Standing waves of nonlinear Schrödinger equations

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1 Introduction

When intense light propagates along an optical fiber or through diverse materials, nonlinear effects can appear, namely the refraction index becomes dependent on the light intensity [1]. This can be observed with light-emitting diodes (LED) or small lasers such as those used to read compact disks.

The propagation of a light wave in presence of nonlinear effects can be described by the nonlinear Schrödinger equation

$$i\partial_t\psi(\mathbf{r},t) = -\Delta\psi(\mathbf{r},t) + g(|\psi|^2)\psi(\mathbf{r},t)$$

where g is a real-valued function characterizing the nonlinearity of the interaction between light and matter. ψ is the wave function and is related to the slow variations of the electromagnetic field of the wave propagating at a given frequency. The term Δ arises from the diffraction or the dispersion of the wave ; the nonlinearity is due to the refraction index n depending on light intensity $I = |\psi|^2$. When n is of the form $n = n_0 + \alpha I$, the propagation is described by the famous cubic Schrödinger equation

$$i\partial_t\psi(\mathbf{r},t) = -\Delta\psi(\mathbf{r},t) - |\psi(\mathbf{r},t)|^2\psi(\mathbf{r},t).$$

We are interested in a special class of solutions, called standing waves, of the form

$$\psi(\mathbf{r},t) = e^{i\lambda^2 t}\Phi(\mathbf{r})$$

for some $\lambda \in \mathbb{R}$, where the spatial profile is time-independent. The function Φ satisfies

$$-\Delta\Phi + \lambda^2\Phi - |\Phi|^2\Phi = 0.$$

If Φ is a positive, radial, smooth and exponentially decaying solution of this equation, it is said to be a *ground state*.

Our focus in this paper will be to discuss the existence of a ground state solution of

$$-\Delta u + \lambda^2 u - |u|^{2\sigma}u = 0$$

where $\lambda > 0$, $\sigma > 0$ and its stability.

2 Existence of a ground state

Our aim in this section is to prove this existence result:

Theorem 2.1

Suppose $d \geq 3$, $\lambda^2 > 0$ and $0 < \sigma < \dfrac{2}{d-2}$. Then,

$$-\Delta u + \lambda^2 u - |u|^{2\sigma} u = 0 \tag{2.1}$$

has a positive, spherically symmetric solution $u \in \mathcal{C}^2(\mathbb{R}^d)$. In addition, u and its derivatives up to order 2 have an exponential decay at infinity. This solution minimizes the action

$$\mathcal{S}(u) = \frac{1}{2}T(u) - V(u)$$

among all $H^1(\mathbb{R}^d)$-solutions of (2.1), when

$$T(u) = \int_{\mathbb{R}^d} |\nabla u|^2 dx$$

and

$$V(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2}|u|^{2\sigma+2} - \frac{\lambda^2}{2}|u|^2 \right) dx.$$

Remark. Here and throughout the report, by solution we mean nontrivial solution.

Define

$$g : s \in \mathbb{R} \mapsto |s|^{2\sigma}s - \lambda^2 s \qquad \text{and} \qquad G : s \in \mathbb{R} \mapsto \frac{1}{2\sigma+2}|s|^{2\sigma+2} - \frac{\lambda^2}{2}s^2$$

its primitive such that $G(0) = 0$.

Take $d \geq 3$ and $0 < \sigma < \dfrac{2}{d-2}$. Let us start by deriving a variational formulation for (2.1). Suppose $u \in H^2(\mathbb{R}^d)$ satisfies (2.1). Then, multiplying by $v \in H^1(\mathbb{R}^d)$ and integrating over $\mathbb{R}^d$ yields

$$\int_{\mathbb{R}^d} \left(\nabla u \cdot \nabla v + \lambda^2 uv - |u|^{2\sigma} uv\right) \mathrm{d}x = 0,$$

which is meaningful since $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^d)$ by Sobolev's embedding theorem (see theorem A.1 and its corollary in Appendix A). Indeed, by Hölder's inequality,

$$\int_{\mathbb{R}^d} \left||u|^{2\sigma} uv\right| \mathrm{d}x \leq \left\||u|^{2\sigma+1}\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)} \|v\|_{L^{2\sigma+2}(\mathbb{R}^d)} = \|u\|^{2\sigma+1}_{L^{2\sigma+2}(\mathbb{R}^d)} \|v\|_{L^{2\sigma+2}(\mathbb{R}^d)} < \infty.$$

Therefore, we'll say that $u \in H^1(\mathbb{R}^d)$ is a solution of (2.1) if either

$$\Delta u - \lambda^2 u + |u|^{2\sigma} u = 0 \qquad \text{in } \mathscr{D}'(\mathbb{R}^d)$$

or

$$\forall v \in H^1(\mathbb{R}^d), \qquad \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, \mathrm{d}x = \int_{\mathbb{R}^d} \left(|u|^{2\sigma} uv - \lambda^2 uv\right) \mathrm{d}x. \tag{2.2}$$

2.1 A necessary condition

Lemma 2.2

For any $d \geq 1$, $\sigma > 0$, any solution u of (2.1) in $H^1(\mathbb{R}^d) \cap H^2_{loc}(\mathbb{R}^d)$ such that $G(u) \in L^1(\mathbb{R}^d)$ satisfies Pohozaev's identity

$$\frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla u|^2 dx = d \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2} |u|^{2\sigma+2} - \frac{\lambda^2}{2} u^2 \right) dx.$$

$\triangleright$ We follow the proof given by O. Kavian [12]. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$ and

$$\chi(x) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| \geq 2 \end{cases}$$

and for $n \in \mathbb{N}^*$, let $\chi_n : x \in \mathbb{R}^d \mapsto \chi\left(\frac{x}{n}\right)$.

Let $u \in H^1(\mathbb{R}^d) \cap H^2_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$-\Delta u = |u|^{2\sigma} u - \lambda^2 u = g(u).$$

For a given $1 \leq i \leq d$, by multiplying this by $x_i \chi_n \partial_i u$, we have

$$-\int_{\mathbb{R}^d} (\Delta u) x_i \chi_n \partial_i u \, \mathrm{d}x = \int_{\mathbb{R}^d} x_i \chi_n g(u) \partial_i u \mathrm{d}x \tag{2.3}$$

Integrating by parts the right term in (2.3) yields

$$\begin{aligned} \int_{\mathbb{R}^d} x_i \chi_n g(u) \partial_i u \mathrm{d}x &= \int_{\mathbb{R}^d} x_i \chi_n \partial_i (G(u)) \mathrm{d}x \\ &= -\int_{\mathbb{R}^d} \chi_n G(u) \mathrm{d}x - \int_{\mathbb{R}^d} x_i \partial_i \chi_n G(u) \mathrm{d}x. \end{aligned}$$

Since

- $G(u) \in L^1(\mathbb{R}^d)$ by hypothesis,
- $\chi_n(x) \xrightarrow[n\to+\infty]{} 1$ and $0 \leq \chi_n \leq 1$,
- $\partial_i \chi_n(x) = \frac{1}{n} \partial_i \chi\left(\frac{x}{n}\right) \xrightarrow[n\to+\infty]{} 0$ and $|x_i \partial_i \chi_n(x)| = \left|\frac{x_i}{n} \partial_i \chi\left(\frac{x}{n}\right)\right| \leq 2 \left\|\nabla \chi\right\|_\infty$.

the dominated convergence theorem ensures that

$$\int_{\mathbb{R}^d} x_i \chi_n g(u) \mathrm{d}x \xrightarrow[n\to+\infty]{} -\int_{\mathbb{R}^d} G(u) \mathrm{d}x. \tag{2.4}$$

Integrating by parts the left term in (2.3) yields

$$\begin{aligned}-\int_{\mathbb{R}^d}(\Delta u)x_i\chi_n\partial_i u\,\mathrm{d}x &= \int_{\mathbb{R}^d}\nabla u\cdot\nabla(x_i\partial_i u\chi_n)\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}^d}x_i\chi_n\partial_i(|\nabla u|^2)\mathrm{d}x+\int_{\mathbb{R}^d}\chi_n|\partial_i u|^2\mathrm{d}x+\int_{\mathbb{R}^d}x_i\partial_i u\nabla u\cdot\nabla\chi_n\mathrm{d}x\end{aligned}$$

The dominated convergence theorem proves that

$$\int_{\mathbb{R}^d}|\partial_i u|^2\chi_n\mathrm{d}x\xrightarrow[n\to+\infty]{}\int_{\mathbb{R}^d}|\partial_i u|^2\mathrm{d}x\qquad\text{and}\qquad\int_{\mathbb{R}^d}x_i\partial_i u\nabla u\cdot\nabla\chi_n\mathrm{d}x\xrightarrow[n\to+\infty]{}0.$$

Furthermore, integrating by part the remaining term yields

$$\frac{1}{2}\int_{\mathbb{R}^d}x_i\chi_n\partial_i(|\nabla u|^2)\mathrm{d}x=-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\chi_n\mathrm{d}x-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2x_i\partial_i\chi_n\mathrm{d}x$$

and, as before,

$$\frac{1}{2}\int_{\mathbb{R}^d}x_i\chi_n\partial_i(|\nabla u|^2)\mathrm{d}x\xrightarrow[n\to+\infty]{}-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x$$

so that

$$-\int_{\mathbb{R}^d}(\Delta u)x_i\chi_n\partial_i u\,\mathrm{d}x\xrightarrow[n\to+\infty]{}\int_{\mathbb{R}^d}|\partial_i u|^2-\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x. \tag{2.5}$$

Injecting (2.4) and (2.5) into (2.3) yields

$$\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x-\int_{\mathbb{R}^d}|\partial_i u|^2\mathrm{d}x=\int_{\mathbb{R}^d}G(u)\mathrm{d}x.$$

By summing this over $1\leq i\leq d$ we prove that

$$\frac{d-2}{2}\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x=d\int_{\mathbb{R}^d}G(u)\mathrm{d}x=d\int_{\mathbb{R}^d}\left(\frac{1}{2\sigma+2}|u|^{2\sigma+2}-\frac{\lambda^2}{2}u^2\right)\mathrm{d}x.$$

□

Corollary 2.3

For $d\geq 3$, (2.1) has no solution $u\neq 0$ in $H^1(\mathbb{R}^d)\cap H^2_{loc}(\mathbb{R}^d)$ such that $G(u)\in L^1(\mathbb{R}^d)$ when $\sigma\geq\dfrac{2}{d-2}$.

▷ Suppose u is such a solution. The calculation leading to (2.2) is still valid with $v=u$ thanks to the hypothesis on $G(u)$ and yields

$$\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x=\int_{\mathbb{R}^d}(|u|^{2\sigma+2}-\lambda^2u^2)\mathrm{d}x.$$

But thanks to the previous lemma, we also have

$$\int_{\mathbb{R}^d}|\nabla u|^2\mathrm{d}x=\frac{2d}{d-2}\int_{\mathbb{R}^d}\left(\frac{1}{2\sigma+2}|u|^{2\sigma+2}-\frac{\lambda^2}{2}u^2\right)\mathrm{d}x$$

so that

$$\frac{2\sigma+2-\sigma d}{(d-2)(\sigma+1)}\int_{\mathbb{R}^d}|u|^{2\sigma+2}\mathrm{d}x=\frac{2\lambda^2}{d-2}\int_{\mathbb{R}^d}|u|^2\mathrm{d}x>0$$

which contradicts $2\sigma+2-\sigma d\leq 0$. □

In the rest of this section, we will consider $d \geq 3$ and $0 < \sigma < \dfrac{2}{d-2}$ unless stated otherwise.

2.2 The constrained minimization method

Following the steps of H. Berestycki and P.-L. Lions [4], let us consider the constrained minimization problem

$$\text{minimize } \{T(w),\, w \in H^1(\mathbb{R}^d),\, V(w) = 1\} \tag{2.6}$$

where the functionals T and V are defined by

$$\forall w \in H^1(\mathbb{R}^d), \qquad T(w) = \int_{\mathbb{R}^d} |\nabla w|^2 \mathrm{d}x$$

and

$$\forall w \in H^1(\mathbb{R}^d), \qquad V(w) = \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2}|w|^{2\sigma+2} - \frac{\lambda^2}{2}|w|^2 \right) \mathrm{d}x.$$

Let us indeed assume that T and V are of class $\mathcal{C}^1(\mathbb{R}^d)$ and consider a solution u^* of (2.6). Then, there exists a Lagrange multiplier[1] θ such that $\frac{1}{2}DT(u^*) = \theta DV(u^*)$. This yields

$$-\Delta u^* = \theta(|u^*|^{2\sigma}u^* - \lambda^2 u^*) \qquad \text{in } \mathscr{D}'(\mathbb{R}^d). \tag{2.7}$$

We will prove below that $\theta > 0$. Then, with $u = u^*_{\sqrt{\theta}}(x) = u^*\left(\dfrac{x}{\sqrt{\theta}}\right)$, we have

$$-\Delta u = |u|^{2\sigma}u - \lambda^2 u$$

so u is a solution of (2.1).

Theorem 2.4

Suppose $d \geq 3$ and $0 < \sigma < \dfrac{2}{d-2}$. Then the minimization problem (2.6) has a solution $u^* \in H^1(\mathbb{R}^d)$ which is positive, spherically symmetric, and decreases with $r = |x|$. Furthermore, there exists a Lagrange multiplier $\theta > 0$ such that u^* satisfies (2.7). Hence $u^*_{\sqrt{\theta}}$ is a solution of (2.1).

We will follow this steps:

1. Proof that $\{w \in H^1(\mathbb{R}^d),\, V(w) = 1\} \neq \varnothing$;
2. Selection of an adequate minimizing sequence;
3. Estimates for the minimizing sequence;
4. Passage to the limit;
5. Conclusion.

[1] See theorem B.1 in Appendix B.

Step 1: $\{w \in \ H^1(\mathbb{R}^d),\, V(w)=1\} \neq \varnothing$. Let $\zeta > 0$ be such that

$$\frac{1}{2\sigma+2}\zeta^{2\sigma+2} - \frac{\lambda^2}{2}\zeta^2 > 0,$$

$R > 1$ and define

$$w_R : x \in \mathbb{R}^d \mapsto \left\{ \begin{array}{cc} \zeta & \text{for } |x| \leq R \\ \zeta(R+1-r) & \text{for } r = |x| \in [R, R+1] \\ 0 & \text{for } |x| \geq R+1. \end{array} \right.$$

We have $w_R \in H^1(\mathbb{R}^d)$ and

$$V(w_R) \geq \left(\frac{1}{2\sigma+2}\zeta^{2\sigma+2} - \frac{\lambda^2}{2}\zeta^2\right)|B(0,R)| - (|B(0,R+1)| - |B(0,R)|) \max_{s\in[0,\zeta]} \left|\frac{1}{2\sigma+2}s^{2\sigma+2} - \frac{\lambda^2}{2}s^2\right|.$$

Since $|B(0,R)| = CR^d$, we have $|B(0,R+1)| - |B(0,R)| \leq CR^{d-1}$ for some other constant $C > 0$. This shows that there exist $C, C' > 0$ such that

$$V(w_R) \geq CR^d - C'R^{d-1}$$

so for $R > 1$ large enough, we have $V(w_R) > 0$. Then, by a scale change $w_{R,\alpha}(x) = w_R\left(\frac{x}{\alpha}\right)$, we have $V(w_{R,\alpha}) = \alpha^d V(w_R)$. Thus, for an appropriate choice of $\alpha > 0$, we have $V(w_{R,\alpha}) = 1$.

Step 2: Selection of an adequate minimizing sequence. There exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $H^1(\mathbb{R}^d)$ such that $V(u_n) = 1$ and

$$\lim_{n\to+\infty} T(u_n) = \inf\{T(w),\, w \in H^1(\mathbb{R}^d),\, V(w) = 1\} = I \geq 0.$$

Let $\bar{u}_n$ be the Schwarz symmetrization of $|u_n|$[1]. For any $n \geq 0$, we have $\bar{u}_n \in H^1(\mathbb{R}^d)$, $V(\bar{u}_n) = 1$ and $I \leq T(\bar{u}_n) \leq T(u_n)$. So $(\bar{u}_n)_{n\in\mathbb{N}}$ is a minimizing sequence. For the rest of this proof, we replace u_n by $\bar{u}_n$ so that for all n, u_n is nonnegative, spherically symmetric and nonincreasing with $r = |x|$.

Step 3 : Estimates for (u_n). Let us prove that $(\|u_n\|_{H^1(\mathbb{R}^d)})_{n\in\mathbb{N}}$ is bounded. We will use the following lemma.

