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ECOLE DOCTORALE N° 601
Mathématiques et Sciences et Technologies
de l'Information et de la Communication
Spécialité : *Mathématiques et leurs interactions*

Par
Fabrice GRELA

Tests minimax et adaptatifs pour la détection et la localisation d'une rupture dans un processus de Poisson

Thèse présentée et soutenue à l'Université Rennes 2, le 1er Décembre 2021
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Rapporteurs avant soutenance :

Ery ARIAS-CASTRO	Professeur, University of California
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Composition du Jury :

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CONTENTS

CHAPTER

ONE

INTRODUCTION (EN FRANÇAIS)

Les processus ponctuels sont un type particulier de processus stochastiques dont une réalisation est un ensemble de points isolés du temps et/ou de l'espace. Comme détaillé dans l'introduction du livre de Daley et Vere-Jones [DVJ07], l'analyse formelle des processus ponctuels a historiquement permis l'étude des données de survie au sein d'une population puis l'émergence de la théorie du renouvellement et des problèmes de comptage avec les travaux de Poisson [Poi37]. Aujourd'hui, les processus ponctuels sont des objets mathématiques très étudiés en probabilité et en statistique car ils décrivent de nombreux phénomènes aléatoires, notamment ceux évoluant dans le temps, dans des domaines très variés allant par exemple des sciences de l'information aux neurosciences, en passant par l'économie, les sciences humaines et sociales, l'écologie, ou encore l'épidémiologie. Pouvoir définir des conditions de détectabilité des comportements anormaux au sein des observations de ces processus et construire des procédures pour détecter ces anomalies offrent ainsi un large champ d'applications en science des données.

L'objectif de cette introduction est de présenter le problème de la détection de ruptures dans la loi d'un processus de Poisson : nous expliquons en quoi ce problème est intéressant pour modéliser et étudier des phénomènes concrets, nous proposons un aperçu des principaux résultats obtenus historiquement et au cours de cette thèse et présentons certains outils mathématiques utiles pour la résolution de ce problème dans les prochains chapitres. Plus précisément, nous rappelons dans une première section ce qu'est un processus de Poisson, un problème de détection de ruptures et présentons des motivations en épidémiologie et en cybersécurité pour le problème particulier de la détection de ruptures dans la loi d'un processus de Poisson, puis nous spécifions le modèle statistique étudié dans cette thèse. Dans une deuxième section, nous énonçons certains résultats théoriques de statistique sur lesquels sont basés la plupart des raisonnements dans la suite de ce manuscrit, à savoir, la notion de quantile, des rappels sur les tests statistiques simples et multiples ainsi que leurs critères d'optimalité respectifs. Nous proposons dans une troisième section un état de l'art pour les deux problèmes étudiés dans cette thèse : le problème de la détection et de la localisation de ruptures dans l'intensité d'un processus de Poisson. Enfin, dans une quatrième section, nous donnons un aperçu des prochains chapitres et des principales contributions de cette thèse. Certaines remarques de cette introduction, volontairement informelles, visent à proposer un éclairage intuitif du problème de la détection de ruptures dans la loi d'un processus de Poisson et des outils mathématiques mis en œuvre dans le reste du manuscrit.

I Détection de ruptures dans la loi d'un processus de Poisson : motivations et modèle étudié

I.1 Le processus de Poisson

Dans cette thèse, nous étudions un processus de comptage $(N_t)_{t\geq 0}$, appelé processus de Poisson, du nom du mathématicien français Siméon Denis Poisson (1781-1840), permettant de modéliser l'évolution temporelle de phénomènes aléatoires de type ponctuel. Pour tout $t \geq 0$, la quantité N_t compte le nombre d'occurrences, aussi appelées *points* ou *instants de saut* du processus, d'un évènement ou d'un phénomène aléatoire pouvant survenir à n'importe quel instant, dans l'intervalle de temps $[0,t]$.

Définition 1.1 (Processus de Poisson homogène)**.**
Un processus de comptage $(N_t)_{t\geq 0}$ est appelé processus de Poisson homogène d'intensité $\lambda > 0$ s'il vérifie les propriétés suivantes :
1. $N_0 = 0$,
2. $(N_t)_{t\geq 0}$ est à accroissements indépendants,
3. Le nombre de points ou d'instants de saut dans un intervalle de temps quelconque de longueur T est une variable aléatoire de loi de Poisson de paramètre λT.

Il est important de remarquer que le nombre de points observés dans un intervalle dépend seulement de la taille de l'intervalle, et pas de sa localisation sur la droite réelle : on dit que les accroissements du processus sont *stationnaires*.

Pour tout n dans $\mathbb{N}$, on note X_n le temps de la n-ième occurrence (avec la convention $X_0 = 0$) : le processus $(X_n)_{n\in\mathbb{N}}$ est alors appelé *processus des temps d'arrivées* et les variables aléatoires $X_0 < X_1 < X_2 < \dots$ sont appelées, comme vu ci-dessus, les *points* ou les *instants de saut* du processus. Ces instants de saut caractérisent entièrement le processus et on a

$$N_t = \sum_{n\geq 1} \mathbb{1}_{X_n \leq t}, \quad t \geq 0 \ .$$

Proposition 1.2.
Soit $(N_t)_{t\geq 0}$ un processus de Poisson homogène d'intensité λ. Pour n dans $\mathbb{N}$ et conditionnellement à l'évènement $\{N_t = n\}$, le n-uplet $(X_1,\dots,X_n)$ a même loi de probabilité que le n-uplet ordonné correspondant à n variables aléatoires indépendantes et identiquement distribuées (i.i.d.), de loi uniforme sur $[0,t]$.

Lorsque l'intensité du processus de Poisson varie dans le temps, on doit s'affranchir de la propriété de stationnarité des accroissements du processus, on parle alors de processus de Poisson inhomogène.

Définition 1.3 (Processus de Poisson inhomogène)**.**
Soit λ une fonction strictement positive et localement intégrable. Un processus de comptage $\{N_t,\ t \geq 0\}$ est appelé processus de Poisson inhomogène de fonction intensité (ou simplement d'intensité) λ, s'il vérifie les propriétés suivantes :
1. $N_0 = 0$,
2. $(N_t)_{t\geq 0}$ est à accroissements indépendants,
3. Le nombre de points ou d'instants de saut dans un intervalle de temps quelconque I est une variable aléatoire de loi de Poisson de paramètre $\int_I \lambda(t)dt$, où dt est la mesure de Lebesgue.

Comme pour le cas homogène, on dispose d'une propriété sur la loi conditionnelle des points d'un processus de Poisson inhomogène.

Proposition 1.4.
Soit $(N_t)_{t\geq 0}$ un processus de Poisson inhomogène d'intensité λ. Pour n dans $\mathbb{N}$ et conditionnellement à $\{N_t = n\}$, le n-uplet $(X_1,\dots,X_n)$ a même loi que le n-uplet ordonné correspondant à n variables aléatoires i.i.d. de densité $\lambda/\int_0^t \lambda(s)ds$ sur $[0,t]$.

I.2 Le problème de la détection de ruptures

À partir de l'observation d'un processus de Poisson sur un intervalle de temps fixé et borné, on s'intéresse au problème de la détection d'une rupture dans la loi du processus, caractérisée par un changement abrupt sous forme de saut, éventuellement transitoire, dans son intensité. Le problème plus général de la détection de ruptures dans la loi de processus stochastiques intéresse les mathématiciennes et mathématiciens depuis les années 1940-1950 avec les travaux fondateurs de Wald [Wal45], Girshick et Rubin [GR52], Page [Pag54] ou Fisher [Fis58]. La littérature sur la détection de ruptures étant particulièrement riche et abondante, nous ne pourrons pas présenter un état de l'art exhaustif. On trouvera par exemple des aperçus plus complets et le détail de certaines techniques de détection dans les ouvrages de référence de Basseville et Nikirov [BN93], Carlstein et al. [CMS94], Csörgö et Horváth [CH97], Brodsky et Darkhovsky [BD13b, BD13a], Tartakovsky et al. [TNB14a], ainsi qu'une bibliographie structurée de différents articles de recherche sur la détection de ruptures dans un papier de Lee [Lee10].

Il existe deux approches permettant de formaliser mathématiquement l'étude statistique des problèmes de détection de ruptures : le cadre en-ligne (ou on-line) et le cadre hors-ligne (ou off-line).

Dans le cadre en-ligne, le problème de détection de ruptures est traité par une approche séquentielle : la taille de l'échantillon ou la durée de l'observation n'est pas fixée au préalable et les données sont évaluées au fur et à mesure qu'elles sont recueillies (en temps réel, par point ou par lot). L'analyse et le traitement des données sont effectués instantanément avant l'arrivée de nouvelles observations et la détection de ruptures repose alors sur le choix d'une statistique et du seuil qu'elle doit atteindre pour signaler une détection. Cette démarche est typique de l'analyse séquentielle, l'échantillonnage ou l'observation du processus est stoppé selon une règle d'arrêt établie au préalable, dès que des résultats significatifs sont notés. Autrement dit, il s'agit de détecter une rupture le plus rapidement possible une fois qu'elle est passée. Des critères d'optimalité permettent de mesurer les performances des procédures de détection en-ligne selon la nature des observations : on trouvera une présentation des critères usuels d'optimalité dans les articles de Lai [Lai01], Moustakides [Mou08] ou Polunchenko et Tartakovsky [PT12]. On pourra également consulter le livre de Siegmund [Sie85] pour une présentation détaillée des outils de l'analyse séquentielle.
En opposition à la formulation séquentielle ou en-ligne du problème de détection, la formulation hors-ligne offre un cadre rétrospectif à la détection de ruptures. Dans le cadre hors-ligne, les observations sont fixées et la détection de ruptures est traitée a posteriori. Deux questions distinctes se posent alors :
(1) détecter un nombre donné de ruptures ou estimer le nombre de ruptures dans la loi à partir des observations,
(2) estimer un ou plusieurs paramètres des ruptures (instants de rupture, tailles des sauts,...) une fois les ruptures détectées.
Comme l'expliquent Niu et al. dans [NHZ16], ces deux questions peuvent être formulées ou interprétées comme des problèmes de tests statistiques simples ou multiples. Si ces deux questions sont généralement traitées simultanément dans la littérature, les performances des procédures de tests et les aspects d'optimalité ne sont pas toujours explicites ou étudiés théoriquement.

Pour les formulations en-ligne et hors-ligne des problèmes de détection, on peut considérer un modèle bayésien ou un modèle non bayésien pour la description des ruptures. Dans les modèles bayésiens, les paramètres des ruptures (comme les instants de saut par exemple) sont supposés aléatoires et on dispose d'informations a priori sur leur loi alors que dans les modèles fréquentistes (ou non bayésiens), ils sont déterministes et inconnus.

Dans cette thèse, nous abordons la question de la détection optimale, au sens du minimax, d'une rupture (éventuellement transitoire) dans la loi d'un processus de Poisson pour une formulation hors-ligne et non bayésienne du problème. Nous proposons une étude minimax non asymptotique des problèmes de tests et une démarche progressive dans la construction de procédures de tests simples optimales pour la détection de ruptures puis dans la construction de procédures de tests multiples optimales pour la localisation d'une rupture.

I.3 Des applications en épidémiologie et en cybersécurité

Afin de motiver ce problème de détection d'un changement abrupt, transitoire ou non, dans l'intensité d'un processus de Poisson, plusieurs applications peuvent être envisagées. Nous nous concentrons sur deux d'entre-elles, en épidémiologie et en cybersécurité.

Tout d'abord, la détection de ruptures dans le taux d'apparitions de certains évènements est un objectif important en santé public ou en surveillance médicale (voir les travaux de Woodall [Woo06], de Shmueli et Burkom [SB10] et de Tsui et al. [TCG+08]). Les méthodes de détection sont souvent basées sur l'observation de séries temporelles ou de processus de comptage comme dans [RBC+07, SZ10]. Le nombre de données observées ponctuellement dans le temps est généralement modélisé par une variable aléatoire discrète de loi de Poisson (voir par exemple les articles de Levin et Kline [LK85] et de Sonesson [Son07]) : une rupture dans la moyenne de la loi indique alors qu'un évènement anormal s'est produit. Généralement, on suppose que le processus d'intérêt est un processus de Poisson (voir par exemple [SB03, Kul97, LW94]) et que les données sont observées sur des intervalles de temps de longueur déterministe puis agrégées (on pourra consulter les articles [SWC13, RPW15] pour la description de ce modèle).

Comme l'explique Han et al. [HTAK10] et selon le schéma des observations, les méthodes usuelles pour détecter des ruptures dans la loi des processus en épidémiologie reposent essentiellement sur trois types de statistiques : les statistiques de scan (voir par exemple [GB99, NW06, WMJ+08]), les statistiques de sommes cumulées (ou statistiques CUSUM) comme dans [LK85, Son07] et les statistiques des moyennes mobiles pondérées exponentiellement (ou statistiques EWMA) utilisées dans [SSJT14]. Il est à noter que les méthodes les plus populaires dans la littérature traitant de la détection de ruptures dans la loi de processus observés en certains points du temps sont celles basées sur la statistique CUSUM. Cette statistique, introduite par Page dans les années 1950 [Pag54] a été très largement étudiée en pratique et plusieurs résultats d'optimalité ont été démontrés : on citera par exemple les articles dédiés de Lorden [Lor71], Moustakides [Mou86], Ritov [Rit90], Beibel [Bei96] et Shiryayev [Shi96]. Si historiquement les approches basées sur la statistique CUSUM sont issues de l'analyse séquentielle pour des problèmes de détection en-ligne, elles peuvent être adaptées pour des formulations hors-ligne (on renverra par exemple à l'article de Galeano [Gal07] pour la détection de ruptures dans la loi d'un processus de Poisson et à la review de Yu [Yu20] pour la détection de ruptures dans un cadre gaussien).

Les questions de cybersécurité, qui ont connu un essor considérable ces dernières décennies, sont un enjeu important pour les gouvernements, les industries, la finance, la défense et la vie privée. Les comportements malveillants et les tentatives d'intrusion comme les campagnes de spam, de phishing et les virus informatiques sont des faits récurrents et deviennent de plus en plus difficiles à détecter. Des évènements anormaux produisent généralement des changements soudains dans le trafic au sein d'un réseau et ces attaques peuvent donc être détectées en remarquant un changement dans le nombre moyen de paquets observés dans l'intensité du trafic. On pourra consulter l'ouvrage de Basseville et al. [TNB14b] pour plus de précision sur le modèle. Enfin, les paquets au sein du trafic internet sont souvent modélisés par un processus de Poisson (voir les articles [PT12], [CCLS03], [KMFB04], [VSO09] ou [SKR08]) : la détection d'une anomalie s'interprète alors comme un problème de détection de rupture dans l'intensité d'un processus de Poisson.

Un problème particulier, étudié par exemple par Soltani et al. [SGTH15], [SGTH20] et Wang et al. [Wan18], concerne les communications au sein d'un canal de paquets. Au sein d'un canal sur lequel un émetteur autorisé envoie des paquets à un destinataire autorisé selon un processus de Poisson (supposé homogène), un émetteur secret (non autorisé) souhaite communiquer des informations à un destinataire secret sans être détecté par un adversaire qui surveille le canal. Plusieurs modèles de communications secrètes ont été étudiés par les auteurs cités. Par exemple dans [SGTH15], l'émetteur secret est restreint à insérer des paquets ou à ralentir le flux de paquets pour transmettre ses informations. La question de la détectabilité de ces stratégies de communications secrètes s'interprète donc comme l'optimalité de la détection d'un segment dans l'intensité d'un processus de Poisson et est donc directement reliée au point de vue minimax des problèmes de tests décrit dans cette thèse.

Enfin, d'autres auteurs ont considéré un processus de Poisson pour modéliser les occurrences de certaines cyberattaques ou d'intrusions au sein d'un système informatique : une modification soudaine de l'intensité du processus témoigne alors d'un changement dans le régime d'attaque (voir les articles de Daras [Dar14] et Holm [Hol13] pour une description plus détaillée de ce modèle).

I.4 Notations et modèle statistique étudié : formulation hors-ligne et non bayésienne

Modèle statistique

On considère un processus de Poisson (éventuellement inhomogène) $N = (N_t)_{t\in[0,1]}$, observé sur l'intervalle de temps fixé $[0,1]$, d'intensité λ définie par rapport à une mesure Λ sur $[0,1]$ et dont la loi est notée P_λ. Comme dans [FLRB11] et [FLRB13], on suppose que la mesure Λ sur $[0,1]$ vérifie $d\Lambda(t) = Ldt$, où L est un réel strictement positif fixé et dt est la mesure de Lebesgue sur $[0,1]$. Le choix de cette mesure par rapport à laquelle est définie l'intensité du processus nous permettra de comparer nos résultats d'optimalité avec ceux obtenus dans d'autres modèles fréquentiels asymptotiques ou non-asymptotiques. On peut en effet remarquer que si L est un entier naturel, l'hypothèse sur la mesure Λ revient à considérer le processus N comme le processus agrégé de L processus de Poisson indépendants et d'intensité commune λ définie par rapport à la mesure de Lebesgue : faire tendre L vers l'infini pourrait donc définir une asymptotique pour notre modèle. Dans toute la suite, insistons cependant sur le fait que nous travaillons à L fixé, c'est-à-dire dans un cadre non asymptotique.

On rappelle qu'une rupture dans la loi du processus de Poisson est ici caractérisée par un saut dans son intensité λ, c'est-à-dire que λ est de la forme

$$\lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t),\ t \in [0,1]\ ,$$

et qu'une rupture transitoire dans la loi du processus est caractérisée par un segment dans son intensité, c'est-à-dire que λ est de la forme

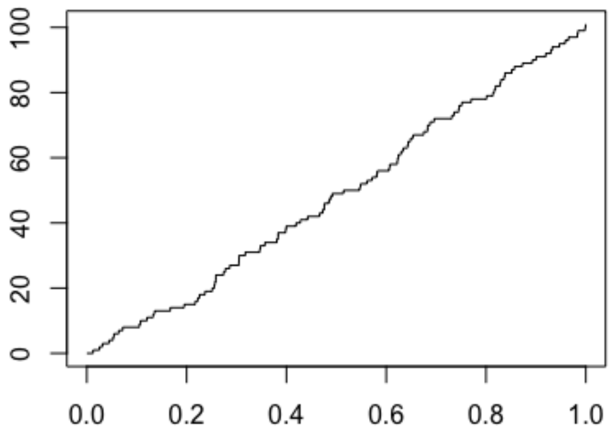
$$\lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell]}(t),\ t \in [0,1]\ ,$$

où

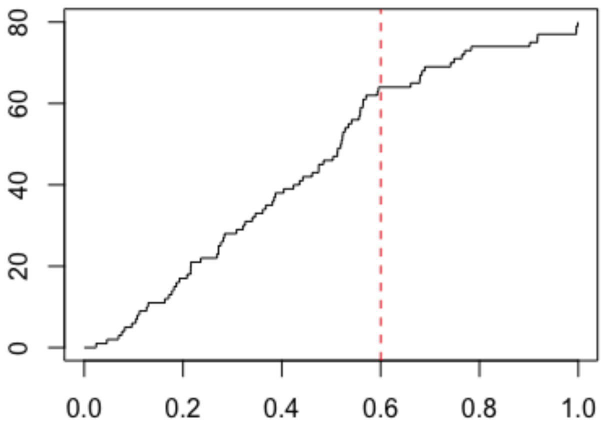
- $\lambda_0 \in \mathbb{R}_*^+$ est l'intensité de référence du processus,
- $\delta \in (-\lambda_0, +\infty) \setminus \{0\}$ est la taille ou l'amplitude du saut (ou du segment),
- $\tau \in (0,1)$ est l'instant de rupture,
- $\ell \in (0, 1-\tau)$ est la longueur du segment.

De manière heuristique, il semble difficile de détecter des sauts dans l'intensité qui interviennent "proche du bord" (c'est-à-dire dont les instants de rupture sont proches de 0 ou de 1) et de faibles amplitudes (c'est-à-dire dont la taille est "petite"). De même, il semble difficile de détecter des segments dont la longueur est proche de 0 ou 1 pour de faibles amplitudes de sauts (voir Figure 1.1d). L'ensemble des paramètres de la rupture pour lesquels on peut espérer détecter un saut ou un segment via une procédure de test est appelé *région de détectabilité*. Comme nous le verrons dans la suite, cette région de détectabilité sera quantifiée par les critères d'optimalité au sens du minimax que nous définissons dans la prochaine section.

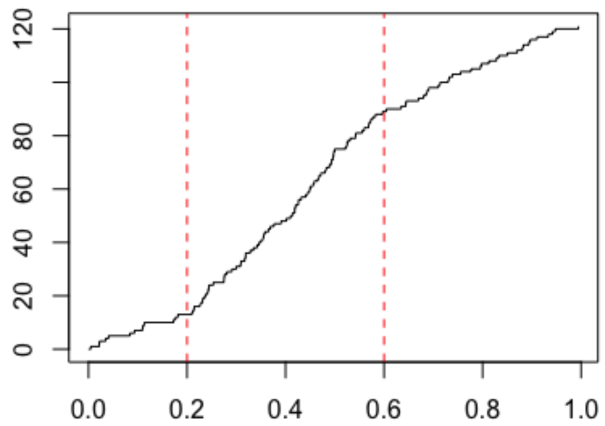
Figure 1.1 – Trajectoires d'un processus de Poisson selon la forme de l'intensité (intensité constante, avec un saut ou un segment) définie par rapport à $d\Lambda(t) = Ldt$ avec $L = 100$. On suppose que l'intensité de référence λ_0 est égale à 1. En abscisse, le temps t et en ordonnée la valeur du processus N_t.



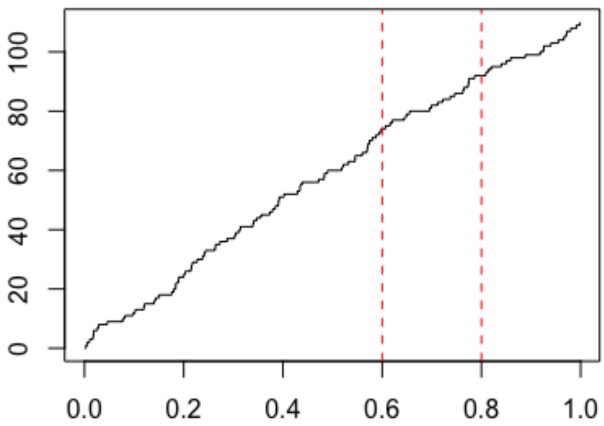
(a) Processus de Poisson homogène - intensité constante (égale à $\lambda_0 L = 100$)



(b) Processus de Poisson inhomogène - un saut non transitoire dans l'intensité en $\tau = 0.6$ de taille $\delta = -0.6$



(c) Processus de Poisson inhomogène - un saut dans l'intensité en $\tau = 0.2$, de longueur $\ell = 0.4$ et de taille $\delta = 0.8$



(d) Processus de Poisson inhomogène - un segment dans l'intensité en $\tau = 0.6$, de longueur $\ell = 0.2$ et de taille $\delta = -0.2$

Notations

Introduisons à présent quelques notations utilisées dans toute la suite de ce manuscrit.

➤ Pour un processus de Poisson $N = (N_t)_{t\in[0,1]}$, l'accroissement $N_{\tau_2} - N_{\tau_1}$ est noté $N(\tau_1, \tau_2]$ pour tout τ_1, τ_2 dans $[0,1]$ et E_λ et Var_λ désignent respectivement l'espérance et la variance sous la loi P_λ, c'est-à-dire lorsque N est d'intensité λ par rapport à $d\Lambda(t) = Ldt$.

➤ La distance notée d_2 est la métrique usuelle de $\mathbb{L}_2([0,1])$. On considère aussi le produit scalaire et la norme de $\mathbb{L}_2([0,1])$ que l'on note respectivement $\langle \cdot, \cdot \rangle_2$ et $\|\cdot\|_2$.

➤ Pour des réels x et y, $x \vee y$ (respectivement $x \wedge y$) désigne le maximum (respectivement le minimum) entre x et y, et la fonction signe, notée sgn, est définie par $\mathrm{sgn}(x) = \mathbb{1}_{x>0} - \mathbb{1}_{x<0}$ et $\mathrm{sgn}(0) = 0$.

➤ Pour un ensemble fini Θ, $|\Theta|$ désigne le cardinal de Θ.

II Des tests simples aux tests multiples pour la détection et la localisation d'une rupture. Optimalité au sens du minimax

II.1 Quantiles

Les valeurs critiques des tests simples que nous construisons dans les prochains chapitres sont définies à partir des quantiles des statistiques de test. La notion de quantile est donc primordiale pour toute la suite de ce manuscrit, nous en rappelons sa définition et certaines de ses propriétés.

Pour une variable aléatoire réelle X, on appelle *fonction de répartition* de X la fonction $F : \mathbb{R} \to [0,1]$ définie par

$$F(t) = \mathbb{P}(X \leq t), \quad \forall t \in \mathbb{R} \ .$$

On appelle *inverse généralisée* de la fonction F, la fonction F^{-1} définie par

$$F^{-1}(u) = \inf\{t \in \mathbb{R},\ F(t) \geq u\}, \quad \forall u \in [0,1] \ .$$

Par construction, la fonction F^{-1} est croissante et vérifie $F^{-1}(0) = -\infty$. De plus, si $\{t \in \mathbb{R},\ F(t) \geq u\} = \varnothing$ alors $u = 1$ et $F^{-1}(u) = +\infty$ (avec la convention $\inf(\varnothing) = +\infty$).
Si la fonction F est bijective de $\mathbb{R}$ dans $(0,1)$, alors elle est continue sur $\mathbb{R}$ et l'inverse généralisée de F coïncide avec l'inverse de F sur $(0,1)$. L'inverse généralisée reste cependant définie même si F n'est pas bijective, en particulier lorsque F est la fonction de répartition d'une loi discrète. On peut à présent introduire la notion de quantile.

Définition 1.5 (Quantile)**.**
Soit X une variable aléatoire réelle de fonction de répartition F dont l'inverse généralisée est notée F^{-1}. Pour u dans $(0,1)$, la quantité $F^{-1}(u)$ s'appelle le quantile d'ordre u ou le u-quantile (de la loi) de X.

Remarquons que la fonction de répartition F d'une variable aléatoire réelle X et son inverse généralisée F^{-1} vérifient, pour tout réel t et pour tout u dans $[0,1]$, la propriété suivante

$$F^{-1}(u) \leq t \Longleftrightarrow u \leq F(t) \ . \tag{1.1}$$

Cette propriété nous permet notamment d'obtenir des contrôles sur les quantiles. Illustrons cela en établissant des contrôles sur les quantiles d'une loi de Poisson.

Lemme 1.6 (Contrôles des quantiles d'une loi de Poisson)**.**
Le u-quantile $p_\xi(u)$ d'une loi de Poisson de paramètre $\xi > 0$ vérifie

$$-\sqrt{\xi/u} + \xi \leq p_\xi(u) \leq \sqrt{\xi/(1-u)} + \xi \ . \tag{1.2}$$

Démonstration.
Soit N une variable aléatoire de loi de Poisson de paramètre ξ. D'après l'inégalité de Bienaymé-Chebyshev, on a pour tout $\varepsilon > 0$,

$$\mathbb{P}\left(N \leq -\sqrt{\frac{\xi}{u-\varepsilon}} + \xi\right) \leq u - \varepsilon < u \quad \text{et} \quad \mathbb{P}\left(N > \sqrt{\frac{\xi}{u}} + \xi\right) \leq u \ .$$

Ainsi, d'après la propriété (1.1),

$$-\sqrt{\frac{\xi}{u-\varepsilon}} + \xi < p_\xi(u) \leq \sqrt{\frac{\xi}{1-u}} + \xi \ ,$$

et on obtient le résultat en faisant tendre ε vers zéro. □

Les contrôles du quantile d'une loi de Poisson déterminés par le lemme précédent, basés sur l'inégalité de Bienaymé-Chebyshev, sont assez grossiers : des contrôles plus fins peuvent être obtenus en utilisant d'autres inégalités de concentration (voir par exemple le Lemme 2.25) ou des inégalités exponentielles.

II.2 Tests statistiques simples pour la détection de ruptures et vitesse de séparation minimax

Dans ce travail, le problème de la détection d'une rupture (éventuellement transitoire) dans la loi d'un processus de Poisson est vu comme un problème de test simple de l'hypothèse nulle (H_0) "$\lambda \in \mathcal{S}_0$" contre l'alternative (H_1) "$\lambda \in \mathcal{S}_1$" dans le modèle décrit précédemment. $\mathcal{S}_0$ désigne soit l'ensemble formé par une intensité constante fixée, dans le cas où l'intensité de référence est connue, soit l'ensemble de toutes les intensités constantes sur $[0,1]$, dans le cas où l'intensité de référence du processus est inconnue. L'ensemble $\mathcal{S}_1$ désigne, quant à lui, un ensemble d'intensités alternatives définies comme des fonctions positives et constantes par morceaux avec un saut ou un segment.

On rappelle qu'une *fonction de test* ou un *test* plus simplement (non randomisé) est une statistique ϕ à valeurs dans $\{0,1\}$, telle que l'hypothèse nulle (H_0) est rejetée au profit de (H_1) si $\phi(N) = 1$. Prendre la décision de rejeter ou non une hypothèse au profit d'une autre, c'est prendre le risque de commettre une erreur.

Dans notre cas, le *risque de première espèce* d'un test ϕ est l'application définie sur $\mathcal{S}_0$ par $\lambda \mapsto P_\lambda(\phi(N) = 1)$. Pour un réel α dans $[0,1]$, on dira qu'un test ϕ est de *niveau* α si son risque de première espèce maximal est inférieur ou égal à α :

$$\sup_{\lambda \in \mathcal{S}_0} P_\lambda(\phi(N) = 1) \leq \alpha \quad ,$$

et on dira qu'il est de *taille* α si son risque de première espèce maximal est égal à α :

$$\sup_{\lambda \in \mathcal{S}_0} P_\lambda(\phi(N) = 1) = \alpha \quad .$$

Le *risque de deuxième espèce* d'un test ϕ est l'application définie sur $\mathcal{S}_1$ par $\lambda \mapsto P_\lambda(\phi(N) = 0)$, et sa *puissance* est l'application définie sur $\mathcal{S}_1$ par $\lambda \mapsto P_\lambda(\phi(N) = 1)$.

Il existe une première notion d'optimalité en théorie des tests basée sur le principe de Neyman et Pearson. Elle consiste à rechercher parmi les tests de niveau α, ceux dont la puissance est la plus grande possible. Pour deux tests ϕ_α et ϕ'_α de niveau α, on dit que ϕ_α est *uniformément plus puissant (UPP)* que le test ϕ'_α si pour tout λ dans $\mathcal{S}_1$, $P_\lambda(\phi_\alpha(N) = 1) \geq P_\lambda(\phi'_\alpha(N) = 1)$. On dit alors qu'un test est *uniformément plus puissant parmi les tests de niveau* α, s'il est de niveau α et s'il est uniformément plus puissant que tout test de niveau α. Cependant, si les tests du rapport de vraisemblance sont des tests uniformément plus puissant dans certains cas particuliers, il n'existe pas de tests optimaux en ce sens dans la majorité des situations. On pourra consulter l'ouvrage de Lehmann et Romano [LR05] pour une présentation de cette théorie des tests de Neyman-Pearson et des tests uniformément plus puissants, ainsi que sa généralisation à des problèmes plus complexes (tests sans biais, tests avec paramètres de nuisance...). Nous allons maintenant définir une autre notion d'optimalité en théorie des tests basée sur le principe dit *minimax*.

La notion d'optimalité minimax que nous allons définir ici est en accord avec le principe de Neyman-Pearson et le point de vue non asymptotique que nous adoptons dans notre modèle de détection de ruptures. Ainsi, dans un premier temps, étant donné un niveau α dans $(0,1)$ pour l'erreur de première espèce, tout test non randomisé ϕ doit être de niveau α. Dans un second temps, étant donné un niveau β dans $(0,1)$ pour l'erreur de seconde espèce, tout test de niveau α doit atteindre sur la classe d'alternatives considérée, la vitesse de séparation minimax de niveaux α et β définie comme suit.
En considérant la métrique usuelle d_2 de $\mathbb{L}_2([0,1])$ et un test ϕ_α de niveau α de (H_0) contre (H_1), on définit la *vitesse de séparation uniforme de niveau* β de ϕ_α sur la classe d'alternative $\mathcal{S}_1$ (pour la distance d_2) par

$$\mathrm{SR}_\beta^{\mathcal{S}_0}(\phi_\alpha, \mathcal{S}_1) = \inf\left\{ r > 0, \sup_{\lambda \in \mathcal{S}_1,\ d_2(\lambda, \mathcal{S}_0) \geq r} P_\lambda(\phi_\alpha(N) = 0) \leq \beta \right\} \quad . \tag{1.3}$$

Autrement dit, la vitesse de séparation uniforme de niveau β pour un test (de niveau α) ϕ_α s'interprète comme le plus petit réel $r > 0$ qui garanti le contrôle de l'erreur de première espèce par α et le contrôle de l'erreur de seconde espèce par β pour toute alternative λ de $\mathcal{S}_1$ à distance au moins r de l'hypothèse nulle (c'est-à-dire des intensités constantes). De manière heuristique et pour un test ϕ_α donné, il s'agit de la plus petite distance entre les hypothèses qui permet de bien "séparer" l'hypothèse nulle de l'hypothèse alternative, au sens où les erreurs de première et de deuxième espèce sont contrôlées par des niveaux fixés a priori. La *vitesse de séparation minimax de niveaux α et β* sur $\mathcal{S}_1$ (pour d_2) est alors définie par

$$\mathrm{mSR}_{\alpha,\beta}^{\mathcal{S}_0}(\mathcal{S}_1) = \inf_{\phi_\alpha,\ \sup_{\lambda\in\mathcal{S}_0} P_\lambda(\phi_\alpha(N)=1)\leq\alpha} \mathrm{SR}_\beta^{\mathcal{S}_0}(\phi_\alpha,\mathcal{S}_1) \quad , \tag{1.4}$$

où l'infimum est pris sur tous les tests non randomisés de niveau α.

Remarque 1.1 (Notations).
Lorsqu'il n'y a pas d'ambiguïté sur le choix de $\mathcal{S}_0$, on notera simplement $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$ (respectivement $\mathrm{SR}_\beta(\phi_\alpha,\mathcal{S}_1)$) pour désigner $\mathrm{mSR}_{\alpha,\beta}^{\mathcal{S}_0}(\mathcal{S}_1)$ (respectivement $\mathrm{SR}_\beta^{\mathcal{S}_0}(\phi_\alpha,\mathcal{S}_1)$).

Heuristiquement, la vitesse de séparation minimax donne la distance minimale entre les intensités de l'alternative $\mathcal{S}_1$ et l'ensemble $\mathcal{S}_0$ nécessaire pour détecter convenablement la rupture, au sens où il existe, pour chacune de ces alternatives "suffisamment éloignées" de $\mathcal{S}_0$, un test de niveau α dont l'erreur de seconde espèce est contrôlée par β. Enfin, on dira qu'un test ϕ_α de niveau α est un *test minimax de niveau β* sur $\mathcal{S}_1$ si $\mathrm{SR}_\beta(\phi_\alpha,\mathcal{S}_1)$ est égale à $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$, à une éventuelle constante multiplicative près qui peut dépendre de α, β et d'autres paramètres de $\mathcal{S}_1$, mais pas de L. Ces définitions, introduites par Baraud [Bar02], traduisent dans un cadre non asymptotique les critères minimax asymptotiques pour les problèmes de tests définis dans les travaux d'Ingster [Ing82, Ing84, Ing93], et qui ont aujourd'hui plusieurs extensions dans la littérature : on citera par exemple le critère asymptotique minimax avec constantes de séparation exactes présenté dans [LT00].

Le calcul de la vitesse de séparation minimax pour un problème de test s'obtient en deux étapes : le calcul d'une borne inférieure puis d'une borne supérieure pour $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$. Une borne inférieure minimax pour la vitesse de séparation se déduit de la théorie de la décision par des arguments bayésiens classiques énoncés par Le Cam, et explicités par Ingster [Ing82, Ing84, Ing93] et Baraud [Bar02] respectivement dans des cadres asymptotique et non asymptotique. Pour le cas particulier des problèmes de test sur un processus de Poisson, ces arguments sont résumés dans les deux lemmes techniques suivants, prouvés dans [FLRB11]. Le Lemme 1.8 étant fondamental dans la suite du manuscrit pour le calcul de bornes inférieures pour les vitesses de séparation minimax, on rappellera sa preuve.

Lemme 1.7.
Pour $r > 0$ et pour tout sous-espace $\mathcal{S}$ de $\mathbb{L}_2([0,1])$, soit $\mathcal{S}_r = \{\lambda \in \mathcal{S},\ d_2(\lambda,\mathcal{S}_0) \geq r\}$. Pour α dans $(0,1)$, on définit

$$\rho_\alpha(\mathcal{S}_r) = \inf_{\phi_\alpha,\sup_{\lambda\in\mathcal{S}_0} P_\lambda(\phi_\alpha(N)=1)\leq\alpha} \ \sup_{\lambda\in\mathcal{S}_r} P_\lambda(\phi_\alpha(N) = 0) \ .$$

(i) Si $\rho_\alpha(\mathcal{S}_r) \geq \beta$ alors $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}) \geq r$.
(ii) Pour tous sous-ensembles $\mathcal{S}'$ et $\mathcal{S}$ de $\mathbb{L}_2([0,1])$ vérifiant $\mathcal{S}' \subset \mathcal{S}$, on a $\rho_\alpha(\mathcal{S}'_r) \leq \rho_\alpha(\mathcal{S}_r)$ et $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}') \leq \mathrm{mSR}_{\alpha,\beta}(\mathcal{S})$.

Lemme 1.8 (Lemme baysésien pour les bornes inférieures minimax)**.**
Soit μ une mesure de probabilité sur $\mathcal{S}_r$ et soit P_μ la loi de mélange définie par

$$P_\mu = \int P_\lambda d\mu(\lambda) \ .$$

Alors

$$\rho_\alpha(\mathcal{S}_r) \geq 1-\alpha-\frac{1}{2}\left(\inf_{\lambda_0\in\mathcal{S}_0} E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right]-1\right)^{1/2} .$$

Par conséquent, si $\alpha+\beta<1$ *et* $\inf_{\lambda_0\in\mathcal{S}_0} E_{\lambda_0}\left[\left(dP_\mu/dP_{\lambda_0}\right)^2\right] \leq 1+4(1-\alpha-\beta)^2$, *alors*

$$\rho_\alpha(\mathcal{S}_r) \geq \beta \quad et \quad \mathrm{mSR}_{\alpha,\beta}(\mathcal{S}) \geq r .$$

Démonstration (voir [FLRB11]).
L'infimum (respectivement le supremum) pris sur tous les tests simples ϕ_α de niveau α de l'hypothèse nulle (H_0) sera simplement noté $\inf_{\phi_\alpha}$ (respectivement $\sup_{\phi_\alpha}$). En utilisant les notations du Lemme 1.7, on a pour tout λ_0 dans $\mathcal{S}_0$,

$$\begin{aligned}
\rho_\alpha(\mathcal{S}_r) &\geq \inf_{\phi_\alpha}\int P_\lambda(\phi_\alpha=0)d\mu(\lambda)\\
&\geq 1-\sup_{\phi_\alpha}\int P_\lambda(\phi_\alpha=1)d\mu(\lambda)\\
&= 1-\sup_{\phi_\alpha} P_\mu(\phi_\alpha=1)\\
&\geq 1-\sup_{\phi_\alpha}\left(|P_\mu(\phi_\alpha=1)-P_{\lambda_0}(\phi_\alpha=1)|+|P_{\lambda_0}(\phi_\alpha=1)|\right)\\
&\geq 1-\alpha-V(P_\mu,P_{\lambda_0}) ,
\end{aligned}$$

où $V(P_\mu,P_{\lambda_0})$ désigne la distance en variation totale entre les mesures de probabilité P_μ et P_{λ_0}. D'après le Lemme de Scheffé (voir Lemme 2.1 de [Tsy08]), puis de l'inégalité de Cauchy-Schwarz, on a donc

$$\begin{aligned}
\rho_\alpha(\mathcal{S}_r) &\geq 1-\alpha-\frac{1}{2}E_{\lambda_0}\left[\left|\frac{dP_\mu}{dP_{\lambda_0}}-1\right|\right]\\
&\geq 1-\alpha-\frac{1}{2}E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}-1\right)^2\right]^{1/2} .
\end{aligned}$$

Comme $E_{\lambda_0}[dP_\mu/dP_{\lambda_0}]=1$, on obtient finalement pour tout λ_0 dans $\mathcal{S}_0$,

$$\rho_\alpha(\mathcal{S}_r) \geq 1-\alpha-\frac{1}{2}\sqrt{E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right]-1} ,$$

ce qui achève la preuve. □

Une fois qu'une borne inférieure est calculée pour $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$, nous construisons une procédure de test minimax, c'est-à-dire une procédure dont la vitesse de séparation est du même ordre que les bornes inférieures préalablement déterminées, offrant ainsi des bornes supérieures pour les vitesses de séparation minimax.

