Lévy's Construction of Brownian Motion.

S'expose avec brio en vingt minutes en anglais. En français et en quinze minutes il faut parler un peu plus vite et passer rapidement sur le début : ne pas réécrire la définition et commencer tout de suite par un dessin pour expliquer la construction (voir l'annexe de la leçon 262).

Definition (Brownian Motion)**.** A standard Brownian Motion on $[0,1]$ is a stochastic process $\{B_t,\ 0 \le t \le 1\}$ with $B_0 = 0$ and which satisfies the following properties :

(i) For all $t, s \in [0,1]$ such that $t+s \in [0,1]$, $B_{t+s} - B_s$ is $\mathcal{N}(0,t)$

(ii) If $0 \le t_1 < t_2 < \ldots < t_n \le 1$, then the increments B_{t_1}, $B_{t_2} - B_{t_1}$, $\ldots$, $B_{t_n} - B_{t_{n-1}}$ are independent.

Theorem (Lévy)**.** *It exists.*

Proof. The idea is to construct the Brownian Motion on the set of dyadic numbers :

$$\mathcal{D} = \bigcup_{n \in \mathbf{N}} \mathcal{D}_n \quad \text{where} \quad \mathcal{D}_n = \left\{ \frac{k}{2^n},\ k = 0, 1, \ldots, 2^n \right\}$$

and to interpolate in between.

For $n \in \mathbf{N}$, let us consider the Schauder tent functions f_{nk} defined by :

$$f_{nk}(t) = \begin{cases} 2^{-(n+2)/2} & \text{if } t = (k+\frac{1}{2})2^{-n} \\ 0 & \text{if } t \notin (k2^{-n}, (k+1)2^{-n}) \\ \text{linear} & \text{in between} \end{cases}$$

Suppose that we can construct a sequence $(X, X_{nk})_{n,k}$ of i.i.d $\mathcal{N}(0,1)$ random variables. We define for $n \in \mathbf{N}$:

$$F_n(t) = \sum_{k=0}^{2^n - 1} X_{nk} f_{nk}(t).$$

Step 1. *Almost surely the series*

$$B_t = tX + \sum_{n=0}^{+\infty} F_n(t)$$

converges uniformly for $0 \le t \le 1$.

Let us define for $m \le 0$ the partial sums

$$B_t^{(m)} = tX + \sum_{n=0}^{m} F_n(t),\ m \ge 0$$

it is sufficient to show that with probability one $(B_t^{(m)})_m$ converges uniformly on $[0,1]$. Note that since the $f_{n,k}$ have disjoint supports :

$$\sup_{t \in [0,1]} |B_t^{(m)} - B_t^{(m-1)}| = \|F_m\|_\infty \le 2^{-(m+2)/2} \max\{|X_{mk}|,\ k = 0, \ldots, 2^m - 1\}.$$

The normal distribution has the following property which is an easy consequence of a change of variables [1].

Lemma. *Let Y be a $\mathcal{N}(0,\sigma^2)$ random variable. Then for $\lambda > 0$:*

$$\mathbf{P}(|Y| \geq \lambda) \leq \sqrt{\frac{2}{\pi}}\frac{\sigma}{\lambda}e^{-\frac{\lambda^2}{2\sigma^2}}.$$

As a consequence,

$$\begin{aligned}\mathbf{P}\left(2^{-\frac{m+2}{2}}\max\{|X_{mk}|,\ k=0,\dots,2^m-1\} > \frac{1}{m^2}\right) &\leq 2^m\mathbf{P}\left(|X_{m1}| \geq \frac{2^{-(m+2)/2}}{m^2}\right)\\ &\leq \frac{1}{\sqrt{2\pi}}2^{\frac{m}{2}-1}m^2e^{-\frac{2^{m+1}}{m^4}}.\end{aligned}$$

Since the right-hand side is summable, by the Borel-Cantelli lemma :

$$\sup_{t\in[0,1]}|B_t^{(m+1)} - B_t^{(m)}| \leq \frac{1}{m^2} \text{ for large enough } m \text{ a.s.}$$

and the uniform convergence follows. Note that since the functions F_n are continuous, so is $t \mapsto B_t$ almost surely.

Step 2. *$\{B_t,\ 0 \leq t \leq 1\}$ is a standard Brownian motion on $[0,1]$.*

We will prove by induction on $m \geq 0$:

$(i)_m$ For all $t,s \in \mathcal{D}_m$ such that $t+s \in [0,1]$, $B_{t+s} - B_t$ is $\mathcal{N}(0,t)$

$(ii)_m$ If $0 \leq t_1 < t_2 < \ldots < t_n \leq 1$ with $t_i \in \mathcal{D}_m$, then the increments B_{t_1}, $B_{t_2} - B_{t_1}$, ..., $B_{t_n} - B_{t_{n-1}}$ are independent.

We will then deduce (i) and (ii) on the set $\mathcal{D}$ and then on $[0,1]$ by density of the dyadic numbers.

Clearly $(i)_m$ and $(ii)_m$ are true for $m=0$ since X is a $\mathcal{N}(0,1)$ random variable and $\mathcal{D}_0 = \{0,1\}$. Let us define for $m \geq 1$ and $k \in \{0,\dots 2^m - 1\}$, the increment :

$$\Delta_{mk} = B_{(k+1)2^{-m}} - B_{k2^{-m}} = B^{(m-1)}_{(k+1)2^{-m}} - B^{(m-1)}_{k2^{-m}}.$$

Fix $m \geq 1$ and suppose that $(i)_m$ and $(ii)_m$ are true. We are going to prove that the increments $\Delta_{m+1,k}$ are gaussian $\mathcal{N}(0,2^{-(m+1)})$ and independent. In fact, we are going to prove a little bit more : for a given $m \in \mathbf{N}$, the vector $(\Delta_{mk})_{k\in\{0,\dots,2^m-1\}}$ is gaussian with mean zero and covariance matrix $2^{-(m+1)}I_{2^m}$.