[1]We refer the reader to [13] and [3], for proofs, we only state here useful properties. Let $f \in L^1(\mathbb{R}^d)$, then $\bar{f}$ is a radial nonincreasing (in r), measurable function such that for any $\alpha > 0$,

$$|\{\bar{f} \geq \alpha\}| = |\{|f| \geq \alpha\}.$$

For every continuous function F such that $F(f)$ is integrable, we have

$$\int_{\mathbb{R}^d} F(f)\mathrm{d}x = \int_{\mathbb{R}^d} F(\bar{f})\mathrm{d}x$$

If $f \in H^1(\mathbb{R}^d)$, then $\bar{f} \in H^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |\nabla \bar{f}|^2 \mathrm{d}x \leq \int_{\mathbb{R}^d} |\nabla f|^2 \mathrm{d}x.$$

Lemma 2.5

Let $\ell = \dfrac{d+2}{d-2}$*. For any* $\varepsilon > 0$*, there exists* $C_\varepsilon > 0$ *such that*

$$\forall s \in \mathbb{R}, \qquad \frac{1}{2\sigma+2}|s|^{2\sigma+2} \le C_\varepsilon |s|^{\ell+1} + \varepsilon \frac{\lambda^2}{2} s^2. \tag{2.8}$$

▷ For $0 \le s \le \left(\varepsilon\lambda^2\right)^{\frac{1}{2\sigma}}$, we have

$$s^{2\sigma+1} \le \varepsilon\lambda^2 s.$$

Therefore,

$$\forall 0 \le s \le \left(\varepsilon\lambda^2\right)^{\frac{1}{2\sigma}}, \forall C > 0, \qquad s^{2\sigma+1} \le C s^\ell + \varepsilon\lambda^2 s. \tag{2.9}$$

Moreover, since $\sigma < \dfrac{2}{d-2}$, $s \mapsto s^{2\sigma+1-\ell}$ is nonincreasing and we have, with $C_\varepsilon = \left(\varepsilon\lambda^2\right)^{\frac{2\sigma+1-\ell}{2\sigma}}$,

$$\forall s \ge \left(\varepsilon\lambda^2\right)^{\frac{1}{2\sigma}}, \qquad s^{2\sigma+1} \le C_\varepsilon s^\ell.$$

Thus,

$$\forall s \ge \left(\varepsilon\lambda^2\right)^{\frac{1}{2\sigma}}, \qquad s^{2\sigma+1} \le C_\varepsilon s^\ell + \varepsilon\lambda^2 s. \tag{2.10}$$

Combining (2.9) and (2.10) yields

$$\forall s \ge 0, \qquad s^{2\sigma+1} \le C_\varepsilon s^\ell + \varepsilon\lambda^2 s.$$

Integrating this yields, for some constant $C_\varepsilon > 0$

$$\forall s \ge 0, \qquad \frac{1}{2\sigma+2} s^{2\sigma+2} \le C_\varepsilon s^{\ell+1} + \varepsilon\frac{\lambda^2}{2} s^2.$$

Therefore,

$$\forall s \in \mathbb{R}, \qquad \frac{1}{2\sigma+2}|s|^{2\sigma+2} \le C_\varepsilon |s|^{\ell+1} + \varepsilon\frac{\lambda^2}{2}|s|^2.$$

□

$T(u_n) \xrightarrow[n\to+\infty]{} I$ so $\|\nabla u_n\|_{L^2(\mathbb{R}^d)}$ is bounded. By Sobolev's embedding theorem A.1, this implies that $\|u_n\|_{L^{2^*}(\mathbb{R}^d)}$ is bounded by some constant $C > 0$, where $2^* = \dfrac{2d}{d-2} = \ell + 1$. Since, $V(u_n) = 1$, we have

$$\int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u_n|^{2\sigma+2} \mathrm{d}x = \int_{\mathbb{R}^d} \frac{\lambda^2}{2}|u_n|^2 \mathrm{d}x + 1$$

and by (2.8),

$$\int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u_n|^{2\sigma+2} \mathrm{d}x \le C + \varepsilon\frac{\lambda^2}{2}\int_{\mathbb{R}^d} |u_n|^2 \mathrm{d}x$$

so that, after choosing, for example, $\varepsilon = \frac{1}{2}$,

$$\|u_n\|_{L^2} = \left(\int_{\mathbb{R}^d} |u_n|^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C$$

for some constant $C > 0$.

Thus, $\|u\|_{H^1(\mathbb{R}^d)}$ is bounded. From $\|u_n\|_{L^2(\mathbb{R}^d)}$ and $\|u_n\|_{L^{2^*}(\mathbb{R}^d)}$ being bounded, Hölder's inequality and interpolation yield that $\|u_n\|_{L^p(\mathbb{R}^d)} \leq C$ for all $2 \leq p \leq 2^*$ and some $C > 0$, as follows. For $2 \leq p \leq 2^*$, there exists $0 \leq \alpha \leq 1$ such that

$$\frac{1}{p} = \frac{\alpha}{2} + \frac{1-\alpha}{2^*} \qquad \textit{ie.} \qquad \alpha = \frac{2(2^* - p)}{p(2^* - 2)}.$$

Then, by Hölder's inequality,

$$\begin{aligned} \|u_n\|_{L^p(\mathbb{R}^d)} &= \left\| |u_n|^{\alpha p} |u_n|^{(1-\alpha)p} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{p}} \\ &\leq \left\| |u_n|^{\alpha p} \right\|_{L^{\frac{2}{\alpha p}}(\mathbb{R}^d)}^{\frac{1}{p}} \left\| |u_n|^{(1-\alpha)p} \right\|_{L^{\frac{2^*}{(1-\alpha)p}}(\mathbb{R}^d)}^{\frac{1}{p}} \\ &= \|u_n\|_{L^2(\mathbb{R}^d)}^{\alpha} \|u_n\|_{L^{2^*}(\mathbb{R}^d)}^{1-\alpha}. \end{aligned}$$

Step 4: Passage to the limit. Let us first note that $u_n(x) \xrightarrow[|x|\to+\infty]{} 0$ uniformly with respect to n. This is an immediate consequence of the following radial lemma.

Lemma 2.6

If $u \in L^p(\mathbb{R}^d)$, $1 \leq p < +\infty$, is a radial nonincreasing function, then

$$\forall x \neq 0, \qquad |u(x)| \leq |x|^{-\frac{d}{p}} \left(\frac{d}{|\mathcal{S}^{d-1}|}\right)^{\frac{1}{p}} \|u\|_{L^p(\mathbb{R}^d)}$$

▷ For all $r = |x| > 0$, we have

$$\|u\|_{L^p(\mathbb{R}^d)}^p \geq |\mathcal{S}^{d-1}| \int_0^r \left|u\left(\frac{sx}{r}\right)\right|^p s^{d-1} \mathrm{d}s \geq |\mathcal{S}^{d-1}|\, |u(x)|^p \frac{r^d}{d}$$

because u is nonincreasing. □

Applying this lemma to u_n yields

$$\forall n \in \mathbb{N}, \forall x \neq 0, \qquad |u_n(x)| \leq C|x|^{-\frac{d}{2}}$$

which indeed implies that $u_n(x) \xrightarrow[|x|\to+\infty]{} 0$ uniformly.

Let us now prove a convergence result for the sequence (u_n).

Lemma 2.7

Let (u_n) be a bounded sequence of $H^1(\mathbb{R}^d)$. Then, up to extraction of a subsequence, (u_n) converges weakly in $H^1(\mathbb{R}^d)$ and almost everywhere in $\mathbb{R}^d$ to a function u^*.

$\triangleright$ $(H^1(\mathbb{R}^d), \|.\|_{H^1(\mathbb{R}^d)})$ is a Hilbert space so, since (u_n) is bounded in $H^1(\mathbb{R}^d)$, for some increasing function $\varphi : \mathbb{N} \to \mathbb{N}$, $u_{\varphi(n)}$ converges weakly to u^* in $H^1(\mathbb{R}^d)$. Then, for $k \in \mathbb{N}^*$, $u_{\varphi(n)}{}_{|B(0,k)}$ converges weakly to $u^*{}_{|B(0,k)}$ in $H^1(B(0,k))$. This implies that, for some increasing function $\psi_k : \mathbb{N} \to \mathbb{N}$ and $v \in L^2(B(0,k))$,

$$u_{\varphi\circ\psi_k(n)}{}_{|B(0,k)} \xrightarrow[n\to\infty]{L^2(B(0,k))} v$$

and $v = u^*{}_{|B(0,k)}$. Then, for some increasing function $\tau_k : \mathbb{N} \to \mathbb{N}$,

$$u_{\varphi\circ\psi_k\circ\tau_k(n)}{}_{|B(0,k)} \xrightarrow[n\to+\infty]{} u^*{}_{|B(0,k)} \text{ a.e in } B(0,k).$$

With $\alpha_k = \varphi \circ \psi_k \circ \tau_k$ and $\Phi(n) = \alpha_1 \circ \cdots \circ \alpha_n(n)$, we have

$$\forall k \in \mathbb{N}, \qquad u_{\Phi(n)}{}_{|B(0,k)} \xrightarrow[n\to+\infty]{} u^*{}_{|B(0,k)} \text{ a.e. in } B(0,k)$$

so that $u_{\Phi(n)}$ converges weakly in $H^1(\mathbb{R}^d)$ and almost everywhere in $\mathbb{R}^d$ to u^*. $\square$

Recall that $u^* \in H^1(\mathbb{R}^d)$ is spherically symmetric and nonincreasing with r.

Let $P(s) = |s|^{2\sigma+2}$ and $Q(s) = s^2 + |s|^\ell$. Then

$$\frac{P(s)}{Q(s)} \xrightarrow[|s|\to+\infty]{} 0 \qquad \frac{P(s)}{Q(s)} \xrightarrow[s\to0]{} 0;$$

since $\|u_n\|_{L^p(\mathbb{R}^d)} \le C$ for all $2 \le p \le 2^*$, we have

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} Q(u_n)(x)\mathrm{d}x < +\infty;$$

and we have seen that

$$P(u_n) \xrightarrow[n\to+\infty]{} P(u^*) \text{ a.e. in } \mathbb{R}^d$$

and

$$u_n(x) \xrightarrow[|x|\to\infty]{} 0 \text{ uniformly in } n \in \mathbb{N}.$$

Therefore, the compactness lemma C.1 of Strauss (see Appendix C) applies and

$$\int_{\mathbb{R}^d} P(u_n(x))\mathrm{d}x \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^d} P(u^*(x))\mathrm{d}x$$

ie.

$$\int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u_n|^{2\sigma+2}\mathrm{d}x \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u^*|^{2\sigma+2}\mathrm{d}x.$$

Then,

$$\begin{aligned}\int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u^*|^{2\sigma+2}\mathrm{d}x &= \liminf_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \frac{1}{2\sigma+2}|u_n|^{2\sigma+2}\mathrm{d}x \\ &= \liminf_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \frac{\lambda^2}{2}|u_n|^2\mathrm{d}x + 1 \\ &\underset{\text{Fatou}}{\geq} \frac{\lambda^2}{2}\int_{\mathbb{R}^d} |u^*|^2\mathrm{d}x + 1\end{aligned}$$

so that $V(u^*) \geq 1$. Moreover, by Fatou's lemma, $T(u^*) \leq I$.

Suppose that $V(u^*) > 1$. Set $u_\alpha(x) = u^*\left(\frac{x}{\alpha}\right)$. Then, for some $0 < \alpha < 1$,

$$V(u_\alpha) = \alpha^d V(u^*) = 1.$$

But

$$I \leq T(u_\alpha) = \alpha^{d-2}T(u^*) \leq \alpha^{d-2}I.$$

Therefore, $I = 0$ and $T(u^*) = 0$ so $u^* = 0$ which contradicts $V(u^*) \geq 1$.

This proves that $V(u^*) = 1$ and $T(u^*) = I > 0$ so that u^* is a solution of the minimization problem (2.6).

Step 5: conclusion. We prove in Appendix B that V and T are $\mathcal{C}^1$ functionals on $H^1(\mathbb{R}^d)$. By the Lagrange multipliers theorem[1] B.1, there exists $\theta \in \mathbb{R}$ such that $\frac{1}{2}DT(u^*) = \theta DV(u^*)$. Note that $\theta \neq 0$, since $\theta = 0$ implies that $T(u^*)$ is an extremal value for T on $H^1(\mathbb{R}^d)$. This means $T(u^*) = 0$ and $u^* = 0$. Let us show that $\theta > 0$. Suppose that $\theta < 0$. Note that $DV(u^*) \neq 0$, since $DV(u^*) = 0$ implies $|u^*|^{2\sigma}u^* - \lambda^2 u^* = 0$ and therefore

$$V(u^*) = \int_{\mathbb{R}^d} \frac{-\lambda^2\sigma}{2\sigma+2}|u^*|^2\mathrm{d}x < 0$$

which contradicts $V(u^*) = 1$. Thus, $DV(u^*)$ is a nontrivial linear form over $\mathscr{D}(\mathbb{R}^d)$ so there exists a function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that

$$\langle DV(u^*), \varphi\rangle = \int_{\mathbb{R}^d} \left(|u^*|^{2\sigma}u^* - \lambda^2 u^*\right)\varphi\mathrm{d}x > 0.$$

Then

$$T(u^* + \varepsilon\varphi) = T(u^*) + \varepsilon\langle DT(u^*), \varphi\rangle \left(1 + \underset{\varepsilon\to 0}{o}(\varepsilon)\right) = T(u^*) + 2\varepsilon\theta\langle DV(u^*), \varphi\rangle \left(1 + \underset{\varepsilon\to 0}{o}(1)\right)$$

and

$$V(u^* + \varepsilon\varphi) = V(u^*) + \varepsilon\langle DV(u^*), \varphi\rangle \left(1 + \underset{\varepsilon\to 0}{o}(1)\right).$$

Thus, for a small $\varepsilon > 0$ and $v = u^* + \varepsilon\varphi$, we have $T(v) < T(u^*) = I$ and $V(v) > V(u^*) = 1$. Then, by a scale change, there exists $0 < \alpha < 1$ such that $V(v_\alpha) = 1$ and $T(v_\alpha) < I$, which is absurd. Hence $\theta > 0$.

Then u^* satisfies, at least in $\mathscr{D}'(\mathbb{R}^d)$, the equation

$$-\Delta u^* = \theta\left(|u^*|^{2\sigma}u^* - \lambda^2 u^*\right).$$

Therefore, as we have seen above, $u^*_{\sqrt{\theta}}$ is a solution of (2.1).

[1] with $E = H^1(\mathbb{R}^d)$, $n = 1$, $f = T$ and $\Phi = V$

2.3 Regularity of spherically symmetric solutions

We can now state general results on the regularity of solutions of (2.1), and especially of radial solutions. More precisely, we begin by proving the following lemma.

Lemma 2.8

If $u \in H^1(\mathbb{R}^d)$ is a solution of (2.1), then $u \in \mathcal{C}^1(\mathbb{R}^d) \cap W^{2,s}_{loc}(\mathbb{R}^d)$ for any $s > 1$.

$\triangleright$ We need to gain an order of derivation in our control of u. Indeed, saying that u is a solution, we assume that $u \in H^1(\mathbb{R}^d)$ but we have no *a priori* control on second-order derivatives. We will not prove such a result and refer the reader to Agmon's work [2] (p. 444) for proof of the following theorem.

Theorem 2.9

Let Ω be a bounded open set of $\mathbb{R}^d$ with boundary of class $\mathcal{C}^{2m}$ and L be an elliptic linear differential operator of order $2m$ with coefficients $a_\ell \in \mathcal{C}^\ell(\overline{\Omega})$. Let $u \in L^q(\Omega)$ for some $q > 1$, and $f \in L^p(\Omega)$ for $p > 1$. Suppose that for all functions $v \in \mathcal{C}^{2m}(\overline{\Omega}) \cap \mathring{W}^{m,p}(\Omega)$,

$$\int uL(v)dx = \int fvdx.$$

Then $u \in W^{2m,p}(\Omega) \cap \mathring{W}^{m,p}(\Omega)$ and

$$\|u\|_{W^{2m,p}(\Omega)} \leq C\left(\|f\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right),$$

where C is a constant depending only on Ω, L, d and p.

We recall that a linear differential operator L of order m on a domain Ω in $\mathbb{R}^d$ can be written

$$L = \sum_{|\alpha| \leq m} a_\alpha(x)\partial^\alpha.$$

L is said to be *elliptic* if

$$\forall x \in \Omega, \forall \xi \in \mathbb{R}^d \backslash \{0\}, \qquad \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0.$$

This theorem is valid for linear problems. To adapt this to our situation, we consider the problem

$$\begin{cases} -\Delta v + \lambda^2 v = |u|^{2\sigma} u & \text{in } \Omega \\ v = u & \text{in } \partial\Omega. \end{cases}$$

u is a solution of this problem and we can apply regularity theorem 2.9. By a bootstrap argument, we will show that $u \in W^{2,s}(\mathbb{R}^d)$ for all $s > 1$. We adapt here an argument presented in [11] (p. 248).

