Malliavin calculcus with respect to a Hawkes process

Dorian Cacitti-Holland *[1], Laurent Denis †[1], and Alexandre Popier‡[1]

[1]Laboratoire Manceau de Mathématiques, Le Mans Université, Avenue O. Messiaen, 72085 Le Mans cedex 9, France.

February 17, 2025

Summary

*Dorian.Cacitti_Holland.Etu@univ-lemans.fr
†Laurent.Denis@univ-lemans.fr
‡Alexandre.Popier@univ-lemans.fr

1 Introduction

In this paper we aim to develop a local Malliavin calculus with respect to a Hawkes process. Malliavin calculus is a mathematical framework used to study the smoothness of random variables and functionals of stochastic processes, especially those driven by Brownian motion. The central concept is the Malliavin derivative, a type of derivative that extends the classical notion of differentiation to the space of random variables. This approach allows one to analyze the regularity and differentiability of random processes, providing powerful tools for studying stochastic differential equations (SDEs) and probabilistic systems. The key ideas of Malliavin calculus were introduced by Paul Malliavin in the 1970s. There is a huge literature on this subject, encompassing Malliavin calculus for Lévy processes (see among others [16, 2, 9, 4] and the references therein).

Hawkes processes have been introduced by Alan Hawkes in the 1970s as a class of self-exciting point processes. They are widely used to model events or occurrences where the occurrence of one event increases the likelihood of subsequent events in the near future. These processes have been used to model earthquakes, and for some time now, they have been experiencing a renewed interest due to their applications in finance and actuarial science [1, 7, 13].

A Hawkes process is characterized by an intensity function that depends on both a baseline intensity and a history of past events. More specifically, the intensity at time t is given by:

$$\lambda^*(t) = \lambda + \int_{(0,t]} \mu(t-s)dN_s$$

where λ is the baseline intensity, μ is a function that describes the impact of past events, and N is the counting process that represents the occurrences of events. The key feature of Hawkes processes is that the function μ is typically a non-negative function, which means that past events increase the probability of future events — hence the term "self-exciting."

Recently a Malliavin calculus with respect to the Hawkes process N is developed in [12] and [11]. Roughly speaking, the main ingredient consists to perturb the system by adding a particle (or a jump) (see [12, Lemma 3.5]) leading to a expansion formula for functionals of the Hawkes process [12, Theorem 3.13]). Let us mention that the derivative operator is not local. With these results, they are able to develop a Stein method (see [11]) and to compute some prices of financial or insurance derivatives (see [12, Section 4]).

Our method is different and follows the approach of Carlen-Pardoux [6]. We perturb the jump times and formally differentiate with respect to these jump times. This allows us to define a local derivative, satisfying the chain rule. We apply it to the study of absolute continuity for the law of Hawkes functionals and to the computation of Greeks.

Breakdown. The first step of our construction of a Malliavin derivative with respect to a Hawkes process is to define a directional derivative with respect to a function $m \in \mathcal{H}$, where $\mathcal{H}$ is the ad hoc Cameron-Martin space. An integration by parts formula is obtained thanks to the absolute continuity property of the law of the perturbed jump times w.r.t. the initial probability measure (see Proposition 2.7, Theorem 2.13, Proposition 2.15 and Corollary 2.16).

The second step is to define the Malliavin derivative D in all directions by considering a Hilbert basis of $\mathcal{H}$. We obtain a local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ which admits a carré du champ Γ and a gradient D (see Proposition 3.1). Therefore we get similar properties to the directional derivative as the chain rule. Moreover we are interested in the associated divergence operator δ, for which we get an explicit expression of $\delta(u)$ when u is predictable (see Proposition 3.4, Remark 3.5 and Corollary 3.7).

We then establish an absolute continuity criterion: conditionally to $\Gamma[F] = (\Gamma[F_i, F_j])_{1\leq i,j\leq d} \in GL_d(\mathbb{R})$, the random vector $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$ admits a absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^d$ (see Theorem 4.5 and Corollary 4.6).

This criterion is firstly applied to the solution of stochastic differential equation driven by the Hawkes process (see Theorem 5.9, Corollary 5.11 and Proposition 5.13). As a second application, we compute Greeks for a financial payoff when the underlying process is driven by the Hawkes process (see Proposition 5.17).

2 Framework and directional derivation

2.1 Setting and first notations

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the space of càdlàg trajectories

$$\omega(t) = \sum_i i 1_{[t_i, t_{i+1})},$$

with $0 < t_1 < \cdots < t_i < \cdots$.

We define

$$N_t(\omega) = \sum_{s\leq t} \Delta\omega_s, \quad t \geq 0,$$

the process which counts the jumps between 0 and t, where $\Delta w_s = w_s - w_{s-}$ and $w_{s-} = \lim_{u\to s^-} w_u$.

We assume that, under $\mathbb{P}$, $(N_t)_{t\in\mathbb{R}_+}$ is a Hawkes process with conditional intensity

$$\lambda^*(t) = \lambda + \int_{(0,t)} \mu(t-s) dN_s = \lambda + \sum_{i=1}^{+\infty} \mu(t-T_i) 1_{\{t>T_i\}} = \lambda + \sum_{i=1}^{N_t - 1} \mu(t-T_i).$$

Throughout this paper, we suppose that

Assumption 1.

- $\lambda \in (0, +\infty)$
- $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *differentiable with bounded derivative and such that*

$$\|\mu\|_1 = \int_0^{+\infty} \mu(t)dt < 1.$$

We introduce $(T_i)_{i\in\mathbb{N}^*}$ the jump instants of the Hawkes process N, and, for any $n \in \mathbb{N}^*$, $0 < t_1 < \cdots < t_n$ and $s \in \mathbb{R}_+^*$:

$$\lambda^*(s; t_1, \cdots, t_n) = \lambda + \sum_{i=1}^{n-1} \mu(s - t_i) 1_{\{s>t_i\}}.$$

Thus, for any $n \in \mathbb{N}^*$, on the event $\{N_t = n\}$,

$$\lambda^*(t) = \lambda^*(t; T_1, \cdots, T_n).$$

We consider the $\mathbb{P}$-complete right continuous filtration $(\mathcal{F}_t)_{0\le t\le T}$ generated by the Hawkes process N where $T \in \mathbb{R}_+^*$ is a fixed time horizon.

We apply the same approach as in [6] to define the directional derivative using the reparametrization of time with respect to a function in a Cameron-Martin space.

Let $L^2([0,T])$ be the usual space of square integrable function on $[0,T]$ with respect to the Lebesgue measure and $\mathcal{H}$ be the closed subspace of $L^2([0,T])$ orthogonal to the constant functions, i.e.,

$$\mathcal{H} = \left\{ m \in L^2([0,T]) \quad \int_0^T m(s)ds = 0 \right\}. \tag{1}$$

We denote $\widehat{m} = \int_0^{\cdot} m(s)ds$ for every $m \in \mathcal{H}$, then $\widehat{m}(0) = \widehat{m}(T) = 0$. In a natural way, $\mathcal{H}$ inherits the Hilbert structure of $L^2([0,T])$ and we denote by $\| \|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the norm and the scalar product on it. From now on in this section, we fix a function $m \in \mathcal{H}$. The condition $\int_0^T m(s)ds = 0$ ensures that the change of intensity that we are about to define simply shifts the jump times without affecting the total number of jumps. Let us define

$$\widetilde{m}_\varepsilon(s) = \begin{cases} -\frac{1}{3\varepsilon} & \text{if } m(s) \le -\frac{1}{3\varepsilon}; \\ m(s) & \text{if } \frac{-1}{3\varepsilon} \le m(s) \le \frac{1}{3\varepsilon}; \\ \frac{1}{3\varepsilon} & \text{if } m(s) \ge \frac{1}{3\varepsilon}. \end{cases}$$

and $m_\varepsilon \in \mathcal{H}$ such that

$$m_\varepsilon(s) = \widetilde{m}_\varepsilon(s) - \frac{1}{T}\int_0^T \widetilde{m}_\varepsilon(s)ds. \tag{2}$$

We remark that $\frac{1}{3} \le 1 + \varepsilon m_\varepsilon(s) \le \frac{5}{3}$ (since $-\frac{1}{3\varepsilon} \le \widetilde{m}_\varepsilon(s) \le \frac{1}{3\varepsilon}$).

Lemma 2.1. *We have the following convergence*

$$||m - m_\varepsilon||_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Proof. We have $||m - m_\varepsilon||_{\mathcal{H}} \leq ||m - \widetilde{m}_\varepsilon||_{\mathcal{H}} + ||\widetilde{m}_\varepsilon - m_\varepsilon||_{\mathcal{H}}$ with for the first term, for almost every $s \in [0, T]$,

$$m(s) - \widetilde{m}_\varepsilon(s) = \begin{cases} m(s) + \dfrac{1}{3\varepsilon} & \text{if} \quad m(s) \leq -\dfrac{1}{3\varepsilon} \\ 0 & \text{if} \quad -\dfrac{1}{3\varepsilon} \leq m(s) \leq \dfrac{1}{3\varepsilon} \\ m(s) - \dfrac{1}{3\varepsilon} & \text{if} \quad \dfrac{1}{3\varepsilon} \leq m(s). \end{cases} \tag{3}$$

Thus

$$|m(s) - \widetilde{m}_\varepsilon(s)| \leq \left(|m(s)| + \frac{1}{3\varepsilon} \right) 1_{\{|m(s)| \geq \frac{1}{3\varepsilon}\}} \leq 2|m(s)| 1_{\{|m(s)| \geq \frac{1}{3\varepsilon}\}}.$$

Therefore, by dominated convergence theorem,

$$||m - \widetilde{m}_\varepsilon||_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Indeed $|m(s)|^2 1_{\{|m(s)| \geq \frac{1}{3\varepsilon}\}} \underset{\varepsilon \to 0}{\longrightarrow} 0$ and $|m(s)|^2 1_{\{|m(s)| \geq \frac{1}{3\varepsilon}\}} \leq |m(s)|^2 \in L^1([0, T])$.
Now for the second term, as $\int_0^T m(s) ds = 0$ and by Cauchy-Schwarz inequality,

$$\begin{aligned} ||\widetilde{m}_\varepsilon - m_\varepsilon||_{\mathcal{H}} &= \frac{1}{T} \left| \int_0^T \widetilde{m}_\varepsilon(s) ds \right| = \frac{1}{T} \left| \int_0^T (\widetilde{m}_\varepsilon(s) - m(s)) ds \right| \\ &\leq \frac{1}{\sqrt{T}} ||\widetilde{m}_\varepsilon - m||_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0. \end{aligned}$$

□

We define the reparametrization of time with respect to m_ε as follow

$$\tau_\varepsilon(s) = s + \varepsilon \widehat{m}_\varepsilon(s) = \int_0^s (1 + \varepsilon m_\varepsilon(u)) du, \quad s \in \mathbb{R}_+.$$

Notice that $\tau_\varepsilon(0) = 0, \tau_\varepsilon(T) = T$, and since $1 + \varepsilon m_\varepsilon(s) \in \left[\frac{1}{3}, \frac{5}{3}\right] \subset \mathbb{R}_+^*$, τ_ε is an increasing function hence invertible so the number and the order of jump times between 0 and T remain unchanged. Moreover, a direct calculation gives

$$\forall s \in [0, T], \quad \tau_\varepsilon^{-1}(s) = \int_0^s \frac{1}{1 + \varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))} du.$$

Let $\mathcal{T}_\varepsilon : \Omega \to \Omega$ be the map defined by, for any $\omega \in \Omega$,

$$(\mathcal{T}_\varepsilon(\omega))(s) = \omega(\tau_\varepsilon(s)),$$

$$\mathcal{T}_\varepsilon F = F \circ \mathcal{T}_\varepsilon \text{ for all } F \in L^2(\Omega),$$

and $\mathbb{P}^\varepsilon$ be the probability measure $\mathbb{P}\mathcal{T}_\varepsilon^{-1}$ defined on $\mathcal{F}_T$.

2.2 Directional derivation

Definition 2.2. *We denote*

$$\mathbb{D}_m^0 = \left\{ F \in L^2(\Omega) : \frac{\partial \mathcal{T}_\varepsilon F}{\partial \varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon F - F) \text{ in } L^2(\Omega) \text{ exists} \right\}.$$

For $F \in \mathbb{D}_m^0$, $D_m F$ is defined as the limit

$$D_m F = \frac{\partial \mathcal{T}_\varepsilon F}{\partial \varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon F - F). \tag{4}$$

Definition 2.3. *Let define the set $\mathcal{S}$ of "smooth" functions. We say that a map $F : \Omega \to \mathbb{R}$ belongs to $\mathcal{S}$ if there exists $a \in \mathbb{R}, d \in \mathbb{N}^*$ and for any $n \in \{1, \cdots, d\}$, a function $f_n : \mathbb{R}^n \to \mathbb{R}$ such that:*

1. *The random variable F can be written*
$$F = a 1_{\{N_T = 0\}} + \sum_{n=1}^{d} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T = n\}}. \tag{5}$$
2. *For any $n \in \{1, \cdots, d\}$, the function f_n is smooth with bounded derivatives of any order.*

Remark 2.4. *The space $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.*

Here are some basic properties of directional derivatives on $\mathcal{S}$.

Lemma 2.5. *Let $j \in \mathbb{N}^*$ and $\overline{T}_j = T_j \wedge T$. Then $\overline{T}_j \in \mathbb{D}_m^0$ and*

$$D_m \overline{T}_j = -\widehat{m}(T_j).$$

Proof. We first remark that, for any $\omega \in \Omega$,

$$T_j(\omega \circ \tau_\varepsilon) \quad = \quad \tau_\varepsilon^{-1}(T_j(\omega)).$$

Then

$$\begin{aligned}
|\mathcal{T}_\varepsilon \overline{T}_j(\omega) - \overline{T_j}(\omega) + \varepsilon \widehat{m}(\overline{T_j})(\omega)| &= \left| (\overline{T}_j \circ \mathcal{T}_\varepsilon)(\omega) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt \right| \\
&= \left| \overline{T}_j(\omega \circ \tau_\varepsilon) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt \right| = \left| \tau_\varepsilon^{-1}(\overline{T}_j(\omega)) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt \right| \\
&= \left| \tau_\varepsilon^{-1}(\overline{T}_j(\omega)) - \tau_\varepsilon(\tau_\varepsilon^{-1}(\overline{T}_j(\omega))) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt \right| \\
&= \left| \tau_\varepsilon(s_\varepsilon) - s_\varepsilon - \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} m(t)dt \right|, \quad \text{with } s_\varepsilon = \tau_\varepsilon^{-1}(\overline{T}_j(\omega)) \\
&\leq \left| \tau_\varepsilon(s_\varepsilon) - s_\varepsilon - \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} m_\varepsilon(t)dt \right| + \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} |m_\varepsilon(t) - m(t)|dt \\
&\leq \varepsilon \left| \int_{s_\varepsilon}^{\tau_\varepsilon(s_\varepsilon)} m_\varepsilon(t)dt \right| + \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} |m_\varepsilon(t) - m(t)|dt \\
&\leq \varepsilon \sqrt{\tau_\varepsilon(s_\varepsilon) - s_\varepsilon} \sqrt{\int_0^T |m_\varepsilon(t)|^2 dt} + \varepsilon \int_0^T |m_\varepsilon(t) - m(t)|dt.
\end{aligned}$$

We have

$$
\begin{aligned}
|\tau_\varepsilon(s_\varepsilon) - s_\varepsilon| &= \overline{T}_j(\omega) - s_\varepsilon = \int_0^{\overline{T}_j(\omega)} \left(1 - \frac{1}{1+\varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))}\right) du \\
&\le \int_0^T \left(1 - \frac{1}{1+\varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))}\right) du \underset{\varepsilon\to 0}{\longrightarrow} 0.
\end{aligned}
$$

Moreover $\lim_{\varepsilon\to 0}\int_0^T |m_\varepsilon(t) - m(t)|dt = 0$ and $\int_0^T |m_\varepsilon(t)|^2 dt$ is bounded so that we get by a dominated convergence argument that $\overline{T}_j$ belongs to $\mathbb{D}_m^0$ and $D_m\overline{T}_j = -\widehat{m}(\overline{T}_j)$. □

Proposition 2.6. *Let $n \in \mathbb{N}^*$ and $f : \mathbb{R}^n \to \mathbb{R}$ a function of class C^1. Then $f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)$ belongs to $\mathbb{D}_m^0$ and*

$$
D_m f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) = -\sum_{j=1}^n \frac{\partial f}{\partial t_j}(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)\, \widehat{m}(\overline{T}_j).
$$

Thus $\mathcal{S} \subset \mathbb{D}_m^0$ and for any $F \in \mathcal{S}$ of the form (5)*,*

$$
D_m F = -\sum_{n=1}^d \sum_{j=1}^n \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j) 1_{\{N_T = n\}}.
$$

Proof. By the definition of D_m given by (4) we have

$$
\begin{aligned}
D_m f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) &= \frac{\partial \mathcal{T}_\varepsilon f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)}{\partial \varepsilon}|_{\varepsilon=0} \\
&= \frac{\partial}{\partial \varepsilon} f(\mathcal{T}_\varepsilon \overline{T}_1, \mathcal{T}_\varepsilon \overline{T}_2, \cdots, \mathcal{T}_\varepsilon \overline{T}_n)|_{\varepsilon=0} = \sum_{j=1}^n \frac{\partial f}{\partial t_j} \frac{\partial}{\partial \varepsilon} \mathcal{T}_\varepsilon \overline{T}_j|_{\varepsilon=0} \\
&= \sum_{j=1}^n \frac{\partial f}{\partial t_j} D_m \overline{T}_j = -\sum_{j=1}^n \frac{\partial f}{\partial t_j}(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)\widehat{m}(\overline{T}_j)
\end{aligned}
$$

where the last equality is due to Lemma 2.5. We deduce the last assertion by linearity using the fact that $N_T \circ \mathcal{T}_\varepsilon = N_T$. □

Proposition 2.7.

1. *If $F, G \in \mathcal{S}$ then $FG \in \mathcal{S}$ and $D_m(FG) = (D_m F)G + F(D_m G)$.*
2. *We have the chain rule: If $F_1, F_2, \cdots, F_n \in \mathcal{S}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function then*

$$
\Phi(F_1, F_2, \cdots, F_n) \in \mathcal{S}
$$

and

$$
D_m \Phi(F_1, F_2, \cdots, F_n) = \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(F_1, F_2, \cdots, F_n) D_m F_j.
$$

Proof.

