

Robustness of timed automata:
computing the maximally-permissive strategies

Emily Clement^{1,2,3} and
Thierry Jéron² Nicolas Markey² David Menré³

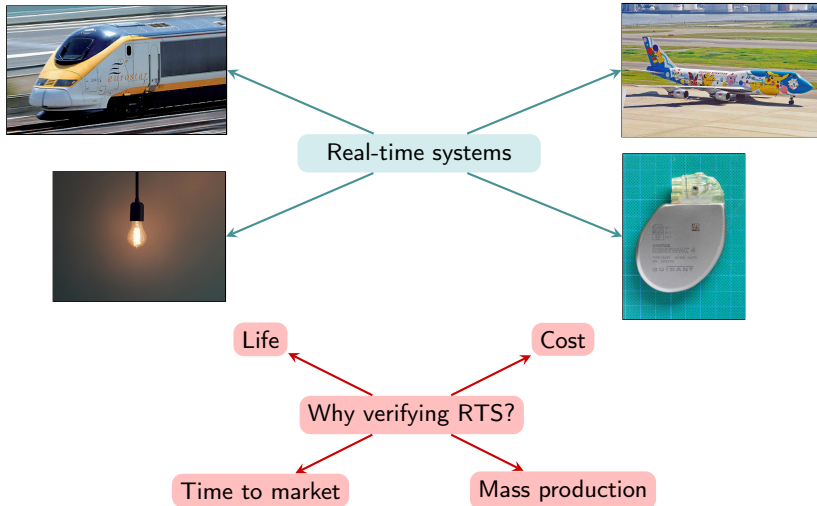
¹Université Paris Cité, CNRS, IRIF, F-75013, Paris, France

²IRISA, Inria & CNRS & Univ. Rennes, France

³Mitsubishi Electric R&D Centre Europe – Rennes, France: MERCE

November 17 2023

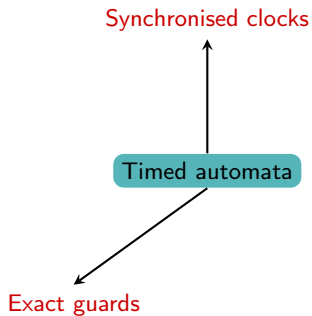
Why verifying real-time systems?

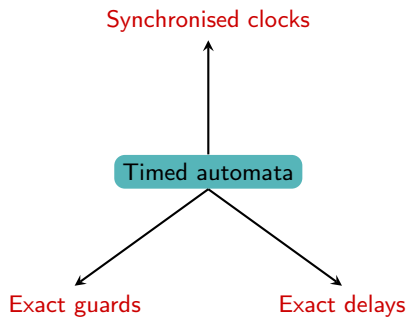


Synchronised clocks



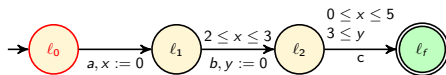
Timed automata



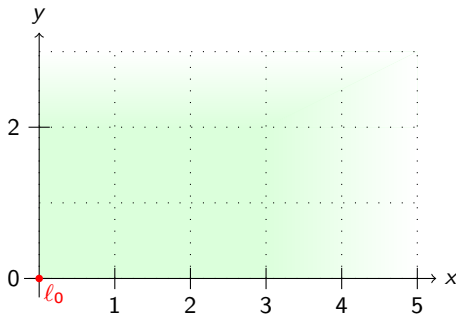


Example of perturbed semantics: clock drifting¹

- Timed automaton \mathcal{A}



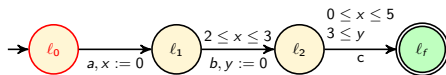
- Run with desynchronised clocks: $0.9 \cdot \dot{y} \leq \dot{x} \leq 1.1 \cdot \dot{y}$



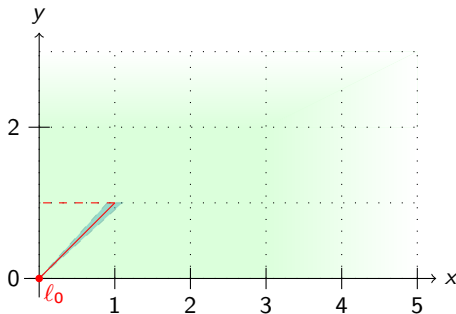
¹Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: clock drifting¹

- Timed automaton \mathcal{A}



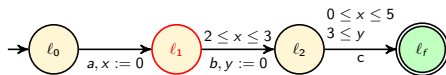
- Run with desynchronised clocks: $0.9 \cdot \dot{y} \leq \dot{x} \leq 1.1 \cdot \dot{y}$



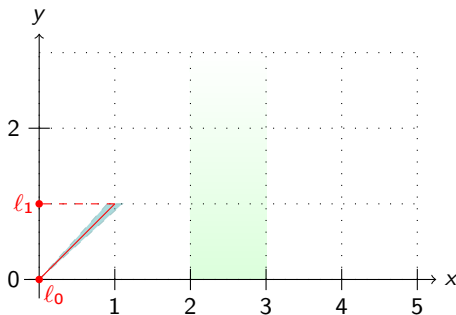
¹Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: clock drifting¹

- Timed automaton \mathcal{A}



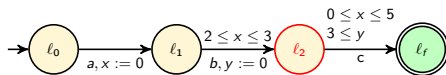
- Run with desynchronised clocks: $0.9 \cdot \dot{y} \leq \dot{x} \leq 1.1 \cdot \dot{y}$



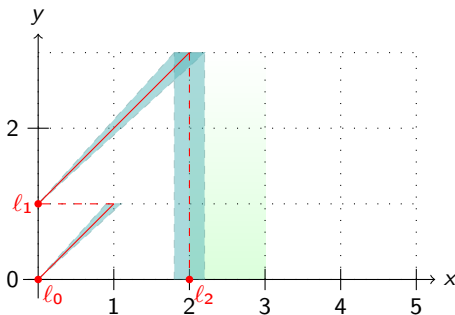
¹Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: clock drifting¹

- Timed automaton \mathcal{A}



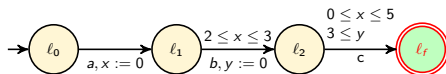
- Run with desynchronised clocks: $0.9 \cdot \dot{y} \leq \dot{x} \leq 1.1 \cdot \dot{y}$



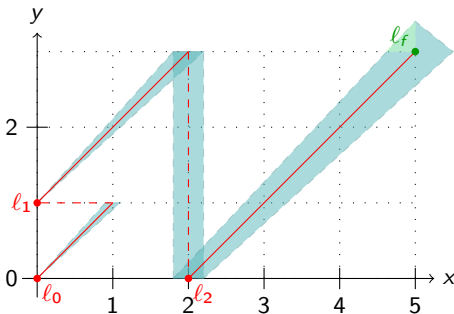
¹Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: clock drifting¹

- Timed automaton \mathcal{A}

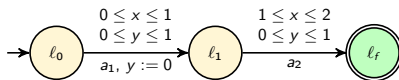


- Run with desynchronised clocks: $0.9 \cdot \dot{y} \leq \dot{x} \leq 1.1 \cdot \dot{y}$



¹Puri, 'Dynamical Properties of Timed Automata', 2000.

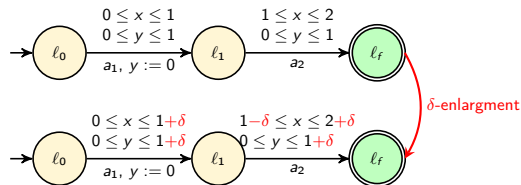
- Timed automaton \mathcal{A} :



²Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: guard enlargement²

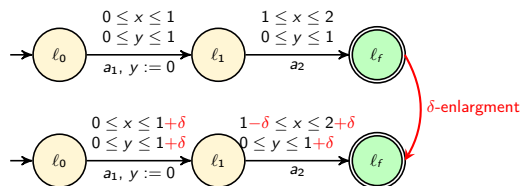
- Timed automaton \mathcal{A} :



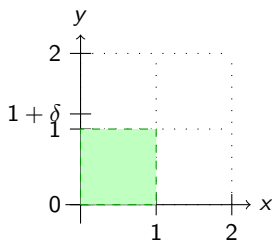
²Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: guard enlargement²

- Timed automaton \mathcal{A} :



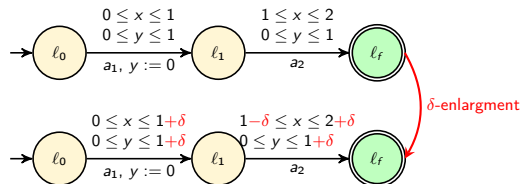
- Representation of an enlarged guard $0 \leq x \leq 1 \wedge 0 \leq y \leq 1$



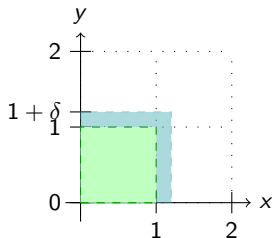
²Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: guard enlargement²

- Timed automaton \mathcal{A} :



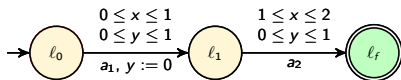
- Representation of an enlarged guard $0 \leq x \leq 1 \wedge 0 \leq y \leq 1$



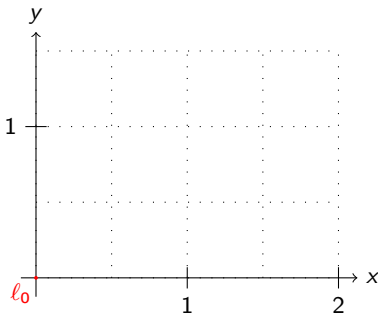
²Puri, 'Dynamical Properties of Timed Automata', 2000.

Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :



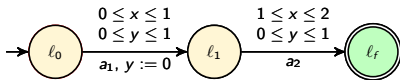
- Run with delay perturbations of at most $\delta = 0.2$



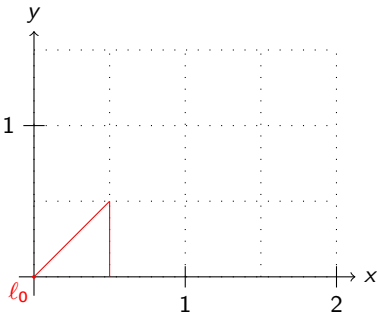
³Bouyer, Fang and Markey, 'Permissive strategies in timed automata and games', 2015.

Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :



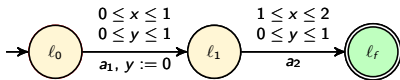
- Run with delay perturbations of at most $\delta = 0.2$



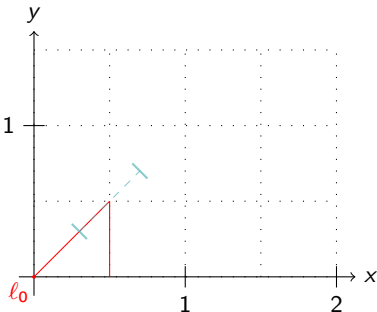
³Bouyer, Fang and Markey, 'Permissive strategies in timed automata and games', 2015.

Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :

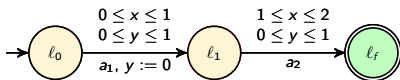


- Run with delay perturbations of at most $\delta = 0.2$

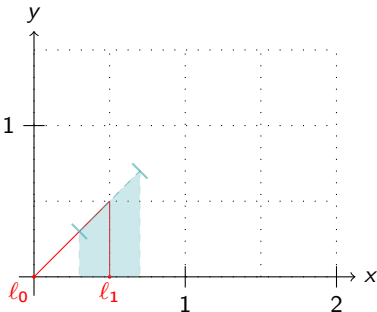


Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :

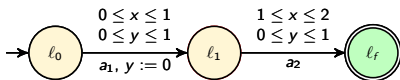


- Run with delay perturbations of at most $\delta = 0.2$

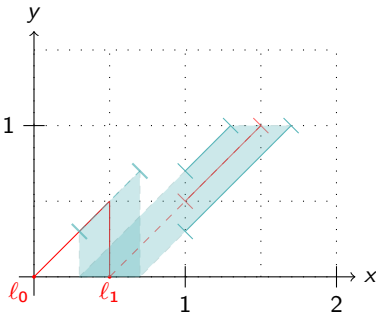


Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :



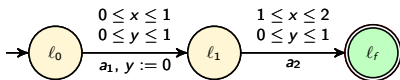
- Run with delay perturbations of at most $\delta = 0.2$



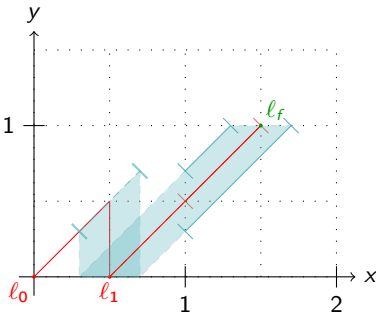
³Bouyer, Fang and Markey, 'Permissive strategies in timed automata and games', 2015.
Emily Clement

Example of perturbed semantics: delay perturbation³

- Timed automaton \mathcal{A} :

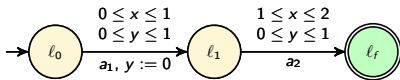


- Run with delay perturbations of at most $\delta = 0.2$

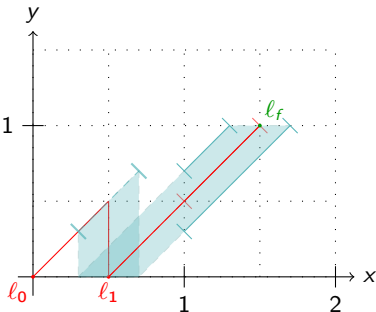


³Bouyer, Fang and Markey, 'Permissive strategies in timed automata and games', 2015.
Emily Clement

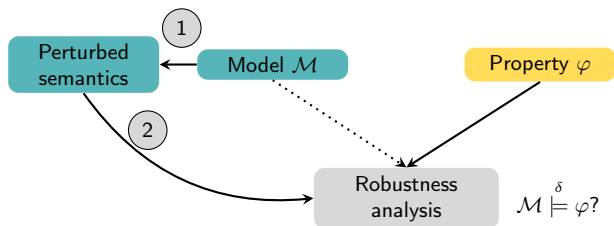
- Timed automaton \mathcal{A} :

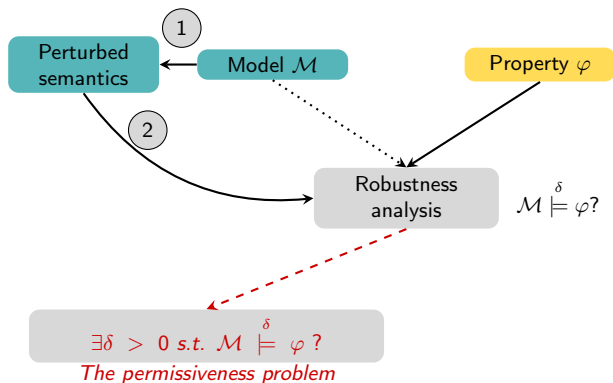


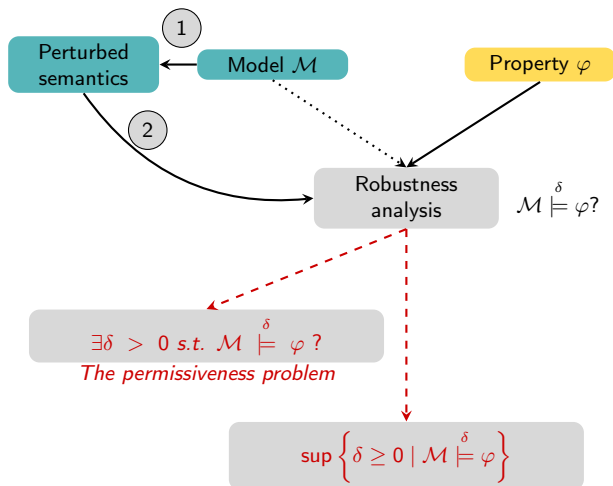
- Run with delay perturbations of at most $\delta = 0.2$

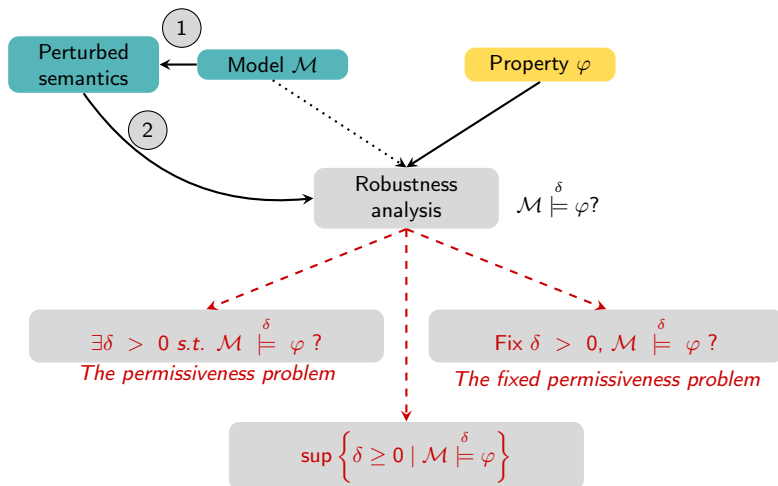


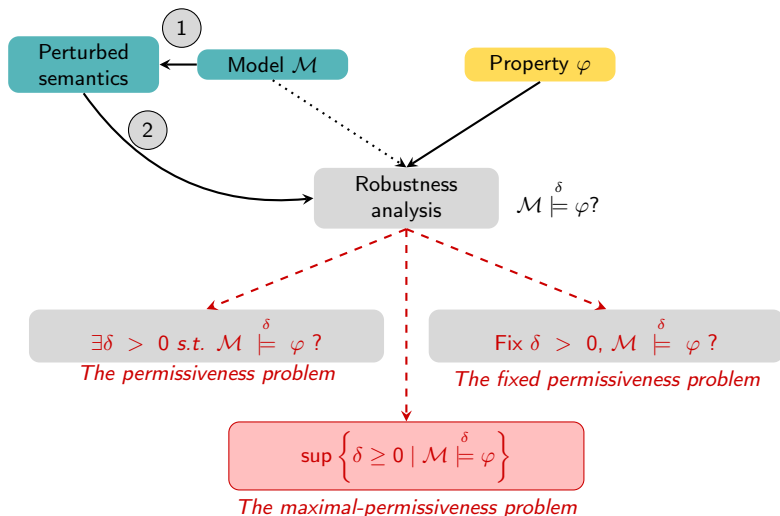
³Bouyer, Fang and Markey, 'Permissive strategies in timed automata and games', 2015.
 Emily Clement Robustness of timed automata

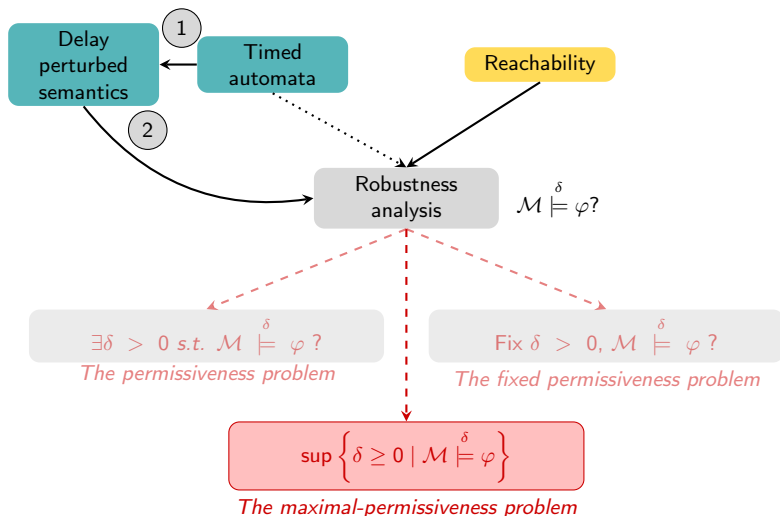






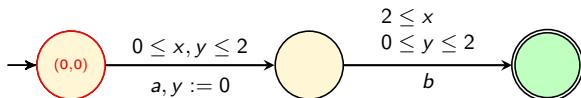






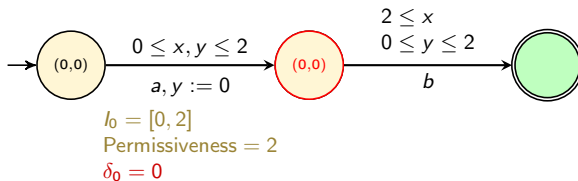
Our permissive semantics: a turn-based game

