and the partial derivative* $\dfrac{\partial f}{\partial y}$ *is locally uniformly bounded :*

$$\forall M\in\mathbb{R}_+^*, \quad \exists C_M\in\mathbb{R}_+^*, \quad \forall y\in[-M,M], s\in[0,T], x\in\mathbb{R}^m, z\in\mathbb{R}^d, \quad \left|\frac{\partial f}{\partial y}(s,x,y,z)\right| \leq C_M.$$

2. *The function* $f$ *is of class* $C^1$ *with respect to* $x$ *and the partial derivative are locally uniformly polynomial growth with respect to* $y$ *:*

$$\exists \ell\in[1,+\infty), \quad \forall M\in\mathbb{R}_+^*, \quad \exists C_M\in\mathbb{R}_+^*, \quad \forall y\in[-M,M], s\in[0,T], x\in\mathbb{R}^m, z\in\mathbb{R}^d,$$

$$\left|\frac{\partial f}{\partial x_i}(s,x,y,z)\right| \leq C_M(1+|x|^\ell+|z|^\ell).$$

3. *The function $a$ verifies the same condition as $b$ in Assumption 1, namely $a$ is bounded and of class $C^1$ with respect to $x$ and the partial derivatives of the function $a$ are polynomial growth: there exists $C \in \mathbb{R}_+^*$ such that*

$$\exists \ell \in [1, +\infty), \quad \forall s \in [0, T], x \in \mathbb{R}^m, \quad \left| \frac{\partial a}{\partial x_i}(s, x) \right| \leq C(1 + |x|^\ell).$$

Remark that since $f$ is Lipschitz continuous w.r.t. $z$ (Assumption 2), its partial derivative w.r.t. $z$ is bounded. Moreover we can choose a constant $\ell$ as the maximum between the quoted constants in Assumptions 1, 2, 4 and 5. Let us state our main result.

**Theorem 1.** *Assume that the generator $F$ admits the structure given by* (13) *and that Assumptions* 1 *to* 5 *hold. If for $q \leq 3$ there exist $0 \leq \alpha < \dfrac{2(q-1)}{q+1}$, $C \in \mathbb{R}_+^*$ and $\ell \geq 1$ such that*

$$\forall (s, x, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^d, \quad |f(s, x, 0, z) - f(s, x, 0, 0)| \leq C(1 + |x|^\ell)|z|^\alpha, \tag{14}$$

*then the minimal supersolution $(Y, Z)$ to the BSDE* (5) *built in Proposition 3 satisfies $\mathbb{P}$-almost surely: $\liminf_{t \to T} Y_t = \xi$.*

**Remark 3.** *The growth assumption of $f$ w.r.t. $z$ is new and never appears in the existing literature. Note that for $q > 3$, Condition* (14) *holds with $\alpha = 1$ from Assumption 2, as in [23]. Let us again emphasize that this growth assumption does not concern the linear part $\langle a, z \rangle$ of $F$.*

The rest of this section concerns the proof of this result. Evoke that $(Y^n, Z^n)$ is the solution of the BSDE (9) and satisfies the properties of Proposition 2, whereas the limit $(Y, Z)$ is given in Proposition 3.

## 3.1 Malliavin differentiability of the couple $(Y^n, Z^n)$

Our main tool is the Malliavin calculus. As explained in the introduction, the Malliavin differentiability of $(Y, Z)$ relies on the differentialbility of $X$. If $b$ is Lipschitz continuous (or of class $C^1$ with bounded derivative), we could directly apply Proposition 5 below to the SDE (4) and keep the initial system (4)-(5).

With our more general setting, we can apply [19, Theorem 3.3], which states that $X$ has a Malliavin derivative $D_\theta X$. But the integrability properties of $D_\theta X$ may be lost and these properties are crucial to obtain the Malliavin derivative of $Y^n$ (see Proposition 6).

### 3.1.1 A transformation of the system (4)-(5)

To circumvent this issue, we modify the system using Girsanov's transformation. From Assumption 1, $\sigma$ is uniformly elliptic thus $(\sigma\sigma^*)(s, x)$ is invertible for any $s \in [0, T], x \in \mathbb{R}^m$. The SDE (4) can be written:

$$\begin{aligned} X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\ &= x_0 + \int_0^t \sigma(s, X_s) \left[ dW_s + \sigma^*(s, X_s)(\sigma(s, X_s)\sigma^*(s, X_s))^{-1} b(s, X_s)ds \right] = x_0 + \int_0^t \sigma(s, X_s)d\widetilde{W}_s. \end{aligned}$$

Moreover $\sigma^*(\sigma\sigma^*)^{-1}b$ is bounded. Hence according to the Girsanov theorem, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for any $t \in [0, T]$

$$\widetilde{W}_t = W_t + \int_0^t (\sigma^*(\sigma\sigma^*)^{-1})(s, X_s)b(s, X_s)ds$$

defines a $\mathbb{Q}$-Brownian motion. Since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, $\mathbb{P}$-a.s. convergence is equivalent to a $\mathbb{Q}$-a.s. convergence. In other words (3) can be proved $\mathbb{Q}$ or $\mathbb{P}$ almost surely.

Let us emphasize that $W$ and $\widetilde{W}$ generate the same filtration. Indeed, since $X$ is the solution of (4), $X$ and $\widetilde{W}$ are adapted to the filtration $\mathbb{F}^W$ of $W$. By the strong uniqueness of the solution of the SDE $dX = \sigma(\cdot, X)d\widetilde{W}$, $X$ is also $\mathbb{F}^{\widetilde{W}}$ measurable, which means that

$$W_t = \widetilde{W}_t - \int_0^t (\sigma^*(\sigma\sigma^*)^{-1})(s, X_s)b(s, X_s)ds$$

is also measurable w.r.t. $\mathbb{F}^{\widetilde{W}}$. Hence the filtrations coincide

Concerning the BSDE (5), we obtain

$$\begin{aligned} dY_t &= -F(t,X_t,Y_t,Z_t)dt + \langle (\sigma^*(\sigma\sigma^*)^{-1})(t,X_t)b(t,X_t), Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle \\ &= -f(t,X_t,Y_t,Z_t)dt - \langle \underbrace{a(t,X_t) - (\sigma^*(\sigma\sigma^*)^{-1})(t,X_t)b(t,X_t)}_{=:\tilde{a}(t,X_t)}, Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle \\ &= -f(t,X_t,Y_t,Z_t)dt - \langle \widetilde{a}(t,X_t), Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle. \end{aligned}$$

The term $\widetilde{a} = a - \sigma^*(\sigma\sigma^*)^{-1}b$ satisfies the same assumptions as $a$ and $b$: $\widetilde{a}$ is bounded and of class $C^1$ with polynomial growth derivatives.

Hence applying the Girsanov theorem leads to

$$X_t = x_0 + \int_0^t \sigma(s,X_s)d\widetilde{W}_s \tag{15}$$

$$\begin{aligned} Y_t &= g(X_T) + \int_t^T f(s,X_s,Y_s,Z_s)ds + \int_t^T \langle \widetilde{a}(s,X_s), Z_s\rangle ds - \int_t^T \langle Z_s, d\widetilde{W}_s\rangle \\ &= g(X_T) + \int_t^T \widetilde{F}(s,X_s,Y_s,Z_s)ds - \int_t^T \langle Z_s, d\widetilde{W}_s\rangle \end{aligned} \tag{16}$$

with $\widetilde{a} = a - \sigma^*(\sigma\sigma^*)^{-1}b$. Thus, even if it means considering $\mathbb{Q}, \tilde{W}, \tilde{a}$ instead of $\mathbb{P}, W, a$, we can assume

$$b = 0.$$

Now we can compute the Malliavin derivative of $X$. Indeed under Conditions (1) on $\sigma$, according to [20, Theorem 2.2.1], we have :

**Proposition 5.** *The SDE* (15) *admits a unique solution $X$ in $S^\infty(0,T)$ such that:*

1. *For all $t \in [0,T]$, $X_t^i \in \mathbb{D}^{1,\infty}$ and for all $p \in [1,+\infty[$,*
$$\sup_{0\le\theta\le t} \mathbb{E}\left(\sup_{\theta\le s\le T} |D_\theta X_s^i|^p\right) < +\infty. \tag{17}$$

2. *The process $DX^i$ satisfies the linear SDE*
$$D_\theta X_t^i = \sigma_i(\theta,X_\theta) + \sum_{j=1}^d \sum_{k=1}^m \int_\theta^t \frac{\partial \sigma_i^j}{\partial x_k}(s,X_s) D_\theta X_s^k dW_s^j, \quad 0\le\theta\le t\le T,$$
*where we note $\sigma = (\sigma_i^j)_{1\le i\le m, 1\le j\le d}$, and $D_\theta X_t^i = 0, \quad 0 \le t < \theta \le T$.*

**Remark 4.** *When $b$ is of class $C^1$ but with bounded derivatives, the previous result holds for the SDE* (4) *with*

$$D_\theta X_t^i = \sigma_i(\theta,X_\theta) + \sum_{k=1}^m \int_\theta^t \frac{\partial b_i}{\partial x_k}(s,X_s) D_\theta X_s^k ds + \sum_{j=1}^d \sum_{k=1}^m \int_\theta^t \frac{\partial \sigma_i^j}{\partial x_k}(s,X_s) D_\theta X_s^k dW_s^j,$$

*without needing to assume $b = 0$ by considering $\mathbb{Q}, \widetilde{W}, \widetilde{a}$.*

### 3.1.2 Differentiability for the BSDE

We show the Malliavin differentiability of the couple $(Y^n, Z^n)$, due to [18, Theorem 5.1 and Application 6.1]. The next result is based on Proposition 5, in particular on the estimates (17) on $D_\theta X$.

