Continuity problem for BSDE and IPDE with singular terminal condition

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Abstract

We study the behavior at the terminal time of the minimal supersolution of backward stochastic differential equation with singular terminal condition by using the associated integro-partial differential equation. We prove that if there are jumps (i.e. the operator of the PDE is non local), we observe a propagation of the singularity, contrary to the continuous case (local operator). We distinguish different cases of driver and terminal condition. The Riccati case is central because for quadratic and subquadratic generators the associated solution is not continuous at the terminal time, while the solutions for stronger non linearity are continuous. Finally we study the consequence for the numerical scheme.

Keywords. Riccati equation, backward stochastic differential equation, integro partial differential equation, singular terminal condition, implicit numerical scheme.

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1 Introduction

The notion of backward stochastic differential equations (BSDEs) was first introduced by Bismut in [10] in the linear setting and by Pardoux & Peng in [34] for non linear equation. One particular interest for the study of BSDE is the application to partial differential equations (PDEs). Indeed as proved by Pardoux & Peng in [35], BSDEs can be seen as generalization of the Feynman-Kac formula for non linear PDEs. Roughly speaking, if we can solve a system of two SDEs with one forward in time and one backward in time, then the solution is a deterministic function and is a (weak) solution of the related PDE. This is a method of characteristics to solve parabolic PDE. The converse assertion can be proved provided the solution of the PDE is enough regular to apply Itô's formula (see [15, Chapter 6]). Since then a large literature has been developped on this topic (see in particular the books [11], [15], [36] and the references therein). The extension to quasi-linear PDEs or to fully non linear PDEs has been already developed (see among other [31], [44] or [47]).

Among all semi-linear PDEs, a particular form has been widely studied:

$$\frac{\partial u}{\partial t}(t,x) + \Delta u(t,x) - u(t,x)|u(t,x)|^{q-1} = 0. \tag{1}$$

Baras & Pierre [8], Marcus & Veron [32] (and many other papers) have given existence and uniqueness results for this PDE. In [32] it is shown that every positive solution of (1) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S}, \mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^d$ and μ a non-negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}$:

$$\lim_{t\to T} \int_{\mathcal{R}} u(t,x)\varphi(x)dx = \int_{\mathcal{R}} \varphi(x)d\mu(x), \quad \forall \varphi \in C_c(\mathcal{R}).$$

The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^d, A \text{ Borel}, \begin{cases} \nu(A) = \infty & \text{if } A \cap \mathcal{S} \neq \varnothing \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view, Dynkin & Kuznetsov [13] and Le Gall [30] have proved similar results for the PDE (1) in the case $1 < q \leq 2$ using the theory of superprocesses. Now if we want to represent the solution u of (1) using an FBSDE (F means forward), we have to deal with a *singular* terminal condition ξ in the BSDE, which means that $\mathbb{P}(\xi = +\infty) > 0$. This singular case and the link between the solution of the BSDE with singular terminal condition and the viscosity solution of the PDE (1) have been studied first in [38] and developed in [40].

Besides PDEs motivation, BSDEs are a powerful tool to solve stochastic optimal control problems (see e.g. the survey article [16] or the books [37, 46]). In [5] and [28], it is proved that BSDEs with singular terminal condition provide a purely probabilistic solution of a stochastic control problem with a terminal constraint on the controlled process, motivated by models of optimal portfolio liquidation under stochastic price impact. On liquidation models see, e.g. [3, 4, 17, 22, 19, 23, 26], among many others. The related BSDEs are of the following form

$$-dY_t = -\frac{Y_t|Y_t|^{q-1}}{\eta_t}dt + \lambda_t dt - Z_t dW_t \tag{2}$$

with $\lim_{t\to T} Y_t = +\infty$ on $\mathcal{S}$. Parameter η is a measure of the illiquidity of the market, whereas λ penalizes the size of the remaining position of the portfolio. Here the singular set $\mathcal{S}$ corresponds to the scenarios with mandatory liquidation. The important feature is that the previous BSDE is directly related to a PDE similar to (1) and this link and PDEs technics have been used in [20, 21, 24, 42] to solve the same optimal liquidation problem.

In the standard L^p setting (see [11, 36]), the solution of the BSDE, with terminal condition ξ, is càdlàg[1] on $[0,T]$ and verifies

$$\lim_{t\to T} Y_t = \xi. \tag{3}$$

[1]French acronym for right-continuous with left-limits

When ξ is not integrable, in particular if $\mathbb{P}(\xi = +\infty) > 0$, the classical notion of solution has to be adapted. As proved in [28], the minimal solution only satisfies: a.s.

$$\liminf_{t \to T} Y_t \geq \xi = Y_T.$$

Therefore it is called a super-solution in [28].

We refer to the problem of establishing that a candidate solution satisfies (3) as the "continuity problem". In the PDE's context, if it is quite immediately that under weak conditions, there exists a minimal (viscosity) solution u such that

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \geq u(T,x_0).$$

The continuity at time T is not obvious.

As explained in details in [1, Section 1.1], solving this problem is crucial to ensure:

- Uniqueness of the solution of the BSDE,
- Tight control for the liquidation problem (no extra liquidation cost or no strict super-hedging),
- Condition in optimal targeting problem [7].

From [39], it is known that the existence of the limit at time T essentially depends on the generator of the BSDE. The solution is càdlàg on $[0,T]$ provided we can control the growth of the generator w.r.t. y. But replacing $\geq$ by $=$ is more delicate and has been studied in [39, 43, 33, 1].

If some partial results are available for general condition ξ, the more accurate results are given in the Markovian case, that is when $\xi = g(X_T)$, where X is a diffusion process and g is a function defined on $\mathbb{R}^d$ with values in $[0,+\infty]$[2]. In this case, the corresponding trace is $\mathcal{S} = \{g = +\infty\}$ and the measure μ has a density w.r.t. the Lebesgue measure given by g. In the rest of this paper, we only consider this Markovian framework.

Let us now distinguish two different cases. In the first one, the forward diffusion process X is continuous. Then the related PDE is a semi-linear parabolic PDE with only local differential operator, as for example Equation (1). Then in [39], it is proved that if the generator is sufficiently non linear ($q > 3$ for PDE (1) and BSDE (2)), Condition (3) holds. Otherwise Malliavin's calculus is a useful tool to prove that (3) holds under some uniform ellipticity condition on the matrix diffusion of X. It has been done in [38] for the specific generator related to PDE (1) ; the general case is studied in another paper, which is in the final stages of writing for upcoming submission. To summarize, with non degenerate diffusion matrix, for continuous diffusion process X or equivalently for parabolic PDE, the continuity property holds and this property is coherent with the results obtained in [32, 14, 30] for the PDE (1).

In the second case, X is also driven by a Poisson process (or more generally by a Poisson random measure). Then the corresponding integro-partial differential equation (IPDE) (11) has a non-local integral operator. This kind of IPDE with terminal singularity has not been studied with analytical methods and is only considered (to our best knowledge) in [40]. From [39, 40], if the generator is sufficiently non linear, continuity property again holds. In other words, the behaviors with or without jumps (or with local or non-local operators) are the same.

The goal of this paper is to provide an explicit example for which continuity property fails. We study the BSDE (2)

$$Y_t = g(N_T) - \int_t^T Y_s |Y_s|^{q-1} ds - \int_t^T U_s d\widetilde{N}_s, \quad 0 \leq t \leq T, \tag{4}$$

where N is a Poisson process with intensity λ and $\widetilde{N}$ is the compensated Poisson process: $\widetilde{N}_t = N_t - \lambda t$. We show that:

- The value $q = 2$ is critical. We construct an example for which $\mathcal{S} = \{g = +\infty\} = [x_0, \infty)$ and the minimal solution of the BSDE (4) or of the related PDE is the function $t \mapsto 1/(T-t)$. Hence the continuity problem (3) does not hold whatever $g\mathbf{1}_{g<+\infty}$ is. We also prove that the lack of continuity is due to the jump part of X ; adding a Brownian part does not change this fact.

[2] The non-negativity of g is not necessary but it simplifies the presentation of the results.

- For $q < 2$, the solution of the BSDE (4) (or of the PDE) explodes at time T. The solution is compared with the solution for $q = 2$ and the first one is greater than the second. Again this behavior does not depend on the terminal value.
- For $2 < q$, continuity property holds. Note that for $q > 3$, the result is already proved in [39].

The main novelty of the paper is the lack of continuity at time T for BSDEs (resp. for PDEs), when there are jumps (resp. when the operator is non-local). Here the regularity of the terminal condition (or of the trace in the context of [32]) does not influence the behavior of the solution. Somehow the solution forgets the terminal constraint. This property has never been observed in the literature.

The paper is organized as follows. In Section 2, we present the setting for BSDEs and PDEs and the known results. Since our goal is to provide an example of discontinuity, the setting is not the most general (see [11, 27, 36, 45] for the wider framework of BSDEs with or without jumps).

Section 3 studies in details the quadratic case (the generator is $y \mapsto -y|y|$) when the forward process is the Poisson process. The terminal condition is equal to $+\infty$ on an interval $[x_0, +\infty)$ for a fixed threshold x_0 and is finite on the complement of this interval. Our main result is in Theorem 2 and Corollary 1: any approximating sequence of the solution (of the BSDE or of the PDE) converges to $u : t \mapsto 1/(T-t)$ on $[0, T)$, whatever the value of the terminal condition on $(-\infty, x_0)$ is. In other words, for this singular terminal condition, there is only one solution equal to u on $[0, T)$. As a consequence, the solution of the BSDE or of the PDE does not depend on the terminal condition. Theorem 3 shows that adding a diffusion part does not change the result. The discontinuity comes from the jump part and cannot be overcome by the smoothing effect of the diffusion part.

In Section 4, we deduce that the quadratic case is critical. With less non linear generators ($q < 2$), the discontinuity holds (Theorem 4), whereas for more non linear generators ($q > 2$), continuity property holds (Proposition 7).

The non-decreasing Poisson process X has a tendency to go into the singular set $\mathcal{S} = [x_0, \infty)$, which intuitively explains the observed discontinuity. Indeed in Section 5, we show that if $\mathcal{S} = (-\infty, x_0]$, then continuity again holds. Here the Poisson process tends to exit from $\mathcal{S}$.

To illustrate this final discontinuity, we also study the related numerical scheme in Section 6. In our setting, we solve an ordinary Ricatti differential equation. The implicit Euler scheme is well posed and approximates the solution for bounded terminal condition with standard convergence rate. We prove that the same scheme can be used in the singular case and that it behaves according to the theoretical analysis, that is it forgets the terminal value and explodes at time T, when the discretization step tends to zero.

Let us emphasize that most of the results of this paper are true if we work with a compound Poisson process with positive jumps that are bounded away from zero. Nonetheless the extension to more general Poisson random measures or to multi-dimensional processes are left for further research.

2 Framework and definitions

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$. We assume that this set supports a one-dimensional Brownian motion W and a Poisson process N with intensity λ. The filtration $\mathbb{F}$ is generated by W and N. The compensated process $\tilde{N} = (\tilde{N}_t - \lambda t, t \geq 0)$ is a martingale w.r.t. $\mathbb{F}$.

For a given $T \geq 0$, we denote by $\mathcal{P}$ the predictable σ-field on $\Omega \times [0, T]$. On $\Omega \times [0, T]$, a function that is $\mathcal{P}$-measurable, is called predictable. $\mathcal{D}$ (resp. $\mathcal{D}(0, T)$) is the set of all predictable processes on $[0, +\infty)$ (resp. on $[0, T]$).

Now to define the solution of our BSDE, let us introduce the following spaces for $p \geq 1$.

- $\mathbb{D}^p(0, T)$ is the space of all adapted processes X with right-continuous with left limits paths, such that $\mathbb{E}\left(\sup_{t\in[0,T]} |X_t|^p\right) < \infty$.
- $\mathbb{H}^p(0, T)$ denotes the subspace of all processes $X \in \mathcal{D}(0, T)$ such that $\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right]$ is finite.
- Finally
$$\mathbb{S}^p(0, T) = \mathbb{D}^p(0, T) \times \mathbb{H}^p(0, T) \times \mathbb{H}^p(0, T).$$

We consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s d\widetilde{N}_s. \tag{5}$$

Here, the random variable ξ is $\mathcal{F}_T$-measurable with values in $\mathbb{R}$ and the generator $f : \Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a random function, measurable with respect to $Prog \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ where $Prog$ denotes the sigma-field of progressive subsets of $\Omega \times [0,T]$. The unknowns are (Y, Z, U) such that Y is progressively measurable and càdlàg with values in $\mathbb{R}$, $Z \in \mathcal{D}(0,T)$ such that a.s. $\int_0^T |Z_s|^2 ds < +\infty$ and $U \in \mathcal{D}(0,T)$.

2.1 Existence of a solution for the BSDE

The next conditions on f are very standard in the BSDE theory (see for example [11, 36]). For notational convenience we will denote $f_t^0 = f(t, 0, 0, 0)$.

- The function $y \mapsto f(t, y, z, u)$ is continuous and monotone: there exists $\mu \in \mathbb{R}$ such that a.s. and for any $t \in [0,T]$, $(z, u) \in \mathbb{R}^2$

$$(f(t, y, z, u) - f(t, y', z, u))(y - y') \leq \mu(y - y')^2. \tag{A1}$$

- For every $n > 0$ the function

$$\sup_{|y| \leq n} |f(t, y, 0, 0) - f_t^0| \in L^1((0,T) \times \Omega). \tag{A2}$$

- f is Lispchitz continuous in z, uniformly w.r.t. all parameters: there exists $L > 0$ such that for any (t, y, u), z and z': a.s.

$$|f(t, y, z, u) - f(t, y, z', u)| \leq L|z - z'|. \tag{A3}$$

- There exists a progressively measurable process $\kappa = \kappa^{y,z,u,v} : \Omega \times [0,T] \to \mathbb{R}$ such that

$$f(t, y, z, u) - f(t, y, z, v) \leq (u - v)\kappa_t^{y,z,u,v} \tag{A4}$$

  with $\mathbb{P} \otimes Leb$-a.e. for any (y, z, u, v), $-1 \leq \kappa_t^{y,z,u,v}$ and $|\kappa_t^{y,z,u,v}| \leq \vartheta$, where ϑ is a constant.

- There exists $\varrho > 1$ such that

$$\mathbb{E} \int_0^T |f_s^0|^\varrho ds < \infty. \tag{A5}$$

In [27, 45], it is proved that under Conditions (A1)-(A5) and if $\xi \in L^\varrho(\Omega)$, then the BSDE (5) has a unique solution (Y, Z, U) in $\mathbb{S}^\varrho(0,T)$. Moreover the comparison principle holds: roughly speaking, if $\xi' \geq \xi$ and $f' \geq f$, then $Y' \geq Y$.

Now if ξ is not integrable or if $\mathbb{P}(\xi = +\infty) > 0$, to ensure the existence of a solution which is finite before time T, we suppose that there exists a constant $q > 1$ and a positive constant η such that for any $y \geq 0$

$$f(t, y, z, u) \leq -\eta y|y|^{q-1} + f(t, 0, z, u). \tag{A6}$$

Definition 1. *The generator f satisfies Condition* **(A)** *if all assumptions* (A1)–(A6) *hold.*

Example 1. *The function $f(t, y, z, u) = -y|y|^{q-1}$ satisfies Condition* **(A)**.

In [28], the following result is proved.