- ▷ Player
- ▷ Opponent



Our permissive semantics: a turn-based game

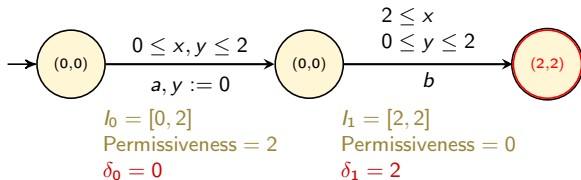
- ▷ Player
- ▷ Opponent



Our permissive semantics: a turn-based game

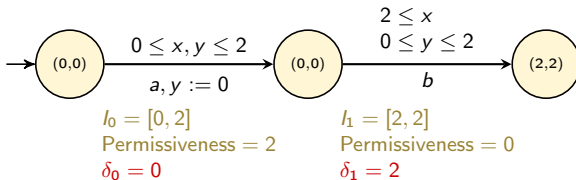
▷ Player

▷ Opponent



Our permissive semantics: a turn-based game

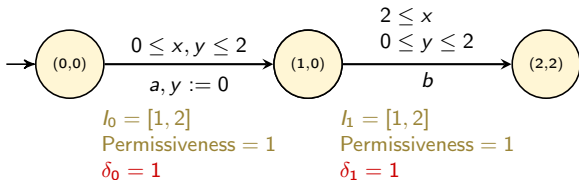
- ▶ **Player** : maximises the permissiveness
- ▶ **Opponent** : minimises the permissiveness



Permissiveness of the run : $\min(|l_0|, |l_1|) = \min(2, 0) = 0$

Our permissive semantics: a turn-based game

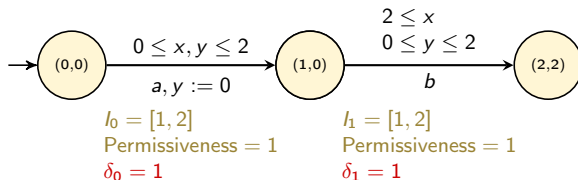
- ▶ **Player** : maximises the permissiveness
- ▶ **Opponent** : minimises the permissiveness



Permissiveness of the run : $\min(|l_0|, |l_1|) = \min(1, 1) = 1$

Our permissive semantics: a turn-based game

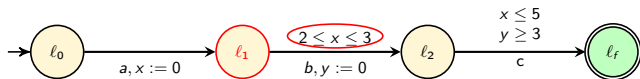
- ▶ **Player** : maximises the permissiveness
- ▶ **Opponent** : minimises the permissiveness



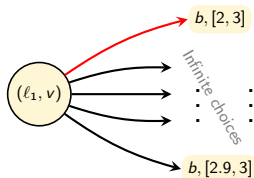
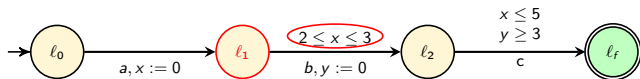
Permissiveness of the run : $\min(|l_0|, |l_1|) = \min(1, 1) = 1$

- ▶ **Opponent** : worst-case environment
- ▶ Our goal: compute the **player** best strategy, whatever the **opponent** decides

Permissiveness: an infinite number of choices



Permissiveness: an infinite number of choices

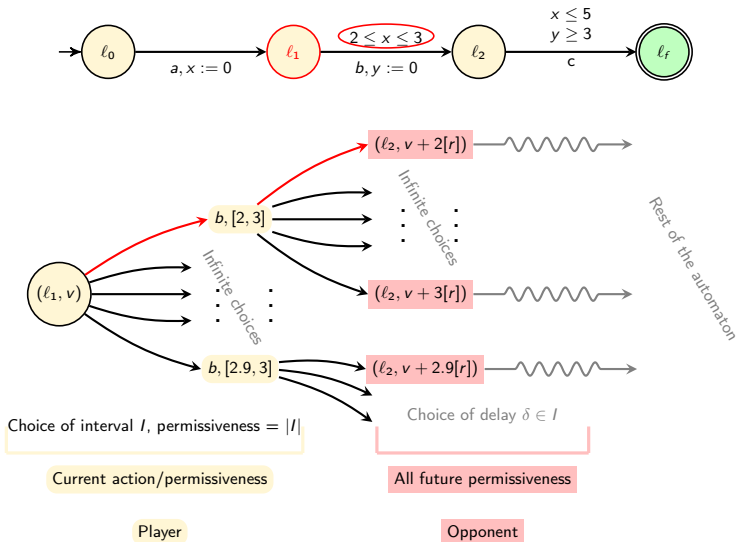


Choice of interval I , permissiveness = $|I|$

Current action/permissiveness

Player

Permissiveness: an infinite number of choices



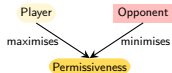
Goal: compute the maximal permissiveness

• Permissive semantics: a turn-based game

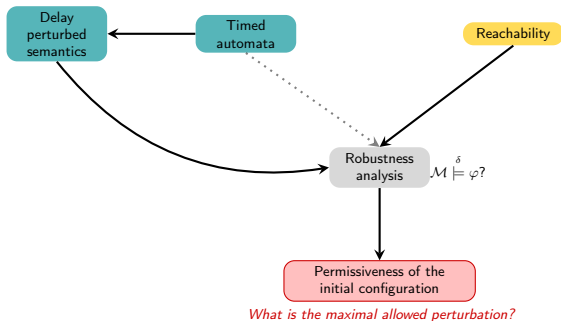
Player : Interval & action: (I, a)

Opponent : delay $\delta \in I$

• Permissiveness



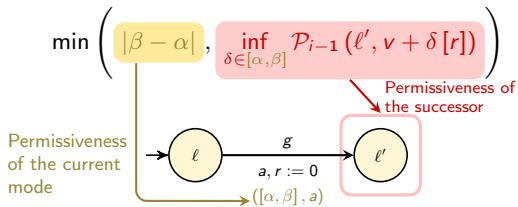
$\min(|I_0|, |I_1|, |I_2|, \dots)$



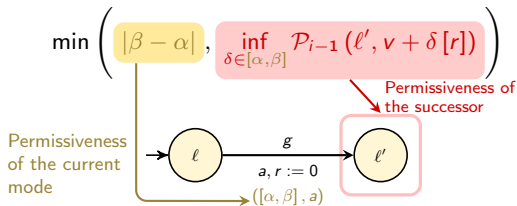
The maximal-permissiveness problem⁴

⁴Clement, Jérón, Markey and Mentré, 'Computing Maximally-Permissive Strategies in Acyclic Timed Automata', 2020.

- Strategy of the player: maximises



- Strategy of the player: maximises



- Opponent strategy lemma (linear case):

player : $([\alpha, \beta], a) \xrightarrow{\text{Opponent's best strategy}} \alpha \text{ or } \beta$

$$\inf_{\delta \in [\alpha, \beta]} \mathcal{P}_{i-1}(l', v + \delta[r]) = \min \left(\mathcal{P}_{i-1}(l', v + \alpha[r]), \mathcal{P}_{i-1}(l', v + \beta[r]) \right)$$

- A recursive function

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, \nu)} \left(\min \left(\beta - \alpha, \inf_{\delta \in [\alpha, \beta]} \mathcal{P}_{i-1}(\ell', \nu + \delta[r]) \right) \right)$$

- A recursive function

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, \nu)} \left(\min \left(\beta - \alpha, \inf_{\delta \in [\alpha, \beta]} \mathcal{P}_{i-1}(\ell', \nu + \delta[r]) \right) \right)$$

For linear timed automata:

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, \nu)} \left(\min \left(\beta - \alpha, \mathcal{P}_{i-1}(\ell', \nu + \alpha[r]), \mathcal{P}_{i-1}(\ell', \nu + \beta[r]) \right) \right)$$

$$\lim_{i \rightarrow +\infty} \mathcal{P}_i(\ell, \nu) = \text{the permissiveness on } (\ell, \nu)$$

- A recursive function

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, \nu)} \left(\min \left(\beta - \alpha, \inf_{\delta \in [\alpha, \beta]} \mathcal{P}_{i-1}(\ell', \nu + \delta[r]) \right) \right)$$

For linear timed automata:

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, \nu)} \left(\min \left(\beta - \alpha, \mathcal{P}_{i-1}(\ell', \nu + \alpha[r]), \mathcal{P}_{i-1}(\ell', \nu + \beta[r]) \right) \right)$$

$\lim_{i \rightarrow +\infty} \mathcal{P}_i(\ell, \nu) =$ the permissiveness on (ℓ, ν)

limit reached in d_ℓ steps

The permissiveness function

- A recursive function

$$\mathcal{P}_i(\ell, v) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, v)} \left(\min \left(\beta - \alpha, \inf_{\delta \in [\alpha, \beta]} \mathcal{P}_{i-1}(\ell', v + \delta[r]) \right) \right)$$

For linear timed automata:

$$\mathcal{P}_i(\ell, v) = \sup_{([\alpha, \beta], a) \in \text{p-moves}(\ell, v)} \left(\min \left(\beta - \alpha, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right) \right)$$