1. We assume that $F, G \in \mathcal{S}$. Then

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}$$

and

$$G = b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1, \cdots, T_n)1_{\{N_T=n\}}.$$

So

$$FG = ab1_{\{N_T=0\}} + \sum_{n=1}^{d} (f_n \times g_n)(T_1, \cdots, T_n)1_{\{N_T=n\}}.$$

Thus $FG \in \mathcal{S}$ and

$$\begin{aligned}
D_m(FG) &= abD_m1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m[(f_n \times g_n)(T_1, \cdots, T_n)1_{\{T=n\}}] \\
&= 0 - \sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial (f_n \times g_n)}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&= -\sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)g_n(T_1, \cdots, T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&\quad -\sum_{n=1}^{d}\sum_{j=1}^{n} f_n(T_1, \cdots, T_n)\frac{\partial g_n}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&= \sum_{n=1}^{d}\left(-\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j)\right) g_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \\
&\quad +\sum_{n=1}^{d}\left(-\sum_{j=1}^{n} \frac{\partial g_n}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j)\right) f_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \\
&= \sum_{n=1}^{d} D_m f_n(T_1, \cdots, T_n)g_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \\
&\quad +\sum_{n=1}^{d} D_m g_n(T_1, \cdots, T_n)f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}.
\end{aligned}$$

Moreover

$$\begin{aligned}
(D_mF)G &= \left(aD_m1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m[f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}]\right) \\
&\quad \times \left(b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1, \cdots, T_n)1_{\{N_T=n\}}\right) \\
&= 0 + \sum_{n=1}^{d} D_m f_n(T_1, \cdots, T_n)g_n(T_1, \cdots, T_n)1_{\{N_T=n\}}
\end{aligned}$$

and

$$F(D_mG) = \sum_{n=1}^{d} D_m g_n(T_1, \cdots, T_n) f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}.$$

Thus we obtain

$$D_m(FG) = (D_mF)G + F(D_mG).$$

2. We assume that $F, \cdots, G \in \mathcal{S}$ and $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ smooth. Then

$$\begin{aligned}
\Phi(F,G) &= \Phi\left(a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}, b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1,\cdots,T_n)\right) \\
&= \Phi(a,b)1_{\{N_T=0\}} + \sum_{n=1}^{d} \Phi(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n))1_{\{N_T=n\}} \\
&= \Phi(a,b)1_{\{N_T=0\}} + \sum_{n=1}^{d} (\Phi(f_n,g_n))(T_1,\cdots,T_n))1_{\{N_T=n\}}.
\end{aligned}$$

Thus $\Phi(F,G) \in \mathcal{S}$ and, according to the Proposition 2.6,

$$\begin{aligned}
& D_m(\Phi(f_n,g_n))(T_1,\cdots,T_n) \\
&= -\sum_{j=1}^{n} \frac{\partial \Phi(f_n,g_n)}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j) \\
&= -\sum_{j=1}^{n} \left(\frac{\partial \Phi}{\partial x}(f_n,g_n)\frac{\partial f_n}{\partial t_j} + \frac{\partial \Phi}{\partial y}(f_n,g_n)\frac{\partial g_n}{\partial t_j}\right)(T_1,\cdots,T_n)\widehat{m}(T_j) \\
&= \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)\left(-\sum_{j=1}^{n}\frac{\partial f_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right) \\
&\quad + \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)\left(-\sum_{j=1}^{n}\frac{\partial g_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right) \\
&= \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)D_m f_n(T_1,\cdots,T_n) \\
&\quad + \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)D_m g_n(T_1,\cdots,T_n).
\end{aligned}$$

Then

$$\begin{aligned}
D_m(\Phi(F,G)) &= \Phi(a,b)D_m 1_{\{N_T=0\}} \\
&\quad + \sum_{n=1}^{d} \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}} \\
&\quad + \sum_{n=1}^{d} \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)D_m g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\frac{\partial \Phi}{\partial x}(F,G)D_mF \\
&= \frac{\partial \Phi}{\partial x}\left(a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}, b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\right) \\
&\quad \times \left(aD_m 1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\right) \\
&= \sum_{n=1}^{d} \frac{\partial \Phi}{\partial x}(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n)) D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}
\end{aligned}$$

and

$$\frac{\partial \Phi}{\partial y}(F,G)D_mG = \sum_{n=1}^{d} \frac{\partial \Phi}{\partial y}(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n)) D_m g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.$$

Thus

$$D_m(\Phi(F,G)) = \frac{\partial \Phi}{\partial x}(F,G)D_mF + \frac{\partial \Phi}{\partial y}(F,G)D_mG.$$

The case with more than two random variables is left to the reader.

□

2.3 Absolute continuity of $\mathbb{P}^\varepsilon$ w.r.t. $\mathbb{P}$

Evoke that $\mathbb{P}^\varepsilon$ is defined at the end of Section 2.1. Let $\mathbb{E}^\varepsilon$ be the expectation under the probability $\mathbb{P}^\varepsilon$.

Let $k \in \mathbb{N}^*$ and $0 < t_1 < t_2 < \cdots < t_k$, knowing $T_1 = t_1, \cdots, T_k = t_k$ the process $(N_t - N_{t_k})_{t \geq t_k}$ is an inhomogeneous Poisson process with intensity $\lambda^*(s; t_1, \cdots, t_k)$. We deduce the following conditional link for $t \geq t_k$:

$$\begin{aligned}
\mathbb{P}(T_{k+1} > t | T_1 = t_1, \cdots, T_k = t_k) &= \mathbb{P}(N_t - N_{t_k} = 0 | T_1 = t_1, \cdots, T_k = t_k) \\
&= e^{-\int_{t_k}^{t} \lambda^*(s; t_1, \cdots, t_k)\, ds}.
\end{aligned}$$

Thus the density of T_{k+1} knowing $T_1 = t_1, \cdots, T_k = t_k$ is

$$t \longmapsto \lambda^*(t; t_1, \cdots, t_k) e^{-\int_{t_k}^{t} \lambda^*(s; t_1, \cdots, t_k)\, ds} 1_{\{t > t_k\}}.$$

We deduce that $(T_1, \cdots, T_k, T_{k+1})$ admits for density

$$(t_1, \cdots, t_{k+1}) \longmapsto \left(\prod_{i=1}^{k+1} \lambda^*(t_i; t_1, \cdots, t_k)\right) e^{-\int_0^{t_{k+1}} \lambda^*(s; t_1, \cdots, t_k)\, ds} 1_{\{0 < t_1 < \cdots < t_{k+1}\}},$$

where we used the fact that $\lambda^*(s; t_1, \cdots, t_k) = \lambda^*(s; t_1, \cdots, t_{i-1})$ if $s \leq t_i$.

Let $n \in \mathbb{N}^*$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}^\varepsilon[f(T_1, \cdots, T_n)1_{\{N_T=n\}}] \\
&= \mathbb{E}[(f \circ \Phi_\varepsilon^{-1})(T_1, \cdots, T_n)1_{\{T_n \le T < T_{n+1}\}}] \\
&= \iint_{0<t_1<\cdots<t_n\le T<t_{n+1}} (f \circ \Phi_\varepsilon^{-1})(t_1, \cdots, t_n) \left(\prod_{i=1}^{n+1} \lambda^*(t_i; t_1, \cdots, t_n) \right) \\
&\quad \times e^{-\int_0^{t_{n+1}} \lambda^*(s;t_1,\cdots,t_n)\, ds} dt_1 \cdots dt_{n+1} \\
&= \iint_{0<t_1<\cdots<t_n\le T} (f \circ \Phi_\varepsilon^{-1})(t_1, \cdots, t_n) \prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) dt_1 \cdots dt_n \\
&\quad \times \int_T^\infty \lambda^*(t_{n+1}; t_1, \cdots, t_n) e^{-\int_0^{t_{n+1}} \lambda^*(s;t_1,\cdots,t_n)\, ds} dt_{n+1} \\
&= \iint_{0<t_1<\cdots<t_n\le T} (f \circ \Phi_\varepsilon^{-1})(t_1, \cdots, t_n) \varphi_n(t_1, \cdots, t_n) dt_1 \cdots dt_n \\
&= \iint_{0<u_1<\cdots<u_n\le T} f(u_1, \cdots, u_n)(\varphi_n \circ \Phi_\varepsilon)(u_1, \cdots, u_n) |\det J_{\Phi_\varepsilon}| du_1 \cdots du_n \\
&= \iint_{0<u_1<\cdots<u_n\le T} f(u_1, \cdots, u_n)(\varphi_n \circ \Phi_\varepsilon)(u_1, \cdots, u_n) \\
&\quad \times \prod_{i=1}^{n} (1 + \varepsilon m_\varepsilon(u_i)) du_1 \cdots du_n \\
&= \mathbb{E}\left[f(T_1, \cdots, T_n)1_{\{N_T=n\}} Z_n^\varepsilon\right],
\end{aligned}$$

where

$$\Phi_\varepsilon(u_1, \cdots, u_n) = (u_1 + \varepsilon \widehat{m}_\varepsilon(u_1), \cdots, u_n + \varepsilon \widehat{m}_\varepsilon(u_n)), \tag{6}$$

$$\varphi_n(t_1, \cdots, t_n) = \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^T \lambda^*(s;t_1,\cdots,t_n) ds}, \tag{7}$$

$\det J_{\Phi_\varepsilon}$ denotes the determinant of the Jacobian matrix of Φ_ε and

$$Z_n^\varepsilon = \frac{(\varphi_n \circ \Phi_\varepsilon)(T_1, \cdots, T_n)}{\varphi_n(T_1, \cdots, T_n)} \prod_{i=1}^{n} (1 + \varepsilon m_\varepsilon(T_i)). \tag{8}$$

Let us emphasize that Assumption 1 is used to ensure that: $\lambda^*(s; t_1, \cdots, t_n) \ge \lambda > 0$. This yields:

Proposition 2.8. *$\mathbb{P}^\varepsilon$ is absolutely continuous with respect to $\mathbb{P}$ with density*

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} = \sum_{n=0}^{+\infty} Z_n^\varepsilon 1_{\{N_T=n\}} := G^\varepsilon.$$

Remark 2.9. *This series converges in L^1 uniformly in ε because for $f = 1$*

$$\mathbb{E}[G^\varepsilon] = \sum_{n=0}^{+\infty} \mathbb{E}[Z_n^\varepsilon 1_{\{N_T=n\}}] = \sum_{n=0}^{+\infty} \mathbb{E}^\varepsilon[1_{\{N_T=n\}}] = 1.$$

Remark 2.10. *In case of the Poisson process, φ_n is constant, equal to λ^n and we have again the result obtained in [6] for standard Poisson processes*

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} = \prod_{i=1}^{N_T}(1+\varepsilon m_\varepsilon(T_i)).$$

2.4 Limit behavior of the density G^ε when $\varepsilon \to 0$

We begin with a first result about the limits when ε tends to 0.

Proposition 2.11. *For any $n \in \mathbb{N}^*$, a.s.*

$$\lim_{\varepsilon\to 0} Z_n^\varepsilon = \lim_{\varepsilon\to 0} G^\varepsilon = 1.$$

Proof. Firstly we have the almost surely convergences

$$\forall i \in \{1,\cdots,n\}, \quad \varepsilon m_\varepsilon(T_i) \xrightarrow[\varepsilon\to 0]{a.s.} 0.$$

Indeed we use (2) and (3) to get

$$m(s) - m_\varepsilon(s) \quad = \quad m(s) - \widetilde{m}_\varepsilon(s) + \frac{1}{T}\int_0^T (m(r) - \widetilde{m}_\varepsilon(r))dr.$$

As $T_i < +\infty$ a.s., for $\varepsilon \in \mathbb{R}_+^*$ small enough,

$$\begin{aligned} |m(T_i) - m_\varepsilon(T_i)| &= \left|\frac{1}{T}\int_0^T (m(r) - \widetilde{m}_\varepsilon(r))dr\right| \le \frac{1}{T}\int_0^T |m(r) - \widetilde{m}_\varepsilon(r)|dr \\ &\le \frac{1}{\sqrt{T}}\|m - \widetilde{m}_\varepsilon\|_{\mathcal{H}} \xrightarrow[\varepsilon\to 0]{} 0. \end{aligned}$$

See Lemma 2.1. Thus

$$\varepsilon m_\varepsilon(T_i) = \varepsilon m(T_i) + \varepsilon(m_\varepsilon(T_i) - m(T_i)) \xrightarrow[\varepsilon\to 0]{a.s.} 0.$$

So we have the convergence of the product in (8)

$$\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i)) \xrightarrow[\varepsilon\to 0]{a.s.} 1.$$

Secondly for the convergence of (6), for a.e. $s \in [0,T]$,

$$|\widehat{m}(s) - \widehat{m}_\varepsilon(s)| \quad \le \quad \int_0^T |m(r) - m_\varepsilon(r)|dr \le \sqrt{T}\|m - m_\varepsilon\|_{\mathcal{H}} \xrightarrow[\varepsilon\to 0]{} 0.$$

In particular we have the uniform convergence on $[0,T]$ of $\widehat{m}_\varepsilon$ to $\widehat{m}$. Then, for any $i \in \{1,\cdots,n\}$,

$$\varepsilon\widehat{m}_\varepsilon(T_i) = \varepsilon\widehat{m}(T_i) + \varepsilon(\widehat{m}_\varepsilon(T_i) - \widehat{m}(T_i)) \xrightarrow[\varepsilon\to 0]{a.s.} 0.$$

Moreover

$$
\begin{aligned}
\varphi_n(\Phi_\varepsilon(T_1,\cdots,T_n)) &= \varphi_n(T_1+\varepsilon\widehat{m}_\varepsilon(T_1),\cdots,T_n+\varepsilon\widehat{m}_\varepsilon(T_n)) \\
&= \prod_{j=1}^{n}\lambda^*(T_j+\varepsilon\widehat{m}_\varepsilon(T_j);T_1+\varepsilon\widehat{m}_\varepsilon(T_1),\cdots,T_n+\varepsilon\widehat{m}_\varepsilon(T_n)) \\
&\quad\times\exp\left(-\int_0^T\lambda^*(s;T_1+\varepsilon\widehat{m}_\varepsilon(T_1),\cdots,T_n+\varepsilon\widehat{m}_\varepsilon(T_n))ds\right) \\
&= \prod_{j=1}^{n}\left(\lambda+\sum_{i=1}^{j-1}\mu(T_j-T_i+\varepsilon(\widehat{m}_\varepsilon(T_j)-\widehat{m}_\varepsilon(T_i)))\right) \\
&\quad\times\exp\left(-\lambda T-\sum_{i=1}^{n-1}\int_0^{T-T_i-\varepsilon\widehat{m}_\varepsilon(T_i)}\mu(u)du\right).
\end{aligned}
$$

Thus, as the function μ is continuous,

$$\varphi_n(\Phi_\varepsilon(T_1,\cdots,T_n))\xrightarrow[\varepsilon\to 0]{a.s.}\varphi_n(T_1,\cdots,T_n).$$

Therefore the a.s. convergence of Z_n^ε is proved. We have the same convergence for G^ε by dominated convergence theorem because

$$\sum_{n=1}^{+\infty}\mathbb{E}[Z_n^\varepsilon 1_{\{N_T=n\}}]=\sum_{n=1}^{+\infty}\mathbb{E}[1_{\{N_T=n\}}]=1<+\infty.$$

This achieves the proof. □

2.5 Integration by parts in Bismut's way

Proposition 2.12. *Under our setting*

$$\frac{\partial G^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0} = \int_{(0,T]}(\psi(m,s)+\widehat{m}(s)\mu(T-s)+m(s))dN_s$$

where

$$
\begin{aligned}
\psi(m,s) &= \frac{1}{\lambda^*(s)}\int_{(0,s)}(\widehat{m}(s)-\widehat{m}(t))\mu'(s-t)dN_t \qquad (9)\\
&= \frac{\sum_{i=1}^{N_s-1}(\widehat{m}(s)-\widehat{m}(T_i))\mu'(s-T_i)}{\lambda+\sum_{i=1}^{N_s-1}\mu(T_i-s)}.
\end{aligned}
$$

Proof. Let $n\in\mathbb{N}^*$ and let's work on the event $\{N_T=n\}$. From (8), we have

$$Z_n^\varepsilon = \frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\ldots,T_n)}{\varphi_n(T_1,\ldots,T_n)}\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i)).$$

Thus, for any $\varepsilon\in\mathbb{R}_+^*$, according to Proposition 2.11, Z_n^ε a.s. converges to $Z_n^0=1$ and

$$
\begin{aligned}
\frac{Z_n^\varepsilon-1}{\varepsilon} &= \frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\ldots,T_n)-\varphi_n(T_1,\ldots,T_n)}{\varepsilon\varphi_n(T_1,\ldots,T_n)}\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i)) \\
&\quad+\frac{1}{\varepsilon}\left(\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))-1\right). \qquad (10)
\end{aligned}
$$

For the first term

$$\begin{aligned}
&\frac{(\varphi_n \circ \Phi_\varepsilon)(T_1, \dots, T_n) - \varphi_n(T_1, \dots, T_n)}{\varepsilon \varphi_n(T_1, \dots, T_n)} \\
= \quad & \frac{\varphi_n(T_1 + \varepsilon \widehat{m}_\varepsilon(T_1), \dots, T_n + \varepsilon \widehat{m}_\varepsilon(T_n)) - \varphi_n(T_1, \dots, T_n)}{\varepsilon \varphi_n(T_1, \dots, T_n)} \\
= \quad & \sum_{i=1}^n \int_0^1 \frac{\partial \varphi_n}{\partial t_i}(T_1 + \alpha\varepsilon \widehat{m}_\varepsilon(T_1), \dots, T_n + \alpha\varepsilon \widehat{m}_\varepsilon(T_n)) \frac{\varepsilon \widehat{m}_\varepsilon(T_i)}{\varepsilon \varphi_n(T_1, \cdots, T_n)} d\alpha \\
= \quad & \sum_{i=1}^n \frac{\widehat{m}_\varepsilon(T_i)}{\varphi_n(T_1, \cdots, T_n)} \int_0^1 \frac{\partial \varphi_n}{\partial t_i}(T_1 + \alpha\varepsilon \widehat{m}_\varepsilon(T_1), \dots, T_n + \alpha\varepsilon \widehat{m}_\varepsilon(T_n)) d\alpha \\
\xrightarrow[\varepsilon \to 0]{a.s.} \quad & \sum_{i=1}^n \frac{\widehat{m}(T_i)}{\varphi_n(T_1, \dots, T_n)} \frac{\partial \varphi_n}{\partial t_i}(T_1, \dots, T_n)
\end{aligned}$$

where the almost surely convergence is justified by, for any $i \in \{1, ..., n\}, \widehat{m}_\varepsilon(T_i) \xrightarrow[\varepsilon \to 0]{a.s.} \widehat{m}(T_i)$ and $\dfrac{\partial \varphi_n}{\partial t_i}$ bounded because μ admits a bounded derivative. And, as in the proof of Proposition 2.11,

$$\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) \xrightarrow[\varepsilon \to 0]{a.s.} 1.$$