Indeed, $u \in H^1(\mathbb{R}^d)$ so by Sobolev's embedding theorem, $u \in L^{2^*}(\mathbb{R}^d)$. Let

$$p_1 = 2^* = \frac{2d}{d-2}.$$

Then $f = |u|^{2\sigma}u \in L^{\frac{p_1}{2\sigma+1}}(\mathbb{R}^d)$ and by regularity theorem 2.9, $u \in W^{2,\frac{p_1}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^d)$. Then, by Sobolev's embedding theorem, $u \in L^{p_2}_{\text{loc}}(\mathbb{R}^d)$ with

$$\frac{1}{p_2} = \frac{2\sigma+1}{p_1} - \frac{2}{d} \qquad \Longleftrightarrow \qquad p_2 = \frac{p_1 d}{(2\sigma+1)d - 2p_1}$$

if $(2\sigma+1)d > 2p_1$, or $u \in L^s_{\text{loc}}(\mathbb{R}^d)$ for all $s > 1$ otherwise. If $(2\sigma+1)d > 2p_1$, then, since $\sigma < \dfrac{2}{d-2}$,

$$p_2 > p_1.$$

We have $f = |u|^{2\sigma}u \in L^{\frac{p_2}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^d)$ so by regularity theorem 2.9, $u \in W^{2,\frac{p_2}{2\sigma+1}}_{\text{loc}}(\mathbb{R}^d)$. Then, by Sobolev's embedding theorem, $u \in L^{p_3}_{\text{loc}}(\mathbb{R}^d)$ with

$$p_3 = \frac{p_2 d}{(2\sigma+1)d - 2p_2}$$

if $(2\sigma+1)d > 2p_2$, or $u \in L^s_{\text{loc}}(\mathbb{R}^d)$ for all $s > 1$ otherwise.

And we repeat this method until after a finite number of steps, we find $u \in L^s_{\text{loc}}(\mathbb{R}^d)$ for all $s > 1$. Indeed, for a real number $p_1 > \sigma d$, if we define by induction

$$p_{i+1} = \begin{cases} \dfrac{p_i d}{(2\sigma+1)d - 2p_i} & \text{if } (2\sigma+1)d > 2p_i \\ +\infty & \text{otherwise,} \end{cases}$$

then, by induction, for all $i \in \mathbb{N}$, $p_i > \sigma d$ so that $\dfrac{p_{i+1}}{p_i} > 1$. Two cases arise. First, if p_i becomes greater than $\dfrac{(2\sigma+1)d}{2}$, then $p_i = \infty$ for large i. Otherwise, for all $i \in \mathbb{N}$, $(2\sigma+1)d > 2p_i$. Then (p_i) is an increasing bounded sequence. Therefore, it converges to some $p > \sigma d$ (since (p_i) is increasing) such that

$$p = \frac{dp}{(2\sigma+1)d - 2p} \qquad \Longleftrightarrow \qquad p = \sigma d$$

which is absurd.

Thus, for large i, $p_i = +\infty$ and by Sobolev's embedding theorem, $u \in L^s_{\text{loc}}(\mathbb{R}^d)$ for all $s > 1$. Applying regularity theorem 2.9 yields $u \in W^{2,s}_{\text{loc}}(\mathbb{R}^d)$. For large s, we have

$$2 - \frac{d}{s} > 1 \qquad \text{and} \qquad 2 - \frac{d}{s} \notin \mathbb{N}.$$

Therefore, by Sobolev's embedding theorem, $u \in \mathcal{C}^1(\mathbb{R}^d)$. □

With this additional regularity, lemma 2.2 applies and we have the following corollary.

Corollary 2.10

Any $u \in H^1(\mathbb{R}^d)$ solution of (2.1) satisfies Pohozaev's identity:

$$\frac{d-2}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx = d\int_{\mathbb{R}^d}\left(\frac{1}{2\sigma+2}|u|^{2\sigma+2} - \frac{\lambda^2}{2}u^2\right)dx. \tag{2.11}$$

Now, if u is also supposed to be spherically symmetry, then we have even more regularity. We will follow in the rest of this paragraph the proofs given in [4].

Lemma 2.11

If u is a spherically symmetric solution of (2.1)*, then* $u \in \mathcal{C}^2(\mathbb{R}^d)$.

Remark. By an abuse of notation, we will write $u(r)$ for $u(x)$ with $|x| = r$.

▷ Since u has spherical symmetry, we can consider $u = u(r)$ and (2.1) can be written

$$-u''(r) - \frac{d-1}{r}u'(r) = |u(r)|^{2\sigma}u(r) - \lambda^2 u(r) \qquad \forall r \in \mathbb{R}_+^* \tag{2.12}$$

This shows that u'' is a meaningful continuous function, except possibly at 0. Define $v(r) = |u(r)|^{2\sigma}u(r) - \lambda^2 u(r)$. Then v is continuous on $\mathbb{R}_+$ and (2.12) can be written

$$-\frac{\mathrm{d}}{\mathrm{d}r}(r^{d-1}u'(r)) = r^{d-1}v(r).$$

Integrating from 0 to r yields

$$r^{d-1}u'(r) = -\int_0^r s^{d-1}v(s)\mathrm{d}s$$

and, putting $t = \dfrac{s}{r}$,

$$u'(r) = -r\int_0^1 t^{d-1}v(rt)\mathrm{d}t.$$

Therefore, $u'(0) = 0$ and

$$\frac{u'(r)}{r} = -\int_0^1 t^{d-1}v(rt)\mathrm{d}t \underset{r\to 0}{\longrightarrow} -\frac{v(0)}{d}$$

by, for example, the dominated convergence theorem (since v is continuous). This proves that $u''(0)$ exists and $u''(0) = -\dfrac{v(0)}{d}$. But thanks to (2.12),

$$u''(r) \underset{r\to 0}{\longrightarrow} -\frac{v(0)}{d}.$$

Thus $u \in \mathcal{C}^2(\mathbb{R}^d)$. □

Note that since u is nonnegative, from this result and the maximum principle, we can derive that u is positive. Furthermore, u and its derivatives can be controlled for large r, as stated in the following theorem, that includes the previous lemma.

Theorem 2.12

If u is a spherically symmetric solution of (2.1), then $u \in \mathcal{C}^2(\mathbb{R}^d)$ and its derivatives up to order 2 have an exponential decay at infinity ie., for some $C, \delta > 0$ and $|\alpha| \leq 2$,

$$\forall x \in \mathbb{R}^d, \qquad |D^\alpha u(x)| \leq Ce^{-\delta|x|}.$$

$\triangleright$ The previous lemma ensures that $u \in \mathcal{C}^2(\mathbb{R}^d)$. Define $v(r) = r^{\frac{d-1}{2}} u(r)$. Then v satisfies

$$\forall r > 0, \qquad v'''(r) = \left(q(r) + \frac{b}{r^2}\right) v(r)$$

where $q(r) = \lambda^2 - |u(r)|^{2\sigma}$ and $b = \dfrac{(d-1)(d-3)}{4}$. Since, $u(r) \xrightarrow[r \to +\infty]{} 0$,

$$q(r) + \frac{b}{r^2} \xrightarrow[r \to +\infty]{} \lambda^2$$

and there exists $r_0 > 0$ such that

$$\forall r > r_0, \qquad q(r) + \frac{b}{r^2} \geq \frac{\lambda^2}{2}.$$

Let $w = v^2$. Then

$$\forall r > 0, \qquad \frac{1}{2} w''(r) = v'(r)^2 + \left(q(r) + \frac{b}{r^2}\right) w(r)$$

so that, because $w \geq 0$,

$$\forall r > r_0, \qquad w''(r) \geq \lambda^2 w(r)$$

and $w'' \geq 0$.

Define now $z(r) = e^{-\lambda r}(w'(r) + \lambda w(r))$. Then $z'(r) = e^{-\lambda r}(w''(r) - \lambda^2 w(r)) \geq 0$ and z is nondecreasing on $]r_0, +\infty[$.

Suppose there exists $r_1 > r_0$ such that $z(r_1) > 0$, and therefore, $z(r) \geq z(r_1) > 0$ for all $r \geq r_1$. Then

$$\forall r \geq r_1, \qquad w'(r) + \lambda w(r) \geq z(r_1) e^{\lambda r}$$

so that $w' + \lambda w$ is not integrable on $]r_1, +\infty[$. But, since $u \in H^1(\mathbb{R}^d)$, v^2 and vv' are integrable near ∞ so that w' and w are also integrable: contradiction. Thus, $z(r) \leq 0$ for all $r > r_0$. Then,

$$\forall r > r_0, \qquad \frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\lambda r} w(r)\right) = e^{2\lambda r} z(r) \leq 0.$$

Therefore, for some constant $C > 0$, $w(r) \leq Ce^{-\lambda r}$, which yields

$$\forall r > r_0, \qquad |u(r)| \leq Cr^{-\frac{d-1}{2}} e^{-\frac{\lambda r}{2}}. \tag{2.13}$$

To obtain the exponential decay of u', observe that u' satisfies

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^{d-1} u'(r)\right) = -r^{d-1}|u(r)|^{2\sigma} u(r) + \lambda^2 r^{d-1} u(r)$$

so that, integrating over $[r,R]$, we have

$$R^{d-1}u'(R)-r^{d-1}u'(r)=-\int_r^R s^{d-1}|u(s)|^{2\sigma}u(s)\mathrm{d}s+\lambda^2\int_r^R s^{d-1}u(s)\mathrm{d}s. \tag{2.14}$$

Letting $R\to+\infty$ shows that $r^{d-1}u'(r)$ has a limit, say $\ell\in\mathbb{R}$, as $r\to+\infty$, so that

$$u'(r)=\frac{\ell}{r^{d-1}}+\underset{r\to+\infty}{o}\left(\frac{1}{r^{d-1}}\right).$$

Integrating over $[r,+\infty[$ yields

$$-u(r)=\int_r^{+\infty}u'(s)\mathrm{d}s=\int_r^{+\infty}\frac{\ell}{s^{d-1}}\mathrm{d}s+\underset{r\to+\infty}{o}\left(\int_r^{+\infty}\frac{\mathrm{d}s}{s^{d-1}}\right)=\frac{\ell}{(d-2)r^{d-2}}+\underset{r\to+\infty}{o}\left(\frac{1}{r^{d-2}}\right).$$

The exponential decay of u (2.13) then forces $\ell=0$.

Therefore, letting $R\to+\infty$ in (2.14) and applying (2.13) yields

$$|r^{d-1}u'(r)|=-\int_r^{+\infty}s^{-\frac{(d-1)(2\sigma-1)}{2}}e^{-\frac{\lambda s}{2}(2\sigma+1)}\mathrm{d}s+\int_r^{+\infty}s^{\frac{d-1}{2}}e^{-\frac{\lambda s}{2}}\mathrm{d}s.$$

But, these two integrals have exponential decay as $r\to+\infty$. For example, for $0<\delta<\dfrac{\lambda}{2}$,

$$e^{\delta r}\int_r^{+\infty}s^{\frac{d-1}{2}}e^{-\frac{\lambda s}{2}}\mathrm{d}s=\int_{\mathbb{R}}s^{\frac{d-1}{2}}e^{-\delta(s-r)}e^{\left(\frac{\lambda}{2}-\delta\right)s}\mathbf{1}_{\{s\geq r\}}(s)\mathrm{d}s$$

and, since $\dfrac{\lambda}{2}>\delta$,

$$s^{\frac{d-1}{2}}e^{-\delta(s-r)}e^{-\left(\frac{\lambda}{2}-\delta\right)s}\mathbf{1}_{\{s\geq r\}}(s)\xrightarrow[r\to+\infty]{}0,$$

$$\left|s^{\frac{d-1}{2}}e^{-\delta(s-r)}e^{-\left(\frac{\lambda}{2}-\delta\right)s}\mathbf{1}_{\{s\geq r\}}(s)\right|\leq s^{\frac{d-1}{2}}e^{-\left(\frac{\lambda}{2}-\delta\right)s}\mathbf{1}_{\{s\geq 0\}}(s)\in L^1(\mathbb{R})$$

so by the dominated convergence theorem,

$$\int_r^{+\infty}s^{\frac{d-1}{2}}e^{-\frac{\lambda s}{2}}\mathrm{d}s=\underset{r\to+\infty}{o}\left(e^{-\delta r}\right).$$

Therefore, u' has exponential decay as $r\to+\infty$.

Finally, (2.12) implies the exponential decay for u''. □

2.4 Minimization of the action

We begin by proving that the ground state u obtained in section 2.2 is in fact equal to the minimization solution u^*.

Recall that

$$-\Delta u^*=\theta(u^*)^{2\sigma+1}-\lambda^2\theta u^*.$$

Since u, thus u^*, is of class $\mathcal{C}^2$, multiplying by u^* and integrating by parts yield

$$T(u^*)=\theta\int_{\mathbb{R}^d}|u^*(x)|^{2\sigma+2}\mathrm{d}x-\lambda^2\theta\int_{\mathbb{R}^d}|u^*(x)|^2\mathrm{d}x.$$

Together with

$$1 = V(u^*) = \frac{1}{2\sigma+2}\int_{\mathbb{R}^d} |u^*(x)|^{2\sigma+2} - \frac{\lambda^2}{2}\int_{\mathbb{R}^d} |u^*(x)|^2 \mathrm{d}x$$

and

$$\theta = \frac{d-2}{2d} T(u^*),$$

this implies

$$\int_{\mathbb{R}^d} |u^*(x)|^{2\sigma+2}\mathrm{d}x = \frac{2(2\sigma+2)}{\sigma d - 2\sigma} \qquad \text{and} \qquad \int_{\mathbb{R}^d} |u^*(x)|^2\mathrm{d}x = \frac{2(2\sigma+2-\sigma d)}{\lambda^2(\sigma d - 2\sigma)}. \tag{2.15}$$

Pohozaev's identity (2.11) and (2.15) yield

$$T(u^*) = \frac{2d}{d-2}$$

and $\theta = 1$ so that $u^* = u$.

Let us conclude the proof of theorem 2.1 and prove that u minimizes the action

$$\mathcal{S}(u) = \frac{1}{2}T(u) - V(u)$$

among all $H^1(\mathbb{R}^d)$-solutions of (2.1).

Theorem 2.13

Let u denote the solution of (2.1) *obtained in section 2.2. Then, for any solution $v \in H^1(\mathbb{R}^d)$ of* (2.1), *we have*

$$0 < \frac{2}{d-2} = \mathcal{S}(u) \le \mathcal{S}(v).$$

▷ We adpat the proof given in [4]. If $v \in H^1(\mathbb{R}^d)$ is another solution of (2.1), then, by Pohozaev's identity (2.11), we have

$$T(v) = \frac{2d}{d-2} V(v)$$

Then $V(v) \neq 0$ and, with

$$\alpha = V(v)^{-\frac{1}{d}} = \left(\frac{d-2}{2d}\right)^{-\frac{1}{d}} T(v)^{-\frac{1}{d}}, \qquad v_\alpha = v\left(\frac{\cdot}{\alpha}\right),$$

we have $V(v_\alpha) = 1$ and $T(v_\alpha) = \alpha^{d-2} T(v)$ so that

$$T(v) = \left(\frac{d-2}{2d}\right)^{\frac{d-2}{2}} T(v_\alpha)^{\frac{d}{2}}.$$

Then, since u minimizes $T(w)$ under the constraint $V(w) = 1$,

$$\mathcal{S}(v) = \frac{1}{d}T(v) = \frac{1}{d}\left(\frac{d-2}{2d}\right)^{\frac{d-2}{2}} T(v_\alpha)^{\frac{d}{2}} \ge \frac{1}{d}\left(\frac{d-2}{2d}\right)^{\frac{d-2}{2}} T(u)^{\frac{d}{2}} = \frac{2}{d-2} = \mathcal{S}(u).$$

□

3 Orbital stability of ground states

We come back now to the study of the initial-valued nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t\psi(x,t) + \Delta\psi(x,t) - |\psi(x,t)|^{2\sigma}\psi(x,t) = 0 & (x,t) \in \mathbb{R}^d \times \mathbb{R}_+ \\ \psi(\cdot,0) = \psi_0 & \text{in } H^1(\mathbb{R}^d). \end{cases} \tag{3.1}$$

This equation has been widely studied. We shall make use of the following theorem, proved by J. Ginibre and G. Velo [10].

Theorem 3.1

Let $0 < \sigma < \dfrac{2}{d}$ and $\psi_0 \in H^1(\mathbb{R}^d)$. Then equation (3.1) has a unique solution ψ in $\mathcal{C}_b(\mathbb{R}, H^1(\mathbb{R}^d))$.