$\lim_{i \rightarrow +\infty} \mathcal{P}_i(\ell, v) =$ the permissiveness on (ℓ, v)

limit reached in d_ℓ steps

- Goal of our algorithm

Compute $v \mapsto \mathcal{P}_{d_\ell}(\ell, v)$ knowing $v \mapsto \mathcal{P}_{d_\ell-1}(\ell', v)$

- Goal: find the α and β that maximises:

$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

- Goal: find the α and β that maximises:

$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

Steps of the algorithm (linear timed automata)

- Goal: find the α and β that maximises:

$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

- $v \mapsto \mathcal{P}_i(\ell, v)$: a 2-Lipschitz piecewise-affine function

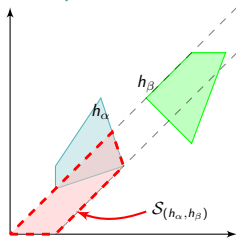
Steps of the algorithm (linear timed automata)

- Goal: find the α and β that maximises:

$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

- $v \mapsto \mathcal{P}_i(\ell, v)$: a 2-Lipschitz piecewise-affine function

- Steps: for each couple of cells (h_α, h_β)



(a) Step 1: compute $S(h_\alpha, h_\beta)$

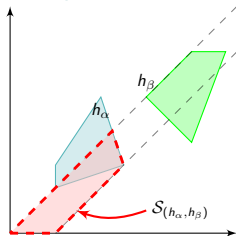
Steps of the algorithm (linear timed automata)

- Goal: find the α and β that maximises:

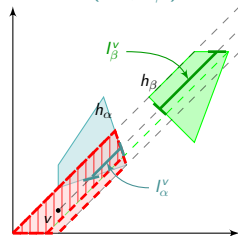
$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

- $v \mapsto \mathcal{P}_i(\ell, v)$: a 2-Lipschitz piecewise-affine function

- Steps: for each couple of cells (h_α, h_β)



(a) Step 1: compute $S(h_\alpha, h_\beta)$



(b) Step 2: compute the possible α and β

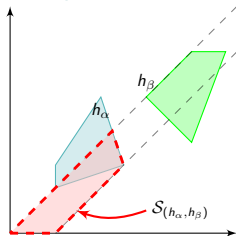
Steps of the algorithm (linear timed automata)

- Goal: find the α and β that maximises:

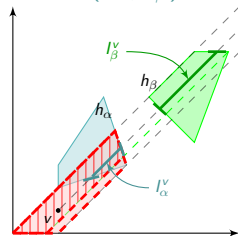
$$\min \left(|\beta - \alpha|, \mathcal{P}_{i-1}(\ell', v + \alpha[r]), \mathcal{P}_{i-1}(\ell', v + \beta[r]) \right)$$

- $v \mapsto \mathcal{P}_i(\ell, v)$: a 2-Lipschitz piecewise-affine function

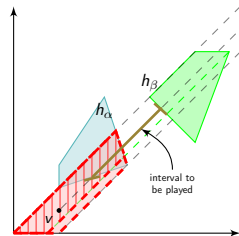
- Steps: for each couple of cells (h_α, h_β)



(a) Step 1: compute $S(h_\alpha, h_\beta)$

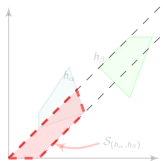


(b) Step 2: compute the possible α and β

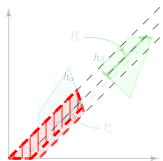


(c) Step 3: compute the optimal α and β

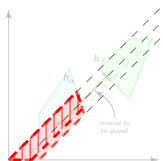
Example



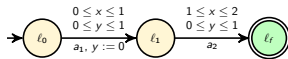
(a) Step 1

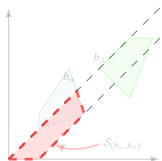


(b) Step 2

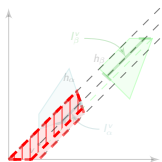


(c) Step 3

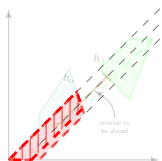




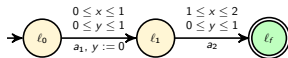
(a) Step 1



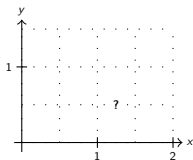
(b) Step 2



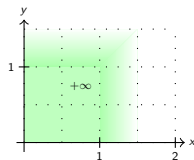
(c) Step 3



- Permissiveness on ℓ_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(\ell_f, v + \alpha), \mathcal{P}_0(\ell_f, v + \beta))$



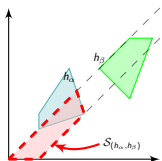
(a) Permissiveness on ℓ_1



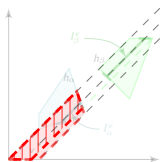
(b) Permissiveness on ℓ_f

Fixing the cells of arrival of the successors h_α and h_β : \mathbb{R}^2

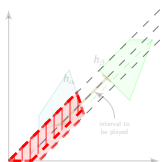
Example



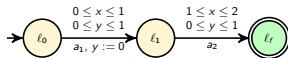
(a) Step 1



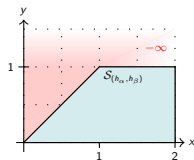
(b) Step 2



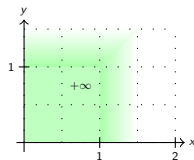
(c) Step 3



- Permissiveness on ℓ_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(\ell_f, v + \alpha), \mathcal{P}_0(\ell_f, v + \beta))$



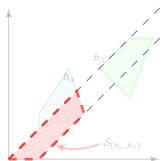
(a) Permissiveness on ℓ_1



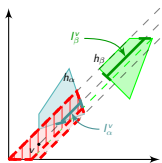
(b) Permissiveness on ℓ_f

Step 1: computing $\mathcal{S}(h_\alpha, h_\beta)$ (Fourier-Motzkin algorithm)

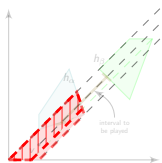
Example



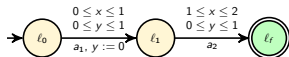
(a) Step 1



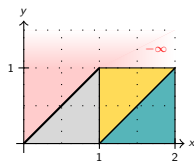
(b) Step 2



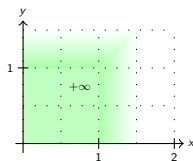
15 / 31 (c) Step 3



- Permissiveness on l_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(l_f, v + \alpha), \mathcal{P}_0(l_f, v + \beta))$



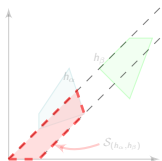
(a) Permissiveness on l_1



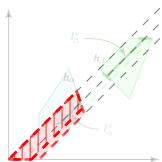
(b) Permissiveness on l_f

Step 2: computing the intervals of α and β (Fourier-Motzkin algorithm)

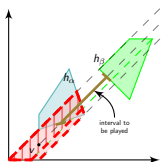
$$I_{\alpha}^V = I_{\beta}^V = [\max(0, 1 - x), \min(2 - x, 1 - y)]$$



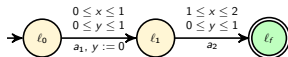
(a) Step 1



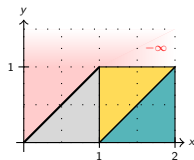
(b) Step 2



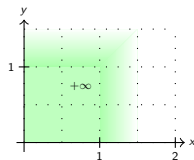
15 / 31 (c) Step 3



- Permissiveness on l_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(l_f, v + \alpha), \mathcal{P}_0(l_f, v + \beta))$

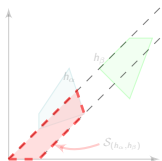


(a) Permissiveness on l_1

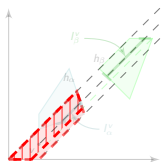


(b) Permissiveness on l_f

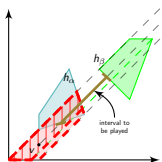
Step 3: computing the optimal α and β , s.t $\alpha \leq \beta$, that maximises...



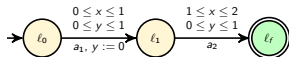
(a) Step 1



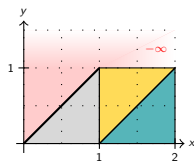
(b) Step 2



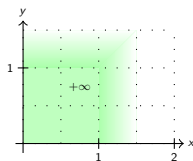
15 / 31 (c) Step 3



- Permissiveness on l_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(l_f, v + \alpha), \mathcal{P}_0(l_f, v + \beta))$



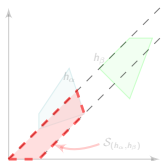
(a) Permissiveness on l_1



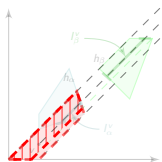
(b) Permissiveness on l_f

Step 3: computing the optimal α and β , s.t $\alpha \leq \beta$, that maximises...

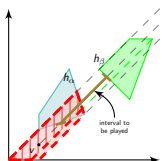
$$\min(\beta - \alpha, \mathcal{P}_{i-1}(l', v + \alpha[r]), \mathcal{P}_{i-1}(l', v + \beta[r]))$$



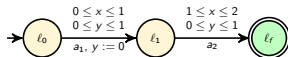
(a) Step 1



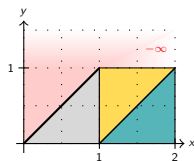
(b) Step 2



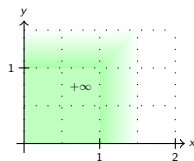
15 / 31 (c) Step 3



- Permissiveness on l_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(l_f, v + \alpha), \mathcal{P}_0(l_f, v + \beta))$



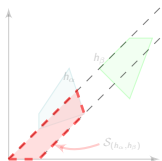
(a) Permissiveness on l_1



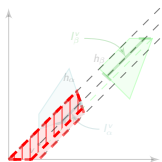
(b) Permissiveness on l_f

Step 3: computing the optimal α and β , s.t $\alpha \leq \beta$, that maximises...