Rappelons que nous déterminons des vitesses de séparation minimax pour la métrique d_2 de $\mathbb{L}_2([0,1])$. Pour une alternative λ de $\mathcal{S}_1$ de la forme $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell]}$ avec $\lambda_0\in\mathbb{R}_*^+$, $\delta\in(-\lambda_0,+\infty)\setminus\{0\}$, $\tau\in(0,1)$ et $\ell\in(0,1-\tau]$, on calcule donc la distance $d_2(\lambda,\mathcal{S}_0)$ (aussi appelée énergie du signal ou rapport signal sur bruit dans la littérature du traitement du signal ou des images). Notons que pour $\ell=1-\tau$, l'intensité λ

est un élément de l'alternative pour le problème de détection d'une rupture (non transitoire) dans la loi du processus de Poisson. Dans le cas où λ_0 est un paramètre connu, on a $\mathcal{S}_0 = \{\lambda_0\}$ et donc

$$d_2(\lambda, \{\lambda_0\}) = |\delta|\sqrt{\ell} \ .$$

Dans le cas où l'intensité de référence λ_0 est inconnue, $\mathcal{S}_0$ désigne l'ensemble des intensités constantes sur $[0,1]$ et la distance de λ à l'ensemble $\mathcal{S}_0$ est définie de manière usuelle par $d_2(\lambda, \mathcal{S}_0) = \inf_{\lambda_0 \in \mathcal{S}_0} d_2(\lambda, \lambda_0)$. On a alors

$$d_2(\lambda, \mathcal{S}_0) = |\delta|\sqrt{\ell(1-\ell)} \ .$$

On peut remarquer que si un test uniformément plus puissant existe pour un problème de test, alors il atteint la vitesse de séparation minimax sur la classe d'alternatives considérée. On dispose en effet du résultat suivant :

Lemme 1.9.
Soient $\mathcal{S}_0$ et $\mathcal{S}_1$ deux sous-espaces de $\mathbb{L}_2([0,1])$ et considérons le problème de test de (H_0)"$\lambda \in \mathcal{S}_0$" contre (H_1)"$\lambda \in \mathcal{S}_1$". Soit α dans $(0,1)$ et supposons qu'il existe un test ϕ_α uniformément plus puissant pour ce problème de test. Alors, pour la distance d_2 et pour tout β dans $(0,1)$, $\mathrm{SR}_\beta(\phi_\alpha, \mathcal{S}_1)$ est une borne inférieure pour $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$.

Démonstration.
La preuve de ce lemme est immédiate : comme ϕ_α est un test uniformément plus puissant, on a pour tout $r > 0$ et pour tout test ϕ'_α de niveau α,

$$\sup_{\lambda \in \mathcal{S}_1,\ d_2(\lambda, \mathcal{S}_0) \geq r} P_\lambda(\phi_\alpha = 0) \leq \sup_{\lambda \in \mathcal{S}_1,\ d_2(\lambda, \mathcal{S}_0) \geq r} P_\lambda(\phi'_\alpha = 0) \ .$$

On a donc $\mathrm{SR}_\beta(\phi_\alpha, \mathcal{S}_1) \leq \mathrm{SR}(\phi'_\alpha, \mathcal{S}_1)$ pour tout test ϕ'_α de niveau α. Alors, $\mathrm{SR}_\beta(\phi_\alpha, \mathcal{S}_1) \leq \mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_1)$ ce qui achève la preuve. □

Enfin, pour les différents choix de $\mathcal{S}_0$ (selon que l'intensité de référence du processus est connue ou pas), nous nous intéressons à plusieurs classes d'alternatives $\mathcal{S}_1$ dont les éléments sont des intensités λ avec un saut ou segment de position et de taille connues ou inconnues. Suivant la terminologie introduite dans les travaux de Spokoiny [Spo96], nous menons dans les chapitres 2 et 3 une étude minimax adaptative complète de ce problème de détection de ruptures : quand au moins un des paramètres de l'alternative est inconnu, les procédures de tests minimax correspondantes sont dites *minimax adaptatives* par rapport à ces paramètres inconnus.

II.3 Tests multiples pour la détection et la localisation d'une rupture et vitesse de séparation minimax par famille

En plus de sa détection, on peut aussi se poser la question de la localisation d'une rupture dans la loi du processus de Poisson sur l'intervalle $[0,1]$. Chercher simultanément à détecter un saut dans l'intensité et estimer l'instant de saut peut être vu comme une réponse complète à un problème de détection de rupture formulé dans un cadre hors-ligne. Comme l'expliquent Niu et al. dans [NHZ16], les problèmes d'estimation de l'instant de rupture peuvent être formulés comme des problèmes de tests multiples.

Commençons par une heuristique illustrant les principales difficultés de la théorie des tests multiples. Lorsque l'on teste plusieurs hypothèses nulles simultanément, on réalise un test multiple. Une procédure de test multiple s'interprète alors comme un procédé permettant de prendre une décision pour chaque hypothèse nulle : on décide de la rejeter ou de ne pas la rejeter. Par rapport aux tests simples, si la démarche statistique est similaire pour chaque test individuellement, la multiplicité des tests entraine des spécificités. Par exemple, le principe de dissymétrie d'un test d'hypothèses simples n'est pas toujours vérifié dans un problème de tests multiples. C'est notamment le cas lorsque les hypothèses nulles forment une partition de l'espace d'état du paramètre d'intérêt : ne pouvant pas être toutes vraies simultanément, elles ne sont plus "privilégiées" contrairement aux hypothèses nulles des tests simples. De plus, contrôler individuellement le niveau de chaque test par α peut ne plus suffire à garantir un contrôle de l'erreur globale de la procédure de test multiple par α. Pour s'en convaincre, supposons que l'on teste indépendamment n vraies hypothèses nulles simultanément au niveau α dans $(0,1)$. La probabilité d'obtenir au moins un faux positif, c'est-à-dire de rejeter (à tort) une vraie hypothèse nulle, est donc égale à

$$\begin{aligned}\mathbb{P}(\text{"obtenir au moins un faux positif"}) &= 1 - \mathbb{P}(\text{"obtenir aucun faux positif"}) \\ &= 1 - (1-\alpha)^n \text{ par indépendance.}\end{aligned}$$

Ainsi, même si toutes les hypothèses nulles testées sont vraies, plus leur nombre augmente et plus on a de chances de réaliser au moins un faux positif. Le problème de la multiplicité est donc un enjeu majeur dans un problème de test multiple.

On suppose que l'intensité λ appartient à l'ensemble $\overline{\mathcal{S}}$ des fonctions de la forme $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$ avec $\lambda_0 > 0$, $\delta \in (-\lambda_0, +\infty)\backslash\{0\}$ et $\tau \in (0,1]$. Remarquons que $\overline{\mathcal{S}}$ contient en particulier les intensités constantes : lorsque $\tau = 1$, on a par convention $\lambda = \lambda_0$. Selon que la taille du saut δ est connue ou non, les problèmes de détection et de localisation de l'instant de rupture τ sont formulés ici comme des problèmes de tests multiples.

Suivant la terminologie de Goeman et Solari [GS10], on considère pour un entier non nul M une *collection d'hypothèses* $\mathcal{H}$ définie par $\mathcal{H} = \{H_k,\ k \in \{1,\ldots,M\}\}$, où l'hypothèse H_k est un sous-ensemble de $\overline{\mathcal{S}}$. On appelle *ensemble des vraies hypothèses nulles* (ou plus simplement *ensemble des hypothèses vraies*) l'ensemble $\mathcal{T}(\lambda) = \{H_k \in \mathcal{H} :\ \lambda \in H_k\}$ et *ensemble des fausses hypothèses nulles* (ou *ensemble des hypothèses fausses*), $\mathcal{F}(\lambda) = \mathcal{H} \setminus \mathcal{T}(\lambda)$. Le but est de tester simultanément pour toute hypothèse H_k de $\mathcal{H}$, H_k contre $\overline{\mathcal{S}} \setminus H_k$, c'est-à-dire "$H_k$ est vraie sous P_λ" contre "H_k est fausse sous P_λ", ou encore "$H_k \in \mathcal{T}(\lambda)$" contre "$H_k \in \mathcal{F}(\lambda)$". Une *procédure de test multiple* $\mathcal{R}$ associée à la collection d'hypothèses $\mathcal{H}$ est une statistique, dépendant de l'observation N, définie comme un ensemble d'hypothèses simples rejetées $\mathcal{R} \subset \mathcal{H}$, dont l'objectif est d'inférer $\mathcal{F}(\lambda)$. Autrement dit, une procédure de test multiple est un ensemble d'hypothèses nulles rejetées, c'est-à-dire un ensemble d'hypothèses nulles qui ont été jugées fausses par la procédure. Dans le Chapitre 4, on utilisera un abus de notation en considérant $\mathcal{T}(\lambda)$, $\mathcal{F}(\lambda)$ et $\mathcal{R}$ comme des sous-ensembles de $\{1,\ldots,M\}$ au lieu de $\mathcal{H}$ afin de simplifier certaines expressions. Ainsi, $\mathcal{T}(\lambda)$, $\mathcal{F}(\lambda)$ et $\mathcal{R}$ peuvent respectivement être définis comme l'ensemble des indices des hypothèses nulles vraies, fausses et rejetées.

Toujours selon l'approche de Neyman-Pearson, on s'intéresse d'abord à l'erreur de première espèce, liée aux vraies hypothèses nulles rejetées et donc aux éléments de l'ensemble $\mathcal{R} \cap \mathcal{T}(\lambda)$. Il existe plusieurs critères pour quantifier l'erreur de première espèce d'une procédure de test multiple. Les deux principaux critères, que l'on rencontre majoritairement dans la littérature, sont le *Family-Wise Error Rate* et le *False Discovery Rate*.
Le critère du *Family-Wise Error Rate* (FWER) est défini sur la base de la probabilité de rejeter au moins une hypothèse nulle à tort (c'est-à-dire d'obtenir au moins un faux positif) :

$$\mathrm{FWER}(\mathcal{R}) = \sup_{\lambda \in \overline{\mathcal{S}}} P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing)\ .$$

Le critère du *False Discovery Rate* (FDR) est plus récent et introduit par Benjamini et Hochberg [BH95]. Le FDR se définit sur la base de l'espérance de la proportion d'erreurs commises dans l'ensemble des hypothèses rejetées (c'est-à-dire la moyenne du taux moyen de faux positifs) :

$$\mathrm{FDR}(\mathcal{R}) = \sup_{\lambda \in \overline{\mathcal{S}}} E_\lambda \left[\frac{|\mathcal{R} \cap \mathcal{T}(\lambda)|}{|\mathcal{R}| \vee 1} \right] .$$

L'étude du FDR présente une difficulté majeure due au caractère aléatoire du dénominateur à l'intérieur de l'espérance. La procédure de Benjamini-Hochberg (définie dans [BH95]) permet d'obtenir un contrôle pour le FDR sous des conditions de dépendances particulières entre les tests (voir par exemple les articles de Benjamini et Yekutieli [BY01], de Guo et Rao [GR08] ou le compte rendu de lecture de Roquain [Roq11]). Notons que le contrôle du FDR est un contrôle plus faible que celui du FWER.

Dans toute la suite de ce manuscrit, nous considérons le critère du FWER pour le contrôle de l'erreur de première espèce. Comme nous l'avons vu, contrôler chaque test individuellement au niveau α n'est pas suffisant a priori pour assurer le contrôle du FWER au même niveau α. Il est alors nécessaire de modifier le niveau de chaque test, on parle alors de niveaux individuels ajustés ou corrigés. Historiquement, la correction de Bonferroni (voir [Bon35]) est la première méthode permettant de corriger le niveau de chaque test afin de contrôler le FWER. L'idée de cette correction est intuitive, il s'agit de diviser le niveau individuel de chaque test par le nombre total d'hypothèses nulles testées. Cependant, lorsqu'il y a une forte dépendance entre les tests ou que le nombre de vraies hypothèses nulles est grand devant le nombre total d'hypothèses testées, le FWER obtenu après une correction de Bonferroni peut être très inférieur à α. Cela signifie donc que peu d'hypothèses nulles sont rejetées par la procédure de Bonferroni, on dit que la procédure est conservative. Depuis, de nombreuses procédures moins conservatives ont été développées, comme celle de Holm [Hol79] ou les procédures de type min$-p$ décrites dans l'ouvrage de Dudoit et van der Laan [DvdL08], afin de contrôler le FWER (et donc le FDR). On pourra trouver un panorama détaillé de ces méthodes dans le compte rendu de lecture de Roquain [Roq11].

Pour définir un critère pour l'erreur de seconde espèce, une première approche consiste à considérer des quantités analogues à celles utilisées pour les critères de l'erreur de première espèce. En effet, on s'intéresse aux fausses hypothèses nulles qui ne sont pas rejetées et donc aux éléments de $(\mathcal{H} \setminus \mathcal{R}) \cap \mathcal{F}(\lambda)$. Comme pour les tests simples et afin de définir une notion de puissance pour une procédure de test multiple, on peut aussi s'intéresser aux fausses hypothèses nulles rejetées par la procédure, c'est-à-dire aux éléments de $\mathcal{R} \cap \mathcal{F}(\lambda)$. La diversité des critères pour l'erreur de première espèce conduit alors à plusieurs notions de puissance comme par exemple la puissance complète (probabilité de rejeter que des fausses hypothèses nulles), la puissance minimale (probabilité de rejeter au moins une fausse hypothèse nulle) ou la puissance moyenne (moyenne de la proportion de fausses hypothèses nulles rejetées dans l'ensemble des fausses hypothèses nulles). On trouvera une étude de ces notions de puissance pour les tests multiples dans [BH95], [CLLM11] et [WTW11]. Si les performances des procédures de tests multiples en terme d'erreur de deuxième espèce sont souvent étudiées en pratique, très peu de critères de comparaison ou d'optimalité liés à l'erreur de deuxième espèce ont été définis et analysés en théorie. À notre connaissance, seuls les articles de Lehmann, Romano et Shaffer [LRS05] et Romano, Shaikh et Wolf [RSW11] fournissent des résultats d'optimalité de type *maximin*. Par analogie avec la notion de risque pour un test simple, qui est défini comme la somme de l'erreur de première espèce et de l'erreur de deuxième espèce, d'autres auteurs définissent une notion de risque pour une procédure de test multiple et établissent des inégalités oracles (on pourra consulter par exemple [GW02], [SC07] ou [ACC17]). Nous allons maintenant définir un autre critère lié à l'erreur de deuxième espèce conduisant à un critère d'optimalité au sens du minimax : la *vitesse de séparation minimax par famille*.

La *vitesse de séparation minimax par famille* ou *minimax Family-Wise Separation Rate* (mFWSR), introduite par Fromont, Lerasle et Reynaud-Bouret [FLRB16], est un critère lié à l'erreur de seconde espèce à la base d'une théorie minimax pour les tests multiples dont le FWER est contrôlé par un niveau α dans $(0,1)$. Pour définir ce critère, qui vient généraliser la notion de vitesse de séparation uniforme (voir Chapitre 1

Section II.2), les auteurs sont partis de l'observation que la plupart des tests (simples) adaptatifs au sens du minimax sont construits sur un principe général d'agrégation, pouvant être relié aux procédures de Bonferroni, de Holm, ou de type min$-p$ (on renvoie à [FLRB16] pour plus de détails).
Pour la distance d_2 associée à la norme usuelle $\|\cdot\|_2$ de $\mathbb{L}_2([0,1])$ et $r>0$, on définit pour tout λ de $\overline{\mathcal{S}}$,

$$\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H} : \ d_2(\lambda, H_k) \geq r\} \ .$$

La définition de $\mathcal{F}_r(\lambda)$ fait intervenir la distance $d_2(\lambda, H_k)$ entre λ et l'ensemble H_k égale à $d_2(\lambda, H_k) = \inf_{\lambda' \in H_k} d_2(\lambda, \lambda')$. De manière heuristique, $\mathcal{F}_r(\lambda)$ désigne l'ensemble des hypothèses H_k "éloignées" de λ qui devraient donc être rejetées par la procédure.

Pour α et β dans $(0,1)$, une classe d'intensités $\mathcal{S} \subset \overline{\mathcal{S}}$ et une procédure de tests multiples $\mathcal{R}$ dont le FWER sur $\overline{\mathcal{S}}$ est contrôlé par α, la *vitesse de séparation par famille de niveau* β de $\mathcal{R}$ sur $\mathcal{S}$ est définie par

$$\begin{aligned}\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}) &= \inf\{r > 0 : \ \inf_{\lambda \in \mathcal{S}} P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) \geq 1 - \beta\} \\ &= \inf\{r > 0 : \ \sup_{\lambda \in \mathcal{S}} P_\lambda(\mathcal{F}_r(\lambda) \cap (\mathcal{H} \setminus \mathcal{R})) \leq \beta\} \ .\end{aligned}$$

Intuitivement, le FWSR correspond à la plus petite distance qui permet de rejeter les hypothèses "éloignées" de λ avec grande probabilité. La *vitesse de séparation par famille minimax de niveaux* α *et* β *sur* $\mathcal{S}$ correspondante est alors définie par

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}) = \inf_{\mathcal{R}:\ \mathrm{FWER}(\mathcal{R}) \leq \alpha} \mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}) \ ,$$

où l'infimum est pris sur toutes les procédures de tests multiples dont le FWER est contrôlé par α sur $\overline{\mathcal{S}}$. On dit qu'une procédure de test multiple $\mathcal{R}$ dont le FWER est contrôlé par α sur $\overline{\mathcal{S}}$ est *minimax de niveau* β sur $\mathcal{S}$ si $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S})$ est du même ordre que $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S})$.
Remarquons que si $\mathcal{H}$ est réduite à une seule hypothèse nulle H_0, $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}) = \mathrm{mSR}_{\alpha,\beta}(\mathcal{S})$. L'approche minimax définie ici pour les tests multiples peut donc être vue comme une généralisation de l'approche minimax classique pour les tests d'une hypothèse nulle. Le Lemme 1.10 montre d'autres liens entre ces deux approches.

Le calcul de la vitesse de séparation minimax par famille pour un problème de test multiple s'obtient, comme pour le calcul de la vitesse de séparation minimax pour un problème de test simple, en deux étapes : le calcul d'une borne inférieure puis d'une borne supérieure pour $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S})$. Afin d'obtenir une borne inférieure pour la vitesse de séparation minimax par famille, on peut utiliser un résultat prouvé dans l'article de Fromont, Lerasle et Reynaud-Bouret [FLRB16] qui établit un lien entre le critère minimax pour les procédures de tests multiples et le critère minimax pour les procédures de test simple.

Lemme 1.10.
Si la collection d'hypothèses $\mathcal{H}$ est fermée (c'est-à-dire que pour tout $H_k, H_{k'} \in \mathcal{H}$, on a $H_k \cap H_{k'} \in \mathcal{H}$), on obtient en notant $\cap\mathcal{H} = \cap_{k=1}^M H_k$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}) \geq \mathrm{mSR}_{\alpha,\beta}^{\cap\mathcal{H}}(\mathcal{S}) \ .$$

On rappelle (voir (1.3)) que

$$\mathrm{mSR}_{\alpha,\beta}^{\cap\mathcal{H}}(\mathcal{S}) = \inf_{\phi_\alpha} \inf \left\{ r > 0 : \ \sup_{\lambda \in \mathcal{S},\ d(\lambda, \cap\mathcal{H}) \geq r} P_\lambda(\phi_\alpha = 0) \leq \beta \right\} \ ,$$

où le premier infimum est pris sur tous les tests simples ϕ_α de niveau α de l'hypothèse nulle (H_0) "$\lambda \in \cap\mathcal{H}$", c'est-à-dire tels que $\sup_{\lambda \in \cap\mathcal{H}} P_\lambda(\phi_\alpha = 1) \leq \alpha$.

De façon heuristique, le Lemme 1.10 traduit l'idée naturelle selon laquelle tester plusieurs hypothèses nulles simultanément est a priori "plus difficile" que de tester une seule hypothèse nulle. Néanmoins, ce résultat n'est vrai que sous des conditions particulières (sur la distance considérée ou sur les hypothèses), on trouvera un contre-exemple pour lequel le mFWSR est plus petit que le mSR dans [FLRB16]. On dispose également d'un résultat de monotonie pour le FWSR :

Lemme 1.11.
Pour tous sous-ensembles $\mathcal{S}'$ *et* $\mathcal{S}$ *de* $\mathbb{L}_2([0,1])$ *vérifiant* $\mathcal{S}' \subset \mathcal{S}$, *on a* $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}') \leq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S})$.

Une fois qu'une borne inférieure est calculée pour $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}')$, nous construisons une procédure de test multiple minimax, c'est-à-dire une procédure dont la vitesse de séparation par famille est du même ordre que la borne inférieure préalablement déterminée, offrant ainsi une borne supérieure pour la vitesse de séparation minimax par famille.

Enfin, toujours suivant la terminologie de Fromont, Lerasle et Reynaud-Bouret [FLRB16], nous menons dans le Chapitre 4 une étude minimax adaptative de ce problème de localisation d'une rupture : quand au moins un des paramètres de la rupture est inconnu, les procédures de tests multiples correspondantes sont dites *minimax adaptatives* par rapport à ces paramètres inconnus.

III Un état de l'art...

III.1 ...pour le problème de la détection d'une rupture

Cette sous-section, présentant un état de l'art pour le problème de la détection d'une rupture, est très largement inspirée de l'introduction de l'article de Fromont, Grela et Le Guével [FGLG21].

Tests d'hypothèses pour des processus de Poisson et la détection de ruptures

Les articles qui traitent de la détection d'une rupture dans la loi d'un processus de Poisson sont majoritairement formulés dans un cadre en-ligne et dédiés à la construction de règles d'arrêt optimales. On pourra par exemple citer [PS02], [HJ04], [BZ06], et [BDK06] pour des modèles bayésiens et [DKW53a], [DKW53b], [MHT11] ou [EKLS15, EKLS+17] pour des modèles non bayésiens.

Dans le cadre hors-ligne, une part considérable des travaux de recherche concerne la construction de procédures de tests asymptotiques. Tout d'abord, il est à noter qu'une partie des procédures hors-ligne sont établies via une approche bayésienne comme celles décrites dans [AR86b], [RA86] et [Raf94] pour la détection d'une seule rupture, et celles de [Gre95], [YYK01] ou [SZ12] pour la détection de plusieurs ruptures. Les procédures de test construites pour une description bayésienne des ruptures sont très sensibles aux lois de probabilité choisies a priori sur les instants de saut et/ou sur les valeurs de l'intensité après les sauts. D'autre part, il existe de nombreuses procédures hors-ligne non bayésiennes basées sur les propriétés des processus de Poisson. Depuis les premières procédures définies par Neyman et Pearson [NP28], Sukhatme [Suk36], Maguire, Pearson et Wynn [MPW52], de nombreux auteurs ont construit des tests de détection basés sur la distribution exponentielle des inter-arrivées du processus de Poisson comme [MFP85], [Wor86], [Sie88], ou plus récemment [AJ07]. Comme nous l'avons rappelé (voir Proposition 1.2), pour tout entier n et sachant que $N_1 = n$, les points d'un processus de Poisson homogène sont indépendants et uniformément distribués sur $[0,1]$. Ainsi, tout test d'uniformité sur $[0,1]$ dans un modèle à densité peut être directement appliqué conditionnellement à N_1, ou être adapté pour tester l'homogénéité d'un processus de Poisson. Plus proche des tests que nous proposons dans ce travail, plusieurs autres procédures pour la détection d'une rupture sont basées ou inspirées des statistiques plus classiques du rapport de vraisemblance et de Cramér von-Mises ou Kolmogorov-Smirnov, avec différentes stratégies de pénalisation ou de pondération comme celles proposées par Rubin [Rub61], Lewis [Lew65], ou Kendall et Kendall [KS80].

Les premières études d'optimalité pour le problème de détection d'une rupture ont été proposées par Deshayes et Picard [Des84, DP85] pour des tests de Kolmogorov-Smirnov et du rapport de vraisemblances pondérés. Dans ces travaux, la notion d'optimalité repose sur le critère asymptotique non local de Bahadur [Bah67] et Brown [Bro71], et sur le critère asymptotique local introduit par Le Cam [LC70].

Plus récemment, certains auteurs se sont concentrés sur le problème plus général de la détection d'une rupture dans la loi d'un processus de Poisson inhomogène. On pourra citer notamment Dachian, Kutoyants et Yang [DKY16b], Farinetto [Far17] pour une étude non bayésienne ainsi que [DY15] et [Yan20] pour une approche bayésienne de ce problème.

Il convient maintenant de présenter certains travaux dont les résultats sont indirectement reliés à notre problème de détection de ruptures. Parmi ces travaux, il est important de mentionner le papier fondateur de Davies [Dav77] sur les tests d'ajustement pour des processus de Poisson. Les tests d'ajustement permettent de tester si les données ponctuelles observées sont bien une réalisation du processus considéré. Plusieurs tests d'ajustement ou d'homogénéité pour des processus de Poisson contre une alternative où l'intensité du processus est croissante ont été comparés dans des études expérimentales, on citera notamment les papiers de Bain et al. [BEW85], Engelhardt et al. [EGW90], de Cohen et Sackrowitz [CS93] et de Ho [Ho93], [Ho95]. Les résultats expérimentaux ont montré que le test de Laplace et le Z test, initialement introduits par Cox [Cox55] et Crown [Cro74], et dont des extensions ont été proposées dans [Peñ98], [AP99], et [BDNN04], sont les plus performants pour la détection d'un saut positif. Fazli et Kutoyants [FK05], Fazli [Faz07], et plus récemment Dachian, Kutoyants et Yang [DKY16a], se sont intéressés au problème du test d'ajustement où l'hypothèse nulle correspond à un processus de Poisson inhomogène donné, et où les alternatives correspondent à une famille paramétrique de processus de Poisson inhomogènes. Enfin, le problème de test qu'un processus ponctuel soit un processus de Poisson homogène d'intensité connue contre des alternatives où le processus appartient à une famille de processus ponctuels auto-excités (processus de Hawkes) ou de processus ponctuels auto-corrigés (dit également processus de relâchement de stress) à été traité dans [DK06] et [DK09].

Notons que les procédures de tests mentionnées précédemment n'ont pas été étudiées sous l'angle des vitesses de séparation minimax que nous considérons dans nos travaux.

Quelques études minimax reliées au problème de la détection de ruptures dans la loi d'un processus de Poisson

À présent, nous allons présenter certains travaux qui traitent de l'optimalité, au sens du minimax, des problèmes de tests sur des processus de Poisson ou des problèmes de détection de ruptures dans d'autres modèles statistiques.

Concernant les tests d'ajustement ou d'homogénéité minimax pour des processus de Poisson, on peut notamment citer les travaux de Ingster et Kutoyants [IK07] et Fromont, Laurent et Reynaud-Bouret [FLRB11], où les hypothèses alternatives correspondent à des processus de Poisson dont les intensités non paramétriques appartiennent à des espaces de Sobolev ou de Besov. Cependant, les procédures de tests définies dans ces travaux ne sont pas dédiées à la détection de ruptures dans l'intensité d'un processus de Poisson. De nombreuses études minimax ont été menées pour le problème spécifique de la détection de ruptures mais dans des modèles statistiques différents.

En utilisant l'astuce de conditionnement par le nombre de points total observés sur un intervalle de temps, on peut traiter le problème de détection de ruptures dans la loi d'un processus de Poisson comme un cas particulier d'un problème de détection de ruptures dans un modèle de densité. De cette manière, Rivera et Walther [RW13] proposent deux tests permettant de détecter un saut transitoire positif basés sur des statistiques de scan et d'agrégation pour le rapport de vraisemblance. Cependant, des difficultés de déconditionnement font que les résultats d'optimalité présentés dans leur article ne sont pas directement transposables à notre notion d'optimalité au sens du minimax. Dans le modèle de densité, Dümbgen et Walther [DW08] ont également traité le problème de la détection d'une augmentation ou d'une diminution locale de la densité ou du taux de défaillance (pour des études de fiabilité). Les auteurs prouvent que leurs procédures de tests, basées sur des approches d'agrégation de statistiques d'ordre ou d'accroissements locaux, satisfont des propriétés minimax asymptotiques adaptatives.

La bibliographie la plus complète sur les tests minimax pour la détection d'une rupture, transitoire ou non, concerne le modèle gaussien classique. Dans ce modèle, on observe un vecteur gaussien $(Y_1, \dots, Y_n)$ dont la matrice de variance-covariance est égale à $\sigma^2 I_n$ où I_n désigne la matrice identité et σ^2 un réel positif connu. Dans les résultats que nous énoncerons dans la suite, on supposera que $\sigma^2 = 1$. La détection d'une rupture (éventuellement transitoire) est ici caractérisée par la détection d'un saut ou d'un segment dans le vecteur moyenne du vecteur gaussien. Comme dans notre modèle de processus de Poisson, plusieurs hypothèses peuvent être faites sur la connaissance ou non de la valeur de référence du vecteur moyenne et/ou de la taille du saut. De manière générale, dans un problème de détection de rupture où l'hypothèse nulle correspond à l'absence de comportement anormal, la méconnaissance de la loi sous l'hypothèse nulle engendre des difficultés dans la calibration des tests de détection (voir par exemple [ACCTW18]). Détaillons les principaux résultats obtenus à ce jour et à notre connaissance.

La première étude minimax de ce problème de détection de ruptures a été menée par Arias-Castro et al. en 2005 [ACDH05] dans un cadre asymptotique. Les auteurs établissent les vitesses de séparation minimax asymptotiques pour la détection d'un segment (avec un saut positif) à partir d'une moyenne connue égale à zéro et en considérant la métrique ℓ_2 sur le vecteur moyenne. Cette métrique, associée à la distance $\mathbb{L}_2$ entre les deux lois de probabilités gaussiennes correspondantes, fait aussi référence au rapport signal sur bruit ou à l'énergie du signal qui sont souvent mentionnés dans l'analyse de modèles de régression. Sous l'alternative, les coordonnées du vecteur Gaussien $(Y_1, \dots, Y_n)$ vérifient dans ce cas $\mathbb{E}[\,Y_i\,] = \delta \mathbb{1}_{\{i \in [\tau, \tau+\ell)\}}$ avec $\tau \in \{1, \dots, n\}$, $\ell \in \{1, \dots, n+1-\tau\}$ et $\delta > 0$: l'instant de rupture τ, la longueur du segment ℓ et la taille du saut δ étant des paramètres inconnus. Les auteurs établissent une vitesse de séparation asymptotique de l'ordre de $\sqrt{\log n}$ avec une constante exacte égale à $\sqrt{2}$. Plus précisément, ils prouvent qu'aucun test ne peut convenablement détecter la rupture si la condition sur la distance $\delta\sqrt{\ell} \geq \sqrt{2(1+\eta)\log n}$ avec $\eta > 0$ n'est pas satisfaite et construisent des tests minimax adaptatifs asymptotiques basés sur l'agrégation des statistiques de Neyman-Pearson $\ell^{-1/2}\sum_{i=\tau}^{\tau+\ell-1} Y_i$ sur tous les intervalles possibles $[\tau, \tau+\ell)$ ou tous les intervalles dyadiques $[k2^j+1, (k+1)2^j)$. Ce test agrégé va donc rejeter l'hypothèse nulle (c'est-à-dire l'hypothèse selon laquelle le vecteur moyenne du vecteur gaussien observé est nul), s'il existe au moins un couple (τ, ℓ) ou (k, j) pour lequel la statistique de Neyman-Pearson correspondante est "trop grande", relativement à la valeur critique du test. Cela justifie la terminologie de statistiques de scan (notamment utilisée dans la littérature concernant le traitement du signal ou des images) ou de statistiques agrégées (notamment employée dans la littérature sur les problèmes de tests minimax).

Quelques années plus tard, Chan et Walther [CW13] ont construit trois autres tests, basés sur la même statistique de Neyman-Pearson, mais combinés à des schémas d'agrégation différents (avec l'ajout d'une pénalisation par exemple). Ils démontrent que leurs tests sont consistants dès que la condition $|\delta|\sqrt{\ell} \geq \sqrt{2\log(n/\ell)} + b_n$ avec $b_n \to +\infty$ est satisfaite, ce qui améliore légèrement la borne inférieure démontrée par Arias-Castro et al. ; en effet, les auteurs autorisent ici $\ell/n := \ell_n/n$ à tendre vers 0 lorsque n tend vers $+\infty$. À noter qu'une borne inférieure similaire, mais énoncée dans un cadre non asymptotique, a récemment été prouvée par Verzelen et al. [VFLRB21]. Cependant, les conditions énoncées par Chan et Walther [CW13] et Verzelen et al. [VFLRB21] pour la détection d'un segment dans la moyenne ne sont pas directement transposables au cadre minimax considéré, la longueur inconnue ℓ du segment apparaissant dans le membre de droite de l'inégalité sur la distance $|\delta|\sqrt{\ell}$.

Dans le cas particulier où la taille du saut est connue et égale à 1, Brunel [Bru14] a également établi une vitesse de séparation (dans un cadre asymptotique) de l'ordre de $\sqrt{\log n/n}$ en construisant un test basé sur une statistique de scan pénalisée $\sum_{i=\tau}^{\tau+\ell-1} Y_i - \ell/2$ qui est consistant dès que la condition $\ell/\log n \to +\infty$ est satisfaite.

Toujours en considérant la métrique ℓ_2 sur le vecteur moyenne, Gao et al. [GHZ20] se sont intéressés à la détection d'un saut (c'est-à-dire une rupture non transitoire) dans la moyenne d'un vecteur gaussien. Dans cet article, la valeur initiale (ou valeur de référence) de la moyenne n'est plus supposée connue et les vitesses de séparation sont établies. Sous l'alternative, les coordonnées du vecteur gaussien $(Y_1,\ldots,Y_n)$ vérifient dans ce cas $\mathbb{E}[\,Y_i\,] = \mu_0 + \delta\mathbb{1}_{\{i\in[\tau,n)\}}$ avec $(\mu_0,\delta)\in\mathbb{R}^2$ et $\tau\in\{2,\ldots,n\}$: l'intensité de référence μ_0, l'instant de rupture τ et la taille du saut δ étant des paramètres inconnus. Ils ont obtenu une borne inférieure pour la vitesse de séparation minimax de l'ordre de $\sqrt{\log\log n}$. Plus précisément, ils ont démontré qu'aucun test ne peut convenablement détecter un saut dans la moyenne si la condition $|\delta|\sqrt{(\tau-1)(n+1-\tau)/n} \geq c\sqrt{\log\log n}$ n'est pas satisfaite, où c est une constante positive vérifiant $c<\sqrt{2}$. Verzelen et al. [VFLRB21] ont établi une borne inférieure non asymptotique égale à $\sqrt{2(1-\varepsilon)(1-n^{-1/2})\log\log n}$ pour une constante positive ε et pour n suffisamment grand. Cette borne inférieure (non asymptotique) améliore le résultat de Gao et al. car elle fait apparaître la constante optimale pour la vitesse de séparation minimax. Pour le calcul d'une borne supérieure, Gao et al. [GHZ20] se sont inspirés du test asymptotique de Csörgö et Horváth [CH97], basé sur la statistique de scan $\max_{1\leq\tau\leq n}\sqrt{n/\tau(n-\tau)}|\sum_{i=1}^{\tau} Y_i - (k/n)\sum_{i=1}^{n} Y_i|$, dont ils démontrent l'optimalité à partir d'inégalités asymptotiques du type Loi du Logarithme Itérée. Notons que cette statistique de scan est très proche des statistiques CUSUM qui sont très largement utilisées dans la littérature traitant des problèmes de détection d'une rupture depuis les travaux Hinkley's [HH70], mais aussi dans les problèmes plus généraux de détection de plusieurs ruptures où elles sont au cœur des approches de segmentation binaire. Verzelen et al. [VFLRB21] ont proposé quant à eux un test basé sur le maximum d'une statistique CUSUM pénalisée, avec une pénalisation qui dépend de la localisation de la rupture, et dont la vitesse de séparation est d'ordre optimal $\sqrt{2\log\log n}$. Ce résultat, associé à celui sur la borne inférieure, prouve donc que la constante exacte pour la vitesse de séparation minimax du problème de détection d'une rupture dans la moyenne d'un vecteur gaussien est $\sqrt{2}$, comme pour la détection d'un segment.

Dans des modèles gaussiens plus complexes, la question de la détection d'un segment dans la moyenne a notamment été étudiée d'un point de vue minimax et asymptotique par Enikeeva et Harchaoui [EH19], Enikeeva et al. [EMW18], Liu et al. [LGS21] et Enikeeva et al. [EMPW20]. Ces modèles intègrent notamment des propriétés d'hétéroscédasticité, de dépendance ou de parcimonie dans les alternatives. Évoquons également des problèmes de détection de rupture dans des modèles plus généraux de séries temporelles ou de champs aléatoires gaussiens : on pourra par exemple consulter [ACCD11, ACBLV18] et les références citées dans ces articles.

III.2 ...pour le problème de la localisation d'une rupture

Nous commençons cette section par présenter un panorama des principales méthodes pour la localisation de ruptures dans la loi d'un processus stochastique. La littérature étant particulièrement abondante pour les modèles gaussiens, nous illustrons ces méthodes pour un modèle gaussien classique, dans la continuité du modèle présenté dans le Chapitre 1 Section III.1. Dans un second temps, nous détaillerons les principaux résultats obtenus dans la littérature pour le problème particulier de la localisation de ruptures dans la loi d'un processus de Poisson.

Localisation de plusieurs instants de rupture dans un modèle gaussien

Dans ce paragraphe, considérons un modèle de ruptures multiples (Multiple Change-Point Model en anglais ou MCM) formulé dans un cadre gaussien classique. On observe un vecteur gaussien $(Y_1, \ldots, Y_n)$ tel que $Y_i = m_i + \varepsilon_i$ avec $\varepsilon_i \sim \mathcal{N}(0,1)$ et $m_1 = m_2 = \ldots = m_{\tau_1 - 1} \neq m_{\tau_1} = m_{\tau_1+1} = \ldots = m_{\tau_2-1} \neq m_{\tau_2} = \ldots = m_{\tau_J - 1} \neq m_{\tau_J} = \ldots = m_n$. Autrement dit, le vecteur moyenne $m = (m_1, \ldots, m_n)$ est constant par morceaux et les sauts interviennent aux temps $\tau = (\tau_1, \ldots, \tau_J)$. Dans ce modèle, m et τ sont des paramètres inconnus et le nombre de ruptures J est supposé petit devant n. L'objectif est de construire un estimateur $\hat{\tau}$ et des intervalles de confiance simultanés pour le vecteur τ. De manière intuitive, la localisation de ruptures n'est possible que si les ruptures peuvent être détectées. Comme nous l'avons vu dans la sous-section précédente III.1, la question de la détectabilité des ruptures ou de la région de détectabilité s'exprime souvent par une condition sur les distances entre les instants de rupture et les tailles des sauts. Par exemple, l'article de Jeng et al. [JCL10] présente une procédure de sélection basée sur le rapport de vraisemblance qui permet de détecter et localiser un nombre inconnu de segments dans la moyenne sous certaines conditions de détectabilité, généralisant les résultats de [ACDH05].

Comme l'expliquent Niu et al. [NHZ16], le problème de la localisation des ruptures peut être formulé comme un problème de test multiple : on teste pour chaque j de $\{2, \ldots, n\}$ si j est un instant de saut. Dans une formulation hors-ligne du problème, Hao, Niu et Zhang [HNZ13] ont généralisé le critère du FDR et ont testé des hypothèses définies par une fenêtre glissante : pour tout j, on teste donc l'absence ou la présence d'une rupture sur l'ensemble $\{j+1-h, \ldots, j+h\}$ où h est un entier fixé bien choisi. Li et al. [LMS⁺16] formulent aussi le problème de détection et localisation des ruptures comme un problème de test multiple et construisent une procédure qui assure le contrôle du FDR. En particulier, la méthode proposée par les auteurs permet de détecter un segment dans la moyenne lorsque la condition de détectabilité (asymptotique) de Chan et Walther [CW13] est vérifiée. Enfin, ils démontrent que les estimateurs issus de cette procédure sont minimax au sens du risque $\mathbb{L}_p$ $(1 \leq p < \infty)$.

Dans un compte rendu de lecture pour le problème de détection et de localisation de ruptures abordé via des techniques d'estimation, Yu [Yu20] dresse les principales étapes pour la localisation des ruptures dans un modèle gaussien et présente les principaux résultats obtenus. Le plus souvent, deux approches sont utilisées pour fournir des estimateurs pour les instants de rupture : celle basée sur la minimisation d'un critère des moindres carrés pénalisé (ou plus généralement d'un critère de coût pénalisé) et celle basée sur les statistiques de type CUSUM.

L'estimateur construit à partir de la minimisation d'un critère des moindres carrés est égal à l'estimateur du maximum de vraisemblance de τ si le nombre J de ruptures est connu. Quand le nombre de ruptures est inconnu et doit être estimé, on ajoute une pénalité à l'estimateur des moindres carrés comme le Critère d'Information Bayésien (voir [Yao88, YA89]), des pénalisations inspirées de la théorie de la sélection de modèle (voir [Leb05]), des pénalités de type Lasso comme le Fused-Lasso [TSR⁺05] ou des pénalités qui dépendent de J et des distances entre les ruptures consécutives (voir [ZS07, ZS12]). Couplé à un algorithme de programmation dynamique, historiquement proposé par Fisher [Fis58] et Bellman [Bel61], ces estimateurs basés sur un critère pénalisé permettent alors de retrouver les instants de rupture. Depuis, de nombreux algorithmes de programmation dynamique ont été développés pour la détection de ruptures, on pourra consulter par exemple [Rig10, KFE12].