For the second term

$$\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1 \quad = \quad \varepsilon \sum_{i=1}^n m_\varepsilon(T_i) + \varepsilon^2 \sum_{1 \le i < j \le n} m_\varepsilon(T_i) m_\varepsilon(T_j) + \dots + \varepsilon^n \prod_{i=1}^n m_\varepsilon(T_i).$$

Thus

$$\frac{\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1}{\varepsilon} \quad = \quad \sum_{i=1}^n m_\varepsilon(T_i) + \varepsilon \sum_{1 \le i < j \le n} m_\varepsilon(T_i) m_\varepsilon(T_j) + \dots + \varepsilon^{n-1} \prod_{i=1}^n m_\varepsilon(T_i).$$

with, for any $i \in \{1, \dots, n\}, m_\varepsilon(T_i) \xrightarrow[\varepsilon \to 0]{a.s.} m(T_i)$. Thus

$$\frac{\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1}{\varepsilon} \xrightarrow[\varepsilon \to 0]{a.s.} \sum_{i=1}^n m(T_i).$$

Therefore

$$\frac{Z_n^\varepsilon - Z_n^0}{\varepsilon} \xrightarrow[\varepsilon \to 0]{a.s.} \sum_{i=1}^n \frac{\widehat{m}(T_i)}{\varphi_n(T_1, \dots, T_n)} \frac{\partial \varphi_n}{\partial t_i}(T_1, \dots, T_n) + \sum_{i=1}^n m(T_i)$$

with, for any $i_0 \in \{1, \ldots, n\}$ and $0 < t_1 < \cdots < t_n \leq T$,

$$
\begin{aligned}
& \frac{1}{\varphi_n(t_1, \ldots, t_n)} \frac{\partial \varphi_n}{\partial t_{i_0}}(t_1, \ldots, t_n) \\
= \; & \frac{\partial}{\partial t_{i_0}} \ln(\varphi_n(t_1, \ldots, t_n)) \\
= \; & \sum_{i=1}^{n} \frac{\partial}{\partial t_{i_0}} \ln(\lambda^*(t_i; t_1, \ldots, t_n)) - \frac{\partial}{\partial t_{i_0}} \left(\int_0^T \lambda^*(s; t_1, \ldots, t_n) ds \right) \\
= \; & \frac{1}{\lambda^*(t_{i_0}; t_1, \ldots t_{i_0-1})} \frac{\partial \lambda^*}{\partial t_{i_0}}(t_{i_0}; t_1, \ldots, t_{i_0-1}) + \sum_{i=i_0+1}^{n} \frac{1}{\lambda^*(t_i; t_1, \ldots t_{i-1})} \frac{\partial \lambda^*}{\partial t_{i_0}}(t_i; t_1, \ldots, t_{i-1}) \\
& - \frac{\partial}{\partial t_{i_0}} \left(\int_0^T \lambda^*(s; t_1, \ldots, t_n) ds \right).
\end{aligned}
$$

Evoke that for any $i \in \{i_0 + 1, ..., n\}$ and $s \in [0, T]$,

$$
\begin{aligned}
\lambda^*(t_{i_0}; t_1, \ldots, t_{i_0-1}) &= \lambda + \sum_{j=1}^{i_0-1} \mu(t_{i_0} - t_j), \\
\lambda^*(t_i; t_1, \ldots, t_{i-1}) &= \lambda + \sum_{j=1}^{i-1} \mu(t_i - t_j), \\
\lambda^*(s; t_1, \cdots, t_n) &= \lambda + \sum_{j=1}^{n} \mu(s - t_j) 1_{\{s > t_j\}}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\partial \lambda^*}{\partial t_{i_0}}(t_{i_0}; t_1, \ldots, t_{i_0-1}) &= \sum_{j=1}^{i_0-1} \mu'(t_{i_0} - t_j), \\
\frac{\partial \lambda^*}{\partial t_{i_0}}(t_i; t_1, \ldots, t_{i-1}) &= -\mu'(t_i - t_{i_0}), \\
\int_0^T \lambda^*(s; t_1, \ldots, t_n) ds &= \lambda T + \sum_{j=1}^{n} \int_{t_j}^{T} \mu(s - t_j) ds \\
&= \lambda T + \sum_{j=1}^{n} \int_0^{T - t_j} \mu(s) ds, \\
\frac{\partial}{\partial t_{i_0}} \int_0^T \lambda^*(s; t_1, \ldots, t_n) ds &= -\mu(T - t_{i_0}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \frac{1}{\varphi_n(t_1, \ldots, t_n)} \frac{\partial \varphi_n}{\partial t_{i_0}}(t_1, \ldots, t_n) \\
= \; & \frac{\sum_{j=1}^{i_0-1} \mu'(t_{i_0} - t_j)}{\lambda^*(t_{i_0}; t_1, \ldots, t_n)} - \sum_{i=i_0+1}^{n} \frac{\mu'(t_i - t_{i_0})}{\lambda^*(t_i; t_1, \ldots, t_n)} + \mu(T - t_{i_0}).
\end{aligned}
$$

Then

$$\begin{aligned}
&\sum_{i_0=1}^{n} \frac{\widehat{m}(T_{i_0})}{\varphi_n(T_1,\ldots,T_n)} \frac{\partial \varphi_n}{\partial t_{i_0}}(T_1,\ldots,T_n) \\
= \;& \sum_{i_0=1}^{n} \widehat{m}(T_{i_0}) \frac{\sum_{j=1}^{i_0-1} \mu'(T_{i_0}-T_j)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} - \sum_{i_0=1}^{n} \widehat{m}(T_{i_0}) \sum_{i=i_0+1}^{n} \frac{\mu'(T_i - T_{i_0})}{\lambda^*(T_i;T_1,\ldots,T_n)} \\
&+ \sum_{i_0=1}^{n} \widehat{m}(T_{i_0})\mu(T-t_{i_0}) \\
= \;& \sum_{i_0=1}^{n} \sum_{j=1}^{i_0-1} \widehat{m}(T_{i_0}) \frac{\mu'(T_{i_0}-T_j)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} - \sum_{i_0=1}^{n} \sum_{i=1}^{i_0-1} \widehat{m}(T_i) \frac{\mu'(T_{i_0}-T_i)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} \\
&+ \sum_{i_0=1}^{n} \widehat{m}(T_{i_0})\mu(T-T_{i_0}) \\
= \;& \sum_{i_0=1}^{n} \left(\frac{1}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} \sum_{j=1}^{i_0-1} (\widehat{m}(T_{i_0}) - \widehat{m}(T_i))\mu'(T_{i_0}-T_i)) + \widehat{m}(T_{i_0})\mu(T-T_{i_0}) \right).
\end{aligned}$$

Finally

$$\frac{Z_n^\varepsilon - Z_n^0}{\varepsilon} \xrightarrow[\varepsilon\to 0]{a.s.} \sum_{i_0=1}^{n} \left(\frac{1}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} \sum_{j=1}^{i_0-1} (\widehat{m}(T_{i_0}) - \widehat{m}(T_i))\mu'(T_{i_0}-T_i) + \widehat{m}(T_{i_0})\mu(T-T_{i_0}) + m(T_{i_0}) \right).$$

Then we would like the same result for $\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$. We have, according to Proposition 2.11,

$$\begin{aligned}
\frac{\partial G^\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} 1_{\{N_T=n\}} &= \lim_{\varepsilon\to 0} \frac{G^\varepsilon - 1}{\varepsilon} 1_{\{N_T=n\}} = \lim_{\varepsilon\to 0} \frac{G^\varepsilon 1_{\{N_T=n\}} - 1_{\{N_T=n\}}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{Z_n^\varepsilon 1_{\{N_T=n\}} - 1_{\{N_T=n\}}}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{Z_n^\varepsilon - 1}{\varepsilon} 1_{\{N_T=n\}} \\
&= \frac{\partial Z_n^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} &= \sum_{n=1}^{+\infty} \frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}} \\
&= \sum_{n=1}^{+\infty} \frac{\partial Z_n^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}}. \\
&= \sum_{i_0=1}^{N_T} \left(\frac{1}{\lambda^*(T_{i_0};T_1,\ldots,T_{N_T})} \sum_{j=1}^{i_0-1} (\widehat{m}(T_{i_0}) - \widehat{m}(T_j))\mu'(T_{i_0}-T_j) + \widehat{m}(T_{i_0})\mu(T-T_{i_0}) + m(T_{i_0}) \right) \\
&= \int_{(0,T]} (\psi(m,s)dN_s + \widehat{m}(s)\mu(T-s) + m(s))dN_s
\end{aligned}$$

where ψ is given by (9). □

Theorem 2.13. *For any $F \in \mathcal{S}$,*

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} F\right].$$

Proof. We consider $F = f_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \in \mathcal{S}$. Then, as the vectors $(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n))$ and $(T_1, \cdots, T_n)$ are in the compact set $[0,T]^n$ and the function f_n is smooth, there exists a constant $C = C(f_n, n, T) \in \mathbb{R}_+^*$ such that

$$\begin{aligned}
\left|\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right| &= \frac{|f_n(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n)) - f_n(T_1, \cdots, T_n)|}{\varepsilon} 1_{\{N_T=n\}} \\
&\le C\frac{\|(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n)) - (T_1, \cdots, T_n)\|}{\varepsilon} 1_{\{N_T=n\}} \\
&= C\frac{\|(\tau_\varepsilon^{-1}(T_1) - T_1, \cdots, \tau_\varepsilon^{-1}(T_n) - T_n)\|}{\varepsilon} 1_{\{N_T=n\}} \\
&= C \sup_{1\le i\le n} \frac{|\tau_\varepsilon^{-1}(T_i) - T_i|}{\varepsilon} 1_{\{N_T=n\}} \\
&\overset{(*)}{\le} C \int_0^T |m_\varepsilon(s)| ds 1_{\{N_T=n\}} \\
&\le C \int_0^T |m_\varepsilon(s) - m(s)| ds 1_{\{N_T=n\}} + C \int_0^T |m(s)| ds \mathbf{1}_{\{N_T=n\}} \\
&\le C^2 \mathbf{1}_{\{N_T=n\}} + C \int_0^T |m(s)| ds 1_{\{N_T=n\}}
\end{aligned}$$

where the last inequality is true for ε small enough because $m_\varepsilon \xrightarrow[\varepsilon\to 0]{\mathcal{H}} m$ according to Lemma 2.1, and the inequality $(*)$ is due to the equality

$$T_i = \tau_\varepsilon(\tau_\varepsilon^{-1}(T_i)) = \tau_\varepsilon^{-1}(T_i) + \varepsilon \int_0^{\tau_\varepsilon^{-1}(T_i)} m_\varepsilon(s) ds.$$

Thus, by dominated convergence theorem and $\mathbb{E}[\mathcal{T}_\varepsilon F] = \mathbb{E}^\varepsilon[F] = \mathbb{E}[G^\varepsilon F]$,

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\lim_{\varepsilon\to 0} \frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} F\right].$$

Hence, as $F = f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}$,

$$\begin{aligned}
\mathbb{E}[D_m F] &= \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}\right] \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{G^\varepsilon \mathbf{1}_{\{N_T=n\}} - 1}{\varepsilon} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}\right] \\
&= \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{Z_n^\varepsilon - 1}{\varepsilon} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}\right] = \lim_{\varepsilon\to 0} \mathbb{E}\left[\frac{Z_n^\varepsilon - 1}{\varepsilon} F\right].
\end{aligned}$$

Let us come back to the definition (10) of the growth rate of Z_n^ε. For the last term in (10), we have

$$\frac{1}{\varepsilon}\left[\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1\right] \xrightarrow[\varepsilon\to 0]{a.s.} \sum_{i=1}^n m(T_i)$$

and

$$\left|\frac{1}{\varepsilon}\left[\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))-1\right]\right| \leq \sum_{i=1}^{n}|m_\varepsilon(T_i)|+\varepsilon\sum_{1\leq i<j\leq n}|m_\varepsilon(T_i)m_\varepsilon(T_j)|+\cdots+\varepsilon^{n-1}\prod_{i=1}^{n}|m_\varepsilon(T_i)|.$$

We know that $(T_1,\cdots,T_{n+1})$ admits for density

$$(t_1,\cdots,t_{n+1})\longmapsto\left(\prod_{j=1}^{n+1}\lambda^*(t_j;t_1,\cdots,t_n)\right)e^{-\int_0^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)\,ds}1_{\{0<t_1<\cdots<t_{n+1}\}}.$$

Thus, for any Borel function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}[f(T_1,\cdots,T_n)\mid N_T=n]=\frac{\mathbb{E}[f(T_1,\cdots,T_n)1_{\{T_n\leq T<T_{n+1}\}}]}{\mathbb{P}(N_T=n)}\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\quad\times\left(\int_{t_n}^{+\infty}\lambda^*(t_{n+1};t_1,\cdots,t_n)e^{-\int_{t_n}^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)ds}dt_{n+1}\right)1_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\quad\times\left[-e^{-\int_{t_n}^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)ds}\right]_{t_n}^{+\infty}1_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\quad\times\left(1-e^{-\int_{t_n}^{+\infty}\lambda^*(s;t_1,\cdots,t_n)ds}\right)1_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n.
\end{aligned}$$

Therefore $(T_1,\cdots,T_n)$ knowing $\{N_T=n\}$ admits for density

$$\begin{aligned}
(t_1,\cdots,t_n)\quad\longmapsto\quad&\frac{1}{\mathbb{P}(N_T=n)}\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})\,ds}\\
&\times\left(1-e^{-\int_{t_n}^{+\infty}\lambda^*(s;t_1,\cdots,t_n)ds}\right)1_{\{0<t_1<\cdots<t_n<T\}}
\end{aligned}$$

with, for any $0<t_1<\cdots<t_n<T$ and any $j\in\{1,\cdots,n\}$,

$$\lambda^*(t_j;t_1,\cdots,t_{n-1}) \;=\; \lambda+\sum_{i=1}^{j-1}\mu(t_j-t_i)\leq\lambda+n\|\mu\|_\infty.$$

Thus the density of $(T_1,\cdots,T_n)$ knowing $\{N_T=n\}$ is bounded by

$$(t_1,\cdots,t_n)\longmapsto\frac{1}{\mathbb{P}(N_T=n)}\,(\lambda+n\|\mu\|_\infty)^n\,1_{\{0<t_1<\cdots<t_n<T\}}.$$

Therefore, for any $I \subset \{1, ..., n\}$,

$$\begin{aligned}
\mathbb{E}\left[\left|\prod_{i\in I} m_\varepsilon(T_i)\right|\right] &\leq \frac{1}{\mathbb{P}(N_T = n)} (\lambda + n\|\mu\|_\infty)^n \\
&\quad \times \int_{[0,T]^n} \left|\prod_{i\in I} m_\varepsilon(t_i)\right| 1_{\{0<t_1<\cdots<t_n<T\}} dt \\
&\leq \frac{1}{\mathbb{P}(N_T = n)} (\lambda + n\|\mu\|_\infty)^n \int_{[0,T]^n} \prod_{i\in I} |m_\varepsilon(t_i)| dt \\
&= \frac{1}{\mathbb{P}(N_T = n)} (\lambda + n\|\mu\|_\infty)^n \, T^{n-|I|} \prod_{i\in I} \int_0^T |m_\varepsilon(t_i)| dt_i \\
&= \frac{1}{\mathbb{P}(N_T = n)} (\lambda + n\|\mu\|_\infty)^n \, T^{n-|I|} \left(\int_0^T |m_\varepsilon(t)| dt\right)^{|I|} \\
&\leq \frac{1}{\mathbb{P}(N_T = n)} (\lambda + n\|\mu\|_\infty)^n \, T^{n-|I|} \left(\int_0^T |m(t)| dt + 1\right)^{|I|}.
\end{aligned}$$

The last inequality is justified by the convergence proved in Lemma 2.1.