The time-dependent nonlinear Schrödinger equation has phase and translation symmetries *ie.* if $\psi(x,t)$ is a solution, then so is $\psi(x+x_0,t)e^{i\gamma}$ for any $x_0 \in \mathbb{R}^d, \gamma \in \mathbb{R}$. We define the orbit of a function $u \in H^1(\mathbb{R}^d)$ under the action of these symmetries by

$$G_u = \{u(\cdot + x_0)e^{i\gamma},\, (x_0,\gamma) \in \mathbb{R}^d \times \mathbb{R}\}.$$

We will say that a ground state is *orbitally stable* if initial data being near the ground state orbit implies that the corresponding standing wave solution at all later times remains near the ground state orbit. To measure the deviation of a solution from G_u, we consider the functional

$$\rho_\lambda(\psi(\cdot,t),G_u)^2 = \inf_{\substack{x_0\in\mathbb{R}^d \\ \gamma\in[0,2\pi]}} \left(\left\|\nabla\psi(\cdot+x_0,t)e^{i\gamma} - \nabla u\right\|^2_{L^2(\mathbb{R}^d)} + \lambda^2 \left\|\psi(\cdot+x_0,t)e^{i\gamma} - u\right\|^2_{L^2(\mathbb{R}^d)} \right).$$

We will prove the following theorem:

Theorem 3.2

Let $d \geq 3$ and $0 < \sigma < \dfrac{2}{d}$. Let $\psi(x,t)$ be the unique solution of (3.1). Then u is orbitally stable, ie. , for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if

$$\rho_\lambda(\psi_0, G_u) < \delta(\varepsilon)$$

then

$$\forall t > 0, \qquad \rho_\lambda(\psi(\cdot,t),G_u) < \varepsilon.$$

Unless stated otherwise, we assume in the rest of this report $d \geq 3$ and $0 < \sigma < \dfrac{2}{d}$.

Following M. Weinstein's proof [18], define the energy

$$\mathscr{E}(\psi) = \mathscr{H}(\psi) + \lambda^2\mathscr{N}(\psi)$$

where

$$\mathscr{H}(\psi) = \int_{\mathbb{R}^d} \left(|\nabla\psi(x,t)|^2 - \frac{1}{\sigma+1}|\psi(x,t)|^{2\sigma+2} \right) \mathrm{d}x$$

and

$$\mathscr{N}(\psi) = \int_{\mathbb{R}^d} |\psi(x,t)|^2 \mathrm{d}x.$$

By a result of Ginibre and Velo [10], $\mathscr{H}$ and $\mathscr{N}$ are conserved in time. Let us prove it formally. First, multiplying (3.1) by $\overline{\psi}$ and integrating yields

$$\int_{\mathbb{R}^d} \left(i\overline{\psi}\partial_t \psi + \overline{\psi}\Delta\psi + |\psi|^{2\sigma+2} \right) \mathrm{d}x = 0.$$

Thus, considering the imaginary part, we have

$$\int_{\mathbb{R}^d} \mathrm{Re}\ (\overline{\psi}\partial_t\psi) \mathrm{d}x = 0$$

so that

$$\int_{\mathbb{R}^d} \partial_t |\psi|^2 \mathrm{d}x = 0$$

and $\mathscr{N}(\psi)$ is conserved.

Similarly, multiplying (3.1) by $\partial_t\overline{\psi}$ and integrating yields

$$\int_{\mathbb{R}^d} \left(i|\partial_t\psi|^2 + \partial_t\overline{\psi}\Delta\psi + |\psi|^{2\sigma}\psi\partial_t\overline{\psi} \right) \mathrm{d}x = 0.$$

By taking the real part, we have

$$\int_{\mathbb{R}^d} \left(-\partial_t |\nabla\psi|^2 + |\psi|^{2\sigma}\partial_t |\psi|^2 \right) \mathrm{d}x = 0$$

ie.

$$\int_{\mathbb{R}^d} \partial_t \left(|\nabla\psi|^2 - \frac{1}{\sigma+1}|\psi|^{2\sigma+2} \right) \mathrm{d}x = 0$$

and $\mathscr{H}$ is conserved.

We are going to estimate $\mathscr{E}$ in terms of ρ_λ. Define $h = f + ig$ by

$$\psi(x + x_0, t)e^{i\gamma} = u(x) + h(x,t).$$

Then,

$$\mathscr{E}(\psi_0) - \mathscr{E}(u) = \mathscr{E}(\psi(\cdot,t)) - \mathscr{E}(u) = \mathscr{E}(\psi(\cdot + x_0, t)e^{i\gamma}) - \mathscr{E}(u) = \mathscr{E}(u+h) - \mathscr{E}(u).$$

But

$$\mathscr{N}(u+h) - \ = \int_{\mathbb{R}^d} (u^2 + 2uh + f^2 + g^2) \mathrm{d}x,$$

$$\int_{\mathbb{R}^d} |\nabla(u+h)|^2 \,\mathrm{d}x = \int_{\mathbb{R}^d} \left(|\nabla u|^2 + 2\nabla u \cdot \nabla f + |\nabla f|^2 + |\nabla g|^2 \right) \mathrm{d}x$$

and

$$|u+h|^{2\sigma+2} = |u|^{2\sigma+2} + (2\sigma+2)u^{2\sigma+1}f + (\sigma+1)(2\sigma+1)u^{2\sigma}f^2 + (\sigma+1)u^{2\sigma}g^2 + R$$

with $|R| \leq C|h|^3$.

If $\sigma \geq \dfrac{1}{2}$, then $d < 4$ *ie.* $d = 3$. Then, by Sobolev's embedding theorem, $H^1(\mathbb{R}^d) \hookrightarrow L^3(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |u+h|^{2\sigma+2} \leq \int_{\mathbb{R}^d} \left(|u|^{2\sigma+2} + (2\sigma+2)u^{2\sigma+1}f + (\sigma+1)(2\sigma+1)u^{2\sigma}f^2 + (\sigma+1)u^{2\sigma}g^2\right) \mathrm{d}x + C\,\|h\|^3_{H^1(\mathbb{R}^d)}.$$

If $0 < \sigma < \dfrac{1}{2}$, we don't necessarily have $H^1(\mathbb{R}^d) \hookrightarrow L^3(\mathbb{R}^d)$ and we have to be more precise. Let us show that $R \leq C|h|^{2\sigma+2}$. Writing $\tilde{f} = \dfrac{f}{u}$, $\tilde{g} = \dfrac{g}{u}$ and $\tilde{h} = \dfrac{h}{u}$, we have

$$\begin{aligned}
\frac{|R|}{|h|^{2\sigma+2}} &= \frac{\left||1+\tilde{h}|^{2\sigma+2} - 1 - (2\sigma+2)\tilde{f} - (\sigma+1)(2\sigma+2)\tilde{f}^2 - (\sigma+1)\tilde{g}^2\right|}{|\tilde{h}|^{2\sigma+2}} \\
&= \frac{\left|(1+2\tilde{f}+\tilde{f}^2+\tilde{g}^2)^{\sigma+1} - 1 - 2(\sigma+1)\tilde{f} - (\sigma+1)(2\sigma+1)\tilde{f}^2 - (\sigma+1)\tilde{g}^2\right|}{\left(\tilde{f}^2+\tilde{g}^2\right)^{\sigma+1}}.
\end{aligned}$$

This defines a function $\tau(\tilde{f},\tilde{g})$. τ is a continuous function on $\mathbb{R}^2\backslash\{0\}$. One can verify that

$$\tau(\tilde{f},\tilde{g}) \xrightarrow[|\tilde{h}|\to+\infty]{} 1$$

and

$$\tau(\tilde{f},\tilde{g}) \underset{|\tilde{h}|\to 0}{=} \mathcal{O}\left(|\tilde{h}|^{1-2\sigma}\right) \underset{|\tilde{h}|\to 0}{=} o(1).$$

This proves $|R| \leq C|h|^{2\sigma+2}$. Then, since $H^1(\mathbb{R}^d) \hookrightarrow L^{2\sigma+2}(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} |u+h|^{2\sigma+2} \leq \int_{\mathbb{R}^d} \left(|u|^{2\sigma+2} + (2\sigma+2)u^{2\sigma+1}f + (\sigma+1)(2\sigma+1)u^{2\sigma}f^2 + (\sigma+1)u^{2\sigma}g^2\right) \mathrm{d}x + C\,\|h\|^{2\sigma+2}_{H^1(\mathbb{R}^d)}.$$

Thus, for any $0 < \sigma < \dfrac{2}{d}$,

$$\mathscr{E}(\psi_0) - \mathscr{E}(u) \geq (L_+f,f) + (L_-g,g) - C\,\|h\|^{2+\theta}_{H^1(\mathbb{R}^d)} \tag{3.2}$$

where $\theta > 0$,

$$(a,b) = \int_{\mathbb{R}^d} a(x)b(x)\mathrm{d}x,$$

and L_+ and L_- are the linearized Schödinger operators:

$$L_+ = -\Delta + \lambda^2 - (2\sigma+1)u^{2\sigma} \qquad L_- = -\Delta + \lambda^2 - u^{2\sigma}.$$

3.1 Further properties of ground states and the linearized operators

We begin by proving that the ground states obtained in section 2 are solutions of another minimization problem.

Theorem 3.3

For $v \in H^1(\mathbb{R}^d)\backslash\{0\}$, define the functional

$$J(v) = \frac{\|\nabla v\|_{L^2(\mathbb{R}^d)}^{\sigma d} \|v\|_{L^2(\mathbb{R}^d)}^{2\sigma+2-\sigma d}}{\|v\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2}}.$$

For $d > 2$ and $0 < \sigma < \dfrac{2}{d-2}$, if u is the previously constructed ground state solution of

$$-\Delta u + \lambda^2 u - u^{2\sigma+1} = 0$$

then

$$\forall v \in H^1(\mathbb{R}^d), \qquad J(u) \leq J(v).$$

▷ We adapt here an idea from Jérôme Vétois.

Let $v \in H^1(\mathbb{R}^d)\backslash\{0\}$. Let $\alpha = \left(\dfrac{\lambda^2(2\sigma+2)}{2\sigma+2-\sigma d}\right)^{\frac{1}{2\sigma}} \dfrac{\|v\|_{L^2(\mathbb{R}^d)}^{\frac{1}{\sigma}}}{\|v\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{\frac{2\sigma+2}{2\sigma}}}$ so that, with $v_\alpha = \alpha v$,

$$\frac{\lambda^2 \|v_\alpha\|_{L^2(\mathbb{R}^d)}^2}{2\sigma+2-\sigma d} = \frac{\|v_\alpha\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2}}{2\sigma+2}.$$

Let $\beta = \left(\dfrac{\lambda^2(\sigma d - 2\sigma)}{2(2\sigma+2-\sigma d)}\right)^{\frac{1}{d}} \|v_\alpha\|_{L^2(\mathbb{R}^d)}^{\frac{2}{d}}$ so that, with $v_{\alpha,\beta}(x) = v_\alpha(\beta x)$,

$$V(v_{\alpha,\beta}) = \frac{1}{2\sigma+2} \|v_{\alpha,\beta}\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2} - \frac{\lambda^2}{2} \|v_{\alpha,\beta}\|_{L^2(\mathbb{R}^d)}^2 = 1.$$

Then, thanks to the remark in the beginning of paragraph 2.4, we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla v_{\alpha,\beta}\|_{L^2(\mathbb{R}^d)}^2 = \left(\frac{2(2\sigma+2-\sigma d)}{\lambda^2(\sigma d - 2\sigma)}\right)^{1-\frac{2}{d}} \left(\frac{\lambda^2(2\sigma+2)}{2\sigma+2-\sigma d}\right)^{\frac{2}{\sigma d}} \frac{\|v\|_{L^2(\mathbb{R}^d)}^{\frac{4}{\sigma d}+\frac{4}{d}-2} \|\nabla v\|_{L^2(\mathbb{R}^d)}^2}{\|v\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{\frac{2(2\sigma+2)}{\sigma d}}}$$

so that

$$\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\sigma d} \leq \left(\frac{2(2\sigma+2-\sigma d)}{\lambda^2(\sigma d - 2\sigma)}\right)^{\frac{\sigma d}{2}-\sigma} \frac{\lambda^2(2\sigma+2)}{2\sigma+2-\sigma d} J(v) = \|u\|_{L^2(\mathbb{R}^d)}^{\sigma d - 2\sigma} \frac{\|u\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+2}}{\|u\|_{L^2(\mathbb{R}^d)}^2} J(v)$$

and

$$J(u) \leq J(v).$$

□

From this we deduce some properties of operator L_+.

Theorem 3.4

L_+ has exactly one negative eigenvalue.

$\triangleright$ We give a detailed proof following Weinstein's arguments [18]. A proof similar to the one in Appendix B proves that J is a $\mathcal{C}^2$ functional on $H^1(\mathbb{R}^d)$. Since u is a minimum of J, by computing the second-order variation of J, we have, for any $v \in H^1(\mathbb{R}^d)$,

$$\begin{aligned} 0 \leq\ & (L_+v,v) - \frac{(2-\sigma d)(d-2)}{2d}\left(\int_{\mathbb{R}^d} v\Delta u \mathrm{d}x\right)^2 \\ & -\lambda^2\sigma(d-2)\left(\int_{\mathbb{R}^d} uv\mathrm{d}x\right)\left(\int_{\mathbb{R}^d} v\Delta u\mathrm{d}x\right) - \frac{\lambda^4\sigma^2(d-2)^2}{2(2\sigma+2-\sigma d)}\left(\int_{\mathbb{R}^d} uv\mathrm{d}x\right)^2. \end{aligned}$$

or $(Tv,v)\geq 0$ where

$$\begin{aligned} T(v) =\ & L_+v - \frac{(2-\sigma d)(d-2)}{2d}\left(\int_{\mathbb{R}^d} v\Delta u\mathrm{d}x\right)\Delta u \\ & -\lambda^2\sigma(d-2)\left(\int_{\mathbb{R}^d} uv\mathrm{d}x\right)\Delta u - \frac{\lambda^4\sigma^2(d-2)^2}{2(2\sigma+2-\sigma d)}\left(\int_{\mathbb{R}^d} uv\mathrm{d}x\right)u. \end{aligned} \tag{3.3}$$

Since $0<\sigma<\dfrac{2}{d}<\dfrac{2}{d-2}$, the operator

$$r_1 : v \mapsto \frac{\lambda^4\sigma^2(d-2)^2}{2(2\sigma+2-\sigma d)}\left(\int_{\mathbb{R}^d} uv\mathrm{d}x\right)u$$

is nonnegative. Thus, $T+r_1$ is nonnegative. This can be written

$$\forall v \in H^1(\mathbb{R}^d), \qquad ((L_+ + r_2)v,v) \geq 0$$

where r_2 is an operator of rank one: $\mathrm{Ra}(r_2) = \mathrm{Span}(\Delta u)$. This implies that L_+ has at most one negative eigenvalue.

Indeed, suppose by contradiction that $\alpha<\beta<0$ are eigenvalues of L_+. Let h_α, h_β be corresponding eigenfunctions, with $\|h_\alpha\|_{L^2(\mathbb{R}^d)} = \|h_\beta\|_{L^2(\mathbb{R}^d)} = 1$. Let $\ell \in H^1(\mathbb{R}^d)^*$ such that $r_2(\cdot) = \ell(\cdot)\Delta u$. Then,

$$0 \leq ((L_+ + r_2)h_\alpha, h_\alpha) = \alpha + \ell(h_\alpha)(\Delta u, h_\alpha)$$

and

$$0 \leq ((L_+ + r_2)h_\beta, h_\beta) = \beta + \ell(h_\beta)(\Delta u, h_\beta).$$

If either $\ell(h_\alpha), \ell(h_\beta), (\Delta u, h_\alpha)$ or $(\Delta u, h_\beta)$ is equal to 0, then we have a contradiction. Suppose that none of these are 0. Let $\gamma = \dfrac{\ell(h_\beta)}{\ell(h_\alpha)}$. Then, since L_+ is self-adjoint, $(h_\alpha, h_\beta)=0$ so that,

$$\begin{aligned} 0 \leq\ & ((L_+ + r_2)(\gamma h_\alpha - h_\beta), \gamma h_\alpha - h_\beta) \\ \leq\ & \gamma^2\alpha + \beta \\ & +\ell(h_\alpha)(\Delta u, h_\alpha)\left(\gamma^2 - \gamma\left(\frac{(\Delta u, h_\beta)}{(\Delta u, h_\alpha)} + \frac{\ell(h_\beta)}{\ell(h_\alpha)}\right) + \frac{\ell(h_\beta)}{\ell(h_\alpha)}\frac{(\Delta u, h_\beta)}{(\Delta u, h_\alpha)}\right) \\ \leq\ & \gamma^2\alpha + \beta \end{aligned}$$

Since $\alpha, \beta < 0$, we have $\gamma^2\alpha + \beta < 0$ and this leads to a contradiction.

Let us now prove that

$$\mu = \inf\{(L_+v, v),\ v \in H^1(\mathbb{R}^d),\ \|v\|_{L^2(\mathbb{R}^d)} = 1\}$$

is a negative eigenvalue for L_+.

First note that $(L_+u, u) = -2\sigma \|u\|^{2\sigma+2}_{L^{2\sigma+2}(\mathbb{R}^d)} < 0$ so that $\mu < 0$.