Plutôt que résoudre le problème d'optimisation globale d'une fonction de coût pénalisée, on peut aussi appliquer de façon itérative une méthode de segmentation locale afin de détecter une rupture à chaque étape. Historiquement, l'utilisation de la statistique CUSUM a permis à Hinkley [Hin70] de détecter une seule rupture au sein d'un vecteur gaussien en maximisant la statistique sur tous les instants de rupture possibles. L'algorithme de Segmentation Binaire, introduit et étudié par Scott et Knott [SK74] et Vostrikova [Vos81] pour la détection de plusieurs instants de rupture, est une méthode qui fonctionne par dichotomie : on applique l'algorithme de Hinkley sur tout l'intervalle puis récursivement, sur chaque segment délimité par les instants de rupture estimés. L'algorithme s'arrête quand plus aucune rupture n'est détectée. Plusieurs extensions ont été proposées récemment, notamment celui de Wild Binary Segmentation [Fry14, Fry18, WYR20] et celui de Circular Binary Segmentation [OVLW04, VO07].

Une méthode hybride a été proposée par Frick, Munk et Sieling [FMS14] : l'algorithme SMUCE (Simultaneous Multiscale Change-Point Estimator) qui consiste à estimer les ruptures en résolvant un problème d'optimisation globale à partir d'une agrégation de statistiques locales. Une extension de cette méthode a été introduite dans [PSM17] pour un modèle gaussien hétéroscédastique.

Dans un second temps, les estimateurs construits par les méthodes précédentes peuvent être corrigés ou ajustés avec des méthodes de *post-processing* : les instants de rupture estimés et jugés aberrants sont supprimés [Fry18] ou raffinés par une maximisation locale d'une statistique CUSUM (voir par exemple la Section 3.2 de [Fry14]). On citera également l'article de Lin et al. [LSRT16] consacré essentiellement aux liens entre les erreurs d'estimation ℓ_2 et les vitesses de détection ainsi qu'à la procédure de Fused-Lasso pour laquelle une étape de post-processing est proposée.

Enfin, citons quelques autres méthodes populaires pour la localisation de ruptures dans la loi de processus stochastiques. L'algorithme de Screening and Ranking (SaRa), qui peut être vu comme une modification de l'algorithme de Segmentation Binaire, a été proposé par Niu et Zhang [NZ12] et on trouvera le contrôle du FDR via le SaRa et la construction d'intervalles de confiance pour les instants de rupture dans [HNZ13]. Enfin, Garreau et Arlot [GA+18] ont mené une étude minimax d'un algorithme basé sur des noyaux (Kernel Change-Point algorithm) et on pourra consulter l'article de Arlot, Celisse et Harchaoui [ACH19] pour une description de l'algorithme.

D'autres études plus récentes abordent le problème de la localisation de plusieurs ruptures sous l'angle de l'estimation minimax, on renvoie par exemple aux articles [WYR18, GHZ20, WYR20, VFLRB21] pour plus de détails.

Localisation d'une rupture dans un modèle gaussien

À présent, spécifions certains résultats propres à l'estimation d'une seule rupture dans un modèle gaussien classique. On observe un vecteur gaussien $(Y_1,\dots,Y_n)$ tel que $Y_i = m_i + \varepsilon_i$ avec $\varepsilon_i \sim \mathcal{N}(0,1)$ et $m_i = \mu_0 + \delta \mathbb{1}_{\{i\in[\tau,n)\}}$ avec $\mu_0 \in \mathbb{R}$, $\delta \in \mathbb{R}^*$ et $\tau \in \{2,\dots,n\}$. Dans ce modèle, l'intensité de référence μ_0, l'instant de rupture τ et la taille du saut δ sont des paramètres inconnus. L'objectif est de construire un estimateur $\hat{\tau}$ de τ tel que $|\hat{\tau}-\tau|$ soit "le plus petit possible". On notera m le vecteur moyenne $(m_1,\dots,m_n)$ et $\hat{m}$ un estimateur de m. Remarquons qu'à partir d'un estimateur de m, on peut déduire un estimateur de τ, δ et μ_0 et que réciproquement, tout estimateur $\hat{\tau}$ de τ permet de construire un estimateur de m par une stratégie de plug-in :

$$\widehat{m_i} = \mathbb{1}_{1\le i\le \hat{\tau}-1}\frac{1}{\hat{\tau}-1}\sum_{j=1}^{\hat{\tau}-1} Y_j + \mathbb{1}_{\hat{\tau}\le i\le n}\frac{1}{n-\hat{\tau}+1}\sum_{j=\hat{\tau}}^{n} Y_j \quad .$$

Ainsi, toute procédure d'estimation de la rupture peut être interprétée comme un estimateur de m ou de τ. L'estimateur des moindres carrés de m, qui est égal à l'estimateur du maximum de vraisemblance, est donné par :

$$\mathrm{argmin}_{m'} \sum_{i=1}^{n} (Y_i - m')^2.$$

La statistique CUSUM évaluée en τ' est égale à la différence pondérée des moyennes empiriques sur $[1,\tau')$ et $[\tau',n]$:

$$\sqrt{\frac{(\tau'-1)(n+1-\tau')}{n}}\left|\frac{\sum_{i=\tau'}^{n} Y_i}{n+1-\tau'} - \frac{\sum_{i=1}^{\tau'-1} Y_i}{\tau'-1}\right| \quad .$$

Un estimateur $\hat{\tau}$ est alors construit en maximisant la statistique CUSUM sur toutes les valeurs possibles pour τ'. Dans le prochain paragraphe, $\hat{\tau}$ désignera ainsi l'estimateur de τ construit à partir de la statistique CUSUM ou à partir de l'estimateur des moindres carrés.

Les premiers travaux qui traitent de ce problème d'estimation, menés dans [Hin70, HH70] dans un cadre asymptotique et pour une taille de saut δ connue, ont établi que $|\hat{\tau}-\tau| = \mathcal{O}_P(1)$. Plus tard, les articles de Dümbgen [Dum91] et Csörgö et Horváth [CH97] ont prouvé des résultats d'optimalité au sens du minimax dans un cadre asymptotique en utilisant la loi limite de $|\hat{\tau}-\tau|$ sous l'hypothèse $|\delta|\sqrt{1-\tau}/\sqrt{\log\log n} \to +\infty$ et pour des instants de rupture particuliers, proportionnels à n. Plus précisément, ils ont prouvé que $|\hat{\tau}-\tau| = \mathcal{O}_P(1/\delta^2)$, ce qui fournit une borne minimax d'après la Proposition 10 de [WS18] dès lors que $\delta^2 = \mathcal{O}(1)$. D'un point de vue non asymptotique, Gao et al. [GHZ20] ont étudié le risque minimax quadratique et ont obtenu un risque optimal (au sens du minimax) de l'ordre de $\log\log(n)$. En particulier, ils ont montré qu'avec grande probabilité, $|\hat{\tau}-\tau| \leq c\log\log(n)/\delta^2$ sous l'hypothèse $|\delta|\sqrt{1-\tau} \geq c'\sqrt{\log\log n}$, où c et c' sont des constantes positives. Récemment, Verzelen et al. [VFLRB21] ont démontré que $|\hat{\tau}-\tau| \leq c(1/\delta^2 + x)$ avec probabilité plus grande que $1-\exp(-c'x\delta^2)$ où c et c' sont des constantes positives, et que cette borne est optimale. L'estimateur de τ qui atteint cette borne est construit par la minimisation d'un critère des moindres carrés pénalisé, avec un nouveau choix de pénalisation. Enfin, Verzelen et al. [VFLRB21] construisent des intervalles de confiance non asymptotiques pour τ également basés sur une statistique des moindres carrés pénalisée et un estimateur pour la taille du saut δ.

Estimation des instants de rupture dans la loi d'un processus de Poisson

Formulée dans un cadre en-ligne, la question de la détection de ruptures repose essentiellement sur le choix d'une règle d'arrêt. La procédure calcule récursivement la statistique de détection pour chaque nouvelle observation, et lorsque la statistique dépasse un certain seuil, la procédure marque un temps d'arrêt pour signaler qu'une rupture s'est produite. La performance des temps d'arrêt ainsi construits est évaluée selon différents critères basés sur l'écart entre les vrais instants de rupture et les temps d'arrêt considérés. Le critère historique de Lorden [Lor71] est un critère minimax qui consiste à minimiser le pire retard à la détection pour une valeur donnée de la durée moyenne avant une fausse alarme. Depuis, de nombreux auteurs ont proposé des modifications du critère de Lorden : on citera par exemple le critère d'optimalité de Pollack et Siegmund [PS75], le critère de Lai [Lai98] ou celui de Moustakides [Mou08]. Concernant le problème particulier de la localisation de ruptures dans un processus de Poisson, on renvoie aux articles [PS02], [HJ04], [BZ06], et [BDK06] pour des modèles bayésiens et [DKW53a], [DKW53b], [MHT11] ou [EKLS15, EKLS+17] pour des modèles non bayésiens.

Pour la formulation hors-ligne, la localisation des instants de rupture dans la loi d'un processus de Poisson repose sur des techniques d'estimation usuelles, le plus souvent dans une approche bayésienne comme dans [RA86, Raf94, CGS92]. Lorsque le nombre de ruptures est inconnu, Shen et Zhang [SZ12] ont proposé une sélection de modèle basée sur la minimisation d'un critère d'information bayésien (BIC) modifié afin de construire des intervalles de confiance bayésiens, aussi appelés intervalles de crédibilité pour les instants de rupture. Récemment, Farinetto, Kutoyants et Top [FKT19] ont étudié un estimateur bayésien pour la localisation d'une rupture : la loi, la consistance et l'efficacité de cet estimateur ont été étudiées dans un cadre asymptotique. Dans l'esprit des tests multiples, Young Yang et Kuo [YYK01] ont proposé une procédure de segmentation binaire pour localiser les instants des ruptures et les tailles des sauts au sein d'un processus de Poisson : à chaque étape de la procédure de segmentation binaire, on teste l'absence de rupture dans l'intensité contre l'alternative formée des intensités avec une seule rupture dans la loi du processus de Poisson. Les tests simples correspondants sont basés sur le facteur de Bayes ou l'approximation BIC de ce facteur et permettent d'obtenir des estimateurs pour les instants de rupture. Afin d'obtenir des estimés sur les intensités et les instants de saut, des algorithmes de type Monte-Carlo par Chaîne de Markov (MCMC) ont été proposés par plusieurs auteurs comme Green [Gre95] ou plus récemment [NM19].

Pour des descriptions non bayésiennes du problème, les techniques d'estimation des instants de saut utilisées dans la littérature reposent essentiellement sur l'estimateur du maximum de vraisemblance et les statistiques CUSUM, dont les propriétés sont généralement démontrées dans un cadre asymptotique. Dans l'article de Deshayes [Des84], les lois asymptotiques de l'estimateur du maximum de vraisemblance sont établies selon différentes hypothèses et asymptotiques sur les paramètres de la rupture. Toujours dans un cadre asymptotique, Akman et Raftery [AR86a] ont construit des procédures d'estimation basées sur l'agrégation des différences normalisées entre les intensités moyennes, dans l'esprit des statistiques CUSUM. En supposant que l'instant de rupture n'est pas trop proche des bords de l'intervalle d'observation, les auteurs ont établi la consistance faible de l'estimateur ainsi que sa loi asymptotique, conduisant à la construction d'intervalles de confiance asymptotiques pour l'instant de rupture. Dans son manuscrit de thèse, Loader [Loa90] traite entre autres du problème de l'estimation de l'instant de rupture et de la construction d'intervalles de confiance asymptotiques (pour l'instant de rupture et pour le couple formé de l'instant de rupture et de la taille du saut) à partir de la statistique du rapport de vraisemblance, conditionnellement au nombre de points observés et en supposant que l'instant de rupture n'est pas trop proche du bord. Dans l'esprit des tests multiples, Galeano [Gal07] a proposé un algorithme de segmentation binaire associé à une statistique CUSUM dont la loi asymptotique a été établie. De plus, la consistance et la loi limite de l'estimateur de l'instant de rupture correspondant ont aussi été prouvées, permettant ainsi d'obtenir des intervalles de confiance pour les instants de rupture.

Présentons à présent quelques travaux indirectement reliés à la localisation d'une rupture abrupte dans la loi d'un processus de Poisson. Tout d'abord, Amiri et Dachian [AD21] se sont intéressés à la localisation d'une rupture non abrupte dans la loi d'un processus de Poisson, c'est-à-dire lorsque le saut dans l'intensité d'une valeur constante à une autre n'est pas soudain mais intervient sur un petit intervalle de temps et où la hauteur du saut est caractérisée par une fonction régulière. À partir de l'observation de n processus de Poisson indépendants dont l'intensité possède une rupture non abrupte, des estimateurs de l'intervalle sur lequel a lieu la rupture sont construits. Lorsque la longueur de l'intervalle tend vers zéro plus lentement que $1/n$ quand n tend vers l'infini, l'estimateur du maximum de vraisemblance et les estimateurs bayésiens sont consistants, asymptotiquement normaux et asymptotiquement efficaces. Si la longueur de l'intervalle tend vers zéro plus rapidement que $1/n$ quand n tend vers l'infini, le modèle est non régulier et se comporte comme un modèle de rupture abrupte dans la loi : les estimateurs bayésiens sont consistants, convergents (à la vitesse $1/n$) et sont asymptotiquement efficaces. Un autre problème, plus général, a été étudié dans [CKT18]. Dans ce modèle, l'intensité du processus de Poisson contient une rupture de longueur connue et de hauteur caractérisée par une fonction connue. L'intensité de référence constante étant fixée au préalable, seul l'instant de saut est inconnu. Le comportement asymptotique et la consistance de l'estimateur du maximum de vraisemblance et des estimateurs bayésiens sont établis ainsi que l'efficacité asymptotique des estimateurs bayésiens. Enfin, Plummer et Chen [PC14] ont proposé une approche bayésienne pour la localisation de ruptures dans la loi d'un processus de Poisson composé et un algorithme de type "fenêtre glissante" est utilisé pour identifier les ruptures multiples.

Procédures de tests multiples pour les processus de Poisson

Yang [Yan20] s'est intéressé au cas d'une rupture dans l'intensité d'un processus de Poisson à partir d'une intensité de référence qui n'est plus constante mais est une fonction qui varie en fonction du temps. L'intensité du processus dépend alors de deux paramètres inconnus : l'instant de saut et un paramètre dans l'intensité de référence. À partir de l'observation de n trajectoires indépendantes d'un tel processus de Poisson, un problème de test multiple est étudié afin de décrire les paramètres de la rupture. En particulier, deux hypothèses nulles sont testées : une hypothèse nulle porte sur la valeur du paramètre de l'intensité de référence et l'autre porte sur l'instant de rupture. Une étude asymptotique de ce problème de test multiple est menée à partir du critère du FWER pour l'erreur de première espèce et une notion de puissance pour l'erreur de seconde espèce. Afin de réduire les effets de la dépendance entre les tests, la procédure de test Bayes Multiples est proposée et comparée (théoriquement et numériquement) à trois autres procédures (Bayes Factor, Wald's Multiple et Bayesian Estimator Multiple).

Enfin, Picard, Reynaud-Bouret et Roquain [PRBR18] ont défini un nouveau cadre théorique traitant des problèmes de test multiple sur un continuum d'hypothèses, c'est-à-dire sur un nombre infini et non dénombrable d'hypothèses nulles. Ce travail, pertinent pour l'observation d'un processus en temps continu, propose des procédures de scan sur des fenêtres glissantes qui assurent le contrôle non asymptotique du FWER et du FDR dont les définitions ont été adaptées au continuum d'hypothèses. Cette théorie est appliquée aux problèmes particuliers du test d'homogénéité et du test de comparaison de l'intensité de deux échantillons de processus de Poisson.

IV Aperçu de la thèse et contributions

Chapitre 2 : Tests minimax et adaptatifs pour la détection d'une rupture éventuellement transitoire dans un processus de Poisson d'intensité de référence connue

et

Chapitre 3 : Tests minimax et adaptatifs pour la détection d'une rupture éventuellement transitoire dans un processus de Poisson d'intensité de référence inconnue

Article de référence :
[FGLG21] *Minimax and adaptive tests for detecting abrupt and possibly transitory changes in a Poisson process*

Ces deux chapitres portent sur la détection optimale d'un saut ou d'un segment dans l'intensité d'un processus de Poisson. Nous formulons ce problème de détection comme un problème de tests simples et présentons une étude minimax non asymptotique. Nous établissons ainsi les vitesses de séparation minimax pour :

- la détection d'une rupture éventuellement transitoire dans l'intensité d'un processus de Poisson à partir d'une intensité de référence connue (Chapitre 2) et inconnue (Chapitre 3),
- la détection d'une rupture dans la loi d'un processus de Poisson caractérisée par un saut dans son intensité, ou la détection d'une rupture transitoire dans la loi caractérisée par un segment dans son intensité,
- la détection d'une rupture (éventuellement transitoire) de taille et/ou de longueur et/ou de localisation inconnue(s) dans la loi d'un processus de Poisson.

À notre connaissance, aucun de ces résultats minimax n'avait été établi pour un processus de Poisson (voir le modèle statistique décrit dans le Chapitre 1 Section I.4).

Ces chapitres offrent un panorama complet des différentes vitesses de séparation minimax pour ces problèmes de détection, précisant ainsi les transitions de phase dans les vitesses. En considérant connu ou inconnu chaque paramètre de la rupture l'un après l'autre, nous exhibons les paramètres responsables des différentes transitions dans les vitesses et précisons ainsi le coût de l'adaptation minimax de chacun de ces paramètres. Cette étude est menée à la fois lorsque l'intensité de référence du processus de Poisson est connue (dans le Chapitre 2) et lorsqu'elle est inconnue (dans le Chapitre 3). Nous montrons notamment une transition de phase dans l'ordre des vitesses de séparation minimax de $\sqrt{\log\log L/L}$ pour la détection d'un saut à $\sqrt{\log L/L}$ pour la détection d'un segment, lorsque la taille et la localisation de la rupture ainsi que la longueur du segment sont inconnues. Si cette transition de phase dans les vitesses est similaire à celle observée dans le modèle gaussien, certaines vitesses de séparation que nous établissons ne sont pas connues, à notre connaissance, même dans le cadre gaussien plus classique.

Plus précisément, pour la détection d'un segment dans l'intensité d'un processus de Poisson, nous avons prouvé que la vitesse de séparation est d'ordre $\sqrt{\log L/L}$ lorsque la localisation et la longueur du segment sont inconnues, quelle que soit la connaissance de la taille du saut. La vitesse est d'ordre $\sqrt{\log\log L/L}$ lorsque seul l'instant de rupture est connu, et d'ordre $\sqrt{1/L}$ dans tous les autres cas. Pour la détection d'un saut dans l'intensité, les vitesses que nous avons obtenues sont cohérentes avec les résultats asymptotiques ou non asymptotiques disponibles en modèle gaussien classique : la vitesse de séparation est d'ordre $\sqrt{\log\log L/L}$ lorsque la taille et la position du saut sont inconnues et d'ordre $\sqrt{1/L}$ dans tous les autres cas. Notons que nous avons obtenu ces vitesses dans un cadre non asymptotique, dans le cas où l'intensité de référence du processus de Poisson est connue et dans le cas où elle est inconnue.

Afin d'obtenir des bornes inférieures pour les vitesses de séparation minimax des différents problèmes de test, nous utilisons principalement le Lemme 1.8 issue de la théorie bayésienne, le lemme de Girsanov (voir le Lemme 2.1) et certaines propriétés des processus de Poisson. Une démarche progressive a aussi été proposée pour la construction de tests minimax ou minimax adaptatifs. Nos procédures de tests sont basées à la fois sur des statistiques linéaires proches des statistiques CUSUM utilisées en modèle gaussien, inspirées et adaptées des tests de Neyman-Pearson utilisés lorsque tous les paramètres de la rupture (éventuellement transitoire) sont connus, et à la fois sur de nouvelles statistiques quadratiques. Par leur construction, ces nouvelles statistiques quadratiques nous ont semblé plus appropriées et naturelles pour estimer la distance d_2 entre l'intensité λ et l'ensemble $\mathcal{S}_0$ considérée dans ce travail. Cette discussion sur le choix de la statistique soulève de nouvelles questions théoriques sur l'agrégation de tests bilatéraux, illustrées par la mise en œuvre de ces procédures de tests sur des données simulées dans une étude expérimentale.

Lorsque des paramètres de la rupture sont inconnus, les propriétés d'adaptation au sens du minimax de nos tests sont obtenues par des stratégies d'agrégation et de scan, toutes différentes selon la nature des paramètres inconnus. Les valeurs critiques associées à ces statistiques d'agrégation et de scan ont également dû être ajustées afin de garantir le niveau non asymptotique α et les propriétés d'optimalité au sens du minimax des procédures de tests correspondantes. Dans le Chapitre 3, le fait que l'intensité de référence du processus soit inconnue complique l'ajustement des valeurs critiques : une astuce de conditionnement, également décrite dans [FLRB11], a été utilisée pour remédier à cette difficulté.

L'obtention de contrôles non asymptotiques pour ces valeurs critiques est le point le plus délicat dans les preuves des bornes supérieures des vitesses de séparation. Nous avons dû utiliser des inégalités exponentielles et des inégalités de concentration très variées. Dans le Chapitre 2, lorsque l'intensité de référence du processus de Poisson est connue, ces inégalités exponentielles vont de résultats sur les suprema de processus de Poisson avec drift, datant du milieu du vingtième siècle, obtenues par Pyke [Pyk59] à des résultats très récents démontrés par Le Guével [LG21] pour des suprema de processus de comptage et leurs martingales ou martingales carrées associées. Dans le Chapitre 3, lorsque l'intensité de référence est inconnue, nous avons utilisé des inégalités exponentielles pour le suprema ou le module d'oscillation de processus empiriques et de U-statistiques établies par Mason, Shorack et Wellner [SW86] et Houdré et Reynaud-Bouret [HRB03], ou des inégalités de concentration obtenues à partir des inégalités de Bernstein et de Bennett (énoncées dans [BDR15]) que nous avons combinées à des techniques de chaînage.

Chapitre 4 : Tests multiples minimax et adaptatifs pour la détection et la localisation d'une rupture dans un processus de Poisson d'intensité de référence connue

Article de référence :

Minimax and adaptive multiple tests for detecting and localising an abrupt change in the intensity of a Poisson process, avec Fromont Magalie et Le Guével Ronan. En préparation.

Ce chapitre porte sur la détection et la localisation optimale d'un saut dans l'intensité d'un processus de Poisson. Nous formulons ce problème de détection et de localisation comme un problème de tests multiples et présentons une étude minimax et non asymptotique. Nous établissons ainsi les vitesses de séparation minimax par famille pour :

- la détection et la localisation d'une rupture dans l'intensité d'un processus de Poisson à partir d'une intensité de référence connue,
- la détection et la localisation d'une rupture de taille connue, bornée inférieurement et inconnue dans la loi d'un processus de Poisson.

À notre connaissance, aucun de ces résultats minimax n'avaient été établis pour un processus de Poisson (voir le modèle statistique décrit dans le Chapitre 1 Section I.4).

Nous proposons un panorama des différentes vitesses de séparation minimax par famille pour ces problèmes de localisation d'une rupture, précisant ainsi les transitions de phase dans les vitesses. En considérant connue ou inconnue la taille de la rupture, nous précisons le coût de l'adaptation minimax pour ce paramètre. Cette étude de détection et de localisation simultanée d'une rupture, menée lorsque l'intensité de référence du processus de Poisson est connue, peut être vue comme un prolongement du problème de détection dans un cadre hors-ligne mené dans le Chapitre 2. Nous montrons notamment une transition de phase dans l'ordre des vitesses de séparation minimax par famille de $\sqrt{1/L}$ lorsque la taille est connue à $\sqrt{\log\log L/L}$ lorsqu'elle est inconnue. À partir des procédures de tests multiples minimax, nous construisons aussi des estimateurs et des intervalles de confiance de taille minimale pour l'instant de rupture.

Plus précisément, nous avons prouvé que la vitesse de séparation minimax par famille est d'ordre $\sqrt{1/L}$ lorsque la taille du saut est connue et lorsqu'elle est minorée par une valeur donnée. Si la taille du saut est complètement inconnue, la vitesse est dégradée d'un facteur $\sqrt{\log\log L}$ ce qui est cohérent avec le coût supplémentaire dû à l'adaptation en la taille du saut observé dans le problème de détection (voir Chapitre 2). Il est à noter que l'on peut retrouver les résultats obtenus pour les vitesses de séparation minimax du Chapitre 2 à partir des résultats démontrés pour les tests multiples et les vitesses de séparation minimax par famille. Ce fait est conforme à l'intuition : avant de pouvoir prétendre localiser une rupture, il faut être en mesure de la détecter. Néanmoins, il est remarquable que la multiplicité des hypothèses dans la formulation des problèmes de tests multiples ne dégrade pas l'ordre de la vitesse de la mFWSR.

Afin d'obtenir des bornes inférieures pour les vitesses de séparation minimax par famille des différents problèmes de test, nous utilisons le Lemme 1.10. Ce lemme établit un lien entre le critère minimax pour les procédures de tests multiples et le critère minimax pour les procédures de tests simples correspondantes, nous permettant ainsi de minorer les vitesses de séparation minimax par famille par les vitesses de séparation minimax obtenues pour le problème de détection d'une rupture dans le Chapitre 2. Pour prouver que ces bornes inférieures sont d'ordre optimal, nous construisons des procédures de tests multiples dont le FWER est contrôlé par un niveau fixé au préalable et dont le FWSR atteint (à une constante multiplicative près) les bornes inférieures. Pour chaque hypothèse nulle, on considère le problème de test simple correspondant et utilisons des statistiques similaires à celles utilisées pour le problème de détection d'une rupture (voir Chapitre 2). Ainsi, lorsque la taille du saut est connue ou bornée inférieurement, nous utilisons une statistique linéaire qui prend en compte la connaissance de la taille ou de la taille minimale du saut. Lorsque la taille est inconnue, nous utilisons une statistique linéaire et une statistique quadratique adaptées de celles définies dans le Chapitre 2. Les propriétés d'adaptation au sens du minimax de nos procédures de tests multiples sont obtenues par des stratégies d'agrégation et de scan, différentes selon la connaissance de la taille du saut. Les valeurs critiques associées à ces statistiques d'agrégation et de scan ont du être ajustées afin de garantir le contrôle non asymptotique du FWER et les propriétés d'optimalité au sens du minimax des procédures de tests multiples. Nous avons notamment prouvé que la vitesse de séparation par famille de chacune de nos procédures de tests multiples ne dépend pas du nombre d'hypothèses nulles testées. Ce fait remarquable offre un intérêt à un futur développement d'une théorie minimax pour les tests multiples d'un continuum d'hypothèses.

Les contrôles non asymptotiques pour les valeurs critiques des procédures de tests multiples sont obtenus grâce aux mêmes outils que ceux utilisés dans le Chapitre 2, à savoir des inégalités exponentielles pour les suprema de processus de Poisson avec drift démontrées par Pyke [Pyk59] et pour les suprema de processus de comptage et leurs martingales ou martingales carrées associées démontrées par Le Guével [LG21]. La prise en compte de la multiplicité des tests sans dégrader l'ordre de la vitesse est le point délicat dans les preuves des bornes supérieures des vitesses de séparation par famille. Nous avons dû adapter les résultats obtenus par Pyke [Pyk59] sur les suprema de processus de Poisson avec drift et utiliser de nouvelles inégalités exponentielles démontrées par Le Guével [LG21] pour les modules d'oscillations de martingales et martingales carrées définies à partir d'un processus de Poisson. Couplées à une partition dyadique de l'intervalle $[0,1]$, ces inégalités exponentielles pour les modules d'oscillations nous ont permis d'obtenir un contrôle assez fin de la probabilité qui intervient dans le calcul des vitesses de séparation par famille sur chaque élément de la partition, garantissant ainsi le bon ordre dans la vitesse.

Enfin, nous avons établi des liens entre les procédures de tests multiples minimax pour la localisation de la rupture et des intervalles de confiance de taille minimale pour l'instant de rupture. S'il est bien connu qu'un test simple de niveau α est intrinsèquement relié à une région de confiance de niveau $1-\alpha$, une correspondance similaire peut être faite pour notre problème de localisation de rupture entre une procédure de test multiple dont le FWER est contrôlé par α et un intervalle de confiance de niveau $1-\alpha$ pour l'instant de rupture. L'apport de la connaissance du FWSR d'une procédure de test multiple nous a ensuite permis d'obtenir des bornes sur la taille minimale des intervalles de confiance pour l'instant de rupture. Associés aux procédures de tests multiples minimax établies au préalable, ces résultats nous ont finalement permis de construire un estimateur puis un intervalle de confiance de taille minimale pour l'instant de saut.

CHAPTER

TWO

MINIMAX AND ADAPTIVE TESTS FOR DETECTING ABRUPT AND POSSIBLY TRANSITORY CHANGES IN A POISSON PROCESS WITH A KNOWN BASELINE INTENSITY

As a first step of work, and because this also addresses particular applications, we are here interested in the problem of detecting an abrupt change in the intensity of the Poisson process N, when its baseline is assumed to be known, equal to a positive constant function λ_0 on $[0,1]$. For the sake of simplicity, the constant function λ_0 and its value on $[0,1]$ are often confused in the following. The null hypothesis of the present section can therefore be expressed as (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$", while the alternative hypothesis varies according to the height, length and location of the intensity jump or bump knowledge. All the possible sets of alternatives according to whether each parameter of the change (height, location and length) is known or not, including the special case where the change is non transitory (jump detection), are handled. For each sets of alternatives, lower bounds for minimax separation rates are provided, as a preliminary basis for corresponding upper bounds (when appropriate, that is when at least the height or the length is unknown). As explained above, these upper bounds are obtained by constructing minimax or minimax adaptive tests, which are mainly based on aggregation of either linear or quadratic statistics, coupled with adjusted critical values. A simulation study is presented in Chapter 2 Section VIII, whose aim is to compare linear and quadratic type tests, and also to compare them with standard tests used to detect nonhomogeneity of Poisson processes in practice. Proofs of the core results are postponed to Chapter 2 Section IX, and proofs of technical results mostly based on exponential inequalities and devoted to quantiles and critical values upper bounds are postponed to Chapter 2 Section X. All along the chapter, we will introduce some positive constants denoted by $C(\alpha,\beta,\ldots)$ and $L_0(\alpha,\beta,\ldots)$, meaning that they depend on $(\alpha,\beta,\ldots)$. Though they are denoted in the same way, they may vary from one line to another. When they appear in the main results about lower and upper bounds, we do not intend to precisely evaluate them. However, some possible, probably pessimistic, explicit expressions for them are proposed in the proofs.

In order to further cover the full range of alternatives in a unified notation, we introduce for δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$, τ^* in $(0,1)$, ℓ^* in $(0,1-\tau^*]$ the set $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ of intensities with a change of height δ^*, location τ^* and length ℓ^* from λ_0:

$$[\textbf{Alt.1}] \quad \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0] = \{\lambda : [0,1] \to (0,+\infty),\ \forall t \in [0,1]\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t)\}\ . \tag{2.1}$$

Testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$" falls within the scope of the Neyman-Pearson fundamental lemma and a most powerful test exists, thus achieving the minimax separation rate over $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$. Details are provided below. Notice that $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ contains only one alternative $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}$ but we still use the notation $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ in the following to standardise the notations of the alternative sets.

Then, when the question of adaptivity with respect to unknown parameters is tackled, the unknown parameters are replaced by single, double or a triple dots in the notation $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$.

Recall that for any alternative intensity $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell]}$ with δ in $(-\lambda_0, +\infty) \setminus \{0\}$, τ in $(0,1)$, and ℓ in $(0, 1-\tau]$, $d_2(\lambda, \mathcal{S}_0[\lambda_0]) = |\delta|\sqrt{\ell}$. Hence, as soon as λ has a known change height $\delta = \delta^*$ and a known change length $\ell = \ell^*$, the distance $d_2(\lambda, \mathcal{S}_0[\lambda_0])$ is fixed, equal to $|\delta^*|\sqrt{\ell^*}$. The β-uniform separation rate of any level α test over $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ or $\mathcal{S}_{\delta^*,\cdot\cdot,\ell^*}[\lambda_0]$ as defined by (1.3) is therefore either 0 or $+\infty$ (with the usual convention $\inf \varnothing = +\infty$), as well as the minimax separation rate. In these only two cases, studying our tests from the minimax point of view would have no sense. Nevertheless, once having ensured that their first kind error rate is at most α, in order to follow the same line as the minimax results obtained in the other cases, we establish conditions expressed as a sufficient minimal distance $d_2(\lambda, \mathcal{S}_0[\lambda_0])$, guaranteeing that their second kind error rate is at most equal to some prescribed level β.

I Uniformly most powerful detection of a possibly transitory change with known location and length

Let us now give more details about the above problem of testing the simple null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus the simple alternative hypothesis (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$" with $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ defined by (2.1). Notice that for any λ in $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$, then

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) = |\delta^*|\sqrt{\ell^*} \ .$$

Given α in $(0,1)$, Neyman-Pearson tests of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$" of size α can be constructed. To this end, we recall Girsanov's lemma (see [Bré81] for a proof).

Lemma 2.1 (Girsanov)**.**
Let $N = (N_t)_{t\in[0,1]}$ be an inhomogeneous Poisson process with jump locations $(X_j)_{j\geq 1}$, with bounded intensity λ with respect to some measure Λ on $[0,1]$, and with distribution denoted by P_λ under the probability $\mathbb{P}$. Assume that λ_0 is a bounded nonnegative function such that for every $j \geq 1$, $\lambda_0(X_j) > 0$ $\mathbb{P}$-almost surely. Then

$$\frac{dP_\lambda}{dP_{\lambda_0}}(N) = \exp\left[\int_0^1 \ln\left(\frac{\lambda(t)}{\lambda_0(t)}\right) dN_t - \int_0^1 (\lambda(t) - \lambda_0(t)) d\Lambda_t\right] \ .$$

From this fundamental lemma, we deduce the likelihood ratio, for λ in $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$,

$$\frac{dP_\lambda}{dP_{\lambda_0}}(N) = \exp\left[\ln\left(1 + \frac{\delta^*}{\lambda_0}\right) N(\tau^*, \tau^* + \ell^*] - \delta^* \ell^* L\right] \ , \tag{2.2}$$

which leads to the following size α Neyman-Pearson tests:

$$\begin{cases} \phi^-_{1,\alpha}(N) & = \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] < p_{\lambda_0 \ell^* L}(\alpha)} + \gamma^-(\alpha) \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] = p_{\lambda_0 \ell^* L}(\alpha)} & \text{if } \delta^* < 0 \ , \\ \phi^+_{1,\alpha}(N) & = \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] > p_{\lambda_0 \ell^* L}(1-\alpha)} + \gamma^+(1-\alpha) \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] = p_{\lambda_0 \ell^* L}(1-\alpha)} & \text{if } \delta^* > 0 \ , \end{cases} \tag{2.3}$$

where $p_\xi(u)$ denotes the u-quantile of the Poisson distribution with parameter ξ, and

$$\gamma^-(u) = \frac{u - P_{\lambda_0}(N(\tau^*, \tau^* + \ell^*] < p_{\lambda_0 \ell^* L}(u))}{P_{\lambda_0}(N(\tau^*, \tau^* + \ell^*] = p_{\lambda_0 \ell^* L}(u))}, \quad \gamma^+(u) = 1 - \gamma^-(u) \ . \tag{2.4}$$

Proposition 2.2 (Second kind error rates control for [**Alt.1**]).
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$, τ^* in $(0,1)$ and ℓ^* in $(0, 1-\tau^*]$. Considering the problem of testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$", let $\phi_{1,\alpha}$ be the test $\phi^-_{1,\alpha}$ if $\delta^* < 0$, $\phi^+_{1,\alpha}$ if $\delta^* > 0$ (see (2.3) for definitions of the tests). The test $\phi_{1,\alpha}$ is a most powerful test of size α. Moreover, $P_\lambda(\phi_{1,\alpha}(N) = 0) \leq \beta$ as soon as λ belongs to $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ with

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{\sqrt{(\lambda_0 + \delta^*)/\beta} + \sqrt{\lambda_0/\alpha}}{\sqrt{L}} \ . \tag{2.5}$$

Comments.
Notice first that the same result holds with $P_\lambda(\phi_{1,\alpha}(N) = 0)$ replaced by the second kind error rate $E_\lambda[1 - \phi_{1,\alpha}(N)]$. Then, as explained above, studying the present test from the minimax point is not really relevant. One can however notice that the uniform separation rate of a most powerful test necessarily provides the minimax separation rate over any set of alternatives. Since for λ in $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$, $d_2(\lambda, \mathcal{S}_0[\lambda_0]) = |\delta^*|\sqrt{\ell^*}$, the above proposition implies that if $L \geq \left(\sqrt{(\lambda_0 + \delta^*)/\beta} + \sqrt{\lambda_0/\alpha}\right)^2 / \left(\delta^{*2}\ell^*\right)$, then $P_\lambda(\phi_{1,\alpha}(N) = 0) \leq \beta$. Therefore, in this case, the β-uniform separation rate of $\phi_{1,\alpha}$ over $\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$ is equal to 0, and consequently, the (α,β)-minimax separation rate $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0])$.

Let us now consider the question of adaptation with respect to the change height only. To this end, we introduce, for τ^* in $(0,1)$ and ℓ^* in $(0, 1-\tau^*]$ the set

$$[\mathbf{Alt.2}] \quad \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0] = \{\lambda : [0,1] \to (0,+\infty), \exists \delta \in (-\lambda_0, +\infty) \setminus \{0\}, \forall t \in [0,1]\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t)\} \ , \tag{2.6}$$

and we consider the problem of testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$".

The following result gives a nonasymptotic lower bound for the (α,β)-minimax separation rate over the set of alternatives $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$ of the parametric order $L^{1/2}$, which is obtained from a now classical Bayesian approach that originates in Le Cam's theory and Ingster's work [Ing93] in an asymptotic perspective, and that has been next adapted to the nonasymptotic perspective by Baraud [Bar02]. The main points of this approach are recalled in Chapter 1 Section II.2, and the complete proof can be found in Section IX of Chapter 2.

Proposition 2.3 (Minimax lower bound for [**Alt.2**]).
Let α, β in $(0,1)$ such that $\alpha + \beta < 1$, $\lambda_0 > 0$, τ^* in $(0,1)$ and ℓ^* in $(0, 1-\tau^*]$. For all $L \geq 1$, the following lower bound holds:

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \ \textit{with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

In order to prove that this lower bound is sharp with respect to L, we introduce a first bilateral test based on the counting statistic $N(\tau^*, \tau^* + \ell^*]$ of the above Neyman-Pearson tests. So let

$$\phi^{(1)}_{2,\alpha}(N) = \mathbb{1}_{\{N(\tau^*,\tau^*+\ell^*] > p_{\lambda_0\ell^*L}(1-\alpha_1)\}} + \gamma^+(1-\alpha_1)\mathbb{1}_{\{N(\tau^*,\tau^*+\ell^*] = p_{\lambda_0\ell^*L}(1-\alpha_1)\}} + \mathbb{1}_{\{N(\tau^*,\tau^*+\ell^*] < p_{\lambda_0\ell^*L}(\alpha_2)\}} + \gamma^-(\alpha_2)\mathbb{1}_{\{N(\tau^*,\tau^*+\ell^*] = p_{\lambda_0\ell^*L}(\alpha_2)\}} \ , \tag{2.7}$$

where α_1 and α_2 in $(0,1)$ are determined by

$$\alpha_1 + \alpha_2 = \alpha \quad \text{and} \quad E_{\lambda_0}[N(\tau^*, \tau^* + \ell^*]\phi^{(1)}_{2,\alpha}(N)] = \alpha E_{\lambda_0}[N(\tau^*, \tau^* + \ell^*]] \ . \tag{2.8}$$

Since our testing problem amounts to a problem of testing "$\delta = 0$" versus "$\delta \neq 0$" in the exponential model $dP_\lambda/dP_{\lambda_0}(N) = \exp[\ln(1 + \delta/\lambda_0)N(\tau^*, \tau^* + \ell^*] - \delta\ell^* L]$, applying the result of Chapter 4.2 in [LR05] allows to see that $\phi^{(1)}_{2,\alpha}$ is an Uniformly Most Powerful Unbiased (UMPU) test of size α. We also prove that it is β-minimax (up to a possible multiplicative constant) over the set of alternatives $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$ with the following result.

Proposition 2.4 (Minimax upper bound for [**Alt**.2]).
Let $L \geq 1$, α,β in $(0,1)$, $\lambda_0 > 0$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. The test $\phi^{(1)}_{2,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$" defined by (2.7)-(2.8) is of level α, that is $P_{\lambda_0}(\phi^{(1)}_{2,\alpha}(N) = 1) \leq \alpha$ (it is even of size α, that is $E_{\lambda_0}[\phi^{(1)}_{2,\alpha}(N)] = \alpha$). Moreover, there exists $C(\alpha,\beta,\lambda_0,\tau^*,\ell^*) > 0$ such that

$$\mathrm{SR}_\beta(\phi^{(1)}_{2,\alpha}, \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]) \leq \frac{C(\alpha,\beta,\lambda_0,\ell^*)}{\sqrt{L}} \quad ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]) \leq C(\alpha,\beta,\lambda_0,\ell^*)/\sqrt{L}$.