**Proposition 6.** *Under Conditions* 1 *to* 5*, the solution $(Y^n, Z^n)$ of the truncated BSDE (9) is in $L^2([0,T], \mathbb{D}^{1,2}\times\mathbb{D}^{1,2})$. Moreover for all $0 \le t < \theta \le T$, $D_\theta Y_t^n = 0$, $D_\theta Z_t^n = 0$ and, for all $0 \le \theta \le t \le T$,*

$$\begin{aligned} D_\theta Y_t^n = D_\theta \xi^n - \int_t^T D_\theta Z_s^n dW_s \\ + \int_t^T \left( \frac{\partial F^n}{\partial y}(s,X_s,Y_s^n,Z_s^n) D_\theta Y_s^n + \sum_{i=1}^d \frac{\partial F^n}{\partial z_i}(s,X_s,Y_s^n,Z_s^n) D_\theta Z_s^{i,n} + D_\theta F^n(s,X_s,Y_s^n,Z_s^n) \right) ds. \end{aligned} \tag{18}$$

*Proof.* Note that $0 \leq Y_t^n \leq Cn(T+1)$. Indeed Proposition 2 holds if $a$ is replaced by $\widetilde{a}$ (only the constant $C$ is modified). Thus we can assume that the driver $F^n$ admits a bounded partial derivative w.r.t. $y$. Indeed with assumptions 5, we can consider a function $\widetilde{f}$ with bounded partial derivative w.r.t. $y$ and which coincides with the function $f$ for $y \in [0, Cn(T+1)]$. Replacing $f$ by $\widetilde{f}$ in (9) leads to the same solution $(Y^n, Z^n)$. In this case we deduce the assumptions of the application 6.1 of [18]: $(A_1)$ due to 1.1-2, $(A_2)(i)$ due to 3.3 and $(A_2)(ii)$ due to 2.3, 5 and because

$$\mathbb{E}\left(\int_0^T |F^n(t,X_t,0,0)|^2 dt\right) \leq \mathbb{E}\left(\int_0^T |f(t,X_t,0,0)\wedge n|^2 dt\right) \leq Tn^2 < +\infty.$$

□

Then we deduce the representation of $Z^n$ as the Malliavin derivative of $Y^n$:

**Corollary 1.** *We have for any $t \in [0,T]$, $D_t Y_t^n = Z_t^n$.*

*Proof.* According to [15, Lemma 2.4]:

$$D_\theta Y_t^n = \nabla Y_t^n (\nabla X_\theta)^{-1} \sigma(\theta, X_\theta) \mathbb{1}_{\{\theta \leq t\}},$$

where $\nabla X, \nabla Y^n, \nabla Z^n$ are the notations of the variational equation associated to the FBSDE on $[t,T]$

$$\begin{cases} X_s^{t,x} &= x + \int_t^s \sigma(r, X_r) dW_r \\ Y_s^{n,t,x} &= g_n(X_T^{t,x}) + \int_s^T F^n(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) dr - \int_s^T Z_r^{n,t,x} dW_r, \end{cases}$$

with $g_n = \varphi_n \circ g$. In other words for each $i \in \{1, \ldots, m\}$, $(\nabla_i X, \nabla_i Y^n, \nabla_i Z^n)$ are the solutions of the FBSDE

$$\begin{cases} \nabla_i X_s &= e_i + \sum_{j=1}^d \int_t^s \nabla_x \sigma^j(r, X_r^{t,x}) \nabla_i X_r dW_r^j \\ \nabla_i Y_s^n &= \nabla_x g_n(X_T) \nabla_i X_T + \int_s^T \Big( \nabla_x F^n(r, X_r, Y_r^n, Z_r^n) \\ & \quad + \frac{\partial F^n}{\partial y}(r, X_r, Y_r^n, Z_r^n) + \langle \nabla_z F^n(r, X_r, Y_r^n, Z_r^n), \nabla_i Z_r^n \rangle \Big) dr - \int_s^T \nabla_i Z_r^n dW_r, \end{cases}$$

where $(e_1, \ldots, e_m)$ is the canonical basis of $\mathbb{R}^m$. Note that $\nabla_x g_n$ and $\nabla_x F_r^n$ make sense because we have truncated with a smooth function $\varphi_n$, and note that $\nabla X$ satisfies a linear SDE with initial condition $\nabla X_t = I_m$, thus $\nabla X_s \in GL_m(\mathbb{R})$ for any $s \in [t,T]$, $\mathbb{P}$-a.s. and $(\nabla X_s)^{-1}$ makes sense. Moreover, according to [32, Lemma 5.2.3]:

$$Z_t^n = \nabla Y_t^n (\nabla X_t)^{-1} \sigma(t, X_t).$$

Therefore we deduce the desired result. □

We also deduce the following results from [22, Proposition 16]. The difference with [22] is the dependence with respect to $z$ of the driver $F$ but the proof uses the same arguments. In particular due to the fact that we are in a Markovian framework, the property that $Y_t^n = u^n(t, X_t)$ with a deterministic function $u^n$ (solution of the partial differential equation associated to the truncated BSDE) holds (see [10, Theorem 4.1]).

**Corollary 2.** *For all $\varphi \in C_c^2(\mathbb{R}^m)$ (set of functions of class $C^2$ with compact support), there exists a measurable function $\psi : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ such that, for all $t \in [0,T]$,*

$$\mathbb{E}(|Y_t^n \psi(t, X_t)|) < +\infty, \quad \mathbb{E}(\langle Z_t^n, \nabla\varphi(X_t)\sigma(t,X_t)\rangle) = -\mathbb{E}(Y_t^n \psi(t, X_t))$$

*and the function $\psi$ is given by the formula*

$$\psi(t,x) = \sum_{i=1}^d (\nabla\varphi\sigma)_i(t,x) \frac{div(p\sigma_i)(t,x)}{p(t,x)} + tr\left(\nabla^2\varphi(x)(\sigma\sigma^*)(t,x)\right) + \sum_{i=1}^d \langle \nabla\varphi(x), ((\nabla\sigma_i)\sigma_i)(t,x)\rangle,$$

*with $p(t,\cdot)$ the density of $X_t$.*

**Corollary 3.** *For all $\varphi \in C_c^2(\mathbb{R}^m)$, there exists a measurable function $\overline{\psi} : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ such that, for all $t \in [0,T]$,*

$$\mathbb{E}(|Y_t^n \overline{\psi}(t, X_t)|) < +\infty, \quad \mathbb{E}\left(\varphi(X_t)\langle a(t,X_t), Z_t^n\rangle\right) = -\mathbb{E}\left(Y_t^n \overline{\psi}(t,X_t)\right)$$

*and the function $\overline{\psi}$ is given by*

$$\overline{\psi}(t,x) = \sum_{i=1}^d \left( \varphi(x) a_i(t,x) \frac{div(p\sigma_i)(t,x)}{p(t,x)} + a_i(t,x)\langle \nabla\varphi(x), \sigma_i(t,x)\rangle + \varphi(x)\langle \nabla a_i(t,x), \sigma_i(t,x)\rangle \right).$$

From now on and in the rest of this part 3, we work either with (4)-(5) under $\mathbb{P}$ or with (15)-(16) under $\mathbb{Q}$, with the same notations, so that the statements of Corollaries 2 and 3 hold.

### 3.2 Central equation

To study the limit behavior of the process $Y$ at time $T$, we consider the term $\varphi(X_t)Y_t^n$ for every function $\varphi$ regular with support included in the complementary of the singular set $\mathcal{S}$ and we study the behavior at time $T$ of this term. We suppose w.l.o.g. that $f$ satisfies Conditions 2 with $\chi = 0$, that is $f$ is non-increasing w.r.t. $y$ (see [13, Remark 1]). First we use Itô's formula and the previous corollaries to deduce:

**Proposition 7.** *Under Conditions 1 to 5, for all $\varphi \in C_c^2(\mathbb{R}^m)$, we have for any $n$ and any $t$*

$$\begin{aligned}
&\mathbb{E}(\varphi(X_T)Y_T^n) - \mathbb{E}(\varphi(X_t)Y_t^n) + \mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right) \\
&= \mathbb{E}\left(\int_t^T Y_s^n \overline{\Psi}(s,X_s)ds\right) - \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right) \\
&\quad - \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right),
\end{aligned} \tag{19}$$

*with*

$$\overline{\Psi}(t,x) = \Psi(t,x) + \overline{\psi}(t,x) = \mathcal{L}(\varphi)(s,x) - \psi(t,x) + \overline{\psi}(t,x),$$

*$\psi$ and $\overline{\psi}$ being given in Corollaries 2 and 3, and $\mathcal{L}$ by (8) with $b = 0$.*

*Proof.* We have, thanks to the Itô formula,

$$\begin{aligned}
Y_T^n\varphi(X_T) &= Y_t^n\varphi(X_t) + \int_t^T Y_s^n \mathcal{L}(\varphi)(s,X_s)ds + \sum_{i=1}^m\sum_{j=1}^d \int_t^T Y_s^n \frac{\partial\varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)dW_s^j \\
&\quad - \int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds + \int_t^T \varphi(X_s)Z_s^n dW_s + \sum_{i=1}^m\sum_{j=1}^d \int_t^T Z_s^{j,n}\frac{\partial\varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)ds.
\end{aligned}$$

But the appearing stochastic integrals are true martingales because $\frac{\partial\varphi}{\partial x_i}$, $\varphi$ and $\sigma$ are bounded and $(Y^n, Z^n) \in S^2(0,T) \times H^2(0,T)$. Thus, by applying the expectation,

$$\begin{aligned}
\mathbb{E}(Y_T^n\varphi(X_T)) &= \mathbb{E}(Y_t^n\varphi(X_t)) + \mathbb{E}\left(\int_t^T Y_s^n\mathcal{L}(\varphi)(s,X_s)ds\right) - \mathbb{E}\left(\int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds\right) \\
&\quad + \sum_{i=1}^m\sum_{j=1}^d \mathbb{E}\left(\int_t^T Z_s^{j,n}\frac{\partial\varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)ds\right).
\end{aligned}$$

Furthermore, due to the Fubini theorem and to Corollary 2,

$$\mathbb{E}(Y_T^n\varphi(X_T)) = \mathbb{E}(Y_t^n\varphi(X_t)) + \mathbb{E}\left(\int_t^T Y_s^n\mathcal{L}(\varphi)(s,X_s)ds\right) - \mathbb{E}\left(\int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds\right) - \mathbb{E}\left(\int_t^T Y_s^n\psi(s,X_s)d$$