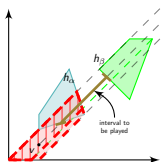
$$\min(\beta - \alpha, +\infty, +\infty)$$



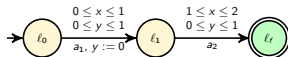
(a) Step 1



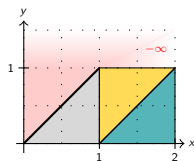
(b) Step 2



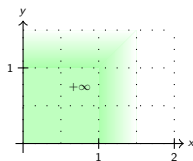
15 / 31 (c) Step 3



- Permissiveness on ℓ_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(\ell_f, v + \alpha), \mathcal{P}_0(\ell_f, v + \beta))$



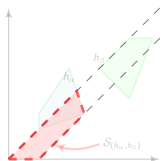
(a) Permissiveness on ℓ_1



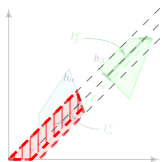
(b) Permissiveness on ℓ_f

Step 3: computing the optimal α and β , s.t $\alpha \leq \beta$, that maximises...

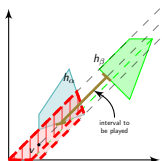
$$\min(\beta - \alpha, +\infty, +\infty) \text{ (Technical lemma)}$$



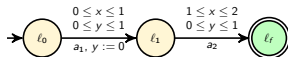
(a) Step 1



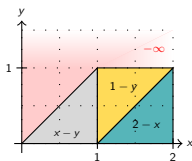
(b) Step 2



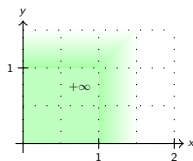
15 / 31 (c) Step 3



- Permissiveness on ℓ_1 : maximise $\min(\beta - \alpha, \mathcal{P}_0(\ell_f, v + \alpha), \mathcal{P}_0(\ell_f, v + \beta))$



(a) Permissiveness on ℓ_1

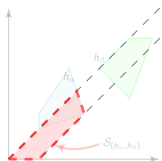


(b) Permissiveness on ℓ_f

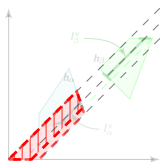
Step 3: computing the optimal α and β , s.t $\alpha \leq \beta$, that maximises...

$$\min(\beta - \alpha, +\infty, +\infty) \quad \text{(Technical lemma)}$$

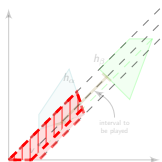
$$\alpha^* = \max(0, 1 - x), \beta^* = \min(2 - x, 1 - y)$$



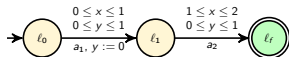
(a) Step 1



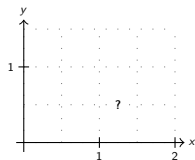
(b) Step 2



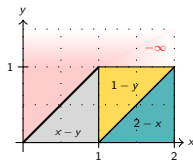
(c) Step 3




- Permissiveness on l_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(l_1, v + \alpha), \mathcal{P}_1(l_1, v + \beta))$

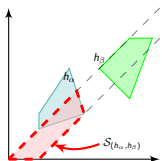


(a) Permissiveness on l_0

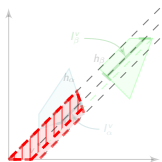


(b) Permissiveness on l_1

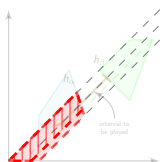
Let us fix $h_\alpha = h_\beta =$ 



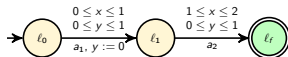
(a) Step 1



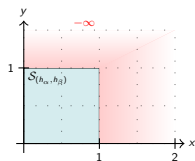
(b) Step 2



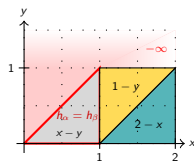
(c) Step 3



- Permissiveness on l_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(l_1, v + \alpha), \mathcal{P}_1(l_1, v + \beta))$



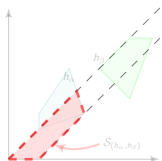
(a) Permissiveness on l_0



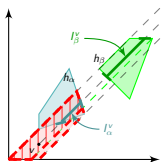
(b) Permissiveness on l_1

Step 1: Computing the corresponding entry set $\mathcal{S}(h_\alpha, h_\beta)$

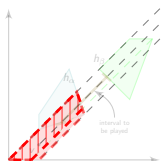
(Fourier-Motzkin algorithm)



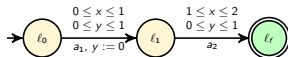
(a) Step 1



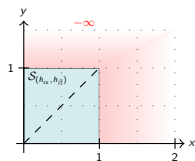
(b) Step 2



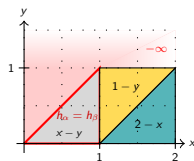
15 / 31 (c) Step 3



- Permissiveness on ℓ_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(\ell_1, v + \alpha), \mathcal{P}_1(\ell_1, v + \beta))$



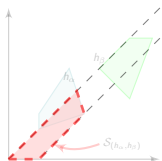
(a) Permissiveness on ℓ_0



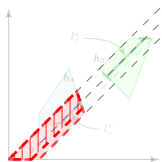
(b) Permissiveness on ℓ_1

Step 2: Computing the set of possible α and β (Fourier-Motzkin algorithm):

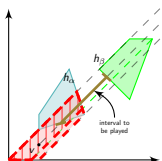
$$I_\alpha^V = I_\beta^V = [0, \min(1 - x, 1 - y)]$$



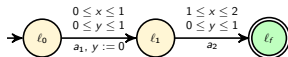
(a) Step 1



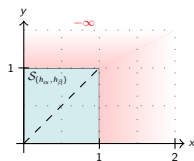
(b) Step 2



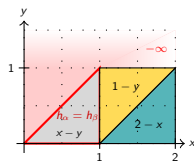
15 / 31 (c) Step 3



- Permissiveness on l_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(l_1, v + \alpha), \mathcal{P}_1(l_1, v + \beta))$



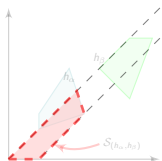
(a) Permissiveness on l_0



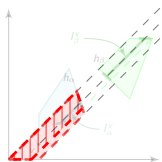
(b) Permissiveness on l_1

Step 3: For $y \leq x$: Computing the optimal α and β in $[0, 1 - x]^2$, s.t. $\alpha \leq \beta$, that maximise:

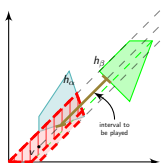
$$\min(\beta - \alpha, 1 \cdot \alpha + x, 1 \cdot \beta + x) \quad \text{(Technical lemma)}$$



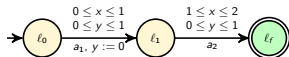
(a) Step 1



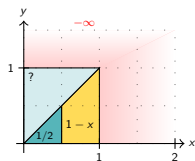
(b) Step 2



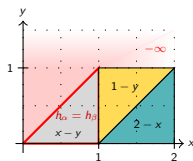
15 / 31 (c) Step 3



- Permissiveness on l_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(l_1, v + \alpha), \mathcal{P}_1(l_1, v + \beta))$



(a) Permissiveness on l_0

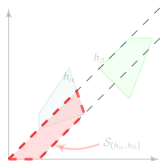


(b) Permissiveness on l_1

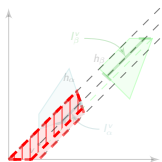
- Step 3:** For $y \leq x$: Computing the optimal α and β in $[0, 1 - x]^2$, s.t. $\alpha \leq \beta$, that maximise:

$$\min(\beta - \alpha, 1 \cdot \alpha + x, 1 \cdot \beta + x) \quad \text{(Technical lemma)}$$

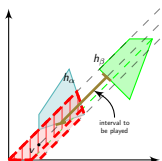
Example



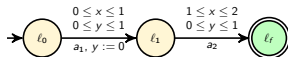
(a) Step 1



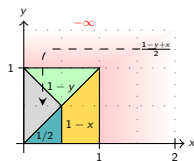
(b) Step 2



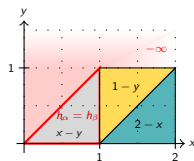
15 / 31 (c) Step 3



- Permissiveness on l_0 : maximise $\min(\beta - \alpha, \mathcal{P}_1(l_1, v + \alpha), \mathcal{P}_1(l_1, v + \beta))$



(a) Permissiveness on l_0



(b) Permissiveness on l_1

- Step 3:** Same for $x \leq y$: Computing the optimal α and β in $[0, 1 - y]^2$, s.t $\alpha \leq \beta$, that maximise:

$$\min(\beta - \alpha, 1 \cdot \alpha + x, 1 \cdot \beta + x) \quad \text{(Technical lemma)}$$

- The algorithm
 - ▷ Limited to acyclic timed automata

- The algorithm

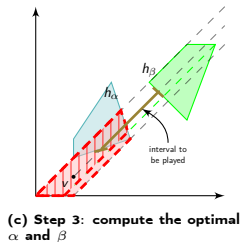
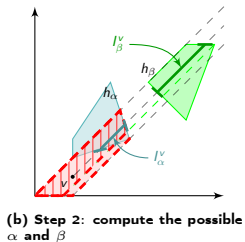
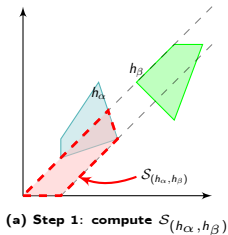
- ▷ Limited to acyclic timed automata
- ▷ Upper bound time complexity: **non-elementary**
- ▷ Complexity: grows with the number of cells, of clocks and d_ℓ

- The algorithm

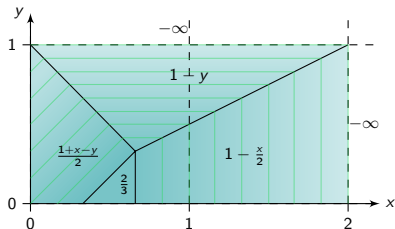
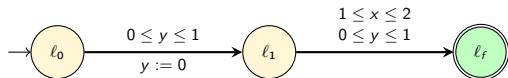
- ▷ Limited to acyclic timed automata
- ▷ Upper bound time complexity: **non-elementary**
- ▷ Complexity: grows with the number of cells, of clocks and d_ℓ

- Causes of the high complexity

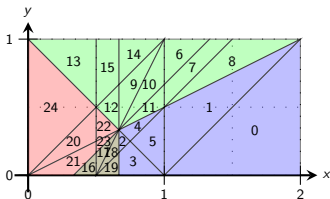
- ▷ Overtiling
- ▷ Exploration of all couple of cells (h_α, h_β)



- Example of overtaking



(a) Permissiveness on l_0



(b) Permissiveness computed by our tool

- Solutions

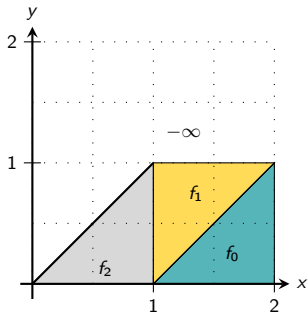
- For linear case: we proved that union of polyhedra associated with the same affine function is **convex**
 - For general case: try merging and roll-back if necessary.