Theorem 1 (Theorem 1 in [28])**.** *Under Condition* **(A)** *and if ξ and f^0 are non-negative, then there exists a process (Y, Z, U) such that*

- *(Y, Z, U) belongs to $\mathbb{S}^\varrho(0,t)$ for any $t < T$.*
- *Y is non-negative;*
- *For all $0 \leq s \leq t < T$:*

$$Y_s = Y_t + \int_s^t f(t, Y_r, Z_r, U_r)dr - \int_s^t Z_r dW_r - \int_s^t U_s d\tilde{N}_s.$$

- (Y, Z, U) *is a super-solution in the sense that a.s.*

$$\liminf_{t \to T} Y_t \geq \xi. \tag{6}$$

Any process $(\tilde{Y}, \tilde{Z}, \tilde{U})$ *satisfying the previous four items is called* ***super-solution*** *of the BSDE* (5) *with singular terminal condition* ξ*. Finally the process* (Y, Z, U) *is the minimal super-solution, in the sense that for any other supersolution, a.s. for any* t, $\tilde{Y}_t \geq Y_t$*.*

Note that this result holds in the more general framework with Poisson random measure and general filtration.

As explained in the introduction, Condition (6) is too weak to ensure uniqueness of the solution and is interpreted as an extra cost for liquidation in finance. Instead of (6), we want to have (3):

$$\lim_{t \to T} Y_t = \xi.$$

It is proved in [39, Section 3] that the existence of a left-limit at time T for Y only depends on f. A sufficient condition is the existence of a non-increasing, of class C^1 and concave function h and of a positive constant $\widetilde{\eta}$ such that for any $y \geq 0$

$$\widetilde{\eta} h(y) \leq f(t, y, z, u) - f(t, 0, z, u).$$

In this paper, we only discuss if a.s.

$$\liminf_{t \to T} Y_t = \xi. \tag{7}$$

If some partial results have been obtained for the non-Markovian setting ([43, 33, 2]), more complete results have been obtained in the Markovian setting.

2.2 Markovian setting

For $x \in \mathbb{R}$, we consider the forward SDE: for any $0 \leq t \leq T$

$$X_t = x + \int_0^t b(r, X_r) dr + \int_0^t \sigma(r, X_r) dW_r + \int_0^t \beta(r, X_{r-}) d\tilde{N}_r. \tag{8}$$

The coefficients $b : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\beta : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy:

1. b, σ and β are jointly continuous w.r.t. (t, x) and Lipschitz continuous w.r.t. x uniformly in t, i.e. there exists a constant K such that for any $t \in [0, T]$, for any x and y in $\mathbb{R}$: a.s.

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| + |\beta(t, x) - \beta(t, y)| \leq K|x - y|$$

2. b and σ growth at most linearly:

$$|b(t, x)| + |\sigma(t, x)| \leq C_{b,\sigma}(1 + |x|).$$

3. β is bounded w.r.t. t and x: there exists a constant C_β such that

$$|\beta(t, x)| \leq C_\beta.$$

Under these assumptions, the forward SDE (8) has a unique strong solution X (see [41]).

We assume that

$$\xi = g(X_T)$$

where the function g is defined on $\mathbb{R}$ with values in $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$. We denote

$$\mathcal{S} = \{x \in \mathbb{R} \quad s.t. \quad g(x) = +\infty\}$$

the set of singularity points for the terminal condition induced by g. We suppose that $\mathcal{S}$ is closed and that for all closed set $\mathcal{K} \subset \mathbb{R} \setminus \mathcal{S}$

$$g(X_T) \mathbf{1}_{\mathcal{K}}(X_T) \in \ L^1(\Omega, \mathcal{F}_T, \mathbb{P}).$$

In [39, Theorem 4.5], under Condition **(A)** on f and under this setting for $\xi = g(X_T)$, it is proved that (7) holds provided that in (A6) $q > 3$. There are also some technical conditions between the jumps of X and the singular set $\mathcal{S}$; these conditions are discussed in Sections 3 and 5. If $\mathbb{F}$ is only generated by W (in particular $\beta = 0$ in (8)), then (7) holds for any $q > 1$, provided that the diffusion coefficient σ is uniformly elliptic. Indeed in this case, we can use the representation of the process Z as the Malliavin derivative of Y.

The rest of this paper shows that the presence of jumps can completely destroy (7), that is for $q \leq 2$, it is possible to have a.s.

$$\liminf_{t\to T} Y_t = +\infty,$$

even if $\mathbb{P}(\xi = +\infty) < 1$.

2.3 Related PDEs

A key feature of BSDE is the link with parabolic PDE. Let us now define for $(t,x) \in [0,T] \times \mathbb{R}$, the forward SDE: for any $0 \leq t \leq s \leq T$

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r + \int_t^s \beta(r, X_{r-}^{t,x})d\tilde{N}_r. \tag{9}$$

The coefficients b, σ and β still satisfy the previously mentioned conditions: Lipschitz continuity w.r.t. x and at most linear growth. Then (9) has a unique strong solution $X^{t,x}$ belonging in $\mathbb{D}^p(0,T)$ for any $p > 1$. Together with the SDE (9), we solve the BSDE: for any $0 \leq t \leq s \leq T$

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r - \int_t^T U_r^{t,x}d\tilde{N}_r. \tag{10}$$

Now the generator $f : [0,T] \times \mathbb{R}^4 \to \mathbb{R}$ is a deterministic function, such that the random function $f^X(s,y,z,u) = f(s, X_s^{t,x}, y, z, u)$ satisfies Condition **(A)** uniformly w.r.t. x. The function $g : \mathbb{R} \to \mathbb{R}$ is measurable and non-negative and $\xi = g(X_T^{t,x})$. Hence we can apply the previous results to ensure the existence of a minimal super-solution $(Y^{t,x}, Z^{t,x}, U^{t,x})$.

To make the link with IPDE, we also suppose that the generator f verifies some extra regularity assumptions (see [9, 40]):

- f is locally Lipschitz continuous w.r.t. y: for all $R > 0$, there exists L_R such that for any y and y' and any (t,x,z,u)
$$|y| \leq R, |y'| \leq R \Longrightarrow |f(t,x,y,z,u) - f(t,x,y',z,u)| \leq L_R|y - y'|.$$
- The function $u \in \mathbb{R} \mapsto f(t,x,y,z,u)$ is non-decreasing for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^3$:
$$\forall u \leq u', \quad 0 \leq f(t,x,y,z,u') - f(t,x,y,z,u) \leq \vartheta(u' - u).$$
ϑ is the constant of Condition (A4).
- $(t,x) \mapsto f(t,x,y,z,u)$ is continuous and for all $R > 0$, $t \in [0,T]$, $|x| \leq R$, $|x'| \leq R$, $|y| \leq R$, $(z,u) \in \mathbb{R}^2$,
$$|f(t,x,y,z,u) - f(t,x',y,z,u)| \leq \omega_R(|x - x'|(1 + |z|)),$$
where $\omega_R(s)$ tends to 0 when $s \searrow 0$.
- $x \mapsto f(t,x,0,0,0)$ is of at most polynomial growth.

The generator of Example 1 satisfies these conditions. Now from [9, Proposition 2.5 and Theorem 3.4]:

Proposition 1. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and with polynomial growth, the function $u(t,x) = Y_t^{t,x}$ is the unique continuous viscosity solution of the IPDE:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u,u'\sigma, \mathcal{B}(t,x,u)) = 0 \\ u(T,x) = g(x) \end{cases} \tag{11}$$

(among the functions with polynomial growth). Moreover if g is bounded, u is also bounded.

In the previous IPDE, we have:

- $\mathcal{L}$ is the local second-order differential operator, due to the continuous part of the forward SDE:

$$\mathcal{L}(t,x,\phi) = \frac{1}{2}\sigma^2(t,x)\phi''(x) + b(t,x)\phi'(x) \ ;$$

- $\mathcal{I}$ is a non local differential operator and comes from the jump part of the forward SDE:

$$\mathcal{I}(t,x,\phi) = \phi(t,x+\beta(t,x)) - \phi(t,x) - \phi'(t,x)\beta(t,x) \ ;$$

- $\mathcal{B}$ is also a non local operator coming from the generator of the BSDE:

$$\mathcal{B}(t,x,\phi) = \phi(t,x+\beta(t,x)) - \phi(t,x).$$

From [40], we obtain the next statement[3]

Proposition 2. *If $g : \mathbb{R} \to [0,+\infty]$ is a continuous function such that for any compact set $\mathfrak{K}$ in $\mathbb{R} \setminus \mathcal{S}$, $g(X_T)\mathbf{1}_{\mathfrak{K}}(X_T)$ is integrable, where $\mathcal{S} = \{x \in \mathbb{R},\ g(x) = \infty\}$, then u is the minimal non-negative viscosity solution of* (11)*, such that:*

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \geq g(x_0) \tag{12}$$

holds.

The continuity problem for BSDE can be written here: does the minimal viscosity solution u satisfy

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0) \ ?$$

A natural question concerns the regularity of the solution u. In [40, Section 4.3], it is proved that if b and σ are bounded functions and σ is uniformly elliptic, if f is Hölder continuous w.r.t. (t,x), then $u \in C^{1,2}([0,T)\times\mathbb{R})$ (see [40, Lemmas 5 and 6, Proposition 5]).

3 Quadratic case with right barrier

In this section, we still assume that $X = N$ is the Poisson process ($\sigma = 0$ and $\beta = 1$), denoted $X^{t,x}$ if we want to emphasize that it starts at time t from point x:

$$X_s^{t,x} = x + \int_t^s \lambda dr + \int_t^s d\tilde{N}_r = x + N_s - N_t.$$

We study the quadratic case: $f(s,y,z,u) = -y|y|$ (Example 1 with $q = 2$), so the BSDEs (2) and (5) become

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\tilde{N}_s, \quad 0 \leq t \leq T. \tag{13}$$

We note $Y^{t,x}$ the solution of the BSDE whose dynamics is that of the BSDE (13) on $[t,T]$ when $X = X^{t,x}$. Moreover we consider the following function g : for $x_0 \in \mathbb{R}$ and $\varphi : \mathbb{R} \to [0,+\infty)$

$$g(x) = (+\infty)\mathbb{1}_{\{x \geq x_0\}} + \varphi(x)\mathbb{1}_{\{x < x_0\}}. \tag{14}$$

For this case, it is obvious that the singularity set $\mathcal{S} = [x_0,\infty)$ has a compact and regular boundary $\{x_0\}$ and obviously if $x \geq x_0$, $x+1 > x_0$. In other words it satisfies the technical conditions (called **(E)** in [39]) mentioned in Section 2.2. But $q = 2$ is too small to apply some known result about the continuity at time T. Let us remark that φ plays a role only if X starts below x_0.

Let us evoke some properties for this BSDE and the truncated BSDE: for any $K > 0$ and for $0 \leq t \leq T$

$$Y_t^K = g(X_T) \wedge K - \int_t^T Y_s^K |Y_s^K| ds - \int_t^T U_s^K d\tilde{N}_s. \tag{15}$$

[3]Note that continuity of the minimal solution is not guaranteed in this proposition.

From Section 2.1 and [28], there exists a unique solution (Y^K, U^K) for (15) and a minimal solution (Y, U) for (13) such that Y is the increasing limit of Y^K and since g is non-negative: a.s.

$$\forall t \in [0,T], \quad 0 \le Y_t^K \le Y_t \le \frac{1}{T-t}.$$

Note that these estimates do not depend on g. Moreover a.s.

$$\lim_{t \to T} Y_t \ge \xi = g(X_T).$$

The existence of the limit follows from [39, Theorem 3.1].

Finally the related IPDE (11) is: for any $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1) \\ u(T,x) = g(x), \end{cases} \tag{16}$$

and the truncated PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda u^K(t,x+1) \\ u^K(T,x) = g(x) \wedge K. \end{cases} \tag{17}$$

If $g : \mathbb{R} \to [0,+\infty]$ is continuous, then there exists a unique continuous viscosity solution u^K for (17) and a minimal viscosity solution u for (16) such that u is the increasing limit of u^K and for any (t,x)

$$0 \le u^K(t,x) \le u(t,x) \le \frac{1}{T-t}.$$

Recall that $u^K(t,x) = Y_t^{K,t,x}$ and $u(t,x) = Y_t^{t,x}$.

We are going to show that we have

$$\lim_{t \to T} Y_t > \xi = Y_T, \quad Y_t = \frac{1}{T-t} \quad \forall t \in [0,T).$$

In other words we show that there exist cases for which the solution is non continuous at time T. Contrary to [39, 40], the main changements are the quadratic driver (corresponding to $q = 2$ in (A6)) and simple jumps associated to a Poisson process (without jumps, for uniformly elliptic diffusions, we have continuity whatever the power q).

3.1 Solving the PDE and behavior at time T

Here we are going to resolve the PDEs (16) and (17), without the help of BSDE's theory. Let us state some results concerning the ODE:

$$\begin{cases} y'(t) - \lambda y(t) - y(t)|y(t)| + \lambda \psi(t) = 0, \quad 0 \le t \le T, \\ y(T) = \chi \in \mathbb{R}. \end{cases} \tag{18}$$

Lemma 1. *If $\psi \in C^0([0,T])$, then there exists a unique bounded solution y. Moreover if $\chi \ge 0$ and if for any t $\psi(t) \ge 0$, then $y(t) \ge 0$ for any t. The solution satisfies:*

$$\forall t < T, \quad y(t) \le \frac{1}{(T-t)^2} \int_t^T \left[\lambda \psi(s)(T-s)^2 + 1\right] e^{-\lambda(s-t)} ds.$$

As a consequence, if $\psi(t) \le 1/(T-t)$, then the same estimate holds for y.

Proof. The function $(t,y) \mapsto \lambda y + y|y| - \lambda \psi(t)$ is continuous w.r.t. t and is locally Lipschitz continuous w.r.t. y. Hence there exists a unique solution of the ODE, defined on an interval $(\tau, T]$. We also have

$$y'(t) = (\lambda + |y(t)|)y(t) - \lambda \psi(t).$$

So for $\tau < t \leq T$

$$y(t) = \chi \exp\left(-\int_t^T (\lambda + |y(s)|)ds\right) + \lambda \int_t^T \psi(s) \exp\left(-\int_t^s (\lambda + |y(u)|)du\right) ds.$$

Thus:

$$|y(t)| \leq |\chi| + \lambda \int_t^T |\psi(s)|ds \leq |y(T)| + \lambda \int_0^T |\psi(s)|ds.$$

As $\psi \in C^0([0,T])$, the function y is bounded on $(\tau, T]$, independently of τ. Hence there exists a global solution defined on $[0,T]$. And if χ and ψ are non-negative, the function y is also non-negative.

