Classification of random groups

Emily Clement[1]
Joint work with: John Mackay[2]

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Short bio

- **L3, M1, M2 (Maths), M2 (Computer science)** — **ENS Rennes** — **2014 - 2018**

- **PhD in computer science** — **Université de Rennes & MERCE** — **10/18 - 03/22**

Directors: *Nicolas Markey, Thierry Jéron*. Advisor: *David Mentré*

- **Post-doc in CS & Robotics** — **Sorbonne Université (ISIR)** — **04/22 - 05/23**

Director: *Nicolas Perrin-Gilbert*. Collabor: *Philipp Schlehuber-Caissier*

- **ATER in Computer Science** — **Université Paris Cité (IRIF)** — **09/23 - 09/24**

Collaborators: *Sylvain Schmitz, Marie Fortin, Jeremy Ledent, Uli Fahrenberg, Hugo Bazille, Amazigh Amrane, Krzysztof Ziemański , Damien Bussato-Gaston, John Mackay*

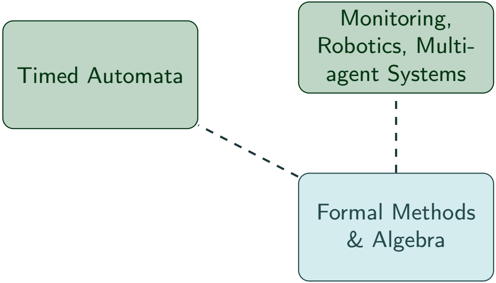
- **Post-doc in Computer Science** — **Sorbonne Nord (LIPN)** — **10/24 - now**

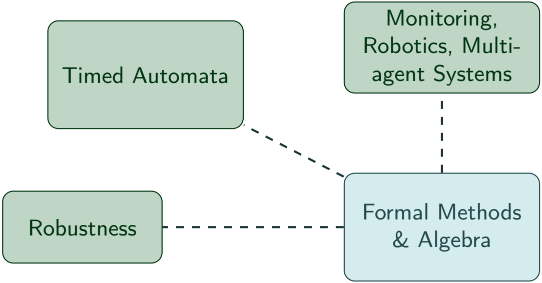
Director: *Étienne André*.