- ▶ Redundancy in the technical lemma

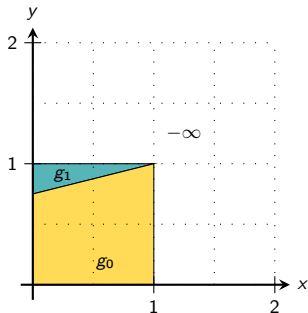
- ▷ Redundancy in the technical lemma
- ▷ Maximisation/Minimisation (when comparing candidate permissiveness functions)

The causes of the overtling

- ▶ Redundancy in the technical lemma
- ▶ Maximisation/Minimisation (when comparing candidate permissiveness functions)
- Example of maximisation: computing $\max(f, g)$



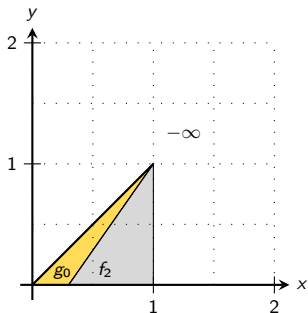
(a) $f: f_0(x, y) = 2 - x$, $f_1(x, y) = 1 - y$,
 $f_2(x, y) = x - y$



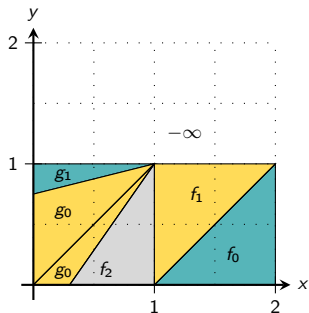
(b) $g: g_0(x, y) = (1 - x) / 2$, $g_1(x, y) = 1 - y$

The causes of the overtiling

- ▷ Redundancy in the technical lemma
- ▷ Maximisation/Minimisation (when comparing candidate permissiveness functions)
- Example of maximisation: computing $\max(f, g)$



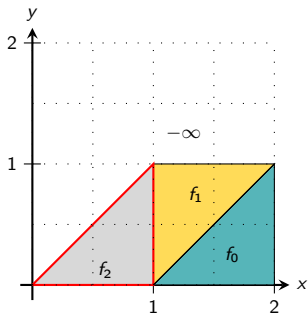
(a) Wrong result



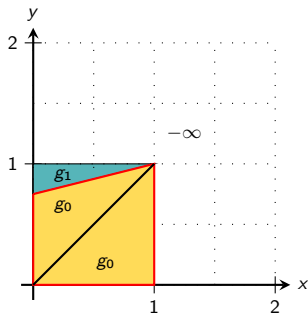
(b) Correct overtiled result $\max(f, g)$

The causes of the overtiling

- ▷ Redundancy in the technical lemma
- ▷ Maximisation/Minimisation (when comparing candidate permissiveness functions)
- Example of maximisation: computing $\max(f, g)$

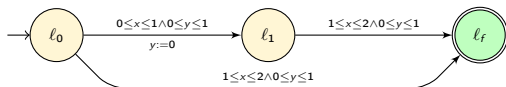


(a) f

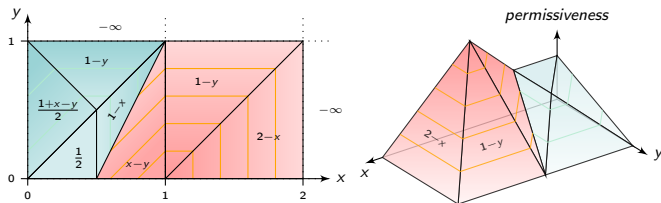


(b) Overtiled representation of g

- Example of an acyclic timed automata



- Its permissiveness on l_0



- Changes: Not concave anymore !

Optimal Strategy of the opponent for mone $([\alpha, \beta], a)$ is not necessary α, β .

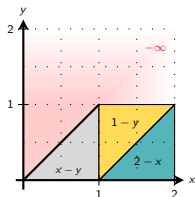
- Solution:

Binary and levelled permissiveness problem.

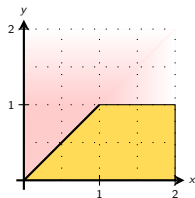
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



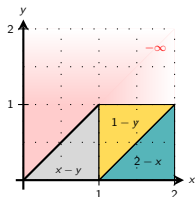
- Binary permissiveness, $p = 0$



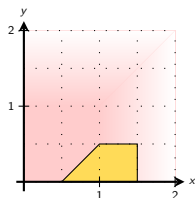
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



- Binary permissiveness, $p = 1/2$

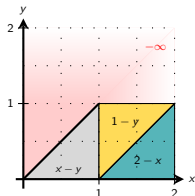


- Some reductions

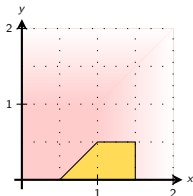
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



- Binary permissiveness, $p = 1/2$



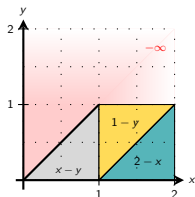
- Some reductions

- ▷ **Linear Lemma**: for linear TA, $\mathcal{S}(p, \ell)$ is a polyhedron.

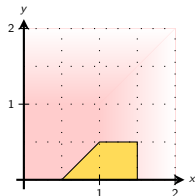
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



- Binary permissiveness, $p = 1/2$



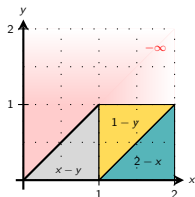
- Some reductions

- ▷ **Linear Lemma**: for linear TA, $\mathcal{S}(p, \ell)$ is a polyhedron.

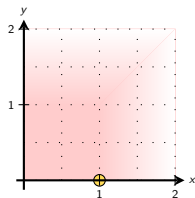
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



- Binary permissiveness, $p = 1$



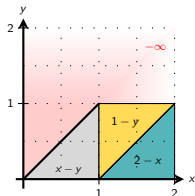
- Some reductions

▷ **Linear Lemma**: for linear TA, $\mathcal{S}(p, \ell)$ is a polyhedron.

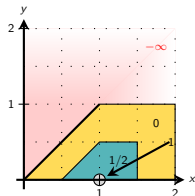
- Binary permissiveness principle

Fix $p \geq 0$ and ℓ , compute $\mathcal{S}(p, \ell) = \{v \mid \mathcal{P}_{d_\ell}(\ell, v) \geq p\}$

- Permissiveness function



- Levelled permissiveness for $\{0, 1/2, 1\}$



- Some reductions

▷ **Linear Lemma**: for linear TA, $\mathcal{S}(p, \ell)$ is a polyhedron.

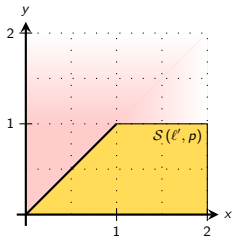
▷ **Levelled permissiveness** $\xrightarrow{\text{reduces}}$ **binary permissiveness**

$$\mathcal{P}_i(\ell, \nu) = \sup_{([\alpha, \alpha + \rho], a) \in \mathbf{p}\text{-moves}(\ell, \nu)} \min(\beta - \alpha, \mathcal{P}_{i-1}(\ell', \nu + \alpha[r]), \mathcal{P}_{i-1}(\ell', \nu + \beta[r]))$$

$$\mathcal{B}_i(\ell, v) = \sup_{([\alpha, \alpha+p], a) \in \mathbf{p}\text{-moves}(\ell, v)} \inf_{\delta \in [\alpha, \alpha+p]} \left(\mathbb{1}_{v+\delta[r] \in \mathcal{S}_{i-1}(p, \ell'_a)} \right)$$

- Steps of the algorithm

- ▶ **Step 1:** Compute set of future enabled valuations $\mathcal{S}(\ell', p)$

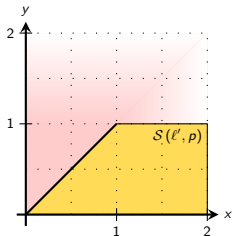


- ▶ **Step 2:** Compute the valuations v such that there exists $\alpha \geq 0$ that verifies:

$$\mathcal{B}_i(\ell, v) = \sup_{([\alpha, \alpha+p], a) \in \mathbf{p}\text{-moves}(\ell, v)} \inf_{\delta \in [\alpha, \alpha+p]} \left(\mathbb{1}_{v+\delta[r] \in \mathcal{S}_{i-1}(p, \ell'_a)} \right)$$

- Steps of the algorithm

- ▶ **Step 1:** Compute set of future enabled valuations $\mathcal{S}(\ell', p)$

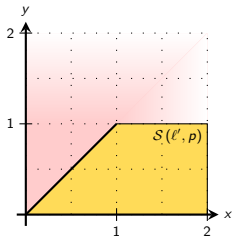


- ▶ **Step 2:** Compute the valuations v such that there exists $\alpha \geq 0$ that verifies:
 - $[\alpha, \alpha + p]$ is an enabled move.
 - the successors are in $\mathcal{S}(\ell', p)$.

$$\mathcal{B}_i(\ell, \nu) = \sup_{([\alpha, \alpha+p], a) \in \mathbf{p}\text{-moves}(\ell, \nu)} \inf_{\delta \in [\alpha, \alpha+p]} \left(\mathbb{1}_{\nu + \delta[r] \in \mathcal{S}_{i-1}(p, \ell'_a)} \right)$$

- Steps of the algorithm

- ▶ **Step 1:** Compute set of future enabled valuations $\mathcal{S}(\ell', p)$



- ▶ **Step 2:** Compute the valuations ν such that there exists $\alpha \geq 0$ that verifies:
 - $[\alpha, \alpha + p]$ is an enabled move.
 - the successors are in $\mathcal{S}(\ell', p)$.