It is interesting to notice that the test $\phi^{(1)}_{2,\alpha}$ can also be written as

$$\phi^{(1)}_{2,\alpha}(N) = \mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*] > \bar{p}_{\lambda_0\ell^*L}(1-\alpha_1)\}} + \gamma^+(1-\alpha_1)\mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*] = \bar{p}_{\lambda_0\ell^*L}(1-\alpha_1)\}}$$
$$+ \mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*] < \bar{p}_{\lambda_0\ell^*L}(\alpha_2)\}} + \gamma^-(\alpha_2)\mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*] = \bar{p}_{\lambda_0\ell^*L}(\alpha_2)\}} \quad , \tag{2.9}$$

where $\bar{N}(\tau^*,\tau^*+\ell^*] = N(\tau^*,\tau^*+\ell^*] - \lambda_0\ell^*L$ and $\bar{p}_{\lambda_0\ell^*L}(u)$ is the u-quantile of this recentered Poisson variable under (H_0). The statistic $\bar{N}(\tau^*,\tau^*+\ell^*]$ being an unbiased estimator of $L\langle\lambda - \lambda_0, \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}\rangle_2$ when λ belongs to $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$, it is intuitively natural to reject (H_0) "$\lambda = \lambda_0$" when it is too small or too large, with critical values determining what "too small" or "too large" means based on the quantiles of $\bar{N}(\tau^*,\tau^*+\ell^*]$ under (H_0).

Since the distance considered in our minimax criteria is the $\mathbb{L}_2$-distance d_2, a more natural idea here is to construct a test based on a statistic estimating $d_2(\lambda,\lambda_0)$ or $d_2^2(\lambda,\lambda_0)$ when $\lambda \in \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$. Such an unbiased estimator of $d_2^2(\lambda,\lambda_0)$ is in fact given by the quadratic statistic $T_{\tau^*,\tau^*+\ell^*}(N)$, where for τ_1,τ_2 such that $0 \leq \tau_1 < \tau_2 \leq 1$,

$$T_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)}\left(N(\tau_1,\tau_2]^2 - (1+2\lambda_0 L(\tau_2-\tau_1))N(\tau_1,\tau_2] + \lambda_0^2 L^2(\tau_2-\tau_1)^2\right) \quad . \tag{2.10}$$

This leads us to consider a second minimax test, defined by

$$\phi^{(2)}_{2,\alpha}(N) = \mathbb{1}_{T_{\tau^*,\tau^*+\ell^*}(N) > t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha)} \quad , \tag{2.11}$$

where $t_{\lambda_0,\tau_1,\tau_2}(u)$ denotes the u-quantile of the distribution of $T_{\tau_1,\tau_2}(N)$ under (H_0).

Proposition 2.5 (Alternate minimax upper bound for [**Alt**.2]).
Let $L \geq 1$, α,β in $(0,1)$, $\lambda_0 > 0$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. The test $\phi^{(2)}_{2,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$" defined by (2.11) is of level α, that is $P_{\lambda_0}(\phi^{(2)}_{2,\alpha}(N) = 1) \leq \alpha$, and there exists $C(\alpha,\beta,\lambda_0,\tau^*,\ell^*) > 0$ such that $\mathrm{SR}_\beta(\phi^{(2)}_{2,\alpha}, \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]) \leq C(\alpha,\beta,\lambda_0,\ell^*)/\sqrt{L}$.

Comments.
Both tests $\phi^{(1)}_{2,\alpha}$ and $\phi^{(2)}_{2,\alpha}$ are therefore β-minimax (up to a possible multiplicative constant) over the set of alternatives $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$, where the height of the change is unknown, with an optimal uniform separation rate of the expected parametric order $1/\sqrt{L}$. Despite the bilateral form of the test $\phi^{(1)}_{2,\alpha}$ and the apparent unilateral form of $\phi^{(2)}_{2,\alpha}$, these tests have in fact very close links that we detail in the following section.

Notice that the present study involves the particular non transitory change or jump detection problem, with a known change location, taking $\ell^* = 1-\tau^*$. The study of the general non transitory change or jump detection problem (of unknown location) is conducted in Chapter 2 Section V.1 as a particular case of change detection problem, with unknown location and length.

II Choice of the test statistics and links between the corresponding procedures

Let us focus on the test $\phi^{(2)}_{2,\alpha}$ and confront it the UMPU test $\phi^{(1)}_{2,\alpha}$. To this end, we notice that

$$\phi^{(2)}_{2,\alpha}(N) = \mathbb{1}_{N(\tau^*,\tau^*+\ell^*]-\lambda_0\ell^* L>\frac{1}{2}\left(1+\sqrt{1+4\lambda_0\ell^* L+4\ell^* L^2 t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha)}\right)} \\ + \mathbb{1}_{N(\tau^*,\tau^*+\ell^*]-\lambda_0 L\ell^*<\frac{1}{2}\left(1-\sqrt{1+4\lambda_0\ell^* L+4\ell^* L^2 t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha)}\right)} \ .$$

The test $\phi^{(2)}_{2,\alpha}$ can thus be clearly related to the test $\phi^{(1)}_{2,\alpha}$ expressed with the recentered statistic $\bar{N}(\tau^*,\tau^*+\ell^*]$ as in (2.9), with the only difference that its critical values are differently chosen. The choice made for $\phi^{(2)}_{2,\alpha}$ has two main consequences: a negative one, which is that the test is not UMPU contrary to $\phi^{(1)}_{2,\alpha}$, and a positive one which is that it does not involve calibration of two levels α_1 and α_2 as in $\phi^{(1)}_{2,\alpha}$ where they have to satisfy (2.8), equivalent to

$$E_{\lambda_0}\left[\left(\bar{N}(\tau^*,\tau^*+\ell^*]-\bar{p}_{\lambda_0\ell^* L}(1-\alpha_1)\right)\mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*]-\bar{p}_{\lambda_0\ell^* L}(1-\alpha_1)>0\}}\right] \\ + E_{\lambda_0}\left[\left(\bar{N}(\tau^*,\tau^*+\ell^*]-\bar{p}_{\lambda_0\ell^* L}(\alpha_2)\right)\mathbb{1}_{\{\bar{N}(\tau^*,\tau^*+\ell^*]-\bar{p}_{\lambda_0\ell^* L}(\alpha_2)<0\}}\right] \\ + \alpha_1\bar{p}_{\lambda_0\ell^* L}(1-\alpha_1)+\alpha_2\bar{p}_{\lambda_0\ell^* L}(\alpha_2) = 0 \ .$$

This calibration, which is already not so easy to execute when using such single bilateral tests, leads to an additional difficulty when considering their aggregation. Correcting the individual levels of some tests of the form of $\phi^{(2)}_{2,\alpha}$ when they are aggregated is quite a classical question, that can be solved by using principles of Bonferroni or min-p multiple tests. This question becomes particularly tricky for tests of the form of $\phi^{(1)}_{2,\alpha}$: the arbitrary choice of $\alpha_1 = \alpha_2$ makes the aggregation more convenient in the following, at the price to slightly degrade the final power of the aggregated test, especially when the distribution of the involved recentered statistics $\bar{N}(\tau_1,\tau_2]$ is not symmetric with respect to 0. More details are given in Chapter 2 Section VI.

This last consequence, together with the fact that the test statistic $T_{\tau_1,\tau_2}(N)$ is an unbiased estimator of $\|\Pi_{V_{\tau_1,\tau_2}}(\lambda-\lambda_0)\|_2^2$ where $V_{\tau_1,\tau_2} = \text{Vect}\left(\varphi_{(\tau_1,\tau_2]} = \mathbb{1}_{(\tau_1,\tau_2]}/\sqrt{\tau_2-\tau_1}\right)$ and $\Pi_{V_{\tau_1,\tau_2}}$ denotes the orthogonal projection onto V_{τ_1,τ_2} in $\mathbb{L}_2([0,1])$, which can be a starting point to generalise the present detection tests to more complex change-point detection problems (for instance in nonconstant baseline intensities), has led us to keep both types of tests all along our study whenever possible. Moreover, the tools used to obtain the minimax results for both tests are very different, hence the proofs for the two tests can be viewed as real alternate proofs of minimax separation rates upper bounds.

III Minimax detection of a transitory change with known length

The present section is dedicated to the problem of testing (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus alternatives where the length of the change from the baseline intensity is known, with adaptation with respect to the change location, and with or without adaptation with respect to the height of the change. We therefore introduce, for ℓ^* in $(0,1)$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$, the two following sets:

$$\textbf{[Alt.3]}\ \mathcal{S}_{\delta^*,\cdot,\ell^*}[\lambda_0] = \left\{\lambda:[0,1]\to(0,+\infty),\ \exists\tau\in(0,1-\ell^*),\ \forall t\in[0,1]\ \lambda(t) = \lambda_0+\delta^*\mathbb{1}_{(\tau,\tau+\ell^*]}(t)\right\}\ , \tag{2.12}$$

$$\textbf{[Alt.4]}\ \mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0] = \left\{\lambda\,[0,1]\to(0,+\infty),\ \exists\delta\in(-\lambda_0,+\infty)\setminus\{0\},\ \exists\tau\in(0,1-\ell^*),\right. \\ \left.\forall t\in[0,1]\quad \lambda(t) = \lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell^*]}(t)\right\}\ . \tag{2.13}$$

As seen in section I, the knowledge of the change height δ^* is not necessary to construct an Uniformly Most Powerful (UMP) test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$" as the test statistic, which is the exhaustive statistic in the considered exponential model, does not depend on the value of δ^*. This enables to directly extend it to an UMPU test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{.,\tau^*,\ell^*}[\lambda_0]$" based on the same exhaustive statistic $N(\tau^*, \tau^* + \ell^*]$.

The only significant question is hence the one of adaptation to the change location τ^*.
A natural approach to handle this question is to take the same linear and quadratic statistics as the ones used for testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$" or (H_1) "$\lambda \in \mathcal{S}_{.,\tau^*,\ell^*}[\lambda_0]$", but making τ^* varying in the whole set of possible change locations, or an appropriate restricted set of possible change locations. This approach, known as statistics scanning in the signal and image processing literature or statistics aggregation in the minimax testing literature, has close connections with multiple tests that were investigated in [FLRB16] (see Chapter 2 Section VI), and that will be exploited in Chapter 4 dedicated to the change localisation problem.

We therefore first introduce the following linear statistic based aggregated tests:

$$\left\{ \begin{array}{ll} \phi^{(1)-}_{3,\alpha}(N) = & \mathbb{1}_{\left\{\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*] < p^-_{\lambda_0,\ell^*}(\alpha)\right\}} \ , \\ \phi^{(1)+}_{3,\alpha}(N) = & \mathbb{1}_{\left\{\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*] > p^+_{\lambda_0,\ell^*}(1-\alpha)\right\}} \ , \end{array} \right. \tag{2.14}$$

where $p^-_{\lambda_0,\ell^*}(u)$ and $p^+_{\lambda_0,\ell^*}(u)$ respectively denote the u-quantiles of the distributions of $\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ and $\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ under (H_0). Notice that the tests $\phi^{(1)-}_{3,\alpha}$ and $\phi^{(1)+}_{3,\alpha}$ corresponds to the general likelihood ratio tests for the one-sided alternatives.
From these unilateral tests, we construct the bilateral test

$$\phi^{(1)}_{4,\alpha}(N) = \phi^{(1)-}_{3,\alpha/2}(N) \vee \phi^{(1)+}_{3,\alpha/2}(N) \ , \tag{2.15}$$

intended to address the change height adaptation issue.
Finally, considering $M = \lceil 2/\ell^* \rceil$ and $u_\alpha = \alpha/\lceil (1-\ell^*)M \rceil$, the test statistic $T_{k/M,k/M+\ell^*}(N)$ defined by (2.10) and its u-quantile $t_{\lambda_0,k/M,k/M+\ell^*}(u)$ under (H_0), we introduce the quadratic statistic based aggregated test

$$\phi^{(2)}_{3/4,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{0,\ldots,\lceil(1-\ell^*)M\rceil-1\}} \left(T_{\frac{k}{M},\frac{k}{M}+\ell^*}(N) - t_{\lambda_0,\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right) > 0\right\}} \ . \tag{2.16}$$

As the set $\mathcal{S}_{\delta^*,.,\ell^*}[\lambda_0]$ defined in (2.12) is composed of alternatives with known change height δ^* and length ℓ^*, the distance between any of its elements and $\mathcal{S}_0[\lambda_0] = \{\lambda_0\}$ is fixed, equal to $|\delta^*|\sqrt{\ell^*}$. Therefore, it is not discussed from the minimax point of view. We only provide sufficient conditions for the tests $\phi^{(1)+}_{3,\alpha}$, $\phi^{(1)-}_{3,\alpha}$ and $\phi^{(2)}_{3/4,\alpha}$ to have a second kind error rate controlled by a prescribed level β under P_λ when $\lambda \in \mathcal{S}_{\delta^*,.,\ell^*}[\lambda_0]$. The proofs of these results, postponed to Chapter 2 Section IX, mainly rely on sharp bounds for the quantiles $p^-_{\lambda_0,\ell^*}(\alpha)$, $p^+_{\lambda_0,\ell^*}(1-\alpha)$ and $t_{\lambda_0,k/M,k/M+\ell^*}(1-u_\alpha)$, that are deduced from two very recent exponential inequalities for the supremum of a counting process and the oscillation modulus of the square martingale associated with a counting process due to Le Guével [LG21]. Recall that the technical proofs of such quantiles bounds are detailed in Chapter 2 Section X.

Proposition 2.6 (Second kind error rate control for [**Alt.3**]).
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ and ℓ^* in $(0,1)$. Considering the problem of testing (H_0) v.s. (H_1) "$\lambda \in \mathcal{S}_{\delta^*,.,\ell^*}[\lambda_0]$", let $\phi^{(1/2)}_{3,\alpha}$ be one of the tests $\phi^{(1)+}_{3,\alpha}$ or $\phi^{(2)}_{3/4,\alpha}$ if $\delta^* > 0$, and one of the tests $\phi^{(1)-}_{3,\alpha}$ or $\phi^{(2)}_{3/4,\alpha}$ if $\delta^* < 0$ (see (2.14) and (2.16) for definitions of the tests). The test $\phi^{(1/2)}_{3,\alpha}$ is of level α, that is $P_{\lambda_0}(\phi^{(1/2)}_{3,\alpha}(N) = 1) \leq \alpha$. Moreover, there exists $C(\alpha,\beta,\lambda_0,\delta^*,\ell^*) > 0$ such that $P_\lambda\left(\phi^{(1/2)}_{3,\alpha}(N) = 0\right) \leq \beta$ as soon as λ belongs to $\mathcal{S}_{\delta^*,.,\ell^*}[\lambda_0]$ with

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{C(\alpha,\beta,\lambda_0,\delta^*,\ell^*)}{\sqrt{L}} \ .$$

Comments.
Remarking that for λ in $\mathcal{S}_{\delta^*,\cdot,\ell^*}[\lambda_0]$, $d_2(\lambda, \mathcal{S}_0[\lambda_0]) = |\delta^*|\sqrt{\ell^*}$, Proposition 2.6 leads to exhibit a sufficient minimal value $L_0(\alpha,\beta,\lambda_0,\delta^*,\ell^*)$ for L so that the second kind error rates of the above tests are controlled by β. Anecdotally, it furthermore shows that if $L \geq L_0(\alpha,\beta,\lambda_0,\delta^*,\ell^*)$, the β-uniform separation rate of the above tests over $\mathcal{S}_{\delta^*,\cdot,\ell^*}[\lambda_0]$ is equal to 0, as well as the (α,β)-minimax separation rate $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\cdot,\ell^*}[\lambda_0])$.

Turning now to the change height adaptation issue, the lower bound for $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0])$ given in Proposition 2.3 directly leads, using the monotonicity property of the minimax separation rate recalled in Lemma 1.7, to the following lower bound for $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0])$.

Corollary 2.7 (Minimax lower bound for [**Alt**.4]).
Let α,β in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$ and ℓ^* in $(0,1)$. For $L \geq 1$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \quad \textit{with } C_{\alpha,\beta} = 1+4(1-\alpha-\beta)^2 \ .$$

Proposition 2.8 (Minimax upper bounds for [**Alt**.4]).
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0>0$, and ℓ^* in $(0,1)$. Let $\phi^{(1/2)}_{4,\alpha}$ be one of the tests $\phi^{(1)}_{4,\alpha}$ and $\phi^{(2)}_{3/4,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0]$" defined respectively by (2.15) and (2.16). Then $\phi^{(1/2)}_{4,\alpha}$ is of level α, that is $P_{\lambda_0}(\phi^{(1/2)}_{4,\alpha}(N)=1) \leq \alpha$. Moreover, there exists $C(\alpha,\beta,\lambda_0,\ell^*)>0$ such that

$$\mathrm{SR}_\beta\left(\phi^{(1/2)}_{4,\alpha}, \mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0]\right) \leq \frac{C(\alpha,\beta,\lambda_0,\ell^*)}{\sqrt{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0]\right) \leq C(\alpha,\beta,\lambda_0,\ell^*)/\sqrt{L}$.

Comments.
The proof of Proposition 2.8 mainly relies as the proof of Proposition 2.6 on the quantile controls of lemmas 2.28 and 2.27, respectively deduced from theorems 8 and 6 in [LG21]. This result with Corollary 2.7 means that the tests $\phi^{(1)}_{4,\alpha}$ and $\phi^{(2)}_{3/4,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot,\ell^*}[\lambda_0]$" are β-minimax. Moreover and importantly, regarding the results obtained for [**Alt**.2], Proposition 2.8 with Corollary 2.7 also means that minimax adaptation with respect to the change location can be achieved with a minimax separation rate of the parametric order, hence without any additional price to pay (possibly except multiplicative constants), as soon as the only change length is known. This may contrast with the common idea (maybe spread by results in the jump detection problem where adaptation to the change location is equivalent to adaptation to the change length) that adaptation to the change location is the main cause of an unavoidable logarithmic cost. Here, by considering all the cases separately and step by step, we aim at precisely exhibiting the various regimes of minimax separation rates: this allows us in particular to specify - where relevant - the price to pay for adaptation to the different alternative parameters.

IV Minimax detection of a transitory change with known location

In this section, we consider the problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus alternative hypotheses where the location of the change from the baseline intensity is known, with adaptation with respect to the change length, and with or without adaptation with respect to the height of the change. Contrary to the study of Chapter 2 Section III, while adaptation to the change length only can be done without any incidence on the minimax separation rate order, adaptation to the change height in addition to the change length has a non-negligible impact. We therefore examine these two questions in two separate subsections.

IV.1 Known change height

Let us first investigate the problem of testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]$", where the set $\mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]$ is defined for δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$ and τ^* in $(0,1)$ by

$$[\textbf{Alt.5}] \quad \mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0] = \Big\{\lambda : [0,1] \to (0,+\infty),\ \exists \ell \in (0,1-\tau^*),\ \forall t \in [0,1]\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell]}(t)\Big\}\ . \quad (2.17)$$

As explained above, we will see that the minimax separation rate over this alternative set remains unchanged, of the parametric order $L^{-1/2}$. A lower bound is easily obtained from the key arguments given in Chapter 1 Section II.2. Therefore the major point here is the construction of a minimax adaptive test, which has to take the knowledge of the change height δ^* into account. In order to determine the most relevant way to integrate this knowledge, we have used an exact expression for the probability distribution function as well as an exponential inequality for the supremum of Poisson processes with shift, both due to Pyke [Pyk59, Equation (6) and Theorem 3]. This has led to a new procedure which is rather atypical regarding the other tests of this paper, and which can be related to Brunel's [Bru14] scan test in the Gaussian set-up.

Proposition 2.9 (Minimax lower bound for [**Alt.5**]).
Let α,β in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$, δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$ and τ^* in $(0,1)$. For all $L \geq \lambda_0 \log C_{\alpha,\beta}/(\delta^{*2}(1-\tau^*))$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]\right) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \quad \text{with } C_{\alpha,\beta} = 1+4(1-\alpha-\beta)^2\ .$$

Let us now introduce the aggregated test

$$\phi_{5,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) > s^+_{\lambda_0,\delta^*,\tau^*,L}(1-\alpha)\right\}}\ , \quad (2.18)$$

where $S_{\delta^*,\tau_1,\tau_2}(N)$ is the statistic defined for $0 \leq \tau_1 < \tau_2 \leq 1$ by

$$S_{\delta^*,\tau_1,\tau_2}(N) = \mathrm{sgn}(\delta^*)\Big(N(\tau_1,\tau_2] - \lambda_0 L(\tau_2-\tau_1)\Big) - \frac{|\delta^*|L(\tau_2-\tau_1)}{2}\ , \quad (2.19)$$

and $s^+_{\lambda_0,\delta^*,\tau^*,L}(u)$ is the u-quantile of $\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N)$ under (H_0).

Lemma 2.29 provides a control of the quantile $s^+_{\lambda_0,\delta^*,\tau^*,L}(1-\alpha)$, which is deduced from Pyke's results [Pyk59], and which is the main argument to prove that $\phi_{5,\alpha}$ has an uniform separation rate of parametric order $L^{-1/2}$ and thus show that the lower bound of Proposition 2.9 is sharp.

Proposition 2.10 (Minimax upper bound for [**Alt.5**]).
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0>0$, δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$ and τ^* in $(0,1)$. Let $\phi_{5,\alpha}$ be the test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]$" defined by (2.18). Then $\phi_{5,\alpha}$ is of level α, that is $P_{\lambda_0}(\phi_{5,\alpha}(N)=1) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,\lambda_0,\delta^*)>0$ such that

$$\mathrm{SR}_\beta\left(\phi_{5,\alpha},\mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]\right) \leq \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}\ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\delta^*,\tau^*,\dots}[\lambda_0]\right) \leq C(\alpha,\beta,\lambda_0,\delta^*)/\sqrt{L}$.

IV.2 Unknown change height

Now addressing the question of adaptation to the change height together with the change length, we consider for τ^* in $(0,1)$ the alternative set

$$\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0] = \left\{\lambda : [0,1] \to (0,+\infty),\ \exists\delta \in (-\lambda_0,+\infty)\setminus\{0\},\ \exists\ell \in (0,1-\tau^*),\ \forall t \in [0,1]\ \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}(t)\right\} . \tag{2.20}$$

A first preliminary result in fact shows that this set of alternatives is too large to be relevantly studied from the minimax point of view: the minimax separation rate is infinite over it.

Lemma 2.11.
Let α and β in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$, and τ^* in $(0,1)$. For the problem of testing (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0]$", with $\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0]$ defined by (2.20), one has $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0]) = +\infty$.

This preliminary result leads us to consider, for $R>\lambda_0$, the restricted set of alternatives

$$\begin{aligned}[\mathbf{Alt.6}]\ \mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists\delta \in (-\lambda_0,R-\lambda_0]\setminus\{0\},& \\ \exists\ell \in (0,1-\tau^*),\ \forall t\in[0,1]\quad \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}(t)\Big\}& . \end{aligned} \tag{2.21}$$

For the problem of testing (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]$", we then obtain the following lower bound.

Proposition 2.12 (Minimax lower bound for [**Alt.6**])**.**
Let α,β in $(0,1)$ with $\alpha+\beta<1/2$, $\lambda_0>0$, $R>\lambda_0$, τ^* in $(0,1)$. There exists $L_0(\alpha,\beta,\lambda_0,R)>0$ such that for $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]) \geq \sqrt{\frac{\lambda_0\log\log L}{L}} .$$

Let us now assume that $L\geq 3$. In order to prove that the above lower bound is of sharp order (with respect to L), we construct two aggregated tests: a first one based on a linear statistic and a second one based on quadratic statistic as in Chapter 2 Section III.
We thus consider the discrete subset of $(0,1-\tau^*)$ of the dyadic form

$$\left\{\ell_{\tau^*,k} = (1-\tau^*)2^{-k};\ k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}\right\} ,$$

and the corrected level $u_\alpha = \alpha/\lfloor\log_2(L)\rfloor$, which allow to define the two following tests:

$$\phi^{(1)}_{6,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}]-p_{\lambda_0\ell_{\tau^*,k}L}\left(1-\frac{u_\alpha}{2}\right)\right)>0\right\}} \vee \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(p_{\lambda_0\ell_{\tau^*,k}L}\left(\frac{u_\alpha}{2}\right)-N(\tau^*,\tau^*+\ell_{\tau^*,k}]\right)>0\right\}} , \tag{2.22}$$

where $p_\xi(u)$ stands for the u-quantile of the Poisson distribution of parameter ξ as in (2.3), and

$$\phi^{(2)}_{6,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(T_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N)-t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right)>0\right\}} , \tag{2.23}$$

where $T_{\tau_1,\tau_2}(N)$ is the quadratic statistic (2.10), and $t_{\lambda_0,\tau_1,\tau_2}(u)$ its u-quantile under (H_0).

Proposition 2.13 (Minimax upper bound for [**Alt.6**])**.**
Let α and β in $(0,1)$, $\lambda_0>0$, $R>\lambda_0$ and τ^* in $(0,1)$. Let $\phi^{(1/2)}_{6,\alpha}$ be one of the tests $\phi^{(1)}_{6,\alpha}$ and $\phi^{(2)}_{6,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]$" respectively defined by (2.22) and (2.23). Then $\phi^{(1/2)}_{6,\alpha}$ is of level α, that is $P_{\lambda_0}\left(\phi^{(1/2)}_{6,\alpha}(N)=1\right)\leq\alpha$. Moreover, there exists $C(\alpha,\beta,\lambda_0,R)>0$ such that

$$\mathrm{SR}_\beta\left(\phi^{(1/2)}_{6,\alpha},\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]\right) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\log\log L/L}$.

Comments.
The proofs of these upper bounds are mainly based on a sharp control of the quantiles $p_{\lambda_0 \ell_{\tau^*,k}L}(u)$ derived from the Cramér-Chernov inequality (see Lemma 2.25), and of the quantile $t_{\lambda_0,\tau_1,\tau_2}(u)$ already used in the proof of Proposition 2.6. This result, combined with its corresponding lower bound, brings out a first phase transition in the minimax separation rates orders, from the parametric rate order $1/\sqrt{L}$ to $\sqrt{\log\log L/L}$. This means that adaptation with respect to both height and length of the bump has a $\sqrt{\log\log L}$ cost, while adaptation to only one of these parameters does not cause any additional price, nor adaptation to both height and location as noticed above. Though a comparable phase transition has already been observed in Gaussian models when dealing with the jump detection problem (where adaptation with respect to the location is equivalent to adaptation with respect to the length), up to our knowledge, such results did not appear yet in the bump detection literature.

V Minimax detection of a possibly transitory change with unknown location and length

In this section, we address the final problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus alternative hypotheses where both location and length of the change from the baseline intensity are unknown, distinguishing the case where the change is transitory from the particular case where it is not transitory.

Still adopting the minimax point of view, we will see that when considering the transitory change detection problem, adaptation to both change location and length has a minimax separation rate cost of order $\sqrt{\log L}$, and this whether the change height is known or not. This highly contrasts with the study of the particular non transitory change or jump detection problem, which makes two different regimes of minimax separation rates appear, with a maximal cost of order $\sqrt{\log\log L}$ for change height adaptation.

Let us underline that the non transitory change or jump detection problem can be viewed as perfectly symmetrical to the above transitory change with known location detection problem (see Chapter 2 Section IV). In the first problem, one can consider that the length of the change is unknown but the endpoint of the change is known, while in the second problem the length of the change is unknown but the starting point of the change is known. The study of the first non transitory change detection problem will therefore use very similar arguments as the study of the second transitory change with known location detection problem, finally leading to the same minimax separation rates. This is why we conduct it first here.

V.1 Non transitory change

In order to investigate the problem of detecting a non transitory change or jump with unknown location, but known height, we introduce for δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ the alternative set

$$[\textbf{Alt.7}] \quad \mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0] = \{\lambda : [0,1] \to (0+\infty),\ \exists \tau \in (0,1),\ \forall t \in [0,1]\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}(t)\} \ . \tag{2.24}$$

This allows us to formalise the considered detection problem as a problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus the alternative (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]$", with the corresponding minimax lower bound stated below.

Proposition 2.14 (Minimax lower bound for [**Alt.7**]).
Let α and β in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. For all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \ \ \textit{with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

Following the study and the notation of Chapter 2 Section IV, we define the test

$$\phi_{7,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N) > s^+_{\lambda_0,\delta^*,L}(1-\alpha)\right\}} \ , \tag{2.25}$$

where $S_{\delta^*,\tau_1,\tau_2}(N)$ is the statistic defined by (2.19) and $s^+_{\lambda_0,\delta^*,L}(u)$ stands for the u-quantile of $\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N)$ under (H_0).

Proposition 2.15 (Minimax upper bound for [**Alt.7**]).
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$. Let $\phi_{7,\alpha}$ be the test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]$" defined by (2.25). Then $\phi_{7,\alpha}$ is of level α, that is $P_{\lambda_0}(\phi_{7,\alpha}(N) = 1) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,\lambda_0,\delta^*) > 0$ such that

$$\mathrm{SR}_\beta(\phi_{7,\alpha}, \mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]) \leq \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]) \leq C(\alpha,\beta,\lambda_0,\delta^*)/\sqrt{L}$.

Let us now tackle the question of adaptation with respect to the change height and therefore introduce to this end a preliminary alternative set

$$\mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0] = \{\lambda : \exists \delta \in (-\lambda_0, +\infty) \setminus \{0\}, \ \exists \tau \in (0,1), \ \forall t \in [0,1] \ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\} \ . \tag{2.26}$$

As in Chapter 2 Section IV, we underline that the minimax separation rate over this set is infinite.

Lemma 2.16.
Let α,β in $(0,1)$ such that $\alpha+\beta < 1$. For the problem of testing (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0]$", with $\mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0]$ defined by (2.26), one has $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0]) = +\infty$.

We thus consider for $R > \lambda_0$ the more suitable set of alternatives bounded by R, defined by

$$[\mathbf{Alt.8}]\ \mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0,R] = \left\{\lambda : [0,1] \to (0,R], \ \exists \delta \in (-\lambda_0, R-\lambda_0]\setminus\{0\}, \ \exists \tau \in (0,1), \ \forall t \in [0,1] \ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\right\} \ . \tag{2.27}$$

Considering the problem of testing (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0,R]$", we then obtain the following lower bound.

Proposition 2.17 (Minimax lower bound for [**Alt.8**]).
Let α,β in $(0,1)$ with $\alpha+\beta < 1/2$, $\lambda_0 > 0$ and $R > \lambda_0$. There exists $L_0(\alpha,\beta,\lambda_0,R) > 0$ such that for $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot,1-\cdot}[\lambda_0,R]) \geq \sqrt{\frac{\lambda_0 \log\log L}{L}} \ .$$

Again, following the study and the notation of Chapter 2 Section IV, we assume now that $L \geq 3$, we consider the discrete subset of $(0,1)$ of the dyadic form

$$\left\{\tau_k = 1 - 2^{-k}; \ k \in \{1, \ldots, \lfloor \log_2 L \rfloor\}\right\} \ ,$$

and we set $u_\alpha = \alpha/\lfloor \log_2(L) \rfloor$, which allows to define the two following tests:

$$\phi^{(1)}_{8,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(N(\tau_k,1] - p_{\lambda_0(1-\tau_k)L}\left(1-\frac{u_\alpha}{2}\right)\right)>0\right\}} \vee \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(p_{\lambda_0(1-\tau_k)L}\left(\frac{u_\alpha}{2}\right) - N(\tau_k,1]\right)>0\right\}} \ , \tag{2.28}$$

where $p_\xi(u)$ stands for the u-quantile of the Poisson distribution of parameter ξ, and

$$\phi^{(2)}_{8,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(T_{\tau_k,1}(N) - t_{\lambda_0,\tau_k,1}(1-u_\alpha)\right)>0\right\}} \ , \tag{2.29}$$

where $T_{\tau_1,\tau_2}(N)$ is the quadratic statistic (2.10) and $t_{\lambda_0,\tau_1,\tau_2}(u)$ its u-quantile under (H_0).

Proposition 2.18 (Minimax upper bound for [**Alt.8**]).
Let α and β in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. Let $\phi_{8,\alpha}^{(1/2)}$ be one of the tests $\phi_{8,\alpha}^{(1)}$ and $\phi_{8,\alpha}^{(2)}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot\cdot,1-\cdot\cdot}[\lambda_0,R]$" respectively defined by (2.28) and (2.29). Then $\phi_{8,\alpha}^{(1/2)}$ is of level α, that is $P_{\lambda_0}\left(\phi_{8,\alpha}^{(1/2)}(N)=1\right) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,\lambda_0,R) > 0$ such that

$$\mathrm{SR}_\beta\left(\phi_{8,\alpha}^{(1/2)}, \mathcal{S}_{\cdot,\cdot\cdot,1-\cdot\cdot}[\lambda_0,R]\right) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot\cdot,1-\cdot\cdot}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\log\log L/L}$.

V.2 Transitory change

In this subsection, we address the transitory change detection problem, focusing here on the question of adaptation to unknown location and length. As explained above, we will see that minimax adaptation to these both parameters has the most important cost in the present study, whose order is as large as $\sqrt{\log L}$, so that adaptation to the height will have no additional cost.

Let us first give lower bounds for the minimax separation rates, focusing on the case where the change height is known since the general case where all three parameters, location, length and height of the change are unknown then follows easily.
Hence, we introduce for δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$ the alternative set

$$[\textbf{Alt.9}]\ \mathcal{S}_{\delta^*,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0] = \Big\{\lambda : [0,1] \to (0,+\infty),\ \exists\tau\in(0,1),\ \exists\ell\in(0,1-\tau),\ \forall t\in[0,1]\ \lambda(t) = \lambda_0 + \delta^*\mathbb{1}_{(\tau,\tau+\ell]}(t)\Big\}\ . \tag{2.30}$$

Proposition 2.19 (Minimax lower bound for [**Alt.9**]).
Let α,β in $(0,1)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. There exists $L_0(\alpha,\beta,\lambda_0,\delta^*) > 0$ such that for all $L \geq L_0(\alpha,\beta,\lambda_0,\delta^*)$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]) \geq \sqrt{\frac{\lambda_0\log L}{2L}}\ .$$

Now considering the very general alternative set

$$\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot}[\lambda_0] = \Big\{\lambda : [0,1] \to (0,+\infty),\ \exists\delta\in(-\lambda_0,+\infty)\setminus\{0\},\ \exists\tau\in(0,1),$$
$$\exists\ell\in(0,1-\tau),\ \forall t\in[0,1]\quad \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau,\tau+\ell]}(t)\Big\}\ , \tag{2.31}$$

since it contains $\mathcal{S}_{\cdot,\tau^*,\cdot\cdot\cdot}[\lambda_0]$ defined by (2.20) for any τ^* in $(0,1)$, Lemma 2.11 straightforwardly leads to an infinite minimax separation rate lower bound.

Corollary 2.20.
Let α,β in $(0,1)$ such that $\alpha+\beta<1$. For the problem of testing (H_0) "$\lambda\in\mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda \in \mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$", with $\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$ defined by (2.31), one has $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]) = +\infty$.

We therefore restrict the alternative set to the one defined for $R \geq \lambda_0$ by

$$[\textbf{Alt.10}]\ \mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0,R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists\delta\in(-\lambda_0,R-\lambda_0]\setminus\{0\},\ \exists\tau\in(0,1),\ \exists\ell\in(0,1-\tau),$$
$$\forall t\in[0,1]\quad \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau,\tau+\ell]}(t)\Big\}\ , \tag{2.32}$$

and deal with the problem of testing (H_0) "$\lambda\in\mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus (H_1) "$\lambda\in\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0,R]$". This alternative set $\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0,R]$ includes $\mathcal{S}_{R-\lambda_0,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$ defined by (2.30) when $R>\lambda_0$, and $\mathcal{S}_{-\lambda_0/2,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$ when $\lambda_0 = R$. Therefore, Proposition 2.19 has the following direct corollary, whose proof as well as the proof of Corollary 2.20 is omitted for simplicity.

Corollary 2.21 (Minimax lower bound for [**Alt10**]).
Let α,β in $(0,1)$ with $\alpha+\beta<1$, $\lambda_0>0$ and $R\geq\lambda_0$. There exists $L_0(\alpha,\beta,\lambda_0,R)>0$ such that for all $L\geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,R]\right)\geq\sqrt{\frac{\lambda_0\log L}{2L}}\ .$$

In order to prove that the above lower bounds are sharp, we secondly construct two novel minimax adaptive tests according to a scanning aggregation principle again. Since the lower bound shows an additional cost for adaptation to change location and length of order $\sqrt{\log L}$ instead of at most $\sqrt{\log\log L}$ when dealing with adaptation to only one of these parameters, we do not necessarily need to consider a dyadic set of aggregated tests. More precisely, setting $u_\alpha=2\alpha/(\lceil L\rceil(\lceil L\rceil+1))$, we define

$$\phi^{(1)}_{9/10,\alpha}(N)=\mathbb{1}_{\left\{\max_{k\in\{0,\dots,\lceil L\rceil-1\},k'\in\{1,\dots,\lceil L\rceil-k\}}\left(N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]-p_{\lambda_0 k'L/\lceil L\rceil}\left(1-\frac{u_\alpha}{2}\right)\right)>0\right\}}$$
$$\vee\,\mathbb{1}_{\left\{\max_{k\in\{0,\dots,\lceil L\rceil-1\},k'\in\{1,\dots,\lceil L\rceil-k\}}\left(p_{\lambda_0 k'L/\lceil L\rceil}\left(\frac{u_\alpha}{2}\right)-N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]\right)>0\right\}}\ ,\qquad(2.33)$$

and

$$\phi^{(2)}_{9/10,\alpha}(N)=\mathbb{1}_{\left\{\max_{k\in\{0,\dots,\lceil L\rceil-1\},k'\in\{1,\dots,\lceil L\rceil-k\}}\left(T_{\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(N)-t_{\lambda_0,\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(1-u_\alpha)\right)>0\right\}}\ .\qquad(2.34)$$

Proposition 2.22 (Minimax upper bound for [**Alt.10**]).
Let α,β in $(0,1)$, $\lambda_0>0$ and $R\geq\lambda_0$. Let $\phi^{(1/2)}_{9/10,\alpha}$ be one of the tests $\phi^{(1)}_{9/10,\alpha}$ and $\phi^{(2)}_{9/10,\alpha}$ of (H_0) versus (H_1) "$\lambda\in\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,R]$" respectively defined by (2.33) and (2.34). Then $\phi^{(1/2)}_{9/10,\alpha}$ is of level α, that is $P_{\lambda_0}\left(\phi^{(1/2)}_{9/10,\alpha}(N)=1\right)\leq\alpha$. Moreover, there exists a constant $C(\alpha,\beta,\lambda_0,R)>0$ such that

$$\mathrm{SR}_\beta\left(\phi^{(1/2)}_{9/10,\alpha},\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,R]\right)\leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log L}{L}}\ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,R]\right)\leq C(\alpha,\beta,\lambda_0,R)\sqrt{\log L/L}$.

As in particular $\mathcal{S}_{\delta^*,\cdot\cdot,\cdots}[\lambda_0]$ is included in $\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,\lambda_0+\delta^*]$ for any $\delta^*>0$ and $\mathcal{S}_{\cdot,\cdot\cdot,\cdots}[\lambda_0,\lambda_0]$ for any δ^* in $(-\lambda_0,0)$, Proposition 2.22 has the following immediate corollary, which closes the study of possibly transitory change in a known baseline intensity detection.

Corollary 2.23 (Minimax upper bound for [**Alt.9**]).
Let α,β in $(0,1)$, $\lambda_0>0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. Let $\phi^{(1/2)}_{9/10,\alpha}$ be one of the tests $\phi^{(1)}_{9/10,\alpha}$ and $\phi^{(2)}_{9/10,\alpha}$ of (H_0) versus (H_1) "$\lambda\in\mathcal{S}_{\delta^*,\cdot\cdot,\cdots}[\lambda_0]$" respectively defined by (2.33) and (2.34). Then $\phi^{(1/2)}_{9/10,\alpha}$ is of level α, that is $P_{\lambda_0}\left(\phi^{(1/2)}_{9/10,\alpha}(N)=1\right)\leq\alpha$. Moreover, there exists $C(\alpha,\beta,\lambda_0,\delta^*)>0$ such that

$$\mathrm{SR}_\beta\left(\phi^{(1/2)}_{9/10,\alpha},\mathcal{S}_{\delta^*,\cdot\cdot,\cdots}[\lambda_0]\right)\leq C(\alpha,\beta,\lambda_0,\delta^*)\sqrt{\frac{\log L}{L}}\ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}_{\delta^*,\cdot\cdot,\cdots}[\lambda_0]\right)\leq C(\alpha,\beta,\lambda_0,\delta^*)\sqrt{\log L/L}$.

Comment.
The upper bounds in Proposition 2.22 and Corollary 2.23, combined with their corresponding lower bounds, bring out a second phase transition in the minimax separation rate orders, from the rate order

$\sqrt{\log\log L/L}$ when considering adaptation with respect to both bump height and length to $\sqrt{\log L/L}$, obtained when dealing with adaptation to at least bump location and length (with no additional cost when adapting to the bump height). As comparable minimax separation rates were already known in Gaussian models with [ACDH05] and [Bru14], these results were more expected that some of the above ones.