We arrive at the equality

$$\mathbb{E}(Y_T^n\varphi(X_T)) = \mathbb{E}(Y_t^n\varphi(X_t)) - \mathbb{E}\left(\int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds\right) + \mathbb{E}\left(\int_t^T Y_s^n\Psi(s,X_s)ds\right),$$

with $\Psi(s,x) = \mathcal{L}(\varphi)(s,x) - \psi(s,x)$. But we also have

$$\begin{aligned}\mathbb{E}\left(\int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds\right) &= \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right)\\ &+ \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right) + \mathbb{E}\left(\int_t^T \varphi(X_s)\langle a(s,X_s), Z_s^n\rangle ds\right)\\ &+ \mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right),\end{aligned}$$

with, according to Corollary 3 and the Fubini theorem,

$$\mathbb{E}\left(\int_t^T \varphi(X_s)\langle a(s,X_s), Z_s^n\rangle ds\right) = -\mathbb{E}\left(\int_t^T Y_s^n\overline{\psi}(s,X_s)ds\right).$$

Therefore

$$\begin{aligned}&\mathbb{E}(\varphi(X_T)Y_T^n) - \mathbb{E}(\varphi(X_t)Y_t^n) + \mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right) = \mathbb{E}\left(\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right)\\ &- \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right) - \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right),\end{aligned}$$

with $\overline{\Psi} = \Psi + \overline{\psi}$, which achieves the proof. ☐

## 3.3 Control of the different terms in the central equation

The set $\mathcal{S}$ is closed (Assumption 3), so $\mathcal{S}^c = \{x \in \mathbb{R}^m,\ g(x) < +\infty\}$ is open. We consider any $\zeta \in C_c^2(\mathbb{R}^m)$ such that $0 \leq \zeta \leq 1,\ \zeta_{|\mathcal{S}} = 0$ and $\varphi = \zeta^\beta$ with

$$\beta \geq 2q_* = \frac{2q}{q-1} > \frac{q}{q-1} > 1. \tag{20}$$

This power $\beta$ will be usefull when we will have to differentiate the function $\varphi$, in particular in Lemma 4.

Thanks to the study of the behavior of different terms in the central equation (19) with the function $\varphi = \zeta^\beta$, we can pass to the limit as $n$ tends to $\infty$ in Equation (19).

**Proposition 8.** *Under conditions of Theorem 1, as $n$ tends to $+\infty$, we obtain: for any $t \in (0,T]$*

$$\begin{aligned}&\mathbb{E}(\varphi(X_T)\xi) - \mathbb{E}(\varphi(X_t)Y_t) + \mathbb{E}\left(\int_t^T \varphi(X_s)f(s,X_s,0,0)ds\right)\\ &= \mathbb{E}\left(\int_t^T Y_s\overline{\Psi}(s,X_s)ds\right) - \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s,Z_s) - f(s,X_s,0,Z_s))ds\right)\\ &\quad - \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))ds\right).\end{aligned} \tag{21}$$

*In this equation, all terms are finite.*

*Proof.* We are going to study each term of the central equation (19) given in Proposition 7:

1. For the terms $\mathbb{E}(\varphi(X_T)Y_T^n)$ and $\mathbb{E}(\varphi(X_t)Y_t^n)$, convergence is obtained by monotone convergence theorem
$$\mathbb{E}(\varphi(X_T)Y_T^n) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\varphi(X_T)\xi), \quad \mathbb{E}(\varphi(X_t)Y_t^n) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}(\varphi(X_t)Y_t).$$

Indeed if $\mathcal{K}$ is the support of $\varphi$,

$$0 \leq \varphi(X_T)\xi = \varphi(X_T)g(X_T)\mathbb{1}_{\{X_T \in \mathcal{K}\}} \leq g(X_T)\mathbb{1}_{\mathcal{K}}(X_T) \in L^1(\Omega),$$

thanks to Assumption 3, and also $\varphi(X_t)Y_t \in L^1(\Omega)$ because $\varphi$ is bounded and $Y \in \mathcal{S}^\infty(0,T-)$ (see Proposition 3).

2. For the term $\mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right)$, by construction of the $\varphi_n$ and monotone convergence,

$$\mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left(\int_t^T \varphi(X_s)f(s,X_s,0,0)ds\right).$$

Indeed

$$0 \leq \mathbb{E}\left(\int_t^T \varphi(X_s)|f(s,X_s,0,0)|ds\right) \leq CT\left(1 + \mathbb{E}\left(\sup_{0\leq s\leq T}|X_s|^\ell\right)\right) < +\infty.$$

3. For the term $\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right)$, we have

$$|\varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))| = \varphi(X_s)\frac{|f(s,X_s,0,Z_s^n) - f(s,X_s,0,0)|}{|Z_s^n|^\alpha}\mathbb{1}_{\{|Z_s^n|\neq 0\}}|Z_s^n|^\alpha.$$

with, by Condition (14) on the function $f$

$$\frac{|f(s,X_s,0,Z_s^n) - f(s,X_s,0,0)|}{|Z_s^n|^\alpha}\mathbb{1}_{\{|Z_s^n|\neq 0\}} \leq C(1+|X_s|^\ell).$$

Since $\varphi$ is bounded, we obtain:

$$|\varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))| \leq C(1+|X_s|^\ell)|Z_s^n|^\alpha. \tag{22}$$

Thus, according to Proposition 5 and Lemma 3 in the appendix, we deduce that there exist $C \geq 0$ and $\nu > 0$ such that

$$\mathbb{E}\left(\int_0^T |\varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))|^{1+\frac{\nu}{2}}\right) ds \leq C.$$

Indeed we have by Hölder inequality with $p = \dfrac{1+\nu}{1+\frac{\nu}{2}} > 1$

$$\mathbb{E}\left(\int_0^T |\varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))|^{1+\frac{\nu}{2}}ds\right) \leq C\mathbb{E}\left(\int_0^T (1+|X_s|^\ell)^{1+\frac{\nu}{2}}|Z_s^n|^{\alpha(1+\frac{\nu}{2})}ds\right)$$
$$\leq C\left(\mathbb{E}\left(\int_0^T (1+|X_s|^\ell)^{(1+\frac{\nu}{2})p_*}ds\right)\right)^{\frac{1}{p_*}}\left(\mathbb{E}\left(\int_0^T |Z_s^n|^{\alpha(1+\nu)}ds\right)\right)^{\frac{1}{p}} \leq C.$$

The sequence of processes $(\varphi(X)(f(\cdot,X,0,Z^n) - f(\cdot,X,0,0)))_{n\in\mathbb{N}}$ is bounded in $L^{1+\frac{\nu}{2}}(\Omega\times[0,T])$. Hence this sequence is uniformly integrable and we can deduce that for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $n$

$$\mathbb{E}\left(\int_{T-\delta_0}^T |\varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))|ds\right) \leq \varepsilon.$$

Furthermore, again with Lemma 3 the same arguments prove:

$$\mathbb{E}\left(\int_0^T \varphi(X_s)|f(s,X_s,0,Z_s) - f(s,X_s,0,0)|ds\right) \leq C\mathbb{E}\left(\int_0^T |Z_s|^\alpha ds\right) < +\infty.$$

Then there exists $\delta_1 \in ]0, \delta_0]$ such that

$$\mathbb{E}\left(\int_{T-\delta_1}^{T} |\varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))|ds\right) \leq \varepsilon.$$

Now for any $p > 1$, the sequence $(Z^n)_{n\in\mathbb{N}}$ converges in $H^p(0, T-\delta_1)$ to $Z$ (Proposition 3). Therefore since $f$ is a Lipschitz continuous function w.r.t. $z$,

$$\lim_{n\to+\infty} \mathbb{E}\left(\int_0^{T-\delta_1} \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right)$$
$$= \mathbb{E}\left(\int_0^{T-\delta_1} \varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))ds\right).$$

Hence there exists $N \in \mathbb{N}$ such that for any $n \geq N$

$$\left|\mathbb{E}\left(\int_0^{T-\delta_1} \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right) - \mathbb{E}\left(\int_0^{T-\delta_1} \varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))ds\right)\right| \leq \varepsilon.$$

With $f(s,X_s,0,0) = f^0(s)$, we deduce that

$$\left|\mathbb{E}\int_0^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f^0(s))ds - \mathbb{E}\int_0^T \varphi(X_s)(f(s,X_s,0,Z_s) - f^0(s))ds\right| \leq 3\varepsilon$$

Thus

$$\lim_{n\to+\infty} \mathbb{E}\left(\int_0^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f^0(s))ds\right) = \mathbb{E}\left(\int_0^T \varphi(X_s)(f(s,X_s,0,Z_s) - f^0(s))ds\right).$$

4. For the term $\mathbb{E}\left(\int_t^T Y_s^n \overline{\Psi}(s,X_s)ds\right)$, we have,

$$Y_s^n \overline{\Psi}(s,X_s) = \left(\eta(s,X_s)^{\frac{1}{q}} Y_s^n \varphi(X_s)^{\frac{1}{q}}\right) \times \left(\overline{\Psi}(s,X_s)\eta(s,X_s)^{-\frac{1}{q}}\varphi(X_s)^{-\frac{1}{q}}\mathbb{1}_{\varphi(X_s)>0}\right).$$

Thus, by Hölder's inequality,

$$\mathbb{E}\left(\int_t^T \left|Y_s^n \overline{\Psi}(s,X_s)\right| ds\right) \leq \left(\mathbb{E}\int_t^T \eta(s,X_s)(Y_s^n)^q \varphi(X_s)ds\right)^{\frac{1}{q}} \left(\mathbb{E}\int_t^T \Gamma(s,X_s)ds\right)^{\frac{1}{q*}}, \quad (23)$$

with

$$\Gamma(s,x) = |\overline{\Psi}(s,x)|^{q*}\eta(s,x)^{-\frac{1}{q-1}}\varphi(x)^{-\frac{1}{q-1}}\mathbb{1}_{\{\varphi(x)>0\}}. \quad (24)$$