Research topics

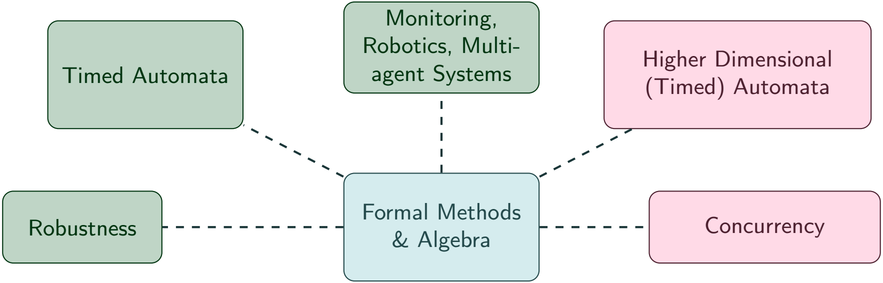
Formal Methods
& Algebra

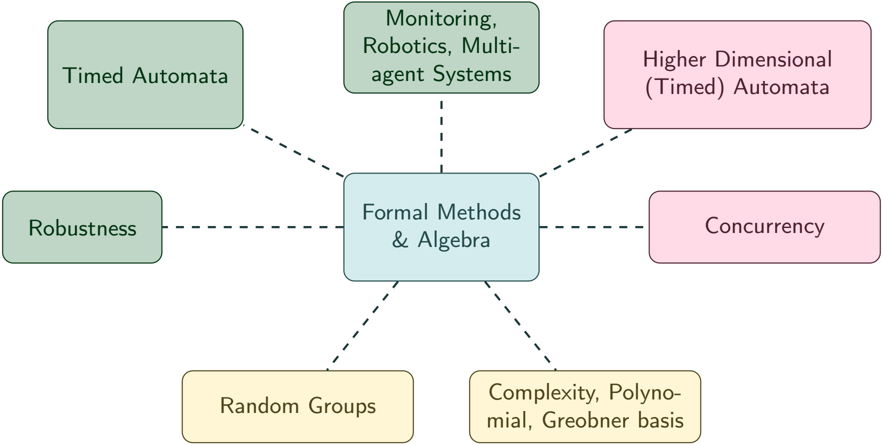
Research topics



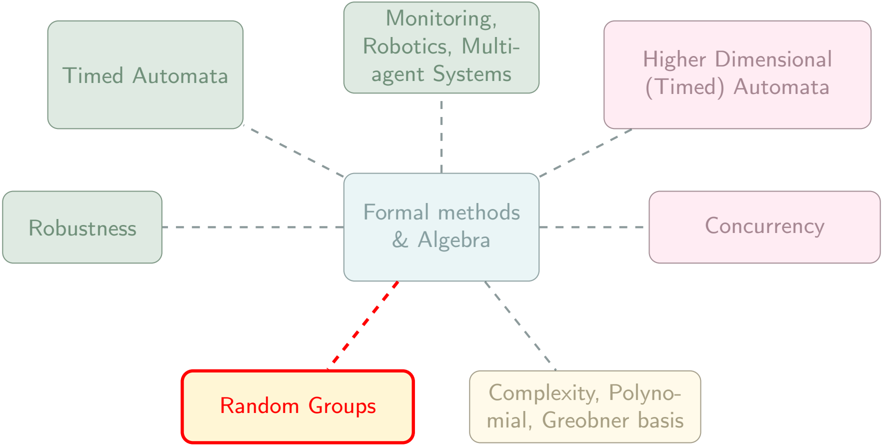

Timed Automata
Monitoring, Robotics, Multi-agent Systems
Robustness
Formal Methods & Algebra

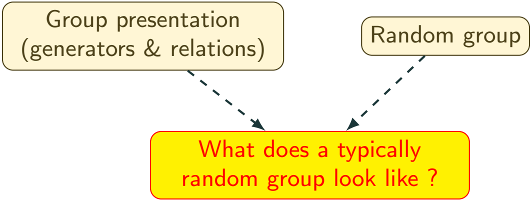
Research topics




Timed Automata
Monitoring, Robotics, Multi-agent Systems
Higher Dimensional (Timed) Automata
Robustness
Formal Methods & Algebra
Concurrency
Random Groups
Complexity, Polynomial, Greobner basis

Research topics





Motivations

- Trivial groups, Finite groups, Free groups
- Hyperbolic groups: "Most of random group are Hyperbolic" (Gromov[1])
  - Decidability of problems (word problem)
  - Representation with Automata
  - Application: networks[2]

[1] Gromov, "Hyperbolic Groups", 1987.
[2] Chepoi, Dragan, and Vaxès, "Core congestion is inherent in hyperbolic networks", 2017.

Generators, Words & Relations

- Generators $S = \{a, b\}$, $S = \{a, b, c\}$.

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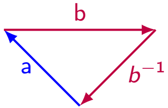
Examples of relations:

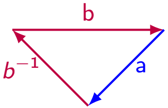
Unreduced

$abb^{-1} = e$



Cyclically unreduced

$b^{-1}ab = e$



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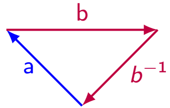

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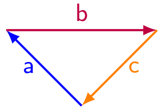

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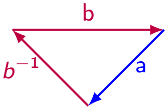

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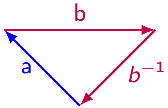

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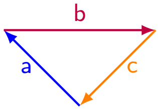

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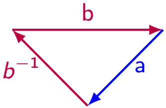

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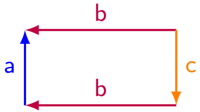

Group presentation

- No relations, only generators $\langle a, b \rangle$: words made with letters a, a^{-1}, b, b^{-1}.

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$$\boxed{G = \langle S_+ \mid R \rangle} \; s.t. \begin{cases} S_+ \subseteq \Sigma & \text{(generator)} \\ R \subseteq (S_+ \cup S_-)^* & \text{(relations)} \end{cases}$$

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    $$\boxed{\langle a \mid a^2 = e\rangle}$$

    Elements are exactly e and a

- $G = \langle S \mid R \rangle$, parameters:

Parameters

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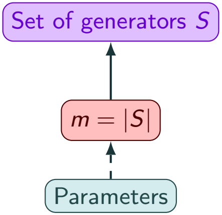
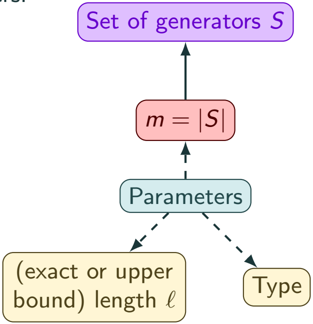


Random group and geometric properties

- $G = \langle S \mid R \rangle$, parameters:



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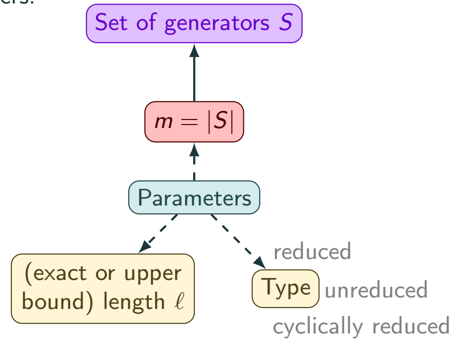

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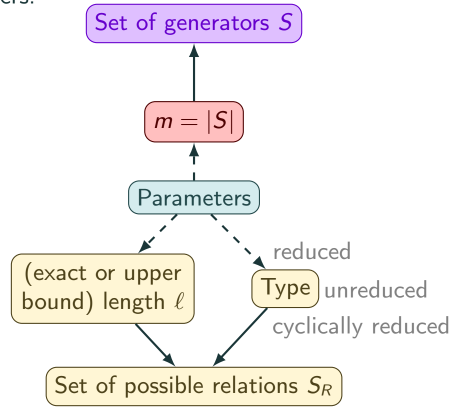

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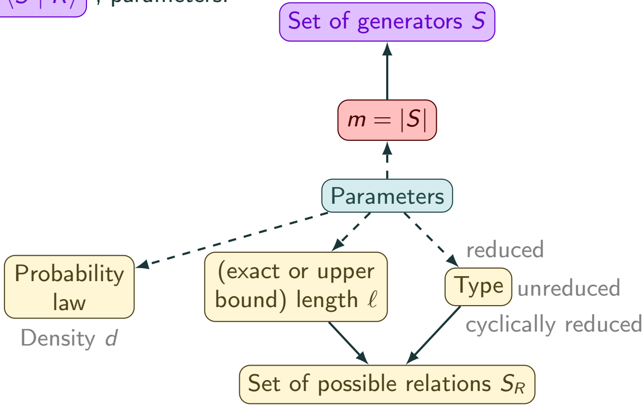

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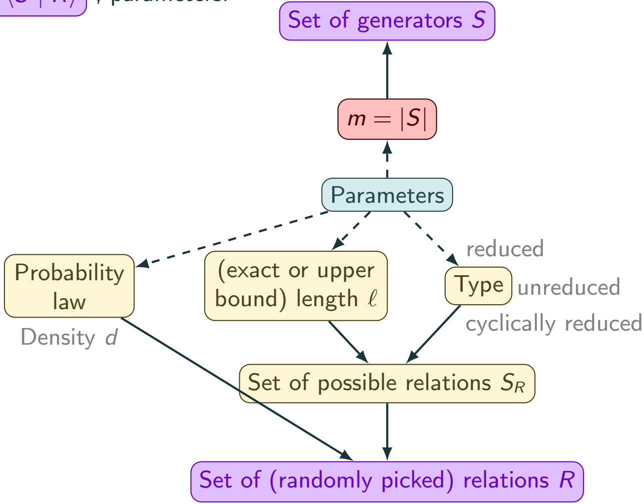

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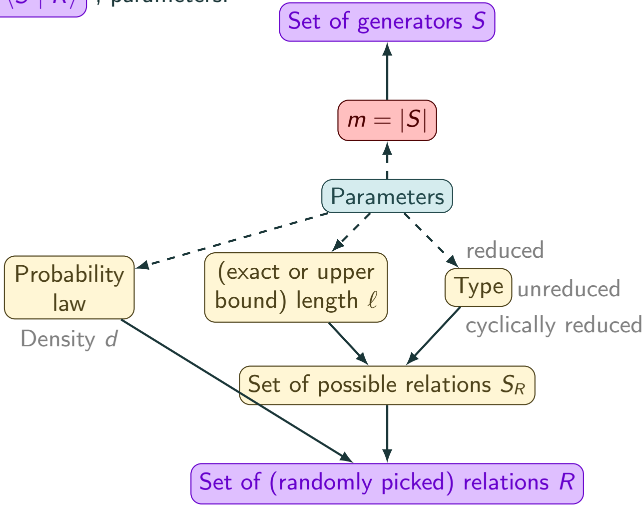

- $\mathcal{P}$ occurs *a.a.s.* (w.r.t. ℓ, or m...) if $\mathbb{P}(\mathcal{P}) \xrightarrow[m\to+\infty]{} 1$

- Motivation: geometric properties

Finite ?

Free ?

What does a typically random group look like ?

Trivial ?

Hyperbolic ?

Motivation & our model: random triangular groups

- Motivation: geometric properties

Finite ?

Free ? **What does a typically random group look like ?** Trivial ?

Hyperbolic ?