We prove that this minimization problem has a solution $v_* \in H^1(\mathbb{R}^d)$. We adapt the ideas developed in [17] (p. 478). Let (v_n) be a minimizing sequence and set $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$

$$\mu \leq (L_+v_n, v_n) = \int_{\mathbb{R}^d} |\nabla v_n|^2 \mathrm{d}x + \lambda^2 \int_{\mathbb{R}^d} v_n^2 \mathrm{d}x - (2\sigma + 1) \int_{\mathbb{R}^d} u^{2\sigma} v_n^2 \mathrm{d}x < \mu + \varepsilon$$

so that, since $\mu < 0$, $\|v_n\|_{L^2(\mathbb{R}^d)} = 1$ and u is bounded,

$$1 \leq \|v_n\|_{H^1(\mathbb{R}^d)} \leq \mu + \varepsilon + 1 - \lambda^2 + (2\sigma + 1) \int_{\mathbb{R}^d} u^{2\sigma} v_n^2 \mathrm{d}x \leq C \tag{3.4}$$

for some constant C. Thus, $(\|v_n\|_{H^1(\mathbb{R}^d)})$ is uniformly bounded. By lemma 2.7, there exists $v_* \in H^1(\mathbb{R}^d)$ such that, up to extraction of a subsequence, (v_n) converges weakly in $H^1(\mathbb{R}^d)$ and almost everywhere in $\mathbb{R}^d$ to v_*. Moreover (v_n) is bounded in $L^p(\mathbb{R}^d)$ for $2 \leq p \leq 2^*$.

Note that since (v_n) converges weakly to v_* in $H^1(\mathbb{R}^d)$, $v_n \rightharpoonup v_*$ and $\nabla v_n \rightharpoonup \nabla v_*$ in $L^2(\mathbb{R}^d)$. To see this, for any $w_1, w_2 \in L^2(\mathbb{R}^d)$, define

$$\Lambda : v \in H^1(\mathbb{R}^d) \mapsto \int_{\mathbb{R}^d} v w_1 \mathrm{d}x + \int_{\mathbb{R}^d} v w_2 \mathrm{d}x.$$

Then $\Lambda \in H^1(\mathbb{R}^d)'$ so, by weak convergence of (v_n), $\Lambda(v_n) \xrightarrow[n\to+\infty]{} \Lambda(v_*)$. Taking alternatively w_1 and w_2 equal to 0 yields the result.

Since (v_n^2) is bounded in $L^{\frac{2^*}{2}}(\mathbb{R}^d)$ and (v_n^2) converges almost everywhere to v_*^2, by lemma C.4 in Appendix C, (v_n^2) converges weakly to v_*^2 in $L^{\frac{2^*}{2}}(\mathbb{R}^d)$. Thus,

$$\int_{\mathbb{R}^d} u^{2\sigma} v_n^2 \mathrm{d}x \xrightarrow[n\to+\infty]{} \int_{\mathbb{R}^d} u^{2\sigma} v_*^2 \mathrm{d}x.$$

Then, passage to the limit in (3.4) yields, since $\mu < 0$,

$$0 \leq \varepsilon - \lambda^2 + (2\sigma + 1) \int_{\mathbb{R}^d} u^{2\sigma} v_*^2 \mathrm{d}x.$$

Since ε is arbitrary, this implies that $v_* \neq 0$.

By Fatou's lemma, $\|v_*\|_{L^2(\mathbb{R}^d)} \leq 1$. Let $w_* = \dfrac{v_*}{\|v_*\|_{L^2(\mathbb{R}^d)}}$. Then $w \in H^1(\mathbb{R}^d)$, $\|w\|_{L^2(\mathbb{R}^d)} = 1$. Let $\zeta \in L^2(\mathbb{R}^d)$, $\|\zeta\|_{L^2(\mathbb{R}^d)} = 1$. Since (∇v_n) converges weakly to ∇v_* in $L^2(\mathbb{R}^d)$,

$$(\zeta, \nabla v_*) = \liminf_{n\to+\infty} (\zeta, \nabla v_n) \leq \liminf_{n\to+\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)}.$$

Maximizing over all such ζ we obtain

$$\|\nabla v_*\|_{L^2(\mathbb{R}^d)} \leq \liminf_{n\to+\infty} \|\nabla v_n\|_{L^2(\mathbb{R}^d)}.$$

Thus, we have

$$(L_+ v_*, v_*) \leq \liminf_{n\to+\infty}(L_+ v_n, v_n) = \mu$$

and

$$(L_+ w_*, w_*) \leq \mu \|v_*\|_{L^2(\mathbb{R}^d)} \leq \mu$$

with $\|w_*\|_{L^2(\mathbb{R}^d)} = 1$ which implies $w_* = v_*$ and $(L_+ v_*, v_*) = \mu$.

Now $v \in H^1(\mathbb{R}^d) \mapsto (L_+ v, v)$ is a $\mathcal{C}^1$ functional on $H^1(\mathbb{R}^d)$. Lagrange multipliers theorem B.1 implies that there exists $\theta \in \mathbb{R}$ such that

$$L_+ v_* = \theta v_*.$$

Therefore, since $\|v_*\|_{L^2(\mathbb{R}^d)} = 1$,

$$\mu = (L_+ v_*, v_*) = \theta$$

and μ is the only negative eigenvalue. □

Another useful consequence is the following theorem.

Theorem 3.5

We have

$$\inf_{(u,v)=0} (L_+ v, v) = 0.$$

▷ Note that $L_+ \partial_{x_j} u = 0$ and $(\partial_{x_j} u, u) = 0$ so that the infimum is nonpositive.

Now, using the same notation as in the previous proof, we have

$$\forall v \in H^1(\mathbb{R}^d), \qquad T(v,v) \geq 0$$

where, for some constants $C_1, C_2, C_3 > 0$ for $\sigma < \dfrac{2}{d}$,

$$T(v) = L_+ v - C_1(\Delta u, v)\Delta u - C_2(u,v)\Delta u - C_3(u,v)u.$$

Thus, for $(u,v) = 0$, this implies

$$(L_+ v, v) \geq C_2(u,v)(\Delta u, v)^2 \geq 0.$$

□

We will need the following description of the kernel of L_+.

Theorem 3.6

We have

$$\operatorname{Ker} L_+ = \operatorname{Span}(\partial_{x_j} u,\ 1 \leq j \leq d).$$

▷ We sketch the proof, using arguments in [7] or [17].

We admit that since L_+ is has spherical symmetry, any solution of $L_+v = 0$ can be decomposed as

$$v = \sum_{k\geq 0}\sum_{j\in\Sigma_k} v_{k,j}(r)Y_{k,j}(\hat{x})$$

where $r = |x|$, $\hat{x} = \frac{x}{r}$, $Y_{k,j}$ denote spherical harmonics

$$-\Delta_{S^{n-1}}Y_{k,j} = \mu_k Y_{k,j} \qquad \mu_k = k(k+d-2),$$

Σ_k is finite and

$$\forall k \geq 0, \qquad A_k v_{k,j} = \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{d-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \lambda^2 - (2\sigma+1)u^{2\sigma} + \frac{\mu_k}{r^2}\right)v_{k,j} = 0.$$

For details, we refer the reader to [8].

Note that $\nabla u = u'(r)\hat{x}$ and $A_1 u' = 0$. Let v satisfy $A_1 v = 0$. Define the Wronskian $W(r) = r^{d-1}(v(r)u'(r) - v'(r)u(r))$. Then we have $W'(r) = 0$ so that $W(r) = \lim_{r\to+\infty} W(r)$. Since we want for the solutions to satisfy $v(r), v'(r) \xrightarrow[r\to+\infty]{} 0$, we have $W(r) = 0$ and $v = Cu$ for some constant C. This implies that $A_1 \geq 0$.

For $k \geq 2$, $A_k = A_1 + \frac{\mu_k - \mu_1}{r^2}$ so that A_k is a positive operator. Thus, $A_k v = 0$ has no nonzero solution.

For $k = 0$, we refer the reader to [7] and Sturm-Liouville theory to prove there is no nonzero solution of $A_0 v = 0$. □

Let us now give a useful result about the operator L_-. We will need the following theorem. For proof, we refer the reader to [16] (p. 236).

Theorem 3.7

Suppose $H = -\Delta + V$ is a self-adjoint and bounded from below operator with $\mathcal{C}^\infty(\mathbb{R}^d)$ as a core. If $E_0 = \min\sigma(H)$ is an eigenvalue, it is simple.

Theorem 3.8

L_- is a nonnegative operator. Furthermore,

$$\operatorname{Ker} L_- = \operatorname{Span}(u).$$

▷ As noticed in [15] (p. 73), it is easily checked that

$$L_- = -\frac{1}{u}\operatorname{div}\left(u^2 \operatorname{grad}\left(\frac{1}{u}\cdot\right)\right)$$

so $(L_-v, v) \geq 0$ for all $v \in H^1(\mathbb{R}^d)$. Thus, the spectrum of L_- is included in $\mathbb{R}_+$. But $u \in \operatorname{Ker} L_-$ so, by the theorem 3.7, 0 is a simple eigenvalue, so $\operatorname{Ker} L_- = \operatorname{Span}(u)$. □

3.2 Constrained variational problems for L_+ and L_-

We can now go further is our lower estimate (3.2) of $\mathscr{E}(\psi_0) - \mathscr{E}(u)$ for some choice of x_0 and γ.

Using the methods developed by J. Bona [5] for the KdV equation, one can prove ([18]) that there exist functions $x_0(t), \gamma(t)$ that minimize

$$\left\|\nabla\psi(\cdot + x_0, t)e^{i\gamma} - \nabla u\right\|_{L^2(\mathbb{R}^d)}^2 + \lambda^2 \left\|\psi(\cdot + x_0, t)e^{i\gamma} - u\right\|_{L^2(\mathbb{R}^d)}^2 \tag{3.5}$$

and the resulting function $h(x,t) = \psi(x + x_0, t)e^{i\gamma} - u(x)$ has continuous H^1 norm.

Minimization of (3.5) yields (by approximation using smooth functions)

$$\forall 1 \leq j \leq d, \qquad \int_{\mathbb{R}^d} u^{2\sigma} u(x) \partial_{x_j}(x) f(x,t) \mathrm{d}x \tag{3.6}$$

and

$$\int_{\mathbb{R}^d} u^{2\sigma+1}(x) g(x,t) \mathrm{d}x = 0. \tag{3.7}$$

These constraints lead to lower estimates for $(L_+ f, f)$ and $(L_- g, g)$.

Lemma 3.9

There exists $C > 0$ such that if $v \in H^1(\mathbb{R}^d)$ satisfies (3.7), ie. $(u^{2\sigma+1}, v) = 0$, then

$$(L_- v, v) \geq C(v, v).$$

▷ Define

$$\mu = \inf\{(L_- v, v),\ v \in H^1(\mathbb{R}^d), \|v\|_{L^2(\mathbb{R}^d)} = 1, (u^{2\sigma+1}, v) = 0\}.$$

The arguments detailed in the last part of the proof of theorem 3.4 also show that, if $\mu = 0$, it is attained for some function $v_* \neq 0$. Since L_- is nonnegative, this implies $v_* \in \mathrm{Ker}(L_-) = \mathrm{Span}(u)$ by theorem 3.8. Since $\|v_*\|_{L^2(\mathbb{R}^d)} = 1$, we have $v_* = \dfrac{u}{\|u\|_{L^2(\mathbb{R}^d)}}$. Then, $(u^{2\sigma+1}, v_*) = 0$ implies $\left\|u^{\sigma+1}\right\|_{L^2(\mathbb{R}^d)} = 0$ which contradicts $u > 0$.

Thus, $\mu > 0$ and

$$\forall v \in H^1(\mathbb{R}^d) \text{ s.t. } (u^{2\sigma+1}, v) = 0, \qquad (L_- v, v) \geq \mu(v, v) > 0.$$

□

Corollary 3.10

There exists $C > 0$ such that for $v \in H^1(\mathbb{R}^d)$, if $(u^{2\sigma+1}, v) = 0$ then

$$(L_- v, v) \geq C \|v\|_{H^1(\mathbb{R}^d)}^2.$$

▷ By the previous lemma, there exists $C > 0$ such that

$$\forall v \in H^1(\mathbb{R}^d) \text{ s.t. } (u^{2\sigma+1}, v) = 0, \qquad (L_- v, v) \geq C \|v\|_{L^2(\mathbb{R}^d)}^2.$$

Moreover, for $v \in H^1(\mathbb{R}^d)$,

$$(L_- v, v) \geq \|\nabla v\|^2_{L^2(\mathbb{R}^d)} - \sup u^{2\sigma} \|v\|^2_{L^2(\mathbb{R}^d)} .$$

So, for any $C' > 0$,

$$(C'+1)(L_- v, v) \geq C' \|\nabla v\|^2_{L^2(\mathbb{R}^d)} + (C - C' \sup u^{2\sigma}) \|v\|^2_{L^2(\mathbb{R}^d)} .$$

For $0 < C' \leq \dfrac{C}{1+\sup u^{2\sigma}}$, we get

$$(L_- v, v) \geq \frac{C'}{C'+1} \|v\|^2_{H^1(\mathbb{R}^d)} .$$

□

To give a lower estimate for $(L_+ g, g)$, we further require that the solution ψ has the same L^2-norm as the ground state *ie.*

$$\int_{\mathbb{R}^d} |\psi(x,t)|^2 \mathrm{d}x = \int_{\mathbb{R}^d} u(x)^2 \mathrm{d}x.$$

This condition will be relaxed later.

This implies that

$$\int_{\mathbb{R}^d} |u + f + ig|^2 \mathrm{d}x = \int_{\mathbb{R}^d} u^2 \mathrm{d}x$$

which yields

$$(u, f) = -\frac{1}{2} \left[(f,f) + (g,g) \right] . \tag{3.8}$$

We have the following lower estimate for $(L_+ f, f)$ in [18].

Lemma 3.11

There exists constants $c_1, c_2, c_3 > 0$ such that if $f \in H^1(\mathbb{R}^d)$ satisfies (3.6) and (3.8), then

$$(L_+ f, f) \geq c_1 \|f\|^2_{H^1(\mathbb{R}^d)} - c_2 \|\nabla h\|_{L^2(\mathbb{R}^d)} \|h\|^2_{L^2(\mathbb{R}^d)} - c_3 \|h\|^4_{L^2(\mathbb{R}^d)} .$$

▷ Without loss of generality, we can consider $\|u\|_{L^2(\mathbb{R}^d)} = 1$. We write

$$f = f_{/\!/} + f_\perp$$

where

$$f_{/\!/} = (u,f)u = -\frac{1}{2} \left[(f,f) + (g,g) \right] u$$

and

$$f_\perp = f - (u,f)u = f + \frac{1}{2} \left[(f,f) + (g,g) \right] u \in u^\perp .$$

Then,

$$(L_+ f, f) = (L_+ f_{/\!/}, f_{/\!/}) + 2(L_+ f_{/\!/}, f_\perp) + (L_+ f_\perp, f_\perp).$$

First, we consider the functional $\frac{(L_+f_\perp, f_\perp)}{(f_\perp, f_\perp)}$ with the constraints $(u, f_\perp) = 0$ and (3.6). By theorem 3.5, it is nonnegative. The arguments detailed in the last part of the proof of theorem 3.4 also show that, if the infimum is zero, it is attained for some function $f_* \neq 0$. Then, since L_+ is nonnegative on $u^\perp$, $f_* \in \mathrm{Ker}(L_+)$ so $f_* = c \cdot \nabla u$ for some $c \in \mathbb{R}^d$. Then

$$\forall 1 \leq j \leq d, \qquad 0 = \int_{\mathbb{R}^d} u^{2\sigma} \partial_{x_j} u f_* \mathrm{d}x = \sum_{i=1}^{d} c_i \int_{\mathbb{R}^d} u^{2\sigma} (\partial_{x_j} u)(\partial_{x_i} u) \mathrm{d}x$$

But, for $i \neq j$,

$$\int_{\mathbb{R}^d} u^{2\sigma} (\partial_{x_j} u)(\partial_{x_i} u) \mathrm{d}x = \int_{\mathbb{R}^d} u^{2\sigma}(r) \frac{u'(r)^2}{r^2} x_i x_j \mathrm{d}x = 0.$$

It follows that for all $1 \leq j \leq d$,

$$c_j \int_{\mathbb{R}^d} \left(u^\sigma \partial_{x_i} u\right)^2 \mathrm{d}x = 0$$

and $c_j = 0$, which contradicts $f_* \neq 0$.

Therefore, the infimum is positive and we have

$$(L_+ f_\perp, f_\perp) \geq b(f_\perp, f_\perp) = b\left[(f,f) - (f_{/\!/}, f_{/\!/})\right] = b\left[(f,f) - \frac{1}{4}\left((f,f) + (g,g)\right)^2\right]$$

for some $b > 0$.