Given that the function $\Psi$ involves the density $p$ of the process $X$ which has a singularity in $t = 0$, we consider $\varepsilon > 0$ and verifiy that $\Gamma(\cdot, X(\cdot)) \in L^1([\varepsilon, T] \times \Omega)$. The fact to consider $[\varepsilon, T]$ is not a problem because we study the behavior at time $T$, that is when $t$ tends to $T$. We have, according to Lemma 4 in the appendix, for any $t \geq \varepsilon$,

$$\mathbb{E}\left(\int_t^T \Gamma(s,X_s)ds\right) < +\infty.$$

According to the previous points,

$$\mathbb{E}(\varphi(X_T)Y_T^n) - \mathbb{E}(\varphi(X_t)Y_t^n) + \mathbb{E}\left(\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right) + \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right)$$

is the term of convergent sequence. Moreover from our equation of interest (19), it is equal to

$$-\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right) + \mathbb{E}\left(\int_t^T Y_s^n \overline{\Psi}(s,X_s)ds\right).$$

Hence there exists a constant $C$ such that

$$\left(\mathbb{E}\left(\int_t^T \Gamma(s,X_s)ds\right)\right)^{\frac{1}{q_*}} \leq C$$

and

$$-\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right) + \mathbb{E}\left(\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right) \leq C.$$

From the second assumption of 4,

$$\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right) \leq -\mathbb{E}\left(\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right).$$

Hence for any $n$ and $t$

$$\mathbb{E}\left(\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right) + \mathbb{E}\left(\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right) \leq C.$$

Let's introduce some notations to understand the behavior of sequences

$$u_n = \mathbb{E}\left(\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right), \quad v_n = \mathbb{E}\left(\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right).$$

We have $u_n + v_n \leq C$, and, by the inequality (23) and our choice of $C$, $|v_n| \leq C u_n^{\frac{1}{q}}$. Thus $u_n \leq C - v_n \leq C + u_n^{\frac{1}{q}}$, i.e. $u_n - u_n^{\frac{1}{q}} \leq C$. But, noting $h_q(x) = x - x^{\frac{1}{q}}$, the set $\left\{x \in \mathbb{R}_+, x - x^{\frac{1}{q}} \leq C\right\} = h_q^{-1}([0,C])$ is bounded. Indeed $\lim_{+\infty} h_q = +\infty$ and $[0,C]$ is compact. Thus $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence by $C \in \mathbb{R}_+^*$. Therefore the sequence

$$\gamma_n = \varphi(X)^{\frac{1}{q}}\eta(\cdot,X)^{\frac{1}{q}}Y^n$$

is bounded in $L^q([0,T]\times\Omega)$ and nondecreasing. Thus we have the convergence of $\gamma_n$ to $\gamma$ in $L^q([0,T]\times\Omega)$. But we also have the almost sure convergence: a.s. for all $s \in [0,T]$,

$$\gamma_n = \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s^n \underset{n\to+\infty}{\longrightarrow} \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s.$$

Thus, by limit uniqueness, $\gamma(s) = \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s$. For any $t \geq \varepsilon$,

$$\mathbb{E}\left(\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right) = \mathbb{E}\left(\int_t^T \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s^n\frac{\overline{\Psi}(s,X_s)}{\varphi(X_s))^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}}\mathbb{1}_{\{\varphi(X_s)>0\}}ds\right)$$

$$\underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left(\int_t^T \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s\frac{\overline{\Psi}(s,X_s)}{\varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}}\mathbb{1}_{\{\varphi(X_s)>0\}}ds\right) = \mathbb{E}\left(\int_t^T Y_s\overline{\Psi}(s,X_s)ds\right)$$

because

$$\frac{\overline{\Psi}(s,X_s)}{\varphi(X_s)^{\frac{1}{q}}\eta(s)^{\frac{1}{q}}}\mathbb{1}_{\{\varphi(X_s)>0\}} \in L^{q_*}([\varepsilon,T]\times\Omega).$$

Indeed from Lemma 4

$$\left|\frac{\overline{\Psi}(s,X_s)}{\varphi(X_s)^{\frac{1}{q}}\eta(s)^{\frac{1}{q}}}\mathbb{1}_{\{\varphi(X_s)>0\}}\right|^{q_*} = \Gamma(s,X_s) \in L^1([\varepsilon,T]\times\Omega).$$

For the remaining term $\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right)$, $f$ is supposed to be nonincreasing with respect to $y$ (see the beginning of Section 3.2) and, by construction of the processes $Y^n$, the sequence $(Y^n)_{n\in\mathbb{N}^*}$ increasingly converges to $Y$. Thus the monotone convergence theorem gives us the convergence

$$\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n)-f(s,X_s,0,Z_s^n))ds\right) \underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s,Z_s)-f(s,X_s,0,Z_s))ds\right).$$

Coming back to the central equation (19), we also deduce the convergence of the sequence

$$\begin{aligned}&\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n)-f(s,X_s,0,Z_s^n))ds\right)\\ &\underset{n\to+\infty}{\longrightarrow} -\mathbb{E}(\varphi(X_T)\xi)+\mathbb{E}(\varphi(X_t)Y_t)-\mathbb{E}\left(\int_t^T \varphi(X_s)f(s,X_s,0,0)ds\right)+\mathbb{E}\left(\int_t^T Y_s\overline{\Psi}(s,X_s)ds\right)\\ &\qquad -\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s)-f(s,X_s,0,0))ds\right)\end{aligned}$$

where the limit is finite due to the previous estimates, which achieves the proof of this proposition. □

**Remark 5.** *The Malliavin calculus allows us to control each linear term which involves $Z_t^n = D_tY_t^n$. Even if $a=0$, $Z^n$ appears in the cross variation. For the term with the increment of $f$ with respect to $z$*

$$\mathbb{E}\left(\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n)-f(s,X_s,0,0))ds\right),$$

*even if we could linearize it (see Section 5.1 and the proof of Proposition 4 in the Appendix) and get $\langle l_s^n, Z_s^n\rangle$, we don't have an expression of the Malliavin derivative of $l^n$. This is the reason why we add this regularity condition* (14)*.*

### 3.4 Conclusion about the BSDE

The different terms in Equation (21) of Proposition 8 are integrable. Thus, when $t$ tends to $T$, we obtain

$$\mathbb{E}(\varphi(X_T)\xi) = \lim_{t\to T}\mathbb{E}(\varphi(X_t)Y_t).$$

So, according to the Fatou lemma and Proposition 3,

$$\mathbb{E}(\varphi(X_T)\xi) \geq \mathbb{E}\left(\liminf_{t\to T}\varphi(X_t)Y_t\right) = \mathbb{E}\left(\varphi(X_T)\liminf_{t\to T}Y_t\right) \geq \mathbb{E}(\varphi(X_T)\xi).$$

Therefore inequalities above are equalities. Thus for every function $\varphi$ whose support is included in $\{\xi<+\infty\}$, we have

$$\varphi(X_T)\liminf_{t\to T}Y_t = \xi\varphi(X_T).$$

Thus, on $\{\xi<+\infty\}$, $\liminf_{t\to T}Y_t=\xi$. Since we already know that this equality holds on $\{\xi=\infty\}$, this achieves the proof of Theorem 1.

## 4 Corollaries and applications

### 4.1 General comparison theorem and generalization of Theorem 1

We can consider two singular terminal conditions $\xi_1$ and $\xi_2$ and two drivers $F_1, F_2 : [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ which satisfy comparison inequalities. Thus we obtain this following corollary about the associated minimal solutions.

**Proposition 9.** *We assume that the Asumptions* 2 *and* 4 *hold for the two different drivers $F_1, F_2$ instead of the driver $F$. We also suppose that the terminal conditions are non-negative and that the following inequalities hold:*

$$\xi_1 \leq \xi_2, \tag{25}$$

$$\forall (t,x,y,z)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d,\quad F_1(t,x,y,z)\leq F_2(t,x,y,z). \tag{26}$$

*Thus the minimal solutions $(Y^1,Z^1),(Y^2,Z^2)$ (in the sense of Proposition 3), associated to the BSDE (5) with respectively the paramaters $(F_1,\xi_1),(F_2,\xi_2)$, satisfy the comparison principle: a.s.*

$$\forall t \in [0,T], \quad Y^1_t \le Y^2_t. \tag{27}$$

*Proof.* Let us consider $(Y^{1,n},Z^{1,n})$ the solution of the BSDE with terminal condition $\xi_1^n = \varphi_n(\xi_1)$ and generator

$$F_1^n(s,x,y,z) = F_1(s,x,y,z) - F_1(s,x,0,0) + \varphi_n(F_1(s,x,0,0))$$

where the functions $(\varphi_n)$ are given by (10). For all $\varepsilon > 0$ and $0 \le t \le T-\varepsilon < T$, we have

$$\begin{aligned} Y^2_t - Y^{1,n}_t &= Y^2_{T-\varepsilon} - Y^{1,n}_{T-\varepsilon} + \int_t^{T-\varepsilon} F_2(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^2_s,Z^2_s)ds \\ &+ \int_t^{T-\varepsilon} F_1(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^{1,n}_s,Z^{1,n}_s)ds \\ &+ \int_t^{T-\varepsilon} F_1(s,X_s,0,0) - \varphi_n(F_1(s,X_s,0,0))ds - \int_t^{T-\varepsilon} Z^2_s - Z^{1,n}_s dW_s. \end{aligned}$$

Note that we can split the next term:

$$\begin{aligned} \int_t^{T-\varepsilon} F_1(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^{1,n}_s,Z^{1,n}_s)ds &= \int_t^{T-\varepsilon} F_1(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^{1,n}_s,Z^2_s)ds \\ &+ \int_t^{T-\varepsilon} F_1(s,X_s,Y^{1,n}_s,Z^2_s) - F_1(s,X_s,Y^{1,n}_s,Z^{1,n}_s)ds. \end{aligned}$$

and use a classical linearization trick (see among other [32, Theorem 4.2.3])

$$\int_t^{T-\varepsilon} F_1(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^{1,n}_s,Z^{1,n}_s)ds = \int_t^{T-\varepsilon} \alpha^n_s(Y^2_s - Y^{1,n}_s)ds + \int_t^{T-\varepsilon} \beta^n_s(Z^2_s) - Z^{1,n}_s)ds.$$