Fourier-Motzkin

- Upper bound complexity for..

Binary permissiveness algorithm:

$$\mathcal{O}\left((4c_g)^{2 \cdot d_\ell}\right)$$

longest path between
 ℓ and a goal location

maximal number of constraints of any guard

- Upper bound complexity for..

Binary permissiveness algorithm:

$$\mathcal{O}\left((4c_g)^{2 \cdot d_\ell}\right)$$

longest path between
 ℓ and a goal location

maximal number of constraints of any guard

Levelled permissiveness algorithm $\{p_0, \dots, p_m\}$:

$$\mathcal{O}\left((m+1)(4c_g)^{2 \cdot d_\ell}\right)$$

number of levels

- Changes for acyclic timed automata

- ▶ The set of winning valuation is no longer a **unique** polyhedron in general
- ▶ Principle: for a fixed p , find if we can find an enabled interval $[\alpha, \alpha + p]$ that crosses only polyhedra of $\mathcal{S}_{i-1}(\ell', p)$:

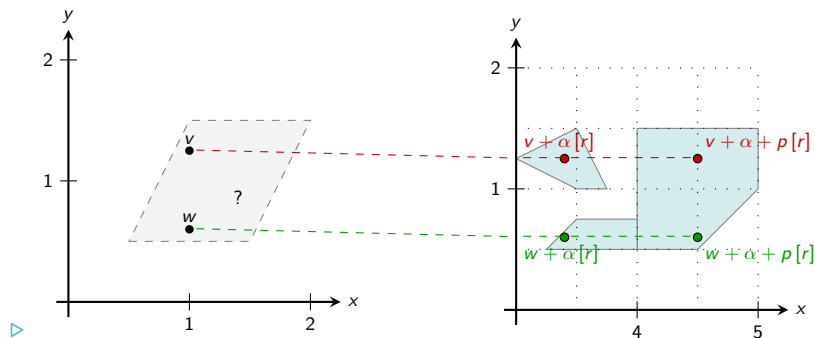
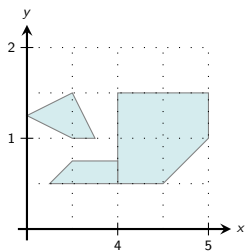


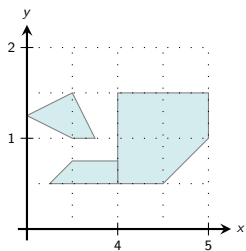
Figure: $\mathcal{S}_{i-1}(\ell', p)$

Intuition to tackle this issue: good representation of polyhedra

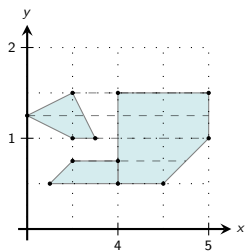


(a) Original representation

Intuition to tackle this issue: good representation of polyhedra

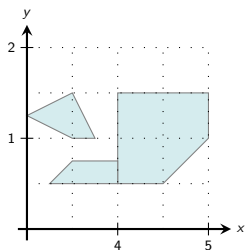


(a) Original representation

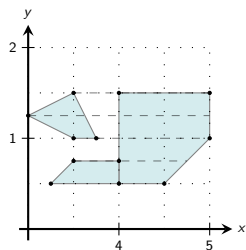


(b) Partition with respect to the vertices

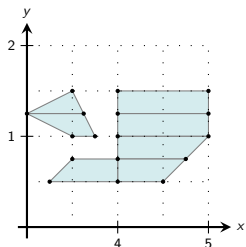
Intuition to tackle this issue: good representation of polyhedra



(a) Original representation

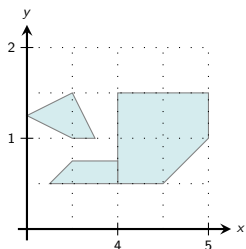


(b) Partition with respect to the vertices

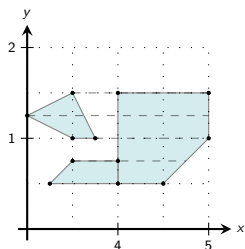


(c) 'sliced' representation of $S_{i-1}(\ell', \rho)$

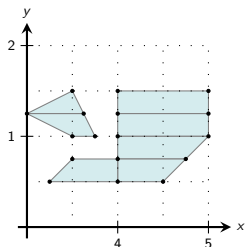
Intuition to tackle this issue: good representation of polyhedra



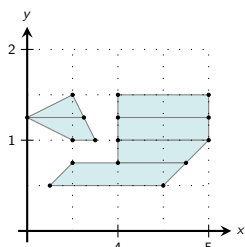
(a) Original representation



(b) Partition with respect to the vertices



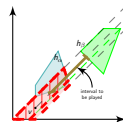
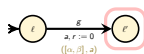
(c) 'sliced' representation of $S_{i-1}(\ell', \rho)$



(d) Merging polyhedra in the same 'slice'

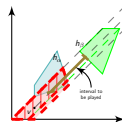
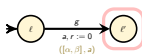
- Symbolic algorithm

- ▶ A non-elementary algorithm
- ▶ Restricted to acyclic timed automata
- ▶ Exact result



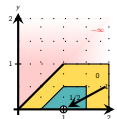
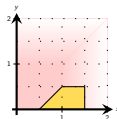
- Symbolic algorithm

- ▷ A non-elementary algorithm
- ▷ Restricted to acyclic timed automata
- ▷ Exact result



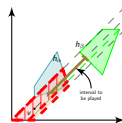
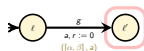
- Binary and levelled permissiveness

- ▷ A doubly-exponential algorithm
- ▷ Linear w.r.t the number of levels
- ▷ Restricted to linear timed automata
- ▷ Controlled approximation of the permissiveness



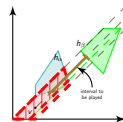
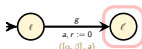
- Symbolic algorithm

- ▷ Cyclic TA
- ▷ Implementation of acyclic TA



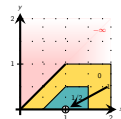
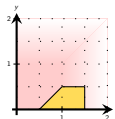
- Symbolic algorithm

- ▷ Cyclic TA
- ▷ Implementation of acyclic TA



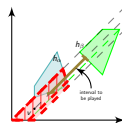
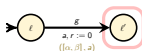
- Binary and levelled permissiveness

- ▷ Acyclic (and cyclic) TA



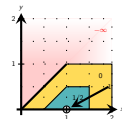
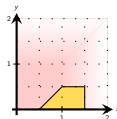
- Symbolic algorithm

- ▶ Cyclic TA
- ▶ Implementation of acyclic TA



- Binary and levelled permissiveness

- ▶ Acyclic (and cyclic) TA



- Binary and levelled permissiveness

- ▶ A structure to represent (and merge) polyhedra with respect to time vector r

- The principle of the algorithm

$$x + \alpha - 1 \geq 0$$

$$x + y + \alpha - 3 \geq 0$$

System of linear equations (S)

- The principle of the algorithm

$$x + \alpha - 1 \geq 0$$

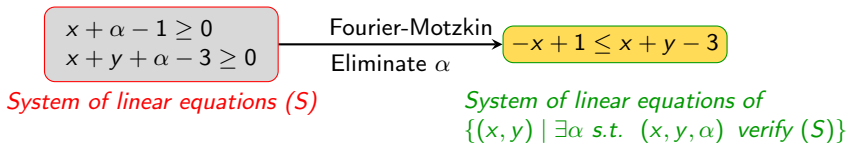
$$x + y + \alpha - 3 \geq 0$$

Fourier-Motzkin

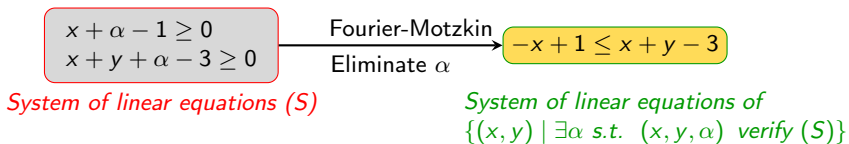
Eliminate α

System of linear equations (S)

- The principle of the algorithm

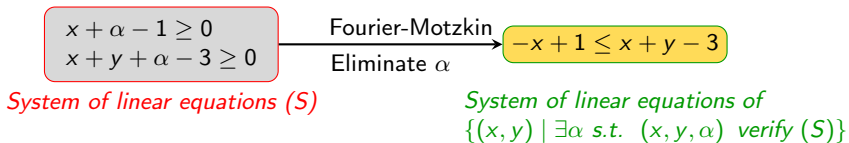


- The principle of the algorithm



- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.