Furthermore,

$$(L_+ f_{/\!/}, f_{/\!/}) = \frac{1}{4}\left((f,f) + (g,g)\right)^2 (L_+ u, u).$$

Moreover,

$$(L_+ f_\perp, f_{/\!/}) = -\frac{1}{2}\left[(f,f) + (g,g)\right](L_+ f_\perp, u).$$

But

$$\begin{aligned} (L_+ f_\perp, u) &= (\nabla f_\perp, \nabla u) - (2\sigma+1)(f_\perp, u^{2\sigma+1}) \\ &\underset{\text{Hölder}}{\leq} C \left\|\nabla h\right\|_{L^2(\mathbb{R}^d)} + C' \left\|u^{2\sigma+1}\right\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} \left\|f_\perp\right\|_{L^{2^*}(\mathbb{R}^d)} \\ &\underset{\text{Sobolev}}{\leq} C \left\|\nabla h\right\|_{L^2(\mathbb{R}^d)} \end{aligned}$$

so

$$(L_+ f_\perp, f_{/\!/}) \geq -b' \left\|h\right\|^2_{L^2(\mathbb{R}^d)} \left\|\nabla h\right\|_{L^2(\mathbb{R}^d)}$$

for some $b' > 0$.

Putting all this together and using a similar trick as in the proof of corollary 3.10 to get $\|f\|^2_{H^1(\mathbb{R}^d)}$ give the announced result. □

Throughout this section, we thus proved the following theorem.

Theorem 3.12

For $x_0(t)$ and $\gamma(t)$ minimizing (3.5), if $\|\psi(\cdot,t)\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)}$, we have

$$(L_+f, f) + (L_-g, g) \geq c_1 \|h\|^2_{H^1(\mathbb{R}^d)} - c_2 \|h\|^3_{H^1(\mathbb{R}^d)} - c_3 \|h\|^4_{H^1(\mathbb{R}^d)} . \qquad (3.9)$$

3.3 Conlusion

Suppose, as in the previous paragraph, that

$$\|\psi(\cdot,t)\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)} . \qquad (3.10)$$

. Since

$$\sqrt{\min(\lambda^2,1)}\, \|h(\cdot,t)\|_{H^1(\mathbb{R}^d)} \leq \rho_\lambda(\psi(\cdot,t), G_u) \leq \sqrt{\max(\lambda^2,1)}\, \|h(\cdot,t)\|_{H^1(\mathbb{R}^d)} ,$$

injecting (3.9) into (3.2), we have, for $x_0(t), \gamma(t)$ minimizing (3.5),

$$\mathscr{E}(\psi_0) - \mathscr{E}(u) \geq a(\rho_\lambda(\psi(\cdot,t), G_u)) \qquad (3.11)$$

with

$$a(s) = c_1 s^2 (1 - c_2 s^\theta - c_3 s - c_4 s^2)$$

for some constants $c_1, c_2, c_3, c_4, \theta > 0$. What is really needed from a is that a is positive near 0.

We can now prove theorem 3.2 under the assumption (3.10). Let $\varepsilon > 0$ sufficiently small. $\mathscr{E}$ is continuous on $H^1(\mathbb{R}^d)$ so there exists $\delta(\varepsilon) > 0$ such that

$$\rho_\lambda(\psi_0, G_u) < \delta(\varepsilon) \qquad \Longrightarrow \qquad \mathscr{E}(\psi_0) - \mathscr{E}(u) < a(\varepsilon).$$

Since $\mathscr{E}$ is constant in time, (3.11) implies that

$$\forall t > 0, \qquad a(\rho_\lambda(\psi(\cdot,t), G_u)) < a(\varepsilon).$$

Since $\rho_\lambda(\psi(\cdot,t), G_u)$ is continuous in time, for ε and $\delta(\varepsilon)$ sufficiently small, this implies that

$$\rho_\lambda(\psi(\cdot,t), G_u) < \varepsilon.$$

Therefore we have proved the following theorem.

Theorem 3.13

Let $d \geq 3$, $0 < \sigma < \dfrac{2}{d}$ and $\psi_0 \in H^1(\mathbb{R}^d)$ such that $\|\psi_0\|_{L^2(\mathbb{R}^d)} = \|u\|_{L^2(\mathbb{R}^d)}$. Let $\psi(x,t)$ be the unique solution of (3.1). Then u is orbitally stable, ie. , for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if

$$\rho_\lambda(\psi_0, G_u) < \delta(\varepsilon)$$

then

$$\forall t > 0, \qquad \rho_\lambda(\psi(\cdot,t), G_u) < \varepsilon.$$

We will now relax assumption (3.10). First note that with $u_\alpha = \alpha^{\frac{1}{\sigma}} u(\alpha\cdot)$ and $\alpha = \left(\frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}}{\|u\|_{L^2(\mathbb{R}^d)}}\right)^{\frac{2\sigma}{2-\sigma d}}$, we have a ground state u_α satisfying

$$-\Delta u_\alpha + \alpha^2 u_\alpha - u_\alpha^{2\sigma+1} = 0$$

such that $\|u_\alpha\|_{L^2(\mathbb{R}^d)} = \|\psi_0\|_{L^2(\mathbb{R}^d)}$.

We will need the following lemma:

Lemma 3.14

The function

$$A_u : \begin{array}{ccc} \mathbb{R}_+ & \to & H^1(\mathbb{R}^d) \\ \alpha & \mapsto & u_\alpha = \alpha^{\frac{1}{\sigma}} u(\alpha\cdot) \end{array}$$

is continuous near $\alpha = 1$.

$\triangleright$ It suffices to prove this for $\tilde{A}_u : \mathbb{R}_+ \to L^2(\mathbb{R}^d)$. To do so, one can first prove it for $\mathcal{C}_c^\infty$ functions, using the uniform continuity of such functions, and then easily extend it to any function in $L^2(\mathbb{R}^d)$. □

Therefore, there exists $\delta_1(\varepsilon)$ such that

$$|\alpha - 1| < \delta_1(\varepsilon) \qquad \Longrightarrow \qquad \|u_\alpha - u\|_{H^1(\mathbb{R}^d)} < \frac{\varepsilon}{2}.$$

But, for $\alpha = \left(\frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}}{\|u\|_{L^2(\mathbb{R}^d)}}\right)^{\frac{2\sigma}{2-\sigma d}}$, applying lemma B.2 yields

$$\begin{aligned}
|\alpha - 1| &= \frac{\|\psi_0\|_{L^2(\mathbb{R}^d)}^{\frac{2\sigma}{2-\sigma d}} - \|u\|_{L^2}^{\frac{2\sigma}{2-\sigma d}}}{\|u\|_{L^2}^{\frac{2\sigma}{2-\sigma d}}} \\
&\leq \frac{C\left(\|\psi_0\|_{L^2(\mathbb{R}^d)}^{\frac{2\sigma+2-\sigma d}{2-\sigma d}} + \|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\sigma+2-\sigma d}{2-\sigma d}}\right)}{\|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\sigma}{2-\sigma d}}} \left| \|\psi_0\|_{L^2(\mathbb{R}^d)} - \|u\|_{L^2(\mathbb{R}^d)} \right| \\
&\leq C\left| \left\|\psi_0(\cdot + x_0)e^{i\gamma}\right\|_{L^2(\mathbb{R}^d)} - \|u\|_{L^2(\mathbb{R}^d)} \right| \\
&\leq C\left\|\psi_0(\cdot + x_0)e^{i\gamma} - u\right\|_{H^1(\mathbb{R}^d)} \\
&\leq C\rho_\lambda(\psi_0, G_u)
\end{aligned}$$

for x_0, γ minimizing (3.5).

Suppose

$$\rho_\lambda(\psi_0, G_u) \leq \delta_2(\varepsilon)$$

with $\delta_2(\varepsilon) \leq \frac{\delta_1(\varepsilon)}{C}$. Then

$$\|u_\alpha - u\|_{H^1(\mathbb{R}^d)} < \frac{\varepsilon}{2}.$$

We can now conclude the proof. Note that, for every $(x_0, \gamma) \in \mathbb{R}^d \times \mathbb{R}$, we have

$$\left\| \psi(\cdot + x_0, t) e^{i\gamma} - u \right\|_{H^1(\mathbb{R}^d)} \leq \left\| \psi(\cdot + x_0, t) e^{i\gamma} - u_\alpha \right\|_{H^1(\mathbb{R}^d)} + \| u_\alpha - u \|_{H^1}$$

so that, taking the infimum over x_0, γ,

$$C \rho_\lambda(\psi(\cdot, t), G_u) \leq C' \rho_\alpha(\psi(\cdot, t), G_{u_\alpha}) + \frac{\varepsilon}{2}. \tag{3.12}$$

But, as we have seen before,

$$\rho_\alpha(\psi_0, G_{u_\alpha}) \leq C \left\| \psi_0(\cdot + x_0) e^{i\gamma} - u_\alpha \right\|_{H^1(\mathbb{R}^d)}$$

for some constant $C > 0$ and any $(x_0, \gamma) \in \mathbb{R}^d \times \mathbb{R}$. Therefore, for (x_0, γ) minimizing (3.5),

$$\begin{aligned} \rho_\alpha(\psi_0, G_{u_\alpha}) &\leq \left\| \psi_0(\cdot + x_0) e^{i\gamma} - u \right\|_{H^1(\mathbb{R}^d)} + \| u - u_\alpha \|_{H^1(\mathbb{R}^d)} \\ &\leq C \rho_\lambda(\psi_0, G_u) + \frac{\varepsilon}{2} \\ &\leq C \delta_2(\varepsilon) + \frac{\varepsilon}{2} \\ &\leq \delta(\varepsilon) \end{aligned}$$

for $\delta_2(\varepsilon)$ small enough. Applying theorem 3.13 with the ground state u_α instead of u, and injecting the result in (3.12), we get

$$\rho_\lambda(\psi_0, G_u) \leq \delta_2(\varepsilon) \qquad \Longrightarrow \qquad \rho_\lambda(\psi(\cdot, t), G_u) \leq C\varepsilon$$

and theorem 3.2 is proved.

A Sobolev's embeddings

In this section, we take $d \geq 2$ and give useful embeddings of Sobolev spaces. For the sake of concision, we refer the reader to [6] for proofs.

Theorem A.1 (*Sobolev, Gagliardo, Nirenberg*)

For $1 \leq p < d$,

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d) \qquad \text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{d}$$

and there exists a constant $C = C(p,d) > 0$ such that

$$\forall u \in W^{1,p}(\mathbb{R}^d), \qquad \|u\|_{L^{p^*}} \leq C \|\nabla u\|_{L^p}$$

and the embedding $W^{1,p} \hookrightarrow L^{p^*}$ is continuous.

Remark. If there exists a constant $C > 0$ such that

$$\forall u \in W^{1,p}(\mathbb{R}^d), \qquad \|u\|_{L^q} \leq C \|\nabla u\|_{L^p} \tag{A.1}$$

for some $1 \leq q \leq \infty$, then, for $\lambda > 0$ and $u \in W^{1,p}(\mathbb{R}^d)$, since $u_\lambda : x \in \mathbb{R}^d \mapsto u\left(\frac{x}{\lambda}\right)$ is in $W^{1,p}(\mathbb{R}^d)$, we have

$$\|u_\lambda\|_{L^q} \leq C \|\nabla u_\lambda\|_{L^p} \tag{A.2}$$

ie.

$$\|u\|_{L^q} \leq C \lambda^{\frac{d}{p} - \frac{d}{q} - 1} \|\nabla u\|_{L^p} \qquad \forall \lambda > 0 \tag{A.3}$$

which implies $\frac{d}{p} - \frac{d}{q} - 1 = 0$ *ie.* $q = p^*$.

To prove this theorem, we will need the following lemma:

Lemma A.2

Let $d > 2$ and $f_1, f_2, \ldots, f_d \in L^{d-1}(\mathbb{R}^{d-1})$. For $x \in \mathbb{R}^d$ and $1 \leq i \leq d$, let

$$\tilde{x_i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \in \mathbb{R}^{d-1}. \tag{A.4}$$

Then, the function

$$f : x \in \mathbb{R}^d \mapsto f_1(\tilde{x_1}) f_2(\tilde{x_2}) \ldots f_d(\tilde{x_d}) \tag{A.5}$$

is in $L^1(\mathbb{R}^d)$ and

$$\|f\|_{L^1(\mathbb{R}^d)} \leq \prod_{i=1}^{d} \|f_i\|_{L^{d-1}(\mathbb{R}^{d-1})}. \tag{A.6}$$

▷ We prove this result by induction on d.

– The case $d = 2$ amounts to Hölder's inequality that we consider as a know fact.

– Suppose that the statement holds for some $d \geq 2$. Let $f_1, \ldots, f_{d+1} \in L^d(\mathbb{R}^d)$. Let x_{d+1} be given. By Hölder's inequality,

$$\int_{\mathbb{R}^d} |f(x)| \mathrm{d}x_1 \ldots \mathrm{d}x_d \leq \|f_{d+1}\|_{L^d(\mathbb{R}^d)} \|f_1 \ldots f_d(x_{d+1})\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)}. \tag{A.7}$$

where

$$\|f_1 \dots f_d(x_{d+1})\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f_1 \dots f_d|^{\frac{d}{d-1}} \mathrm{d}x_1 \dots \mathrm{d}x_d \right)^{\frac{d-1}{d}} (x_{d+1}). \tag{A.8}$$

The induction hypothesis applied to the functions $|f_1|^{\frac{d}{d-1}}, \dots, |f_d|^{\frac{d}{d-1}}$ yields

$$\begin{aligned} \|f_1 \dots f_d(x_{d+1})\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} &= \left\| |f_1|^{\frac{d}{d-1}} \dots |f_d|^{\frac{d}{d-1}} (x_{d+1}) \right\|_{L^1}^{\frac{d-1}{d}} \\ &\leq \left(\prod_{i=1}^{d} \left\| f_i^{\frac{d}{d-1}} (x_{d+1}) \right\|_{L^{d-1}(\mathbb{R}^{d-1})} \right)^{\frac{d-1}{d}} \\ &\leq \prod_{i=1}^{d} \|f_i(x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}. \end{aligned} \tag{A.9}$$

We allow now x_{d+1} to vary. For $1 \leq i \leq d$, the function $x_{d+1} \mapsto \|f_i(x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}$ is in $L^d(\mathbb{R})$. Then by Hölder's inequality[1], $x_{d+1} \in \mathbb{R} \mapsto \prod_{i=1}^{d} \|f_i(x_{d+1})\|_{L^d(\mathbb{R}^{d-1})}$ is in $L^1(\mathbb{R})$ and

$$\left\| \prod_{i=1}^{d} \|f_i(x_{d+1})\|_{L^d(\mathbb{R}^{d-1})} \right\|_{L^1(\mathbb{R})} \leq \prod_{i=1}^{d} \left\| \|f_i(x_{d+1})\|_{L^d(\mathbb{R}^{d-1})} \right\|_{L^d(\mathbb{R})} = \prod_{i=1}^{d} \|f_i\|_{L^d(\mathbb{R}^d)}. \tag{A.10}$$

Thus, we have

$$\int_{\mathbb{R}^{d+1}} |f(x)| \mathrm{d}x \leq \prod_{i=1}^{d+1} \|f_i\|_{L^d(\mathbb{R}^d)}. \tag{A.11}$$

□

Let us begin the proof of theorem A.1 by the case $p = 1$ with $u \in \mathcal{C}_c^1(\mathbb{R}^d)$. We have

$$|u(x_1, \dots, x_d)| = \left| \int_{-\infty}^{x-1} \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d) \mathrm{d}t \right| \leq \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_1}(t, x_2, \dots, x_d) \right| \mathrm{d}t \tag{A.12}$$

and, similarly, pour $1 \leq i \leq d$,

$$|u(x)| \leq \int_{-\infty}^{+\infty} \left| \frac{\partial u}{\partial x_i}(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_d) \right| \mathrm{d}t \underset{\text{def}}{=} f_i(\tilde{x_i}). \tag{A.13}$$

So

$$|u(x)|^d \leq \prod_{i=1}^{d} f_i(\tilde{x_i}). \tag{A.14}$$

[1]In the form: If $f_1, \dots, f_k$ are such that $f_i \in L^{p_i}(\mathbb{R}^d)$ with $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_k} \leq 1$, then $f = f_1 \dots f_k \in L^p(\mathbb{R}^d)$ and

$$\|f\|_{L^p} \leq \|f_1\|_{L^{p_1}} \dots \|f_k\|_{L^{p_k}}.$$

By the previous lemma, since $f_i^{\frac{1}{d-1}} \in L^{d-1}(\mathbb{R}^{d-1})$,

$$\int_{x\in\mathbb{R}^d} \prod_{i=1}^d f_i^{\frac{1}{d-1}} \mathrm{d}x_1 \dots \mathrm{d}x_d \leq \prod_{i=1}^d \left\| f_i^{\frac{1}{d-1}} \right\|_{L^{d-1}(\mathbb{R}^{d-1})} = \prod_{i=1}^d \|f_i\|_{L^1(\mathbb{R}^{d-1})}^{\frac{1}{d-1}} = \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}}, \tag{A.15}$$

so

$$\int_{\mathbb{R}^d} |u(x)|^{\frac{d}{d-1}} \mathrm{d}x \leq \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d-1}}. \tag{A.16}$$

From this, we get

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}}. \tag{A.17}$$

Let $t \geq 1$. Applying A.17 with $|u|^{t-1}|u$ instead of u yields

$$\|u\|_{L^{\frac{td}{d-1}}(\mathbb{R}^d)}^t \leq t \prod_{i=1}^d \left\| |u|^{t-1} \frac{\partial u}{\partial x_i} \right\|_{L^1(\mathbb{R}^d)}^{\frac{1}{d}} \leq t \, \|u\|_{L^{\frac{p(t-1)}{p-1}}(\mathbb{R}^d)}^{t-1} \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^d)}^{\frac{1}{d}} \tag{A.18}$$

by Hölder's inequality.