For the first term in (10), we have, due to $\varepsilon m_\varepsilon(T_i) \xrightarrow[\varepsilon\to 0]{a.s.} 0$, for any ε small enough,

$$\left|\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i))\right| \leq 2^n$$

and

$$\begin{aligned}
&\frac{(\varphi_n \circ \Phi_\varepsilon)(T_1, \cdots, T_n) - \varphi_n(T_1, \cdots, T_n)}{\varepsilon} \\
&= \sum_{k=1}^n \int_0^1 \frac{\partial \varphi_n}{\partial t_k} (T_1, \ldots, T_{k-1}, T_k + \alpha\varepsilon \widehat{m}_\varepsilon(T_k), T_{k+1} + \varepsilon \widehat{m}_\varepsilon(T_{k+1}), \ldots) \, d\alpha \; \widehat{m}_\varepsilon(T_k).
\end{aligned}$$

Let us define

$$\psi_n(t_1, \cdots, t_n) = \prod_{i=1}^n \lambda^*(t_i; t_1, \cdots, t_n).$$

Thus

$$\varphi_n(t_1, \cdots, t_n) = \psi_n(t_1, \cdots, t_n) e^{-\int_0^T \lambda^*(s; t_1, \cdots, t_n) ds}.$$

Therefore, as μ is differentiable,

$$\begin{aligned}
&\frac{1}{\varphi_n(T_1, \cdots, T_n)} \frac{\partial \varphi_n}{\partial t_k} (T_1, \ldots, T_{k-1}, T_k + \alpha\varepsilon \widehat{m}_\varepsilon(T_k), T_{k+1} + \varepsilon \widehat{m}_\varepsilon(T_{k+1}), \ldots) \\
&= \left[\frac{1}{\psi_n(T_1, \cdots, T_n)} \frac{\partial \psi_n}{\partial t_k} (T_1, \ldots, T_{k-1}, T_k + \alpha\varepsilon \widehat{m}_\varepsilon(T_k), T_{k+1} + \varepsilon \widehat{m}_\varepsilon(T_{k+1}), \ldots) \right. \\
&\qquad \left. - \frac{\partial}{\partial t_k} \int_0^T \lambda^*(s; T_1, \ldots, T_{k-1}, T_k + \alpha\varepsilon \widehat{m}_\varepsilon(T_k), T_{k+1} + \varepsilon \widehat{m}_\varepsilon(T_{k+1}), \ldots) ds\right]
\end{aligned}$$

with, for the first term

$$\psi_n(t_1,\cdots,t_n)=\prod_{i=1}^{n}\left(\lambda+\sum_{j=1}^{i-1}\mu(t_i-t_j)1_{\{t_i>t_j\}}\right).$$

Thus

$$|\psi_n(t_1,\cdots,t_n)|\geq\lambda^n$$

and for any $0<t_1<\cdots<t_n<T$,

$$\frac{\partial\psi_n}{\partial t_k}(t_1,\cdots,t_n)=\sum_{\ell=1}^{n}\left[\sum_{j=1}^{\ell-1}\frac{\partial}{\partial t_k}(\mu(t_\ell-t_j)1_{\{t_\ell>t_j\}})\prod_{i=1,i\neq\ell}^{n}\left(\lambda+\sum_{i=1}^{j-1}\mu(t_i-t_j)1_{\{t_i>t_j\}}\right)\right].$$

Therefore

$$\begin{aligned}\left|\frac{\partial\psi_n}{\partial t_k}(t_1,\cdots,t_n)\right| &\leq \sum_{\ell=1}^{n}\left[\sum_{j=1}^{\ell-1}\|\mu'\|_\infty\prod_{i=1,i\neq\ell}^{n}\left(\lambda+\sum_{i=1}^{j-1}\|\mu\|_\infty\right)\right]\\ &\leq n^2\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}.\end{aligned}$$

Now for the second term

$$\left|\frac{\partial\lambda^*}{\partial t_k}(t_1,\cdots,t_n)\right|\leq n\|\mu'\|_\infty.$$

Thus, as, for any $\alpha\in\ ]0,1[$, $T_1<\cdots<T_{k-1}<T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k)<T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1})<\cdots$,

$$\begin{aligned}&\left|\frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\cdots,T_n)-\varphi_n(T_1,\cdots,T_n)}{\varepsilon}\frac{1}{\varphi_n(T_1,\cdots,T_n)}\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))\right|\\ \leq&\ 2^n\left(\frac{n^2\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}}{\lambda^n}+Tn\|\mu'\|_\infty\right)\sum_{k=1}^{n}|\widehat{m}_\varepsilon(T_k)|\\ \leq&\ n^22^n\left(\frac{n\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}}{\lambda^n}+T\|\mu'\|_\infty\right)\end{aligned}$$

where the last inequality is due to the uniform convergence of $\widehat{m}_\varepsilon$ to 0.

Therefore we get a control (independant of ε)

$$\left|\frac{Z_n^\varepsilon-1}{\varepsilon}\right|\leq C_{n,T,\lambda,\mu}.$$

Thus, by dominated convergence theorem,

$$\mathbb{E}[D_mF]\ =\ \mathbb{E}\left[\lim_{\varepsilon\to0}\frac{Z_n^\varepsilon-1}{\varepsilon}F\right]=\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}F\right].$$

Then, by linearity, we deduce the same equality for any $F\in\mathcal{S}$. □

Remark 2.14. *For $F=1$ in $\mathbb{E}[D_mF]=\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}F\right]$ we get: $\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\right]=0$.*

Note that we can also prove this property with the expression of the Proposition 2.12:

$$
\begin{aligned}
& \mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}\right] \\
= \;& \mathbb{E}\left[\int_0^T (\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))\lambda^*(s)ds\right] \\
= \;& \mathbb{E}\left[\int_0^T \left(\int_{(0,s)} (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)dN_t + (m(s) + \widehat{m}(s)\mu(T-s))\lambda^*(s)\right) ds\right] \\
= \;& \int_0^T \int_0^s (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)\mathbb{E}[\lambda^*(t)]dtds + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))\mathbb{E}[\lambda^*(s)]ds \\
= \;& \int_0^T \int_0^s (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)g(t)dtds + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \;& \int_0^T \left(\int_t^T (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)ds\right) g(t)dt + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \;& \int_0^T \left([(\widehat{m}(s) - \widehat{m}(t))\mu(s-t)]_t^T - \int_t^T m(s)\mu(s-t)ds\right) g(t)dt \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \;& \int_0^T \left((0 - \widehat{m}(t))\mu(T-t) - 0 - \int_t^T m(s)\mu(s-t)ds\right) g(t)dt \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \;& -\int_0^T \widehat{m}(t)\mu(T-t)g(t)dt - \int_0^T m(s) \underbrace{\left(\int_0^s \mu(s-t)g(t)dt\right)}_{=g(s)-\lambda} ds \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \;& 0
\end{aligned}
$$

where we note $g(s) = \mathbb{E}[\lambda^*(s)]$ which satisfies, according to [14],

$$
g(s) = \lambda + \int_0^s \mu(s-t)g(t)dt.
$$

2.6 Directional Dirichlet space

Proposition 2.15 (and definition of $\mathbb{D}_m^{1,2}$)**.** *The quadratic bilinear form on $L^2(\Omega)$, $(\mathcal{S}, \mathcal{E}_m)$ defined by*

$$
\forall X, Y \in \mathcal{S}, \quad \mathcal{E}_m(X,Y) = \mathbb{E}[D_m X D_m Y],
$$

is closable. We denote by $(\mathbb{D}_m^{1,2}, \mathcal{E}_m)$ its closed extension. As a consequence, D_m is also closable and we still denote by D_m its extension which is well-defined on the whole space $\mathbb{D}_m^{1,2}$.

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$ converging to 0 in $L^2(\Omega)$ and such that

$$\lim_{n,k\to+\infty} \mathcal{E}_m(X_n - X_k) = \lim_{n,k\to+\infty} \mathbb{E}[(D_m X_n - D_m X_k)^2] = 0.$$

Thus $(D_m X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, so it converges to an element Z in $L^2(\Omega)$. Then, let Y be in $\mathcal{S}$, we have by integration by part formula:

$$\begin{aligned} \mathbb{E}[D_m X_n Y] &= \mathbb{E}[D_m(X_n Y)] - \mathbb{E}[X_n D_m Y] \\ &= \mathbb{E}\left[X_n Y \frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}\right] - \mathbb{E}[X_n D_m Y]. \end{aligned}$$

The last equality comes from Theorem 2.13. Since $Y\dfrac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$ and $D_m Y$ belong to $L^2(\Omega)$, we get, by dominated convergence theorem,

$$\forall Y \in \mathcal{S}, \quad \mathbb{E}[ZY] = \lim_{n\to+\infty} \mathbb{E}[D_m X_n Y] = 0.$$

Hence $Z = 0$ by density. We deduce thanks to [5, Proposition 1.3.2] that $(\mathcal{S}, \mathcal{E}_m)$ is closable.
As a consequence, for any $X \in \mathbb{D}_m^{1,2}$, there exists a sequence $(X_n)_{n\in\mathbb{N}} \in \mathcal{S}$ converging to X in $\mathbb{D}_m^{1,2}$ since for all $n, k \in \mathbb{N}$

$$\mathcal{E}_m(X_n - X_k) = \mathbb{E}[|D_m X_n - D_m X_k|^2]$$

we deduce that $(D_m X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2 hence converges to an element that we still denote $D_m X$ and defines in a unique way the extension of D_m to $\mathbb{D}_m^{1,2}$. □

Corollary 2.16. *Proposition 2.7 and Theorem 2.13 remain valid for any* $F \in \mathbb{D}_m^{1,2}$.

3 The local Dirichlet form

3.1 Definition using a Hilbert basis

We would like to define an operator D with domain $\mathbb{D}^{1,2} \subset L^2(\Omega)$ and taking values in $L^2(\Omega; \mathcal{H})$ such that

$$\forall F \in \mathbb{D}^{1,2}, m \in \mathcal{H}, \quad D_m F = \langle DF, m\rangle_{\mathcal{H}} = \int_0^T D_t F m(t) dt.$$

Let $(m_i)_{i\in\mathbb{N}}$ be a Hilbert basis of the space $\mathcal{H}$. Then every function $m \in \mathcal{H}$ can be expressed as

$$m = \sum_{i=0}^{+\infty} \langle m, m_i\rangle_{\mathcal{H}} m_i.$$

We now set

$$\mathbb{D}^{1,2} = \left\{ X \in \bigcap_{i=0}^{+\infty} \mathbb{D}_{m_i}^{1,2}, \quad \sum_{i=0}^{+\infty} \|D_{m_i} X\|_{L^2(\Omega)}^2 < +\infty \right\}$$

and

$$\forall X, Y \in \mathbb{D}^{1,2} \quad \mathcal{E}(X, Y) = \sum_{i=0}^{+\infty} \mathbb{E}[D_{m_i} X D_{m_i} Y].$$

We also note $\mathcal{E}(X) = \mathcal{E}(X, X)$. Then, according to [5, Proposition 4.2.1],

Proposition 3.1. *The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a local Dirichlet form admitting a carré du champ Γ and a gradient D given by, for all $X, Y \in \mathbb{D}^{1,2}$,*

$$\Gamma[X, Y] = \langle DX, DY \rangle_{\mathcal{H}}$$

and

$$DX = \sum_{i=0}^{+\infty} D_{m_i} X m_i \in L^2(\Omega, \mathcal{H}).$$

As a consequence $\mathbb{D}^{1,2}$ is a Hilbert space equipped with the norm

$$\|X\|_{\mathbb{D}^{1,2}}^2 = \|X\|_{L^2(\Omega)}^2 + \mathcal{E}(X).$$

Moreover, as $\mathcal{S}$ is dense in each $\mathbb{D}^{1,2}_{m_i}, i \in \mathbb{N}$, $\mathcal{S}$ is dense in $\mathbb{D}^{1,2}$.

Proof. We start by proving that $(\mathbb{D}^{1,2}, \mathcal{E})$ is a Dirichlet form on $L^2(\Omega)$ in sense of Definition 2.14 of [4]:

- The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a closed form on $L^2(\Omega)$. Let $(X_n)_{n\in\mathbb{N}} \in (\mathbb{D}^{1,2})^{\mathbb{N}}$ such that $\mathbb{E}[(X_n)^2] \underset{n\to+\infty}{\longrightarrow} 0$ and

  $$\lim_{n,k\to+\infty} \sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i} X_n - D_{m_i} X_k)^2] = \lim_{n,k\to+\infty} \mathcal{E}(X_n - X_k) = 0.$$

  Then, for any $i \in \mathbb{N}$,

  $$\lim_{n,k\to+\infty} \mathbb{E}[(D_{m_i} X_n - D_{m_i} X_k)^2] = 0.$$

  However, according to Proposition 2.15, the operator D_{m_i} is closable. Thus

  $$\lim_{n\to+\infty} \mathbb{E}[(D_{m_i} X_n)^2] = 0.$$

  Let $\varepsilon \in \mathbb{R}_+^*$. There exists $n_0 \in \mathbb{N}$ such that, for any $n, k \geq n_0$,

  $$\sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i} X_n - D_{m_i} X_k)^2] \leq \frac{\varepsilon}{4}.$$

  Thus for any $j \in \mathbb{N}$,

  $$\sum_{i=0}^{j} \mathbb{E}[(D_{m_i} X_n - D_{m_i} X_k)^2] \leq \sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i} X_n - D_{m_i} X_k)^2] \leq \frac{\varepsilon}{4}.$$

Therefore

$$\begin{aligned}\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_n)^2] &\leq 2\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] + 2\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_k)^2] \\ &\leq \frac{\varepsilon}{2} + 2\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_k)^2]\end{aligned}$$

with $\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_k)^2] \underset{k\to+\infty}{\longrightarrow} 0$. Thus there exists $k = k_j \in \mathbb{N}$ such that

$$\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_k)^2] \leq \frac{\varepsilon}{4}.$$

Therefore $\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_n)^2] \leq \varepsilon$ and

$$\sum_{i=0}^{+\infty}\mathbb{E}[(D_{m_i}X_n)^2] \leq \varepsilon.$$

To conclude we get

$$\mathcal{E}(X_n) = \sum_{i=0}^{+\infty}\mathbb{E}[(D_{m_i}X_n)^2] \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore $(\mathbb{D}^{1,2}, \mathcal{E})$ is a closed form on $L^2(\Omega)$.

- The closed form $(\mathbb{D}^{1,2}, \mathcal{E})$ satisfies:

$$\forall F \in \mathbb{D}^{1,2}, \quad F \wedge 1 \in \mathbb{D}^{1,2}, \mathcal{E}(F \wedge 1) \leq \mathcal{E}(F).$$

Indeed let $F \in \mathbb{D}^{1,2}$ and $i \in \mathbb{N}$. Thus, by definition of $\mathbb{D}^{1,2}$, $F \in \mathbb{D}^{1,2}_{m_i}$. We have to prove that $F \wedge 1 \in \mathbb{D}^{1,2}_{m_i}$ and that $\mathcal{E}_{m_i}(F \wedge 1) \leq \mathcal{E}_{m_i}(F)$. According to the definition of $\mathbb{D}^{1,2}_{m_i}$ in Proposition 2.15, there exists $(F_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ such that $F_n \underset{n\to+\infty}{\overset{L^2(\Omega)}{\longrightarrow}} F$ and $D_{m_i}F = \lim_{n\to+\infty} D_{m_i}F_n$ exists in $L^2(\Omega)$. For any $n \in \mathbb{N}$, there exists $a_n \in \mathbb{R}, d_n \in \mathbb{N}^*$ and $f_1^n : \mathbb{R} \longrightarrow \mathbb{R}, \ldots, f_{d_n}^n : \mathbb{R}^{d_n} \longrightarrow \mathbb{R}$ smooth functions with bounded derivatives of any order such

$$F_n = a_n 1\{N_T = 0\} + \sum_{m=1}^{d_m} f_m^n(T_1, \cdots, T_m)1_{\{N_T=m\}}.$$

We consider a sequence of smooth functions $(\phi_k)_{k\in\mathbb{N}}$ with bounded derivatives of any order such that we have the uniform convergence of ϕ_k to the function $x \longmapsto x \wedge 1$ and $\|\phi_k'\|_\infty \leq 1$ for any $k \in \mathbb{N}$. Thus

$$\phi_k(F_n) = \phi_k(a_n)1_{\{N_T=0\}} + \sum_{m=1}^{d_m} \phi_k(f_m^n(T_1, \cdots, T_m))1_{\{N_T=m\}} \in \mathcal{S}.$$

Moreover the following convergence in $L^2(\Omega)$ holds:

$$F \wedge 1 = \lim_{n\to+\infty}\lim_{k\to+\infty}\phi_k(F_n).$$

Therefore $F \wedge 1 \in \mathbb{D}^{1,2}_{m_i}$ and

$$D_{m_i}(F \wedge 1) = \lim_{n \to +\infty} \lim_{k \to +\infty} D_{m_i}(\phi_k(F_n))$$

with for any $n, k \in \mathbb{N}$, by the chain rule,

$$D_{m_i}(\phi_k(F_n)) = \phi_k'(F_n) D_{m_i} F_n.$$

Thus

$$\mathcal{E}_{m_i}(F \wedge 1) = \mathbb{E}[(D_{m_i}(F \wedge 1))^2] \leq \mathbb{E}[(D_{m_i} F)^2] = \mathcal{E}_{m_i}(F).$$

Therefore

$$\mathcal{E}(F \wedge 1) = \sum_{i=0}^{+\infty} \mathcal{E}_{m_i}(F \wedge 1) \leq \sum_{i=0}^{+\infty} \mathcal{E}_{m_i}(F) = \mathcal{E}(F).$$

Now we prove that Γ is a carré du champ of the local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ in sense of Definition 2.19 in [4]. Indeed the application Γ is a positive symmetric continuous bilinear form from $\mathbb{D}^{1,2} \times \mathbb{D}^{1,2}$ into $L^1(\Omega)$ such that, for any $X, Y \in \mathbb{D}^{1,2}$,

$$\begin{aligned}
\mathbb{E}[\Gamma[X, Y]] &= \mathbb{E}[\langle DX, DY \rangle_{\mathcal{H}}] \\
&= \mathbb{E}\left[\sum_{i=0}^{+\infty} \langle DX, m_i \rangle \langle DY, m_i \rangle\right] \\
&= \mathbb{E}\left[\sum_{i=0}^{+\infty} D_{m_i} X D_{m_i} Y\right] \\
&= \mathcal{E}(X, Y)
\end{aligned}$$

where we used the fact that the family $(m_i)_{i \in \mathbb{N}}$ is a Hilbert basis in $\mathcal{H}$.

To conclude we prove that the operator D is the gradient of the local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ in sense of its definition at the page 16 in [4]. Indeed:

- For any $X \in \mathbb{D}^{1,2}$, as $(m_i)_{i \in \mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$,

$$\|DX\|^2_{\mathcal{H}} = \left\| \sum_{i=0}^{+\infty} D_{m_i} X m_i \right\|^2_{\mathcal{H}} = \sum_{i=0}^{+\infty} (D_{m_i} X)^2 = \Gamma[X].$$

- Let $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function and $X \in \mathbb{D}^{1,2}$. Then, according to Remark 3.3, $\Phi(X) \in \mathbb{D}^{1,2}$ and $D(\Phi(X)) = \Phi'(X) DX$ where $\Phi'(X)$ is the Lebesgue partial derivative of Φ almost everywhere defined.

□

Corollary 3.2. *For all $n \in \mathbb{N}^*$ and $j \in \{0, ..., n\}$, writing $\overline{T}_j = T_j \wedge T$, we have*

$$D\overline{T}_j = \frac{\overline{T}_j}{T} - 1_{[0, \overline{T}_j]}.$$

As consequence for all

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \in \mathcal{S},$$

we have $F \in \mathbb{D}^{1,2}$ *and*

$$DF = \sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)\left(\frac{T_j}{T} - 1_{[0,T_j]}\right)1_{\{N_T=n\}}.$$

In particular this expression does not depend on the basis $(m_i)_{i\in\mathbb{N}}$.