From our assumption 2, $\beta^n$ is a bounded process (by $K$, uniformly in $n$) and $\alpha^n$ is bounded from above (by $\chi$, uniformly in $n$). Using the expression of the solution of a linear BSDE ([32, Proposition 4.2.1]), we obtain

$$\begin{aligned} Y^2_t - Y^{1,n}_t &= \mathbb{E}\left[ (Y^2_{T-\varepsilon} - Y^{1,n}_{T-\varepsilon})\Gamma^n(t,T-\varepsilon) + \int_t^{T-\varepsilon} \left(F_2(s,X_s,Y^2_s,Z^2_s) - F_1(s,X_s,Y^2_s,Z^2_s)\right)\Gamma^n(t,s)ds \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E}\left[ \int_t^{T-\varepsilon} \left(F_1(s,X_s,0,0) - \varphi_n(F_1(s,X_s,0,0))\right)\Gamma^n(t,s)ds \middle| \mathcal{F}_t \right] \end{aligned}$$

with

$$\Gamma^n(t,s) = \exp\left( \int_t^s \left( \alpha^n_u - \frac{1}{2}(\beta^n_u)^2 \right) du + \int_t^s \beta^n_u dW_u \right).$$

From the definition of $\varphi_n$, the last term is non-negative. From our assumption on $F_1$ and $F_2$, the last but one term is also non-negative. Hence

$$Y^2_t - Y^{1,n}_t \ge \mathbb{E}\left[ (Y^2_{T-\varepsilon} - Y^{1,n}_{T-\varepsilon})\Gamma^n(t,T-\varepsilon) \middle| \mathcal{F}_t \right].$$

From Proposition 2, we know that

$$Y^2_{T-\varepsilon} - Y^{1,n}_{T-\varepsilon} \ge -Cn(T+1)$$

since $Y^2$ is non-negative. $\Gamma^n$ is non-negative and bounded from above by

$$\zeta^n(t,s) = \exp\left( -\int_t^s \frac{1}{2}(\beta^n_u)^2 du + \int_t^s \beta^n_u dW_u \right)$$

which belongs to any $S^p(t,T)$, $p>1$ (see [26]). Hence we can use Fatou's lemma to deduce that

$$Y^2_t - Y^{1,n}_t \ge \mathbb{E}\left[ \liminf_{\varepsilon\to 0} (Y^2_{T-\varepsilon} - Y^{1,n}_{T-\varepsilon})\Gamma^n(t,T) \middle| \mathcal{F}_t \right] \ge \mathbb{E}\left[ (\xi_2 - \varphi_n(\xi_1))\Gamma^n(t,T) \middle| \mathcal{F}_t \right] \ge 0.$$

We used the fact that a.s.

$$\liminf_{t\to T} Y_t^2 \geq \xi_2, \quad \lim_{t\to T} Y_t^{1,n} = \varphi_n(\xi_1).$$

Hence a.s. $Y^2 - Y^{1,n} \geq 0$. Since this inequality holds for all $n$, the same holds for $Y^2 - Y^1$, which achieves the proof.

□

**Remark 6.** *The Markovian setting is not used here. This result holds if the generators are defined on* $[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$ *and if the generators are singular in the sense of [13]: Condition 2-2 is replaced by: for any* $\rho \geq 0$

$$\mathbb{E}\left[\sup_{|y|\leq\rho} |F_i(t,\omega,y,z) - F_i(t,\omega,0,z)|\right] < +\infty, \quad \mathbb{E}\int_0^T (T-s)^{q_*} F_i(s,\omega,0,0)ds < +\infty.$$

*The continuity at time $T$ is also not necessary to compare the minimal solutions. Hence we do not need all assumptions of Theorem 1.*

With the previous proposition, we can generalize Theorem 1. We consider a driver $F : [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ which satisfies the following conditions.

**Assumption 6.** *$F$ satisfies Conditions 2 and 4 and there exists $f$ and $a$ such that*

$$\forall (t,x,y,z) \in [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d, \quad F(t,x,y,z) \leq f(t,x,y,z) + \langle a(t,x), z\rangle,$$

*where the functions $f$ and $a$ verify Assumptions 2, 4 and 5. Moreover* (14) *holds for $f$.*

We can apply Proposition 3 for the BSDE with driver $F$ and obtain a minimal supersolution $(Y^F, Z^F)$.

**Corollary 4.** *Under Assumptions 1, 3 and 6 we have the limit behavior: $\mathbb{P}$-almost surely*

$$\liminf_{t\to T} Y_t^F = \xi.$$

*Proof.* We have, according to Assumption 6 and Proposition 9, a.s. for any $t \in [0,T]$ $Y_t^F \leq Y_t$. Then $\xi \leq \liminf_{t\to T} Y_t^F \leq \liminf_{t\to T} Y_t = \xi$. The proof is complete. □

**Remark 7.** *Instead to suppose $g \geq 0$ and $F(s,x,0,0) \geq 0$, we can assume that for some $\rho > 1$,*

$$\xi^- = (g(X_T))^- \in L^\rho(\Omega), \quad (F(s,X_s,0,0))^- \in L^\rho((0,T)\times\Omega).$$

*Indeed we have by Itô's formula*

$$(Y_t^n)^- \leq \xi^- - \int_t^T (F(s,X_s,Y_s^n,Z_s^n) - F(s,X_s,0,0))\mathbb{1}_{\{Y_s^n\leq 0\}}ds + \int_t^T F(s,X_s,0,0)^- ds - \int_t^T Z_s^n \mathbb{1}_{\{Y_s^n\leq 0\}}dW_s.$$

*Then for any $n$,*

$$(Y_t^n)^- \leq \mathbb{E}\left(\xi^-\mathcal{E}(t,T) + \int_t^T F(s,X_s,0,0)^-\mathcal{E}(t,s)ds \middle| \mathcal{F}_t\right) =: \hat{Y}_t,$$

*with* $\mathcal{E}(t,T) = \exp\left(\chi(T-t) + K(W_T - W_t) - K^2\frac{T-t)}{2}\right)$. *Thus $\hat{Y}_t \in S^{\rho-\varepsilon}(0,T)$ for any $\varepsilon > 0$ and* $-\hat{Y}_t \leq Y_t^n \leq Y_t$. *Furthermore we have* $\lim_{t\to T} Y_t^- = \xi^-$, $\liminf_{t\to T} Y_t \geq \xi$. *So we can limit our study to $Y_t^+$.*

## 4.2 Application to optimal liquidation problem

With the notations of the papers [3, 13] in the Brownian case without jumps, we consider the Markovian BSDE

$$Y_t = \xi - \int_t^T (p-1)\frac{|Y_s|^{q-1}Y_s}{\eta_s^{q-1}}ds + \int_t^T \gamma_s ds + \int_t^T Z_s dW_s, \tag{28}$$

where $\eta_s = \eta(s,X_s)$ and $\gamma_s = \gamma(s,X_s)$. $X$ still denotes the solution of the SDE (4) with Condition 1. Thus, with the assumptions of [3, 13], the driver of this BSDE satisfies Assumption 6. So we have a unique minimal supersolution $(Y,Z)$.

From here we consider the stochastic control problem to minimize the functional

$$J(t,\alpha) = \mathbb{E}\left( \int_t^T (\eta_s|\alpha_s|^p + \gamma_s|\Xi_s|^p)\, ds + \xi|\Xi_T|^p \middle| \mathcal{F}_t \right)$$

over all $\alpha \in \mathcal{A}(t,x)$ where $\mathcal{A}(t,x)$ is the set of admissible controls such that $\Xi$ satisfy the dynamics

$$\Xi_s = x + \int_t^s \alpha_u du \quad t \le s \le T, \quad \alpha \in L^1(t,\infty) \text{ a.s.}$$

Note that there is an implicit constraint on $\Xi_T$: when $\xi = +\infty$, to obtain a finite cost, $\Xi_T$ must be equal to zero. The mandatory liquidation corresponds to the case $\xi = +\infty$ a.s. and is studied in [3].

In [13], it is proved that a minimizer of the functional $J$ is the process $\Xi^*$ given by

$$\Xi^*_s = x \exp\left( - \int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du \right)$$

where $(Y,Z)$ is the minimal supersolution of the BSDE (28). Moreover the value function of this control problem is given by $v(t,x) = |x|^p Y_t$.

Evoke that a.s. $\lim_{t\to T} Y_t \ge \xi$ (see Proposition 3; existence of the limit is given by [23]). This condition is sufficient to solve the control problem. But it means that the value function only satisfies: a.s.

$$\lim_{t\to T} v(t,x) \ge |x|^p \xi.$$

Hence there could be an extra cost, due to the liquidation constrain (or the terminal singularity of the BSDE).

Now if 1, 3 and 6 hold, then Corollary 4 shows that a.s.

$$\lim_{t\to T} Y_t = \liminf_{t\to T} Y_t = \xi \qquad \text{and} \qquad \lim_{t\to T} v(t,x) = |x|^p \xi.$$

Therefore there is no additional cost to minimize our control problem. This result was already proved in [23] but for large values on $q$, that is small values on $p$. Our result states that the equality also holds for any $p > 1$.

**Remark 8.** *The Malliavin calculus has been used to control $Z$ and establish the continuity property of $Y$. In that application to mathematical finance, it could be used to analyze sensitivity with respect to the parameters $\eta$ and $\gamma$. So, it would be natural to study the Malliavin derivability of $Y$ and the convergence of $D.Y_n$ to $D.Y$. This will be the object of further researches.*

## 4.3 Application to partial differential equation

We consider the partial differential equation (7) associated to the FBSDE (4)-(5). where the operator $\mathcal{L}$ is given by (8). Thus, according to the article [24], we consider the function

$$u(t,x) = Y_t^{t,x} \quad \forall (t,x) \in [0,T] \times \mathbb{R}^m,$$

where $Y^{t,x}$ is the unique minimal super-solution of the Markovian FBSDE

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dt + \int_t^s \sigma(r, X_r) dW_r \quad t \le s \le T, \\ Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r dW_r \quad t \le s \le T. \end{cases}$$

We still assume that 1, 3 and 6 hold (plus some regularity condition w.r.t. $t$; see Assumption (C) of [24]). According to [24, Theorem 1], the function $u$ is deterministic and a minimal viscosity solution of the PDE (7) among all non negative solutions satisfying (7), with the terminal constraint

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \ge g(x_0).$$

From [24, Theorem 2], if $q$ is sufficiently large ($q$ is defined in Assumption 4), continuity holds:

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0).$$

Roughly speaking the condition is $q > 3$ (see [24, Remark 2]). Here we prove that under our setting, for any $q > 1$, the previous equality holds a.s. The arguments are almost the same as in [24], together with the arguments of the proof of Theorem 1. Since $F$ depends on the gradient of $u$, to our best knowledge, this result does not exist in the PDE literature.