We can now chose t such that $\dfrac{td}{d-1} = \dfrac{p(t-1)}{p-1}$ *ie.* $t = \dfrac{p(d-1)}{d-p} = \dfrac{d-1}{d} p^* \geq 1$ since $1 \leq p < d$. Then,

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq t \prod_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\mathbb{R}^d)}^{\frac{1}{d}} \tag{A.19}$$

so

$$\forall u \in \mathcal{C}_c^1(\mathbb{R}^d), \qquad \|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u\|_{L^p(\mathbb{R}^d)}. \tag{A.20}$$

Let $u \in W^{1,p}(\mathbb{R}^d)$. There exists a sequence $(u_n) \in \mathcal{C}_c^1(\mathbb{R}^d)$ such that $u_n \xrightarrow{W^{1,p}(\mathbb{R}^d)} u$. We consider, taking subsequences if needed, that $u_n \to u$ almost everywhere. Then, for all n,

$$\|u_n\|^{L^{p^*}(\mathbb{R}^d)} \leq C \|\nabla u_n\|_{L^p(\mathbb{R}^d)}. \tag{A.21}$$

Fatou's lemma yields $u \in L^{p^*}(\mathbb{R}^d)$ and

$$\|u\|_{L^{p^*}} = \left\| \liminf_{n\in\mathbb{N}} u_n \right\|_{L^{p^*}(\mathbb{R}^d)} \leq \liminf_{n\in\mathbb{N}} \|u_n\|_{L^{p^*}(\mathbb{R}^d)} \leq C \liminf_{n\in\mathbb{N}} \|\nabla u_n\|_{L^p(\mathbb{R}^d)} = \|\nabla u\|_{L^p} < \infty \tag{A.22}$$

which concludes the proof.

We deduce from theorem A.1 the following result:

Corollary A.3

Let $1 \leq p < d$. Then

$$\forall q \in [p, p^*], \qquad W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$$

and the embeddings are continuous.

▷ For $p \leq q \leq p^*$, there exists $0 \leq \alpha \leq 1$ such that

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p^*} \qquad ie. \qquad q = \frac{pp^*}{\alpha p^* + (1-\alpha)p} \tag{A.23}$$

Then, by Hölder's inequality,

$$\|u\|_{L^q} = \left\| |u|^{\alpha q} |u|^{(1-\alpha)q} \right\|_{L^1}^{\frac{1}{q}} \leq \| |u|^{\alpha q} \|_{L^{\frac{p}{\alpha q}}(\mathbb{R}^d)}^{\frac{1}{q}} \left\| |u|^{(1-\alpha)q} \right\|_{L^{\frac{p^*}{(1-\alpha)q}}(\mathbb{R}^d)}^{\frac{1}{q}} = \|u\|_{L^p(\mathbb{R}^d)}^{\alpha} \|u\|_{L^{p^*}(\mathbb{R}^d)}^{1-\alpha}. \tag{A.24}$$

Applying Young's inequality, we have:

$$\|u\|_{L^q} \leq \alpha \|u\|_{L^p(\mathbb{R}^d)} + (1-\alpha) \|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \|u\|_{L^p(\mathbb{R}^d)} + \|u\|_{L^{p^*}(\mathbb{R}^d)} \tag{A.25}$$

and by virtue of theorem A.1, there exists $C > 0$ such that

$$\forall u \in W^{1,p}(\mathbb{R}^d), \qquad \|u\|_{L^q(\mathbb{R}^d)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^d)}. \tag{A.26}$$

□

Theorem A.4

Let $m \in \mathbb{N}^*$ and $1 \leq p < \infty$. We have,

if $\frac{1}{p} - \frac{m}{d} > 0$ then $W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ where $\frac{1}{q} = \frac{1}{p} - \frac{m}{d}$;

if $\frac{1}{p} - \frac{m}{d} = 0$ then $W^{m,p}(\mathbb{R}^d) \subset L^q(\mathbb{R}^d)$ for all $p \leq q < \infty$;

if $\frac{1}{p} - \frac{m}{d} < 0$ then $W^{m,p} \subset L^\infty(\mathbb{R}^d)$;

and the respective embeddings are continuous.

Moreover, if $m - \frac{d}{p} > 0$ is not an integer, then $W^{m,p}(\mathbb{R}^d) \subset \mathcal{C}^k(\mathbb{R}^d)$ where $k = \left\lfloor m - \frac{d}{p} \right\rfloor$.

If Ω is an open subset of $\mathbb{R}^d$ of class $\mathcal{C}^1$ with bounded boundary, or $\Omega = \mathbb{R}^d_+$, we can replace $\mathbb{R}^d$ by Ω in the above theorem.

B Lagrange multipliers and regularity of functionals on $H^1(\mathbb{R}^d)$

Langrange multipliers

We state and prove a very useful theorem associated with the study of constrained minimization problems. We follow the proof given in [11] (p. 225).

Theorem B.1 (*Lagrange multipliers*)

Let $(E, \|.\|)$ a Banach space, Ω an open subset of E, $f : \Omega \to \mathbb{R}$ a differentiable function on Ω, and $\Phi : \Omega \to \mathbb{R}^n$ a function of class $\mathcal{C}^1$ on Ω. We write $\Phi = (\Phi_1, \ldots, \Phi_n)$ where, for $1 \leq i \leq n$, $\Phi_i : \Omega \to \mathbb{R}$ is the i^{th}-component of Φ. Let $a \in \mathbb{R}^n$ and suppose $\Phi^{-1}(a) \neq \varnothing$.

If $f(x_0) = \min_{x \in \Phi^{-1}(a)} f(x)$ for some $x_0 \in \Phi^{-1}(a)$ and if $D\Phi(x_0) \in \mathscr{L}(E, \mathbb{R}^n)$ is onto, then there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$Df(x_0) = \lambda_1 D\Phi_1(x_0) + \cdots + \lambda_n D\Phi_n(x_0).$$

$\triangleright$ Since $\dim \left(E/_{\operatorname{Ker} D\Phi(x_0)}\right) = \operatorname{rank} D\Phi(x_0) = n$, there exists a subspace F of E such that $\dim F = n$ and $E = \operatorname{Ker} D\Phi(x_0) \oplus F$. We will write χ the restriction of $D\Phi(x_0)$ to F. χ is an isomorphism between F and $\mathbb{R}^n$.

For x near 0, we define

$$\Psi(x) = \Phi(x_0 + x) - a.$$

It is clear that $\Psi(0) = 0$ and $D\Psi(0) = D\Phi(0)$. Let Π_1 be the projection $E \to \operatorname{Ker} D\Phi(x_0)$ and define

$$h = \chi^{-1} \circ \Psi + \Pi_1$$

in a neighborhood of 0. We have

$$Dh(0) = D(\chi^{-1})(0) \circ D\Psi(0) + \Pi_1 = \chi^{-1} \circ D\Psi(0) + \Pi_1 = \operatorname{Id}_E.$$

Thus, by the inverse function theorem, there exist two neighborhood U_1 and U_2 of 0 such that h is a diffeomorphism between U_1 and U_2.

Let Π_2 be the projection $E \to F$. We have $\Pi_2 \circ h = \chi^{-1} \circ \Psi$, so $\chi \circ \Pi_2 \circ h = \Psi$. Moreover, since $D\Psi(0) = D\Phi(x_0) = \chi \circ \Pi_2$, we have $D\Psi(0) \circ h = \Psi$.

Let us now prove that $\operatorname{Ker} D\Phi(x_0) \subset \operatorname{Ker} Df(x_0)$. Let $v \in \operatorname{Ker} D\Phi(x_0) = \operatorname{Ker} D\Psi(0)$ and let $\gamma_1 : t \in]-\varepsilon, \varepsilon[\mapsto tv \in \operatorname{Ker} D\Psi(0)$ where $\varepsilon > 0$. For ε sufficiently small, $\gamma_1(]-\varepsilon, \varepsilon[) \subset U_2 \cap \operatorname{Ker} D\Psi(0)$. Let then $\gamma_2 = h^{-1} \circ \gamma_1$. Since $D\Psi(0) \circ h = \Psi$, we have $\psi \circ \gamma_2 = D\Psi(0) \circ \gamma_1 = 0$ so

$$\forall |t| < \varepsilon, \qquad \Phi(x_0 + \gamma_2(t)) - a = 0$$

ie. $x_0 + \gamma_2(t) \in \Phi^{-1}(a)$ pour any $|t| < \varepsilon$. Then the path $\gamma_3(t) = x_0 + \gamma_2(t)$ takes its values in $\Phi^{-1}(a)$. We have $\gamma_3(0) = x_0$ and $\gamma'(0) = \gamma_2'(0) = Dh^{-1}(h(0))v = Dh(0)v = v$. Since $f \circ \gamma_3$ has a minimum in 0, we have $(f \circ \gamma_3)'(0) = 0$ *ie.* $Df(x_0)v = 0$ and $v \in \operatorname{Ker} Df(x_0)$.

Next, we note that $(D\Phi_1(x_0)_{|F}, \ldots, D\Phi_n(x_0)_{|F})$ is a basis of F^*. Since this set has n elements and $\dim F^* = \dim F = n$, it suffices to prove that the elements are linearly independent. Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\lambda_1 D\Phi_1(x_0)_{|F} \cdots + \lambda_n D\Phi_n(x_0)_{|F} = 0. \tag{B.1}$$

Since Ran $D\Phi(x_0) = \mathbb{R}^n$, the canonical basis $(e_1, \ldots, e_n)$ lies in Ran $D\Phi(x_0)$ and, for each $1 \leq i \leq n$, there exists $x^i \in F$ such that $D\Phi(x_0)x^i = e_i$, *ie.* $D\Phi_k(x_0)x^i = \delta_{ik}$ for all $1 \leq k \leq n$. Applying (B.1) to such x^i yields λ_i for each $1 \leq i \leq n$.

Finally, since $Df(x_0)_{|F} \in F^*$, there exist $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\forall x \in F, \qquad Df(x_0)x = \lambda_1 D\Phi_1(x_0)x + \cdots + \lambda_n D\Phi_n(x_0)x.$$

Since $\operatorname{Ker} D\Phi(x_0) \subset \operatorname{Ker} Df(x_0)$ and $E = F \oplus \operatorname{Ker} D\Phi(x_0)$, we can conclude that the above equation is in fact valid for any $x \in E$. $\square$

Regularity of some functionals on $H^1(\mathbb{R}^d)$

In order to prove the existence of a solution to equation (2.1), we use the above theorem with $E = H^1(\mathbb{R}^d)$, $f = T$ and $\Phi = V$, where

$$\forall u \in H^1(\mathbb{R}^d), \qquad T(u) = \int_{\mathbb{R}^d} |\nabla u|^2 \mathrm{d}x \qquad V(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2}|u|^{2\sigma+2} - \frac{\lambda^2}{2}|u|^2 \right) \mathrm{d}x.$$

Let us now verify that T and V are indeed $\mathcal{C}^1$ functionals on $(H^1(\mathbb{R}^d), \|.\|_{H^1(\mathbb{R}^d)})$.

For $u, h \in H^1(\mathbb{R}^d)$, we have

$$T(u+h) = T(u) + 2\int_{\mathbb{R}^d} \nabla u \cdot \nabla h \mathrm{d}x + \underbrace{T(h)}_{\underset{h\to 0}{=}\mathcal{O}(\|h\|_{H^1}^2)}.$$

The application $\ell(u) : h \in H^1(\mathbb{R}^d) \mapsto 2\int_{\mathbb{R}^d} \nabla u \cdot \nabla h \mathrm{d}x$ is obviously linear and, by Cauchy-Swcharz's inequality,

$$|\ell(u)h| \leq 2\,\|\nabla u\|_{L^2(\mathbb{R}^d)}\,\|\nabla h\|_{L^2(\mathbb{R}^d)} \leq 2\,\|u\|_{H^1(\mathbb{R}^d)}\,\|h\|_{H^1(\mathbb{R}^d)}$$

and $\ell(u)$ is continuous on $H^1(\mathbb{R}^d)$. This proves that T is differentiable on $H^1(\mathbb{R}^d)$, that $DT(u) = \ell(u)$ for any $u \in H^1(\mathbb{R}^d)$ and that $\|DT(u)\|_{H^1(\mathbb{R}^d)'} \leq \|u\|_{H^1(\mathbb{R}^d)}$. Since $DT : u \in H^1(\mathbb{R}^d) \mapsto DT(u) \in H^1(\mathbb{R}^d)'$ is linear, this proves that T is $\mathcal{C}^1$ on $H^1(\mathbb{R}^d)$.

For $u \in H^1(\mathbb{R}^d)$, we have, for $h \in H^1(\mathbb{R}^d)$,

$$\begin{aligned} V(u+h) &= \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2}|u+h|^{2\sigma+2} - \frac{\lambda^2}{2}|u+h|^2 \right) \mathrm{d}x \\ &= \int_{\mathbb{R}^d} \left(\frac{1}{2\sigma+2}\left(|u|^{2\sigma+2} + (2\sigma+2)|u|^{2\sigma}uh + |h|^2\varepsilon(h)\right) - \frac{\lambda^2}{2}(|u|^2 + 2uh + |h|^2) \right) \mathrm{d}x \end{aligned}$$

where ε is bounded so that

$$V(u+h) = V(u) + \underbrace{\int_{\mathbb{R}^d} (|u|^{2\sigma}u - \lambda^2 u)h\mathrm{d}x}_{\ell(u)h} + \underbrace{\int_{\mathbb{R}^d} |h|^2(1+\varepsilon(h))\mathrm{d}x}_{\underset{h\to 0}{=}\mathcal{O}(\|h\|_{H^1}^2)}.$$

$\ell(u)$ is linear and, for $h \in H^1(\mathbb{R}^d)$, by Sobolev's embedding theorem and Hölder's inequality,

$$\begin{aligned} |\ell(u)h| &\leq \left\||u|^{2\sigma+1}\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)} \|h\|_{L^{2\sigma+2}(\mathbb{R}^d)} + \lambda^2\,\|u\|_{L^2(\mathbb{R}^d)}\,\|h\|_{L^2(\mathbb{R}^d)} \\ &\leq C(\|u\|_{L^{2\sigma+2}(\mathbb{R}^d)}^{2\sigma+1} + \|u\|_{L^2(\mathbb{R}^d)})\,\|h\|_{H^1(\mathbb{R}^d)} \end{aligned}$$

so $\ell(u)$ is continuous. This proves that V is differentiable on $H^1(\mathbb{R}^d)$, that $DV(u) = \ell(u)$ for any $u \in H^1(\mathbb{R}^d)$ and that $\|DV(u)\|_{H^1(\mathbb{R}^d)'} \leq C(\|u\|^{2\sigma+1}_{L^{2\sigma+2}(\mathbb{R}^d)} + \|u\|_{L^2(\mathbb{R}^d)})$.

Let us now consider $u \in H^1(\mathbb{R}^d)$ and a sequence $(u_n)_{n\geq 0} \in H^1(\mathbb{R}^d)^{\mathbb{N}}$ such that

$$u_n \xrightarrow[n\to+\infty]{H^1(\mathbb{R}^d)} u.$$

As before, we can prove that

$$\|DV(u) - DV(u_n)\|_{H^1(\mathbb{R}^d)'} \leq C\left(\left\||u|^{2\sigma}u - |u_n|^{2\sigma}u_n\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)} + \|u-u_n\|_{L^2(\mathbb{R}^d)}\right).$$

In order to prove that DV is continuous, it then suffices to prove that

$$\left\||u|^{2\sigma}u - |u_n|^{2\sigma}u_n\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)} \xrightarrow[n\to 0]{} 0.$$

We will use the following lemma.