Proof. We have

$$\begin{aligned} D\overline{T}_j &= \sum_{i=0}^{+\infty} D_{m_i}\overline{T}_j m_i = -\sum_{i=0}^{+\infty} \widehat{m}_i(\overline{T}_j)m_i = -\sum_{i=0}^{+\infty}\int_0^T m_i(s)1_{\{0\le s\le \overline{T}_j\}}ds m_i \\ &= \sum_{i=0}^{+\infty}\left\langle \frac{\overline{T}_j}{T} - 1_{[0,\overline{T}_j]}, m_i\right\rangle_{\mathcal{H}} m_i = \frac{\overline{T}_j}{T} - 1_{[0,\overline{T}_j]}. \end{aligned}$$

Notice that the term $\dfrac{\overline{T}_j}{T}$ is mandatory to belong to $\mathcal{H}$ defined by (1). □

We get the chain rule for the operator D on $\mathbb{D}^{1,2}$

Remark 3.3. *For any* $F_1, \ldots, F_n \in \mathbb{D}^{1,2}$ *and smooth function* $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, $\Phi(F_1, \ldots, F_n)$ *belongs to* $\mathbb{D}^{1,2}$ *and*

$$D\Phi(F_1, \ldots, F_n) = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j}(F_1, \ldots, F_n)DF_j.$$

Moreover we can extend this result with a Lipschitz function Φ *thanks to Proposition 4.3 where we replace* $\dfrac{\partial\Phi}{\partial x_j}$ *is the Lebesgue partial derivative of* Φ *almost everywhere defined. (référence à trouver dans Bouleau-Hirsch Proposition 2.1.5 ?)*

3.2 Divergence operator by duality

Let $\delta : L^2(\Omega, \mathcal{H}) \longrightarrow L^2(\Omega)$ be the adjoint operator of D. Its domain, $\mathrm{Dom}(\delta)$, is the set of $u \in L^2(\Omega, \mathcal{H})$ such that there exists $c \in \mathbb{R}_+^*$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \left|\mathbb{E}\left[\int_0^T D_t F u_t dt\right]\right| \le c\|F\|_{\mathbb{D}^{1,2}}.$$

Hence, for all $u \in \mathrm{Dom}(\delta)$, $\delta(u)$ is the unique element in $L^2(\Omega)$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF\rangle_{\mathcal{H}}] = \mathbb{E}\left[\int_0^T u_t D_t F dt\right].$$

We now introduce the set $\widetilde{\mathcal{S}}$ of elementary processes u of the form

$$u = \sum_{i=1}^{n} A_i m_i, \quad n \in \mathbb{N}^*, \quad A_i \in \mathbb{D}^{1,2}.$$

Proposition 3.4. *For all u in $\widetilde{\mathcal{S}}$, we have $u \in Dom(\delta)$ and*

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s - \int_0^T D_t(u(t))dt$$

where ψ is defined by (9).

Proof. Let $u = Am_{i_0}$ with $A \in \mathbb{D}^{1,2}$ and $i_0 \in \mathbb{N}$. For any $F \in \mathbb{D}^{1,2}$ we have

$$\begin{aligned}
\left|\mathbb{E}\left[\int_0^T D_t F u_t dt\right]\right| &\leq \mathbb{E}\left[\int_0^T |D_tF||Am_{i_0}(t)|dt\right] \\
&\leq \|F\|_{\mathbb{D}^{1,2}} \sqrt{\mathbb{E}\left[\int_0^T |A|^2|m_{i_0}(t)|^2 dt\right]} \\
&= \|F\|_{\mathbb{D}^{1,2}} \|A\|_{L^2(\Omega)} \|m_{i_0}\|_{L^2(0,T)} = c\,\|F\|_{\mathbb{D}^{1,2}}
\end{aligned}$$

with $c = \|A\|_{L^2(\Omega)} \|m_{i_0}\|_{L^2(0,T)}$. Thus $u \in \mathrm{Dom}(\delta)$ and for any $F \in \mathbb{D}^{1,2}$

$$\begin{aligned}
\mathbb{E}[\delta(u)F] &= \mathbb{E}\left[\int_0^T u_t D_t F dt\right] = \mathbb{E}\left[A\int_0^T m_{i_0}(t) D_t F dt\right] \\
&= \mathbb{E}\left[A\int_0^T m_{i_0}(t) \sum_{i=0}^{+\infty} D_{m_i} F m_i(t) dt\right] \\
&= \mathbb{E}\left[A\sum_{i=0}^{+\infty} D_{m_i} F \int_0^T m_{i_0}(t) m_i(t) dt\right] \\
&= \mathbb{E}\left[A\sum_{i=0}^{+\infty} D_{m_i} F \langle m_{i_0}, m_i\rangle_{\mathcal{H}}\right] = \mathbb{E}[AD_{m_{i_0}}F].
\end{aligned}$$

Thus, integrating by parts:

$$\begin{aligned}
\mathbb{E}[\delta(u)F] &= \mathbb{E}[D_{m_{i_0}}(AF)] - \mathbb{E}[FD_{m_{i_0}}A] \\
&= \mathbb{E}\left[\frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} AF\right] - \mathbb{E}[FD_{m_{i_0}}A] \\
&= \mathbb{E}\left[\left(\frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} A - D_{m_{i_0}}A\right) F\right].
\end{aligned}$$

Therefore, because $\langle DA, m_{i_0}\rangle_{\mathcal{H}} = \sum_{i=0}^{+\infty} D_{m_i}A\langle m_i, m_{i_0}\rangle_{\mathcal{H}} = D_{m_{i_0}}A$,

$$\begin{aligned}
\delta(u) &= \frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} A - D_{m_{i_0}}A \\
&= \left(\int_{(0,T]} (\psi(m_{i_0},s) + \widehat{m_{i_0}}(s)\mu(T-s) + m_{i_0}(s))dN_s\right) A - \int_0^T m_{i_0}(t) D_t A dt \\
&= \int_{(0,T]} (\psi(Am_{i_0},s) + \widehat{Am_{i_0}}(s)\mu(T-s) + Am_{i_0}(s))dN_s - \int_0^T D_t(Am_{i_0}(t))dt \\
&= \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s - \int_0^T D_t(u(t))dt.
\end{aligned}$$

We deduce the result for any $u \in \widetilde{\mathcal{S}}$ by linearity. □

Remark 3.5. *We can retain that:*

1. *For all* $m \in \mathcal{H}$ *and* $A \in \mathbb{D}^{1,2}$,
$$\delta(mA) = \delta(m)A - D_m A.$$
2. *For all* $m \in \mathcal{H}$ *and* $A, F \in \mathbb{D}^{1,2}$,
$$\mathbb{E}[AD_m F] = \mathbb{E}[F\delta(mA)].$$
3. *For all* $u \in \mathcal{H}$ *we have* $Du = 0$ *and*
$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s.$$

Remark 3.6. *Contrary to the standard Malliavin calculus on the Wiener space (see [16]), we do not have a priori the inclusion of* $\mathbb{D}^{1,2} \otimes \mathcal{H}$ *in Dom*(δ) *(see Example 3.4 in [6] where* $\mu = 0$*).*

Corollary 3.7. *If* $u \in L^2(\Omega, \mathcal{H})$ *is a predictable process then*
$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s$$

Proof. We establish this result for an elementary process of the form:
$$u(t) = f_0 1_{[0,t_1]}(t) + \sum_{j=1}^{n-1} f_j(\overline{T}_1, \cdots, \overline{T}_n) 1_{]t_j,t_{j+1}]}(t)$$

where $n \in \mathbb{N}^*$, $t_j = \frac{jT}{n}$, f_0 is a constant, for any $j \in \{1, \cdots, n\}$, f_j is an infinitely differentiable function from $\mathbb{R}^n$ into $\mathbb{R}$ vanishing outside the simplex
$$\Delta_n^j = \{(x_1, \cdots, x_n) \in \mathbb{R}^n,\ 0 < x_1 < \cdots < x_n \leq t_j\}$$
and $f_{n-1} = -f_0 - \sum_{j=1}^{n-2} f_j$. This last condition ensures that u belongs to $L^2(\Omega; \mathcal{H})$. As a consequence, we can rewrite u as
$$u(t) = f_0\left(1_{[0,t_1]}(t) - \frac{T}{n}\right) + \sum_{j=1}^{n-1} f_j(\overline{T}_1, \cdots, \overline{T}_n)\left(1_{]t_j,t_{j+1}]}(t) - \frac{T}{n}\right),$$
this proves that u belongs to $\widetilde{\mathcal{S}}$. We have, by Proposition 3.4 and Corollary 3.2,
$$\begin{aligned} D_t u(t) &= \sum_{j=1}^{n-1}\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n) D_t \overline{T}_i 1_{]t_j,t_{j+1}]}(t) \\ &= \sum_{j=1}^{n-1}\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n)\left(\frac{\overline{T}_i}{T} - 1_{[0,\overline{T}_i]}(t)\right) 1_{]t_j,t_{j+1}]}(t). \end{aligned}$$
Now, since f_j vanishes outside Δ_n^j, we have for any $j \in \{1, \cdots, n-1\}$ and any $i \in \{1, \cdots, n\}$

$$\frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n)1_{[0,\overline{T}_i]}(t)1_{]t_j,t_{j+1}]}(t) = 0.$$

Moreover for any i, as u is valued in $\mathcal{H}$,

$$\int_0^T \sum_{j=1}^q \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n)\frac{\overline{T}_i}{T}1_{]t_j,t_{j+1}]}(t)dt = \frac{\overline{T}_i}{T}\frac{\partial}{\partial x_i}\int_0^T u(t)dt = 0.$$

Therefore

$$\int_0^T D_t u(t)dt = 0,$$

Hence from Proposition 3.4 we deduce that

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s.$$

We conclude by using a density argument. Indeed if $u \in L^2(\Omega, \mathcal{H})$ is a predictable process, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of elementary processes as above converging to u in $L^2(\Omega, \mathcal{H})$ and clearly $\delta(u_n)$ converges to $\int_{(0,T]}(\psi(u,s)+ \widehat{u}(s)\mu(T-s)+u(s))dN_s$. Since δ is a closed operator, we conclude that u belongs to Dom(δ) and that

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s.$$

□

Proposition 3.8. *Let $F \in \mathbb{D}^{1,2}$ and $X \in Dom(\delta)$ such that*

$$F\delta(X) - \int_0^T D_t F X_t dt \in L^2(\Omega),$$

then $FX \in Dom(\delta)$ and

$$\delta(FX) = F\delta(X) - \int_0^T D_t F X_t dt.$$

Proof. For any $G \in \mathcal{S}$

$$\begin{aligned}
\mathbb{E}[\delta(FX)G] &= \mathbb{E}\left[\int_0^T F X_t D_t G dt\right] \\
&= \mathbb{E}\left[\int_0^T X_t(D_t(GF) - GD_tF)dt\right] \\
&= \mathbb{E}\left[\delta(X)GF - G\int_0^T X_t D_t F dt\right] \\
&= \mathbb{E}\left[G\left(F\delta(X) - \int_0^T D_t F X_t dt\right)\right].
\end{aligned}$$

□

In particular if $X = m \in \mathcal{H} \subset \mathrm{Dom}(\delta)$ then $\delta(m) = \int_{(0,T]}(\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s$ and $\int_0^T D_tFm(t)dt = D_mF$. Hence we have

$$\delta(mF) = F\int_{(0,T]}(\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s - D_mF.$$

Remark 3.9. *We do not have the Clark-Ocone formula because for $F = N_T \in \mathbb{D}^{1,2}$ we have $N_T \neq \mathbb{E}[N_T]$ and $D_tN_T = 0$. Indeed, for any $m \in \mathcal{H}$ and $\varepsilon \in \mathbb{R}_+^*, \mathcal{T}_\varepsilon N_T = N_T$.*

4 Absolute continuity criterion

4.1 Local criterion

Lemma 4.1. *The distribution of $(T_1, \cdots, T_n)$ conditionally to $\{N_T = n\}$ has a density*

$$\begin{array}{rccc} k_n : & \mathbb{R}^n & \longrightarrow & \mathbb{R}_+ \\ & (t_1, \cdots, t_n) & \longmapsto & \dfrac{\kappa(t)}{\int_{\mathbb{R}^n}\kappa(s)ds} \end{array}$$

with, for any $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$,

$$\kappa(t) = 1_{\{0<t_1<\cdots<t_n\leq T\}}\left(\prod_{i=1}^n \lambda^*(t_i; t_1, \cdots, t_n)\right) e^{-\int_0^T \lambda^*(s;t_1,\cdots,t_n)ds}.$$

Proof. Let f be a measurable function on $\mathbb{R}^n$. Then

$$\begin{aligned} &\mathbb{E}[f(T_1, \cdots, T_n) \mid N_T = n] \\ &= \frac{\mathbb{E}[1_{\{N_T=n\}}f(T_1, \cdots, T_n)]}{\mathbb{P}(N_T = n)} \\ &= \frac{\mathbb{E}[1_{\{T_n\leq T<T_{n+1}\}}f(T_1, \cdots, T_n)]}{\mathbb{P}(T_n \leq T < T_{n+1})} \\ &= \frac{\int_{0<t_1<\cdots<t_n\leq T<t_{n+1}} f(t_1, \cdots, t_n)\varphi(t_1, \cdots, t_{n+1})dt_1\cdots dt_{n+1}}{\int_{0<t_1<\cdots<t_n\leq T<t_{n+1}} \varphi(t_1, \cdots, t_{n+1})dt_1\cdots dt_{n+1}} \end{aligned}$$

with

$$\begin{aligned} \varphi(t_1, \cdots, t_{n+1}) &= \left(\prod_{i=1}^{n+1} \lambda^*(t_i; t_1, \cdots, t_n)\right) e^{-\int_0^{t_{n+1}} \lambda^*(s;t_1,\cdots,t_n)ds} \\ &= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n)\right) \lambda^*(t_{n+1}; t_1, \cdots, t_n)e^{-\int_0^{t_{n+1}} \lambda^*(s;t_1,\cdots,t_n)ds} \\ &= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n)\right) e^{-\int_0^{t_n} \lambda^*(s;t_1,\cdots,t_n)ds} \\ &\quad \times \left(\lambda + \sum_{i=1}^n \mu(t_{n+1} - t_i)\right) e^{-\int_{t_n}^{t_{n+1}}\left(\lambda+\sum_{i=1}^n \mu(s-t_i)\right)ds}. \end{aligned}$$

Thus

$$\begin{aligned}
&\int_{T<t_{n+1}} \varphi(t_1,\cdots,t_{n+1})dt_{n+1} \\
&= \int_T^{+\infty} \left(\lambda + \sum_{i=1}^n \mu(t_{n+1}-t_i)\right) e^{-\int_{t_n}^{t_{n+1}}\left(\lambda+\sum_{i=1}^n \mu(s-t_i)\right)ds} dt_{n+1} \\
&= e^{-\int_{t_n}^{T}\left(\lambda+\sum_{i=1}^n \mu(s-t_i)\right)ds}.
\end{aligned}$$

Therefore

$$\mathbb{E}[f(T_1,\cdots,T_n) \mid N_T = n] = \frac{\int_{\mathbb{R}^n} f(t)\kappa(t)dt}{\int_{\mathbb{R}^n} \kappa(t)dt} = \int_{\mathbb{R}^n} f(t)k_n(t)dt$$

with, for any $t = (t_1,\cdots,t_n) \in \mathbb{R}^n$,

$$\begin{aligned}
\kappa(t) &= 1_{\{0<t_1<\cdots<t_n\leq T\}} \left(\prod_{i=1}^n \lambda^*(t_i;t_1,\cdots,t_n)\right) e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_n)ds} \\
&\quad \times e^{-\int_{t_n}^{T}\left(\lambda+\sum_{i=1}^n \mu(s-t_i)\right)ds} \\
&= 1_{\{0<t_1<\cdots<t_n\leq T\}} \left(\prod_{i=1}^n \lambda^*(t_i;t_1,\cdots,t_n)\right) e^{-\int_0^{T}\lambda^*(s;t_1,\cdots,t_n)ds},
\end{aligned}$$

which implies the conclusion of the lemma. □

Now fix $n \in \mathbb{N}^*$, as usual $C^\infty(\mathbb{R}^n)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^n$. We consider the following quadratic form on $C^\infty(\mathbb{R}^n)$:

$$e_n(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^n \frac{\partial u}{\partial t_i}(t)\frac{\partial v}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_it_j}{T}\right)k_n(t)dt$$

and

$$e_n(u) = e_n(u,u).$$

Proposition 4.2.

1. *$(C^\infty(\mathbb{R}^n), e_n)$ is closable, its closure (d_n, e_n) defines a local Dirichlet form on $L^2(k_n(t)dt)$ and each $u \in d_n$ is a $\mathcal{B}(\mathbb{R}^n)$-measurable function in $L^2(k_n(t)dt)$ such that for any $i \in \{1,\cdots,n\}$ and for almost all*

   $$\widetilde{t} = (t_1,\cdots,t_{i-1},t_{i+1},\cdots,t_n) \in \mathbb{R}^{n-1},$$

   the function

   $$s \longmapsto u_{\widetilde{t}}^{(i)}(s) = u(t_1,\cdots,t_{i-1},s,t_{i+1},\cdots,t_n)$$

   has an absolutely continuous version $\widetilde{u}_{\widetilde{t}}^{(i)}$ on $[t_{i-1},t_{i+1}]$ such that

   $$\sum_{i,j=1}^n \frac{\partial u}{\partial t_i}(t)\frac{\partial u}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_it_j}{T}\right) \in L^1(k_n(t)dt)$$

   where $\dfrac{\partial u}{\partial t_i} = \dfrac{\partial \widetilde{u}_{\widetilde{t}}^{(i)}}{\partial s}$.

2. *The Dirichlet form (d_n, e_n) admits a carré du champ operator γ_n and a gradient operator $\widetilde{D}^n$ given by*

$$\gamma_n[u,v](t) = \sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t) \frac{\partial v}{\partial t_j}(t) \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right)$$

and

$$\widetilde{D}^n_s u(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial t_i}(t) \left(\frac{t_i}{T} - 1_{[0,t_i]}(s) \right)$$

for all $u, v \in d_n, t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ and $s \in [0,T]$.