- Our model: triangular ($\ell = 3$, exact length) random group, with:
  - unreduced words.
  - Relations picked uniformly, independently with probability $m^{(d-1)\ell}$.

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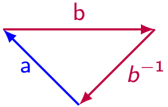
- Example of unreduced relations:

$$abb^{-1} = e$$



- Models:

Parameters \ Words	Reduced, cyclically reduced words	Unreduced words
$m \geq 2, \ell \to +\infty$	Yann Ollivier[3]	Yann Ollivier[4]
$m \to +\infty, \ell = 3$	Antoniuk et al.[5]	Our model

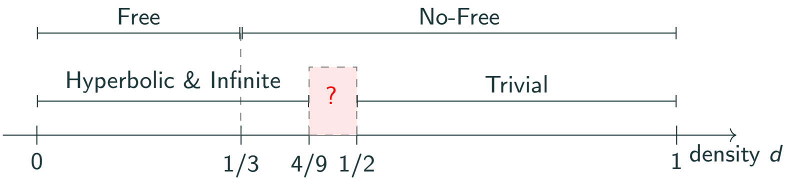
[3] Ollivier, "A january 2005 Invitation to Random Groups", 2005.

[4] Ollivier, "A january 2005 Invitation to Random Groups", 2005.

[5] Antoniuk, Friedgut, and Łuczak, "A sharp threshold for collapse of the random triangular group", 2014; Antoniuk, Łuczak, and Swicatkowski, "Collapse of random triangular groups: a closer look", 2013; Antoniuk, Łuczak, and Świcatkowski, "Random triangular groups at density 1/3", 2013.

Our contribution

- Our model: triangular ($\ell = 3$, exact length) random group, with unreduced words. Relations picked uniformly, independently with probability $m^{(d-1)\ell}$.
- Our results when $m \to +\infty$:



Let $G = \langle S \mid R \rangle$. G is:

- Finite: $|G| < +\infty$
- Trivial: all word from $\langle S \mid R \rangle$ are equal to e_G
- Free: for some set of generators $S' \subseteq G$, all elements of G have a unique representation (as sequence of S').

Representation of empty word

For any word $w = e_G$:

$$w = \prod_{i=1}^{N(w)} g_i \cdot r_i^{\pm 1} \cdot g_i^{-1}, g_i \in S, r_i \in R$$

- $N(w)$: the minimal number of relations needed to write w.
- Example: $a^4 = a^2 a^2$ in $\langle a \mid a^2 \rangle$, $N(a) = 2$

Word problem

Given G a group, determine the set of word w *s.t.* $w = e_G$?

Hyperbolic groups: linear bound of N

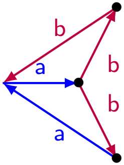
G is **hyperbolic** iff $\exists C > 0, \forall w \in G$ *s.t.* $w = e_G$, $\boxed{N(w) \leq C\,|w|}$.

Examples

- Hyperbolic: Finite groups, $\mathbb{Z}$
- Non-hyperbolic: $\mathbb{Z}^2$

Geometric point of view: Van Kampen Diagram $\mathcal{D}$

- Intuition: "Glue" relations together to form a word equal to e.
- $S = \{a, b\}$, Relations $R = \{abb, aba\}$, $G = \langle S \mid R \rangle$.
- Is $bba^{-1}b^{-1} = e_G$?



$$bba^{-1}b^{-1} = a^{-1}(abb)aa^{-1}(aba)^{-1}a = e_G$$

So is $a^{-1}b$...

- Hyperbolicity caracterisation: $\exists C > 0$, *s.t.* for any $\mathcal{D}$, $|\mathcal{D}| \leq C\,|\delta\mathcal{D}|$

Our contribution

What relation exists *a.a.s.* in our model?

Relation of type i

- Type i: contains exactly i different letters (inverse does not count).
- abb, aba^{-1}: type 2, aaa^{-1}: type 1, abc: type 3.

Counting relations

- $\mathbb{E}[X_i] \asymp m^{2d-3+i}$
- When $d < 2/3$, $\mathbb{E}[X_1] \xrightarrow[\ell \to +\infty]{} 0$
- When $d < 1/3$, $i = 1, 2$, $\mathbb{E}[X_i] \xrightarrow[\ell \to +\infty]{} 0$.