VI Choice of individual levels for aggregated tests and links with multiple tests

Except the tests $\phi^-_{1,\alpha}$, $\phi^+_{1,\alpha}$, $\phi^{(1)}_{2,\alpha}$ and $\phi^{(2)}_{2,\alpha}$ that are classical single tests, all the tests introduced in the above study are based on aggregation principles. Among them, we can essentially distinguish two kinds of such aggregated tests.

The first aggregated test type is of the form

$$\phi_{agg1,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\theta\in\Theta} S_\theta(N) > s^+_{\lambda_0}(1-\alpha)\right\}} \text{ or } \mathbb{1}_{\left\{\sup_{\theta\in\Theta} S_\theta(N) > s^+_{\lambda_0}(1-\alpha/2)\right\}} \vee \mathbb{1}_{\left\{\inf_{\theta\in\Theta} S_\theta(N) < s^-_{\lambda_0}(\alpha/2)\right\}},$$

where:

- θ is one possible parameter or couple of parameters among the location τ or length ℓ of the bump/jump in the alternative intensity, and Θ is a subset of possible values for θ,
- S_θ is a statistic designed to test (H_0) "$\lambda = \lambda_0$" versus (H_1) "λ has a jump or a bump with parameter or parameters θ", such that $\sup_{\theta\in\Theta} S_\theta(N)$ and $\inf_{\theta\in\Theta} S_\theta(N)$ have computable, exactly or by a Monte Carlo method, u-quantiles $s^+_{\lambda_0}(u)$ and $s^-_{\lambda_0}(u)$ under (H_0).

The tests $\phi^{(1)-}_{3,\alpha}$, $\phi^{(1)+}_{3,\alpha}$, $\phi^{(1)}_{4,\alpha}$, $\phi_{5,\alpha}$ and $\phi_{7,\alpha}$ can all be written in this way.

The second aggregated test type is of the form

$$\phi_{agg2,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(S_\theta(N) - s_{\lambda_0,\theta}(1-u_\alpha)\right)>0\right\}} \text{ or } \mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(S_\theta(N) - s_{\lambda_0,\theta}(1-u_\alpha/2)\right)>0\right\}} \vee \mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(s_{\lambda_0,\theta}(u_\alpha/2) - S_\theta(N)\right)>0\right\}},$$

where:

- θ is one possible parameter or couple of parameters among the location τ or length ℓ of the bump/jump in the alternative intensity, and Θ is a finite subset of possible values for θ,
- S_θ is a statistic designed to test (H_0) "$\lambda = \lambda_0$" versus (H_1) "λ has a jump or a bump with parameter or parameters θ", with computable $(1-u)$-quantiles $s_{\lambda_0,\theta}(1-u)$ under (H_0),
- u_α is an individual, adjusted and smaller than α, level of test.

The tests $\phi^{(2)}_{3/4,\alpha}$, $\phi^{(1)}_{6,\alpha}$, $\phi^{(2)}_{6,\alpha}$, $\phi^{(1)}_{8,\alpha}$, $\phi^{(2)}_{8,\alpha}$, $\phi^{(1)}_{9/10,\alpha}$, $\phi^{(2)}_{9/10,\alpha}$ can all be written in this way.

Both $\phi_{agg1,\alpha}$ and $\phi_{agg2,\alpha}$ aggregated test types can thus be expressed as

$$\mathbb{1}_{\left\{\exists\theta\in\Theta,\ S_\theta(N) > c_{\lambda_0,\theta,\alpha}\right\}} \text{ or } \mathbb{1}_{\left\{\exists\theta\in\Theta,\ S_\theta(N) > c^1_{\lambda_0,\theta,\alpha} \text{ or } S_\theta(N) < c^2_{\lambda_0,\theta,\alpha}\right\}},$$

where $c_{\lambda_0,\theta,\alpha} = s^+_{\lambda_0}(1-\alpha)$, $c^1_{\lambda_0,\theta,\alpha} = s^+_{\lambda_0}(1-\alpha/2)$ and $c^2_{\lambda_0,\theta,\alpha} = s^-_{\lambda_0}(\alpha/2)$ do not vary with θ in $\phi_{agg1,\alpha}$ and $c_{\lambda_0,\theta,\alpha} = s_{\lambda_0,\theta}(1-u_\alpha)$, $c^1_{\lambda_0,\theta,\alpha} = s_{\lambda_0,\theta}(1-u_\alpha/2)$ and $c^2_{\lambda_0,\theta,\alpha} = s_{\lambda_0,\theta}(u_\alpha/2)$ vary with θ in $\phi_{agg2,\alpha}$. This therefore means that these tests reject (H_0) when, scanning all the parameters or couples of parameters θ in Θ, at least one single test in the collection $\left\{\mathbb{1}_{S_\theta(N)>c_{\lambda_0,\theta,\alpha}},\ \theta\in\Theta\right\}$ or $\left\{\mathbb{1}_{S_\theta(N)>c^1_{\lambda_0,\theta,\alpha}} \vee \mathbb{1}_{S_\theta(N)<c^2_{\lambda_0,\theta,\alpha}},\ \theta\in\Theta\right\}$ rejects (H_0), which explains the name of scan aggregation principle. All the single tests $\mathbb{1}_{S_\theta(N)>c_{\lambda_0,\theta,\alpha}}$ or $\mathbb{1}_{S_\theta(N)>c^1_{\lambda_0,\theta,\alpha}} \vee \mathbb{1}_{S_\theta(N)<c^2_{\lambda_0,\theta,\alpha}}$ in the considered collections are of level α, but they can be in fact, individually,

very conservative, otherwise their aggregation would not preserve the level α property *in fine*. In the particular case of $\phi_{agg2,\alpha}$, the single tests are of individual level u_α, taken here equal to $u_\alpha = \alpha/|\Theta|$.

A better choice for u_α, leading to a less conservative aggregated test, was first proposed in another context by Baraud et al. [BHL05]. In our context, this choice corresponds to u'_α equal to

$$u'_\alpha = \begin{cases} \sup\left\{u \in (0,1),\ P_{\lambda_0}\left(\sup_{\theta\in\Theta}\left(S_\theta(N) - s_{\lambda_0,\theta}(1-u)\right) > 0\right) \leq \alpha\right\} \text{ or} \\ \sup\Big\{u \in (0,1), \\ \qquad P_{\lambda_0}\left(\sup_{\theta\in\Theta}\left(\left(S_\theta(N) - s_{\lambda_0,\theta}(1-u/2)\right) \vee \left(s_{\lambda_0,\theta}(u/2) - S_\theta(N)\right)\right) > 0\right) \leq \alpha\Big\}\ . \end{cases} \tag{2.35}$$

Since $u_\alpha \leq u'_\alpha$, by definition, $s_{\lambda_0,\theta}(1-u'_\alpha) \leq s_{\lambda_0,\theta}(1-u_\alpha)$, $s_{\lambda_0,\theta}(1-u'_\alpha/2) \leq s_{\lambda_0,\theta}(1-u_\alpha/2)$ and $s_{\lambda_0,\theta}(u_\alpha/2) \leq s_{\lambda_0,\theta}(u'_\alpha/2)$. Any upper bound for $s_{\lambda_0,\theta}(1-u_\alpha)$ or $s_{\lambda_0,\theta}(1-u_\alpha/2)$, or lower bound for $s_{\lambda_0,\theta}(u_\alpha/2)$, such as those used in the proofs of the minimax separation rates upper bounds and deduced from the quantiles bounds of Chapter 2 Section X, therefore remain valid for $s_{\lambda_0,\theta}(1-u'_\alpha)$, $s_{\lambda_0,\theta}(1-u'_\alpha/2)$ or $s_{\lambda_0,\theta}(u'_\alpha/2)$ respectively. As a consequence, all the above tests of type $\phi_{agg2,\alpha}$ but with u'_α instead of u_α, that we can denote by $\phi'_{agg2,\alpha}$, satisfy the same minimax properties as $\phi_{agg2,\alpha}$.

The fact that such adjusted aggregated tests $\phi'_{agg2,\alpha}$ are more powerful than $\phi_{agg2,\alpha}$ is not discernable in minimax results whereas it is clearly noticeable in practice, is a known shortcoming of the present nonasymptotic minimax point of view, where exact constants (making lower and upper bound match, up to a possible negligible term) are not expected, which is not solved yet up to our knowledge in any testing framework. Our simulation study presented in Chapter 2 Section VIII focuses on the performances of adjusted aggregated tests of the form $\phi'_{agg2,\alpha}$.

Let us now turn to the links that can be highlighted between such minimax adaptive, aggregated tests $\phi_{agg2,\alpha}$ or adjusted aggregated tests $\phi'_{agg2,\alpha}$ and multiple tests. The parallel between such aggregated tests and multiple tests has been established in [FLRB16], as the foundation of a minimax theory for multiple tests. Notice that when each single test $\mathbb{1}_{S_\theta(N)>c_{\lambda_0,\theta,\alpha}}$ in the collection $\left\{\mathbb{1}_{S_\theta(N)>c_{\lambda_0,\theta,\alpha}},\ \theta\in\Theta\right\}$ can be interpreted as a test of a null hypothesis $(H_{0,\theta})$ "$\lambda \in \mathcal{S}_{0,\theta}$" versus $(H_{1,\theta})$ "$\lambda \notin \mathcal{S}_{0,\theta}$", with $\mathcal{S}_0 \subset \cap_{\theta\in\Theta}\mathcal{S}_{0,\theta}$, it appears that our first choice of individual level $u_\alpha = \alpha/|\Theta|$ can be related to a Bonferroni type multiple test of the set of hypotheses $\{(H_{0,\theta}),\ \theta\in\Theta\}$, while our second choice u'_α defined by (2.35) can be related to a min-p type multiple test of the same set of hypotheses. As an example, recall that $T_{\tau_1,\tau_2}(N)$ defined by (2.10) is an unbiased estimator of $\|\Pi_{V_{\tau_1,\tau_2}}(\lambda-\lambda_0)\|_2^2$ (see Chapter 2 Section II). Therefore, the single tests $\mathbb{1}_{T_{k/\lceil L\rceil,(k+k')/\lceil L\rceil}(N)>t_{\lambda_0,k/\lceil L\rceil,(k+k')/\lceil L\rceil}(1-u)}$ involved in $\phi^{(2)}_{9/10,\alpha}(N)$ can be viewed as single tests of $(H_{0,(k/\lceil L\rceil,k'/\lceil L\rceil)})$ "$\Pi_{V_{k/\lceil L\rceil,(k+k')/\lceil L\rceil}}(\lambda-\lambda_0) = 0$" versus $(H_{1,(k/\lceil L\rceil,k'/\lceil L\rceil)})$ "$\Pi_{V_{k/\lceil L\rceil,(k+k')/\lceil L\rceil}}(\lambda-\lambda_0) \neq 0$". Our aggregated tests of the form $\phi_{agg2,\alpha}$ are thus clearly aggregated tests constructed from Bonferroni multiple tests, while the corresponding adjusted aggregated tests of the form $\phi'_{agg2,\alpha}$ are constructed from min-p multiple tests of such collections of hypotheses (see [FLRB16] for some detailed study). To conclude, notice that a control of the type I error rate of such aggregated tests ensures directly a control of the FWER for the corresponding multiple tests.

VII Summary and discussion

We present below a summary of the results stated above in a tabular form. Recall (*c.f.* (2.10) and (2.19)) that for $0 \le \tau_1 < \tau_2 \le 1$,

$$T_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)}\left(N(\tau_1,\tau_2]^2 - (1+2\lambda_0 L(\tau_2-\tau_1))N(\tau_1,\tau_2] + \lambda_0^2 L^2(\tau_2-\tau_1)^2\right) ,$$

$S_{\delta^*,\tau_1,\tau_2}(N) = \mathrm{sgn}(\delta^*)(N(\tau_1,\tau_2] - \lambda_0 L(\tau_2-\tau_1)) - |\delta^*|L(\tau_2-\tau_1)/2$, and that $p_\xi(u)$ and $t_{\lambda_0,\tau_1,\tau_2}(u)$ stand for the u-quantiles of the Poisson distribution of parameter ξ and the distribution of $T_{\tau_1,\tau_2}(N)$ under (H_0) respectively.

Non transitory change or jump detection		
Alternative set	$\mathrm{mSR}_{\alpha,\beta}$	Test statistics
$\mathcal{S}_{\delta^*,\tau^*,1-\tau^*}[\lambda_0]$	-	$N(\tau^*,1]$
$\mathcal{S}_{\cdot,\tau^*,1-\tau^*}[\lambda_0]$	$L^{-1/2}$	$N(\tau^*,1]$
		$T_{\tau^*,1}(N)$
$\mathcal{S}_{\delta^*,\cdot\cdot,1-\cdot\cdot}[\lambda_0]$	$L^{-1/2}$	$\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N)$
$\mathcal{S}_{\cdot,\cdot\cdot,1-\cdot\cdot}[\lambda_0,R]$	$\sqrt{\frac{\log\log L}{L}}$	$\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(N(1-2^{-k},1] - p_{\lambda_0 2^{-k}L}(1-u_\alpha/2)\right)$
		$\vee\left(p_{\lambda_0 2^{-k}L}(u_\alpha/2) - N(1-2^{-k},1]\right)$
		$\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(T_{1-2^{-k},1}(N) - t_{\lambda_0,1-2^{-k},1}(1-u_\alpha)\right)$
Transitory change or bump detection		
Alternative set	$\mathrm{mSR}_{\alpha,\beta}$	Test statistics
$\mathcal{S}_{\delta^*,\tau^*,\ell^*}[\lambda_0]$	-	$N(\tau^*,\tau^*+\ell^*]$
$\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$	$L^{-1/2}$	$N(\tau^*,\tau^*+\ell^*]$
		$T_{\tau^*,\tau^*+\ell^*}(N)$
$\mathcal{S}_{\delta^*,\cdot\cdot,\ell^*}[\lambda_0]$	-	$\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$, $\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$
		$\max_{k\in\{0,\ldots,\lceil(1-\ell^*)\lceil 2/\ell^*\rceil\rceil-1\}}\left(T_{\frac{k}{\lceil 2/\ell^*\rceil},\frac{k}{\lceil 2/\ell^*\rceil}+\ell^*}(N) - t_{\lambda_0,\frac{k}{\lceil 2/\ell^*\rceil},\frac{k}{\lceil 2/\ell^*\rceil}+\ell^*}(1-u_\alpha)\right)$
$\mathcal{S}_{\cdot,\cdot\cdot,\ell^*}[\lambda_0]$	$L^{-1/2}$	$\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$, $\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$
		$\max_{k\in\{0,\ldots,\lceil(1-\ell^*)\lceil 2/\ell^*\rceil\rceil-1\}}\left(T_{\frac{k}{\lceil 2/\ell^*\rceil},\frac{k}{\lceil 2/\ell^*\rceil}+\ell^*}(N) - t_{\lambda_0,\frac{k}{\lceil 2/\ell^*\rceil},\frac{k}{\lceil 2/\ell^*\rceil}+\ell^*}(1-u_\alpha)\right)$
$\mathcal{S}_{\delta^*,\tau^*,\cdot\cdot\cdot}[\lambda_0]$	$L^{-1/2}$	$\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N)$
$\mathcal{S}_{\cdot,\tau^*,\cdot\cdot\cdot}[\lambda_0,R]$	$\sqrt{\frac{\log\log L}{L}}$	$\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(N(\tau^*,\tau^*+(1-\tau^*)2^{-k}] - p_{\lambda_0(1-\tau^*)2^{-k}L}(1-u_\alpha/2)\right)$
		$\vee\left(p_{\lambda_0(1-\tau^*)2^{-k}L}(u_\alpha/2) - N(\tau^*,\tau^*+(1-\tau^*)2^{-k}]\right)$
		$\max_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(T_{\tau^*,\tau^*+(1-\tau^*)2^{-k}}(N) - t_{\lambda_0,\tau^*,\tau^*+(1-\tau^*)2^{-k}}(1-u_\alpha)\right)$
$\mathcal{S}_{\delta^*,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$	$\sqrt{\frac{\log L}{L}}$	$\max_{k\in\{0,\ldots,\lceil L\rceil-1\},k'\in\{1,\ldots,\lceil L\rceil-k\}}\left(N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right] - p_{\lambda_0 k'L/\lceil L\rceil}\left(1-\frac{u_\alpha}{2}\right)\right)$
$\mathcal{S}_{\cdot,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0,R]$		$\vee\left(p_{\lambda_0 k'L/\lceil L\rceil}\left(\frac{u_\alpha}{2}\right) - N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]\right)$
		$\max_{k\in\{0,\ldots,\lceil L\rceil-1\},k'\in\{1,\ldots,\lceil L\rceil-k\}}\left(T_{\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(N) - t_{\lambda_0,\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(1-u_\alpha)\right)$

The present overview notably enables to highlight two main phase transitions in minimax separation rates. A phase transition from the smallest parametric rate order $1/\sqrt{L}$ to the intermediate rate order $\sqrt{\log\log L/L}$, due to adaptation to both height and length of the bump when dealing with the bump detection problem (BDP), or both height and location of the jump (which is in fact equivalent to adaptation to the bump length here) when dealing with the jump detection problem (JDP). A similar phase transition was already known in the independent Gaussian model when dealing with the JDP as explained in the introduction (see [GHZ20] and [VFLRB21]). But the tools used in this Gaussian model, mainly based on Law of Iterated Logarithm exponential inequalities could not be used here, which led us to circumvent the difficulty via new exponential inequalities of Le Guével [LG21] combined with a dyadic type scan aggregation approach. Two points which seem important to us here are: first, in the BDP, adaptation to both bump height and location can be conducted without any additional cost as soon as the bump length is known; second, when adaptation to the length in the BDP or the location in the JDP is considered, the knowledge of the bump or jump height suffices to cancel any price to pay for adaptation. Up to our knowledge, such results were not known, even in classical Gaussian models. Constructing minimax adaptive tests actually required a careful analysis of the shifted Poisson process. Then, a phase transition from the intermediate rate order $\sqrt{\log\log L/L}$ to the largest rate order $\sqrt{\log L/L}$, due to adaptation to both position and length of the bump when dealing with the BDP. Notice that this rate is so large that additional adaptation to the height has no supplementary cost. Notice also that similar minimax separation rates were already known in the independent Gaussian model: Arias-Castro et al. [ACDH05] handled the case where the height is unknown, while Brunel [Bru14] handled the case where the height is known, equal to 1 (therefore positive) within the asymptotic perspective, with linear statistics in the spirit of well-known CUSUM statistics. From this angle, our study provides nonasymptotic and Poisson processes counterparts for the Gaussian tools used in [ACDH05] and [Bru14]. But we furthermore introduce, in the unknown height case, a novel scan aggregated quadratic statistic, whose interest is discussed in Chapter 2 Section II.

VIII Simulation study

We study in this section the performance of our minimax adaptive tests from an experimental point of view, by giving estimations of their size and their power for various distributions of the observed Poisson process, characterised by a jump or a bump in its intensity. Motivated by some applications in epidemiology and in cybersecurity, we check the feasibility of our new change-points detection procedures in practice and compare them with existing procedures.

We focus here on the most general problems investigated in this chapter of detecting a change (a jump or a bump) in the intensity λ when the change location and height are unknown. The known baseline intensity of λ, denoted by λ_0, is taken equal to 1 on $[0,1]$ in all the sequel. For several piecewise constant intensities λ with respect to the measure $d\Lambda(t) = Ldt$, where we have chosen $L = 50$, we take a level of test $\alpha = 0,05$.

We compare the estimated powers of our procedures with more classical conditional tests previously studied in practice by other authors. For instance Cohen and Sackrowitz [CS93], and Bain, Engelhardt and Wright [BEW85] considered six well-known tests in the context of detecting increasing intensities of a Poisson process. They showed that two of these six tests, namely the so-called Laplace and Z tests (respectively studied first by Cox [Cox55] and Crow [Cro74]) are more efficient from a practical point of view.
The Laplace test, denoted by (La) is based on the statistic $\sum_{i=1}^{N_1} X_i - q_{N_1}^{(La)}(1-\alpha)$, where $(X_1,\ldots,X_{N_1})$ are the points of the Poisson process N, and for all n in $\mathbb{N}$, $q_n^{(La)}(1-\alpha)$ is the $(1-\alpha)$-quantile of the sum of n independent random variables uniformly distributed on $[0,1]$. The Z test, denoted by (Z), is based on the statistic $2\sum_{i=1}^{N_1}\log(X_i)+q_{N_1}^{(Z)}(\alpha)$, where for every n in $\mathbb{N}$, $q_n^{(Z)}(\alpha)$ is the α-quantile of the chi-square distribution with $2n$ degrees of freedom. Note that these tests were especially designed to test homogeneity versus an increasing trend, with rejection of homogeneity when they take positive values. Therefore, we have had to adapt them to fit our context. More precisely, we decided to reject the null hypotheses

(H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" or (H_0) "$\lambda \in \mathcal{S}_0^u[R]$" when $\mathcal{T}_\alpha^{(La)}(N) > 0$, where

$$\mathcal{T}_\alpha^{(La)}(N) = \left(\sum_{i=1}^{N_1} X_i - q_{N_1}^{(La)}(1-\alpha/2)\right) \vee \left(q_{N_1}^{(La)}(\alpha/2) - \sum_{i=1}^{N_1} X_i\right) , \tag{2.36}$$

for the Laplace test (La), and when $\mathcal{T}_\alpha^{(Z)}(N) > 0$, where

$$\mathcal{T}_\alpha^{(Z)}(N) = \left(2\sum_{i=1}^{N_1} \log(X_i) + q_{N_1}^{(Z)}(\alpha/2)\right) \vee \left(-2\sum_{i=1}^{N_1} \log(X_i) - q_{N_1}^{(Z)}(1-\alpha/2)\right) , \tag{2.37}$$

for the Z test (Z). Notice that a generalised version of the Laplace and the Z tests have been studied by Peña [Peñ98] and Agustin and Peña [AP99]. A simulation study has been performed for these generalised procedures, but we have found that they have very similar estimated powers to the more classical Laplace and Z tests for the considered alternatives. The results are therefore omitted in this section.

The minimax adaptive tests we introduced to detect a change with unknown parameters from such a known intensity are based on two kinds of statistics. The first statistic is simply $N(\tau_1, \tau_2]$, and is therefore of linear nature, while the second statistic, of quadratic nature, is defined by

$$T_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)}\left(N(\tau_1,\tau_2]^2 - (1+2\lambda_0 L(\tau_2-\tau_1))N(\tau_1,\tau_2] + \lambda_0^2 L^2(\tau_2-\tau_1)^2\right) .$$

VIII.1 Detection of a non transitory change or jump

Let us recall that our tests are based on an aggregation principle which involves a scanning of the above linear and quadratic statistics over a discrete subset of possible values for the change location on $(0,1)$. The subset introduced in Chapter 2 Section V.1 is of the dyadic form

$$\Theta_d = \left\{1 - 2^{-k},\ k \in \{1, \ldots, 5\}\right\} .$$

Considering the alternative [**Alt.8**], the test statistic of our first procedure denoted by $(\mathtt{CP1}(\Theta_d))$ is thus

$$\mathcal{T}_{\lambda_0,\alpha}^{(1)}(N) = \max_{\tau\in\Theta_d}\left(\left(N(\tau,1] - p_{\lambda_0(1-\tau)L}\left(1-u_\alpha^{(1)}/2\right)\right) \vee \left(p_{\lambda_0(1-\tau)L}\left(u_\alpha^{(1)}/2\right) - N(\tau,1]\right)\right) ,$$

where $p_{\lambda_0(1-\tau)L}(u)$ is the u-quantile of a Poisson distribution with parameter $\lambda_0(1-\tau)L$ and $u_\alpha^{(1)}$ is defined as in (2.35) by

$$u_\alpha^{(1)} = \sup\left\{u \in (0,1),\ P_{\lambda_0}\left(\max_{\tau\in\Theta_d}\left(\left(N(\tau,1] - p_{\lambda_0(1-\tau)L}(1-u/2)\right) \vee \left(p_{\lambda_0(1-\tau)L}(u/2) - N(\tau,1]\right)\right) > 0\right) \leq \alpha\right\} , \tag{2.38}$$

while the test statistic of our second procedure denoted by $(\mathtt{CP2}(\Theta_d))$ is

$$\mathcal{T}_{\lambda_0,\alpha}^{(2)}(N) = \max_{\tau\in\Theta_d}\left(T_{\tau,1}(N) - t_{\lambda_0,\tau,1}\left(1-u_\alpha^{(2)}\right)\right) ,$$

where $t_{\lambda_0,\tau_1,\tau_2}(1-u)$ is the $(1-u)$-quantile of $T_{\tau_1,\tau_2}(N)$ under (H_0) and

$$u_\alpha^{(2)} = \sup\left\{u \in (0,1),\ P_{\lambda_0}\left(\max_{\tau\in\Theta_d}\left(T_{\tau,1}(N) - t_{\lambda_0,\tau,1}(1-u)\right) > 0\right) \leq \alpha\right\} . \tag{2.39}$$

The null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" is rejected when $\mathcal{T}^{(1)}_{\lambda_0,\alpha}(N) > 0$ for (CP1(Θ_d)), or when $\mathcal{T}^{(2)}_{\lambda_0,\alpha}(N) > 0$ for (CP2(Θ_d)).

Noticing that the tests we use for the bump detection problem (see the following subsection) and the jump detection procedure studied in [VFLRB21] are based on regular grids of possible values of change location instead of the dyadic subset Θ_d, we also consider the same tests but replacing Θ_d by a regular grid, with a cardinality close to the cardinality of Θ_d, namely

$$\Theta_r = \left\{ \frac{2k+1}{10},\ k \in \{0,\ldots,4\} \right\} .$$

The corresponding tests are then respectively denoted by (CP1(Θ_r)) and (CP2(Θ_r)).

For all τ in Θ_d or Θ_r, we have estimated the quantities $u^{(1)}_\alpha$, $u^{(2)}_\alpha$ and $t_{\lambda_0,\tau,1}(1-u^{(2)}_\alpha)$ by classical Monte Carlo methods based on the simulation of 200 000 independent copies of a Poisson random variable with parameter $\lambda_0(1-\tau)L$ or $T_{\tau,1}(N)$ under (H_0). The approximations of $u^{(1)}_\alpha$ and $u^{(2)}_\alpha$ were obtained by dichotomy, such that the probabilities or estimated probabilities occurring in (2.38) and (2.39) are less than α, but as close to α as possible.

VIII.2 Detection of a transitory change or bump

Let us consider the discrete set :

$$\Theta = \left\{ \left(\frac{k}{50}, \frac{k+k'}{50} \right),\ k \in \{0,\ldots,49\},\ k' \in \{1,\ldots,50-k\} \right\} .$$

Considering the alternative [**Alt.10**], the test statistic of our first procedure denoted by (TC1) is

$$\mathcal{T}^{(3)}_{\lambda_0,\alpha}(N) = \max_{(\tau,\tau')\in\Theta} \left(\left(N(\tau,\tau'] - p_{\lambda_0(\tau'-\tau)L}\left(1 - u^{(3)}_\alpha/2\right)\right) \vee \left(p_{\lambda_0(\tau'-\tau)L}\left(u^{(3)}_\alpha/2\right) - N(\tau,\tau']\right)\right) ,$$

where $p_{\lambda_0(\tau'-\tau)L}(u)$ is the u-quantile of a Poisson distribution with parameter $\lambda_0(\tau'-\tau)L$ and $u^{(3)}_\alpha$ is defined, again as in (2.35) by

$$u^{(3)}_\alpha = \sup\Big\{ u \in (0,1),\ P_{\lambda_0}\Big(\max_{(\tau,\tau')\in\Theta} \Big(\big(N(\tau,\tau'] - p_{\lambda_0(\tau'-\tau)L}(1-u/2) \big) \vee \big(p_{\lambda_0(\tau'-\tau)L}(u/2) - N(\tau,\tau'] \big) \Big) > 0 \Big) \le \alpha \Big\} , \quad (2.40)$$

while the test statistic of our second procedure denoted by (TC2) is

$$\mathcal{T}^{(4)}_{\lambda_0,\alpha}(N) = \max_{(\tau,\tau')\in\Theta} \left(T_{\tau,\tau'}(N) - t_{\lambda_0,\tau,\tau'}\left(1-u^{(4)}_\alpha\right)\right) ,$$

where $u^{(4)}_\alpha$ is defined by

$$u^{(4)}_\alpha = \sup\left\{ u \in (0,1),\ P_{\lambda_0}\left(\max_{(\tau,\tau')\in\Theta} \left(T_{\tau,\tau'}(N) - t_{\lambda_0,\tau,\tau'}\left(1-u^{(4)}_\alpha\right)\right) > 0 \right) \le \alpha \right\} . \quad (2.41)$$

The null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" is rejected when $\mathcal{T}^{(3)}_{\lambda_0,\alpha}(N) > 0$ for (TC1), or when $\mathcal{T}^{(4)}_{\lambda_0,\alpha}(N) > 0$ for (TC2).

As above, for all (τ,τ') in Θ, the quantities $u^{(3)}_\alpha$, $u^{(4)}_\alpha$ and $t_{\lambda_0,\tau,\tau'}(1-u^{(4)}_\alpha)$ have been estimated by Monte Carlo methods based on the simulation of 200 000 independent copies of a Poisson random variable with parameter $\lambda_0(\tau'-\tau)L$ or $T_{\tau,\tau'}(N)$ under (H_0). The approximations of $u^{(3)}_\alpha$ and $u^{(4)}_\alpha$ have been obtained by dichotomy.

VIII.3 Simulation results

We compare the tests (La) and (Z) with (CP1(Θ_d)), (CP2(Θ_d)), (CP1(Θ_r)) and (CP2(Θ_r)) when addressing the jump detection problem (described in VIII.1), and with (TC1) and (TC2) when addressing the bump detection problem (described in VIII.2).

Estimated sizes
We first study the size of each test via simulation of 10 000 independent homogeneous Poisson processes of intensity $\lambda_0 = 1$ w.r.t. Λ on $[0,1]$. The probabilities of first kind error of all the considered tests were simply estimated by the number of rejections divided by 10 000. The results are given in Table 2.1.

Table 2.1 – Estimated sizes

La	Z
0.049	0.049
CP1(Θ_d)	CP2(Θ_d)
0.046	0.048
CP1(Θ_r)	CP2(Θ_r)
0.049	0.048
TC1	TC2
0.046	0.048

Notice that the estimated sizes of our tests always remain below the target 0.05, as expected from the definitions of $u_\alpha^{(1)}$, $u_\alpha^{(2)}$, $u_\alpha^{(3)}$ and $u_\alpha^{(4)}$. It is in particular interesting to see that the Monte Carlo estimation, which is calibrated according to a balance precision/running time, does not affect here the first kind error rate control property.

Estimated powers
For both testing problems, we study the estimated power of each test under various alternatives.

Let us start with the jump detection problem. We consider alternative intensities $\lambda_{\tau,\delta}$ defined for all t in $[0,1]$ by

$$\lambda_{\tau,\delta}(t) = 1 + \delta \mathbb{1}_{(\tau,1]}(t) \ , \tag{2.42}$$

where $\delta \in \{-0.8,-0.6,-0.4,-0.2,0.4,0.8,1.2,1.6,2\}$, and $\tau = 0.5$ (Table 2.2), $\tau = 0.9$ (Table 2.3) and $\tau = 0.95$ (Table 2.4). For each alternative, 1 000 independent inhomogeneous Poisson processes with intensity $\lambda_{\tau,\delta}$ w.r.t. Λ on $[0,1]$ have been simulated. The power of the considered tests has then been simply estimated by the number of rejections divided by 1 000, leading to the results gathered in Tables 2.2-2.4.

Let us now turn to the bump detection problem. We have considered alternative intensities $\lambda_{\tau,\ell,\delta}$ defined for all t in $[0,1]$ by

$$\lambda_{\tau,\ell,\delta}(t) = 1 + \delta \mathbb{1}_{(\tau,\tau+\ell]}(t) \ , \tag{2.43}$$

where $\tau = 0.2$, $\delta \in \{-0.8,-0.6,-0.4,-0.2,0.4,0.8,1.2,1.6,2\}$ and $\ell = 0.1$ (Table 2.5), $\ell = 0.2$ (Table 2.6) and $\ell = 0.5$ (Table 2.7). For each alternative, we have simulated 1 000 independent inhomogeneous Poisson processes with intensity $\lambda_{0.2,\ell,\delta}$ w.r.t. Λ on $[0,1]$. The powers have been estimated for each test by the number of rejections divided by 1 000, and the results are provided in Tables 2.5-2.7.

Table 2.2 – Estimated probability of detecting a jump with $\tau = 0.5$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.92	0.58	0.27	0.09	0.19	0.56	0.84	0.96	0.99
Z	0.62	0.34	0.15	0.07	0.16	0.44	0.70	0.89	0.97
CP1(Θ_d)	1	0.93	0.44	0.13	0.40	0.90	1	1	1
CP2(Θ_d)	1	0.87	0.35	0.09	0.40	0.90	1	1	1
CP1(Θ_r)	1	0.91	0.43	0.14	0.42	0.90	1	1	1
CP2(Θ_r)	1	0.89	0.39	0.11	0.45	0.91	1	1	1

Table 2.3 – Estimated probability of detecting a jump with $\tau = 0.9$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.13	0.10	0.07	0.06	0.08	0.15	0.24	0.39	0.51
Z	0.07	0.06	0.04	0.06	0.06	0.09	0.12	0.20	0.29
CP1(Θ_d)	0.18	0.11	0.05	0.04	0.13	0.31	0.53	0.74	0.86
CP2(Θ_d)	0.17	0.09	0.04	0.04	0.15	0.33	0.56	0.78	0.88
CP1(Θ_r)	0.44	0.20	0.09	0.06	0.10	0.25	0.49	0.68	0.84
CP2(Θ_r)	0.09	0.08	0.04	0.04	0.13	0.33	0.59	0.79	0.89

Table 2.4 – Estimated probability of detecting a jump with $\tau = 0.95$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.05	0.06	0.06	0.05	0.07	0.07	0.12	0.17	0.23
Z	0.05	0.05	0.05	0.04	0.06	0.05	0.08	0.11	0.13
CP1(Θ_d)	0.06	0.05	0.04	0.04	0.07	0.18	0.29	0.43	0.55
CP2(Θ_d)	0.05	0.04	0.04	0.04	0.09	0.23	0.35	0.50	0.63
CP1(Θ_r)	0.10	0.08	0.06	0.06	0.07	0.11	0.17	0.27	0.37
CP2(Θ_r)	0.04	0.04	0.04	0.05	0.08	0.15	0.24	0.36	0.47

Table 2.5 – Estimated probability of detecting a bump with $\ell = 0.1$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.08	0.06	0.06	0.05	0.06	0.07	0.11	0.14	0.19
Z	0.07	0.06	0.05	0.05	0.05	0.05	0.06	0.04	0.05
TC1	0.13	0.08	0.06	0.04	0.08	0.17	0.28	0.48	0.71
TC2	0.06	0.05	0.04	0.04	0.09	0.20	0.31	0.54	0.74

Table 2.6 – Estimated probability of detecting a bump with $\ell = 0.2$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.14	0.10	0.07	0.05	0.06	0.08	0.17	0.26	0.37
Z	0.08	0.07	0.06	0.05	0.04	0.04	0.03	0.05	0.04
TC1	0.52	0.21	0.10	0.06	0.12	0.34	0.65	0.88	0.96
TC2	0.18	0.07	0.05	0.05	0.14	0.38	0.71	0.90	0.97

Table 2.7 – Estimated probability of detecting a bump with $\ell = 0.5$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.18	0.10	0.07	0.07	0.04	0.04	0.06	0.07	0.09
Z	0.15	0.08	0.07	0.07	0.04	0.03	0.04	0.06	0.05
TC1	0.99	0.74	0.27	0.08	0.28	0.79	0.98	1	1
TC2	0.98	0.56	0.15	0.05	0.34	0.83	0.99	1	1

Comments

1. It first arises that our procedures have estimated powers significantly larger than the Laplace and the Z tests for both testing problems corresponding to alternatives [**Alt.8**] and [**Alt.10**] in most cases. The lower performances of the Laplace and the Z tests may be due to the fact that their construction does not take the knowledge of λ_0 into account. Moreover, among our testing procedures, it is to note that both procedures based on the linear statistics (CP1) and (TC1) are mostly more powerful than (CP2) and (TC2) in the case of negative change heights whereas the procedures based on the quadratic statistics (CP2) and (TC2) are slightly more powerful when positive change heights occur. More precisely, the better performances of the linear statistics to detect negative jumps mainly concern the bump detection and the better performances of the quadratic statistics to detect positive jumps mainly concern the jump detection when the change-point is close to 1.
2. The comparison of the estimated powers of the different testing procedures confirm the intuition that detecting a bump is harder than detecting a jump. Moreover, the performances of each test are very different according to the sign of the change height. In both jump and bump detection problems, it is easier to detect a change with large negative jumps than a change with large positive jumps, except for the cases where $\ell = 0.1$ in the bump detection problem and $\tau \geq 0.9$ in the abrupt change detection problem. Indeed, in these cases, the estimated powers are close to the size of the tests for small negative change heights. Note that this capability of our tests to better detect a jump or a bump with negative height than with positive height can be explained by the fact that the significant parameter to evaluate the detectability of a (transitory or not) change is not the change height itself but the ratio between the minimum and the maximum values of the intensity. In other words, this means that it is easier to detect an intensity increasing from 1 to 2 than an intensity increasing from 100 to 101 whereas in both cases, the jump height is equal to one. In the same way, it is hence easier to detect an intensity decreasing from 1 to 0.8 than an intensity increasing from 1 to 1.2.
3. By comparing the estimated powers of our jump detection procedures based on the dyadic and regular sets Θ_d and Θ_r, one can take note that using the dyadic set is as expected more relevant when the jump is close to the observation interval ending point, that is the most difficult to detect.

4. Finally, for the bump detection problem, we have to notice that complementary experiments showed that the estimated powers of (TC1) and (TC2) are equivalent for a same value of change length whatever the values of the change location, which is not always true for the procedures (La) and (Z): for a same change length, these procedures are more powerful when the change location is close to 1.

IX Proofs of the main results

Notation

As explained in the Chapter 1, the main tools to prove our nonasymptotic minimax separation rates upper bounds are exponential inequalities. Many of these exponential inequalities involve the function g defined for every $x > -1$ by

$$g(x) = (1+x)\log(1+x) - x \ . \tag{2.44}$$

The inverse function of g restricted to $]0,+\infty[$ is denoted by g^{-1} and is defined on $]0,+\infty[$. For any $x>0$, one has:

$$g^{-1}(x) \leq 2x/3 + \sqrt{2x} \ . \tag{2.45}$$

The inverse function of g restricted to $]-1,0[$ is denoted by $g^{-1}_{|]-1,0[}$ and is defined on $]0,1[$. For all x in $]0,1[$, one has $g^{-1}_{|]-1,0[}(x) \geq -\sqrt{2x}$ and in particular

$$g^{-1}_{|]-1,0[}(x) \geq -g^{-1}(x) \ . \tag{2.46}$$

These upper bounds for g^{-1} and $g^{-1}_{|]-1,0[}$ are deduced from a lower bound of g stated in [SW86, Proposition 11.1].

Proof of Proposition 2.2

The first statement of Proposition 2.2 directly results from the Neyman-Pearson fundamental lemma and Girsanov's lemma recalled in Lemma 2.1.
Assume that $\delta^* > 0$ and notice that the assumption (2.5) leads to

$$\delta^*\ell^* L \geq \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}} + \sqrt{\frac{\lambda_0 \ell^* L}{\alpha}} \ . \tag{2.47}$$

From (2.47), the quantile bound (1.2) and the Bienayme-Chebyshev inequality, we obtain

$$\begin{aligned} P_\lambda(\phi^+_{1,\alpha}(N)=0) &\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] \leq \sqrt{\frac{\lambda_0\ell^* L}{\alpha}} + \lambda_0\ell^* L\right) \\ &\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] \leq (\lambda_0^*+\delta^*)\ell^* L - \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}}\right) \\ &\leq \beta \ . \end{aligned}$$

Assume now that $-\lambda_0^* < \delta^* < 0$ and notice that the assumption (2.5) leads to

$$\delta^*\ell^* L \leq -\sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}} - \sqrt{\frac{\lambda_0 \ell^* L}{\alpha}} \ . \tag{2.48}$$

As above, using (2.48), (1.2) and the Bienayme-Chebyshev inequality, we obtain

$$\begin{aligned} P_\lambda(\phi^-_{1,\alpha}(N)=0) &\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] \geq -\sqrt{\frac{\lambda_0\ell^*L}{\alpha}}+\lambda_0\ell^*L\right) \\ &\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] \geq (\lambda_0+\delta^*)\ell^*L+\sqrt{\frac{(\lambda_0+\delta^*)\ell^*L}{\beta}}\right) \\ &\leq \beta \ . \end{aligned}$$

Proof of Proposition 2.3

Let $C_{\alpha,\beta} = 1+4(1-\alpha-\beta)^2$. Let us introduce for $r>0$ the Poisson intensity λ_r defined by

$$\lambda_r(t) = \lambda_0 + \frac{r}{\sqrt{\ell^*}}\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t) \text{ for all } t \text{ in } [0,1] \ .$$

Notice that λ_r belongs to $(\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0])_r = \{\lambda \in \mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0],\ d_2(\lambda,\mathcal{S}_0[\lambda_0]) \geq r\}$, as defined by Lemma 1.7. We get from Lemma 2.1 and Lemma 2.24 that

$$E_{\lambda_0}\left[\left(\frac{dP_{\lambda_r}}{dP_{\lambda_0}}\right)^2(N)\right] = \exp\left(\frac{r^2L}{\lambda_0}\right) \ .$$

Choosing $r = (\lambda_0 \log C_{\alpha,\beta}/L)^{1/2}$ then leads to $E_{\lambda_0}\left[\left(dP_{\lambda_r}/dP_{\lambda_0}\right)^2(N)\right] = C_{\alpha,\beta}$, and thanks to Lemma 1.8, we conclude that $\rho_\alpha\left((\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0])_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]) \geq r$.