3. *The structure $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n(t)dt, d_n, \gamma_n)$ satisfies for every $d \in \mathbb{N}^*, u = (u_1, \cdots, u_d) \in (d_n)^d$,*

$$u_*[\det(\gamma_n[u]) \cdot k_n \nu_n] \ll \nu_d$$

where $\gamma_n[u]$ denotes the matrix $(\gamma_n(u_i, u_j))_{1 \le i,j \le d}$, ν_n (resp. ν_d) the Lebesgue measure on $\mathbb{R}^n$ (resp. $\mathbb{R}^d$) and $u_*[\det(\gamma_n[u]) \cdot k_n \nu_n]$ the image measure defined by, for any $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} (u_*[\det(\gamma_n[u]) \cdot k_n \nu_n])(B) &= [\det(\gamma_n[u]) \cdot k_n \nu_n](u^{-1}(B)) \\ &= \int_{u^{-1}(B)} \det(\gamma_n[u,u](t)) k_n(t) dt. \end{aligned}$$

Proof. We prove this result thanks to [4, Proposition 2.30 and Theorem 2.31] with

$$k = k_n, \quad d = \tilde{d}_n, \quad \xi_{ij}(t) = t_i \wedge t_j - \frac{t_i t_j}{T},$$

where $\tilde{d}_n$ is the set of $\mathcal{B}(\mathbb{R}^n)$-measurable functions $u \in L^2(k_n(t)dt)$ such that for any $i \in \{1, \cdots, n\}$ and for almost all

$$\widetilde{t} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}^{n-1},$$

the function

$$s \longmapsto u^{(i)}_{\widetilde{t}}(s) = u(t_1, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)$$

has an absolutely continuous version $\widetilde{u}^{(i)}_{\widetilde{t}}$ on $[t_{i-1}, t_{i+1}]$ such that

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t) \frac{\partial u}{\partial t_j}(t) \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right) \in L^1(k_n(t)dt)$$

where $\dfrac{\partial u}{\partial t_i} = \dfrac{\partial \widetilde{u}^{(i)}_{\widetilde{t}}}{\partial s}$ and set for any $u, v \in \tilde{d}_n$:

$$e_n(u,v) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t) \frac{\partial v}{\partial t_j}(t) \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right) k_n(t) dt.$$

The function $k = k_n : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is measurable and the functions $\xi_{i,j}$ are symmetric Borel function. We have to check if the two assumptions (HG) of [3] are satisfied.

Condition 1. For any $i \in \{1, ..., n\}$ and ν_{n-1}-a.e. $\bar{t} \in B_{n-1}$ with

$$B_{n-1} = \left\{ \tilde{t} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}^{n-1} \quad \int_{\mathbb{R}} k_{n,\tilde{t}}^{(i)}(s) ds > 0 \right\},$$

we have $0 < t_1 < \cdots < t_{i-1} < t_{i+1} < \cdots < t_n < T$ because else $\kappa_{\bar{t}}^{(i)} = \kappa(t_1, \cdots, t_{i-1}, \cdot, t_{i+1}, \cdots, t_n) = 0$ and $\int_{\mathbb{R}} k_{n,\bar{t}}^{(i)}(s) ds = 0$. Thus, for any $t_i \in]t_{i-1}, t_{i+1}[$,

$$k_{n,\bar{t}}^{(i)}(t_i) = \kappa(t_1, \cdots, t_n) > 0$$

because $\lambda > 0$ and $\mu \geq 0$. Therefore

$$k_n(t_1, \cdots, t_n) = \frac{\kappa(t_1, \cdots, t_n)}{\int_{\mathbb{R}^n} \kappa(t) dt} > 0.$$

In particular $k_n(t_1, \cdots, t_n)$ is invertible and

$$\begin{aligned}(k_n(t_1, \cdots, t_n))^{-1} &= \frac{\int_{\mathbb{R}^n} \kappa(t) dt}{\kappa(t_1, \cdots, t_n)} \\ &= 1_{\{0 < t_1 < \cdots < t_n \leq T\}} \left(\prod_{i=1}^n (\lambda^*(t_i; t_1, \cdots, t_n))^{-1} \right) e^{\int_0^T \lambda^*(s; t_1, \cdots, t_n) ds} \int_{\mathbb{R}^n} \kappa(t) dt.\end{aligned}$$

Therefore $t_i \longmapsto (k_n(t_1, \cdots, t_n))^{-1}$ is integrable on $\mathbb{R}$ because it is only the integral of a continuous function on $[t_{i-1}, t_{i+1}]$ and equal to 0 on $\mathbb{R} \backslash [t_{i-1}, t_{i+1}]$.

Condition 2. For any $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ and any $c \in \mathbb{R}^n$, such that $0 = t_0 < t_1 < t_2 < \cdots < t_n < T = t_{n+1}$,

$$\begin{aligned}\sum_{i,j=1}^n \xi_{ij}(t) c_i c_j &= \sum_{i,j=1}^n \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right) c_i c_j \\ &= \frac{1}{T} \sum_{i=1}^n t_i (T - t_i) c_i^2 + \frac{2}{T} \sum_{1 \leq i < j \leq n} t_i (T - t_j) c_i c_j\end{aligned}$$

This double sum can be split as follows:

$$\begin{aligned}\sum_{i=1}^n t_i (T - t_i) c_i^2 &= \sum_{i=1}^n \sum_{k=1}^i (t_k - t_{k-1}) \sum_{\ell=i}^n (t_{\ell+1} - t_\ell) c_i^2 \\ &= \sum_{k=1}^n (t_k - t_{k-1}) \sum_{\ell=k}^n (t_{\ell+1} - t_\ell) \sum_{i=k}^{\ell} c_i^2 \\ &= \sum_{k=1}^n (t_k - t_{k-1})(t_{k+1} - t_k) c_k^2 + \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{\ell=k+1}^n (t_{\ell+1} - t_\ell) \sum_{i=k}^{\ell} c_i^2\end{aligned}$$

and

$$
\begin{aligned}
2\sum_{1\le i<j\le n} t_i\,(T-t_j)\,c_ic_j &= 2\sum_{1\le i<j\le n}\sum_{k=1}^{i}(t_k-t_{k-1})\sum_{\ell=j}^{n}(t_{\ell+1}-t_\ell)c_ic_j\\
&= 2\sum_{k=1}^{n-1}(t_k-t_{k-1})\sum_{k\le i<j\le n}\sum_{\ell=j}^{n}(t_{\ell+1}-t_\ell)c_ic_j\\
&= 2\sum_{k=1}^{n-1}(t_k-t_{k-1})\sum_{\ell=k+1}^{n}(t_{\ell+1}-t_\ell)\sum_{k\le i<j\le \ell}c_ic_j
\end{aligned}
$$

Coming back to the initial sum, we have

$$
\begin{aligned}
\sum_{i,j=1}^{n}\xi_{ij}(t)c_ic_j &= \frac{1}{T}\sum_{k=1}^{n}(t_k-t_{k-1})(t_{k+1}-t_k)c_k^2\\
&\quad+\frac{1}{T}\sum_{k=1}^{n-1}(t_k-t_{k-1})\sum_{\ell=k+1}^{n}(t_{\ell+1}-t_\ell)\left[\sum_{i=k}^{\ell}c_i^2+2\sum_{k\le i<j\le\ell}c_ic_j\right]\\
&= \frac{1}{T}\sum_{k=1}^{n}(t_k-t_{k-1})(t_{k+1}-t_k)c_k^2\\
&\quad+\frac{1}{T}\sum_{k=1}^{n-1}(t_k-t_{k-1})\sum_{\ell=k+1}^{n}(t_{\ell+1}-t_\ell)\left[\sum_{i,j=k}^{\ell}(c_i+c_j)^2\right]\\
&\ge \frac{1}{T}\sum_{k=1}^{n}(t_k-t_{k-1})(t_{k+1}-t_k)c_k^2.
\end{aligned}
$$

Thus, for any compact $K\subset\{(t_1,\cdots,t_n),\quad 0=t_0<t_1<\cdots<t_n<T=t_{n+1}\}$, there exists $c\in\mathbb{R}_+^*$ such that, for any $(t_1,\cdots,t_n)\in K$,

$$
\sum_{i,j=1}^{n}\xi_{ij}(t)c_ic_j\ge\frac{c^2}{T}\sum_{k=1}^{n}c_k^2.
$$

The hypotheses (HG) of [4, Proposition 2.30] being satisfied, we conclude that $(\tilde{d}_n,e_n)$ is a local Dirichlet hence $(C^\infty(\mathbb{R}^n),e_n)$ is closable and its closure (d_n,e_n) is such that $d_n\subset\tilde{d}_n$. The last assertion is a consequence of [4, Theorem 2.31]. □

Using this result, we consider $\|\cdot\|_{d_n}$ the norm on d_n defined by, for any $u\in d_n$,

$$
\|u\|_{d_n}^2=\|u\|_{L^2(k_n(t)dt)}^2+2e_n(u).
$$

4.2 Global criterion

We remind that for any $F\in L^0(\Omega)$, there exists $a\in\mathbb{R}$ and $f_n:\mathbb{R}^n\longrightarrow\mathbb{R}, n\in\mathbb{N}^*$, measurable such that, $\mathbb{P}$-almost surely,

$$
F=a1_{\{N_T=0\}}+\sum_{n=1}^{+\infty}f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}. \tag{11}
$$

Proposition 4.3. *Let $F \in L^0(\Omega)$ of the form (11). Then $F \in \mathbb{D}^{1,2}$ if and only if $f_n \in d_n$ for any $n \in \mathbb{N}^*$ and*

$$\sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n) < +\infty.$$

In this case

$$\|F\|_{\mathbb{D}^{1,2}}^2 = a^2 \mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Proof. Let $F = a 1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$ be in $\mathcal{S}$. Then as a consequence of Corollary 3.2, F belongs to $\mathbb{D}^{1,2}$ and

$$\mathcal{E}(F, F) = \mathbb{E}\left[\int_0^T |D_s F|^2 \, ds\right] = 2 \sum_{n=1}^{d} P(N_T = n) e_n(f_n).$$

Hence

$$\|F\|_{\mathbb{D}^{1,2}}^2 = a^2 \mathbb{P}(N_T = 0) + \sum_{n=1}^{d} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

We conclude using a density argument. Indeed, if F belongs to $\mathbb{D}^{1,2}$, there exists a sequence $(F^k)_k$ in $\mathcal{S}$ converging to F in $\mathbb{D}^{1,2}$. Now if for any k

$$F^k = a_k 1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n^k(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$$

with $f_n^k \in C^\infty(\mathbb{R}^d)$ and $f_n^k = 0$ for n large, then clearly for all n, k, ℓ:

$$\|f_n^k - f_n^\ell\|_{d_n}^2 \mathbb{P}(N_T = n) \leq \|F^k - F^\ell\|_{\mathbb{D}^{1,2}}^2$$

hence $(f_n^k)_n$ converges to an element f_n in d_n, a^k tends to a real number a and we get that

$$F = a 1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}},$$

and

$$\|F\|_{\mathbb{D}^{1,2}}^2 = \lim_{m \to +\infty} \|F 1_{\{N_T \leq m\}}\|_{\mathbb{D}^{1,2}}^2 = a^2 \mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Conversely, if $F \in L^0(\Omega)$ of the form (11) is such that $f_n \in d_n$ for any $n \in \mathbb{N}^*$ and

$$\sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n) < +\infty,$$

then define for any m, $F^m = a 1_{\{N_T=0\}} + \sum_{n=1}^{m} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$ by approximating each f_n for $n \in \{1, \cdots, m\}$ by a sequence of functions in $C^\infty(\mathbb{R}^d)$ we easily get that F^m belong to $\mathbb{D}^{1,2}$ and

$$\|F^m\|_{\mathbb{D}^{1,2}}^2 = a^2 \mathbb{P}(N_T = 0) + \sum_{n=1}^{m} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Then (F^m) is a Cauchy sequence in $\mathbb{D}^{1,2}$ converging to F in L^2, this ends the proof. $\square$

Remark 4.4. *We can summarize the equalities between the Dirichlet structures* $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{D}^{1,2}, \Gamma)$ *and* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n(t)dt, d_n, \gamma_n), n \in \mathbb{N}^*$*: for any* $F \in \mathbb{D}^{1,2}$ *of the form* (11)

1. $$\|F\|^2_{L^2(\Omega)} = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|^2_{L^2(k_n(t)dt)},$$

2. *For a.e.* $s \in [0, T]$,
$$D_s F = \sum_{n=1}^{+\infty} \widetilde{D}^n_s f_n(T_1, \cdots, T_n) 1_{\{N_T = n\}},$$
with
$$\widetilde{D}^n_s f_n(T_1, \cdots, T_n) = \sum_{i=1}^{n} \frac{\partial f_n}{\partial t_i}(T_1, \cdots, T_n) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s) \right),$$

3. $$\Gamma[F] = \sum_{n=1}^{+\infty} \gamma_n[f_n](T_1, \cdots, T_n) 1_{\{N_T = n\}},$$

4. $$\mathcal{E}(F) = \sum_{n=1}^{+\infty} e_n(f_n) \mathbb{P}(N_T = n),$$

5. $$\|F\|^2_{\mathbb{D}^{1,2}} = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|^2_{d_n} \mathbb{P}(N_T = n).$$

Theorem 4.5. *Let* $d \in \mathbb{N}^*$ *and* $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$*. Then, noting*
$$\Gamma[F] = (\Gamma[F_i, F_j])_{1 \le i,j \le d},$$
the image measure $F_*[\det(\Gamma[F]).\mathbb{P}]$ *is absolutely continuous with respect to the Lebesgue measure* ν_d *on* $\mathbb{R}^d$.

Proof. Let $B \subset \mathbb{R}^d$ such that $\nu_d(B) = 0$. We would like to get
$$\begin{aligned} 0 &= (F_*[\det(\Gamma[F]).\mathbb{P}])(B) = \int_{F^{-1}(B)} \det(\Gamma[F](\omega)) d\mathbb{P}(\omega) \\ &= \int_\Omega \det(\Gamma[F](\omega)) 1_B(F(\omega)) d\mathbb{P}(\omega) = \mathbb{E}\left[\det(\Gamma[F]) 1_B(F)\right]. \end{aligned}$$
But, according to Proposition 4.3, there exist $a \in \mathbb{R}$ and $f_n \in (d_n)^d, n \in \mathbb{N}^*$ such that
$$F = a 1_{\{N_T = 0\}} + \sum_{n=1}^{+\infty} f_n(T_1, \cdots, T_n) 1_{\{N_T = n\}}.$$

Thus

$$\Gamma[F] = \Gamma[F,F] = \sum_{n=1}^{+\infty} \gamma_n[f_n](T_1, \cdots, T_n) 1_{\{N_T=n\}}.$$

In particular

$$\det(\Gamma[F]) 1_{\{N_T=0\}} = 0,$$

for any $n \in \mathbb{N}^*$,

$$\det(\Gamma[F]) 1_{\{N_T=n\}} \quad = \quad \det(\gamma_n[f_n](T_1, \cdots, T_n)) 1_{\{N_T=n\}}$$

and also

$$1_B(F) 1_{\{N_T=n\}} \quad = \quad 1_B(f_n(T_1, \cdots, T_n)) 1_{\{N_T=n\}}.$$

Therefore

$$\det(\Gamma[F]) 1_B(F) \quad = \quad \sum_{n=1}^{+\infty} \det(\gamma_n[f_n](T_1, \cdots, T_n)) 1_B(f_n(T_1, \cdots, T_n)) 1_{\{N_T=n\}}$$

and, according to Lemma 4.1,

$$\begin{aligned}
&\mathbb{E}\left[\det(\Gamma[F]) 1_B(F)\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left[\det(\gamma_n[f_n](T_1, \cdots, T_n) 1_B(f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left[\det(\gamma_n[f_n](T_1, \cdots, T_n) 1_B(f_n(T_1, \cdots, T_n) \mid N_T = n\right] \mathbb{P}(N_T = n) \\
&= \sum_{n=1}^{+\infty} \left(\int_{\mathbb{R}^n} \det(\gamma_n[f_n](t)) 1_B(f_n(t)) k_n(t) dt \right) \mathbb{P}(N_T = n) \\
&= \sum_{n=1}^{+\infty} ((f_n)_*[\det(\gamma_n[f_n] \cdot k_n \nu_n)])(B) \mathbb{P}(N_T = n).
\end{aligned}$$

However, according to Proposition 4.2 applied to $f_n \in (d_n)^d$, the measure $(f_n)_*[\det(\gamma_n[f_n] \cdot k_n \nu_n)]$ is absolutely continuous with respect to ν_d. Thus, for any $n \in \mathbb{N}^*$,

$$((f_n)_*[\det(\gamma_n[f_n] \cdot k_n \nu_n)])(B) = 0$$

and

$$\mathbb{E}\left[\det(\Gamma[F]) 1_B(F)\right] = 0.$$

This concludes the proof. □

Corollary 4.6. *Let $d \in \mathbb{N}^*$ and $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$. Then, conditionally to $\Gamma[F] \in GL_d(\mathbb{R})$, the law of the random variable F is absolutely continuous with respect to the Lebesgue measure ν_d.*

5 Applications

5.1 SDE and density of the solution

We consider the stochastic differential equation

$$X_t = x_0 + \int_0^t f(s, X_s)ds + \int_{(0,t]} g(s, X_{s-})dN_s, \quad 0 \le t \le T, \tag{12}$$

or in the differential form

$$dX_t = f(t, X_t)dt + g(t, X_{t-})dN_t, \quad X_0 = x_0.$$

We assume that:

Assumption 2. *The functions $f, g : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are measurable and satisfy*

1. *For any $t \in [0,T]$, the maps $f(t,\cdot), g(t,\cdot)$ are of class C^1.*
2. $\sup_{t,x}(|\nabla_x f(t,x)| + |\nabla_x g(t,x)|) < +\infty$.
3. *For any $x \in \mathbb{R}^d$, the map $g(\cdot, x)$ is differentiable.*

Remark 5.1. *Since N_T admits moments of any order (See [15]), according to [17, Chapter V.3 Theorem 7 and Chapter V.4 Theorem 10], there exists a unique solution X to* (12) *such that* $\sup_{0\le t\le T} |X_t| \in L^2(\Omega)$.