Now we prove the a priori estimate on y (adaptation of [28]). Let us solve the following linear ODE on the interval $[0, T-\varepsilon]$ for $0 < \varepsilon < T$,

$$(y^\varepsilon)' - \lambda y^\varepsilon - 2\frac{1}{T-t} y^\varepsilon + \frac{1}{(T-t)^2} + \lambda\psi(t), \quad y^\varepsilon(T-\varepsilon) = y(T-\varepsilon).$$

The solution y^ε is given by

$$\begin{aligned} y^\varepsilon(t) &= y(T-\varepsilon)\exp\left(-\int_t^{T-\varepsilon}\left(\lambda + 2\frac{1}{T-s}\right)ds\right) + \lambda\int_t^{T-\varepsilon}\psi(s)\exp\left(-\int_t^s\left(\lambda+2\frac{1}{T-u}\right)du\right)ds \\ &\quad + \int_t^{T-\varepsilon}\frac{1}{(T-s)^2}\exp\left(-\int_t^s\left(\lambda + 2\frac{1}{T-u}\right)du\right)ds \\ &= y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)}\exp\left(-2\int_t^{T-\varepsilon}\frac{1}{T-s}ds\right) + \lambda\int_t^{T-\varepsilon}\psi(s)e^{-\lambda(s-t)}\exp\left(-2\int_t^s\frac{1}{T-u}du\right)ds \\ &\quad + \int_t^{T-\varepsilon}\frac{1}{(T-s)^2}e^{-\lambda(s-t)}\exp\left(-2\int_t^s\frac{1}{T-u}du\right)ds \end{aligned}$$

But

$$\exp\left(-2\int_t^s \frac{1}{T-u}du\right) = \left(\frac{T-s}{T-t}\right)^2.$$

Thus

$$y^\varepsilon(t) = \frac{1}{(T-t)^2}\varepsilon^2 y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)} + \frac{1}{(T-t)^2}\int_t^{T-\varepsilon}\left[\lambda\psi(s) + \frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds.$$

Using the inequality $y^2 \geq 2cy - c^2$ with $c = 1/(T-t)$, we have the inequality between the two generators

$$\lambda y + 2\frac{1}{T-t}y - \frac{1}{(T-t)^2} - \lambda\psi(t) \leq \lambda y + y^2 - \lambda\psi(t).$$

Thus, with the comparison result for backward ODE, we deduce that for any $t \in [0, T-\varepsilon]$

$$y(t) \leq y^\varepsilon(t) = \frac{1}{(T-t)^2}\varepsilon^2 y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)} + \frac{1}{(T-t)^2}\int_t^{T-\varepsilon}\left[\lambda\psi(s) + \frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds.$$

Letting ε go to zero, since the function y is bounded, we deduce that for any $t < T$

$$\begin{aligned} y(t) &\leq \frac{1}{(T-t)^2}\int_t^T \left[\lambda\psi(s) + \frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds \\ &= \frac{1}{(T-t)^2}\int_t^T \left[\lambda\psi(s)(T-s)^2 + 1\right]e^{-\lambda(s-t)}ds. \end{aligned}$$

If $\psi(t) \leq \dfrac{1}{T-t}$, a computation shows that the same estimate holds for y:

$$\begin{aligned} y(t) &\leq \frac{1}{(T-t)^2}\int_t^T \left[\lambda(T-s)+1\right]e^{-\lambda(s-t)}ds \\ &= \frac{1}{(T-t)^2}\left(-\frac{e^{-\lambda(T-t)}}{\lambda} + \frac{\lambda(T-t)+1}{\lambda} + \frac{e^{-\lambda(T-t)}}{\lambda} - \frac{1}{\lambda}\right) = \frac{1}{T-t}. \end{aligned}$$

This achieves the proof of the lemma. □

Remark 1. *Let us emphasize that the mapping $(y(T),\psi)\mapsto y$ is non-decreasing: if $\widehat{\chi}\geq\chi\geq 0$ and $\widehat{\psi}(t)\geq\psi(t)\geq 0$ for any t, then $\widehat{y}(t)\geq y(t)\geq 0$.*

Indeed we have

$$\begin{aligned}\widehat{y}(t)-y(t)&=\widehat{\chi}-\chi-\int_t^T(\lambda\widehat{y}(s)+\widehat{y}(s)|\widehat{y}(s)|-\lambda\widehat{\psi}(s)-\lambda y(s)-y(s)|y(s)|+\lambda\psi(s))ds\\&=\widehat{\chi}-\chi-\int_t^T(\lambda\widehat{y}(s)-\lambda y(s)+\underbrace{\widehat{y}(s)|\widehat{y}(s)|}_{=\widehat{y}(s)^2}-\underbrace{y(s)|y(s)|}_{=y(s)^2})ds+\lambda\int_t^T\widehat{\psi}(s)-\psi(s)ds\\&=\widehat{\chi}-\chi-\int_t^T(\lambda+a(s))\,(\widehat{y}(s)-y(s))ds+\lambda\int_t^T\widehat{\psi}(s)-\psi(s)ds,\end{aligned}$$

with

$$a(s)=\frac{\widehat{y}(s)|\widehat{y}(s)|-y(s)|y(s)|}{\widehat{y}(s)-y(s)}\mathbf{1}_{\widehat{y}(s)\leq y(s)}\geq 0.$$

Thus

$$\begin{aligned}\widehat{y}(t)-y(t)=(\widehat{\chi}-\chi)\exp&\left(-\int_t^T(\lambda+a(s))ds\right)\\&+\lambda\int_t^T(\widehat{\psi}(s)-\psi(s))\exp\left(-\int_t^s(\lambda+a(u))du\right)ds\geq 0.\end{aligned}$$

We begin with the case $x\geq x_0$. We rewrite the PDEs

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x)-\lambda u(t,x)-u(t,x)|u(t,x)|=-\lambda u(t,x+1),\\ u(T,x)=+\infty,\end{cases}\tag{19}$$

and

$$\begin{cases}\dfrac{\partial u^K}{\partial t}(t,x)-\lambda u^K(t,x)-u^K(t,x)|u^K(t,x)|=-\lambda u^K(t,x+1),\\ u^K(T,x)=K.\end{cases}\tag{20}$$

Lemma 2. *On $[0,T]\times[x_0,\infty)$, for any $K>0$, the solutions of (19) and (20) are:*

$$u(t,x)=\frac{1}{T-t},\quad u^K(t,x)=\frac{1}{T-t+\dfrac{1}{K}}.$$

Proof. For the equation (20), we notice that the function $t\longmapsto\dfrac{1}{T-t+\frac{1}{K}}$ satisfies the PDE and is continuous and bounded. By uniqueness of the viscosity solution ([9, Theorem 3.5]), we have

$$u^K(t,x)=\frac{1}{T-t+\dfrac{1}{K}}.$$

Then the minimal solution is the increasing limit of u^K:

$$u(t,x)=\lim_{K\to+\infty}u^K(t,x)=\frac{1}{T-t}.$$

Clearly, it satisfies PDE (19) on $[0,T)\times[x_0,+\infty)$. □

We consider now the case $x \in [x_0 - 1, x_0)$. So $x + 1 \geq x_0$ and we rewrite the PDE (17) for u^K

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda \dfrac{1}{T - t + \dfrac{1}{K}}, \\ \\ u^K(T,x) = \varphi(x) \wedge K. \end{cases}$$

Note that it is an ODE with parameter x.

Lemma 3. *On $[0,T] \times [x_0 - 1, x_0)$, for any $K > 0$, the solution of (17) is*

$$u^K(t,x) = \frac{1}{T - t + \dfrac{1}{K}} - \frac{\mathbf{1}_{\varphi(x)<K}}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2 \left(\dfrac{1}{K - \varphi(x)} + \displaystyle\int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds \right)}, \tag{21}$$

and

$$u(t,x) = \lim_{K \to +\infty} u^K(t,x) = \frac{1}{T-t} \mathbb{1}_{\{t<T\}} + \varphi(x) \mathbb{1}_{\{t=T\}}.$$

Proof. Here x is a fixed parameter in $[x_0 - 1, x_0)$. We begin with the equation in u^K. We recognize a Riccati equation whose a particular solution of the dynamic is $t \longmapsto \dfrac{1}{T - t + \dfrac{1}{K}}$. We make the variable changement

$$u^K(t,x) = \frac{1}{T - t + \dfrac{1}{K}} - w^K(t,x)$$

where w^K is a non-negative function. The sign of w^K comes from the a priori estimate on u^k given by [28, Lemma 1]. So the function $w^K(\cdot, x)$ satisfies the ODE

$$\frac{\partial w^K}{\partial t}(t,x) - \left(\lambda + \frac{2}{T - t + \dfrac{1}{K}} \right) w^K(t,x) + w^K(t,x)^2 = 0.$$

We recognize a Bernoulli equation. We make the variable changement, under reserve of non cancellation,

$$y^K(t,x) = \frac{1}{w^K(t,x)}.$$

So the function $y^K(\cdot, x)$ satisfies a first order linear differential equation. Solving this ODE and going back to u^K, we obtain that if $\varphi(x) < K$

$$u^K(t,x) = \frac{1}{T - t + \dfrac{1}{K}} - \frac{1}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2 \left(\dfrac{1}{K - \varphi(x)} + \displaystyle\int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds \right)}$$

and for $\varphi(x) = K$

$$u^K(t,x) = \frac{1}{T - t + \dfrac{1}{K}}.$$

Let us pass to the limit on K for $t < T$. Since $\varphi(x) < +\infty$, we have

$$\lim_{K \to +\infty} \left(K(T-t)+1\right)^2 \frac{1}{K - \varphi(x) \wedge K} = +\infty$$

whereas

$$0 \leq \int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds \leq \int_t^T \frac{1}{(K(T-s)+1)^2} ds = \frac{T-t}{K(T-t)+1}.$$

Therefore we obtain for any $t < T$.

$$u(t,x) = \lim_{K \to +\infty} u^K(t,x) = \frac{1}{T-t}.$$

□

This function u satisfies the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda \dfrac{1}{T-t}, \\ \\ u(T,x) = \varphi(x). \end{cases} \tag{22}$$

Again it is an ODE with parameter x. Since $\psi = \dfrac{1}{T-\cdot} \notin C^0([0,T])$, we cannot apply Lemma 1. Nonetheless we have :

Lemma 4. *This function $u(\cdot,x) = \dfrac{1}{T-\cdot}$ is the unique non-negative solution of* (22) *defined on* $[0,T)$*.*

Proof. Again x is fixed and we assume there exists a non-negative solution $u(\cdot,x)$ defined on $[0,T)$. So the function $u(\cdot,x)$ satisfies the forward PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)^2 = -\lambda \dfrac{1}{T-t} \\ \\ u(0,x) = a, \end{cases} \tag{23}$$

with $a \geq 0$ defined by $a = u(0,x)$. According to the Cauchy-Lipschitz theorem, this ODE has a unique solution $u(\cdot,x)$ defined on $[0,\tau^a)$ (τ^a also depends on x, but we do not write this dependence). If $a = \dfrac{1}{T}$ then the function $t \longmapsto \dfrac{1}{T-t}$ is solution and well defined on $[0,T)$. In that case

$$u(t,x) = \frac{1}{T-t}.$$

We are going to show it is the only possibility. We suppose by absurd $a \neq \dfrac{1}{T}$. We also have a Riccati equation whose a particular solution is $t \longmapsto \dfrac{1}{T-t}$. So, by applying the previous method, we obtain an explicit solution

$$u(t,x) = \frac{1}{T-t} + \frac{1}{\dfrac{(T-t)^2}{T^2} e^{-\lambda t} \left(c - T^2 \displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds \right)}, \tag{24}$$

with $c = \left(a - \dfrac{1}{T} \right)^{-1} = \left(u(0,x) - \dfrac{1}{T} \right)^{-1}$. Now if $a > \dfrac{1}{T}$ then by divergence of the integral $\displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds$ in T, there exists $\tau^a \in [0,T)$ such that

$$\int_0^{\tau^a} \frac{e^{\lambda s}}{(T-s)^2} ds = \frac{c}{T^2} = \frac{1}{T^2 a - T} > 0.$$

So the function $u(\cdot,x)$ is defined only on $[0,\tau^a)$ with $\tau^a < T$, what contradicts our assumption on u. Now if $a < \dfrac{1}{T}$ then the function $u(\cdot,x)$ is defined on $[0,T)$ and

$$\begin{aligned} u(t,x) &= \frac{1}{T-t} + \frac{1}{\dfrac{(T-t)^2}{T^2} e^{-\lambda t} \left(\dfrac{1}{a - \dfrac{1}{T}} - T^2 \displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds \right)} \\ &= \frac{1}{T-t} + \frac{1}{(T-t)^2 e^{-\lambda t} \dfrac{1}{aT^2 - T} - e^{-\lambda t}(T-t)^2 \displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds}. \end{aligned}$$

Integrating by parts leads to

$$u(t,x) = \frac{1}{T-t}\left[1 + \frac{1}{(T-t)e^{-\lambda t}\left(\dfrac{a}{aT-1} + \lambda\ln(T) + \lambda^2\displaystyle\int_0^t e^{\lambda s}\ln(T-s)ds\right) - 1 - \lambda(T-t)\ln(T-t)}\right]$$

$$= \frac{e^{-\lambda t}\left(\dfrac{a}{1-aT} - \lambda\ln(T) - \lambda^2\displaystyle\int_0^t e^{\lambda s}\ln(T-s)ds\right) + \lambda\ln(T-t)}{(T-t)e^{-\lambda t}\left(\dfrac{a}{1-aT} - \lambda\ln(T) - \lambda^2\displaystyle\int_0^t e^{\lambda s}\ln(T-s)ds\right) + 1 + \lambda(T-t)\ln(T-t)}.$$

Therefore we obtain for any x

$$u(t,x) \underset{t\to T}{\sim} \lambda\ln(T-t). \tag{25}$$

In particular, $u(\cdot,x)$ becomes negative when t tends to T, which contradicts our assumption on u. Thus $a = \dfrac{1}{T}$ and

$$u(t,x) = \frac{1}{T-t}$$

is the only non-negative solution defined on $[0,T)$. □

We suppose now $x \in [x_0-2, x_0-1)$. Then $x+1 \in [x_0-1, x_0)$ and we rewrite the PDE (17)

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)^2 = -\lambda u^K(t,x+1), \\ u^K(T,x) = \varphi(x)\wedge K, \end{cases} \tag{26}$$

where $u^K(t,x+1)$ is given by (21) with $\varphi(x+1)$ instead of $\varphi(x)$:

$$u^K(t,x+1) = \frac{1}{T-t+\dfrac{1}{K}} - \frac{\mathbf{1}_{\varphi(x+1)<K}}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2\left(\displaystyle\int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2}ds + \frac{1}{K-\varphi(x+1)}\right)}.$$

Existence of u^K, solution of (26), is given by Lemma 1 (x is a parameter) since $u^K(\cdot,x+1)$ is a bounded function. Moreover since $u^K(\cdot,x+1)$ is bounded from above by $1/(T-\cdot)$ and is non-decreasing w.r.t. K, and with Remark 1, we have the estimate : for $K \le \widehat{K}$

$$0 \le u^K(t,x) \le u^{\widehat{K}}(t,x) \le \frac{1}{T-t}.$$

Nonetheless we cannot derive the explicit expression of u^K, but we prove that u^K still converges to $t \mapsto 1/(T-t)$. And from our previous result (Lemma 4), this function is the unique non-negative solution of (22) on $[x_0-2, x_0-1)$.

Lemma 5. *For any $x \in [x_0-2, x_0-1)$, the solution $u^K(\cdot,x)$ of (26) converges:*

$$u^K(t,x) \underset{K\to+\infty}{\longrightarrow} \frac{1}{T-t}.$$

Proof. Since $u^K(\cdot,x)$ is a non-decreasing sequence of functions, it converges to some limit function $u(\cdot,x)$ such that for any $t<T$:

$$0 \le u^K(t,x) \underset{K\to+\infty}{\nearrow} u(t,x) \le \frac{1}{T-t}.$$

For any $t<T$

$$u^K(t,x) = u^K(0,x) + \int_0^t \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)| - \lambda u^K(s,x+1)\right] ds.$$

Using dominated convergence theorem and Lemma 3, we can pass to the limit:

$$u(t,x) = u(0,x) + \int_0^t \left[\lambda u(s,x) + u(s,x)|u(s,x)| - \lambda\frac{1}{T-s}\right] ds.$$

Thus u solves the PDE (22): on $[0,T)$

$$\frac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda \frac{1}{T-t}.$$

If we assume

$$u(0,x) = a < \frac{1}{T} - \varepsilon$$

for some $\varepsilon > 0$, then, according to the performed reasoning in the proof of Lemma 4, the solution u is equal to zero in a time $\tau \in [0,T)$ what it cannot be. Thus $u(0,x) = \frac{1}{T}$ and

$$u^K(0,x) \underset{K\to+\infty}{\longrightarrow} \frac{1}{T}.$$

Now we consider the difference function

$$e^K(t,x) = u^K(t,x) - \frac{1}{T-t} = u^K(t,x) - u^\infty(t), \quad 0 \le t < T.$$

By performing the difference between the two PDEs

$$\frac{\partial e^K}{\partial t}(t,x) - \lambda e^K(t,x) - (u^K(t,x)^2 - u^\infty(t)^2) = -\lambda(u^K(t,x+1) - u^\infty(t)),$$

i.e., according to the identity $a^2 - b^2 = (a-b)(a+b)$,

$$\frac{\partial e^K}{\partial t}(t,x) - \left(\lambda + u^K(t,x) + u^\infty(t)\right) e^K(t,x) = -\lambda(u^K(t,x+1) - u^\infty(t)).$$

If we denote

$$a^K(t,x) = \lambda + u^K(t,x) + u^\infty(t), \quad c_K = e^K(0,x) = u^K(0,x) - \frac{1}{T}$$

the difference function e^K is given by

$$e^K(t,x) = \left(c_K - \lambda \int_0^t (u^K(s,x+1) - u^\infty(s)) \exp\left(-\int_0^s a^K(r,x)dr\right) ds\right) \exp\left(\int_0^t a^K(s,x)ds\right).$$

We are going to study the behavior of each term when $K \to +\infty$. We already know that

$$c_K = e^K(0,x) = u^K(0,x) - \frac{1}{T} \underset{K\to+\infty}{\longrightarrow} 0.$$

The term

$$a^K(t,x) = \lambda + u^K(t,x) + u^\infty(t) \in [0, \lambda + 2u^\infty(t)]$$

is bounded w.r.t. K, so $\exp\left(\int_0^t a^K(s,x)ds\right)$ also. Finally for the last, apply the dominated convergence theorem:

$$\int_0^t (u^K(s,x+1) - u^\infty(s)) \exp\left(-\int_0^s a^K(r,x)dr\right) ds \underset{K\to+\infty}{\longrightarrow} 0.$$

Therefore we obtain

$$e^K(t,x) \underset{K\to+\infty}{\longrightarrow} 0,$$

i.e.

$$u^K(t,x) \underset{K\to+\infty}{\longrightarrow} u^\infty(t) = \frac{1}{T-t}.$$

□

Theorem 2. *The solution u^K of (17) converges to the solution u of (16), which is given by*

$$u(t,x) = \frac{1}{T-t}\mathbb{1}_{\{t<T\}} + g(x)\mathbb{1}_{\{t=T\}}, \quad 0 \le t \le T, \quad x \in \mathbb{R}.$$

This solution u is the unique non-negative solution defined on $[0,T]$.