Proof of Proposition 2.4

The first statement of the proposition is straightforward.
As for the second kind error study, let us consider first $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}$ in $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$ with $\delta>0$. From the quantile bound (1.2), one deduces that

$$\begin{aligned} P_\lambda\left(\phi^{(1)}_{2,\alpha}=0\right) &\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] \leq \sqrt{\frac{\lambda_0\ell^*L}{\alpha_1}}+\lambda_0\ell^*L\right) \ , \\ &= P_\lambda\left(N(\tau^*,\tau^*+\ell^*]-(\lambda_0+\delta)\ell^*L \leq -\delta\ell^*L+\sqrt{\frac{\lambda_0\ell^*L}{\alpha_1}}\right) \ . \end{aligned}$$

It remains to find a condition on $d_2(\lambda,\mathcal{S}_0[\lambda_0])$ which will guarantee that

$$-\delta\ell^*L+\sqrt{\frac{\lambda_0\ell^*L}{\alpha_1}} \leq -\sqrt{\frac{(\lambda_0+\delta)\ell^*L}{\beta}} \ , \tag{2.49}$$

so that $P_\lambda\left(\phi^{(1)}_{2,\alpha}=0\right) \leq \beta$ thanks to the Bienayme-Chebyshev inequality.
Let us assume for instance that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0]) \geq 2\sqrt{\frac{\lambda_0}{L}}\left(\frac{1}{\sqrt{\beta}}+\frac{1}{\sqrt{\alpha_1}}\right)+\frac{1}{\beta\sqrt{\ell^*L}} \ .$$

Since $d_2(\lambda, \mathcal{S}_0[\lambda_0]) = \delta\sqrt{\ell^*}$, this implies

$$\delta\sqrt{\ell^*} \geq 2\sqrt{\frac{\lambda_0}{L}}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_1}}\right) + \frac{1}{\beta\sqrt{\ell^*}L} \quad ,$$

whereby

$$\delta\sqrt{\ell^*} - \left(\frac{\delta\sqrt{\ell^*}}{2} + \frac{1}{2\beta\sqrt{\ell^*}L}\right) \geq \sqrt{\frac{\lambda_0}{L}}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_1}}\right) \quad .$$

Using the basic inequality $\sqrt{ab} \leq (a+b)/2$ then leads to

$$\delta\sqrt{\ell^*} - \sqrt{\frac{\delta}{\beta L}} \geq \sqrt{\frac{\lambda_0}{L}}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_1}}\right) \quad ,$$

and (2.49) conveniently follows.
Let us consider now $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}$ in $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$ with δ in $(-\lambda_0, 0)$. From the quantile bound (1.2) again, one deduces that

$$\begin{aligned} P_\lambda\left(\phi^{(1)}_{2,\alpha} = 0\right) &\leq P_\lambda\left(N(\tau^*, \tau^*+\ell^*] \geq -\sqrt{\frac{\lambda_0\ell^*L}{\alpha_2}} + \lambda_0\ell^*L\right) \\ &= P_\lambda\left(N(\tau^*, \tau^*+\ell^*] - (\lambda_0+\delta)\ell^*L \geq -\delta\ell^*L - \sqrt{\frac{\lambda_0\ell^*L}{\alpha_2}}\right) \quad . \end{aligned}$$

As above, it remains to find a condition on $d_2(\lambda, \mathcal{S}_0[\lambda_0])$ which will guarantee that

$$-\delta\ell^*L - \sqrt{\frac{\lambda_0\ell^*L}{\alpha_2}} \geq \sqrt{\frac{(\lambda_0+\delta)\ell^*L}{\beta}} \quad , \tag{2.50}$$

so that $P_\lambda\left(\phi^{(1)}_{2,\alpha} = 0\right) \leq \beta$. Since $d_2(\lambda, \mathcal{S}_0[\lambda_0]) = -\delta\sqrt{\ell^*}$ and $\delta < 0$, the following condition suffices

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \sqrt{\frac{\lambda_0}{L}}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_2}}\right) \quad .$$

Taking

$$C(\alpha, \beta, \lambda_0, \ell^*) = \max\left(2\sqrt{\lambda_0}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_1}}\right) + \frac{1}{\beta\sqrt{\ell^*}}, \sqrt{\lambda_0}\left(\frac{1}{\sqrt{\beta}} + \frac{1}{\sqrt{\alpha_2}}\right)\right)$$

finally allows to conclude the proof.

Proof of Proposition 2.5

The first statement of the proposition is straightforward.
Let us now consider $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}$ in $\mathcal{S}_{\cdot,\tau^*,\ell^*}[\lambda_0]$. Since $T_{\tau^*,\tau^*+\ell^*}(N)$ is centered under (H_0), $t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha) \leq \sqrt{\mathrm{Var}_{\lambda_0}(T_{\tau^*,\tau^*+\ell^*}(N))/\alpha}$. From the variance computation of Lemma 2.26 under (H_0), we derive the upper bound $t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha) \leq (\lambda_0/L)\sqrt{2/\alpha}$.
Moreover, still using Lemma 2.26 but under (H_1) now, one can see that $E_\lambda[T_{\tau^*,\tau^*+\ell^*}(N)] = \delta^2\ell^*$ (recall that $T_{\tau^*,\tau^*+\ell^*}(N)$ is an unbiased estimator of $d_2^2(\lambda, \mathcal{S}_0[\lambda_0] = \delta^2\ell^*)$, and

$$\mathrm{Var}_\lambda(T_{\tau^*,\tau^*+\ell^*}(N)) = \frac{4(\lambda_0+\delta)\delta^2\ell^*}{L} + \frac{2(\lambda_0+\delta)^2}{L^2} \quad .$$

Therefore,

$$\begin{aligned}
P_\lambda(\phi_{2,\alpha}^{(2)}=0) &= P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)\le t_{\lambda_0,\tau^*,\tau^*+\ell^*}(1-\alpha)\right) \ , \\
&\le P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)\le \frac{\lambda_0}{L}\sqrt{\frac{2}{\alpha}}\right) \ , \\
&\le P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)-\delta^2\ell^*\le \frac{\lambda_0}{L}\sqrt{\frac{2}{\alpha}}-\delta^2\ell^*\right) \ .
\end{aligned}$$

Assume now that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\ge\frac{C(\alpha,\beta,\lambda_0,\ell^*)}{\sqrt{L}} \ ,$$

with

$$C(\alpha,\beta,\lambda_0,\ell^*)=\max\left(\sqrt{3\lambda_0\left(\sqrt{\tfrac{2}{\alpha}}+\sqrt{\tfrac{2}{\beta}}\right)},\ \tfrac{6\sqrt{\lambda_0}}{\sqrt{\beta}}+\tfrac{3\sqrt{2}}{\sqrt{\beta\ell^*L}},\ \tfrac{36}{\beta\sqrt{\ell^*L}}\right) \ .$$

This implies

$$\delta^2\ell^*\ge 3\max\left(\frac{\lambda_0}{L}\left(\sqrt{\frac{2}{\alpha}}+\sqrt{\frac{2}{\beta}}\right),\ |\delta|\sqrt{\ell^*}\left(\frac{2}{\sqrt{L}}\sqrt{\frac{\lambda_0}{\beta}}+\frac{1}{L}\sqrt{\frac{2}{\beta\ell^*}}\right),\ \frac{2|\delta|^{3/2}\sqrt{\ell^*}}{\sqrt{\beta L}}\right) \ ,$$

and then

$$\delta^2\ell^*\ge\frac{\lambda_0}{L}\sqrt{\frac{2}{\alpha}}+2\sqrt{\frac{\lambda_0}{\beta L}}|\delta|\sqrt{\ell^*}+\sqrt{\frac{2}{\beta}}\left(\frac{\lambda_0}{L}+\frac{1}{L\sqrt{\ell^*}}|\delta|\sqrt{\ell^*}\right)+\frac{2|\delta|^{3/2}\sqrt{\ell^*}}{\sqrt{\beta L}} \ ,$$

hence, using $\sqrt{\lambda_0+\delta}\le\sqrt{\lambda_0}+\sqrt{|\delta|}$,

$$\delta^2\ell^*\ge\frac{\lambda_0}{L}\sqrt{\frac{2}{\alpha}}+2\sqrt{\frac{\lambda_0+\delta}{L\beta}}|\delta|\sqrt{\ell^*}+\sqrt{\frac{2}{\beta}}\frac{\lambda_0+\delta}{L} \ . \tag{2.51}$$

Therefore,

$$\begin{aligned}
P_\lambda(\phi_{2,\alpha}^{(2)}=0) &\le P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)-\delta^2\ell^*\le\frac{\lambda_0}{L}\sqrt{\frac{2}{\alpha}}-\delta^2\ell^*\right) \\
&\le P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)-\delta^2\ell^*\le-\frac{1}{\sqrt{\beta}}\sqrt{\frac{4(\lambda_0+\delta)\delta^2\ell^*}{L}+\frac{2(\lambda_0+\delta)^2}{L^2}}\right) \\
&\le P_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)-E_\lambda\left[T_{\tau^*,\tau^*+\ell^*}(N)\right]\le-\sqrt{\frac{\mathrm{Var}_\lambda\left(T_{\tau^*,\tau^*+\ell^*}(N)\right)}{\beta}}\right) \\
&\le\beta \ .
\end{aligned}$$

This concludes the proof.

Proof of Proposition 2.6

Let us first give a short proof for the tests $\phi_{3,\alpha}^{(1)+}$ and $\phi_{3,\alpha}^{(1)-}$.
Start by remarking that the first kind error rates control of both tests is straightforward.
Since $g^{-1}(x)\le 2x/3+\sqrt{2x}$ for all $x>0$ (see (2.45)), Lemma 2.28 leads to

$$p_{\lambda_0,\ell^*}^{+}(1-\alpha)\le\lambda_0\ell^*L+2\sqrt{2\lambda_0\log(2/\alpha)L}+4\log(2/\alpha)/3 \ , \tag{2.52}$$

and

$$p^-_{\lambda_0,\ell^*}(\alpha) \geq \lambda_0 \ell^* L - 2\sqrt{2\lambda_0 \log(2/\alpha) L} - 4\log(2/\alpha)/3 \ . \tag{2.53}$$

Let us consider $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau,\tau+\ell^*]}$ in $\mathcal{S}_{\delta^*,\cdot,\ell^*}$ and assume that

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{1}{\sqrt{L}}\left(2\sqrt{\frac{2\lambda_0 \log(2/\alpha)}{\ell^*}} + \sqrt{\frac{\lambda_0+\delta^*}{\beta}}\right) + \frac{4\log(2/\alpha)}{3L\sqrt{\ell^*}} \ . \tag{2.54}$$

If $\delta^* > 0$, the condition (2.54) yields

$$\delta^* \ell^* L \geq 2\sqrt{2\lambda_0 \log(2/\alpha) L} + \frac{4}{3}\log(2/\alpha) + \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}} \ , \tag{2.55}$$

which entails

$$\begin{aligned}
&P_\lambda\left(\max_{t\in[0,1-\ell^*]} N(t,t+\ell^*] \leq p^+_{\lambda_0,\ell^*}(1-\alpha)\right)\\
&\leq P_\lambda\left(\max_{t\in[0,1-\ell^*]} N(t,t+\ell^*] \leq (\lambda_0+\delta^*)\ell^* L - \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}}\right)\\
&\leq P_\lambda\left(N(\tau,\tau+\ell^*] - (\lambda_0+\delta^*)L\ell^* \leq -\sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}}\right)\\
&\leq \beta \ \ \text{with the Bienayme-Chebyshev inequality} \ .
\end{aligned}$$

This concludes the proof for $\phi^{(1)+}_{3,\alpha}$.
If δ^* belongs to $(-\lambda_0, 0)$, the condition (2.54) yields

$$-\delta^* \ell^* L \geq 2\sqrt{2\lambda_0 \log(2/\alpha) L} + \frac{4}{3}\log(2/\alpha) + \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}} \ . \tag{2.56}$$

We get then as above, with (2.53), (2.56) and the Bienayme-Chebyshev inequality,

$$\begin{aligned}
&P_\lambda\left(\min_{t\in[0,1-\ell^*]} N(t,t+\ell^*] \geq p^-_{\lambda_0,\ell^*}(\alpha)\right)\\
&\leq P_\lambda\left(\min_{t\in[0,1-\ell^*]} N(t,t+\ell^*] \geq \lambda_0\ell^* L - 2\sqrt{2\lambda_0\log(2/\alpha)L} - \frac{4}{3}\log(2/\alpha)\right)\\
&\leq P_\lambda\left(\min_{t\in[0,1-\ell^*]} N(t,t+\ell^*] \geq (\lambda_0+\delta^*)\ell^* L + \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}}\right)\\
&\leq P_\lambda\left(N(\tau,\tau+\ell^*] - (\lambda_0+\delta^*)\ell^* L \geq \sqrt{\frac{(\lambda_0+\delta^*)\ell^* L}{\beta}}\right)\\
&\leq \beta \ .
\end{aligned}$$

This concludes the proof for $\phi^{(1)-}_{3,\alpha}$.

Now, let us turn to the test $\phi^{(2)}_{3/4,\alpha}$. As above, start by remarking that the first kind error rate control of this test straightforwardly follows from a basic union bound:

$$\begin{aligned} P_{\lambda_0}\left(\phi^{(2)}_{3/4,\alpha}(N)=1\right) &\leq \sum_{k=0}^{\lceil(1-\ell^*)M\rceil-1} P_{\lambda_0}\left(T_{\frac{k}{M},\frac{k}{M}+\ell^*}(N) > t_{\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right) \\ &\leq \sum_{k=0}^{\lceil(1-\ell^*)M\rceil-1} \frac{\alpha}{\lceil(1-\ell^*)M\rceil} \\ &\leq \alpha \ . \end{aligned}$$

Let λ in $\mathcal{S}_{\delta^*,\cdot\cdot,\ell^*}$ such that $\lambda=\lambda_0+\delta^*\mathbb{1}_{(\tau,\tau+\ell^*]}$ with τ in $[0,1-\ell^*]$, and assume that the following holds:

$$d_2\left(\lambda,\mathcal{S}_0[\lambda_0]\right) \geq \frac{2}{\sqrt{L}}\max\left(8\sqrt{\frac{\lambda_0+|\delta^*|}{\beta}}\ ,\ \left(\frac{4\sqrt{2}(\lambda_0+|\delta^*|)}{\sqrt{\beta}}+8\lambda_0\left(\frac{2}{3}\frac{\log(3/u_\alpha)}{\sqrt{\lambda_0\ell^*L}}+\sqrt{2\log(3/u_\alpha)}\right)^2\right)^{\frac{1}{2}}\right)\ . \quad (2.57)$$

This entails

$$d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) \geq \frac{8\lambda_0}{L}\left(\frac{2}{3}\frac{\log(3/u_\alpha)}{\sqrt{\lambda_0\ell^*L}}+\sqrt{2\log(3/u_\alpha)}\right)^2+\frac{4\sqrt{2}(\lambda_0+|\delta^*|)}{\sqrt{\beta}L}+\frac{8d_2(\lambda,\mathcal{S}_0[\lambda_0])}{\sqrt{L}}\sqrt{\frac{\lambda_0+|\delta^*|}{\beta}}\ . \quad (2.58)$$

Noticing that

$$P_\lambda\left(\phi^{(2)}_{3/4,\alpha}(N)=0\right) \leq \min_{k\in\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}} P_\lambda\left(T_{\frac{k}{M},\frac{k}{M}+\ell^*}(N) \leq t_{\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right)\ ,$$

we only need to exhibit some k_τ in $\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}$ satisfying

$$P_\lambda\left(T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) \leq t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\right) \leq \beta\ .$$

We set $k_\tau=\lfloor\tau M\rfloor$. Since $0<\tau<1-\ell^*$, k_τ actually belongs to $\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}$, and since $M=\lceil 2/\ell^*\rceil$, $k_\tau/M\leq\tau<k_\tau/M+\ell^*/2$. Therefore, using Lemma 2.26 equation (2.117), we get on the one hand

$$E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]=\delta^{*2}\frac{(\ell^*+k_\tau/M-\tau)^2}{\ell^*}\geq\frac{\delta^{*2}\ell^*}{4}\ ,$$

that is

$$E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]\geq\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{4}\ , \quad (2.59)$$

and on the other hand with Lemma 2.26 equation (2.118),

$$\begin{aligned} \mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] &= \frac{4\delta^{*2}\left(\lambda_0\ell^*+\delta^*(k_\tau/M+\ell^*-\tau)\right)}{L}\frac{(k_\tau/M+\ell^*-\tau)^2}{\ell^{*2}}+\frac{2}{L^2}\frac{\left(\lambda_0\ell^*+\delta^*(k_\tau/M+\ell^*-\tau)\right)^2}{\ell^{*2}} \\ &\leq \frac{4(\lambda_0+|\delta^*|)}{L}d_2^2(\lambda,\mathcal{S}_0[\lambda_0])+\frac{2(\lambda_0+|\delta^*|)^2}{L^2}\ . \end{aligned} \quad (2.60)$$

Moreover, Lemma 2.27 entails

$$t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\leq 2\lambda_0^2\ell^*\left(g^{-1}\left(\frac{\log(3/u_\alpha)}{\lambda_0\ell^*L}\right)\right)^2\ ,$$

with $g^{-1}(x) \leq 2x/3 + \sqrt{2x}$ (see (2.45)), which implies, with (2.58), (2.59) and (2.60) that

$$t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha) \leq E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] - \sqrt{\mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]/\beta}\ .$$

We simply conclude the proof for $\phi^{(2)}_{3/4,\alpha}$ with

$$\begin{aligned}
&P_\lambda\left(T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) \leq t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\right)\\
&\leq P_\lambda\left(T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) - E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] \leq -\sqrt{\mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]/\beta}\right)\\
&\leq \beta\ .
\end{aligned}$$

Proof of Proposition 2.8

The control of the first kind error rates of the two tests $\phi^{(1)}_{4,\alpha}$ and $\phi^{(2)}_{3/4,\alpha}$ is straightforward using simple union bounds.

(i) Control of the second kind error rate of $\phi^{(1)}_{4,\alpha}$.

Recall from the proof of Proposition 2.6, equations (2.52) and (2.53), that

$$\left\{\begin{array}{rcl} p^+_{\lambda_0,\ell^*}(1-\alpha/2) & \leq & \lambda_0\ell^*L + 2\sqrt{2\lambda_0\log(4/\alpha)L} + 4\log(4/\alpha)/3\ ,\\ p^-_{\lambda_0,\ell^*}(\alpha/2) & \geq & \lambda_0\ell^*L - 2\sqrt{2\lambda_0\log(4/\alpha)L} - 4\log(4/\alpha)/3\ .\end{array}\right. \tag{2.61}$$

Let us first set λ in $\mathcal{S}_{\cdot,\cdot,\ell^*}$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,\tau+\ell^*]}$ with $\delta > 0$ or δ in $(-\lambda_0, 0)$, τ in $(0, 1-\ell^*)$, and

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{2\sqrt{\lambda_0}}{\sqrt{L}}\left(\frac{1}{\sqrt{\beta}} + 2\sqrt{\frac{2\log(4/\alpha)}{\ell^*}}\right) + \frac{1}{\sqrt{\ell^*}L}\left(\frac{1}{\beta} + \frac{8}{3}\log(4/\alpha)\right)\ . \tag{2.62}$$

The condition (2.62) entails

$$|\delta|\sqrt{\ell^*} \geq \frac{|\delta|\sqrt{\ell^*}}{2} + \frac{\sqrt{\lambda_0}}{\sqrt{L}}\left(\frac{1}{\sqrt{\beta}} + 2\sqrt{\frac{2\log(4/\alpha)}{\ell^*}}\right) + \frac{1}{\sqrt{\ell^*}L}\left(\frac{1}{2\beta} + \frac{4}{3}\log(4/\alpha)\right)\ ,$$

and therefore, using the inequalities $\sqrt{ab} \leq (a+b)/2$ and $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for every $a, b \geq 0$,

$$\begin{aligned}
|\delta|\sqrt{\ell^*} \quad &\geq \quad \sqrt{\frac{|\delta|}{\beta L}} + \frac{\sqrt{\lambda_0}}{\sqrt{L}}\left(\frac{1}{\sqrt{\beta}} + 2\sqrt{\frac{2\log(4/\alpha)}{\ell^*}}\right) + \frac{4}{3}\frac{\log(4/\alpha)}{\sqrt{\ell^*}L}\\
&\geq \quad \sqrt{\frac{|\delta| + \lambda_0}{\beta L}} + 2\sqrt{\frac{2\log(4/\alpha)\lambda_0}{\ell^* L}} + \frac{4}{3}\frac{\log(4/\alpha)}{\sqrt{\ell^*}L}\ .
\end{aligned} \tag{2.63}$$

Then, assuming that $\delta > 0$, we conclude with the following inequalities:

$$\begin{aligned}
P_\lambda\left(\phi^{(1)}_{4,\alpha}(N) = 0\right) \quad &\leq \quad P_\lambda\left(\phi^{(1)+}_{3,\alpha/2}(N) = 0\right)\\
&\leq \quad P_\lambda\left(\max_{t\in[0,1-\ell^*]} N(t, t+\ell^*] \leq p^+_{\lambda_0,\ell^*}(1-\alpha/2)\right)\\
&\leq \quad P_\lambda\left(N(\tau, \tau+\ell^*] \leq \lambda_0\ell^*L + 2\sqrt{2\lambda_0\log(4/\alpha)L} + 4\log(4/\alpha)/3\right) \ \text{ with (2.61)}\\
&\leq \quad P_\lambda\left(N(\tau, \tau+\ell^*] - (\lambda_0^* + \delta)\ell^*L \leq -\sqrt{\frac{(\lambda_0+\delta)\ell^*L}{\beta}}\right) \ \text{ with (2.63)}\\
&\leq \quad \beta \ \text{ with the Bienayme-Chebyshev inequality}\ .
\end{aligned}$$

Assuming now that δ is in $(-\lambda_0, 0)$,

$$\begin{aligned}
P_\lambda\left(\phi^{(1)}_{4,\alpha}(N)=0\right) &\leq P_\lambda\left(\phi^{(1)-}_{3,\alpha/2}(N)=0\right)\\
&\leq P_\lambda\left(\min_{t\in[0,1-\ell^*]} N(t,t+\ell^*]\geq p^-_{\lambda_0,\ell^*}(\alpha/2)\right)\\
&\leq P_\lambda\left(N(\tau,\tau+\ell^*]\geq \lambda_0\ell^* L-2\sqrt{2\lambda_0\log(4/\alpha)L}-4\log(4/\alpha)/3\right) \text{ with (2.61)}\\
&\leq P_\lambda\left(N(\tau,\tau+\ell^*]-(\lambda_0+\delta)\ell^* L\geq\sqrt{\frac{(\lambda_0+|\delta|)\ell^* L}{\beta}}\right) \text{ with (2.63)}\\
&\leq \beta\ .
\end{aligned}$$

(ii) Control of the second kind error rate of $\phi^{(2)}_{3/4,\alpha}$.
Let us now set λ in $\mathcal{S}_{\cdot,\cdot,\ell^*}$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell^*]}$ with $\delta>0$ or δ in $(-\lambda_0,0)$, τ in $(0,1-\ell^*)$, and

$$\begin{aligned}
d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq 2\max\Bigg(&12\frac{\sqrt{\lambda_0}}{\sqrt{\beta L}}+\frac{6\sqrt{2}}{\sqrt{\beta\ell^* L}}\ ,\ \frac{288}{\beta\sqrt{\ell^*}L}\ ,\\
&\frac{\sqrt{3\lambda_0(4\log(3/u_\alpha)+\sqrt{2/\beta})}}{\sqrt{L}}+\frac{2\sqrt{2}\log(3/u_\alpha)}{\sqrt{3\ell^*}L}+\frac{2\sqrt{2}(2\lambda_0)^{1/4}\log^{3/4}(3/u_\alpha)}{\ell^{*1/4}L^{3/4}}\Bigg)\ . \qquad (2.64)
\end{aligned}$$

Notice that (2.64) entails that

$$\begin{aligned}
d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\geq 3\max\Bigg(&d_2(\lambda,\mathcal{S}_0[\lambda_0])\left(\frac{8\sqrt{\lambda_0}}{\sqrt{\beta L}}+\frac{4\sqrt{2}}{\sqrt{\beta\ell^* L}}\right),\ d_2^{3/2}(\lambda,\mathcal{S}_0[\lambda_0])\frac{8}{\ell^{*1/4}\sqrt{\beta L}}\ ,\\
&\frac{16\lambda_0\log(3/u_\alpha)}{L}+\frac{32\log^2(3/u_\alpha)}{9\ell^* L^2}+\frac{32\sqrt{2\lambda_0}\log^{3/2}(3/u_\alpha)}{3\sqrt{\ell^*}L^{3/2}}+\frac{4\lambda_0\sqrt{2}}{\sqrt{\beta}L}\Bigg)\ ,
\end{aligned}$$

and therefore

$$\begin{aligned}
d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\geq\ & d_2(\lambda,\mathcal{S}_0[\lambda_0])\left(\frac{8\sqrt{\lambda_0}}{\sqrt{\beta L}}+\frac{4\sqrt{2}}{\sqrt{\beta\ell^* L}}\right)+d_2^{3/2}(\lambda,\mathcal{S}_0[\lambda_0])\frac{8}{\ell^{*1/4}\sqrt{\beta L}}+\frac{16\lambda_0\log(3/u_\alpha)}{L}\\
&+\frac{32\log^2(3/u_\alpha)}{9\ell^* L^2}+\frac{32\sqrt{2\lambda_0}\log^{3/2}(3/u_\alpha)}{3\sqrt{\ell^*}L^{3/2}}+\frac{4\lambda_0\sqrt{2}}{\sqrt{\beta}L}\ .
\end{aligned}$$

Since $d_2(\lambda,\mathcal{S}_0[\lambda_0])=|\delta|\sqrt{\ell^*}$ and $\sqrt{\lambda_0+|\delta|}\leq\sqrt{\lambda_0}+\sqrt{|\delta|}$, this implies that

$$\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{4}\geq\frac{2\sqrt{\lambda_0+|\delta|}}{\sqrt{\beta L}}d_2(\lambda,\mathcal{S}_0[\lambda_0])+\frac{\sqrt{2}(|\delta|+\lambda_0)}{\sqrt{\beta}L}+\frac{4\lambda_0\log(3/u_\alpha)}{L}+\frac{8\log^2(3/u_\alpha)}{9\ell^* L^2}+\frac{8\sqrt{2\lambda_0}\log^{3/2}(3/u_\alpha)}{3\sqrt{\ell^*}L^{3/2}}\ . \qquad (2.65)$$

Let us now prove that $P_\lambda\left(\phi^{(2)}_{3/4,\alpha}(N)=0\right)\leq\beta$. From the definition (2.16), we notice that

$$P_\lambda\left(\phi^{(2)}_{3/4,\alpha}(N)=0\right)\leq\min_{k\in\{0,\ldots,\lceil(1-\ell^*)M\rceil-1\}}P_\lambda\left(T_{\frac{k}{M},\frac{k}{M}+\ell^*}(N)\leq t_{\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right)\ ,$$

so that we only need to exhibit some k_τ in $\{0,\ldots,\lceil(1-\ell^*)M\rceil-1\}$ such that

$$P_\lambda\left(T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\leq t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\right)\leq\beta\ .$$

As in the proof of Proposition 2.6, we choose $k_\tau = \lfloor \tau M \rfloor$, which leads (see (2.59) and (2.60)) to

$$E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] \geq \frac{d_2^2(\lambda, \mathcal{S}_0[\lambda_0])}{4} \ , \tag{2.66}$$

and

$$\mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] \leq \frac{4(\lambda_0+|\delta|)}{L} d_2^2(\lambda, \mathcal{S}_0[\lambda_0]) + \frac{2(\lambda_0+|\delta|)^2}{L^2} \ . \tag{2.67}$$

From (2.65), (2.66) and (2.67), we derive that

$$E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] \geq \sqrt{\mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]/\beta} + \frac{4\lambda_0 \log(3/u_\alpha)}{L} + \frac{8\log^2(3/u_\alpha)}{9\ell^* L^2} + \frac{8\sqrt{2\lambda_0}\log^{3/2}(3/u_\alpha)}{3\sqrt{\ell^*}L^{3/2}} \ .$$

The conclusion then basically follows from Lemma 2.27, supplemented by the upper bound (2.45), which allows to see that

$$E_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] \geq \sqrt{\mathrm{Var}_\lambda\left[T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]/\beta} + t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha) \ ,$$

and the Bienayme-Chebyshev inequality, which entails

$$P_\lambda\left(T_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) \leq t_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\right) \leq \beta \ .$$

Proof of Proposition 2.9

Let $C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2$, $r = (\lambda_0 \log C_{\alpha,\beta}/L)^{1/2}$ and λ_r defined for all t in $(0,1)$ by

$$\lambda_r(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*, \tau^* + r^2/\delta^{*2}]}(t) \ .$$

Notice that for all $L \geq \lambda_0 \log C_{\alpha,\beta}/(\delta^{*2}(1-\tau^*))$, $r^2/\delta^{*2} \leq 1-\tau^*$ and λ_r belongs to $(\mathcal{S}_{\delta^*,\tau^*,\ldots}[\lambda_0])_r$ in the notation of Lemma 1.7. We get now from Lemma 2.1 and Lemma 2.24

$$E_{\lambda_0}\left[\left(\frac{dP_{\lambda_r}}{dP_0}\right)^2(N)\right] = \exp\left(\frac{r^2 L}{\lambda_0}\right) = C_{\alpha,\beta}.$$

Lemmas 1.8 and 1.7 then entail $\rho_\alpha\left((\mathcal{S}_{\delta^*,\tau^*,\ldots}[\lambda_0])_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\tau^*,\ldots}[\lambda_0]) \geq r$.

Proof of Proposition 2.10

The first kind error rate control is straightforward. As for the second kind error rate control, let $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell]}$ belonging to $\mathcal{S}_{\delta^*,\tau^*,\ldots}[\lambda_0]$ with ℓ in $(0, 1-\tau^*)$ and satisfying

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \ \geq \ \frac{2}{\sqrt{L}} \max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}} \ , \ \sqrt{\delta^* s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}\mathbb{1}_{\{\delta^*>0\}} + \sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}}\mathbb{1}_{\{-\lambda_0<\delta^*<0\}}\right) \ . \tag{2.68}$$

Assume that $\delta^* > 0$ and recall that $s^+_{\lambda_0,\delta^*/2}(1-\alpha)$ defined in Lemma 2.29 is a constant which does not depend on L. The assumption (2.68) implies

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{2}{\sqrt{L}} \max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}} \ , \ \sqrt{\delta^* s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}\right) \ ,$$

which yields

$$\delta^*\ell \geq 4\max\left(\sqrt{\frac{(\lambda_0+\delta^*)\ell}{\beta L}}\,,\ \frac{s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}{L}\right)\ ,$$

hence

$$\frac{\delta^*}{2}\ell L \geq \sqrt{\frac{(\lambda_0+\delta^*)\ell L}{\beta}} + s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\ . \tag{2.69}$$

We get then

$$P_\lambda(\phi_{5,\alpha}(N)=0) \leq P_\lambda\left(\sup_{\ell'\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell'}(N) \leq s^+_{\lambda_0,\delta^*,\tau^*,L}(1-\alpha)\right)\ ,$$

where

$$S_{\delta^*,\tau^*,\tau^*+\ell'}(N) = \operatorname{sgn}(\delta^*)\Big(N(\tau^*,\tau^*+\ell'] - \lambda_0 L\ell'\Big) - |\delta^*|L\ell'/2\ ,$$

as defined by (2.19) and $s^+_{\lambda_0,\delta^*,\tau^*,L}(u)$ is the u-quantile of $\sup_{\ell'\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell'}(N)$ under (H_0). From the quantile upper bound (2.121), we deduce

$$\begin{aligned}
P_\lambda(\phi_{5,\alpha}(N)=0) &\leq P_\lambda\left(\sup_{\ell'\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell'}(N) \leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\right)\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell] - \left(\lambda_0+\frac{\delta^*}{2}\right)\ell L \leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\right)\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell] - (\lambda_0+\delta^*)\ell L \leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha) - \frac{\delta^*}{2}\ell L\right)\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell] - (\lambda_0+\delta^*)\ell L \leq -\sqrt{\frac{(\lambda_0+\delta^*)\ell L}{\beta}}\right) \quad \text{with (2.69)}\\
&\leq \beta\ ,
\end{aligned}$$

with a last line simply following from the Bienayme-Chebyshev inequality.

Assume now that δ^* is in $(-\lambda_0,0)$. The assumption (2.68) implies

$$d_2(\lambda,\mathcal{S}_0[\lambda_0]) \geq \frac{2}{\sqrt{L}}\max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\,,\ \sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}}\right)\ ,$$

which yields

$$|\delta^*|\ell \geq 4\max\left(\sqrt{\frac{(\lambda_0+\delta^*)\ell}{\beta L}}\,,\ \frac{\log(1/\alpha)}{L\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\right)\ .$$

Hence

$$\frac{|\delta^*|}{2}\ell L \geq \sqrt{\frac{(\lambda_0+\delta^*)\ell L}{\beta}} + \frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\ ,$$

and then

$$\left(\lambda_0-\frac{|\delta^*|}{2}\right)\ell L - \frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))} - (\lambda_0+\delta^*)\ell L \geq \sqrt{\frac{(\lambda_0+\delta^*)\ell L}{\beta}}\ . \tag{2.70}$$

We conclude with the following inequalities:

$$
\begin{aligned}
P_\lambda\Big(\phi_{5,\alpha}(N)=0\Big) &\le P_\lambda\left(\sup_{\ell'\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell'}(N)\le s^+_{\lambda_0,\delta^*,\tau^*,L}(1-\alpha)\right)\\
&\le P_\lambda\left(\left(\lambda_0-\frac{|\delta^*|}{2}\right)\ell L - N(\tau^*,\tau^*+\ell]\le \frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\right) \text{ with (2.121)}\\
&\le P_\lambda\left(N(\tau^*,\tau^*+\ell]\ge \left(\lambda_0-\frac{|\delta^*|}{2}\right)\ell L-\frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\right)\\
&\le P_\lambda\left(N(\tau^*,\tau^*+\ell]-(\lambda_0+\delta^*)\ell L\ge\sqrt{\frac{(\lambda_0+\delta^*)\ell L}{\beta}}\right) \text{ with (2.70)}\\
&\le \beta \text{ with the Bienayme-Chebyshev inequality }.
\end{aligned}
$$

Proof of Lemma 2.11

Let $\lambda_0>0$, τ^* in $(0,1)$, and ϕ_α a level-α test of (H_0) "$\lambda\in\mathcal{S}_0[\lambda_0]=\{\lambda_0\}$" versus (H_1) "$\lambda\in\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0]$", with

$$\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0]=\Big\{\lambda:\ \exists\delta\in(-\lambda_0,+\infty)\setminus\{0\},\ \exists\ell\in(0,1-\tau^*),\ \lambda(t)=\lambda_0+\delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}(t)\Big\}\ ,$$

as defined by (2.20).
Let $r>0$ and λ in $\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0]$ satisfying $d_2(\lambda,\mathcal{S}_0[\lambda_0])\ge r$. We compute

$$
\begin{aligned}
P_\lambda(\phi_\alpha(N)=0) &= 1-P_\lambda(\phi_\alpha(N)=1)+P_{\lambda_0}(\phi_\alpha(N)=1)-P_{\lambda_0}(\phi_\alpha(N)=1)\\
&\ge 1-\alpha-|P_{\lambda_0}(\phi_\alpha(N)=1)-P_\lambda(\phi_\alpha(N)=1)|\\
&\ge 1-\alpha-V\Big(P_\lambda,P_{\lambda_0}\Big)\ ,
\end{aligned}
$$

where $V\Big(P_\lambda,P_{\lambda_0}\Big)$ is the total variation distance between the probability measures P_λ and P_{λ_0}. Then, using the Pinsker inequality (see for example Lemma 2.5 in [Tsy08]),

$$P_\lambda(\phi_\alpha(N)=0)\ge 1-\alpha-\sqrt{\frac{K\Big(P_\lambda,P_{\lambda_0}\Big)}{2}}\ ,$$

where $K\Big(P_\lambda,P_{\lambda_0}\Big)$ is the Kullback divergence between the probability measures P_λ and P_{λ_0}. We deduce from Lemma 1.7 that if there exists λ in $\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0]$ such that $d_2(\lambda,\mathcal{S}_0[\lambda_0])\ge r$ satisfying $1-\alpha-\sqrt{K\Big(P_\lambda,P_{\lambda_0}\Big)/2}\ge\beta$, then $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0])\ge r$.
Let us introduce for all ℓ in $(0,1-\tau^*)$, $\lambda_r=\lambda_0+r\ell^{-1/2}\mathbb{1}_{(\tau^*,\tau^*+\ell]}$ in $\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0]$ which satisfies $d_2(\lambda_r,\mathcal{S}_0[\lambda_0])=r$. Then, Lemma 2.1 entails

$$K\Big(P_{\lambda_r},P_{\lambda_0}\Big)=\int\log\left(\frac{dP_{\lambda_r}}{dP_{\lambda_0}}\right)dP_{\lambda_r}=\log\left(1+\frac{r}{\lambda_0\sqrt{\ell}}\right)\left(\lambda_0+\frac{r}{\sqrt{\ell}}\right)\ell L-Lr\sqrt{\ell}\ .$$

Hence choosing ℓ close enough to 0 – which is allowed as long as λ_r is not constrained to be upper bounded by some given constant, $K\Big(P_{\lambda_r},P_{\lambda_0}\Big)\le 2(1-\alpha-\beta)^2$. This entails

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0])\ge r$$

for every $r>0$ and allows to conclude that $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0])=+\infty$.