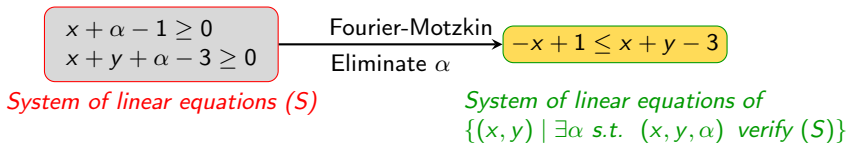
- The principle of the algorithm



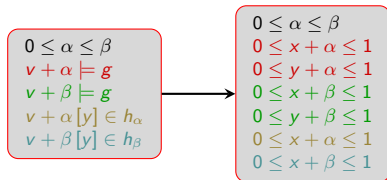
- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.

$$\begin{aligned} 0 &\leq \alpha \leq \beta \\ v + \alpha &\models g \\ v + \beta &\models g \\ v + \alpha[y] &\in h_\alpha \\ v + \beta[y] &\in h_\beta \end{aligned}$$

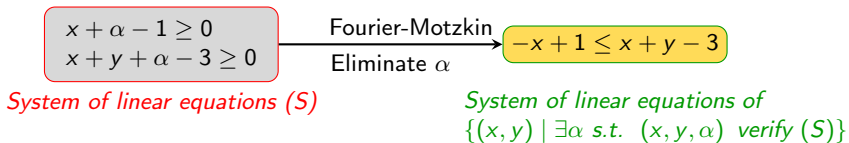
- The principle of the algorithm



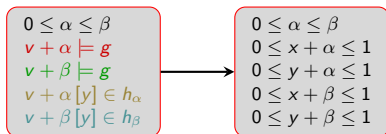
- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.



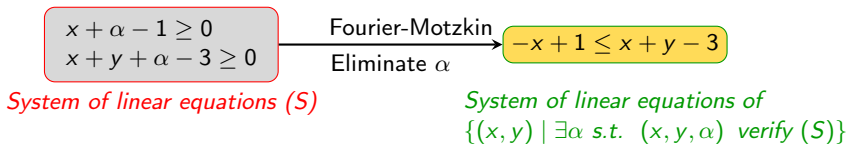
- The principle of the algorithm



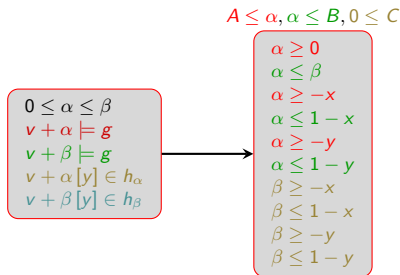
- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.



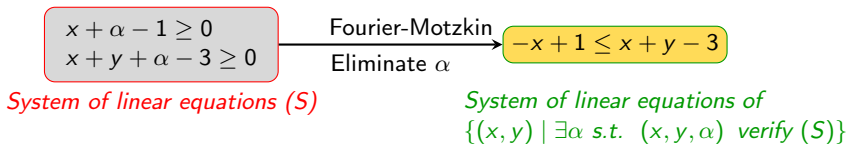
- The principle of the algorithm



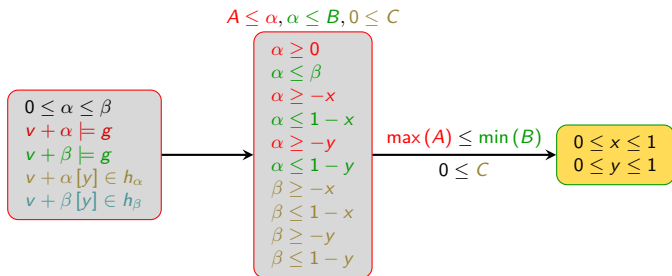
- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.



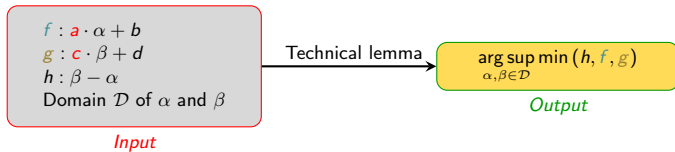
- The principle of the algorithm



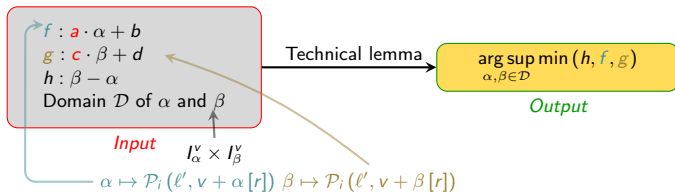
- Computing the entry set $\mathcal{S}_{(h_\alpha, h_\beta)}$: the set of $(x, y) \in \mathbb{R}_+^2$ s.t. $\exists \alpha, \beta$ s.t.



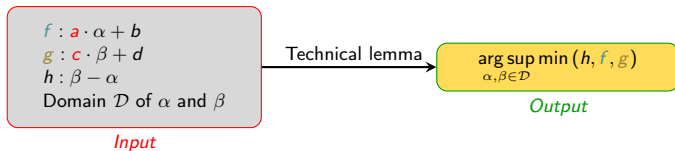
(1st contribution) Technical Lemma



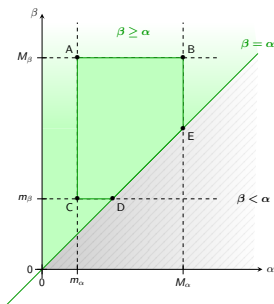
(1st contribution) Technical Lemma



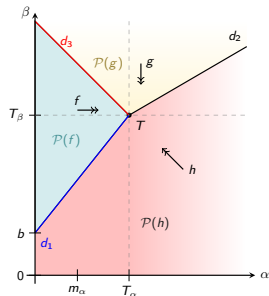
(1st contribution) Technical Lemma



- When $a > 0$ and $c < 0$:



(a) Domain \mathcal{D}



(b) $\min (f, g, h)$ on \mathbb{R}_+^2

(1st contribution) Technical Lemma

$f : a \cdot \alpha + b$
 $g : c \cdot \beta + d$
 $h : \beta - \alpha$
Domain \mathcal{D} of α and β

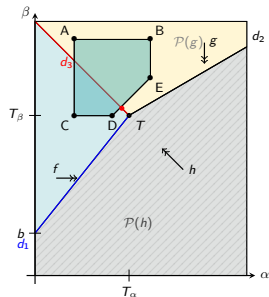
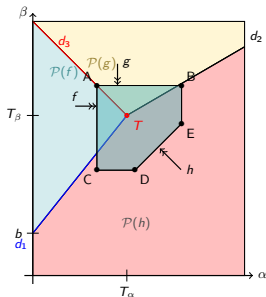
Input

Technical lemma

$\arg \sup_{\alpha, \beta \in \mathcal{D}} \min(h, f, g)$

Output

- When $a > 0$ and $c < 0$:



- ▶ Based on *pplpy*

- ▷ Based on *pplpy*
- ▷ Covers the case of **linear timed automata** with **polyhedral** guards

- ▷ Based on *pplpy*
- ▷ Covers the case of **linear timed automata** with **polyhedral guards**
- ▷ Runtime results on our examples:

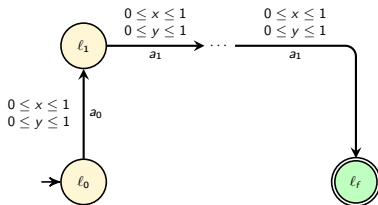
Clocks	Nb. of transitions	Runtime for ℓ_0	Runtime for ℓ_1	Runtime for ℓ_2	Runtime for ℓ_3
2	2	0.82 (2 cells)	0.059 (2 cells)	-	-
2	2	0.071 (6 cells)	0.062 (3 cells)	-	-
2	2	0.73 (24 cells)	0.034 (3 cells)	-	-
2	3	38.16 (582 cells)	1.01 (25 cells)	0.09 (3 cells)	-
3	4	19.95 (234 cells)	0.41 (12 cells)	0.56 (6 cells)	0.14 (6 cells)
3	4	143.48 (1825 cells)	0.39 (12 cells)	0.59 (6 cells)	0.14 (6 cells)

(1st contribution) Implementation

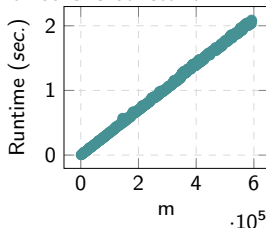
- ▶ Based on *pplpy*
- ▶ Covers the case of **linear timed automata with polyhedral guards**
- ▶ Runtime results on our examples:

Clocks	Nb. of transitions	Runtime for ℓ_0	Runtime for ℓ_1	Runtime for ℓ_2	Runtime for ℓ_3
2	2	0.82 (2 cells)	0.059 (2 cells)	-	-
2	2	0.071 (6 cells)	0.062 (3 cells)	-	-
2	2	0.73 (24 cells)	0.034 (3 cells)	-	-
2	3	38.16 (582 cells)	1.01 (25 cells)	0.09 (3 cells)	-
3	4	19.95 (234 cells)	0.41 (12 cells)	0.56 (6 cells)	0.14 (6 cells)
3	4	143.48 (1825 cells)	0.39 (12 cells)	0.59 (6 cells)	0.14 (6 cells)

- ▶ Runtime results on a case where the number of cells is constant:



(a) A m transitions timed automaton



(b) Runtimes depending on the number of transitions (m)

- Case of two polyhedra

Solved in 2001 by Bemporad, Fukuda and D. Torrisi⁵:

- ▶ Computing convex hull by removing inequalities
- ▶ Solving a linear program

⁵Bemporad, Fukuda and Torrisi, 'Convexity recognition of the union of polyhedra', 2001.

(1st contribution) Computing the union of several polyhedra

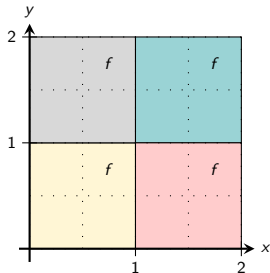
- Case of two polyhedra

Solved in 2001 by Bemporad, Fukuda and D. Torrisi⁵:

- ▶ Computing convex hull by removing inequalities
- ▶ Solving a linear program

- Case of several (≥ 3) polyhedra

Open problem...



⁵Bemporad, Fukuda and Torrisi, 'Convexity recognition of the union of polyhedra', 2001.