Proof. For $x \geq x_0$, it results from Lemma 2. Then we argue by recursion on the intervals $[x_0-k-1, x_0-k)$, with $k \in \mathbb{N}$. The initialization step comes from Lemma 3. We suppose the result for the $x \in [x_0 - k - 1, x_0 - k)$, then, by applying the proof of Lemma 5 we obtain the result for the $x \in [x_0 - k - 2, x_0 - k - 1)$. The recurrence principle allows to conclude. Uniqueness comes from Lemma 4. □

Remark 2. *Of course the same study can be done for $f(t,x,y,u) = -\eta y|y|$ with some constant $\eta > 0$. Then the solution becomes $u(t,x) = \dfrac{1}{\eta(T-t)}$.*

The next result is used below to control the martingale part of the BSDE.

Proposition 3. *For any $K > 0$, the difference $u^K(t,x+1) - u^K(t,x)$ is the sum of a non-negative term and a bounded term controlled for $x < x_0$ by*

$$|x_0 - x| \sup_{y \in [x,x_0)} |\varphi(y+1) - \varphi(y)|.$$

Proof. For $x \geq x_0 - 1$, we have: $u^K(t,x+1) - u^K(t,x) \geq 0$; it is an immediate consequence of Lemmata 2 and 3 and the formulas therein. Now for $x < x_0 - 1$, the difference $\Delta(t,x) = u^K(t,x+1) - u^K(t,x)$ satisfies: for any $t \in [0,T]$

$$\Delta'(t,x) - (\lambda + u^K(t,x+1) + u^K(t,x))\Delta(t,x) + \lambda\Delta(t,x+1) = 0.$$

Hence for any $x < x_0 - 1$:

$$\begin{aligned}\Delta(t,x) &= (\varphi(x+1) - \varphi(x)) \exp\left(-\int_t^T (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) \\ &+ \int_t^T \lambda\Delta(r,x+1) \exp\left(-\int_t^r (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) dr.\end{aligned}$$

Since u^K is non-negative, the first term is bounded by $|\varphi(x+1) - \varphi(x)|$.

Now for $x \in [x_0 - 2, x_0 - 1)$, $x + 1 \in [x_0 - 1, x_0)$, thus $\Delta(r, x+1) \geq 0$. Thus the claim is true on $[x_0 - 2, x_0 - 1)$:

$$\Delta(t,x) = \Delta^+(t,x) + \Gamma(t,x)$$

where $\Delta^+(t,x) \geq 0$ and $|\Gamma(t,x)| \leq |\varphi(x+1) - \varphi(x)|$.

For $x \in [x_0 - 3, x_0 - 2)$,

$$\begin{aligned}&\int_t^T \lambda\Delta(r,x+1) \exp\left(-\int_t^r (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) dr \\ &= \int_t^T \lambda\Delta^+(r,x+1) \exp\left(-\int_t^r (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) dr \\ &+ \int_t^T \lambda\Gamma(r,x+1) \exp\left(-\int_t^r (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) dr.\end{aligned}$$

And

$$\begin{aligned}&\int_t^T \lambda|\Gamma(r,x+1)| \exp\left(-\int_t^r (\lambda + u^K(s,x+1) + u^K(s,x))ds\right) dr \\ &\leq \int_t^T \lambda|\varphi(x+2) - \varphi(x+1)| \exp\left(-\lambda(r-t)\right) dr \leq |\varphi(x+2) - \varphi(x+1)|.\end{aligned}$$

Thus again for $2 < x_0 - x \leq 3$

$$\Delta(t,x) = \Delta^+(t,x) + \Gamma(t,x)$$

where $\Delta^+(t,x) \geq 0$ and $|\Gamma(t,x)| \leq 2\sup_{y\in[x,x_0)} |\varphi(y+1) - \varphi(y)|$.

We conclude by recursion. □

3.2 Consequence for the BSDE (13)

Here we still consider the terminal condition $\xi = g(X_T)$, with g given by (14) and the BSDE (13) is:

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\tilde{N}_s, \quad 0 \leq t < T.$$

We denote by (Y^K, U^K) the solution of the same BSDE (13) with terminal condition $g(X_T) \wedge K$. The first immediate consequence is the next result:

Corollary 1. *A.s. for any $t \in [0, T)$*

$$\lim_{K \to +\infty} Y_t^K = \frac{1}{T-t}, \quad \lim_{K \to +\infty} U_t^K = 0.$$

The solution Y of the BSDE (13) *is given by*

$$Y_t = \frac{1}{T-t} \mathbb{1}_{\{t<T\}} + g(X_T)\mathbb{1}_{\{t=T\}}, \quad 0 \leq t \leq T.$$

Moreover the process U^K is the sum of a non-negative term and of a term controlled by $\Phi(X_{s-})$ where $\Phi(x) = |x_0 - x| \sup_{y \in [x, x_0)} |\varphi(y+1) - \varphi(y)|$.

Proof. We can apply Itô's formula to $u^K(t, X_t)$ (only regularity w.r.t. t is required) to obtain the solution Y^K:

$$\begin{aligned}
Y_t^K = u^K(t, X_t) &= g(X_T) \wedge K - \int_t^T \frac{\partial u^K}{\partial t}(s, X_s)ds - \sum_{t<s\leq T} \left[u^K(s, X_s) - u^K(s, X_{s-})\right] \\
&= g(X_T) \wedge K - \int_t^T [\lambda(u^K(s, X_{s-}+1) - u^K(s, X_{s-})) + u^K(s, X_{s-})^2]ds \\
&\quad - \int_t^T \left[u^K(s, X_{s-}+1) - u^K(s, X_{s-})\right] dN_s \\
&= g(X_T) \wedge K - \int_t^T (Y_s^K)^2 ds - \int_t^T \left[u^K(s, X_{s-}+1) - u^K(s, X_{s-})\right] (dN_s - \lambda ds).
\end{aligned}$$

Hence $U_s^K = u^K(s, X_{s-}+1) - u^K(s, X_{s-})$. The conclusion follows from Proposition 3 and when we pass to the limit when K tends to ∞. ☐

Therefore we do not have the continuity of the process Y at the terminal time T: a.s.

$$\lim_{t \to T} Y_t = +\infty > \xi = Y_T.$$

This property does not depend on a particular choice of φ on $(-\infty, x_0)$. The singularity is propagated by the jumps of the forward process.

Note that Remark 2 still holds for the BSDE.

3.3 When we add a diffusion term

As mentioned in the introduction, if there is no jump and if the diffusion is uniformly elliptic, continuity at time T holds. Here we study the BSDE (13)

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T Z_s dW_s - \int_t^T U_s \tilde{N}(ds),$$

with g given by (14):

$$g(x) = (+\infty)\mathbb{1}_{\{x_0 \leq x\}} + \varphi(x)\mathbb{1}_{\{x<x_0\}},$$

and now we add a diffusion part in X, namely:

$$X_t = x + N_t + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

So the associated PDE (16) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1) \\ \\ u(T,x) = g(x), \end{cases} \tag{27}$$

with

$$\mathcal{L}u(t,x) = b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2 \frac{\partial^2 u}{\partial x^2}(t,x).$$

If $\lambda = 0$, we have a standard parabolic PDE which is studied in [32]. We want to prove that the minimal solution is again $u(t,x) = \dfrac{1}{T-t}$. Compared to Theorem 2, the differential operator $\mathcal{L}$ does not change the behavior of the solution.

First note that $t \mapsto \dfrac{1}{T-t}$ solves the PDE (27) on $[x_0,\infty)$ where $g(x) = +\infty$. We also consider the truncated version of the PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) + \mathcal{L}u^K(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda u^K(t,x+1) \\ \\ u^K(T,x) = g(x) \wedge K. \end{cases} \tag{28}$$

An auxiliary function. Let us consider the following PDE on $[0,T] \times \mathbb{R}$

$$\frac{\partial \overline{u}^K}{\partial t}(t,x) + \mathcal{L}\overline{u}^K(t,x) - \lambda \overline{u}^K(t,x) - \overline{u}^K(t,x)|\overline{u}^K(t,x)| = -\lambda \frac{1}{T-t+\dfrac{1}{K}}, \tag{29}$$

with $\overline{u}^K(T,x) = g(x) \wedge K$. This PDE is related with the BSDE without jumps

$$\overline{Y}^K_t = g(\mathfrak{X}_t) \wedge K - \int_t^T \overline{Y}^K_s |\overline{Y}^K_s| ds - \lambda \int_t^T \overline{Y}^K_s ds + \lambda \int_t^T \frac{ds}{T-s+\dfrac{1}{K}} - \int_t^T \overline{Z}^K_s dW_s, \tag{30}$$

where

$$\mathfrak{X}_t = x + \int_0^t b(\mathfrak{X}_s) ds + \int_0^t \sigma(\mathfrak{X}_s) dW_s.$$

Lemma 6. *There exists a unique solution $(\overline{Y}^K, \overline{Z}^K)$ to the BSDE (30) such that a.s. for any t,*

$$0 \leq \overline{Y}^K_t \leq \frac{1}{T-t+\dfrac{1}{K}}.$$

This sequence converges to $(\overline{Y}, \overline{Z})$ in $\mathbb{S}^p(0, T-\varepsilon)$ for any $p > 1$ and $\varepsilon > 0$ and for any $0 \leq t \leq s < T$

$$\overline{Y}_t = \overline{Y}_s - \int_t^s \overline{Y}^2_r dr - \lambda \int_t^s \overline{Y}_r dr + \lambda \int_t^s \frac{dr}{T-r} - \int_t^s \overline{Z}_r dW_r.$$

Proof. The driver is given by

$$f_K(s,y) = -y|y| - \lambda y + \frac{\lambda}{T-s+\dfrac{1}{K}}.$$

It is continuous and monotone w.r.t. y and bounded w.r.t. s. The terminal condition is bounded. Thus the solution $(\overline{Y}^K, \overline{Z}^K)$ exists and is unique in $\mathbb{S}^\varrho(0,T)$ for any $\varrho > 1$. Comparison principle implies that a.s. for any t,

$$0 \leq \overline{Y}^K_t \leq \frac{1}{T-t+\dfrac{1}{K}},$$

since $f_K(s,y) \geq -y|y| - \lambda y$ and $\left(\frac{1}{T-t+\frac{1}{K}}, 0\right)$ is the solution with terminal value K. Moreover for $K \leq K'$, a.s. $\overline{Y}^K_t \leq \overline{Y}^{K'}_t$. Hence

$$\overline{Y}_t = \lim_{K\to\infty} \overline{Y}^K_t$$

exists and satisfies $0 \leq \overline{Y}_t \leq 1/(T-t)$. The rest of the Lemma can be deduced with the same arguments as in [28, Proposition 3]. □

Proposition 4. *For any $t < T$, $\overline{Y}_t = \frac{1}{T-t}$. In particular*

$$\lim_{t\to T} \overline{Y}_t = +\infty.$$

Proof. The BSDE (30) can be considered as a linear BSDE and thus

$$\begin{aligned}
\overline{Y}^K_t &= \mathbb{E}\left[(g(\mathfrak{X}_t)\wedge K)e^{-\lambda(T-t)}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right) + \lambda\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^s |\overline{Y}^K_r|dr\right)ds\Big|\mathcal{F}_t\right] \\
&\geq \lambda\mathbb{E}\left[\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^s |\overline{Y}^K_r|dr\right)ds\Big|\mathcal{F}_t\right] \\
&\geq \lambda\mathbb{E}\left[\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)ds\Big|\mathcal{F}_t\right] \\
&\geq \lambda e^{-\lambda T}\int_t^T \frac{1}{T-s+\frac{1}{K}}ds\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\Big|\mathcal{F}_t\right] \\
&= \lambda e^{-\lambda T}\ln(K(T-t)+1)\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\Big|\mathcal{F}_t\right].
\end{aligned}$$

Since $\overline{Y}^K$ converges to $\overline{Y}$ and $\overline{Y}$ is finite on $[0,T)$, we deduce that for any $t < T$

$$\lim_{K\to\infty}\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\Big|\mathcal{F}_t\right] = 0,$$

in particular by Fatou's lemma,

$$\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}_r|dr\right)\Big|\mathcal{F}_t\right] = 0. \tag{31}$$

Now consider the difference

$$\begin{aligned}
\frac{1}{T-t+\frac{1}{K}} - \overline{Y}^K_t &= (K - g(\mathfrak{X}_T)\wedge K) - \int_t^T \left(\lambda + \frac{1}{T-s+\frac{1}{K}} + \overline{Y}^K_s\right)\left(\frac{1}{T-s+\frac{1}{K}} - \overline{Y}^K_s\right)ds \qquad (32) \\
&\quad + \int_t^T \overline{Z}^K_s dW_s \\
&= \mathbb{E}\left[\frac{K-g(\mathfrak{X}_T)\wedge K}{K(T-t)+1}e^{-\lambda(T-t)}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\Big|\mathcal{F}_t\right]. \qquad (33)
\end{aligned}$$

Since for $t < T$,

$$0 \leq \frac{K-(g(\mathfrak{X}_T)\wedge K)}{K(T-t)+1} \leq \frac{1}{T-t}, \text{ and } \lim_{K\to+\infty}\frac{K-(g(\mathfrak{X}_T)\wedge K)}{K(T-t)+1} = \frac{1}{T-t}\mathbf{1}_{g(\mathfrak{X}_T)<+\infty},$$

we deduce

$$\lim_{K\to+\infty}\left(\frac{1}{T-t+\frac{1}{K}}-\overline{Y}^K_t\right)=0$$

and that $\overline{Y}_t = 1/(T-t)$, which achieves the proof. □

According to [9, 40], we have the following link between the BSDE and the PDE:

Lemma 7. *If the function $g\wedge K$ is continuous, then there exists a unique bounded and continuous viscosity solution $\overline{u}^K$ to (29). Moreover for any t, x and K*

$$0\le \overline{u}^K(t,x)\le \frac{1}{T-t+\frac{1}{K}}.$$

Finally if b is bounded and σ is uniformly elliptic, that is, there exists $\nu>0$ such that for any (t,x)

$$\nu\le\sigma(t,x)^2\le\frac{1}{\nu},$$

then for any ε, $\overline{u}^K$ belongs to $C^{1,2}([0,T-\varepsilon]\times\mathbb{R})$.