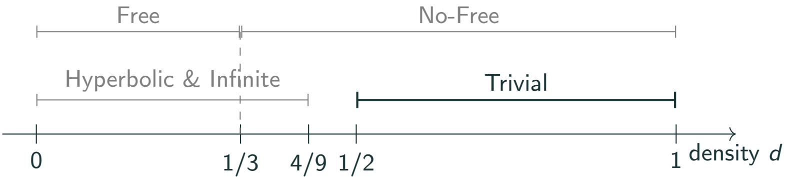
Case of $d < 1/3$

- When $d < 1/3$, *a.a.s.* relations are of type 3
- Type 3 $\Rightarrow$ reduced word of length 3
- Reduction to Antoniuk et al.[6] results:

[6] Antoniuk, Łuczak, and Swicatkowski, "Collapse of random triangular groups: a closer look", 2013; Antoniuk, Łuczak, and Świcatkowski, "Random triangular groups at density 1/3", 2013; Antoniuk, Friedgut, and Łuczak, "A sharp threshold for collapse of the random triangular group", 2014.

Compute the set of trivial generators T (using graph theory)

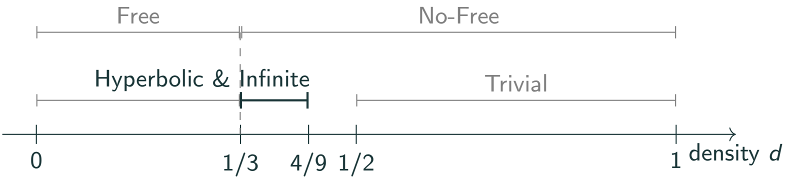
- Initialise $T = \varnothing$
- Relation of type[7] $aaa^{-1} = e$ or $abb^{-1} = e$: add a in T
- Relation of type 3, of the form $[T][T][?]$: add $[?]$ to T.
- Results: $T = S$



[7] up to permutation

Adapt our model to Yann Ollivier's one

Yann Ollivier's model: fixed number of relations.



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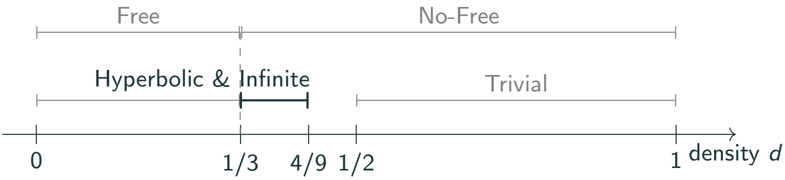
- 1): Give a lower and upper bound of $|R|$.



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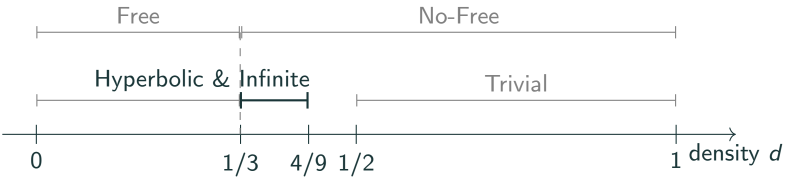
Yann Ollivier's model: fixed number of relations.

- 1): Give a lower and upper bound of $|R|$.
- 2): Give an equivalent model, with varying probability (to pick a relation).



Conclusion

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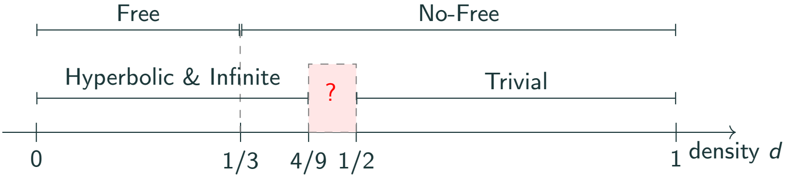
Our results



Current & Future work

What about $4/9 \leq d \leq 1/2$?

Publications (Conférence)

- International conference:
  - Formats'20: **Computing Maximally-Permissive Strategies in Acyclic Timed Automata.**
    Emily Clement, Thierry Jéron, Nicolas Markey, David Mentré
  - Formats'23: **Layered controller synthesis for dynamic multi-agent systems**
    Emily Clement, Nicolas Perrin-Gilbert, Philipp Schlehuber-Caissier
  - Petri Nets'24: **Languages of Higher-Dimensional Timed Automata**
    Amazigh Amrane, Hugo Bazille, Emily Clement, Uli Fahrenberg
  - Ramics'24: **Presenting Interval Pomsets with Interfaces**
    Amazigh Amrane, Hugo Bazille, Emily Clement, Uli Fahrenberg, Krzysztof Ziemiański
- Submitted articles:
  - FOSSACS'25: **Expressivity of Linear Temporal Logic for Pomset Languages of Higher Dimensional Automata**
    Emily Clement, Enzo Erlich, Jérémy Ledent