We consider the deterministic flow Φ defined by the solution of the ordinary differential equation (ODE in short)

$$\Phi_{s,t}(x) = x + \int_s^t f(u, \Phi_{s,u}(x))du, \quad 0 \le s \le t \le T, \quad x \in \mathbb{R}^d.$$

Proposition 5.2. *On the set $\{N_T = 0\}$, we have*

$$X_t = \Phi_{0,t}(x_0), \quad 0 \le t \le T.$$

And, for any $n \in \mathbb{N}^*$, on the set $\{N_T = n\}$, we have

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0), \quad T_n \le t \le T,$$

where

$$\Psi(t,x) = x + g(t,x), \quad 0 \le t \le T, \quad x \in \mathbb{R}^d.$$

Proof. We proceed by induction. On the set $\{N_T = 0\}$ we have

$$X_t = x_0 + \int_0^t f(s, X_s)ds, \quad 0 \le t \le T.$$

Thus for all $t \in [0,T]$, $X_t = \Phi_{0,t}(x_0)$.
On the $\{N_T = 1\}$ we have, for any $t \in [0, T_1)$,

$$X_t = x_0 + \int_0^t f(s, X_s)ds.$$

Thus $X_t = \Phi_{0,t}(x_0)$. Then for any $t \in [T_1, T]$ we have

$$\begin{aligned} X_t &= X_{T_1-} + \int_{T_1}^t f(s, X_s)ds + \int_{[T_1,t]} g(s, X_{s-})dN_s \\ &= X_{T_1-} + \int_{T_1}^t f(s, X_s)ds + g(T_1, X_{T_1-}) \\ &= \Psi(T_1, X_{T_1-}) + \int_{T_1}^t f(s, X_s)ds. \end{aligned}$$

Hence

$$\begin{aligned} X_t &= \Phi_{T_1,t}(\Psi(T_1, X_{T_1-})) = \Phi_{T_1,t}(\Psi(T_1, \Phi_{0,T_1}(x_0))) \\ &= [\Phi_{T_1,t} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0). \end{aligned}$$

We assume that the equality is satisfied on $\{N_T = n\}$ for $n \in \mathbb{N}^*$:

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \Phi_{T_1,T_2}(\cdot) \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0), \quad T_n \leq t \leq T.$$

Let consider the set $\{N_T = n+1\}$. The process X satisfies, for any $t \in [T_{n+1}, T]$,

$$\begin{aligned} X_t &= X_{T_{n+1}-} + \int_{T_{n+1}}^t f(s, X_s)ds + \int_{[T_{n+1},t]} g(s, X_{s-})dN_s \\ &= \Psi(T_{n+1}, X_{T_{n+1}-}) + \int_{T_{n+1}}^t f(s, X_s)ds. \end{aligned}$$

Thus

$$X_T = \Phi_{T_{n+1},T}(\Psi(T_{n+1}, X_{T_{n+1}-})).$$

However, as we are on $\{N_T = n + 1\}$, we are also on $\{N_t = n\}$ for any $t \in [T_n, T_{n+1})$. According to the recurrence hypothesis,

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0).$$

Thus, by continuity of the map Φ with respect to t,

$$X_{T_{n+1}-} = [\Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0).$$

Therefore

$$\begin{aligned} X_T &= \Phi_{T_{n+1},T}(\Psi(T_{n+1}, X_{T_{n+1}-})) \\ &= \Phi_{T_{n+1},T}(\Psi(T_{n+1}, [\Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0))) \\ &= [\Phi_{T_{n+1},T} \circ \Psi(T_{n+1}, \cdot) \circ \Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0). \end{aligned}$$

The statement is proved. □

We continue this section with the same ideas as in [8]. As the following results are formal computations, they are proved in the same way.

Remark 5.3. *The process* $\nabla_x \Phi$ *satisfies, for any* $0 \le s \le t \le T$ *and* $x \in \mathbb{R}^d$,

$$\frac{\partial}{\partial t}\nabla_x \Phi_{s,t}(x) = \nabla_x f(t, \Phi_{s,t}(x))\nabla_x \Phi_{s,t}(x), \quad \nabla_x \Phi_{s,s}(x) = I_d.$$

Thus

$$\nabla_x \Phi_{s,t}(x) = \exp\left(\int_s^t \nabla_x f(u, \Phi_{s,u}(x))du\right), \quad 0 \le s \le t \le T.$$

Definition 5.4. *We define the process* K *as the derivative of the flow generated by* X*, solution of the SDE*

$$K_t = I_d + \int_0^t \nabla_x f(s, X_s)K_s ds + \int_{(0,t]} \nabla_x g(s, X_{s-})K_{s-}dN_s, \quad 0 \le t \le T.$$

From now we assume that:

Assumption 3. *For any* $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\det(I_d + \nabla_x g(t, x)) \neq 0$$

and $(I_d + \nabla_x g)^{-1}$ *is uniformly bounded.*

We now define the process $\widetilde{K}$ as the solution of

$$\begin{aligned}\widetilde{K}_t &= I_d - \int_0^t \widetilde{K}_s \nabla_x f(s, X_s)ds \\ &\quad - \int_{(0,t]} \widetilde{K}_s \nabla_x g(s, X_{s-})(I_d - \nabla_x g(s, X_{s-})(I_d + \nabla_x g(s, X_{s-}))^{-1})dN_s.\end{aligned}$$

Following [4, Proposition 8.7], we have:

Lemma 5.5. *Processes* K *and* $\widetilde{K}$ *satisfy*

$$K_t\widetilde{K}_t = I_d, \quad 0 \le t \le T.$$

Moreover:

$$K_{T_i} = (I_d+\nabla_x g(t, X_{T_i-}))K_{T_i-}, \quad \widetilde{K}_{T_i} = (I_d+\nabla_x g(T_i, X_{T_i-}))^{-1}\widetilde{K}_{T_i-}, \quad i \in \mathbb{N}^*.$$

Definition 5.6. *We define the process* $(K_t^s)_{0\le s\le t\le T}$ *by:*

$$K_t^s = K_t\widetilde{K}_s, \quad 0 \le s \le t \le T.$$

Similarly to [8, Proposition 6.4], we get the following result.

Proposition 5.7. *Let* $\varphi : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *defined by:*

$$\forall (t, x) \in [0, T]\times\mathbb{R}^d, \quad \varphi(t, x) = f(t, x+g(t, x))-(I_d+\nabla_x g(t, x))f(t, x)-\frac{\partial g}{\partial t}(t, x).$$

Then $X_T \in \mathbb{D}^{1,2}$ *and, for a.e.* $s \in [0, T]$,

$$D_s X_T = -\int_{(0,T]} K_T^t \varphi(t, X_{t-})\left(\frac{t}{T} - 1_{[0,t]}(s)\right)dN_t.$$

Moreover

$$\Gamma[X_T] = \int_{(0,T]}\int_{(0,T]} K_T^t \varphi(t, X_{t-})(\varphi(u, X_{u-}))^*(K_T^u)^*\left(u \wedge t - \frac{ut}{T}\right)dN_t dN_u.$$

Thanks to this expression of DX_T and $\Gamma[X_T]$, we can deduce results about the absolute continuity of the law of X_T using Corollary 4.6.

Corollary 5.8. *If we consider the event*

$$\mathcal{C} = \left\{ \det \left(\int_{(0,T]} \int_{(0,T]} K_T^t \varphi(t, X_{t-}) (\varphi(u, X_{u-}))^* (K_T^u)^* \left(u \wedge t - \frac{ut}{T} \right) dN_t dN_u \right) > 0 \right\}$$

then if $\mathbb{P}(\mathcal{C}) > 0$, the law of X_T conditionally to $\mathcal{C}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. It is a direct consequence of Proposition 5.7 and Corollary 4.6. □

Theorem 5.9. *In dimension $d = 1$, if for any $t \in [0, T]$ and $x \in \mathbb{R}$,*

$$\varphi(t, x) = f(t, x + g(t, x)) - f(t, x) - \frac{\partial g}{\partial x}(t, x) f(t, x) - \frac{\partial g}{\partial t}(t, x) \neq 0,$$

then, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. As $d = 1$ we have, according to Proposition 5.7,

$$D_s X_T = - \sum_{i=1}^{N_T} K_T^{T_i} \varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s) \right).$$

Let $\omega \in \Omega$ such that

$$\Gamma[X_T](\omega) = \int_0^T |D_s X_T(\omega)|^2 ds = 0,$$

then, for almost every $s \in [0, T], D_s X_T(\omega) = 0$. Thus, for almost every $s \in [T_{N_T(\omega)}, T]$, we get, writing T_i instead of $T_i(\omega)$,

$$\sum_{i=1}^{N_T(\omega)} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} = 0.$$

Then for almost every $s \in [T_{N_T(\omega)-1}, T_{N_T(\omega)}]$

$$\sum_{i=1}^{N_T(\omega)-1} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} + K_T^{T_{N_T(\omega)}} \varphi(T_{N_T(\omega)}, X_{T_{N_T(\omega)}-}) \left(\frac{T_{N_T(\omega)}}{T} - 1 \right) = 0.$$

Therefore, by subtracting the two equations, we get

$$\varphi(T_{N_T(\omega)}, X_{T_{N_T(\omega)}-}) = 0$$

then

$$\sum_{i=1}^{N_T(\omega)-1} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} = 0.$$

Thus, by considering $s \in [T_{N_T(\omega)-2}, T_{N_T(\omega)-1}]$ then $s \in [T_{N_T(\omega)-3}, T_{N_T(\omega)-2}]$, ..., we get, by successive iterations, for any $i \in \{1, \cdots, N_T(\omega)\}$,

$$\varphi(T_i, X_{T_i-}) = 0.$$

Therefore, by contrapositive, if $\varphi(t,x) \neq 0$ for any $t \in [0,T]$ and $x \in \mathbb{R}$ then $\Gamma[X_T] > 0$ on the set $\{N_T \geq 1\}$. Thus, according to Corollary 4.6, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. □

Example 5.10 (Linear with constant coefficients in dimension $d = 1$)**.** *We consider X the solution of the linear SDE*

$$dX_t = (aX_t + b)dt + (\alpha X_{t-} + \beta)dN_t, \; 0 \leq t \leq T, \quad X_0 = x_0, \tag{13}$$

where $x_0, a, b, \alpha, \beta \in \mathbb{R}$ satisfy

$$a\beta - \alpha b \neq 0.$$

In this case we have, for any $t \in [0,T]$ and $x \in \mathbb{R}$,

$$\varphi(t,x) = a(x + \alpha x + \beta) + b - ax - b - \alpha(ax + b) = a\beta - \alpha b \neq 0.$$

Then, according to Theorem 5.9, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Corollary 5.11. *We assume that $d = 1$ and the parameters f, g do not depend on $t \in [0,T]$. We consider the Wronskian of f and g:*

$$W(f,g) = g' \times f - f' \times g.$$

Thus if the function f is of class C^2 and

$$\forall x \in \mathbb{R}, \quad |W(f,g)(x)| > \frac{1}{2}\|f''\|_\infty \|g\|_\infty^2$$

then, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. As the parameters f and g do not depend on t, we have, for any $t \in [0,T]$ and $x \in \mathbb{R}$,

$$\varphi(t,x) = f(x + g(x)) - f(x) - g'(x)f(x).$$

Then, by Taylor expansion, there exists $y_x \in \mathbb{R}$ such that

$$\varphi(t,x) = g(x)f'(x) + \frac{1}{2}g(y_x)^2 f''(x) - g'(x)f(x) = \frac{1}{2}g(y_x)^2 f''(x) - W(f,g)(x).$$

Thus, if there exists $x \in \mathbb{R}$ such that $\varphi(t,x) = 0$ then, according to the assumption,

$$|W(f,g)(x)| = \frac{1}{2}g(y_x)^2|f''(x)| \leq \frac{1}{2}\|g\|_\infty^2 \|f''\|_\infty < |W(f,g)(x)|$$

which is absurd. Therefore $\varphi(t,x) \neq 0$ for any $t \in [0,T]$ and $x \in \mathbb{R}$. We conclude by using Theorem 5.9. □

Example 5.12. *We consider X the solution of the SDE in dimension $d = 1$*

$$dX_t = \cos(X_t)dt + \sin(X_{t-})dN_t,\ 0 \le t \le T, \quad X_0 = x_0, \tag{14}$$

where $x_0 \in \mathbb{R}$. In particular

$$\forall x \in \mathbb{R}, \quad |W(f,g)(x)| = 1 > \frac{1}{2} = \frac{1}{2}\|\cos''\|_\infty \|\sin\|_\infty^2.$$

Thus, according to Corollary 5.11, conditionally to $\{N_T \ge 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proposition 5.13. *If there exists $\ell \in \mathbb{N}$ such that for any $n \ge \ell, 0 \le t_1 < \cdots < t_n \le T$ and $x_1, \cdots, x_n \in \mathbb{R}^d$, the family $(K_T^{t_i}\varphi(t_i, x_i))_{1\le i\le n}$ spans $\mathbb{R}^d$ then, conditionally to $\{N_T \ge \ell\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.*

Proof. Let $\omega \in \{N_T \ge \ell\}$ such that $\Gamma[X_T](\omega)$ is non invertible. Then, as it is a nonnegative symetric matrix, there exists $u \in \mathbb{R}^d\backslash\{0\}$ such that

$$u^*\Gamma[X_T](\omega)u = \int_0^T (u^* D_s X_T(\omega))^2 ds = 0.$$

Then, according to Proposition 5.7,

$$0 = \int_0^T (u^* D_s X_T(\omega))^2 ds = \int_0^T \left(u^* \sum_{i=1}^{N_T(\omega)} K_T^{T_i}\varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right)\right)^2 ds.$$

Thus, for almost every $s \in [0,T]$,

$$u^* \sum_{i=1}^{N_T(\omega)} K_T^{T_i}\varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right) = 0.$$

We deduce, as in dimension $d = 1$, that, for any $i \in \{1, \dots, N_T(\omega)\}$,

$$u^* K_T^{T_i}\varphi(T_i, X_{T_i-}) = 0$$

which is absurd because $(K_T^{T_i}\varphi(T_i, X_{T_i-}))_{1\le i\le N_T(\omega)}$ spans $\mathbb{R}^d$. □

Remark 5.14. *Necessarily we have $\ell \ge d$.*

Example 5.15 (Linear with constant coefficients in dimension $d \in \mathbb{N}^*$)**.** *We consider X the solution of the linear SDE*

$$dX_t = (AX_t + b)dt + (MX_{t-} + \beta)dN_t,\ 0 \le t \le T, \quad X_0 = x_0, \tag{15}$$

where $x_0, b, \beta \in \mathbb{R}^d$ and $A, M \in \mathbb{R}^{d\times d}$ such that

$$\det(M + I_d) \ne 0, \quad AM = MA$$

and if there exists $\ell \in \mathbb{N}$ such that, for any $n \ge \ell$ and $0 \le t_1 < \cdots < t_n \le T$, the family

$$(\exp(A(T - t_i))(I_d + M)^{n-i}(A\beta - Mb))_{1\le i\le n}$$

spans $\mathbb{R}^d$. In this case we have:

- *The process K defined by Definition 5.4 satisfies*

$$K_t = I_d + \int_0^t AK_s ds + \int_{(0,t]} MK_{s-} dN_s, \quad 0 \le t \le T,$$

i.e.

$$dK_t = AK_s ds + MK_{s-} dN_s, \quad 0 \le t \le T, \quad K_0 = I_d,$$

i.e., for any $t \in [0,T]$, as $AM = MA$,

$$K_t = \exp(At) \prod_{0<s\le t} (I_d + M\Delta N_s) = \exp(At)(I_d + M)^{N_t}.$$

- *As $\det(I_d + M) \neq 0$ then we can consider the process $\widetilde{K} = K^{-1}$ defined just above Lemma 5.5:*

$$\widetilde{K}_t = I_d - \int_0^t A\widetilde{K}_s ds - M(I_d + M)^{-1} \int_{(0,t]} \widetilde{K}_{s-} dN_s, \quad 0 \le t \le T,$$

i.e.

$$d\widetilde{K}_t = -A\widetilde{K}_t dt - M(I_d + M)^{-1}\widetilde{K}_{t-} dN_t, \quad 0 \le t \le T, \quad \widetilde{K}_0 = 1,$$

i.e., for any $t \in [0,T]$, as $AM = MA$,

$$\begin{aligned} \widetilde{K}_t &= \exp(-At) \prod_{0<s\le t} \big(I_d - M(I_d + M)^{-1}\Delta N_s\big) \\ &= \exp(-At)(I_d - M(I_d + M)^{-1})^{N_t} \\ &= \exp(-At)(I_d + M)^{-N_t}. \end{aligned}$$

- *The process (K_t^s) is equal to, for any $0 \le s \le t \le T$,*

$$\begin{aligned} K_t^s &= K_t \widetilde{K}_s \\ &= \exp(A(t-s))(I_d + M)^{N_t - N_s}. \end{aligned}$$

- *According to Proposition 5.7, we consider the function $\varphi : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by, for any $t \in [0,T]$ and $x \in \mathbb{R}$,*

$$\begin{aligned} \varphi(t,x) &= A(x + Mx + \beta) + b - (I_d + M)(Ax + b) \\ &= A\beta - Mb. \end{aligned}$$

Therefore, according to the assumption about the spanning property, we get the following result by Proposition 5.13: conditionally to $\{N_T \ge \ell\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

This is the case for example when $A = I_d$, M is a diagonalizable matrix with d distinct eigenvalues (different to -1 to have $\det(M + I_d) \neq 0$): there exists $\lambda_1, \cdots, \lambda_d$ in $\mathbb{R}\backslash\{-1\}$ (distinct) and $P \in GL_d(\mathbb{R})$ such that

$$M = P \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_d \end{pmatrix} P^{-1},$$

and $v := [P^{-1}(\beta - Mb)]_j = [P^{-1}\beta - DP^{-1}b]_j \neq 0$ *for any* $j \in \{1, \cdots, d\}$*. Thus, for any* $n \geq \ell, 0 \leq t_1 < \cdots < t_n \leq T$ *and* $i \in \{1, \cdots, n\}$,

$$\begin{aligned}
&\exp(A(T-t_i))(I_d+M)^{n-i}(A\beta - Mb) \\
&= e^{T-t_i} P \begin{pmatrix} (1+\lambda_1)^{n-i} & & (0) \\ & \ddots & \\ (0) & & (1+\lambda_d)^{n-i} \end{pmatrix} P^{-1}(\beta - Mb).
\end{aligned}$$

Therefore the family $(\exp(A(T-t_i))(I_d+M)^{n-i}(A\beta - Mb))_{1\leq i\leq n}$ *spans* $\mathbb{R}^d$ *if and only if the family*

$$\left(\begin{pmatrix} (1+\lambda_1)^{n-1}v_1 \\ \vdots \\ (1+\lambda_d)^{n-1}v_d \end{pmatrix}, \cdots, \begin{pmatrix} (1+\lambda_1)v_1 \\ \vdots \\ (1+\lambda_d)v_d \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \right)$$

spans $\mathbb{R}^d$*. The determinant of the last d vectors of this family is related to a Vandermonde determinant:*

$$v_1 \cdots v_d \prod_{1\leq i<j\leq d} (\lambda_j - \lambda_i)$$

which is not null. Therefore, conditionally to $\{N_T \geq \ell\}$*, the law of* X_T *admits an absolutely continuous law with respect to the Lebesgue measure on* $\mathbb{R}^d$*.*

5.2 Application to Greek computation

We consider an asset price whose dynamics is given by

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{N}_t, \quad S_0 = x_0, \tag{16}$$

where N is a Hawkes process with conditional intensity λ^*, r the interest rate, σ the volatility and x_0 the initial wealth. In other words we have

$$dS_t = (r - \sigma\lambda^*(t))S_t dt + \sigma S_{t-} dN_t.$$

Thus, if we write $\alpha_t = r - \sigma\lambda^*(t)$ then the dynamic is equivalent to

$$dS_t = \alpha_t S_t dt + \sigma S_{t-} dN_t, \quad S_0 = x_0. \tag{17}$$

We also consider an European option

$$C = \mathbb{E}[f(S_T)]$$

with f a function which can be not continuous as $f = 1_{[K,+\infty[}$ for example for a binary European option, and we interest to compute Greeks as

$$\Delta = \frac{\partial C}{\partial x_0}, \quad \Gamma = \frac{\partial^2 C}{\partial x_0^2}, \quad \rho = \frac{\partial C}{\partial r}, \quad \nu = \frac{\partial C}{\partial \sigma}.$$

In the sequel x denotes a real number in an interval $]a, b[$ which can be equal to x_0, r or σ.