Proof. The regularity of $\overline{u}^K$ is proved in [40, Section 4.3] and follows from classical results for PDEs, see among others [18, 29]. □

Now evoke that for any (t,x)

$$\overline{u}^K(t,x)=\overline{Y}^{K,t,x}_t.$$

From this proposition and this lemma, we deduce that for $x\in\mathbb{R}$ and $t<T$

$$\lim_{K\to+\infty}\overline{u}^K(t,x)=\lim_{K\to+\infty}\overline{Y}^{K,t,x}_t=\frac{1}{T-t}.$$

Back to the PDEs (28) **and** (27)**.** On $[x_0,+\infty)$, $g(x)\wedge K=K$ and the solution is $u^K(t,x)=\dfrac{1}{T-t+\dfrac{1}{K}}$. Therefore we can deduce that for $x\in[x_0-1,x_0)$ the PDE (28) is

$$\frac{\partial u^K}{\partial t}(t,x)+\mathcal{L}u^K(t,x)-\lambda u^K(t,x)-u^K(t,x)|u^K(t,x)|=-\lambda\frac{1}{T-t+\dfrac{1}{K}},$$

with $u^K(T,x)=g(x)\wedge K$ and thus on $[0,T]\times[x_0-1,x_0)$, $u^K(t,x)=\overline{u}^K(t,x)$. Hence the minimal solution of (27) satisfies for $x\in[x_0-1,x_0)$ and $t<T$: $u(t,x)=\dfrac{1}{T-t}$. In particular

$$\lim_{t\to T}u(t,x)=+\infty>g(x)=u(T,x).$$

Now to handle the case $x\in[x_0-2,x_0-1)$, let us introduce a second auxiliary PDE: on $[0,T]\times\mathbb{R}$

$$\begin{cases}\dfrac{\partial\tilde{u}^K}{\partial t}(t,x)+\mathcal{L}\tilde{u}^K(t,x)-\lambda\tilde{u}^K(t,x)-\tilde{u}^K(t,x)|\tilde{u}^K(t,x)|=-\lambda\overline{u}^K(t,x+1)\\[2ex] \tilde{u}^K(T,x)=g(x)\wedge K.\end{cases}$$

The associated BSDE is

$$\tilde{Y}^K_t=g(\mathfrak{X}_T)\wedge K-\int_t^T\tilde{Y}^K_s|\tilde{Y}^K_s|ds-\lambda\int_t^T\tilde{Y}^K_s ds+\lambda\int_t^T\overline{u}^K(s,\mathfrak{X}_s+1)ds-\int_t^T\tilde{Z}^K_s dW_s. \tag{34}$$

Lemma 8. *There exists a unique solution $(\tilde{Y}^K,\tilde{Z}^K)$ to the BSDE (34) such that a.s. for any t,*

$$0\le\tilde{Y}^K_t\le\frac{1}{T-t+\dfrac{1}{K}}.$$

This sequence converges to $(\tilde{Y},\tilde{Z})$ in $\mathbb{S}^p(0,T-\varepsilon)$ for any $p>1$ and $\varepsilon>0$ and for any $0\le t\le s<T$

$$\tilde{Y}_t=\tilde{Y}_s-\int_t^s\tilde{Y}_r^2dr-\lambda\int_t^s\tilde{Y}_r dr+\lambda\int_t^s\frac{dr}{T-r}-\int_t^s\tilde{Z}_r dW_r.$$

Proof. This proof can be deduced with the same arguments as in Lemma 6. □

Now we state the same result as in Proposition 4:

Proposition 5. *For any $t < T$, a.s. $\tilde{Y}_t = \dfrac{1}{T-t}$. In particular*

$$\lim_{t\to T} \tilde{Y}_t = +\infty.$$

Proof. The proof is rather similar as the proof of Proposition 4. However since the BSDE (34) contains a stochastic term $\overline{u}^K(s, \mathfrak{X}_s + 1)$ (instead of the deterministic and explicit $1/(T-s+1/K)$), new arguments have to be used.

We notice that $(\tilde{Y}^K, \tilde{Z}^K)$ is solution of a linear BSDE:

$$\begin{aligned}\tilde{Y}^K_t &= \mathbb{E}\left[(\exp\left(-\int_t^T |\tilde{Y}^K_s| ds\right) e^{-\lambda(T-t)} g(\mathfrak{X}_T) \wedge K) \Big| \mathcal{F}_t\right] \\ &\quad + \lambda \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s |\tilde{Y}^K_r| dr\right) e^{-\lambda(s-t)} \overline{u}^K(s, \mathfrak{X}_s + 1) ds \Big| \mathcal{F}_t\right] \\ &\geq \lambda e^{-\lambda T} \mathbb{E}\left[\exp\left(-\int_t^T |\tilde{Y}^K_r| dr\right) \int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) ds \Big| \mathcal{F}_t\right],\end{aligned}$$

with for $t < t$, $\tilde{Y}^K_t \leq 1/(T-t) < +\infty$,

$$\int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) ds \underset{K\to+\infty}{\overset{a.s.}{\longrightarrow}} \int_t^T \frac{1}{T-s} ds = +\infty,$$

and

$$\exp\left(-\int_t^T |\tilde{Y}^K_r| dr\right) \underset{K\to+\infty}{\overset{a.s.}{\searrow}} \exp\left(-\int_t^T |\tilde{Y}_r| dr\right).$$

Thus:

$$\exp\left(-\int_t^T |\tilde{Y}^K_r| dr\right) \underset{K\to+\infty}{\overset{a.s.}{\searrow}} 0. \tag{35}$$

Then with $\tilde{Y}^K_s \geq 0$, for any $t < T$

$$\begin{aligned} 0 &\leq \frac{1}{T-t+\frac{1}{K}} - \tilde{Y}^K_t \\ &= K - \int_t^T \frac{1}{(T-s+\frac{1}{K})^2} ds \\ &\quad - g(\mathfrak{X}_T) \wedge K + \int_t^T \tilde{Y}^K_s |\tilde{Y}^K_s| ds + \lambda \int_t^T \tilde{Y}^K_s ds - \lambda \int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) + \int_t^T \tilde{Z}_s dW_s \\ &= K - g(\mathfrak{X}_T) \wedge K - \int_t^T \left(\lambda + \frac{1}{T-s+\frac{1}{K}} + \tilde{Y}^K_s\right)\left(\frac{1}{T-s+\frac{1}{K}} - \tilde{Y}^K_s\right) ds \\ &\quad + \lambda \int_t^T \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s, \mathfrak{X}_s + 1) ds\right) + \int_t^T \tilde{Z}_s dW_s. \end{aligned}$$

Using the explicit formula for the solution of a linear BSDE leads to:

$$
\begin{aligned}
0 &\leq \frac{1}{T-t+\frac{1}{K}} - \tilde{Y}^K_t \\
&= \mathbb{E}\left[e^{-\lambda(T-t)} \frac{\frac{1}{K}}{T-t+\frac{1}{K}} \exp\left(- \int_t^T \tilde{Y}^K_s ds \right) (K - g(\mathfrak{X}_T)\wedge K) \Big| \mathcal{F}_t \right] \\
&\quad + \lambda \mathbb{E}\left[\int_t^T e^{-\lambda(s-t)} \frac{T-s+\frac{1}{K}}{T-t+\frac{1}{K}} \exp\left(- \int_t^s \tilde{Y}^K_r dr \right) \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \Big| \mathcal{F}_t \right] \\
&= e^{-\lambda(T-t)} \frac{1}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_t^T \tilde{Y}^K_s ds \right) \left(1 - \frac{g(\mathfrak{X}_T)\wedge K}{K} \right) \Big| \mathcal{F}_t \right] \\
&\quad + \lambda \frac{1}{T-t+\frac{1}{K}} \mathbb{E}\left[\int_t^T \left(T-s+\frac{1}{K} \right) e^{-\lambda(s-t)} \exp\left(- \int_t^s \tilde{Y}^K_r dr \right) \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \Big| \mathcal{F}_t \right] \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_t^T \tilde{Y}^K_s ds \right) \right. \\
&\qquad \left. + \int_t^T \left(T-s+\frac{1}{K} \right) \exp\left(- \int_t^s \tilde{Y}^K_r dr \right) \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \Big| \mathcal{F}_t \right],
\end{aligned}
$$

with, according to (33),

$$
0 \leq \frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \leq \frac{e^{-\lambda T}}{T-s+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_s^T \overline{Y}^K_r dr \right) \Big| \mathcal{F}_s \right].
$$

Thus:

$$
\begin{aligned}
0 &\leq \frac{1}{T-t+\frac{1}{K}} - \tilde{Y}^K_t \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_t^T \tilde{Y}^K_s ds \right) + \int_t^T \exp\left(- \int_t^s \tilde{Y}^K_r dr \right) \mathbb{E}\left[\exp\left(- \int_s^T \overline{Y}^K_r dr \right) \Big| \mathcal{F}_s \right] ds \Big| \mathcal{F}_t \right] \\
&= \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_t^T \tilde{Y}^K_s ds \right) + \int_t^T \exp\left(- \int_t^s \tilde{Y}^K_r dr \right) \exp\left(- \int_s^T \overline{Y}^K_r dr \right) ds \Big| \mathcal{F}_t \right] \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(- \int_t^T \tilde{Y}^K_s ds \right) + \int_t^T \exp\left(- \int_s^T \overline{Y}^K_r dr \right) ds \Big| \mathcal{F}_t \right].
\end{aligned}
$$

From (31) and (35), we deduce the statement of the proposition. □

Therefore for any $t < T$ and any $x \in \mathbb{R}$,

$$
\lim_{K\to+\infty} \tilde{u}^K(t,x) = \lim_{K\to+\infty} \tilde{Y}^{K,t,x}_t = \frac{1}{T-t}.
$$

And for such $x \in [x_0-2, x_0-1)$ and for all $t \in [0,T]$,

$$
u^K(t,x) = \tilde{u}^K(t,x).
$$

We deduce for $x \in [x_0-2, x_0-1)$

$$
\lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}.
$$

Then, by recurrence, for all $x \in \mathbb{R}$:

$$
u(t,x) = \lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}.
$$

Theorem 3. *The minimal super-solution (Y,Z,U) verifies: a.s.*

$$
\lim_{t\to T} Y_t = +\infty > \xi = g(X_T).
$$

Proof. From Lemma 7, the solutions u^K are smooth. Thus $Y^K_t = u^K(t,X_t)$. Passing through the limit on K, we deduce that $Y_t = 1/(T-t)$ a.s. The conclusion follows immediately. □

4 Other generators

We now use the results of the previous section, to show that the quadratic case is pivotal. We consider the BSDE (5) and we assume that the generator f satisfies Condition **(A)**, such that existence of a solution is guaranteed.

We suppose that X is the Poisson process and that the terminal condition is still given by (14). We denote by $(Y^{(2),K}, Z^{(2),K}, U^{(2),K}) = (Y^{(2),K}, 0, U^{(2),K})$ the solution of BSDE (13) with terminal condition $g(X_T) \wedge K$.

4.1 For $q < 2$

Let us start with the particular case: $f(t,x,y,u) = -y|y|^{q-1}$ for $1 < q < 2$.

Proposition 6. *The minimal super-solution (Y,U) of the BSDE (5) with generator $y \mapsto -y|y|^{q-1}$ for $1 < q < 2$ satisfies: a.s. for any $t \in [(T-1) \vee 0, T)$*

$$Y_t \geq \frac{1}{T-t}.$$

In particular a.s. $\lim_{t \to T} Y_t = +\infty$.

Proof. $(Y^{(q),K}, U^{(q),K})$ denotes the solution of the BSDE with terminal condition $g(X_T) \wedge K$. Then

$$\begin{aligned} Y_t^{(q),K} - Y_t^{(2),K} &= \int_t^T (-Y_s^{(q),K}|Y_s^{(q),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|)ds \\ &\quad - \int_t^T \left(U_s^{(q),K} - U_s^{(2),K}\right) d\tilde{N}_s \\ &= \int_t^T a_s^K (Y_s^{(q),K} - Y_s^{(2),K}) + \int_t^T (-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|)ds \\ &\quad - \int_t^T \left(U_s^{(q),K} - U_s^{(2),K}\right) d\tilde{N}_s \end{aligned}$$

with, by decrease of the function $y \longmapsto -y|y|^{q-1}$,

$$a_s^K = \frac{-Y_s^{(q),K}|Y_s^{(q),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|^{q-1}}{Y_s^{(q),K} - Y_s^{(2),K}} \mathbf{1}_{\{Y_s^{(q),K} \neq Y_s^{(2),K}\}} \leq 0.$$

The formula for linear BSDE implies that

$$Y_t^{(q),K} - Y_t^{(2),K} = \mathbb{E}\left[\int_t^T \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|\right) \Gamma_{t,s}^K ds \middle| \mathcal{F}_t\right]$$

with $\Gamma_{t,s}^K = \exp\left(\int_t^s a_u^K du\right) \in [0,1]$. In other words

$$Y_t^{(q),K} - Y_t^{(2),K} \geq \mathbb{E}\left[\int_t^T \Gamma_{t,s}^K \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|\right) \mathbf{1}_{[0,1]}(Y_s^{(2),K}) ds \middle| \mathcal{F}_t\right]$$

since for $y \geq 1$, $y|y|^{q-1} = y^q \leq y^2 = y|y|$. By the dominated convergence theorem, we deduce that for any $T-1 < t < T$

$$Y_t^{(q)} - Y_t^{(2)} = \lim_{K \to +\infty} Y_t^{(q),K} - Y_t^{(2),K} \geq 0,$$

that is : a.s. for $(T-1) \vee 0 \leq t < T$

$$Y_t^{(q)} \geq \frac{1}{T-t}.$$

This achieves the proof. □

Our proof shows that more general generators can be considered.