# 5 Appendix

## 5.1 A priori estimates of the solution

*Proof of Proposition 4.* For $z \in \mathbb{R}^d$ and $i \in \{1, \ldots, d\}$ we denote by $z_i$ its $i-$th coordinate and $\varepsilon_i^-$ the map from $\mathbb{R}^d$ into $\mathbb{R}^d$ such that $(\varepsilon_i^- z)_j = z_j$ if $j \neq i$ and $(\varepsilon_i^-)_j = 0$ . Now we set $\varepsilon_0^- z = z$ and define the $\mathbb{R}^d$-valued process $l^n$ by $\forall i \in \{1, \cdots, d\}$, $\forall t \in [0, T]$

$$l_i^n(t) = \frac{f(t, X_t, 0, \varepsilon_0^- \circ \cdots \circ \varepsilon_{i-1}^- Z_t^n) - f(t, X_t, 0, \varepsilon_0^- \circ \cdots \circ \varepsilon_i^- Z_t^n)}{Z_{i,t}^n} \mathbb{1}_{\{Z_{i,t}^n \neq 0\}},$$

so that due to the Lipschitz hypothesis on $h$ (see Assumption 2), $l^n$ is a bounded process and

$$\langle l_t^n, Z_t^n \rangle = f(t, X_t, 0, Z_t^n) - f(t, X_t, 0, 0).$$

The main idea is to linearize the backward equation satisfied by $(Y^n, Z^n)$. To this end, we introduce the process

$$\gamma_t^n = l_t^n + a(t, X_t).$$

Then there exists a constant $C$ which does not depend on $n$ such that $|\gamma_t^n| \leq C$. We consider the driver

$$H^n(t, y, z) = \kappa_t^n - \frac{q_*}{T - t + \frac{1}{n^{q-1}}} y + \langle \gamma_t^n, z \rangle, \quad \text{with} \quad \kappa_t^n = \left( \frac{q_* - 1}{\eta(t, X_t)} \right)^{q_* - 1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} + \varphi_n(f(t, X_t, 0, 0))$$

and denote by $(\mathcal{Y}^n, \mathcal{Z}^n)$ the solution of the BSDE on $[0, T]$ with driver $H^n$ and terminal condition $\mathcal{Y}_T^n = n$.
Hence $(\mathcal{Y}^n, \mathcal{Z}^n)$ is solution of a linear BSDE, so we have

$$\mathcal{Y}_t^n = \mathbb{E} \left( \Gamma_{t,T}^n n + \int_t^T \Gamma_{t,s}^n \kappa_s^n ds \middle| \mathcal{F}_t \right) = \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} \left[ \frac{1}{n^{q-1}} + \mathbb{E} \left( \int_t^T V_{t,s}^n (T - s + \frac{1}{n^{q-1}})^{q_*} \kappa_s^n ds \middle| \mathcal{F}_t \right) \right]$$

where for $t \leq s \leq T$

$$\Gamma_{t,s}^n = \exp \left( - \int_t^s \frac{q_*}{T - u + \frac{1}{n^{q-1}}} du \right) V_{t,s}^n = \left( \frac{T - s + \frac{1}{n^{q-1}}}{T - t + \frac{1}{n^{q-1}}} \right)^{q_*} V_{t,s}^n, \qquad V_{t,s}^n = 1 + \int_t^s V_{t,u}^n \gamma_u^n dW_u.$$

Moreover since $Y_t^n \geq 0$ a.s.

$$\begin{aligned} F^n(t, X_t, Y_t^n, Z_t^n) &= f(t, X_t, Y_t^n, Z_s^n) - f(t, X_t, 0, Z_s^n) + F^n(t, X_t, 0, Z_t^n) \\ &\leq -\eta(t, X_t)(Y_t^n)^q + F^n(t, X_t, 0, Z_t^n) \end{aligned}$$

and, according to the expressions of $F^n, l_t^n, \gamma_t^n, \kappa_t^n$ and $H^n$,

$$\begin{aligned} F^n(t, X_t, 0, Z_t^n) &= f(t, X_t, 0, Z_t^n) + \langle a(t, X_t), Z_t^n \rangle - f(t, X_t, 0, 0) + \varphi_n(f(t, X_t, 0, 0)) \\ &= \langle l_t^n, Z_t^n \rangle + \langle a(t, X_t), Z_t^n \rangle + \varphi_n(f(t, X_t, 0, 0)) = \langle \gamma_t^n, Z_t^n \rangle + \varphi_n(f(t, X_t, 0, 0)) \\ &= \langle \gamma_t^n, Z_t^n \rangle + \kappa_t^n - \left( \frac{q_* - 1}{\eta(t, X_t)} \right)^{q_* - 1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} \\ &= H^n(t, Y_t^n, Z_t^n) + \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y_t^n - \left( \frac{q_* - 1}{\eta(t, X_t)} \right)^{q_* - 1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}}. \end{aligned}$$

It follows that, with $\beta_t = \dfrac{q_* - 1}{\eta(t, X_t)}$,

$$F^n(t, X_t, Y_t^n, Z_t^n) \leq H^n(t, Y_t^n, Z_t^n) - \eta(t, X_t)(Y_t^n)^q - \frac{\beta_t^{q_* - 1}}{(T - t + \frac{1}{n^{q-1}})^{q_*}} + \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y_t^n \leq H^n(t, Y_t^n, Z_t^n),$$

the last inequality comes from the Young inequality since

$$\begin{aligned} &\eta(t, X_t)(Y_t^n)^q + \frac{\beta_t^{q_* - 1}}{(T - t + \frac{1}{n^{q-1}})^{q_*}} - \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y_t^n \\ &= \eta(t, X_t) q \left( \frac{(Y_t^n)^q}{q} + \frac{1}{q_*} \left( \frac{q_* - 1}{(T - t + \frac{1}{n^{q-1}}) \eta(t, X_t)} \right)^{q_*} - (q_* - 1) \frac{Y_t^n}{(T - t + \frac{1}{n^{q-1}}) \eta(t, X_t)} \right) \\ &\geq 0. \end{aligned}$$

The comparison theorem implies $Y^n_t \leq \mathcal{Y}^n_t$ for all $t \in [0,T]$, that is:

$$Y^n_t \leq \frac{1}{(T-t+\frac{1}{n^{q-1}})^{q_*}} \left[ \frac{1}{n^{q-1}} + \mathbb{E}\left( \int_t^T V^n_{t,s}(T-s+\frac{1}{n^{q-1}})^{q_*} \kappa^n_s ds \bigg| \mathcal{F}_t \right) \right]. \tag{29}$$

Recall that $V^n_{t,.}$ belongs to $H^\varrho(0,T)$ for $\varrho \geq 1$ and there exists a constant $K_\varrho$ such that a.s. for any $n$: $\mathbb{E}\left(\int_t^T (V^n_{t,s})^\varrho ds \Big| \mathcal{F}_t\right) \leq C_\varrho$. The process $((T-t+\frac{1}{n^{q-1}})^{q_*}\kappa^n_t)_{\ 0\leq t \leq T}$ belongs to $H^r(0,T)$. Therefore by Hölder inequality we obtain for any $n \geq 1$

$$\mathbb{E}\left( \int_t^T V^n_{t,s}(T-s+\frac{1}{n^{q-1}})^{q_*} \kappa^n_s ds \bigg| \mathcal{F}_t \right) \leq K_r \mathbb{E}\left( \int_t^T ((T-s+\frac{1}{n^{q-1}})^{q_*} \kappa^n_s)^r ds \bigg| \mathcal{F}_t \right)^{\frac{1}{r}}$$
$$\leq K_r \mathbb{E}\left( \int_t^T \left( \left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*}(F(s,X_s,0,0)) \right)^r ds \bigg| \mathcal{F}_t \right)^{\frac{1}{r}}$$

The last inequality comes from the very definition of $\kappa^n$. Thus we obtain the upper bound in (11).

□

Now we have that

$$\frac{1}{\eta(s,x)} \leq C(1+|x|^\ell), \quad F(s,x,0,0) \leq C(1+|x|^\ell).$$

Then for any $r > 1$

$$\mathbb{E}\left( \int_t^T \left( \left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*} F(s,X_s,0,0) \right)^r ds \bigg| \mathcal{F}_t \right) \leq C\mathbb{E}\left( \int_t^T (1+|X_s|^{r(q_*-1)\ell}) ds \bigg| \mathcal{F}_t \right) < +\infty.$$

Hence in (11), we can choose any $r > 1$. Also note that for any $\eta > 0$ and $r > 1$

$$\mathbb{E}\left( \int_0^T (T-s)^{-1+\eta} \left( \left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*} F(s,X_s,0,0) \right)^r ds \right) < +\infty. \tag{30}$$

Passing though the limit in Estimate (11) we deduce (12) for $0 \leq t < T$ and $n \geq 1$.