- First, we can suppose without loss of generality (we'll see why later) that $d := k2^{-(m+1)} \notin \mathcal{D}_m$. Then :

$$\Delta_{m+1,k} = \frac{1}{2}\Delta - 2^{-(m+2)/2}X_{mk'}$$

1. Write that $\mathbf{P}(|Y| \geq \lambda) \leq \frac{2}{\sigma\sqrt{2\pi}}\int_{y\geq\lambda} e^{-\frac{y^2}{2\sigma^2}}\frac{y}{\lambda}dy.$

where (figure 1)

$$\frac{k}{2^{m+1}} = \left(k' + \frac{1}{2}\right) 2^{-m} \text{ and } \Delta = B\left(d + 2^{-(m+1)}\right) - B\left(d - 2^{-(m+1)}\right) \sim \mathcal{N}(0, 2^{-m}).$$

Since $d \pm 2^{-(m+1)} \in \mathcal{D}_{m-1}$, Δ and $X_{mk'}$ are independent and from the induction hypothesis we conclude that Δ_{mk} is gaussian with

$$\mathbf{E}(\Delta_{m+1,k}) = 0 \text{ and } \operatorname{Var}(\Delta_{m+1,k}) = \frac{1}{4}\operatorname{Var}(\Delta) + \frac{1}{2^{m+2}} = 2^{-(m+1)}.$$

The same is true if $d \in \mathcal{D}_m$ and it is easy to see that the vector (Δ_{mk}) is gaussian.

- Two successive increments $\Delta_{m+1,k}$ and $\Delta_{m+1,k-1}$ are independent since

$$\begin{aligned} \operatorname{cov}(\Delta_{m+1,k}, \Delta_{m+1,k-1}) &= \mathbf{E}\left((\frac{1}{2}\Delta - 2^{-(m+2)/2} X_{mk'})(\frac{1}{2}\Delta + 2^{-(m+2)/2} X_{mk'}) \right) \\ &= \frac{1}{4} 2^{-m} - 2^{-(m+2)} = 0 \end{aligned}$$

since Δ and $X_{mk'}$ are independent. The same is true if $d \in \mathcal{D}_m$. Two any increments over disjoint intervals are thus independent by summing independent increments over intervals of the form $(k2^{-(m+1)}, (k+1)2^{-(m+1)})$.

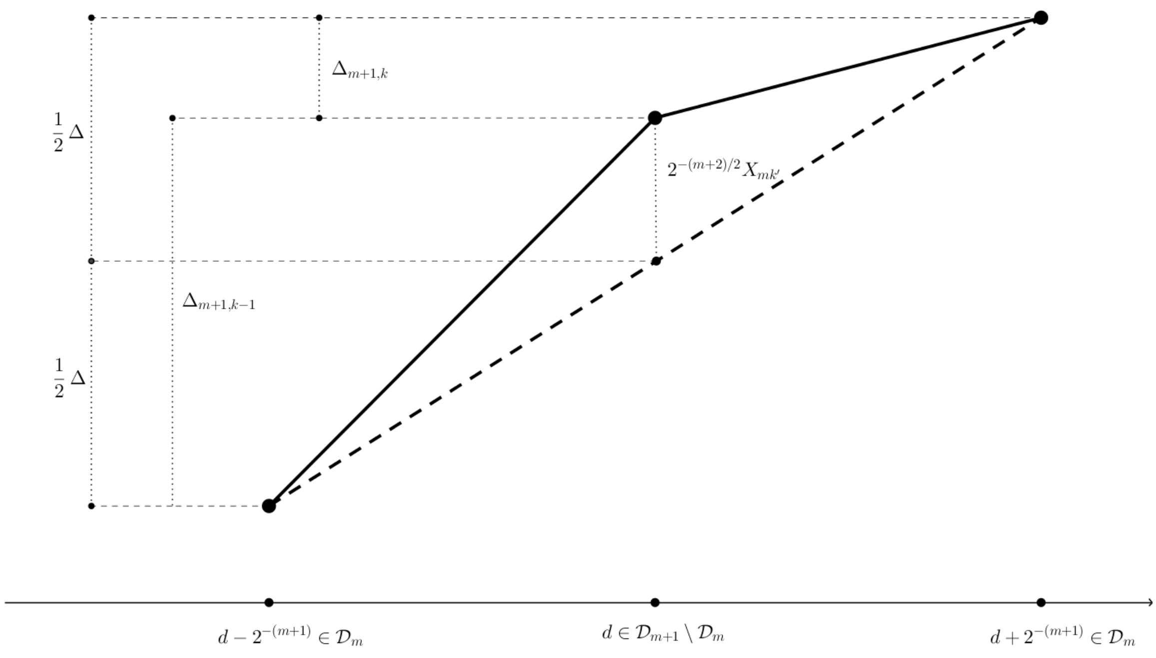

FIGURE 1: Detail around d

Since the set of dyadic numbers is dense in $[0, 1]$ the conclusion follows by writing any vector of increments as the limit of gaussian vectors with mean zero and convergent covariance matrices (use the characteristic functions). $\square$

Reference. J. B. Walsh, *Knowing the Odds : an Introduction to Probability*

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