Proof of Proposition 2.12

Assume that $L \geq 3$ and $\alpha+\beta < 1/2$.
Let $C'_{\alpha,\beta} = 4(1-\alpha-\beta)^2$, $K_{\alpha,\beta,L} = \lceil (\log_2 L)/C'_{\alpha,\beta} \rceil$, and for k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $\lambda_k = \lambda_0 + \delta_k \mathbb{1}_{(\tau^*,\tau^*+\ell_k]}$ with $\ell_k = (1-\tau^*)/2^k$ and $\delta_k = (\lambda_0 \log\log L/(\ell_k L))^{1/2}$. Then $d_2(\lambda_k, \mathcal{S}_0[\lambda_0]) = \sqrt{\lambda_0 \log\log L/L}$ for all k in $\{1,\ldots,K_{\alpha,\beta,L}\}$ and assuming that

$$\frac{\log\log L}{L^{1-1/C'_{\alpha,\beta}}} \leq (R-\lambda_0)^2 \frac{1-\tau^*}{2\lambda_0} \ , \tag{2.71}$$

λ_k belongs to $\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]$. Recall that for any k in $\{1,\ldots,K_{\alpha,\beta,L}\}$ P_{λ_k} denotes the distribution of a Poisson process with intensity λ_k with respect to the measure Λ, and consider κ, a random variable with uniform distribution on $\{1,\ldots,K_{\alpha,\beta,L}\}$, which allows to define the probability distribution μ of λ_κ. From Lemma 1.8, we know that it is enough to prove $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2] \leq 1 + C'_{\alpha,\beta}$ to conclude that $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]) \geq \sqrt{\lambda_0 \log\log L/L}$.
By definition, $(dP_\mu/dP_{\lambda_0})(N) = E_\kappa\left[(dP_{\lambda_\kappa}/dP_{\lambda_0})(N)\right]$ (where E_κ denotes the expectation w.r.t. the uniform variable κ) and therefore

$$\frac{dP_\mu}{dP_{\lambda_0}}(N) = \frac{1}{K_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}} \exp\left(\log\left(1+\frac{\delta_k}{\lambda_0}\right) N(\tau^*,\tau^*+\ell_k] - L\ell_k\delta_k\right) \ .$$

Since $\ell_{k'} < \ell_k$ for all $k' > k$,

$$\begin{aligned}\left(\frac{dP_\mu}{dP_{\lambda_0}}(N)\right)^2 &= \frac{1}{K^2_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}} \exp\left(2\log\left(1+\frac{\delta_k}{\lambda_0}\right) N(\tau^*,\tau^*+\ell_k] - 2L\ell_k\delta_k\right) \\ &\quad + \frac{2}{K^2_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}-1} \sum_{k'=k+1}^{K_{\alpha,\beta,L}} \exp\Bigg(\left(\log\left(1+\frac{\delta_k}{\lambda_0}\right) + \log\left(1+\frac{\delta_{k'}}{\lambda_0}\right)\right) N(\tau^*,\tau^*+\ell_{k'}] + \\ &\qquad + \log\left(1+\frac{\delta_k}{\lambda_0}\right) N(\tau^*+\ell_{k'},\tau^*+\ell_k] - L\ell_k\delta_k - L\ell_{k'}\delta_{k'}\Bigg) \ .\end{aligned}$$

Recall that under (H_0), N is a homogeneous Poisson process with intensity λ_0 with respect to the measure Λ, so

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] = \frac{1}{K^2_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}} \exp\left(\frac{L\ell_k\delta_k^2}{\lambda_0}\right) + \frac{2}{K^2_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}-1} \sum_{k'=k+1}^{K_{\alpha,\beta,L}} \exp\left(\frac{L\ell_{k'}\delta_k\delta_{k'}}{\lambda_0}\right) \ .$$

Then

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] = \frac{\log L}{K_{\alpha,\beta,L}} + \frac{2}{K^2_{\alpha,\beta,L}} \sum_{k=1}^{K_{\alpha,\beta,L}-1} \sum_{k'=k+1}^{K_{\alpha,\beta,L}} \exp\left(2^{\frac{k-k'}{2}}(\log\log L)\right) \ , \tag{2.72}$$

which entails

$$\begin{aligned}E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] &= \frac{\log L}{K_{\alpha,\beta,L}} + \frac{2}{K^2_{\alpha,\beta,L}} \sum_{l=1}^{K_{\alpha,\beta,L}-1} \left(K_{\alpha,\beta,L} - l\right) \exp\left(2^{-\frac{l}{2}}(\log\log L)\right) \\ &\leq C'_{\alpha,\beta} \log 2 + \frac{2}{K^2_{\alpha,\beta,L}} \sum_{l=1}^{K_{\alpha,\beta,L}-1} \left(K_{\alpha,\beta,L} - l\right) \exp\left(2^{-\frac{l}{2}}(\log\log L)\right) \ .\end{aligned}$$

Now taking η such that $0 < \eta < 1 - 1/\sqrt{2}$,

$$
\begin{aligned}
E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] &\leq C'_{\alpha,\beta}\log 2 + \frac{2}{K^2_{\alpha,\beta,L}}\sum_{l=1}^{K_{\alpha,\beta,L}-1}\left(K_{\alpha,\beta,L}-l\right)\exp\left(2^{-\frac{l}{2}}(\log\log L)\right) \\
&= C'_{\alpha,\beta}\log 2 + \frac{2}{K^2_{\alpha,\beta,L}}\sum_{l=1}^{\lfloor(\log L)^\eta\rfloor}\left(K_{\alpha,\beta,L}-l\right)\exp\left(2^{-\frac{l}{2}}(\log\log L)\right) \\
&\quad + \frac{2}{K^2_{\alpha,\beta,L}}\sum_{l=\lfloor(\log L)^\eta\rfloor+1}^{K_{\alpha,\beta,L}-1}\left(K_{\alpha,\beta,L}-l\right)\exp\left(2^{-\frac{l}{2}}(\log\log L)\right) \\
&\leq C'_{\alpha,\beta}\log 2 + \frac{2C'_{\alpha,\beta}\log 2}{\log L}(\log L)^{\eta+\frac{1}{\sqrt{2}}} + \exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) \quad .
\end{aligned}
\tag{2.73}
$$

If we assume now that

$$
\exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) + \frac{2C'_{\alpha,\beta}\log 2}{(\log L)^{1-\eta-\frac{1}{\sqrt{2}}}} \leq 1 + (1-\log 2)C'_{\alpha,\beta} \quad , \tag{2.74}
$$

we finally obtain the expected result, i.e.

$$
E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] \leq 1 + C'_{\alpha,\beta} \quad .
$$

To end the proof, it remains to notice that there exists $L_0(\alpha,\beta,\lambda_0,R) \geq 3$ such that for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$, both assumptions (2.71) and (2.74) hold.

Proof of Proposition 2.13

The control of the first kind error rates of the two tests $\phi^{(1)}_{6,\alpha}$ and $\phi^{(2)}_{6,\alpha}$ is straightforward using simple union bounds.

(i) Control of the second kind error rate of $\phi^{(1)}_{6,\alpha}$.
Let λ in $\mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0,R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}$, with δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$, ℓ in $(0,1-\tau^*)$, and such that

$$
d_2\left(\lambda,\mathcal{S}_0[\lambda_0]\right) \geq \sqrt{2}\max\left(2\sqrt{\frac{R\log(2/u_\alpha)}{3L}}\,,\;2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}} + 2\sqrt{\frac{R}{\beta L}}\,,\;\frac{R}{\sqrt{L}}\right) \quad . \tag{2.75}
$$

Let us prove that $P_\lambda\left(\phi^{(1)}_{6,\alpha}(N) = 0\right) \leq \beta$.

Assume first that δ belongs to $(0, R-\lambda_0]$. Noticing that

$$
P_\lambda\left(\phi^{(1)}_{6,\alpha}(N) = 0\right) \leq \inf_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}} P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}] \leq p_{\lambda_0\ell_{\tau^*,k}L}\left(1-u_\alpha/2\right)\right) \quad ,
$$

one can see that it is enough to exhibit some k in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$ satisfying

$$
P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}] \leq p_{\lambda_0\ell_{\tau^*,k}L}\left(1-u_\alpha/2\right)\right) \leq \beta \quad .
$$

We get from (2.75) that $d_2^2\left(\lambda,\mathcal{S}_0[\lambda_0]\right) \geq 2R^2/L$ which entails $\ell \geq 2/L$, so
$(1-\tau^*)2^{-\lfloor\log_2 L\rfloor} \leq 2(1-\tau^*)/L < 2/L \leq \ell$ and $\ell < (1-\tau^*)2^{-1+1}$. Therefore, one can find k_{τ^*} in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$

satisfying $(1-\tau^*)2^{-k_{\tau^*}} \le \ell < (1-\tau^*)2^{-k_{\tau^*}+1}$. Consider now $\ell_{\tau^*} = (1-\tau^*)2^{-k_{\tau^*}} = \ell_{\tau^*,k_{\tau^*}}$ such that $\ell/2 < \ell_{\tau^*} \le \ell$. We get

$$\delta^2 \ell_{\tau^*} > d_2^2(\lambda, \mathcal{S}_0[\lambda_0])/2 \ . \tag{2.76}$$

Moreover, we deduce from (2.75) that

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \ge \sqrt{2} \max\left(2\sqrt{\frac{\delta \log(2/u_\alpha)}{3L}} \ , \ 2\sqrt{\frac{2\lambda_0 \log(2/u_\alpha)}{L}} + 2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) \ ,$$

and then with (2.76),

$$\delta\sqrt{\ell_{\tau^*}} \ge \max\left(2\sqrt{\frac{\delta \log(2/u_\alpha)}{3L}} \ , \ 2\sqrt{\frac{2\lambda_0 \log(2/u_\alpha)}{L}} + 2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) \ .$$

This entails in particular $\delta\ell_{\tau^*} \ge 4\log(2/u_\alpha)/(3L)$ as well as

$$\delta\ell_{\tau^*} \ge 2\sqrt{\ell_{\tau^*}}\left(\sqrt{\frac{2\lambda_0 \log(2/u_\alpha)}{L}} + \sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) \ .$$

Therefore,

$$\delta\ell_{\tau^*} \ge 2\max\left(\frac{2\log(2/u_\alpha)}{3L} \ , \ \sqrt{\ell_{\tau^*}}\left(\sqrt{\frac{2\lambda_0 \log(2/u_\alpha)}{L}} + \sqrt{\frac{\lambda_0+\delta}{\beta L}}\right)\right) \ ,$$

hence

$$\delta\ell_{\tau^*}L \ge \frac{2}{3}\log(2/u_\alpha) + \sqrt{2\lambda_0\ell_{\tau^*}L\log(2/u_\alpha)} + \sqrt{\frac{(\lambda_0+\delta)\ell_{\tau^*}L}{\beta}} \ . \tag{2.77}$$

On the one hand, since $\ell_{\tau^*} \le \ell$, Lemma 2.24 gives $E_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]] = \mathrm{Var}_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]] = (\lambda_0+\delta)\ell_{\tau^*}L$. On the other hand, Lemma 2.25 gives

$$p_{\lambda_0\ell_{\tau^*}L}(1-u_\alpha/2) \le \lambda_0\ell_{\tau^*}L + \lambda_0\ell_{\tau^*}Lg^{-1}\left(\frac{\log(2/u_\alpha)}{\lambda_0 L\ell_{\tau^*}}\right) \ ,$$

with $g^{-1}(x) \le 2x/3 + \sqrt{2x}$ for all $x > 0$ (see (2.45)), which leads to

$$p_{\lambda_0\ell_{\tau^*}L}(1-u_\alpha/2) \le \lambda_0\ell_{\tau^*}L + \frac{2}{3}\log(2/u_\alpha) + \sqrt{2\lambda_0\ell_{\tau^*}L\log(2/u_\alpha)} \ .$$

Combined with (2.77), these computations yield

$$E_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]] \ge p_{\lambda_0\ell_{\tau^*}L}(1-u_\alpha/2) + \sqrt{\frac{\mathrm{Var}_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]]}{\beta}} \ . \tag{2.78}$$

We conclude with (2.78) and the Bienayme-Chebyshev inequality successively:

$$\begin{aligned}
&P_\lambda\Big(N(\tau^*, \tau^*+\ell_{\tau^*}] \le p_{\lambda_0\ell_{\tau^*}L}(1-u_\alpha/2)\Big) \\
&\quad \le P_\lambda\Big(N(\tau^*, \tau^*+\ell_{\tau^*}] - E_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]] \le -\sqrt{\mathrm{Var}_\lambda[N(\tau^*, \tau^*+\ell_{\tau^*}]]/\beta}\Big) \\
&\quad \le \beta \ .
\end{aligned}$$

Assume now that δ belongs to $(-\lambda_0, 0)$ and notice that

$$P_\lambda\left(\phi^{(1)}_{6,\alpha}(N)=0\right) \le \inf_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}} P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}] \ge p_{\lambda_0\ell_{\tau^*,k}L}(u_\alpha/2)\right) \ .$$

Lemma 2.25 with (2.45) gives

$$p_{\lambda_0\ell_{\tau^*}L}(u_\alpha/2) \ge \lambda_0\ell_{\tau^*}L - \frac{2}{3}\log(2/u_\alpha) - \sqrt{2\lambda_0\ell_{\tau^*}L\log(2/u_\alpha)} \ .$$

The same choice of k_{τ^*} and $\ell_{\tau^*} = \ell_{\tau^*,k_{\tau^*}}$ as in the above case where $\delta \in (0, R-\lambda_0]$ entails

$$|\delta|\ell_{\tau^*}L \ge \lambda_0\ell_{\tau^*}L - p_{\lambda_0\ell_{\tau^*}L}(u_\alpha/2) + \sqrt{\frac{(\lambda_0+\delta)\ell_{\tau^*}L}{\beta}} \ , \tag{2.79}$$

and since $E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*}]] = \mathrm{Var}_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*}]] = (\lambda_0+\delta)\ell_{\tau^*}L = \lambda_0\ell_{\tau^*}L - |\delta|\ell_{\tau^*}L$, we obtain in the same way

$$\begin{aligned} &P_\lambda\Big(N(\tau^*,\tau^*+\ell_{\tau^*}] \ge p_{\lambda_0\ell_{\tau^*}L}(u_\alpha/2)\Big) \\ &\quad\le P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*}] - E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*}]] \ge \sqrt{\mathrm{Var}_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*}]]/\beta}\right) \\ &\quad\le \beta \ . \end{aligned}$$

Finally (2.75) leads in both cases to $P_\lambda\left(\phi^{(1)}_{6,\alpha}(N)=0\right) \le \beta$, which allows to conclude that

$$\mathrm{SR}_\beta\Big(\phi^{(1)}_{6,\alpha}, \mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0,R]\Big) \le \sqrt{2}\max\left(2\sqrt{\frac{R\log(2\lfloor\log_2 L\rfloor/\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2\lfloor\log_2 L\rfloor/\alpha)}{L}} + 2\sqrt{\frac{R}{\beta L}}\ ,\ \frac{R}{\sqrt{L}}\right) \ .$$

(ii) Control of the second kind error rate of $\phi^{(2)}_{6,\alpha}$.
Let λ in $\mathcal{S}_{\cdot,\tau^*,\dots}[\lambda_0,R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}$, with δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$, ℓ in $(0,1-\tau^*)$, and such that

$$\begin{aligned} d_2(\lambda,\mathcal{S}_0[\lambda_0]) \ge \max\Bigg(&4\sqrt{\frac{2\lambda_0\log(3/u_\alpha)}{L}} + 2\sqrt{2\sqrt{\frac{2}{\beta}\frac{R}{L}}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log(3/u_\alpha)}{3L}}\ ,\ 4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}}\ , \\ &16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\Bigg) \ . \end{aligned} \tag{2.80}$$

Let us prove that $P_\lambda\left(\phi^{(2)}_{6,\alpha}(N)=0\right) \le \beta$.

Noticing that

$$P_\lambda\left(\phi^{(2)}_{6,\alpha}(N)=0\right) \le \inf_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}} P_\lambda\left(T_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N) \le t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right) \ ,$$

one can see that it is enough to exhibit some k in $\{1,\dots,\lfloor\log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(T_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N) \le t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right) \le \beta \ .$$

From (2.80), we deduce that $d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) \ge 2R^2/L$ which entails $\ell \ge 2/L$. Therefore, as in the above part (i) of the proof, let k_{τ^*} in $\{1,\dots,\lfloor\log_2 L\rfloor\}$ be such that $(1-\tau^*)2^{-k_{\tau^*}} \le \ell < (1-\tau^*)2^{-k_{\tau^*}+1}$, and consider $\ell_{\tau^*} = (1-\tau^*)2^{-k_{\tau^*}} = \ell_{\tau^*,k_{\tau^*}}$. Then $\ell/2 < \ell_{\tau^*} \le \ell$ and

$$\delta^2\ell_{\tau^*} > \frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{2} \ . \tag{2.81}$$

Moreover, we get from (2.80)

$$d_2(\lambda,\mathcal{S}_0[\lambda_0]) \ge \max\left(4\sqrt{\frac{2\lambda_0\log(3/u_\alpha)}{L}}+2\sqrt{2\sqrt{\frac{2}{\beta}}\frac{R}{L}},\ 2\sqrt{\frac{2\sqrt{2}R\log(3/u_\alpha)}{3L}},\right.$$
$$\left. 4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}},\ 16\sqrt{\frac{R}{\beta L}}\right).$$

On the one hand, this entails in particular $d_2^4(\lambda,\mathcal{S}_0[\lambda_0]) \ge 128R^2\log^2(3/u_\alpha)/(9L^2)$, and then, with (2.81), $d_2^4(\lambda,\mathcal{S}_0[\lambda_0]) \ge 64 d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\log^2(3/u_\alpha)/(9L^2\ell_{\tau^*})$.
On the other hand, this yields

$$d_2^3(\lambda,\mathcal{S}_0[\lambda_0]) \ge \frac{64}{3}d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\ell_{\tau^*}}},$$

using the same arguments. Therefore

$$d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) \ge \max\left(32\frac{\lambda_0\log(3/u_\alpha)}{L}+8\sqrt{\frac{2}{\beta}}\frac{R}{L},\ \frac{64\log^2(3/u_\alpha)}{9L^2\ell_{\tau^*}},\ \frac{64}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\ell_{\tau^*}}},\right.$$
$$\left. 16 d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{R}{\beta L}}\right).$$

Hence

$$\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{2} \ge \frac{4\lambda_0\log(3/u_\alpha)}{L}+\sqrt{\frac{2}{\beta}}\frac{R}{L}+\frac{8\log^2(3/u_\alpha)}{9L^2\ell_{\tau^*}}+\frac{8}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\ell_{\tau^*}}}+2d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{R}{\beta L}}. \tag{2.82}$$

Since $\ell_{\tau^*} \le \ell$, Lemma 2.26 gives $E_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] = \delta^2\ell_{\tau^*}$ and

$$\mathrm{Var}_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] = \frac{4\delta^2(\lambda_0+\delta)\ell_{\tau^*}}{L}+\frac{2(\lambda_0+\delta)^2}{L^2}.$$

This leads with (2.81) to

$$E_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] \ge \frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{2}, \tag{2.83}$$

and

$$\mathrm{Var}_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] \le \frac{4d_2^2(\lambda,\mathcal{S}_0[\lambda_0])R}{L}+\frac{2R^2}{L^2}. \tag{2.84}$$

With (2.83) and (2.84), the inequality (2.82) yields

$$E_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] \ge \frac{4\lambda_0\log(3/u_\alpha)}{L}+\frac{8\log^2(3/u_\alpha)}{9L^2\ell_{\tau^*}}+\frac{8}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\ell_{\tau^*}}}+\sqrt{\frac{\mathrm{Var}_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right]}{\beta}}. \tag{2.85}$$

Moreover, Lemma 2.27 gives

$$t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*}}(1-u_\alpha) \le 2\lambda_0^2\ell_{\tau^*}\left(g^{-1}\left(\frac{\log(3/u_\alpha)}{\lambda_0\ell_{\tau^*}L}\right)\right)^2,$$

where $g^{-1}(x) \leq 2x/3 + \sqrt{2x}$ for all $x > 0$ (see (2.45)), and then

$$t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*}}(1-u_\alpha) \leq \frac{4\lambda_0 \log(3/u_\alpha)}{L} + \frac{8\log^2(3/u_\alpha)}{9L^2\ell_{\tau^*}} + \frac{8}{3}\sqrt{\frac{2\lambda_0 \log^3(3/u_\alpha)}{L^3\ell_{\tau^*}}}$$
$$\leq E_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] - \sqrt{\frac{\mathrm{Var}_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right]}{\beta}} \quad .$$

We obtain with the Bienayme-Chebyshev inequality

$$P_\lambda\Big(T_{\tau^*,\tau^*+\ell_{\tau^*}}(N) \leq t_{\lambda_0,\tau^*,\tau^*+\ell_{\tau^*}}(1-u_\alpha)\Big)$$
$$\leq P_\lambda\left(T_{\tau^*,\tau^*+\ell_{\tau^*}}(N) \leq E_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right] - \sqrt{\frac{\mathrm{Var}_\lambda\left[T_{\tau^*,\tau^*+\ell_{\tau^*}}(N)\right]}{\beta}}\right)$$
$$\leq \beta \ ,$$

which entails $P_\lambda\left(\phi^{(2)}_{6,\alpha}(N) = 0\right) \leq \beta$.
This finally allows to conclude that

$$\mathrm{SR}_\beta\left(\phi^{(2)}_{6,\alpha}, \mathcal{S}_{\cdot,\tau^*,\ldots}[\lambda_0, R]\right) \leq \max\Bigg(4\sqrt{\frac{2\lambda_0 \log(3\lfloor \log_2 L\rfloor/\alpha)}{L}} + 2\sqrt{2\sqrt{\frac{2}{\beta}\frac{R}{L}}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log(3\lfloor\log_2 L\rfloor/\alpha)}{3L}}\ ,$$
$$4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ,\ 16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\Bigg) \ .$$

Proof of Proposition 2.14

Let $C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2$, $r = (\lambda_0 \log C_{\alpha,\beta}/L)^{1/2}$ and λ_r defined for all t in $(0,1)$ by

$$\lambda_r(t) = \lambda_0 + \delta^* \mathbb{1}_{(1-r^2/\delta^{*2},1]}(t) \ .$$

Notice that for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$, we have $r \leq |\delta^*|$ and λ_r belongs to $(\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0])_r$ in the notation of Lemma 1.7. We get now from Lemma 2.1 and Lemma 2.24

$$E_{\lambda_0}\left[\left(\frac{dP_{\lambda_r}}{dP_0}\right)^2(N)\right] = \exp\left(\frac{r^2 L}{\lambda_0}\right) = C_{\alpha,\beta}.$$

Lemmas 1.8 and 1.7 then entail $\rho_\alpha\left((\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0])_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]) \geq r$.

Proof of Proposition 2.15

The first kind error rate control is straightforward. As for the second kind error rate control, let $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}$ belonging to $\mathcal{S}_{\delta^*,\cdot,1-\cdot}[\lambda_0]$ with τ in $(0,1)$ and satisfying

$$d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq \frac{2}{\sqrt{L}}\max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\ ,\ \sqrt{\delta^* s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}\mathbb{1}_{\{\delta^*>0\}} + \sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}}\mathbb{1}_{\{-\lambda_0<\delta^*<0\}}\right) \ . \tag{2.86}$$

Then the proof essentially follows the same line as the one of Proposition 2.10 just replacing ℓ by $(1-\tau)$. Assume that $\delta^*>0$ and recall that $s^+_{\lambda_0,\delta^*/2}(1-\alpha)$ defined in Lemma 2.29 is a constant which does not depend on L. The assumption (2.86) implies that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq \frac{2}{\sqrt{L}}\max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\,,\,\sqrt{\delta^* s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}\right)\,,$$

which entails

$$\frac{\delta^*}{2}(1-\tau)L\geq \sqrt{\frac{(\lambda_0+\delta^*)(1-\tau)L}{\beta}}+s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\;. \tag{2.87}$$

Then we get from the quantile upper bound (2.122)

$$\begin{aligned}
P_\lambda\left(\phi_{7,\alpha}(N)=0\right)&\leq P_\lambda\left(\sup_{\tau'\in(0,1)}S_{\delta^*,\tau',1}(N)\leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\right)\\
&\leq P_\lambda\left(N(\tau,1]-\left(\lambda_0+\frac{\delta^*}{2}\right)(1-\tau)L\leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)\right)\\
&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta^*)(1-\tau)L\leq s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)-\frac{\delta^*}{2}(1-\tau)L\right)\\
&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta^*)(1-\tau)L\leq -\sqrt{\frac{(\lambda_0+\delta^*)(1-\tau)L}{\beta}}\right)\quad\text{with (2.87)}\\
&\leq\beta\;.
\end{aligned}$$

Assume now that δ^* is in $(-\lambda_0,0)$. The assumption (2.86) implies that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq \frac{2}{\sqrt{L}}\max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\,,\,\sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}}\right)\,,$$

which entails

$$\frac{|\delta^*|}{2}(1-\tau)L\geq \sqrt{\frac{(\lambda_0+\delta^*)(1-\tau)L}{\beta}}+\frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\,,$$

and then

$$\left(\lambda_0-\frac{|\delta^*|}{2}\right)(1-\tau)L-\frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}-(\lambda_0+\delta^*)(1-\tau)L\geq\sqrt{\frac{(\lambda_0+\delta^*)(1-\tau)L}{\beta}}\;. \tag{2.88}$$

We conclude with the following inequalities, deduced from (2.122), (2.88) and the Bienayme-Chebyshev inequality successively:

$$\begin{aligned}
P_\lambda\left(\phi_{7,\alpha}(N)=0\right)&\leq P_\lambda\left(\sup_{\tau'\in(0,1)}S_{\delta^*,\tau',1}(N)\leq s^+_{\lambda_0,\delta^*,\tau^*,L}(1-\alpha)\right)\\
&\leq P_\lambda\left(\left(\lambda_0-\frac{|\delta^*|}{2}\right)(1-\tau)L-N(\tau,1]\leq\frac{\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}\right)\\
&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta^*)(1-\tau)L\geq\sqrt{\frac{(\lambda_0+\delta^*)(1-\tau)L}{\beta}}\right)\\
&\leq\beta\;.
\end{aligned}$$

Coming back to the formulation of assumption (2.86), we therefore can take in the statement of Proposition 2.15

$$C(\alpha,\beta,\lambda_0,\delta^*) = 2\max\left(2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\ ,\ \sqrt{\delta^* s^+_{\lambda_0,\frac{\delta^*}{2}}(1-\alpha)}\mathbb{1}_{\{\delta^*>0\}} + \sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))}}\mathbb{1}_{\{-\lambda_0<\delta^*<0\}}\right) .$$

Proof of Lemma 2.16

Let $\lambda_0 > 0$ and ϕ_α a level-α test of the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0[\lambda_0] = \{\lambda_0\}$" versus the alternative (H_1) "$\lambda \in \mathcal{S}_{.,..,1-..}[\lambda_0]$", with $\mathcal{S}_{.,..,1-..}[\lambda_0]$ defined by (2.26). Let us fix some $r > 0$. As in the proof of Lemma 2.11, we can argue that if there exists λ in $\mathcal{S}_{.,..,1-..}[\lambda_0]$ such that $d_2(\lambda, \mathcal{S}_0[\lambda_0]) \geq r$ satisfying $1-\alpha-\sqrt{K\left(P_\lambda,P_{\lambda_0}\right)/2} \geq \beta$, then $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{.,..,1-..}[\lambda_0]) \geq r$.
Let us introduce for all τ in $(0,1)$, $\lambda_r = \lambda_0 + r(1-\tau)^{-1/2}\mathbb{1}_{(\tau,1]}$ in $\mathcal{S}_{.,..,1-..}[\lambda_0]$ which satisfies $d_2(\lambda_r, \mathcal{S}_0[\lambda_0]) = r$. The end of the proof follows the same line as the one of Lemma 2.11, noticing that

$$K\left(P_{\lambda_r},P_{\lambda_0}\right) = \log\left(1+\frac{r}{\lambda_0\sqrt{1-\tau}}\right)\left(\lambda_0+\frac{r}{\sqrt{1-\tau}}\right)(1-\tau)L - Lr\sqrt{1-\tau}\ ,$$

and choosing τ close enough to 1 in order to get $K\left(P_{\lambda_r},P_{\lambda_0}\right) \leq 2(1-\alpha-\beta)^2$, which yields $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{.,..,1-..}[\lambda_0]) \geq r$ for all $r > 0$ and allows to conclude.

Proof of Proposition 2.17

Assume that $L \geq 3$ and $\alpha+\beta < 1/2$. As in the proof of Proposition 2.12, we consider $C'_{\alpha,\beta} = 4(1-\alpha-\beta)^2$, $K_{\alpha,\beta,L} = \lceil(\log_2 L)/C'_{\alpha,\beta}\rceil$ and for k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $\lambda_k = \lambda_0 + \delta_k \mathbb{1}_{(\tau_k,1]}$ with $\tau_k = 1-2^{-k}$ and $\delta_k = (2^k\lambda_0\log\log L/L)^{1/2}$. Then, for every k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $d_2(\lambda_k,\mathcal{S}_0[\lambda_0]) = \sqrt{\lambda_0\log\log L/L}$, and assuming that

$$\frac{\log\log L}{L^{1-1/C'_{\alpha,\beta}}} \leq \frac{(R-\lambda_0)^2}{2\lambda_0}\ , \tag{2.89}$$

λ_k belongs to $\mathcal{S}_{.,.,1-..}[\lambda_0,R]$. The proof then essentially follows the same arguments as the proof of Proposition 2.12. Thus, considering a random variable κ with uniform distribution on $\{1,\ldots,K_{\alpha,\beta,L}\}$ and the probability distribution μ of λ_κ, we aim at proving that $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2] \leq 1+C'_{\alpha,\beta}$, with P_μ defined as in Lemma 1.8, in order to conclude that $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}_{.,..,1-..}[\lambda_0,R]) \geq \sqrt{\lambda_0\log\log L/L}$.
By definition,

$$\frac{dP_\mu}{dP_{\lambda_0}}(N) = \frac{1}{K_{\alpha,\beta,L}}\sum_{k=1}^{K_{\alpha,\beta,L}}\exp\left(\log\left(1+\frac{\delta_k}{\lambda_0}\right)N(\tau_k,1] - L(1-\tau_k)\delta_k\right)\ .$$

Since $\tau_{k'} > \tau_k$ for all $k' > k$,

$$\begin{aligned}\left(\frac{dP_\mu}{dP_{\lambda_0}}(N)\right)^2 = \frac{1}{K^2_{\alpha,\beta,L}}&\sum_{k=1}^{K_{\alpha,\beta,L}}\exp\left(2\log\left(1+\frac{\delta_k}{\lambda_0}\right)N(\tau_k,1] - 2L(1-\tau_k)\delta_k\right)\\ &+\frac{2}{K^2_{\alpha,\beta,L}}\sum_{k=1}^{K_{\alpha,\beta,L}-1}\sum_{k'=k+1}^{K_{\alpha,\beta,L}}\exp\left(\left(\log\left(1+\frac{\delta_k}{\lambda_0}\right)+\log\left(1+\frac{\delta_{k'}}{\lambda_0}\right)\right)N(\tau_{k'},1]\right.\\ &\qquad\left.+\log\left(1+\frac{\delta_k}{\lambda_0}\right)N(\tau_k,\tau_{k'}] - L(1-\tau_k)\delta_k - L(1-\tau_{k'})\delta_{k'}\right)\ ,\end{aligned}$$

hence

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right]=\frac{1}{K_{\alpha,\beta,L}^2}\sum_{k=1}^{K_{\alpha,\beta,L}}\exp\left(\frac{L(1-\tau_k)\delta_k^2}{\lambda_0}\right)+\frac{2}{K_{\alpha,\beta,L}^2}\sum_{k=1}^{K_{\alpha,\beta,L}-1}\sum_{k'=k+1}^{K_{\alpha,\beta,L}}\exp\left(\frac{L(1-\tau_{k'})\delta_k\delta_{k'}}{\lambda_0}\right) .$$

We then obtain the same expression of $E_{\lambda_0}\left[\left(dP_\mu/dP_{\lambda_0}\right)^2\right]$ as in Equation (2.72), and the proof ends exactly as the one of Proposition 2.12, just replacing (2.71) by (2.89) in the final argument.

Proof of Proposition 2.18

This proof is very similar to the one of Proposition 2.13. For the sake of completeness, we nevertheless detail it below.
The control of the first kind error rates of the two tests $\phi_{8,\alpha}^{(1)}$ and $\phi_{8,\alpha}^{(2)}$ is straightforward using simple union bounds.

(i) Control of the second kind error rate of $\phi_{8,\alpha}^{(1)}$.
Let λ in $\mathcal{S}_{.,..,1-..}[\lambda_0,R]$ be such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$, with τ in $(0,1)$, δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$, and such that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq\sqrt{2}\max\left(2\sqrt{\frac{R\log(2/u_\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{R}{\beta L}}\ ,\ \frac{R}{\sqrt{L}}\right) , \tag{2.90}$$

as in (2.75).
We prove here that $P_\lambda\left(\phi_{8,\alpha}^{(1)}(N)=0\right)\leq\beta$, assuming first that δ belongs to $(0,R-\lambda_0]$.
Noticing that

$$P_\lambda\left(\phi_{8,\alpha}^{(1)}(N)=0\right)\leq\inf_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}}P_\lambda\left(N(\tau_k,1]\leq p_{\lambda_0(1-\tau_k)L}(1-u_\alpha/2)\right) ,$$

one can see that it is enough to exhibit some k in $\{1,\dots,\lfloor\log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(N(\tau_k,1]\leq p_{\lambda_0(1-\tau_k)L}(1-u_\alpha/2)\right)\leq\beta .$$

Let $k_\tau=\lfloor-\log_2(1-\tau)\rfloor+1$. Since $0<1-\tau<1$, $k_\tau\geq1$. Moreover, from (2.90), we obtain $d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\geq 2R^2/L$ which entails $(1-\tau)\geq 2/L$ and $k_\tau\leq\lfloor\log_2(L/2)\rfloor+1\leq\lfloor\log_2 L\rfloor$. Consider now $\tau_{k_\tau}=1-2^{-k_\tau}$, which satisfies $(1-\tau)/2\leq 1-\tau_{k_\tau}<1-\tau$ as well as

$$\delta^2(1-\tau_{k_\tau})\geq d_2^2(\lambda,\mathcal{S}_0[\lambda_0])/2 . \tag{2.91}$$

We get from (2.90)

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq\sqrt{2}\max\left(2\sqrt{\frac{\delta\log(2/u_\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) ,$$

which gives with (2.91)

$$\delta\sqrt{1-\tau_{k_\tau}}\geq\max\left(2\sqrt{\frac{\delta\log(2/u_\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) .$$

This entails in particular $\delta\left(1-\tau_{k_\tau}\right)\geq(4/3)\log(2/u_\alpha)/L$ and also

$$\delta\left(1-\tau_{k_\tau}\right)\geq 2\sqrt{1-\tau_{k_\tau}}\left(\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) .$$

Hence,

$$\delta\left(1-\tau_{k_\tau}\right)L \geq \frac{2}{3}\log(2/u_\alpha)+\sqrt{2\lambda_0\left(1-\tau_{k_\tau}\right)L\log(2/u_\alpha)}+\sqrt{\frac{(\lambda_0+\delta)\left(1-\tau_{k_\tau}\right)L}{\beta}}\ . \tag{2.92}$$

On the one hand, since $\tau_{k_\tau} > \tau$, Lemma 2.24 gives $E_\lambda\left[N(\tau_{k_\tau},1]\right] = \mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]\right] = (\lambda_0+\delta)\left(1-\tau_{k_\tau}\right)L$. On the other hand, Lemma 2.25 with the inequality (2.45) give

$$p_{\lambda_0(1-\tau_{k_\tau})L}(1-u_\alpha/2) \leq \lambda_0(1-\tau_{k_\tau})L+\frac{2}{3}\log(2/u_\alpha)+\sqrt{2\lambda_0\left(1-\tau_{k_\tau}\right)L\log(2/u_\alpha)}\ .$$

Combined with (2.92), these computations yield

$$E_\lambda\left[N(\tau_{k_\tau},1]\right] \geq p_{\lambda_0(1-\tau_{k_\tau})L}(1-u_\alpha/2)+\sqrt{\mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]\right]/\beta}\ . \tag{2.93}$$

The Bienayme-Chebyshev then leads to

$$P_\lambda\left(N(\tau_{k_\tau},1] \leq p_{\lambda_0(1-\tau_{k_\tau})L}(1-u_\alpha/2)\right) \leq \beta\ .$$

Assume now that δ belongs to $(-\lambda_0,0)$ and notice that

$$P_\lambda\left(\phi^{(1)}_{8,\alpha}(N)=0\right) \leq \inf_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}} P_\lambda\left(N(\tau_k,1] \geq p_{\lambda_0(1-\tau_k)L}(u_\alpha/2)\right)\ .$$

Lemma 2.25 with (2.45) gives

$$p_{\lambda_0(1-\tau_k)L}(u_\alpha/2) \geq \lambda_0(1-\tau_k)L-\frac{2}{3}\log(2/u_\alpha)-\sqrt{2\lambda_0(1-\tau_k)L\log(2/u_\alpha)}\ .$$

The same choice of k_τ as in the above case where $\delta\in(0,R-\lambda_0]$ thus entails

$$|\delta|\left(1-\tau_{k_\tau}\right)L \geq \lambda_0\left(1-\tau_{k_\tau}\right)L-p_{\lambda_0\left(1-\tau_{k_\tau}\right)L}(u_\alpha/2)+\sqrt{\frac{(\lambda_0+\delta)\left(1-\tau_{k_\tau}\right)L}{\beta}}\ , \tag{2.94}$$

and since $E_\lambda\left[N(\tau_{k_\tau},1]\right] = \mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]\right] = (\lambda_0+\delta)\left(1-\tau_{k_\tau}\right)L = \lambda_0\left(1-\tau_{k_\tau}\right)L-|\delta|\left(1-\tau_{k_\tau}\right)L$, we obtain in the same way

$$\begin{aligned}P_\lambda\left(N(\tau_{k_\tau},1] \geq p_{\lambda_0\left(1-\tau_{k_\tau}\right)L}(u_\alpha/2)\right) &\leq P_\lambda\left(N(\tau_{k_\tau},1]-E_\lambda\left[N(\tau_{k_\tau},1]\right] \geq \sqrt{\mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]\right]/\beta}\right)\\ &\leq \beta\ .\end{aligned}$$

Finally, (2.90) leads in both cases to $P_\lambda\left(\phi^{(1)}_{8,\alpha}(N)=0\right) \leq \beta$, which allows to conclude that

$$\mathrm{SR}_\beta\left(\phi^{(1)}_{8,\alpha},\mathcal{S}_{.,..,1\_..}[\lambda_0,R]\right) \leq \sqrt{2}\max\left(2\sqrt{\frac{R\log(2\lfloor\log_2 L\rfloor/\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2\lfloor\log_2 L\rfloor/\alpha)}{L}}+2\sqrt{\frac{R}{\beta L}}\ ,\ \frac{R}{\sqrt{L}}\right)\ .$$

(ii) Control of the second kind error rate of $\phi^{(2)}_{8,\alpha}$.
Let λ in $\mathcal{S}_{.,..,1\_..}[\lambda_0,R]$ be such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$, with τ in $(0,1)$, δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$, and such that

$$\begin{aligned}d_2(\lambda,\mathcal{S}_0[\lambda_0]) \geq \max\Bigg(&4\sqrt{\frac{2\lambda_0\log(3/u_\alpha)}{L}}+2\sqrt{2\sqrt{\frac{2}{\beta}}\frac{R}{L}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log(3/u_\alpha)}{3L}}\ ,\\ &4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}}\ ,\ 16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\Bigg)\ ,\end{aligned} \tag{2.95}$$

as in (2.80).
Let us prove that this implies that $P_\lambda\left(\phi^{(2)}_{8,\alpha}(N)=0\right)\le\beta$.
Notice that

$$P_\lambda\left(\phi^{(2)}_{8,\alpha}(N)=0\right)\le\inf_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}}P_\lambda\left(T_{\tau_k,1}(N)\le t_{\lambda_0,\tau_k,1}(1-u_\alpha)\right)\ ,$$

to see that one only needs to exhibit some k in $\{1,\dots,\lfloor\log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(T_{\tau_k,1}(N)\le t_{\lambda_0,\tau_k,1}(1-u_\alpha)\right)\le\beta\ ,$$

to obtain the expected result.
As in the above part (i) of the proof, let $k_\tau=\lfloor-\log_2(1-\tau)\rfloor+1$ and $\tau_{k_\tau}=1-2^{-k_\tau}$. From (2.95) which in particular entails $d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\ge 2R^2/L$, we get that k_τ actually belongs to $\{1,\dots,\lfloor\log_2 L\rfloor\}$. Furthermore, by definition,

$$\delta^2\left(1-\tau_{k_\tau}\right)\ge d_2^2(\lambda,\mathcal{S}_0[\lambda_0])/2\ . \tag{2.96}$$

Now, we also deduce from (2.95) that

$$\begin{aligned}d_2(\lambda,\mathcal{S}_0[\lambda_0])\ge\max\Bigg(&4\sqrt{\frac{2\lambda_0\log(3/u_\alpha)}{L}}+2\sqrt{2\sqrt{\frac{2}{\beta}}\frac{R}{L}},\ \sqrt{\frac{\sqrt{2}R\log(3/u_\alpha)}{3L}},\\ &4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}},\ 16\sqrt{\frac{R}{\beta L}}\Bigg)\ .\end{aligned}$$

This entails on the one hand that $d_2^4(\lambda,\mathcal{S}_0[\lambda_0])\ge 128R^2\log^2(3/u_\alpha)/(9L^2)$, and with (2.96), $d_2^4(\lambda,\mathcal{S}_0[\lambda_0])\ge 64d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\log^2(3/u_\alpha)/(9L^2\left(1-\tau_{k_\tau}\right))$. On the other hand, we deduce that

$$d_2^3(\lambda,\mathcal{S}_0[\lambda_0])\ge\frac{64}{3}d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\left(1-\tau_{k_\tau}\right)}}\ .$$

Therefore

$$\begin{aligned}d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\ge\max\Bigg(&32\frac{\lambda_0\log(3/u_\alpha)}{L}+8\sqrt{\frac{2}{\beta}}\frac{R}{L},\ \frac{64\log^2(3/u_\alpha)}{9L^2\left(1-\tau_{k_\tau}\right)},\ \frac{64}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\left(1-\tau_{k_\tau}\right)}},\\ &16d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{R}{\beta L}}\Bigg)\ .\end{aligned}$$

Hence,

$$\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{2}\ge\frac{4\lambda_0\log(3/u_\alpha)}{L}+\sqrt{\frac{2}{\beta}}\frac{R}{L}+\frac{8\log^2(3/u_\alpha)}{9L^2\left(1-\tau_{k_\tau}\right)}+\frac{8}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\left(1-\tau_{k_\tau}\right)}}+2d_2(\lambda,\mathcal{S}_0[\lambda_0])\sqrt{\frac{R}{\beta L}}\ . \tag{2.97}$$

Since $\tau_{k_\tau}>\tau$, Lemma 2.26 gives that $E_\lambda\left[T_{\tau_{k_\tau},1}(N)\right]=\delta^2\left(1-\tau_{k_\tau}\right)$ and

$$\mathrm{Var}_\lambda\left[T_{\tau_{k_\tau},1}(N)\right]=\frac{4\delta^2(\lambda_0+\delta)\left(1-\tau_{k_\tau}\right)}{L}+\frac{2(\lambda_0+\delta)^2}{L^2}\ .$$

From (2.96), we get

$$E_\lambda\left[T_{\tau_{k_\tau},1}(N)\right]\ge\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{2}\ . \tag{2.98}$$

Moreover,

$$\mathrm{Var}_\lambda\left[T_{\tau_{k_\tau},1}(N)\right] \leq \frac{4d_2^2(\lambda, \mathcal{S}_0[\lambda_0])R}{L} + \frac{2R^2}{L^2} \ . \tag{2.99}$$