We consider a class $\mathcal{L}$ of real functions f of the form

$$f(y) = \sum_{i=1}^{n} \Phi_i(y) 1_{A_i}(y), \quad y \in \mathbb{R}, \tag{18}$$

where $n \in \mathbb{N}$, Φ_i is continuous and bounded and A_i is an interval with endpoints in $\mathbb{T} = \mathbb{T}_0 \cup \{-\infty, +\infty\}$ with the set $\mathbb{T}_0 \subset \mathbb{R}$ defined by:

$$\mathbb{T}_0 = \left\{ y \in \mathbb{R} \quad \lim_{n \to +\infty} \sup_{a<x<b} \mathbb{P}\left(F^x \in \left] y - \frac{1}{n}, y + \frac{1}{n} \right[\right) = 0 \right\}.$$

Proposition 5.16. *Let $]a,b[$ be an interval of $\mathbb{R}$. Let $(F^x)_{a<x<b}$ and $(G^x)_{a<x<b}$ be two families of random variables such that the maps $x \in\ ]a,b[\ \longmapsto F^x \in \mathbb{D}^{1,2}$ and $x \in\ ]a,b[\ \longmapsto \mathbb{D}^{1,2}$ are continuously differentiable. Let $m \in \mathcal{H}$ such that for any $x \in\ ]a,b[$ on $\left\{ \frac{\partial F^x}{\partial x} \neq 0 \right\}$*

$$D_m F^x \neq 0$$

and such that $mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x}$ is continuous in x in Dom(δ). Thus for any $f \in \mathcal{L}$ the map $x \longmapsto \mathbb{E}[f(F^x)]$ is continuous differentiable and

$$\frac{\partial}{\partial x} \mathbb{E}[G^x f(F^x)] = \mathbb{E}\left[f(S_T^{x_0}) \delta \left(G^x m \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} \right) \right] + \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x) \right].$$

Proof. We follow the proof of [8, Proposition 7.2]. First if $f \in C_b^1(\mathbb{R})$ then, as the maps $x \longmapsto F^x$ and $x \longmapsto G^x$ are differentiable, the map $x \longmapsto G^x f(F^x)$ is differentiable and, for any $x \in\ ]a,b[$,

$$\frac{\partial}{\partial x}(G^x f(F^x)) \quad = \quad \frac{\partial G^x}{\partial x} f(F^x) + \frac{\partial F^x}{\partial x} f'(F^x) G^x.$$

Then, as $f \in C_b^1(\mathbb{R})$, $D_m f(F^x) = f'(F_x) D_m F^x$, and, as $D_m F^x \neq 0$ on $\left\{ \frac{\partial F^x}{\partial x} \neq 0 \right\}$,

$$\frac{\partial}{\partial x}(G^x f(F^x)) \quad = \quad \frac{\partial G^x}{\partial x} f(F^x) + \frac{\partial F^x}{\partial x} \frac{D_m f(F^x)}{D_m F^x} G^x.$$

Thus, according to Remark 3.5 and $mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} \in \mathrm{Dom}(\delta)$,

$$\begin{aligned} \mathbb{E}\left[\frac{\partial}{\partial x}(G^x f(F^x)) \right] &= \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x) \right] + \mathbb{E}\left[G^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} D_m f(F^x) \right] \\ &= \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x) \right] + \mathbb{E}\left[\delta \left(mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} \right) f(F^x) \right]. \end{aligned}$$

Finally, as the function $x \longmapsto \frac{\partial}{\partial x}(G^x f(F^x))$ is continuous, we have

$$\frac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)] \;=\; \mathbb{E}\left[\frac{\partial}{\partial x}(G^x f(F^x))\right]$$

and the result follows in that case. For the general case ($f \in \mathcal{L}$) we conclude as in [8, Proposition 7.2], using an approximation argument. □

As in [10], if we can apply the Proposition 5.16 with

$$F^x = S_T^x, \quad G^x = 1_{\{N_T>0\}}, \quad x \in \;]a,b[,$$

then we can compute the Delta conditionally to $\{N_T \geq 1\}$

$$\frac{\partial}{\partial x_0}\mathbb{E}\left[1_{\{N_T>0\}} f\left(S_T^{x_0}\right)\right] = \mathbb{E}\left[f(S_T^{x_0})\delta\left(m 1_{\{N_T>0\}}\frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right].$$

The solution of (16)

$$dS_t = S_t \alpha_t dt + \sigma S_{t-} dN_t, \quad S_0 = x_0,$$

is given by

$$S_t^{x_0} = x_0 \exp\left(\int_0^t \alpha_s ds\right)(1+\sigma)^{N_t} = x_0 \exp\left(rt - \sigma\int_0^t \lambda^*(s)ds\right)(1+\sigma)^{N_t}, \quad 0 \leq t \leq T.$$

In particular

$$S_T^{x_0} = x_0 \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T}$$

with

$$\begin{aligned}\Lambda_T &= \int_0^T \lambda^*(s)ds = \lambda T + \sum_{i=1}^{N_T}\int_0^T \mu(s-T_i)1_{\{T_i \leq s\}}ds \\ &= \lambda T + \sum_{i=1}^{N_T}\int_{T_i}^T \mu(s-T_i)ds = \lambda T + \sum_{i=1}^{N_T}\int_0^{T-T_i}\mu(s)ds \\ &= \lambda T + \sum_{i=1}^{N_T}\widehat{\mu}(T-T_i).\end{aligned}$$

Thus the map $x_0 \longmapsto S_T^{x_0}$ is continuously differentiable and

$$\frac{\partial S_T^{x_0}}{\partial x_0} = \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T} = \frac{S_T^{x_0}}{x_0}.$$

To compute $DS_T^{x_0}$ we cannot use the results of Section 5.1 because the paramater of the SDE $f : (t,x) \longmapsto \alpha_t x$ is not deterministic. However we can directly compute $DS_T^{x_0}$. Indeed we have

$$\begin{aligned}\mathcal{T}_\varepsilon \Lambda_T &= \Lambda_T \circ \mathcal{T}_\varepsilon \\ &= \lambda T + \sum_{i=1}^{N_T \circ \mathcal{T}_\varepsilon}\widehat{\mu}(T - T_i \circ \mathcal{T}_\varepsilon)\end{aligned}$$

where $N_T \circ \mathcal{T}_\varepsilon = N_T$ because $\tau_\varepsilon(T) = T$. Therefore

$$\begin{aligned}\frac{\mathcal{T}_\varepsilon \Lambda_T - \Lambda_T}{\varepsilon} &= \sum_{i=1}^{N_T} \frac{\widehat{\mu}(T - T_i \circ \mathcal{T}_\varepsilon) - \widehat{\mu}(T - T_i)}{\varepsilon} \\ &\xrightarrow[\varepsilon \to 0]{L^2(\Omega)} \sum_{i=1}^{N_T} (\widehat{\mu}(T - T_i))' D_m T_i = \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i).\end{aligned}$$

Thus $\Lambda_T \in \mathbb{D}_m^0$ and

$$D_m \Lambda_T = \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i) = \int_{(0,T]} \mu(T - t) \widehat{m}(t) dN_t.$$

Since $D_m N_T = 0$ (see Remark 3.9), we get, by chain rule, $S_T^{x_0} \in \mathbb{D}_m^{1,2}$ and

$$D_m S_T^{x_0} = -\sigma S_T^{x_0} D_m \Lambda_T = -\sigma S_T^{x_0} \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i) = -\sigma S_T^{x_0} \int_{(0,T]} \mu(T - t) \widehat{m}(t) dN_t.$$

Thus we have to choose a function $m \in \mathcal{H}$ such that, for any $t \in [0,T]$, $\widehat{m}(t) = 0$ if and only if $t \in \{0,T\}$. In this case we get

$$1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} = -\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i)} = -\frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T - t) \widehat{m}(t) dN_t}.$$

Finally, according to Remark 3.5, $m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} \in \mathrm{Dom}(\delta)$ because we have $m \in \mathcal{H}$ and $1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} \in \mathbb{D}_m^{1,2}$, and

$$\begin{aligned}&\delta\left(m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right) \\ &= -\delta(m) \frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i)} + D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T} \mu(T - T_i) \widehat{m}(T_i)}\right) \\ &= -\delta(m) \frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T - t) \widehat{m}(t) dN_t} + D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T - t) \widehat{m}(t) dN_t}\right)\end{aligned}$$

with, on $\{N_T > 0\}$,

$$
\begin{aligned}
& D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)}\right) \\
&= -\frac{\sigma x_0 \sum_{i=1}^{N_T} D_m(\mu(T-T_i)\widehat{m}(T_i))}{\sigma^2 x_0^2 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} \\
&= -\frac{\sum_{i=1}^{N_T}[\widehat{m}(T_i)D_m(\mu(T-T_i)) + \mu(T-T_i)D_m\widehat{m}(T_i)]}{\sigma x_0 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} \\
&= \frac{\sum_{i=1}^{N_T} \widehat{m}(T_i)\mu'(T-T_i)D_m T_i}{\sigma x_0 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} - \frac{\sum_{i=1}^{N_T} \mu(T-T_i)m(T_i)D_m T_i}{\sigma x_0 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} \\
&= -\frac{\sum_{i=1}^{N_T} \mu'(T-T_i)\widehat{m}(T_i)^2}{\sigma x_0 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} + \frac{\sum_{i=1}^{N_T} \mu(T-T_i)m(T_i)\widehat{m}(T_i)}{\sigma x_0 \left(\sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\right)^2} \\
&= -\frac{\int_{(0,T]} \mu'(T-s)\widehat{m}(s)^2 dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s\right)^2} + \frac{\int_{(0,T]} \mu(T-s)m(s)\widehat{m}(s)dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s\right)^2},
\end{aligned}
$$

and, using Remark 3.5,

$$
\begin{aligned}
\delta(m) &= \int_{(0,T]} (\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s \\
&= \sum_{j=1}^{N_T} (\psi(m,T_j) + \widehat{m}(T-T_j) + m(T_j)) \\
&= \sum_{j=1}^{N_T} \left(\frac{1}{\lambda^*(T_j)} \int_{(0,T_j)} (\widehat{m}(T_j) - \widehat{m}(t))\mu'(T_j - t)dN_t + \widehat{m}(T-T_j) + m(T_j)\right) \\
&= \sum_{j=1}^{N_T} \left(\frac{\sum_{i=1}^{j-1}(\widehat{m}(T_j) - \widehat{m}(T_i))\mu'(T_j - T_i)}{\lambda + \sum_{i=1}^{j-1} \mu(T_j - T_i)} + \widehat{m}(T-T_j) + m(T_j)\right).
\end{aligned}
$$

Thus we deduce the following expression of the Delta:

Proposition 5.17. *We have*

$$
\begin{aligned}
& \frac{\partial}{\partial x_0}\mathbb{E}[1_{\{N_T \geq 1\}} f(S_T)] \\
&= \mathbb{E}\left[f(S_T^{x_0})\delta\left(m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right] \\
&= -\mathbb{E}\left[\frac{f(S_T^{x_0})\delta(m)1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T-t)\widehat{m}(t)dN_t}\right] - \mathbb{E}\left[\frac{f(S_T^{x_0}) \int_{(0,T]} \mu'(T-s)\widehat{m}(s)^2 dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s\right)^2} 1_{\{N_T>0\}}\right] \\
&\quad + \mathbb{E}\left[\frac{f(S_T^{x_0}) \int_{(0,T]} \mu(T-s)m(s)\widehat{m}(s)dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s\right)^2} 1_{\{N_T>0\}}\right].
\end{aligned}
$$

Remark 5.18. *Any term in the expression of* Δ *can be written from the Hawkes process* N*, the jump instants* $(T_i)_{i\in\mathbb{N}^*}$ *and the parameters* $T, \lambda, \mu, \widehat{\mu}, \mu', m, \widehat{m}$ *and* f*:*

$$\begin{aligned}
S_T^{x_0} &= x_0 \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T}, \\
\Lambda_T &= \lambda T + \sum_{i=1}^{N_T} \widehat{\mu}(T - T_i), \\
\delta(m) &= \sum_{j=1}^{N_T}\left(\frac{\sum_{i=1}^{j-1}(\widehat{m}(T_j) - \widehat{m}(T_i))\mu'(T_j - T_i)}{\lambda + \sum_{i=1}^{j-1}\mu(T_j - T_i)} + \widehat{m}(T - T_j) + m(T_j)\right), \\
\int_{(0,T]} \mu(T-t)\widehat{m}(t)dN_t &= \sum_{i=1}^{N_T} \mu(T - T_i)\widehat{m}(T_i) \\
\int_{(0,T]} \mu'(T-t)\widehat{m}(t)^2 dN_t &= \sum_{i=1}^{N_T} \mu'(T - T_i)\widehat{m}(T_i)^2 \\
\int_{(0,T]} \mu(T-t)m(t)\widehat{m}(t)dN_t &= \sum_{i=1}^{N_T} \mu(T - T_i)m(T_i)\widehat{m}(T_i).
\end{aligned}$$

In other words, if we simulate a sample of Hawkes process then we can approach Δ *conditionally to* $\{N_T > 0\}$*.*

Remark 5.19. *On* $\{N_T = 0\}$*, the process* S^{x_0} *is deterministic and we have to know the derivative of the function* f *to compute* Δ*.*

Remark 5.20. *For the other Greeks we can notice that*

$$\begin{aligned}
\frac{\partial^2 S_T^{x_0}}{\partial x_0^2} &= 0, \\
\frac{\partial S_T^r}{\partial r} &= T S_T^{x_0} \\
\frac{\partial S_T^\sigma}{\partial \sigma} &= \left(-\Lambda_T + \frac{N_T}{1+\sigma}\right) S_T^{x_0}.
\end{aligned}$$

Then we can deduce similar expressions of the other Greeks conditionally to $\{N_T > 0\}$*. For* $\Gamma = \dfrac{\partial^2 C}{\partial x_0^2}$ *we can start by writing*

$$\begin{aligned}
G^{x_0} &= m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}, \\
\frac{\partial^2}{\partial x_0^2}\mathbb{E}[1_{\{N_T>0\}} f(S_T^{x_0})] &= \frac{\partial}{\partial x_0}\mathbb{E}\left[f(S_T^{x_0})\delta\left(G^{x_0}\right)\right] \\
&= \mathbb{E}\left[f(S_T^{x_0})\delta\left(G^x m \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right] + \mathbb{E}\left[f(S_T^{x_0})\frac{\partial}{\partial x_0}\delta\left(G^{x_0}\right)\right]
\end{aligned}$$

where we apply two times Proposition 5.16 with different processes G*.*

References

[1] E. Bacry, I. Mastromatteo, and J.-F. Muzy. Hawkes processes in finance. *Market Microstructure and Liquidity*, 01(01):1550005, 2015.

[2] K. Bichteler, J.-B. Gravereaux, and J. Jacod. *Malliavin calculus for processes with jumps*, volume 2 of *Stochastics Monographs*. Gordon and Breach Science Publishers, New York, 1987.

[3] N. Bouleau and L. Denis. Energy image density property and the lent particle method for poisson measures. *Journal of Functional Analysis*, 257:1144–1174, 08 2009.

[4] N. Bouleau and L. Denis. *Dirichlet forms methods for Poisson point measures and Lévy processes*, volume 76 of *Probability Theory and Stochastic Modelling*. Springer, Cham, 2015. With emphasis on the creation-annihilation techniques.

[5] N. Bouleau and F. Hirsch. *Dirichlet Forms and Analysis on Wiener Space*. De Gruyter, Berlin, New York, 1991.

[6] E. A. Carlen and E. Pardoux. *Differential Calculus and Integration by Parts on Poisson Space*, pages 63–73. Springer Netherlands, Dordrecht, 1990.

[7] D. J. Daley and D. Vere-Jones. *An introduction to the theory of point processes. Vol. II*. Probability and its Applications (New York). Springer, New York, second edition, 2008. General theory and structure.

[8] L. Denis and T. M. Nguyen. Malliavin calculus for markov chains using perturbations of time. *Stochastics*, 88(6):813–840, 2016.

[9] G. Di Nunno, B. Ø ksendal, and F. Proske. *Malliavin calculus for Lévy processes with applications to finance*. Universitext. Springer-Verlag, Berlin, 2009.

[10] Y. El-Khatib and N. Privault. Computations of greeks in a market with jumps via the malliavin calculus. *Finance and Stochastics*, 8, 09 2003.

[11] C. Hillairet, L. Huang, M. Khabou, and A. Réveillac. The Malliavin-Stein method for Hawkes functionals. *ALEA : Latin American Journal of Probability and Mathematical Statistics*, 19(2):1293, 2022.

[12] C. Hillairet, A. Réveillac, and M. Rosenbaum. An expansion formula for Hawkes processes and application to cyber-insurance derivatives *. *Stochastic Processes and their Applications*, 160:89–119, June 2023.

[13] P. Laub, Y. Lee, and T. Taimre. *The Elements of Hawkes Processes*. 01 2021.

[14] P. J. Laub, T. Taimre, and P. K. Pollett. Hawkes processes, 2015.

[15] T. Leblanc. Exponential moments for Hawkes processes under minimal assumptions. *Electron. Commun. Probab.*, 29:Paper No. 55, 11, 2024.

[16] D. Nualart. *The Malliavin calculus and related topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[17] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. Stochastic Modelling and Applied Probability.