**Remark 9.** *When $F$ does not depend on $Z$, then we can choose $r = K_r = 1$.*

In the sequel let us denote by $\Xi$ the process

$$\Xi_t = \frac{K^r_r}{T-t} \mathbb{E}\left( \int_t^T \left( \left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} + (T-s+1)^{q_*} F(s,X_s,0,0) \right)^r ds \bigg| \mathcal{F}_t \right).$$

Thus Estimate (12) can be written: for all $0 \leq t < T$

$$0 \leq Y^n_t \leq Y_t \leq \frac{1}{(T-t)^{q_*-\frac{1}{r}}} \Xi_t^{\frac{1}{r}}. \tag{31}$$

**Lemma 1.** *We have for any $\eta > 0$, $\mathbb{E}\left( \int_0^T (T-s)^{-1+\eta} \Xi_s ds \right) < +\infty$.*

*Proof.* Note that

$$\mathbb{E}(\Xi_s) = K^r_r (T-s)^{-1} \int_s^T \mathbb{E}\left( \left(\frac{q_*-1}{\eta(u,X_u)}\right)^{q_*-1} + (T-u+1)^{q_*} F(u,X_u,0,0) \right)^r du = K^r_r (T-s)^{-1} \int_0^T \theta_u \mathbb{1}_{\{u \geq s\}} du$$

with the continuous deterministic function

$$\theta_u = \mathbb{E}\left( \left(\frac{q_*-1}{\eta(u,X_u)}\right)^{q_*-1} + (T-u+1)^{q_*} F(u,X_u,0,0) \right)^r \leq C(1+\mathbb{E}|X_u|^{r(q_*-1)\ell}).$$

Thus for any $s \in [0,T)$
$$\mathbb{E}(\Xi_s) \le K_r^r \sup_{u\in[0,T]} \theta_u = \Theta < +\infty. \tag{32}$$
Hence by Fubini's theorem
$$\mathbb{E}\left( \int_0^T (T-s)^{-1+\eta} \Xi_s ds \right) \le \Theta \int_0^T (T-s)^{-1+\eta} ds = \frac{\Theta}{\eta} T^\eta.$$
This achieves the proof of the lemma. □

Let us now derive an a priori estimate on $Z^n$.

**Lemma 2.** *For $r > 1, \eta > 0$ and $\varpi = 2q_* - \frac{2}{r} + \frac{2\eta}{r} = \frac{2}{q-1} + 2\left(1-\frac{1}{r}\right) + \frac{2\eta}{r}$, there exists a constant $C$ s.t. for any $n$*
$$\mathbb{E}\left( \left( \int_0^T (T-s)^\varpi |Z_s^n|^2 ds \right)^{\frac{r}{2}} \right) \le C.$$
*The same estimate also holds for $Z$.*

*Proof.* For $\eta > 0$ and $r > 1$, let us define
$$\delta = rq_* - 1 + \eta > 0.$$
We define $c(r) = \frac{r((r-1)\wedge 1)}{2}$ and we apply Itô's formula to $(T-t)^\delta (Y_t^n)^r$ (see [5, Corollary 2.3]). Evoke that $Y^n$ is non-negative. We fix $\varepsilon > 0$ and $\tau = T - \varepsilon$ in the sequel. Hence we have for $0 \le t \le \tau$:
$$\begin{aligned}(T-t)^\delta (Y_t^n)^r &\le \varepsilon^\delta (Y_{T-\varepsilon}^n)^r + \int_t^\tau \delta (T-s)^{\delta-1} (Y_s^n)^r ds + r \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} \mathbb{1}_{\{Y_s^n \neq 0\}} F(s, X_s, Y_s^n, Z_s^n) ds \\ &\quad -r \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} \mathbb{1}_{\{Y_s^n \neq 0\}} Z_s^n dW_s - c(r) \int_t^\tau (T-s)^\delta (Y_s^n)^{r-2} |Z_s^n|^2 \mathbb{1}_{\{Y_s^n \neq 0\}} ds.\end{aligned} \tag{33}$$
The monotonicity condition implies that
$$\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} \mathbb{1}_{\{Y_s^n \neq 0\}} F(s, X_s, Y_s^n, Z_s^n) ds \le \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} \mathbb{1}_{\{Y_s^n \neq 0\}} F(s, X_s, 0, Z_s^n) ds$$
and we use the regularity Lipschitz condition w.r.t. $z$ to obtain:
$$\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} \mathbb{1}_{\{Y_s^n \neq 0\}} F(s, X_s, 0, Z_s^n) ds \le K \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} |Z_s^n| ds + \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} F(s, X_s, 0, 0) ds.$$
Young's inequality leads to:
$$rK \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} |Z_s^n| ds \le \frac{K^2 r^2}{2c(r)} \int_t^\tau (T-s)^\delta (Y_s^n)^r ds + \frac{c(r)}{2} \int_t^\tau (T-s)^\delta (Y_s^n)^{r-2} |Z_s^n|^2 \mathbb{1}_{\{Y_s^n \neq 0\}} ds$$
and
$$r \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} F(s, X_s, 0, 0) ds \le (r-1) \int_t^\tau (T-s)^\delta (Y_s^n)^r ds + \int_t^\tau (T-s)^\delta (F(s, X_s, 0, 0))^r ds.$$
Finally all local martingales involved above in (33) are true martingales. Hence taking the expectation and using the convexity of $x \mapsto |x|^r$ we have:
$$\begin{aligned}&\sup_{t\in[0,\tau]} \mathbb{E}\left((T-t)^\delta (Y_t^n)^r\right) + \frac{c(r)}{2} \mathbb{E}\left( \int_t^\tau (T-s)^\delta (Y_s^n)^{r-2} |Z_s^n|^2 \mathbb{1}_{\{Y_s^n \neq 0\}} ds \right) \\ &\le \varepsilon^\delta \mathbb{E}((Y_{T-\varepsilon}^n)^r) + \mathbb{E}\left( \int_t^\tau \delta (T-s)^{\delta-1} (Y_s^n)^r ds \right) \\ &\quad + \left( 2\frac{K^2 r^2}{2c(r)} + (r-1) \right) \mathbb{E}\left( \int_t^\tau (T-s)^\delta (Y_s^n)^r ds \right) + \mathbb{E}\left( \int_t^\tau (T-s)^\delta (F(s, X_s, 0, 0))^r ds \right).\end{aligned} \tag{34}$$

Let us emphasize that this inequality holds with $(Y,Z)$ instead of $(Y^n,Z^n)$ since $\tau = T-\varepsilon < T$.

Using (31), the second term on the right-hand side can be controlled as follows:

$$\mathbb{E}\left(\int_0^\tau (T-s)^{\delta-1}(Y^n_s)^r ds\right) \leq \mathbb{E}\left(\int_0^T (T-s)^{\delta-1}\frac{1}{(T-s)^{rq_*-1}}\Xi_s ds\right) = \mathbb{E}\left(\int_0^T (T-s)^{-1+\eta}\Xi_s ds\right) < +\infty.$$

The third one satisfies the same estimate:

$$\mathbb{E}\left(\int_0^\tau (T-s)^{\delta}(Y^n_s)^r ds\right) \leq \mathbb{E}\left(\int_0^T (T-s)^{\eta}\Xi_s ds\right) < +\infty.$$

And the last term does not depend on $(Y^n,Z^n)$ and is bounded.

For the first term, since $Y^n$ is bounded by $Cn(T+1)$, we immediately obtain

$$\lim_{\varepsilon\to 0}\varepsilon^\delta \mathbb{E}((Y^n_{T-\varepsilon})^r) = 0.$$

If $Y^n$ is replaced by $Y$, from (32) we have

$$\varepsilon^\delta \mathbb{E}((Y_{T-\varepsilon})^r) \leq \varepsilon^\delta \frac{1}{\varepsilon^{q_* r-1}}\mathbb{E}(\Xi_{T-\varepsilon}) \leq \Theta \varepsilon^{\delta - q_* r+1} = \Theta\varepsilon^\eta$$

and the limit is again equal to zero.

Therefore we can let $\varepsilon$ go to zero in (34) and we can replace every $\tau$ by $T$:

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}\left((T-t)^\delta (Y^n_t)^r\right) + \frac{c(r)}{2}\mathbb{E}\left(\int_t^T (T-s)^\delta (Y^n_s)^{r-2}|Z^n_s|^2 \mathbb{1}_{\{Y^n_s\neq 0\}}ds\right)\\
&\leq \mathbb{E}\left(\int_t^T \delta(T-s)^{\delta-1}(Y^n_s)^r ds\right)\\
&\quad + \left(2\frac{K^2r^2}{2c(r)} + (r-1)\right)\mathbb{E}\left(\int_t^T (T-s)^\delta (Y^n_s)^r ds\right) + \mathbb{E}\left(\int_t^T (T-s)^\delta (F(s,X_s,0,0))^r ds\right).
\end{aligned}$$

The same inequality holds with $(Y,Z)$.

Next, by standard arguments, we can control the quantity $\mathbb{E}\left(\sup_{t\in[0,T]}(T-t)^\delta (Y^n_t)^r\right)$ by the same right-hand side (up to some multiplicative constant). Hence there exists $C$ s.t. for any $n$:

$$\mathbb{E}\left(\sup_{t\in[0,T]}(T-t)^\delta (Y^n_t)^r\right) + \frac{c(r)}{2}\mathbb{E}\left(\int_t^T (T-s)^\delta (Y^n_s)^{r-2}|Z^n_s|^2\mathbb{1}_{\{Y^n_s\neq 0\}}ds\right) \leq C. \tag{35}$$

Then if $r\geq 2$, we use (35) with $r=2$ and the result follows immediately with $\delta = 2q_*-1+\eta$. If $1<r<2$, the conclusion is more tricky. Let us define $M = \sup_{t\in[0,T]}(T-t)^{\frac{\delta}{r}}Y^n_t$ and:

$$\begin{aligned}
&\mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}}|Z^n_s|^2 ds\right)^{\frac{r}{2}}\right) = \mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}}\mathbb{1}_{\{Y^n_s\neq 0\}}|Z^n_s|^2 ds\right)^{\frac{r}{2}}\right)\\
&= \mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}}\left(Y^n_s\right)^{2-r}\left(Y^n_s\right)^{r-2}\mathbf{1}_{Y^n_s\neq 0}|Z^n_s|^2 ds\right)^{\frac{r}{2}}\right)\\
&\leq \mathbb{E}\left(M^{\frac{(2-r)r}{2}}\left(\int_0^T (T-s)^{\delta}\left(Y^n_s\right)^{r-2}\mathbf{1}_{Y^n_s\neq 0}|Z^n_s|^2 ds\right)^{\frac{r}{2}}\right) \leq \left(\mathbb{E}\left(M^r\right)\right)^{\frac{2-r}{2}}\left(\mathbb{E}\left(\int_0^T (T-s)^{\delta}\left(Y^n_s\right)^{r-2}\mathbb{1}_{\{Y^n_s\neq 0\}}|Z^n_s|^2 d\right.\right.\\
&\leq \frac{2-r}{2}\mathbb{E}\left(M^r\right) + \frac{r}{2}\mathbb{E}\left(\int_0^T (T-s)^{\delta}\left(Y^n_s\right)^{r-2}\mathbb{1}_{\{Y^n_s\neq 0\}}|Z^n_s|^2 ds\right) < +\infty.
\end{aligned}$$

where we have used Hölder's and Young's inequality with $\frac{2-r}{2}+\frac{r}{2}=1$. This achieves the proof of the lemma with $\varpi = \dfrac{2\delta}{r}$. □

Evoke that $0 \leq \alpha \leq 1$.