Lemma B.2

For any $p > 0$, there exists a constant $C > 0$ such that

$$\forall a, b \in \mathbb{R}, \qquad ||a|^p a - |b|^p b| \leq C|a-b|\left(|a|^p + |b|^p\right).$$

$\triangleright$ Take $a, b \in \mathbb{R}$. If $b = 0$ or $a = b$ then $C = 1$ satisfies the condition. We can suppose that $b \neq 0$ and $b \neq a$, define $s = \dfrac{a}{b}$ and consider the function

$$f(t) = \frac{|t|^p t - 1}{(|t|^p + 1)|t-1|}.$$

f is continuous on $\mathbb{R}\backslash\{1\}$ and $f(t) \xrightarrow[|t|\to\infty]{} 1$. Moreover, $f(t) \underset{t\to 1}{\sim} \operatorname{sgn}(t-1)\dfrac{p+1}{2}$. Thus f is bounded and there exists $C > 0$ such that

$$\forall t \neq 1, \qquad ||t|^p t - 1| \leq C|t-1|(|t|^p + 1).$$

For $t = s$, we have

$$\left|\left|\frac{a}{b}\right|^p \frac{a}{b} - 1\right| \leq C\left|\frac{a}{b} - 1\right|\left(\left|\frac{a}{b}\right|^p + 1\right)$$

ie.

$$|a|^p a - |b|^p b| \leq C|a-b|(|a|^p + |b|^p).$$

□

This lemma proves that

$$\left\||u|^{2\sigma}u - |u_n|^{2\sigma}u_n\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)} \leq C\left\|(|u|^{2\sigma} + |u_n|^{2\sigma})|u-u_n|\right\|_{L^{\frac{2\sigma+2}{2\sigma+1}}(\mathbb{R}^d)}$$

and, by Hölder's inequality,

$$\begin{aligned}
\left\||u|^{2\sigma}u - |u_n|^{2\sigma}u_n\right\|_{L^{2\frac{2\sigma+2}{2\sigma=1}}(\mathbb{R}^d)} &\leq C\left(\left\||u|^{2\sigma} + |u_n|^{2\sigma}\right\|_{L^{\frac{2\sigma+2}{2\sigma}}(\mathbb{R}^d)}\right)\|u-u_n\|_{L^{2\sigma+2}(\mathbb{R}^d)} \\
&\leq C\left(\left\||u|^{2\sigma}\right\|_{L^{\frac{2\sigma+2}{2\sigma}}(\mathbb{R}^d)} + \left\||u_n|^{2\sigma}\right\|_{L^{\frac{2\sigma+2}{2\sigma}}(\mathbb{R}^d)}\right)\|u-u_n\|_{L^{2\sigma+2}(\mathbb{R}^d)} \\
&\leq C\left(\|u\|^{2\sigma}_{L^{2\sigma+2}(\mathbb{R}^d)} + \|u_n\|^{2\sigma}_{L^{2\sigma+2}(\mathbb{R}^d)}\right)\|u-u_n\|_{L^{2\sigma+2}(\mathbb{R}^d)}.
\end{aligned}$$

By Sobolev's embedding theorem, $\|u_n\|_{L^{2\sigma+2}(\mathbb{R}^d)}$ is bounded and $\|u - u_n\|_{L^{2\sigma+2}(\mathbb{R}^d)} \to 0$, which concludes the proof that V is a $\mathcal{C}^1$ functional on $H^1(\mathbb{R}^d)$.

C Compactness lemmas

To prove the existence of a ground state, we use the following, more general that needed, compactness lemma, due to Strauss [14]. The following proof is due to H. Berestycki and P.-L. Lions [4].

Lemma C.1

Let $P, Q : \mathbb{R} \to \mathbb{R}$ be two continuous functions satisfying

$$\frac{P(s)}{Q(s)} \xrightarrow[|s|\to+\infty]{} 0. \tag{C.1}$$

Let (u_n) be a sequence of measurable functions $\mathbb{R}^d \to \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} |Q(u_n(x))| dx < +\infty. \tag{C.2}$$

and

$$P(u_n(x)) \xrightarrow[n\to+\infty]{} v(x) \text{ a.e. in } \mathbb{R}^d. \tag{C.3}$$

Then for any bounded Borel set B one has

$$\int_B |P(u_n(x)) - v(x)| dx \xrightarrow[n\to+\infty]{} 0.$$

If one further assumes that

$$\frac{P(s)}{Q(s)} \xrightarrow[s\to 0]{} 0 \tag{C.4}$$

and

$$u_n(x) \xrightarrow[|x|\to+\infty]{} 0 \text{ uniformly with respect to } n, \tag{C.5}$$

then $(P(u_n))_{n\in\mathbb{N}}$ converges to v in $L^1(\mathbb{R}^d)$.

To prove this, we will need the following results, that we prove as in [9].

Theorem C.2 (*Egorov*)

Let (E, Σ, μ) be a measure space such that $\mu(E) < \infty$. Let f_n, f be measurable functions such that (f_n) converges to f almost everywhere on E. Then, for all $\varepsilon > 0$, there exists $A \in \Sigma$ such that $\mu(A) < \varepsilon$ and (f_n) converges uniformly to f on $E \backslash A$.

▷ For $n, k \geq 1$ define

$$E_{k,n} = \bigcap_{i\geq n} \left\{ x \in E,\, |f_i(x) - f(x)| \leq \frac{1}{k} \right\}.$$

For $k \geq 1$,

$$\forall n \geq 1, \qquad E_{k,n} \subset E_{k,n+1}$$

so

$$\mu\left(\bigcup_{n\geq 1} E_{k,n}\right) = \lim_{n\to+\infty} \mu(E_{k,n}).$$

Since, (f_n) converges to f almost everywhere on E, for all $k \geq 1$,

$$\lim_{n\to+\infty} \mu(E_{k,n}) = \mu\left(\bigcup_{n\geq 1} E_{k,n}\right) = \mu(E).$$

Then, for $\varepsilon > 0$, since $\mu(E) < \infty$, for any $k \geq 1$, there exists $n_k \geq 0$ such that

$$\mu(E_{k,n_k}) \geq \mu(E) - 2^{-k}\varepsilon.$$

Now, set

$$A = \bigcup_{k\geq 1} (E \backslash E_{k,n_k}).$$

We have $\mu(A) \leq \varepsilon$ and $E\backslash A = \bigcap_{k\geq 1} E_{k,n_k}$ so that

$$\forall k \geq 1, \forall i \geq n_k, \forall x \in E\backslash A, \qquad |f_i(x) - f(x)| \leq \frac{1}{k}$$

and (f_n) converges uniformly to f on $E\backslash A$. □

Lemma C.3

Let (E, Σ, μ) be a measure space such that $\mu(E) < \infty$, $f_n, f \in L^1(E)$ such that $f_n \to f$ almost everywhere in E and (f_n) is uniformly integrable. Then, f_n converges to f in $L^1(E)$.

▷ Let $\varepsilon > 0$ be constant. The uniform integrability implies that there exists $\delta > 0$ such that for all $A \in \Sigma$,

$$|A| < \delta \qquad \Longrightarrow \qquad \sup_{n\in\mathbb{N}} \int_A |f_n| \mathrm{d}\mu < \varepsilon.$$

Indeed, for any $K > 0$,

$$\begin{aligned} \int_A |f_n| \mathrm{d}\mu &= \int_{\{|f_n|\leq K\}\cap A} |f_n| \mathrm{d}\mu + \int_{\{|f_n|\geq K\}\cap A} |f_n| \mathrm{d}\mu \\ &\leq K\mu(A) + \frac{\varepsilon}{2} \end{aligned}$$

for K large enough so that, if $\mu(A) < \varepsilon(2K)^{-1} = \delta$, we have the result.

But, by Egorov's theorem, there exists $A \in \Sigma$ such that $\mu(A) < \delta$ and (f_n) converges uniformly to f in $E\backslash A$. Then,

$$\int_{E\backslash A} |f_n - f| \mathrm{d}x \leq \mu(E) \sup_{x\in B\backslash A} |f_n(x) - f(x)| \xrightarrow[n\to+\infty]{} 0$$

so there exists $n_0 \in \mathbb{N}$ such that

$$\forall n \geq n_0, \qquad \int_{E\setminus A} |f_n - f| \mathrm{d}x \leq \varepsilon.$$

Then, for $n \geq n_0$,

$$\begin{aligned}
\int_E |f_n - f| \mathrm{d}x &\leq \int_{E\setminus A} |f_n - f| \mathrm{d}x + \int_A |f_n| \mathrm{d}x + \int_A |f| \mathrm{d}x \\
&\leq 2\varepsilon + \int_A |f| \mathrm{d}x \\
&\underset{\text{Fatou}}{\leq} 2\varepsilon + \liminf_{n\in\mathbb{N}} \underbrace{\int_A |f_n| \mathrm{d}x}_{\leq \varepsilon} \leq 3\varepsilon
\end{aligned}$$

□

Let us now prove the compactness lemma.

▷ Let us first prove that $P(u_n)$ is uniformly integrable on B. By condition (C.1), there exists $C > 0$ such that

$$\forall x \in \mathbb{R}^d, \qquad |P(u_n(x))| \leq C(1 + |Q(u_n(x))|).$$

Thus, by condition (C.2), $P(u_n) \in L^1(B)$. This implies, by Fatou's lemma:

$$\begin{aligned}
\int_B |v(x)| \mathrm{d}x &= \int_B \left| \liminf_{n\in\mathbb{N}} P(u_n(x)) \right| \mathrm{d}x \\
&\leq \liminf_{n\in\mathbb{N}} \int_B |P(u_n(x))| \mathrm{d}x \\
&\leq \liminf_{n\in\mathbb{N}} C \left(|B| + \int_B |Q(u_n(x))| \mathrm{d}x \right) < \infty,
\end{aligned}$$

because of (C.2), *ie.* $v \in L^1(B)$ as well. Next, define

$$\varphi : K > 0 \mapsto \begin{cases} \inf \{|s|, \, |P(s)| \geq K\} & \text{if } \{|s|, \, |P(s)| \geq K\} \neq \varnothing \\ K & \text{otherwise.} \end{cases}$$

Then, since P is continuous, $\varphi(K) \xrightarrow[K\to+\infty]{} +\infty$. By definition of φ, we have

$$\int_{\{|P(u_n(x))| \geq k\} \cap B} |P(u_n(x))| \mathrm{d}x \leq \int_{\{|u_n(x)| \geq \varphi(K)\} \cap B} |P(u_n(x))| \mathrm{d}x.$$

Now, by condition (C.1), there exists a function ε such that

$$\forall s \in \mathbb{R}, \qquad |P(s)| \leq \varepsilon(s) Q(s)$$

with $\varepsilon(s) \xrightarrow[|s|\to+\infty]{} 0$. Then,

$$\begin{aligned}\int_{\{|P(u_n(x))|\geq K\}\cap B} |P(u_n(x))|\mathrm{d}x &\leq \int_{\{|u_n(x)|\geq\varphi(K)\}\cap B} \varepsilon(u_n(x))|Q(u_n(x))|\mathrm{d}x \\ &\leq C\varepsilon(\varphi(K)) \int_B |Q(u_n(x))|\mathrm{d}x \\ &\leq C\epsilon(K)\end{aligned}$$

with $\epsilon(K) \xrightarrow[K\to+\infty]{} 0$. This shows the uniform integrability and, thanks to lemma C.3, $P(u_n)$ converges to v in $L^1(B)$.

Let us now prove the second part. Let $\varepsilon > 0$. Thanks to conditions (C.4) and (C.5), there exists $R_0 > 0$ such that

$$|x| \geq R_0 \qquad \Longrightarrow \qquad \forall n \in \mathbb{N},\ |P(u_n(x))| \leq \varepsilon|Q(u_n(x))|.$$

Therefore, $P_n(u_n)$ and,by Fatou's lemma, v are in $L^1(\mathbb{R}^d)$ and

$$\int_{|x|\geq R_0\}} |P(u_n(x))|\mathrm{d}x \leq \varepsilon C, \qquad \int_{\{|x|\geq R_0\}} |v(x)|\mathrm{d}x \leq \varepsilon C.$$

But from the first part of the lemma, there exists n_0 such that for $n \geq n_0$,

$$\int_{|x|\leq R_0} |P(u_n(x)) - v(x)|\mathrm{d}x \leq \varepsilon$$

so

$$\forall n \geq n_0, \qquad \int_{\mathbb{R}^d} |P(u_n(x)) - v(x)|\mathrm{d}x \leq (2C+1)\varepsilon.$$

□

To prove the stability of the ground state, we estimate $(L_- v, v)$ for $v \in H^1(\mathbb{R}^d)$ satisfying some constraint. We need the following lemma.

Lemma C.4

Let $1 < p < \infty$ and $(f_n)_{n\in\mathbb{N}}$ be bounded in $L^p(\mathbb{R}^d)$. If (f_n) converges almost everywhere to some function f, then $f \in L^p(\mathbb{R}^d)$ and (f_n) converges weakly to f in $L^p(\mathbb{R}^d)$.

▷ We follow the proof in [11] (p. 256).

We begin by proving that $f \in L^p(\mathbb{R}^d)$. Define

$$\forall n \in \mathbb{N}, \qquad g_n = \inf_{k\geq n} f_k.$$

Then, for any $n \in \mathbb{N}$, g_n is a measurable nonnegative function such that $g_n \leq |f_n|$ almost everywhere. Thus, $g_n \in L^p(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Since $g_n \leq g_{n+1}$ a.e and (g_n) converges a.e to $|f|$, by the monotone convergence theorem, $f \in L^p(\mathbb{R}^d)$.

Now, we prove that it suffices to show that if (f_n) is bounded in $L^p(\mathbb{R}^d)$ and converges weakly to 0 in $L^p(\mathbb{R}^d)$ and a.e to a function f, then, necessarily, $f = 0$ a.e. Indeed suppose that this is true. (f_n) is bounded in $L^p(\mathbb{R}^d)$ so, since $L^p(\mathbb{R}^d)$ is reflexive for $1 < p < \infty$, there exists $h \in L^p(\mathbb{R}^d)$ such that (f_n), up to a subsequence, converges weakly to h in $L^p(\mathbb{R}^d)$. Then, $(f_n - h)$ is a bounded sequence converging weakly to 0 in $L^p(\mathbb{R}^d)$ and a.e to $f - h$. Under our assumption, this implies $h = f$ and the lemma is proved.

We suppose now on that (f_n) is a bounded sequence in $L^p(\mathbb{R}^d)$ converging weakly to 0 and a.e to f. Let $K \subset \mathbb{R}^d$ be a compact set and define

$$S_K = \{x \in K,\, f(x) > 0\}.$$

Since f is measurable, so is S_K. For $n \in \mathbb{N}$, define

$$A_K^n = \{x \in S_K,\, f_n(x) > 0\} \qquad \text{and} \qquad B_K^n = \{x \in S_K,\, f_n(x) < 0\}$$

which are also measurable sets.

Since $(\mathbf{1}_{B_K^n})$ converges to 0 a.e and $\mathbf{1}_{B_K^n} \leq \mathbf{1}_K$ which is integrable, the dominated convergence theorem ensures that $(\mathbf{1}_{B_K^n})$ converges to 0 in $L^1(\mathbb{R}^d)$. Then, by Hölder's inequality

$$\left| \int_{B_K^n} f_n \mathrm{d}x \right| \leq \underbrace{\left(\int_{\mathbb{R}^d} |f_n|^p \mathrm{d}x \right)^{\frac{1}{p}}}_{\text{bounded}} \left(\int_{\mathbb{R}^d} \mathbf{1}_{B_K^n} \mathrm{d}x \right)^{1-\frac{1}{p}} \xrightarrow[n\to+\infty]{} 0.$$

Moreover, since $S_K \subset K$ which is compact, $\mathbf{1}_{S_K} \in L^s(\mathbb{R}^d)$ for any $s \geq 1$. Since (f_n) converges weakly to 0 in $L^p(\mathbb{R}^d)$, this implies that

$$\int_{S_K} f_n \mathrm{d}x \xrightarrow[n\to+\infty]{} 0.$$

Thus,

$$\int_{A_K^n} f_n \mathrm{d}x = \int_{S_K} f_n \mathrm{d}x - \int_{B_K^n} f_n \mathrm{d}x \xrightarrow[n\to+\infty]{} 0.$$

Since $f_n \mathbf{1}_{A_K^n}$ is nonnegative almost everywhere, this means that $(f_n \mathbf{1}_{A_K^n})$ converges to 0 in $L^1(\mathbb{R}^d)$, so, up to a subsequence, $(f_n \mathbf{1}_{A_K^n})$ converges to 0 a.e. Since $(f_n \mathbf{1}_{A_K^n})$ converges a.e to $f \mathbf{1}_{S_K}$, this implies

$$f(x) \leq 0 \qquad \text{a.e in } K.$$

Since K is arbitrary, we have $f \leq 0$ almost everywhere in $\mathbb{R}^d$. Replacing f_n by $(-f_n)$ shows that $f \geq 0$ almost everywhere in $\mathbb{R}^d$, so $f = 0$ a.e in $\mathbb{R}^d$. □

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