Theorem 4. *If Condition* **(A)** *holds, if f^0 is non-negative, if $u \mapsto f(t,y,z,u)$ is non-decreasing and if for $q<2$ and some $R>0$*

$$\forall y > R, \quad f(t,y,z,u) - f(t,0,z,u) \geq -y^q,$$

then the minimal super-solution of the BSDE (5) *verifies: a.s.*

$$\lim_{t\to T} Y_t = +\infty.$$

Proof. If (Y^K, Z^K, U^K) denotes the solution of the BSDE (5) with terminal condition $g(X_T)\wedge K$, then using a standard linearization method :

$$\begin{aligned}
Y_t^K - Y_t^{(2),K} &= \int_t^T \left(f(s, Y_s^K, Z_s^K, U_s^K) + Y_s^{(2),K}|Y_s^{(2),K}| \right) ds - \int_t^T Z_s^K dW_s \\
&\quad - \int_t^T \left(U_s^K - U_s^{(2),K} \right) d\tilde{N}_s \\
&= \int_t^T a_s^K (Y_s^{(q),K} - Y_s^{(2),K}) ds + \int_t^T b_s^K Z_s^K ds - \int_t^T Z_s^K dW_s \\
&\quad + \int_t^T \left(f(s, Y_s^{(2),K}, 0, U_s^K) - f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) \right) ds - \int_t^T \left(U_s^K - U_s^{(2),K} \right) d\tilde{N}_s \\
&\quad + \int_t^T f(s,0,0,0) ds \\
&\quad + \int_t^T \left(f(s, Y_s^{(2),K}, 0, 0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \right) ds \\
&\quad + \int_t^T \left(f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, 0) \right) ds
\end{aligned}$$

with

$$\begin{aligned}
a_s^K &= \frac{f(s, Y_s^K, Z_s^K, U_s^K) - f(s, Y_s^{(2),K}, Z_s^K, U_s^K)}{Y_s^K - Y_s^{(2),K}} \mathbf{1}_{\{Y_s^K \neq Y_s^{(2),K}\}}, \\
b_s^K &= \frac{f(s, Y_s^{(2),K}, Z_s^K, U_s^K) - f(s, Y_s^{(2),K}, 0, U_s^K)}{Z_s^K} \mathbf{1}_{\{Z_s^K \neq 0\}}.
\end{aligned}$$

The process b^K is bounded and the process a^K is bounded from above. Solving this linear BSDE leads to

$$Y_t^K - Y_t^{(2),K} = \mathbb{E}\left[\int_t^T \left(f(s,0,0,0) + A_s^K + B_s^K \right) \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t \right]$$

where $\mathcal{E}^K$ is non-negative and belongs to any $L^\varrho([0,T]\times\Omega)$,

$$B_s^K = f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, 0),$$

and

$$\begin{aligned}
A_s^K &= f(s, Y_s^{(2),K}, 0, 0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \\
&\geq \left(f(s, Y_s^{(2),K}, 0, 0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \leq R\right\}} \\
&\quad + \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \geq R\right\}} \\
&\geq \left(f(s, Y_s^{(2),K}, 0, 0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \leq R\right\}}
\end{aligned}$$

if $R>1$. From our assumptions, with the dominated convergence theorem, we obtain that for every $t \in \left[T - \dfrac{1}{R}, T\right]$

$$\liminf_{K\to\infty} \mathbb{E}\left[\int_t^T A_s^K \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t \right] \geq 0.$$

Moreover

$$|B_s^K| \leq C|U_s^{(2),K}| = C|u^K(s, X_{s-}+1) - u^K(s, X_{s-})| \leq C\frac{2}{T-s}.$$

Thus for any $\varepsilon > 0$

$$\lim_{K\to\infty} \mathbb{E}\left[\int_t^{T-\varepsilon} B_s^K \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t\right] = 0$$

Let us decompose $U_s^{(2),K} = \widehat{U}_s^{(2),K} + \widetilde{U}_s^{(2),K}$, with $\widehat{U}_s^{(2),K} \geq 0$. Then

$$B_s^K = f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, \widetilde{U}_s^{(2),K}) + f(s, Y_s^{(2),K}, 0, \widetilde{U}_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, 0).$$

Since $u \mapsto f(t, y, z, u)$ is non-decreasing, the first term is non-negative, whereas the second is bounded by $C|\widetilde{U}_s^{(2),K}| \leq C\Phi(X_{s-})$ (see Corollary 1). Hence for any $\varepsilon > 0$

$$\mathbb{E}\left[\int_{T-\varepsilon}^T B_s^K \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t\right] \geq -C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \Big| \mathcal{F}_t\right].$$

Evoke that $f(s, 0, 0, 0)$ is also non-negative. Hence

$$Y_t^K \geq Y_t^{(2),K} + \mathbb{E}\left[\int_t^T A_s^K \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^{T-\varepsilon} B_s^K \mathcal{E}_{t,s}^K ds \Big| \mathcal{F}_t\right] - C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \Big| \mathcal{F}_t\right].$$

Then passing to the limit on K gives : a.s. $Y_t \geq Y_t^{(2)} - C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \Big| \mathcal{F}_t\right]$ on $[T - \frac{1}{R}, T]$. Note that

$$\Phi(X_{s-}) \leq |X_{s-} - x_0| \sup_{y \in [X_0, x_0)} |\varphi(y+1) - \varphi(y)|.$$

Letting ε go to zero, we obtain that a.s. $Y_t \geq Y_t^{(2)}$ on $[T - \frac{1}{R}, T]$, which achieves the proof of the proposition. □

4.2 The case $q > 2$

From [39, Section 4] we already know that for $q > 3$, continuity holds: a.s. $\lim_{t\to T} Y_t = g(X_T)$. From the previous section, we also know that continuity fails for $q \leq 2$. In this part, we prove that continuity remains true for $2 < q \leq 3$.

We still consider the terminal condition $g(X_T)$ with g given by (14) and the truncated BSDE is : for $0 \leq t \leq T$

$$Y_t^K = g(X_T) \wedge K - \int_t^T Y_s^K |Y_s^K|^{q-1} ds - \int_t^T U_s^K d\tilde{N}_s. \tag{36}$$

Again from [28], there exists a unique solution (Y^K, U^K) for (36) and a minimal solution (Y, U) for (13) such that Y is the increasing limit of Y and since g is non-negative: a.s.

$$\forall t \in [0, T], \quad 0 \leq Y_t^K \leq Y_t \leq \left(\frac{p-1}{T-t}\right)^{p-1}.$$

Here p is the Hölder conjugate of q. Note that these estimates do not depend on g. Moreover a.s.

$$\varliminf_{t\to T} Y_t \geq \xi = g(X_T).$$

Finally the related IPDE (11) is: for any $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - \lambda u(t, x) - u(t, x)|u(t, x)|^{q-1} = -\lambda u(t, x+1) \\ u(T, x) = g(x), \end{cases} \tag{37}$$

and the truncated PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t, x) - \lambda u^K(t, x) - u^K(t, x)|u^K(t, x)|^{q-1} = -\lambda u^K(t, x+1) \\ u^K(T, x) = g(x) \wedge K. \end{cases} \tag{38}$$

With the same arguments as for Lemma 2, on $[0,T]\times[x_0,\infty)$, the solutions are

$$u(t,x) = \left(\frac{p-1}{T-t}\right)^{p-1}, \quad u^K(t,x) = \left(\frac{p-1}{T-t+K^{1-q}}\right)^{p-1}.$$

Thus on $[0,T]\times[x_0-1,x_0)$, the PDE (38) becomes

$$\frac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)|^{q-1} = -\lambda u^K(t,x+1) = -\lambda\left(\frac{p-1}{T-t+K^{1-q}}\right)^{p-1}.$$

The statement of Lemma 1 remains valid, that is u^K is well-defined, is non-decreasing w.r.t. K and converges to u as K tends to ∞. Moreover for any $t<T$

$$\begin{aligned}
u^K(t,x) &= u^K(T,x) - \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds + \lambda \int_t^T \left(\frac{p-1}{T-s+K^{1-q}}\right)^{p-1} ds \\
&= g(x)\wedge K - \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds \\
&+ \lambda\frac{(p-1)^{p-1}}{2-p}\left((T+K^{1-q})^{2-p} - (T-t+K^{1-q})^{2-p}\right).
\end{aligned}$$

It is equivalent to

$$\begin{aligned}
u^K(t,x) + \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds &= g(x)\wedge K \\
&+ \lambda\frac{(p-1)^{p-1}}{2-p}\left((T+K^{1-q})^{2-p} - (T-t+K^{1-q})^{2-p}\right).
\end{aligned}$$

Here is the key point : $q>2$ implies that $p<2$ or $2-p>0$. Hence we can pass to the limit w.r.t. K and the right-hand side is finite and equal to

$$g(x) + \lambda\frac{(p-1)^{p-1}}{2-p}\left(T^{2-p} - (T-t)^{2-p}\right).$$

By the monotone convergence theorem, the left-hand side converges to

$$u(t,x) + \int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds$$

and is larger than $u(t,x)$. We deduce that

$$u(t,x) + \int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds = g(x) + \lambda\frac{(p-1)^{p-1}}{2-p}\left(T^{2-p} - (T-t)^{2-p}\right)$$

and that $t\longmapsto u(t,x)$ is bounded by $g(x) + \lambda\frac{(p-1)^{p-1}}{2-p}T^{2-p}$. Therefore

$$\lim_{t\to T}\int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds = 0$$

and for $x\in[x_0-1,x_0)$

$$\lim_{t\to T} u(t,x) = g(x).$$

We can iterate these arguments on $[x_0-2,x_0-1)$ since $u^K(t,x+1)\le u(t,x+1)$ and $t\mapsto u(t,x+1)$ is a bounded function. Then by recursion we prove that:

Proposition 7. *If $q>2$, the PDE (37) has a unique solution u, which is equal to $t\mapsto \left(\frac{p-1}{T-t}\right)^{p-1}$ for $x\ge x_0$, such that $t\mapsto u(t,x)$ is bounded for any $x<x_0$ and that $\lim_{t\to T}u(t,x)=g(x)$.*

In other words continuity holds for $q>2$. For the BSDE, from the representation $Y_t^K = u^K(t,X_t)$, we immediately deduce that a.s.

$$\lim_{t\to T} Y_t = g(X_T).$$

Remark 3. *Note that the same result holds if the generator is of the form* $(s,y) \mapsto -y|y|^{q-1} + f^0_s$ *where* f^0 *is a deterministic and integrable function.*

We replace the explicit expression of the solution on $[x_0, +\infty)$, by the solution of the ODE

$$y' = y|y|^{q-1} - f^0_s, \quad y(T) = +\infty$$

which is bounded by

$$\frac{1}{(T-t)^p} \int_0^T \left[(p-1)^{p-1} + (T-s)^p f^0_s \right] ds$$

(see [28]) and is still integrable on $(0,T)$ if $q > 2$.

As a consequence, under condition **(A)** with $q > 2$, and if f^0 is deterministic and integrable (or bounded from above by a deterministic and integrable function), then the solution of the BSDE (5) satisfies a.s.

$$\lim_{t \to T} Y_t = g(X_T).$$

The proof is based on a comparison principle between the solution of (5) and the solution of the BSDE with generator $(s,y) \mapsto -y|y|^{q-1} + f^0_s$.

5 Poisson case with left barrier

As mentioned in Section 2.2, continuity property is proved in [40] under a sufficient condition, which link the set $\mathcal{S}$ to the jumps of X. This assumption is verified for the terminal value given by (14).

Let us show here that this condition is unnecessary. We again consider that X is a Poisson process. Now the function g is defined with $x_0 \in \mathbb{R}$ and $\varphi : \mathbb{R} \longrightarrow [0,\infty)$ a continuous function with polynomial growth:

$$g(x) = (+\infty)\mathbb{1}_{\{x \leq x_0\}} + \varphi(x)\mathbb{1}_{\{x > x_0\}}. \tag{39}$$

Note that $\mathcal{S} = (-\infty, x_0]$ has a compact boundary, but $x \in \mathcal{S}$ does not imply that $x+1 \in \mathcal{S}$. Moreover in general, the truncated function $g \wedge K$ is not continuous at the point x_0.

Nonetheless since the forward process is a Poisson process, if the process is greater than x_0 at some time τ, it remains greater than x_0 after. Let us consider the unique solution of the BSDE:

$$\mathcal{Y}^K_t = \varphi(X_T) \wedge K + \int_t^T f(s, \mathcal{Y}^K_s, \mathcal{U}^K_s) ds - \int_t^T \mathcal{U}^K_s d\tilde{N}_s,$$

where f still verifies Condition **(A)**.

Let us fix $\tau < T$ and consider the $\mathcal{F}_\tau$-measurable set $A_\tau = \{X_\tau > x_0\}$. Then on A_τ, for any $\tau \leq t \leq T$, $X_t \geq X_\tau > x_0$ and $g(X_T) = \varphi(X_T)$. Multiplying the two BSDEs by $\mathbf{1}_{A_\tau}$, we deduce that for any $\tau < t \leq T$, $\mathcal{Y}^K_t = Y^K_t$ on A_τ. Letting K go to $+\infty$ leads to: $Y_t = \mathcal{Y}_t$ on the set A_τ, where $(\mathcal{Y}, \mathcal{U})$ solves the BSDE:

$$\mathcal{Y}_t = \varphi(X_T) + \int_t^T f(s, \mathcal{Y}_s, \mathcal{U}_s) ds - \int_t^T \mathcal{U}_s d\tilde{N}_s.$$

Note that the existence and uniqueness of $(\mathcal{Y}, \mathcal{U})$ is ensured by the growth assumption on φ. In particular a.s. on the set A_τ

$$\lim_{t \to T} Y_t = \varphi(X_T).$$

Now we take a increasing sequence of τ_n converging to T. Since the family A_τ is a non-decreasing family of sets, a.s.

$$\bigcup_{n \in \mathbb{N}} \{X_{\tau_n} > x_0\} = \{X_{T-} > x_0\} = \{X_T > x_0\}$$

since T cannot be a jump time of X. According to the Theorem 1, we deduce that a.s.

$$\lim_{t \to T} Y_t = g(X_T).$$

We proved that

Proposition 8. *The minimal super-solution* (Y, U) *of the BSDE* (5) *with terminal condition* $\xi = g(X_T)$ *for* g *given by* (39)*, satisfies* (3).

With our choice of X, the PDE (11) becomes:

$$\frac{\partial u}{\partial t}(t,x) + \lambda u(t,x+1) - \lambda u(t,x) + f(t,x,u(t,x),\lambda(u(t,x+1)-u(t,x))) = 0. \tag{40}$$

Here the function f satisfies Condition **(A)** (uniformly in x) and the regularity conditions mentioned just before Proposition 1.

Proposition 9. *The minimal viscosity solution u of the PDE* (40)*, with terminal condition g given by* (39) *is continuous on $[0,T] \times \mathbb{R} \setminus \{x_0\}$.*

Proof. For a starting point $x > x_0$ at time t, we have : for any $t \le s \le T$, $X_s^{t,x} > x_0$. Hence a.s. $g(X_T^{t,x}) = \varphi(X_T^{t,x})$. We can apply Proposition 1 to deduce that the quantity $\widehat{u}(t,x) = \mathcal{Y}_t^{t,x}$ solves the PDE (40) with terminal condition φ, and that on $\{x > x_0\}$, $\widehat{u} = u$. In other words, the solution u^K of the PDE (40), with terminal condition $g(x) \wedge K$, converges to $\widehat{u}$ on $(x_0, +\infty)$. And we know that $u = \widehat{u}$ is continuous on $[0,T] \times (x_0, \infty)$.

Now for $x \in (x_0 - 1, x_0]$, to solve the PDE (40), let us consider the ordinary differential equation with parameter x:

$$(y^{K,x})'(t) + \lambda \widehat{u}(t,x+1) - \lambda y^{K,x}(t) + f(t,x,y^{K,x}(t),\lambda(\widehat{u}(t,x+1) - y^{K,x}(t))) = 0$$

with terminal condition $y^K(T) = K$. It is equivalent to

$$y^{K,x}(t) = K + \int_t^T F(s,x,y^{K,x}(s))ds.$$

The generator F satisfies Conditions (A1) to (A5), hence the solution $y^{K,x}$ exists and is unique. Since $\widehat{u} = u$ for $x > x_0$, it is immediate that $\check{u}^K(t,x) = y^{K,x}(t)$ is the solution of (40) with terminal condition $\check{u}^K(T,x) = K$.

Moreover F is continuous w.r.t. x. Hence standard stability estimate on BSDE (see [35, Theorem 2.9] or [25] for the Lipschitz case or [36, Theorem 5.10]) implies that $x \mapsto y^{K,x}$ is also continuous w.r.t. x, uniformly in K. Roughly speaking for (x, x')

$$\begin{aligned}
(y^{K,x}(t) - y^{K,x'}(t))^2 &= 2\int_t^T (y^{K,x}(s) - y^{K,x'}(s))(F(s,x,y^{K,x}(s)) - F(s,x',y^{K,x'}(s)))ds \\
&\le C_{\mu,\vartheta} \int_t^T (y^{K,x}(s) - y^{K,x'}(s))^2 ds + \lambda^2 \int_t^T (\widehat{u}(s,x+1) - \widehat{u}(s,x'+1))^2 ds \\
&+ \int_t^T (F(s,x,y^{K,x}(s)) - F(s,x',y^{K,x}(s)))^2 ds.
\end{aligned}$$

Using regularity condition on f w.r.t. x and Gronwall's lemma, we deduce the regularity of $x \mapsto y^{K,x}$ uniformly w.r.t. K. Hence passing to the limit on K for $\check{u}^K$, $(t,x) \mapsto u(t,x)$ is also continuous on $[0,T] \times (x_0 - 1, x_0]$. Iterating this procedure, we deduce that u is continuous on $[0,T] \times (-\infty, x_0]$. □

At the point x_0, g is not continuous. But if $\lim_{x \to x_0} \varphi(x) = +\infty$, then u is also continuous at x_0.