**Lemma 3.** *If $\alpha < \dfrac{2(q-1)}{q+1}$ for $q \leq 3$ and $\alpha \leq 1$ for $q > 3$, there exist $\nu > 0$ and $C > 0$ such that*

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\left(\int_0^T |Z_s^n|^{\alpha(1+\nu)} ds\right) \leq C.$$

*The same estimate holds for $Z$.*

*Proof.* For $r > 1$, $\alpha \leq 1$ and $\nu > 0$, suppose that $\beta = \alpha(1+\nu) < r \wedge 2$. Then:

$$\int_t^T |Z_s^n|^{\alpha(1+\nu)} ds = \int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2}}}(T-s)^{\frac{\varpi\beta}{2}} |Z_s^n|^\beta ds \leq \left(\int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds\right)^{\frac{2-\beta}{2}} \left(\int_t^T (T-s)^\varpi |Z_s^n|^2 ds\right)^{\frac{\beta}{2}}$$
$$\leq \frac{r-\beta}{r}\left(\int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds\right)^{\frac{(2-\beta)r}{2(r-\beta)}} + \frac{\beta}{r}\left(\int_t^T (T-s)^\varpi |Z_s^n|^2 ds\right)^{\frac{r}{2}}$$

with Young's inequality for $\dfrac{\beta}{r} + \dfrac{r-\beta}{r} = 1$. Therefore

$$\mathbb{E}\left(\int_0^T |Z_s^n|^{\alpha(1+\nu)} ds\right) \leq \frac{r-\beta}{r}\left(\int_0^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds\right)^{\frac{(2-\beta)r}{2(r-\beta)}} + \frac{\beta}{r}\mathbb{E}\left(\int_0^T (T-s)^\varpi |Z_s^n|^2 ds\right)^{\frac{r}{2}}.$$

From Lemma 2, the last term is bounded uniformly in $n$ if for some $\eta > 0$, $\varpi = \dfrac{2}{q-1} + 2\left(1 - \dfrac{1}{r}\right) + \dfrac{2\eta}{r}$. And the first integral is finite if and only if

$$\frac{\varpi\beta}{2-\beta} = \frac{\beta}{2-\beta}\left[\frac{2}{q-1} + 2\left(1-\frac{1}{r}\right) + \frac{2\eta}{r}\right] < 1.$$

But $\alpha < 2\dfrac{q-1}{q+1}$, then $(q+1)\alpha = 2\alpha + \alpha(q-1) < 2(q-1)$, thus $2\alpha < (2-\alpha)(q-1)$, that is $\dfrac{2\alpha}{(q-1)(2-\alpha)} < 1$. Now we can choose $\nu_0 > 0$ such that for any $0 < \nu \leq \nu_0$

$$\frac{\beta}{2-\beta}\frac{2}{q-1} \leq (1+\nu_0)\left(\frac{2-\alpha}{2-\alpha-\alpha\nu_0}\right)\frac{2\alpha}{(q-1)(2-\alpha)} < 1$$

and

$$0 \leq \frac{\alpha}{2-\alpha} \leq \frac{\beta}{2-\beta} = \frac{\alpha}{2-\alpha-\alpha\nu}(1+\nu) \leq \frac{\alpha}{2-\alpha-\alpha\nu_0}(1+\nu_0).$$

We can fix $r > 1$, $0 < \nu < \nu_0$ and $\eta > 0$ such that $r > (1+\nu) \geq (1+\nu)\alpha$ and

$$\frac{\varpi\beta}{2-\beta} = \frac{2\beta}{(q-1)(2-\beta)} + 2\left(1-\frac{1}{r}\right)\frac{\beta}{2-\beta} + \frac{2\eta}{r}\frac{\beta}{2-\beta} < 1.$$

Thus all integrals are finite and the conclusion holds. □

## 5.2 Control in the central equation

**Lemma 4.** *For $\Gamma$ given by (24), we have: $\mathbb{E}\left(\int_t^T \Gamma(s, X_s) ds\right) < +\infty$.*

*Proof.* Evoke that from (20), $\beta \geq 2q_* = \dfrac{2q}{q-1}$. Moreover the compact support of $\zeta$ is denoted by $\mathcal{K}$.

Now

$$
\begin{aligned}
\Gamma(s,X_s) &\le C\left(|\Psi(s,X_s)|^{q_*}+|\overline{\psi}(s,X_s)|^{q_*}\right)\left(1+|X_s|^{\frac{\ell}{q-1}}\right)\varphi(X_s)^{-\frac{1}{q-1}}\\
&\le C\varphi(X_s)^{-\frac{2}{q-1}}\left|\mathrm{tr}\left(\nabla^2\varphi(X_s)\sigma\sigma^*(s,X_s)\right)\right|^{2q_*}+C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d\left|(\nabla\varphi\sigma)_i(s,X_s)\frac{\mathrm{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*}\\
&\quad+C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|\langle\nabla\varphi(X_s),((\nabla\sigma_i)\sigma_i)(s,X_s)\rangle|^{2q_*}+C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d\left|\varphi(X_s)a_i(s,X_s)\frac{\mathrm{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*}\\
&\quad+C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|a_i(s,X_s)|^{2q_*}|\langle\nabla\varphi(X_s),\sigma_i(s,X_s)\rangle|^{2q_*}\\
&\quad+C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|\varphi(X_s)|^{2q_*}|\langle\nabla a_i(s,X_s),\sigma_i(s,X_s)\rangle|^{2q_*}+C(1+|X_s|^{\frac{2\ell}{q-1}})\\
&=: C(A_s^1+A_s^2+A_s^3+A_s^4+A_s^5+A_s^6+A_s^7).
\end{aligned}
$$

Evoke that $\varphi=\zeta^\beta$. For $A_s^1=\varphi(X_s)^{-\frac{2}{q-1}}\left|\mathrm{tr}\left(\nabla^2\varphi(X_s)\sigma\sigma^*(s,X_s)\right)\right|^{2q_*}$, we have

$$
\begin{aligned}
(\nabla^2\varphi(x))_{ij} &= \frac{\partial^2(\zeta^\beta)}{\partial x_i\partial x_j}(x)=\frac{\partial}{\partial x_i}\left(\frac{\partial(\zeta^\beta)}{\partial x_j}(x)\right)=\frac{\partial}{\partial x_i}\left(\beta\frac{\partial\zeta}{\partial x_j}(x)\zeta^{\beta-1}(x)\right)\\
&=\beta\frac{\partial^2\zeta}{\partial x_i\partial x_j}(x)\zeta^{\beta-1}(x)+\beta(\beta-1)\frac{\partial\zeta}{\partial x_i}(x)\frac{\partial\zeta}{\partial x_j}(x)\zeta^{\beta-2}(x)\\
&=\underbrace{\left(\beta\frac{\partial^2\zeta}{\partial x_i\partial x_j}(x)\zeta(x)+\beta(\beta-1)\frac{\partial\zeta}{\partial x_i}(x)\frac{\partial\zeta}{\partial x_j}(x)\right)}_{=:\widetilde{\zeta}_{ij}(x)}\zeta^{\beta-2}(x),
\end{aligned}
$$

with $\widetilde{\zeta}$ bounded. Thus

$$
A_s^1\le C\zeta(X_s)^{-\frac{2\beta}{q-1}+(\beta-2)2q_*}|\sigma(s,X_s)|^{4q_*}\le C\zeta(X_s)^{2(\beta-2q_*)}\left(|X_s|^{4q_*}+C\right)\le C(|X_s|^{4q_*}+C)
$$

because $\beta\ge 2q_*$ from (20). Therefore

$$
\mathbb{E}\left(\int_t^T A_s^1ds\right)\le C\left(\mathbb{E}\left(\sup_{0\le s\le T}|X_s|^{4q_*}\right)+1\right)<+\infty.
$$

For $A_s^2=\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d\left|(\nabla\varphi\sigma)_i(s,X_s)\frac{\mathrm{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*}$, we have

$$
\begin{aligned}
A_s^2 &\le C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|\sigma(s,X_s)|^{2q_*}|\nabla\varphi(X_s)|^{2q_*}\frac{|\mathrm{div}(p\sigma_i)(s,X_s)|^{2q_*}}{p(s,x)^{2q_*}}\\
&\le C\varphi(X_s)^{-\frac{2}{q-1}}\left(\sum_{i=1}^d|\mathrm{div}(p\sigma_i)(s,X_s)|^{2q_*}\right)\frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}}|\nabla\varphi(X_s)|^{q_*}\\
&= C\zeta(X_s)^{2\left(\beta-\frac{q}{q-1}\right)}\left(\sum_{i=1}^d|\mathrm{div}(p\sigma_i)(s,X_s)|^{2q_*}\right)\frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}}|\nabla\zeta(X_s)|^{2q_*}\\
&= C\zeta(X_s)^{2\left(\beta-\frac{q}{q-1}\right)}\left(\sum_{i=1}^d|\mathrm{div}(p\sigma_i)(s,X_s)|^{2q_*}\right)\frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}}|\nabla\zeta(X_s)|^{2q_*}\mathbb{1}_{\{X_s\in\mathcal{K}\}}\le C_{\mathcal{K}},
\end{aligned}
$$

because $\beta>q_*=\dfrac{q}{q-1}$, all functions are regular and we can control $\dfrac{1}{p}$ due to property 2 of Propositon 1. Thus $\mathbb{E}\left(\int_t^T A_s^2ds\right)<+\infty$.

The study of terms $A_3$ to $A_6$ is similar and $A_7$ is direct. □

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