Remark 4. *Note that the arguments can be generalized to any non-decreasing forward process X. For example X can be a Lévy subordinator and the BSDE is driven by the Poisson random measure associated to X.*

6 Associated Euler scheme

We are interesting here in the convergence of the numerical scheme for the ODE: for $t \in [0,T)$

$$\begin{cases} u'(t) - \lambda u(t) - u(t)|u(t)| = -\lambda \dfrac{1}{T-t}, \\ \\ u(T) = \chi \in [0, +\infty). \end{cases}$$

This ODE is the same as the ODE (23), but with a terminal condition, and has been used to solve the PDE (22) and in Lemma 4, we prove that the unique non-negative solution u is given by:

$$u(t) = \frac{1}{T-t}, \quad \forall t < T,$$

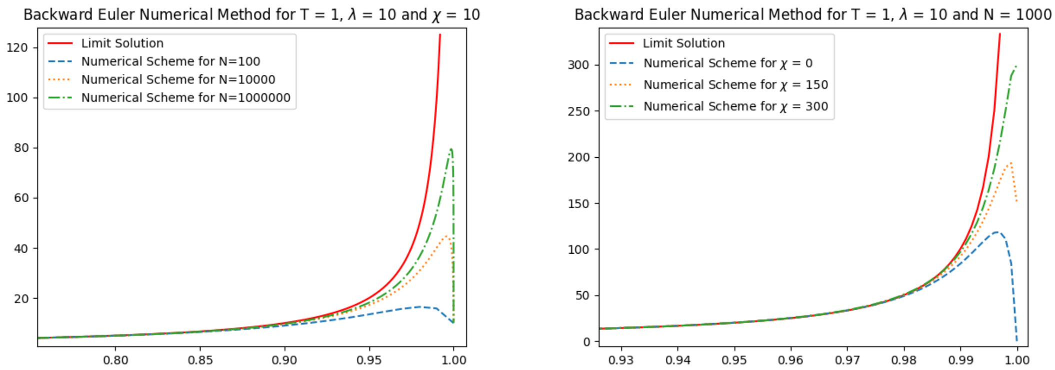

Figure 1: Backward Euler Numerical Method for $T = 1$ and $\lambda = 10$. On the left, $\chi = 10$; on the right $N = 1000$.

whatever χ is. Our aim is to illustrate this behavior for the numerical scheme and to show that the approximating sequence generated to the scheme converges to u for any χ.

We consider a regular subdivision $0 = t_0 < ... < t_N = T$ of the interval $[0, T]$ with a step $h_N = \dfrac{T}{N}$. We use here the implicit Euler method to define the scheme $u_N(t_k)$ by

$$u_N(t_N) = \chi$$

and by the implicit descending recurrence relation

$$u_N(t_{k+1}) = u_N(t_k) - h_N f(t_k, u_N(t_k)),$$

with

$$f(t, u) = \lambda u + u^2 - \lambda \frac{1}{T - t}.$$

We have the convergence on all closed intervals of $[0, T)$:

Theorem 5. *For all* $0 < \alpha < 1$, *we have*

$$\max_{0 \le k \le \lfloor \alpha N \rfloor} \left| u_N(t_k) - \frac{1}{T - t_k} \right| \underset{N \to +\infty}{\longrightarrow} 0.$$

We implemented the scheme. On Figure 1, on the left graph, the terminal value $\chi = 10$ is fixed and N increases. On the interval $[0, 0.8]$, the curves are overlaid on each other. On the right, N is equal to 1000 and χ increases; again on $[0, 0.93]$, the curves are overlaid.

To proof this theorem, firstly we are going to study the behavior of the scheme at the time $t_0 = 0$ thanks to the inferior and superior limits. Secondly we are using the results of convergence of forward schemes. To study the behavior of the scheme, we can explicit its expression.

Lemma 9. *The implicit backward Euler scheme can be written explicit : for all* $k \in [\![0, N-1]\!]$, *we have*

$$u_N(t_k) = \frac{\sqrt{(1 + h_N\lambda)^2 + 4h_N \left(u_N(t_{k+1}) + h_N \lambda \frac{1}{T - t_k} \right)} - (1 + h_N \lambda)}{2h_N} \ge 0.$$

Proof. We prove by recurrence the non negativity of $u_N(t_k)$. For $k = N$, we have $u_N(t_N) = \chi \ge 0$. Then if we assume $u_N(t_{k+1}) \ge 0$ for $k \in [\![0, N-1]\!]$, thus

$$0 \le u_N(t_{k+1}) + \lambda h_N \frac{1}{T - t_k} = u_N(t_k) + h_N \left(\lambda u_N(t_k) + u_N(t_k)|u_N(t_k)| \right) = F(u_N(t_k)),$$

with $F(x) = h_N(\lambda x + x|x|)$. F is non-decreasing with $F(0) = 0$. Thus $u_N(t_k) \ge 0$.

Then we have that $u_N(t_k)$ is a non-negative root of the polynome

$$P = h_N X^2 + (1 + \lambda h_N)X - \left(u_N(t_{k+1}) + \lambda h_N \frac{1}{T - t_k} \right),$$

of discriminant

$$\Delta = (1+\lambda h_N)^2 + 4h_N\left(u_N(t_{k+1}) + \lambda h_N \frac{1}{T-t_k}\right) > 0,$$

so the root are

$$x_1 = \frac{-(1+\lambda h_N)+\sqrt{\Delta}}{2h_N} > 0, \quad x_2 = \frac{-(1+h_N)-\sqrt{\Delta}}{2h_N} < 0.$$

Therefore we obtain the expression of the lemma. □

With this lemma, we have the following inequalities which will be useful to obtain contradictions if we assume the inferior and superior limits are different from the wished limit.

Lemma 10. *For all $k \in [\![0, N]\!]$,*

$$0 \le u_N(t_k) \le \chi + \lambda \sum_{i=1}^{N} \frac{1}{i}.$$

Proof. We show by descending recurrence on $k \in [\![0, N]\!]$ the property

$$u_N(t_k) \le \chi + \lambda \sum_{i=1}^{N-k} \frac{1}{i}.$$

For $k = N$, we directly have

$$u_N(t_k) = u_N(t_N) = \chi.$$

Then we suppose the result at rank $k+1$ for $k \in [\![0, N-1]\!]$. Then, we have the expression of $u_N(t_k)$ in function of $u_N(t_{k+1})$ according to the lemma 9,

$$\begin{aligned} u_N(t_k) &= \frac{\sqrt{(1+h_N\lambda)^2 + 4h_N\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)} - (1+h_N\lambda)}{2h_N} \\ &= \frac{2\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)}{\sqrt{(1+h_N\lambda)^2 + 4h_N\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)} + (1+h_N\lambda)} \\ &\le u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k} \\ &\le \chi + \lambda\sum_{i=1}^{N-k-1}\frac{1}{i} + h_N\lambda\frac{1}{Nh_N - kh_N} = \chi + \lambda\sum_{i=1}^{N-k}\frac{1}{i}. \end{aligned}$$

The recurrence principle allows to conclude and to obtain the inequality of this lemma. □

From this lemma, we deduce the rough estimate that for any $0 \le k \le N$: $u_N(t_k) \le \chi + \lambda(1+\gamma) + \lambda\ln(N)$, with the Euler's constant γ.

Lemma 11. *The inferior limit in $t_0 = 0$ satisfies*

$$\liminf_{N\to+\infty} u_N(t_0) \ge \frac{1}{T}.$$

Proof. We assume by contradiction that $\liminf_{N\to+\infty} u_N(t_0) < \frac{1}{T}$. Then, for all $\varepsilon > 0$, there exists a subsequence of $(u_N(t_0))_{N\in\mathbb{N}}$ which we note $(U_N)_{N\in\mathbb{N}}$, and $N_0 \in \mathbb{N}$ such that

$$\lim_{N\to+\infty} U_N = \ell < \frac{1}{T} - 2\varepsilon, \quad \forall N \ge N_0,\ 0 \le U_N \le \frac{1}{T} - \varepsilon.$$

For all $a \in \left[0, \frac{1}{T} - \varepsilon\right]$ the solution v^a of the ODE (23) with initial condition a is given by (24):

$$v^a(t) = \frac{1}{T-t}\left(1 - \frac{1}{\frac{T-t}{T(1-aT)}e^{-\lambda t} + (T-t)e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{(T-s)^2}ds}\right), \quad t \in [0, T).$$

So, for $\varepsilon < \dfrac{e^{-\lambda T}}{T}$,

$$v^a(t) \le v^{\frac{1}{T}-\varepsilon}(t) \le \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{\varepsilon T^2}e^{-\lambda t} + (T-t)\displaystyle\int_0^t \frac{1}{(T-s)^2}ds}\right)$$
$$= \frac{e^{-\lambda t} - \varepsilon T}{(T-t)e^{-\lambda t} + \varepsilon t T} \le \frac{1}{(T-t)e^{-\lambda T} + \varepsilon t T}$$
$$= \frac{1}{Te^{-\lambda T} + (\varepsilon T - e^{-\lambda T})t} \le \frac{1}{Te^{-\lambda T} + (\varepsilon T - e^{-\lambda T})T} = \frac{1}{\varepsilon T^2},$$

According to (25), we have

$$v^{\frac{1}{T}-\varepsilon}(t) \underset{t\to T}{\sim} \lambda \ln(T-t) < 0.$$

So, for $\eta > 0$, there exists $\tau^\varepsilon \in \ (0,T)$ such that: $v^{\frac{1}{T}-\varepsilon}(\tau^\varepsilon) < -\eta$. Moreover

$$v^a(t) \ge v^0(t) = \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{T}e^{-\lambda t} + (T-t)e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2}ds}\right)$$
$$= -\frac{\lambda e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{T-s}ds}{1 - \lambda(T-t)e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{T-s}ds}.$$

By continuity on the interval $[0,\tau^\varepsilon]$, the function v^0 is bounded from below by a constant $K_\varepsilon < 0$. So, on this interval, each solution v^a is bounded between K_ε and $\dfrac{1}{T^2\varepsilon}$. Then we can consider the ODE (23) starting at time 0 from U_N, with a driver $\tilde f$ which is bounded, of class C^1 with bounded derivative, such that the bounds for $\tilde f$ do not depend on a or N. Thus the associated Euler scheme

$$\begin{cases} w_N(t_0) = U_N \\ w_N(t_{k+1}) = w_N(t_k) + h_N\left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda\dfrac{1}{T-t_k}\right) \end{cases}$$

satisfies the standard conistency and stability results for Euler's scheme (see [6, Theorem 2.4] or [12, Chapter VIII]): there exists a constant $C > 0$ which depends on the driver $\tilde f$, such that

$$\max_{0\le k\le \lfloor \frac{\tau^\varepsilon N}{T}\rfloor} |w_N(t_k) - v^\ell(t_k)| \le C|w_N(t_0) - \ell| + \frac{C}{N}, \quad C > 0.$$

where we have chosen $a = \ell < \dfrac{1}{T} - 2\varepsilon < \dfrac{1}{T} - \varepsilon$. But we have

$$w_N(t_0) = U_N \underset{N\to+\infty}{\longrightarrow} \ell, \quad \frac{1}{N} \underset{N\to+\infty}{\longrightarrow} 0.$$

So, for N big enough,

$$\max_{0\le k\le \lfloor \frac{\tau^\varepsilon N}{T}\rfloor} |w_N(t_k) - v^\ell(t_k)| \le \frac{\eta}{2}.$$

Furthermore

$$v^\ell(\tau^\varepsilon) \le v^{\frac{1}{T}-\varepsilon}(\tau_\varepsilon) < -\eta,$$

the function v^ℓ is continuous on $[0,\tau^\varepsilon]$ and $t^N = \left\lfloor \dfrac{\tau^\varepsilon N}{T}\right\rfloor \dfrac{T}{N} \underset{N\to+\infty}{\longrightarrow} \tau^\varepsilon$. So, for N big enough,

$$v^\ell(t^N) < -\frac{3\eta}{4}$$

Thus

$$w_N(t^N) = w_N(t^N) - v^\ell(t^N) + v^\ell(t^N) \le \max_{0 \le k \le \lfloor \frac{\tau \varepsilon}{T} \rfloor} |w_N(t_k) - v^\ell(t_k)| + v^\ell(t^N) < \frac{\eta}{2} - \frac{3\eta}{4} = -\frac{\eta}{4} < 0.$$

But, after extraction, the sequence $w_N(t_k)$ satisfies the same Euler scheme than $u_N(t_k)$, so, after extraction,

$$u_N\left(t^N\right) = w_N\left(t^N\right) < 0.$$

However the scheme $u_N(t_k)$ cannot be negative according to the lemma 9. We obtain a contradiction, therefore

$$\liminf_{N \to +\infty} u_N(t_0) \ge \frac{1}{T}.$$

□

Lemma 12. *The superior limit in $t_0 = 0$ satisfies*

$$\limsup_{N \to +\infty} u_N(t_0) \le \frac{1}{T}.$$

Proof. We again prove this result by contradiction. We assume

$$\limsup_{N \to +\infty} u_N(t_0) > \frac{1}{T}.$$

Then, for all $\varepsilon > 0$, there exists a subsequence of $(u_N(t_0))_{N \in \mathbb{N}}$, which we note $(U_N)_{N \in \mathbb{N}}$, and $N_0 \in \mathbb{N}$ such that

$$\lim_{N \to +\infty} U_N = \ell > \frac{1}{T} + 2\varepsilon, \quad \forall N \ge N_0,\ U_N \ge \frac{1}{T} + \varepsilon.$$

But, for $a > \frac{1}{T}$, the differential equation (23) does not admit solution on $[0, T)$ (Lemma 4). More precisely, the solution is defined on $[0, \tau)$ with τ defined like the first time in $[0, T)$ such that

$$\frac{1}{Ta - 1} = T \int_0^\tau \frac{e^{\lambda s}}{(T - s)^2} ds.$$

If $a \ge \frac{1}{T} + \varepsilon$, then we have

$$\frac{T}{T - \tau} - 1 = T \int_0^\tau \frac{1}{(T - s)^2} ds \le T \int_0^\tau \frac{e^{\lambda s}}{(T - s)^2} ds = \frac{1}{Ta - 1} \le \frac{1}{\varepsilon T}.$$

So

$$\tau \le \frac{T}{1 + \varepsilon T}.$$

In other words, each solution v^a of the differential equation (23) which satisfies $v^a(0) = a \ge \frac{1}{T} + \varepsilon$, explodes before the time $\frac{T}{1 + \varepsilon T}$. Now we consider the Euler scheme

$$\begin{cases} w_N(t_0) = a \ge \frac{1}{T} + \varepsilon \\ w_N(t_{k+1}) = w_N(t_k) + h_N \left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda \frac{1}{T - t_k} \right), \end{cases}$$

for $0 \le t_k \le \frac{T}{1 + \varepsilon T}$ i.e. $k \in \left[\!\left[0, \left\lfloor \frac{N}{1 + \varepsilon T} \right\rfloor \right]\!\right]$. On this interval we have

$$0 \le \frac{1}{T - t_k} \le \frac{1 + \varepsilon}{\varepsilon T^2} < +\infty.$$

Let us now prove by recursion that for N larger than some constant depending on T and ε, and for any k

$$w_N(t_k) \geq \frac{1}{T-t_k} + \varepsilon.$$

This property holds for $k = 0$. If the property is satisfied for k then

$$\begin{aligned} w_N(t_{k+1}) &= w_N(t_k) + h_N \left(\lambda w_N(t_k) + w_N(t_k)^2 - \lambda \frac{1}{T-t_k} \right) \\ &\geq w_N(t_k) + h_N w_N(t_k)^2 + h_N \lambda \varepsilon \geq w_N(t_k) + h_N w_N(t_k)^2. \end{aligned}$$

But

$$\frac{1}{T-t_{k+1}} = \frac{1}{T-t_k-h_N} = \frac{1}{T-t_k} + \frac{h_N}{(T-t_k)(T-t_k-h_N)}.$$

So

$$\begin{aligned} w_N(t_{k+1}) - \frac{1}{T-t_{k+1}} &\geq w_N(t_k) - \frac{1}{T-t_k} + h_N \left(w_N(t_k)^2 - \frac{1}{(T-t_k)(T-t_k-h_N)} \right) \\ &\geq \varepsilon + h_N \left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} + \frac{1}{(T-t_k)^2} - \frac{1}{(T-t_k)(T-t_k-h_N)} \right) \\ &= \varepsilon + h_N \left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} - \frac{h_N}{(T-t_k)^2(T-t_k-h_N)} \right), \end{aligned}$$

with

$$\frac{1}{(T-t_k)^2(T-t_k-h_N)} = \frac{1}{(T-t_k)^2} \frac{1}{T-t_{k+1}} \leq \frac{(1+\varepsilon)^2}{\varepsilon^2 T^4} \frac{1+\varepsilon}{\varepsilon T^2} = \frac{(1+\varepsilon)^3}{\varepsilon^3 T^6}.$$

So, for N larger than $\varepsilon^5 T^5/(1+\varepsilon)^3$

$$w_N(t_{k+1}) - \frac{1}{T-t_{k+1}} \geq \varepsilon + h_N \left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} - h_N \frac{(1+\varepsilon)^3}{\varepsilon^3 T^6} \right) \geq \varepsilon.$$

Hence the property is proved for any k and the recurrence formula can be rewritten

$$w_n(t_{k+1}) = w_N(t_k) + h_N w_N(t_k)^2 + h_N \lambda \left(w_N(t_k) - \frac{1}{T-t_k} \right) \geq w_N(t_k) + h_N w_N(t_k)^2.$$

If we define $\overline{w}_N(t_k)$ the sequence defined by

$$\begin{cases} \overline{w}_N(t_0) = \dfrac{1}{T} + \varepsilon \\ \overline{w}_N(t_{k+1}) = \overline{w}_N(t_k) + h_N \overline{w}_N(t_k) |\overline{w}_N(t_k)| \end{cases}$$

this sequence is well-defined, non-negative and non-decreasing. From the previous property of $w_N(t_k)$, a direct comparison for the schemes leads to:

$$\overline{w}_N(t_k) \leq w_N(t_k).$$

We consider the sequence

$$y_N(t_k) = \frac{1}{\overline{w}_N(t_k)}.$$

So

$$y_N(t_{k+1}) = \frac{1}{\overline{w}_N(t_{k+1})} = \frac{1}{\overline{w}_N(t_k) + h_N \overline{w}_N(t_k)^2} = \frac{1}{\dfrac{1}{y_N(t_k)} + \dfrac{h_N}{y_N(t_k)^2}} = \frac{y_N(t_k)}{1 + \dfrac{h_N}{y_N(t_k)}},$$

and

$$y_N(t_0) = \frac{1}{\overline{w}_N(t_0)} = \frac{1}{\dfrac{1}{T} + \varepsilon} = \frac{T}{1+\varepsilon T}.$$

Assume that for the biggest k : $\hat{k} = \left\lfloor \dfrac{N}{1+\varepsilon T} \right\rfloor$,

$$y_N(t_{\hat{k}}) > 2\sqrt{T}\sqrt{h_N}.$$

So, since the sequence $y_N(t_k)$ is non-increasing in k, we have

$$y_N(t_0) \geq ... \geq y_N(t_k) \geq y_N(t_{k+1}) \geq ... \geq y_N(t_{\hat{k}}) > 2\sqrt{T}\sqrt{h_N}.$$

Thus, from the inequality $\dfrac{1}{1+u} \leq 1 - u + u^2$,

$$\begin{aligned} y_N(t_{k+1}) = y_N(t_k)\frac{1}{1+\dfrac{h_N}{y_N(t_k)}} &\leq y_N(t_k)\left(1 - \frac{h_N}{y_N(t_k)} + \frac{h_N^2}{y_N(t_k)^2}\right) \\ &= y_N(t_k) - h_N + \frac{h_N^2}{y_N(t_k)} < y_N(t_k) - h_N + \frac{1}{2\sqrt{T}} h_N^{\frac{3}{2}}. \end{aligned}$$

So, by successive iterations,

$$\begin{aligned} y_N(t_k) &< y_N(t_0) - kh_N + k\frac{1}{2\sqrt{T}} h_N^{\frac{3}{2}} \\ &= \frac{T}{1+\varepsilon T} - k\frac{T}{N} + k\frac{T}{N}\frac{1}{2\sqrt{T}}\sqrt{h_N}. \end{aligned}$$

But, for $\hat{k} \geq \dfrac{N}{1+\varepsilon T} - 1$, we have

$$\frac{T}{1+\varepsilon T} - \hat{k}\frac{T}{N} \leq \frac{T}{1+\varepsilon T} - \frac{T}{1+\varepsilon T} + \frac{T}{N} = \frac{T}{N}.$$

Hence with $\hat{k} \leq N$

$$y_N(t_{\hat{k}}) < \frac{T}{N} + \hat{k}\frac{T}{N}\frac{1}{2\sqrt{T}}\sqrt{h_N} \leq \frac{T}{N} + \frac{T}{2\sqrt{T}}\sqrt{h_N} = \sqrt{h_N}\left(\frac{\sqrt{T}}{\sqrt{N}} + \frac{\sqrt{T}}{2}\right) \leq 2\sqrt{T}\sqrt{h_N}$$

what contradicts the assumption. Therefore

$$0 \leq y_N(t_{\hat{k}}) \leq 2\sqrt{T}\sqrt{h_N} = 2T\frac{1}{\sqrt{N}}.$$

Thus

$$w_N(t_{\hat{k}}) \geq \overline{w}_N(t_{\hat{k}}) = \frac{1}{y_N(t_{\hat{k}})} \geq \frac{1}{2T}\sqrt{N}.$$

Now if we consider $a = \ell > \dfrac{1}{T} + 2\varepsilon > \dfrac{1}{T} + \varepsilon$, we can deduce, after extraction,

$$u_N(t_{\hat{k}}) \geq \frac{1}{2T}\sqrt{N},$$

what cannot be according to the lemma 9. Therefore we have

$$\limsup_{N\to+\infty} u_N(t_0) \leq \frac{1}{T}.$$

□

As a consequence of the two previous lemmata, we state

Proposition 10. *We have the limit in $t_0 = 0$:*

$$\lim_{N\to+\infty} u_N(t_0) = \frac{1}{T}.$$

Finally we obtain Theorem 5 by using the convergence results about the forward Euler scheme.

Proof of Theorem 5. We consider the forward numerical scheme

$$\begin{cases} w_N(t_{k+1}) = w_N(t_k) + h_N\left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda\dfrac{1}{T-t_k}\right) \\ \\ w_N(t_0) = u_N(t_0) \end{cases},$$

associated to the differential equation

$$\begin{cases} w'(t) = \lambda w(t) + w(t)|w(t)| - \lambda\dfrac{1}{T-t} = f(t, w(t)), \quad 0 \le t < T \\ \\ w(0) = u_N(t_0). \end{cases}$$

To obtain the exact solution, we have to distinguish if $u_N(t_0) < \dfrac{1}{T}, u_N(t_0) > \dfrac{1}{T}$ or $u_N(t_0) = \dfrac{1}{T}$. The last case is direct because, in that case, the exact solution is $w^N(t) = \dfrac{1}{T-t}$, $0 \le t < T$. In a first time, if

$$u_N(t_0) < \frac{1}{T},$$

then the solution is given by (24):

$$w^N(t) = \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{T(1-u_N(t_0)T)}e^{-\lambda t} + (T-t)e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2}ds}\right).$$

Let us prove that for N large enough (depending on the convergence proved in the previous proposition), w^N is non-negative and bounded from above on $[0, \alpha T]$. Indeed with an integration by part, we obtain

$$w^N(t) = \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{T(1-u_N(t_0)T)}e^{-\lambda t} + 1 - \dfrac{(T-t)e^{-\lambda t}}{T} - (T-t)\lambda e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{T-s}ds}\right).$$

Therefore the function w^N is positive until the first time $t = t(N)$ such that

$$\frac{T-t}{T(1-u_N(t_0)T)}e^{-\lambda t} - \frac{(T-t)e^{-\lambda t}}{T} - (T-t)\lambda e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{T-s}ds = 0,$$

which is equivalent to

$$\frac{1}{\dfrac{1}{u_N(t_0)} - T} = \int_0^t \frac{e^{\lambda s}}{T-s}ds.$$

Thus

$$\frac{1}{\dfrac{1}{u_N(t_0)} - T} = \int_0^t \frac{e^{\lambda s}}{T-s}ds < e^{\lambda T}\int_0^t \frac{ds}{T-s} = e^{\lambda T}\ln\left(\frac{T}{T-t}\right),$$

and we deduce that

$$t(N) > T\left(1 - \exp\left(-\frac{e^{-\lambda T}}{\dfrac{1}{u_N(t_0)} - T}\right)\right) \underset{N\to+\infty}{\longrightarrow} T,$$

because $u_N(t_0) \underset{N\to+\infty}{\longrightarrow} \dfrac{1}{T}$. Therefore for N large enough, the time $t(N)$ is greater than αT, that w^N is positive on $[0, \alpha T]$. Furthermore the function w^N is bounded from above on $[0, \alpha T]$ by

$$C_\alpha = \frac{1}{T - \alpha T},$$

because

$$w^N(t) \le \frac{1}{T-t}, \quad 0 \le t \le \alpha T.$$

Now if $u_N(t_0) > \frac{1}{T}$, then the exact solution w^N is still given by (24), is non-decreasing, but is only defined on $[0,\tau)$ with τ defined by

$$\frac{1}{u_N(t_0) - \frac{1}{T}} = \int_0^\tau \frac{e^{\lambda s}}{(T-s)^2} ds.$$

But $u_N(t_0) \underset{N\to+\infty}{\longrightarrow} \frac{1}{T}$, so $\frac{1}{u_N(t_0) - \frac{1}{T}} \underset{N\to+\infty}{\longrightarrow} +\infty$, thus, for N big enough, we have $\tau > \alpha T$. So the function w^N is defined and continuous on $[0,\alpha T]$ and bounded from above by, for $\varepsilon > 0$ and N large enough such that $u_N(t_0) < \frac{1}{T} + 1$

$$\begin{aligned} C_{\alpha,N} &= w^N(\alpha T) \\ &= \frac{1}{T-\alpha T}\left(1 - \frac{1}{\frac{T-\alpha T}{T(1-u_N(t_0)T)} e^{-\lambda\alpha T} + (T-\alpha T)e^{-\lambda \alpha T}\int_0^{\alpha T}\frac{e^{\lambda s}}{(T-s)^2}ds}\right) \\ &\le \frac{1}{T-\alpha T}\left(1 - \frac{1}{-(1-\alpha)e^{-\lambda\alpha T} + (T-\alpha T)e^{-\lambda \alpha T}\int_0^{\alpha T}\frac{e^{\lambda s}}{(T-s)^2}ds}\right) =: C_\alpha. \end{aligned}$$

Therefore, in each case, the solution is non-negative and bounded from above by some constant C_α on $[0,\alpha T]$.

We can consider the function ψ defined by

$$\psi(w) = \begin{cases} \lambda w & \text{si} \quad w < 0 \\ \lambda w + w^2 & \text{si} \quad 0 \le w \le C_\alpha \\ (\lambda + 2C_\alpha)w - C_\alpha^2 & \text{si} \quad w > C_\alpha. \end{cases}$$

ψ is of class C^1 on $\mathbb{R}$ and Lipschitz continuous with a Lipschitz constant equal to $\lambda + 2C_\alpha$. We consider the function $\tilde{f}$ defined by

$$\tilde{f}(t,w) = \psi(w) - \lambda\frac{1}{T-t}, \quad w \in \mathbb{R},\ 0 \le t < T.$$

This function is equal to f on $[0,T) \times [0, C-\alpha]$ and inherits the regularity property of ψ w.r.t. w. Therefore, with the previous inequalities, the function w^N satisfies the differential equation

$$\begin{cases} w'(t) = \tilde{f}(t,w(t)), \quad 0 \le t < T \\ \\ w(0) = u_N(t_0) \end{cases}$$

with driver $\tilde{f}$. So, according to [6] or [12], there exists a constant C such that

$$\max_{0\le k\le \lfloor \alpha N\rfloor} |w_N(t_k) - w^N(t_k)| \le C(|w_N(t_0) - \underbrace{w^N(t_0)}_{=u_N(t_0)}| + Th_N) = CTh_N.$$

This constant C depends on T, λ and α, but not on N. A direct computation shows that C can be chosen equal to

$$e^{\lambda + 2C_\alpha}\left[\lambda\frac{1}{T^2(1-\alpha)^2} + \lambda^2\frac{1}{T(1-\alpha)} + \left(\lambda^2 + 2\lambda\frac{1}{T(1-\alpha)}\right)C_\alpha + 3\lambda C_\alpha^2 + 2\lambda C_\alpha^3\right],$$

which shows the dependence w.r.t. α. Therefore, since $u_N(t_k) = w_N(t_k)$ by definition of the numerical schemes satisfied by these two sequences,

$$\max_{0\leq k\leq \lfloor \alpha N\rfloor} |u_N(t_k) - w^N(t_k)| \leq CTh_N \underset{N\to+\infty}{\longrightarrow} 0.$$

Therefore all that remains is to study the second term in the inequality

$$\max_{0\leq k\leq \lfloor \alpha N\rfloor} |u_N(t_k) - u(t_k)| \leq \max_{0\leq k\leq \lfloor \alpha N\rfloor} |u_N(t_k) - w^N(t_k)| + \max_{0\leq k\leq \lfloor \alpha N\rfloor} |w^N(t_k) - u(t_k)|,$$

with

$$u(t) = \frac{1}{T-t}, \quad 0 \leq t < T.$$

We have

$$\begin{aligned} |w^N(t_k) - u(t_k)| &= \frac{1}{\dfrac{T-t_k}{T(1-u_N(t_0)T)}e^{-\lambda t_k} + (T-t_k)e^{-\lambda t_k}\displaystyle\int_0^{t_k} \frac{e^{\lambda s}}{(T-s)^2}ds} \\ &\leq \frac{1}{\dfrac{T-t_k}{T(1-u_N(t_0)T)}e^{-\lambda t_k}} = \frac{T(1-u_N(t_0)T)e^{\lambda t_k}}{T-t_k} \\ &\leq \frac{T(1-u_N(t_0)T)e^{-\lambda t_{\lfloor \alpha N\rfloor}}}{T-t_{\lfloor \alpha N\rfloor}}. \end{aligned}$$

So

$$\max_{0\leq k\leq \lfloor \alpha N\rfloor} |w^N(t_k) - u(t_k)| \leq \frac{T(1-u_N(t_0)T)e^{-\lambda t_{\lfloor \alpha N\rfloor}}}{T-t_{\lfloor \alpha N\rfloor}} \underset{N\to+\infty}{\longrightarrow} 0,$$

because

$$u_N(t_0) \underset{N\to+\infty}{\longrightarrow} \frac{1}{T} \quad \text{and} \quad t_{\lfloor \alpha N\rfloor} = \lfloor \alpha N\rfloor \frac{T}{N} \underset{N\to+\infty}{\longrightarrow} \alpha T < T.$$

Finally we have shown

$$\max_{0\leq k\leq \lfloor \alpha N\rfloor} |u_N(t_k) - u(t_k)| \underset{N\to+\infty}{\longrightarrow} 0$$

which achieves the proof of Theorem 5. □

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