With (2.98) and (2.99), the inequality (2.97) yields

$$E_\lambda\left[T_{\tau_{k_\tau},1}(N)\right] \geq \frac{4\lambda_0 \log(3/u_\alpha)}{L} + \frac{8\log^2(3/u_\alpha)}{9L^2\left(1-\tau_{k_\tau}\right)} + \frac{8}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\left(1-\tau_{k_\tau}\right)}} + \sqrt{\frac{\mathrm{Var}_\lambda\left[T_{\tau_{k_\tau},1}(N)\right]}{\beta}} \ .$$

Furthermore, Lemma 2.27 gives

$$\begin{aligned} t_{\lambda_0,\tau_{k_\tau},1}(1-u_\alpha) &\leq \frac{4\lambda_0 \log(3/u_\alpha)}{L} + \frac{8\log^2(3/u_\alpha)}{9L^2\left(1-\tau_{k_\tau}\right)} + \frac{8}{3}\sqrt{\frac{2\lambda_0\log^3(3/u_\alpha)}{L^3\left(1-\tau_{k_\tau}\right)}} \\ &\leq E_\lambda\left[T_{\tau_{k_\tau},1}(N)\right] - \sqrt{\frac{\mathrm{Var}_\lambda\left[T_{\tau_{k_\tau},1}(N)\right]}{\beta}} \ , \end{aligned}$$

and the Bienayme-Chebyshev leads to

$$P_\lambda\left(T_{\tau_{k_\tau},1}(N) \leq t_{\lambda_0,\tau_{k_\tau},1}(1-u_\alpha)\right) \leq \beta \ .$$

This entails the expected result $P_\lambda\left(\phi^{(2)}_{8,\alpha}(N)=0\right) \leq \beta$, and finally allows to conclude that

$$\begin{aligned} \mathrm{SR}_\beta\left(\phi^{(2)}_{8,\alpha}, \mathcal{S}_{\cdot,\cdot\cdot,1-\cdot\cdot}[\lambda_0,R]\right) \leq \max\Bigg(& 4\sqrt{\frac{2\lambda_0\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}} + 2\sqrt{2\sqrt{\frac{2}{\beta}\frac{R}{L}}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log(3\lfloor\log_2 L\rfloor/\alpha)}{3L}}\ , \\ & 4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ,\ 16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\Bigg) \ . \end{aligned}$$

Proof of Proposition 2.19

Let $L \geq 2$. For all k in $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$, let us define $\lambda_k(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau_k,\tau_k+\ell]}(t)$ with $\tau_k = k/L$, and $\ell = \lambda_0 \log L/(2\delta^{*2}L)$. Then λ_k belongs to $\mathcal{S}_{\delta^*,\cdot\cdot,\cdot\cdot\cdot}[\lambda_0]$ for all k in $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$ as soon as

$$\frac{\lceil L^{3/4}\rceil}{L} + \frac{\lambda_0\log L}{2\delta^{*2}L} < 1 \ , \tag{2.100}$$

and it satisfies

$$d_2^2(\lambda_k, \mathcal{S}_0[\lambda_0]) = \lambda_0\log L/(2L) \ .$$

Considering a random variable J uniformly distributed on $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$ and the distribution μ of λ_J and using Lemma 1.8, one can see that it is enough to prove that $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2] \leq 1 + 4(1-\alpha-\beta)^2$ to obtain the expected lower bound. By definition, $(dP_\mu/dP_{\lambda_0})(N) = E_J\left[(dP_{\lambda_J}/dP_{\lambda_0})(N)\right]$, therefore

$$\frac{dP_\mu}{dP_{\lambda_0}}(N) = \frac{1}{\lceil L^{3/4}\rceil}\sum_{k=1}^{\lceil L^{3/4}\rceil}\exp\left(\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_k,\tau_k+\ell] - L\delta^*\ell\right) \ .$$

We then expand the square as

$$\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N) = \frac{1}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil}\exp\left(2\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_k,\tau_k+\ell]-2L\delta^*\ell\right)$$
$$+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}\sum_{k'=k+1}^{\lceil L^{3/4}\rceil}\exp\left(\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_k,\tau_k+\ell]\right.$$
$$\left.+\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_{k'},\tau_{k'}+\ell]-2L\delta^*\ell\right) .$$

For k in $\left\{1,\dots,\lceil L^{3/4}\rceil-1\right\}$, setting $K_0(k)=\max\left(k'\in\left\{k+1,\dots,\lceil L^{3/4}\rceil\right\}:\tau_{k'}<\tau_k+\ell\right)$, we may write

$$\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N) = \frac{1}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil}\exp\left(2\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_k,\tau_k+\ell]-2L\delta^*\ell\right)$$
$$+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}\sum_{k'=K_0(k)+1}^{\lceil L^{3/4}\rceil}\exp\left(\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_k,\tau_k+\ell]\right.$$
$$\left.+\log\left(1+\frac{\delta^*}{\lambda_0}\right)N(\tau_{k'},\tau_{k'}+\ell]-2L\delta^*\ell\right)$$
$$+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}\sum_{k'=k+1}^{K_0(k)}\exp\left(\log\left(1+\frac{\delta^*}{\lambda_0}\right)(N(\tau_k,\tau_{k'}]+N(\tau_{k'},\tau_k+\ell])\right.$$
$$\left.+\log\left(1+\frac{\delta^*}{\lambda_0}\right)(N(\tau_{k'},\tau_k+\ell]+N(\tau_k+\ell,\tau_{k'}+\ell])-2L\delta^*\ell\right) .$$

Under (H_0), N is a homogeneous Poisson process with intensity λ_0 with respect to the measure Λ. Thus

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N)\right] = \frac{\sqrt{L}}{\lceil L^{3/4}\rceil}+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}(\lceil L^{3/4}\rceil-K_0(k))+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}\sum_{k'=k+1}^{K_0(k)}\exp\left(\frac{\delta^{*2}L}{\lambda_0}(\tau_k-\tau_{k'}+\ell)\right)$$
$$=\frac{\sqrt{L}}{\lceil L^{3/4}\rceil}+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}(\lceil L^{3/4}\rceil-K_0(k))+\frac{2\sqrt{L}}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}\sum_{k'=k+1}^{K_0(k)}\exp\left(-\frac{\delta^{*2}}{\lambda_0}(k'-k)\right)$$
$$\leq\frac{\sqrt{L}}{\lceil L^{3/4}\rceil}+\frac{2}{\lceil L^{3/4}\rceil^2}\sum_{k=1}^{\lceil L^{3/4}\rceil-1}(\lceil L^{3/4}\rceil-K_0(k))+\frac{2\sqrt{L}}{\lceil L^{3/4}\rceil}\left(e^{\delta^{*2}/\lambda_0}-1\right)^{-1} .$$

Since $K_0(k)\geq k+1$ for all k in $\left\{1,\dots,\lceil L^{3/4}\rceil-1\right\}$, notice that

$$\sum_{k=1}^{\lceil L^{3/4}\rceil-1}(\lceil L^{3/4}\rceil-K_0(k))\leq\sum_{k=1}^{\lceil L^{3/4}\rceil-2}k\leq\frac{\lceil L^{3/4}\rceil^2}{2} ,$$

hence

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N)\right]\leq 1+\frac{\sqrt{L}}{\lceil L^{3/4}\rceil}\frac{e^{\delta^{*2}/\lambda_0}+1}{e^{\delta^{*2}/\lambda_0}-1} .$$

Finally, assuming that

$$\frac{\sqrt{L}}{\lceil L^{3/4}\rceil}\frac{e^{\delta^{*2}/\lambda_0}+1}{e^{\delta^{*2}/\lambda_0}-1}\leq 4(1-\alpha-\beta)^2 , \tag{2.101}$$

we get

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N)\right] \le 1+4(1-\alpha-\beta)^2 \ .$$

Noticing that there exists $L_0(\alpha,\beta,\lambda_0,\delta^*) \ge 2$ such that for all $L \ge L_0(\alpha,\beta,\lambda_0,\delta^*)$, both assumptions (2.100) and (2.101) hold then allows to end the proof.

Proof of Proposition 2.22

The control of the first kind error rates of the two tests $\phi^{(1)}_{9/10,\alpha}$ and $\phi^{(2)}_{9/10,\alpha}$ is straightforward using simple union bounds.

(*i*) *Control of the second kind error rate of* $\phi^{(1)}_{9/10,\alpha}$.

Let λ in $\mathcal{S}_{.,..,...}[\lambda_0,R]$. We may fix δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$, τ in $(0,1)$, ℓ in $(0,1-\tau)$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell]}$.
We assume by now that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0]) \ge \sqrt{3}\max\left(\sqrt{\frac{4R\log(2/u_\alpha))}{3L}},2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{R}{\beta L}},\frac{R}{\sqrt{L}}\right) , \tag{2.102}$$

and we prove the inequality $P_\lambda\left(\phi^{(1)}_{9/10,\alpha}(N)=0\right) \le \beta$.

Assume first that δ belongs to $(0,R-\lambda_0]$. Noticing that

$$P_\lambda\left(\phi^{(1)}_{9/10,\alpha}(N)=0\right) \le \inf_{k\in\{0,\dots,\lceil L\rceil-1\}}\ \inf_{k'\in\{1,\dots,\lceil L\rceil-k\}} P_\lambda\left(N(k/\lceil L\rceil,(k+k')/\lceil L\rceil] \le p_{\lambda_0 k'L/\lceil L\rceil}(1-u_\alpha/2)\right) \ ,$$

one can see that it is enough to exhibit some k_0 in $\{0,\dots,\lceil L\rceil-1\}$ and k'_0 in $\{1,\dots,\lceil L\rceil-k_0\}$ satisfying

$$P_\lambda\left(N(k_0/\lceil L\rceil,(k_0+k'_0)/\lceil L\rceil] \le p_{\lambda_0 k'_0 L/\lceil L\rceil}(1-u_\alpha/2)\right) \le \beta \ .$$

We get from (2.102) that

$$d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) \ge 3R^2/L > 3\delta^2/\lceil L\rceil \ , \tag{2.103}$$

which entails

$$\ell > 3/\lceil L\rceil \text{ and } \tau < 1-3/\lceil L\rceil \ . \tag{2.104}$$

We therefore can define $k_0=\min(k\in\{0,\dots,\lceil L\rceil-1\},\ \tau\le k/\lceil L\rceil)$ and $k'_0=\max(k'\in\{1,\dots,\lceil L\rceil-k_0\},\ (k_0+k')/\lceil L\rceil \le \tau+\ell)$, so that $\tau \le k_0/\lceil L\rceil < (k_0+k'_0)/\lceil L\rceil \le \tau+\ell$.
Since by definition $k_0/\lceil L\rceil-\tau < 1/\lceil L\rceil$ and $\tau+\ell-(k_0+k'_0)/\lceil L\rceil < 1/\lceil L\rceil$, notice that

$$\frac{k'_0}{\lceil L\rceil} = \ell-\left(\left(\frac{k_0}{\lceil L\rceil}-\tau\right)+\left(\tau+\ell-\frac{k_0+k'_0}{\lceil L\rceil}\right)\right) > \ell-\frac{2}{\lceil L\rceil} \ .$$

This, combined with (2.104) and the expression of $d_2^2(\lambda,\mathcal{S}_0[\lambda_0])=\delta^2\ell$, implies that

$$\delta^2\frac{k'_0}{\lceil L\rceil} > d_2^2(\lambda,\mathcal{S}_0[\lambda_0])-\frac{2\delta^2}{\lceil L\rceil} > \frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{3} \ , \tag{2.105}$$

which yields with (2.102)

$$\delta\sqrt{\frac{k'_0}{\lceil L\rceil}} > \max\left(\sqrt{\frac{4\delta\log(2/u_\alpha))}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) . \tag{2.106}$$

We then deduce on the one hand

$$\frac{\delta k_0'}{\lceil L\rceil} > \frac{4\log(2/u_\alpha)}{3L} \ ,$$

and on the other hand

$$\frac{\delta k_0'}{\lceil L\rceil} \geq \sqrt{\frac{k_0'}{\lceil L\rceil}}\left(2\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+2\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right) \ ,$$

which together can be synthesized in

$$\frac{\delta k_0'}{\lceil L\rceil} > 2\max\left(\frac{2\log(2/u_\alpha)}{3L} \ , \ \sqrt{\frac{k_0'}{\lceil L\rceil}}\left(\sqrt{\frac{2\lambda_0\log(2/u_\alpha)}{L}}+\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right)\right) \ .$$

Hence

$$\frac{\delta k_0' L}{\lceil L\rceil} > \frac{2\log(2/u_\alpha)}{3} + \sqrt{\frac{2\log(2/u_\alpha)\lambda_0 k_0' L}{\lceil L\rceil}} + \sqrt{\frac{(\lambda_0+\delta)k_0' L}{\beta\lceil L\rceil}} \ . \tag{2.107}$$

From Lemma 2.24, we easily deduce that
$E_\lambda\left[N(k_0/\lceil L\rceil,\left(k_0+k_0'\right)/\lceil L\rceil]\right] = \mathrm{Var}_\lambda\left(N(k_0/\lceil L\rceil,\left(k_0+k_0'\right)/\lceil L\rceil]\right) = (\lambda_0+\delta)k_0'L/\lceil L\rceil$, and from Lemma 2.25

$$p_{\lambda_0 k_0' L/\lceil L\rceil}(1-u_\alpha/2) \leq \frac{\lambda_0 k_0' L}{\lceil L\rceil} + \frac{\lambda_0 k_0' L}{\lceil L\rceil} g^{-1}\left(\frac{\log(2/u_\alpha)\lceil L\rceil}{\lambda_0 k_0' L}\right) \ .$$

Using the upper bound (2.45), this leads to

$$p_{\lambda_0 k_0' L/\lceil L\rceil}(1-u_\alpha/2) \leq \frac{\lambda_0 k_0' L}{\lceil L\rceil} + \frac{2\log(2/u_\alpha)}{3} + \sqrt{\frac{2\log(2/u_\alpha)\lambda_0 k_0' L}{\lceil L\rceil}} \ . \tag{2.108}$$

The inequality (2.107) therefore entails

$$E_\lambda\left[N(k_0/\lceil L\rceil,(k_0+k_0')/\lceil L\rceil]\right] > p_{\lambda_0 k_0' L/\lceil L\rceil}(1-u_\alpha/2) + \sqrt{\mathrm{Var}_\lambda\left(N(k_0/\lceil L\rceil,\left(k_0+k_0'\right)/\lceil L\rceil]\right)/\beta} \ . \tag{2.109}$$

We conclude with (2.109) and the Bienayme-Chebyshev inequality:

$$\begin{aligned}
&P_\lambda\left(N(k_0/\lceil L\rceil,(k_0+k_0')/\lceil L\rceil] \leq p_{\lambda_0 k_0' L/\lceil L\rceil}(1-u_\alpha/2)\right)\\
&\leq P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]\right] < -\sqrt{\mathrm{Var}_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]\right)/\beta}\right)\\
&\leq \beta \ .
\end{aligned}$$

Assume now that δ belongs to $(-\lambda_0, 0)$ and notice that we have here

$$P_\lambda\left(\phi^{(1)}_{9/10,\alpha}(N)=0\right) \leq \inf_{k\in\{0,\ldots,\lceil L\rceil-1\}}\ \inf_{k'\in\{1,\ldots,\lceil L\rceil-k\}} P_\lambda\left(N(k/\lceil L\rceil,(k+k')/\lceil L\rceil] \geq p_{\lambda_0 k' L/\lceil L\rceil}(u_\alpha/2)\right) \ .$$

Lemma 2.25 and (2.45) give

$$p_{\lambda_0 k' L/\lceil L\rceil}(u_\alpha/2) \geq \frac{\lambda_0 k_0' L}{\lceil L\rceil} - \frac{2\log(2/u_\alpha)}{3} - \sqrt{\frac{2\log(2/u_\alpha)\lambda_0 k_0' L}{\lceil L\rceil}} \ .$$

The same choice of k_0 and k_0' as in the previous case yields

$$E_\lambda\left[N(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}]\right] = \mathrm{Var}_\lambda\left(N(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}]\right) = (\lambda_0+\delta)k_0'\frac{L}{\lceil L\rceil} = (\lambda_0-|\delta|)k_0'\frac{L}{\lceil L\rceil} \ ,$$

so

$$P_\lambda\left(N(k_0/\lceil L\rceil,(k_0+k_0')/\lceil L\rceil]\geq p_{\lambda_0 k_0' L/\lceil L\rceil}(u_\alpha/2)\right)\leq\beta\ ,$$

in a similar way, notably replacing δ by $|\delta|$ (except when it is involved in $\lambda_0+\delta$) in the rest of the proof. Coming back to the assumption (2.102) and the definition of u_α, one can finally claim that

$$\mathrm{SR}_\beta\left(\phi^{(1)}_{9/10,\alpha},\mathcal{S}_{.,..,..}[\lambda_0,R]\right)\leq\sqrt{3}\max\left(\sqrt{\frac{4R\log(\lceil L\rceil(\lceil L\rceil+1)/\alpha)}{3L}}\ ,\ 2\sqrt{\frac{2\lambda_0\log(\lceil L\rceil(\lceil L\rceil+1)/\alpha)}{L}}+2\sqrt{\frac{R}{\beta L}}\ ,\ \frac{R}{\sqrt{L}}\right)\ ,$$

which ends the proof for the test $\phi^{(1)}_{9/10,\alpha}$.

(ii) Control of the second kind error rate of $\phi^{(2)}_{9/10,\alpha}$.

Let λ in $\mathcal{S}_{.,..,..}[\lambda_0,R]$. There exist δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$, τ in $(0,1)$, ℓ in $(0,1-\tau)$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell]}$. Let us assume now that

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])\geq\max\left(R\sqrt{\frac{3}{L}}\ ,\ 4\sqrt{\frac{3\lambda_0\log(3/u_\alpha)}{L}}+2\sqrt{\frac{3\sqrt{2}R}{\sqrt{\beta}L}}\ ,\ 2\sqrt{\frac{\sqrt{2}R\log(3/u_\alpha)}{L}}\ ,\right.$$
$$\left.32^{1/3}(6\lambda_0)^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}}\ ,\ 24\sqrt{\frac{R}{\beta L}}\right)\ ,\qquad(2.110)$$

and prove that it entails $P_\lambda\left(\phi^{(2)}_{9/10,\alpha}(N)=0\right)\leq\beta$.

As in (i), we begin by noticing that

$$P_\lambda\left(\phi^{(2)}_{9/10,\alpha}(N)=0\right)\leq\inf_{k\in\{0,\ldots,\lceil L\rceil-1\}}\ \inf_{k'\in\{1,\ldots,\lceil L\rceil-k\}}P_\lambda\left(T_{\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(N)\leq t_{\lambda_0,\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}}(1-u_\alpha)\right)\ ,$$

so it suffices to find k_0 in $\{0,\ldots,\lceil L\rceil-1\}$ and k_0' in $\{1,\ldots,\lceil L\rceil-k_0\}$ satisfying

$$P_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\leq t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha)\right)\leq\beta\ ,$$

to obtain $P_\lambda\left(\phi^{(2)}_{9/10,\alpha}(N)=0\right)\leq\beta$.

We get from (2.110) that $d_2^2(\lambda,\mathcal{S}_0[\lambda_0])\geq 3R^2/L>3\delta^2/\lceil L\rceil$ which entails

$$\ell>3/\lceil L\rceil\text{ and }\tau<1-3/\lceil L\rceil\ .\qquad(2.111)$$

We therefore can define $k_0=\min(k\in\{0,\ldots,\lceil L\rceil-1\},\ \tau\leq k/\lceil L\rceil)$ and $k_0'=\max(k'\in\{1,\ldots,\lceil L\rceil-k_0\},\ (k_0+k')/\lceil L\rceil\leq\tau+\ell)$, so that $\tau\leq k_0/\lceil L\rceil<(k_0+k_0')/\lceil L\rceil\leq\tau+\ell$.

As in (i) above, starting from the remark that $k_0/\lceil L\rceil-\tau<1/\lceil L\rceil$ and $\tau+\ell-(k_0+k_0')/\lceil L\rceil<1/\lceil L\rceil$, which implies $k_0'/\lceil L\rceil>\ell-2/\lceil L\rceil$, we obtain, combined with (2.111) and the expression of $d_2^2(\lambda,\mathcal{S}_0[\lambda_0])=\delta^2\ell$:

$$\delta^2\frac{k_0'}{\lceil L\rceil}>d_2^2(\lambda,\mathcal{S}_0[\lambda_0])-\frac{2\delta^2}{\lceil L\rceil}>\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{3}\ .\qquad(2.112)$$

Moreover, we get from (2.110)

$$d_2(\lambda,\mathcal{S}_0[\lambda_0])>\max\left(4\sqrt{\frac{3\lambda_0\log(3/u_\alpha)}{L}}+2\sqrt{\frac{3\sqrt{2}(\lambda_0+\delta)}{\sqrt{\beta}L}}\ ,\ 2\sqrt{\frac{\sqrt{2}|\delta|\log(3/u_\alpha)}{L}}\ ,\right.$$
$$\left.32^{1/3}(6\lambda_0)^{1/6}|\delta|^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}}\ ,\ 24\sqrt{\frac{\lambda_0+\delta}{\beta L}}\right)\ ,$$

which entails

$$d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) > \max\left(\frac{48\lambda_0\log(3/u_\alpha)}{L} + \frac{12\sqrt{2}(\lambda_0+\delta)}{\sqrt{\beta L}}, \frac{32\delta^2\log^2(3/u_\alpha)}{L^2 d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}, \frac{32\sqrt{6\lambda_0}|\delta|\log^{3/2}(3/u_\alpha)}{L^{3/2}d_2(\lambda,\mathcal{S}_0[\lambda_0])},\right.$$
$$\left.\frac{24\sqrt{\lambda_0+\delta}\,d_2(\lambda,\mathcal{S}_0[\lambda_0])}{\sqrt{\beta L}}\right).$$

Then with (2.112),

$$d_2^2(\lambda,\mathcal{S}_0[\lambda_0]) > \max\left(\frac{48\lambda_0\log(3/u_\alpha)}{L} + \frac{12\sqrt{2}(\lambda_0+\delta)}{\sqrt{\beta L}}, \frac{32\log^2(3/u_\alpha)\lceil L\rceil}{3k_0' L^2}, \frac{32\log^{3/2}(3/u_\alpha)}{L^{3/2}}\sqrt{\frac{2\lambda_0\lceil L\rceil}{k_0'}},\right.$$
$$\left.\frac{24\sqrt{\lambda_0+\delta}\,d_2(\lambda,\mathcal{S}_0[\lambda_0])}{\sqrt{\beta L}}\right),$$

hence

$$\frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{3} > \frac{4\lambda_0\log(3/u_\alpha)}{L} + \frac{\sqrt{2}(\lambda_0+\delta)}{\sqrt{\beta L}} + \frac{8\log^2(3/u_\alpha)\lceil L\rceil}{9k_0' L^2} + \frac{8\log^{3/2}(3/u_\alpha)}{3L^{3/2}}\sqrt{\frac{2\lambda_0\lceil L\rceil}{k_0'}}$$
$$+\frac{2\sqrt{\lambda_0+\delta}\,d_2(\lambda,\mathcal{S}_0[\lambda_0])}{\sqrt{\beta L}}\Bigg). \quad (2.113)$$

Using Lemma 2.26, we compute

$$E_\lambda\left[T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right] = \delta^2\frac{k_0'}{\lceil L\rceil} > \frac{d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{3} \quad \text{with (2.112)},$$

and

$$\mathrm{Var}_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right) = \frac{4(\lambda_0+\delta)\delta^2}{L}\frac{k_0'}{\lceil L\rceil} + \frac{2(\lambda_0+\delta)^2}{L^2}$$
$$\leq \frac{4(\lambda_0+\delta)d_2^2(\lambda,\mathcal{S}_0[\lambda_0])}{L} + \frac{2(\lambda_0+\delta)^2}{L^2}.$$

These computations combined with (2.113) leads to

$$E_\lambda\left[T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right] > \frac{4\lambda_0\log(3/u_\alpha)}{L} + \frac{8\log^2(3/u_\alpha)\lceil L\rceil}{9k_0' L^2} + \frac{8\log^{3/2}(3/u_\alpha)}{3L^{3/2}}\sqrt{\frac{2\lambda_0\lceil L\rceil}{k_0'}} + \sqrt{\mathrm{Var}_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right)/\beta}. \quad (2.114)$$

Furthermore, Lemma 2.27 gives

$$t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha) \leq \frac{2\lambda_0^2 k_0'}{\lceil L\rceil}\left(g^{-1}\left(\frac{\log(3/u_\alpha)\lceil L\rceil}{\lambda_0 k_0' L}\right)\right)^2,$$

where $g^{-1}(x) \leq 2x/3 + \sqrt{2x}$ for all $x > 0$ (see (2.45)), and then

$$t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha) \leq \frac{8\log^2(3/u_\alpha)\lceil L\rceil}{9k_0' L^2} + \frac{8\log^{3/2}(3/u_\alpha)}{3L^{3/2}}\sqrt{\frac{2\lambda_0\lceil L\rceil}{k_0'}} + \frac{4\lambda_0\log(3/u_\alpha)}{L}. \quad (2.115)$$

It follows from (2.114) and (2.115) that

$$E_\lambda\left[T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right] > t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha)+\sqrt{\mathrm{Var}_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right)/\beta}\ ,$$

whereby

$$P_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\le t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha)\right)\le P_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\le E_\lambda\left[T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right]-\sqrt{\mathrm{Var}_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\right)/\beta}\right)\ .$$

The Bienayme-Chebyshev inequality allows to conclude that

$$P_\lambda\left(T_{\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(N)\le t_{\lambda_0,\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}}(1-u_\alpha)\right)\le\beta\ .$$

Coming back to the assumption (2.110) and the definition of u_α, one can finally claim that

$$\begin{aligned}\mathrm{SR}_\beta\left(\phi^{(2)}_{9/10,\alpha},\mathcal{S}_{.,..,...}[\lambda_0,R]\right)\le\max\Bigg(&4\sqrt{\frac{3\lambda_0\log(3\lceil L\rceil(\lceil L\rceil+1)/(2\alpha))}{L}}+2\sqrt{\frac{3\sqrt{2}R}{\sqrt{\beta}L}}\,,\ 2\sqrt{\frac{\sqrt{2}R\log(3\lceil L\rceil(\lceil L\rceil+1)/2\alpha)}{L}}\,,\\ &24\sqrt{\frac{R}{\beta L}}\,,\ R\sqrt{\frac{3}{L}}\,,\ 32^{1/3}(6\lambda_0)^{1/6}R^{1/3}\sqrt{\frac{\log(3\lceil L\rceil(\lceil L\rceil+1)/2\alpha)}{L}}\Bigg)\ ,\end{aligned}$$

which ends the proof for the test $\phi^{(2)}_{9/10,\alpha}$.

X Technical results for minimax separation rates upper bounds

Lemma 2.24 (Moments of a Poisson distribution).
Let X be a Poisson distributed random variable with constant intensity $\xi > 0$.

1. *The Laplace transform for X is given by*

$$\forall c\in\mathbb{R},\quad \mathbb{E}[\exp(cX)]=\exp[\xi(\exp(c)-1)].$$

2. *The first moments of X are given by the following formulas:*

$$\mathbb{E}[X]=\xi,\ \mathbb{E}[X^2]=\xi^2+\xi,\ \mathbb{E}[X^3]=\xi^3+3\xi^2+\xi,\ \mathbb{E}[X^4]=\xi^4+6\xi^3+7\xi^2+\xi,$$

   and its central moments are given by $\mathbb{E}[(X-\mathbb{E}[X])^3]=\xi$ and $\mathbb{E}[(X-\mathbb{E}[X])^4]=3\xi^2+\xi$.

Recall that a rough quantile bounds for the Poisson distribution, based on the Bienayme-Chebyshev inequality, is given by Lemma 1.6. The further Lemma provides a more refined control using the Cramér-Chernov inequality.

Lemma 2.25 (Sharp quantile bounds for the Poisson distribution).
The u-quantile $p_\xi(u)$ of the Poisson distribution with parameter ξ satisfies

$$\left(\xi-\xi g^{-1}\left(\frac{\log(1/u)}{\xi}\right)\right)\vee 0\le p_\xi(u)\le\xi+\xi g^{-1}\left(\frac{\log(1/(1-u))}{\xi}\right)\ . \tag{2.116}$$

Proof.
Let N be a Poisson random variable with parameter ξ. The Cramér-Chernov inequality applied to this Poisson random variable gives that for all $x > 0$,

$$\mathbb{P}(N - \xi > x) \le \exp(-\xi g(x/\xi)) \ ,$$

and for $0 < x < \xi$,

$$\mathbb{P}(N - \xi \le -x) \le \exp(-\xi g(-x/\xi)) \ .$$

The first exponential inequality gives

$$\mathbb{P}\left(N > \xi + \xi g^{-1}\left(\frac{\log(1/(1-u))}{\xi}\right)\right) \le u \ ,$$

which leads to the upper bound of Lemma 2.25. The second inequality gives that for all $u > e^{-\xi}$ and all ε such that $u > u - \varepsilon > e^{-\xi}$, using (2.46),

$$\mathbb{P}\left(N \le \xi - \xi g^{-1}\left(\frac{\log(1/(u-\varepsilon))}{\xi}\right)\right) \le \mathbb{P}\left(N \le \xi + \xi g^{-1}_{|]-1,0[}\left(\frac{\log(1/(u-\varepsilon))}{\xi}\right)\right)$$
$$\le u - \varepsilon < u \ .$$

By letting ε tend to zero and by continuity of g^{-1}, this leads to the lower bound of Lemma 2.25 for any $u > e^{-\xi}$. Since $0 < g(1) \le 1$ and g^{-1} is increasing on $]0, +\infty[$, $1 = g^{-1}(g(1)) \le g^{-1}(1)$, hence for $0 \le u \le e^{-\xi}$,

$$g^{-1}\left(\frac{\log(1/u)}{\xi}\right) \ge g^{-1}(1) \ge 1 \ .$$

Therefore in this case $\xi - \xi g^{-1}\left(\frac{\log(1/u)}{\xi}\right) \le 0$. The conclusion follows from the remark that for any u in $[0,1]$, $p_\xi(u) \ge 0$ as some quantile of a Poisson distribution. □

Lemma 2.26 (Expectation and variance of $T_{\tau_1,\tau_2}(N)$)**.**
Let τ_1, τ_2 be in $(0,1)$ such that $0 \le \tau_1 < \tau_2 \le 1$, $\lambda_0 > 0$ and $T_{\tau_1,\tau_2}(N)$ be defined by (2.10). Assume that $N(\tau_1, \tau_2]$ follows a Poisson distribution with parameter Lx with $x > 0$. Then

$$\mathbb{E}[T_{\tau_1,\tau_2}(N)] = \left(\frac{x}{\sqrt{\tau_2 - \tau_1}} - \lambda_0\sqrt{\tau_2 - \tau_1}\right)^2 \ , \tag{2.117}$$

and

$$\mathrm{Var}(T_{\tau_1,\tau_2}(N)) = \frac{4x}{L}\left(\frac{x}{\tau_2 - \tau_1} - \lambda_0\right)^2 + \frac{2}{L^2}\frac{x^2}{(\tau_2 - \tau_1)^2} \ . \tag{2.118}$$

Proof.
Set m_i the moment of order i of $N(\tau_1, \tau_2]$. Since $N(\tau_1, \tau_2]$ follows a Poisson distribution, $m_2 = m_1 + m_1^2$ and

$$\mathbb{E}[T_{\tau_1,\tau_2}(N)] = \frac{1}{L^2(\tau_2 - \tau_1)} m_1^2 - \frac{2\lambda_0}{L} m_1 + \lambda_0^2(\tau_2 - \tau_1),$$

with $m_1 = Lx$. This straightforwardly leads to the first statement of Lemma 2.26. Notice now that

$$T_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2 - \tau_1)}\Big((N(\tau_1, \tau_2] - m_1)^2 + m_1^2 + (2m_1 - 1)(N(\tau_1, \tau_2] - m_1) - m_1\Big) - \frac{2\lambda_0}{L}(N(\tau_1, \tau_2] - m_1) - \frac{2\lambda_0}{L} m_1$$
$$+ \lambda_0^2(\tau_2 - \tau_1) \ ,$$

which entails

$$T_{\tau_1,\tau_2}(N) - \mathbb{E}[T_{\tau_1,\tau_2}(N)] = \frac{1}{L^2(\tau_2-\tau_1)}\Big((N(\tau_1,\tau_2]-m_1)^2 + (2m_1-1)(N(\tau_1,\tau_2]-m_1) - m_1\Big) - \frac{2\lambda_0}{L}(N(\tau_1,\tau_2]-m_1).$$

Considering the moment c_i of order i of the centered variable $N(\tau_1,\tau_2]-m_1$, one obtains

$$\begin{aligned}\mathrm{Var}(T_{\tau_1,\tau_2}(N)) = \frac{1}{L^4(\tau_2-\tau_1)^2}\Big(c_4 + (2m_1-1)^2c_2 + m_1^2 + 2(2m_1-1)c_3 - 2m_1c_2\Big) + \frac{4\lambda_0^2}{L^2}c_2 \\ - \frac{4\lambda_0}{L^3(\tau_2-\tau_1)}(c_3+(2m_1-1)c_2).\end{aligned}$$

Applying Lemma 2.24 finally leads to the second statement of Lemma 2.26. □

Lemma 2.27 (Quantile bound for $T_{\tau_1,\tau_2}(N)$).
Let $\lambda_0 > 0$, u in $(0,1)$ and assume that N is a homogeneous Poisson process of intensity λ_0 with respect to the measure Λ. Let $t_{\lambda_0,\tau_1,\tau_2}(1-u)$ be the $(1-u)$-quantile of $T_{\tau_1,\tau_2}(N)$ defined by (2.10). Then for all $0 \le \tau_1 < \tau_2 \le 1$,

$$t_{\lambda_0,\tau_1,\tau_2}(1-u) \le 2\lambda_0^2(\tau_2-\tau_1)\left(g^{-1}\left(\frac{\log(3/u)}{\lambda_0L(\tau_2-\tau_1)}\right)\right)^2 ,$$

where g is defined by (2.44).

Proof.
Notice first that under the assumption of Lemma 2.27, $T_{\tau_1,\tau_2}(N)$ can be written as

$$T_{\tau_1,\tau_2}(N) = \left(\int_0^1 \frac{\varphi_{(\tau_1,\tau_2]}(t)}{L}(dN_t - \lambda_0Ldt)\right)^2 - \int_0^1\left(\frac{\varphi_{(\tau_1,\tau_2]}(t)}{L}\right)^2 dN_t$$

with $\varphi_{(\tau_1,\tau_2]} = \mathbb{1}_{(\tau_1,\tau_2]}/\sqrt{\tau_2-\tau_1}$. Applying the exponential inequality of Theorem 6 in [LG21], we obtain for all $x > 0$

$$\mathbb{P}(T_{\tau_1,\tau_2}(N) > x) \le 3\exp\left(-\frac{\|H_{\tau_1,\tau_2}\|_{2,L}^2}{\|H_{\tau_1,\tau_2}\|_\infty^2}g\left(\frac{\|H_{\tau_1,\tau_2}\|_\infty}{\|H_{\tau_1,\tau_2}\|_{2,L}^2}\sqrt{\frac{x}{2}}\right)\right) ,$$

where $H_{\tau_1,\tau_2}(t) = \varphi_{(\tau_1,\tau_2]}(t)/L$, g is defined by (2.44) and $\|H_{\tau_1,\tau_2}\|_{2,L}$ is the $\mathbb{L}_2$-norm of H_{τ_1,τ_2} in $\mathbb{L}_2([0,1],\lambda_0Ldt)$, that is $\|H_{\tau_1,\tau_2}\|_{2,L}^2 = \int_0^1 |H_{\tau_1,\tau_2}(t)|^2\lambda_0Ldt = \lambda_0/L$. This leads to

$$\mathbb{P}\Big(T_{\tau_1,\tau_2}(N) > x\Big) \le 3\exp\left(-\lambda_0L(\tau_2-\tau_1)g\left(\frac{1}{\lambda_0\sqrt{\tau_2-\tau_1}}\sqrt{\frac{x}{2}}\right)\right) .$$

We therefore obtain $\mathbb{P}\Big(T_{\tau_1,\tau_2}(N) > x\Big) \le u$ if $x \ge 2\lambda_0^2(\tau_2-\tau_1)\Big(g^{-1}(\log(3/u)/(\lambda_0L(\tau_2-\tau_1)))\Big)^2$ and the result follows. □

Lemma 2.28 (Quantile bound for $\max/\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$).
Let $L \ge 1$. The $(1-\alpha)$-quantile $p^+_{\lambda_0,\ell^*}(1-\alpha)$ of $\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ under (H_0) satisfies

$$p^+_{\lambda_0,\ell^*}(1-\alpha) \le \lambda_0L\ell^* + 2\lambda_0Lg^{-1}\left(\frac{\log(2/\alpha)}{\lambda_0L}\right) .$$

The α-quantile $p^-_{\lambda_0,\ell^*}(\alpha)$ of $\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ under (H_0) satisfies

$$p^-_{\lambda_0,\ell^*}(\alpha) \ge \lambda_0L\ell^* - 2\lambda_0Lg^{-1}\left(\frac{\log(2/\alpha)}{\lambda_0L}\right) ,$$

where g is defined by (2.44).

Proof.
We define $M_s^t = \int_s^t (dN_u - \lambda_0 L du)$ for all $s, t > 0$. Let $x > 0$ be such that

$$x \geq \lambda_0 L\ell^* + 2\lambda_0 L g^{-1}\left(\frac{\log(2/\alpha)}{\lambda_0 L}\right) . \tag{2.119}$$

Since

$$\begin{aligned} P_{\lambda_0}\left(\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*] > x\right) &= P_{\lambda_0}\left(\max_{\tau\in[0,1-\ell^*]} M_\tau^{\tau+\ell^*} > x - \lambda_0 L\ell^*\right) \\ &\leq P_{\lambda_0}\left(\max_{s,t\in[0,1]} |M_s^t| > x - \lambda_0 L\ell^*\right) , \end{aligned}$$

Theorem 8 in [LG21] ensures that

$$P_{\lambda_0}\left(\max_{s,t\in[0,1]} |M_s^t| > x - \lambda_0 L\ell^*\right) \leq 2\exp\left(-\lambda_0 L g\left(\frac{x - \lambda_0 L\ell^*}{2\lambda_0 L}\right)\right) .$$

Then (2.119) entails

$$2\exp\left(-\lambda_0 L g\left(\frac{x - \lambda_0 L\ell^*}{2\lambda_0 L}\right)\right) \leq \alpha ,$$

leading to $P_{\lambda_0}(\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*] > x) \leq \alpha$. The $(1-\alpha)$-quantile $p^+_{\lambda_0,\ell^*}(\alpha)$ of $\max_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ under (H_0) therefore satisfies $p^+_{\lambda_0,\ell^*}(1-\alpha) \leq x$ for every x such that (2.119) holds. In particular,

$$p^+_{\lambda_0,\ell^*}(1-\alpha) \leq \lambda_0 L\ell^* + 2\lambda_0 L g^{-1}\left(\frac{\log(2/\alpha)}{\lambda_0 L}\right) .$$

Let us consider now x in $\mathbb{R}$ and ε in $(0,1)$ satisfying

$$x \leq \lambda_0 L\ell^* - 2\lambda_0 L g^{-1}\left(\frac{\log(2/(\alpha(1-\varepsilon)))}{\lambda_0 L}\right) . \tag{2.120}$$

Using (2.120) and Theorem 8 in [LG21] again, we obtain

$$\begin{aligned} P_{\lambda_0}\left(\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*] \leq x\right) &= P_{\lambda_0}\left(\max_{\tau\in[0,1-\ell^*]} -M_\tau^{\tau+\ell^*} \geq \lambda_0 L\ell^* - x\right) \\ &\leq P_{\lambda_0}\left(\max_{s,t\in[0,1]} |M_s^t| \geq \lambda_0 L\ell^* - x\right) \\ &\leq 2\exp\left(-\lambda_0 L g\left(\frac{\lambda_0 L\ell^* - x}{2\lambda_0 L}\right)\right) \\ &\leq \alpha(1-\varepsilon) \\ &< \alpha . \end{aligned}$$

Thus the α-quantile $p^-_{\lambda_0,\ell^*}(\alpha)$ of $\min_{\tau\in[0,1-\ell^*]} N(\tau,\tau+\ell^*]$ under (H_0) satisfies $p^-_{\lambda_0,\ell^*}(\alpha) > x$ for every x such that (2.120) holds. In particular

$$p^-_{\lambda_0,\ell^*}(\alpha) > \lambda_0 L\ell^* - 2\lambda_0 L g^{-1}\left(\frac{\log(2/(\alpha(1-\varepsilon)))}{\lambda_0 L}\right)$$

for every ε in $(0,1)$. The result then follows by continuity of g^{-1}. □

Lemma 2.29 (Quantile bound for $\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N)$).
Let $\gamma > 0$ and $L \geq 1$. Let $(N_t^{\lambda_0})_{t\geq 0}$ be an homogeneous Poisson process with a known constant intensity $\lambda_0 L > 0$ w.r.t. the Lebesgue measure. Then, the u-quantile of the supremum $\sup_{t\geq 0}(N_t^{\lambda_0} - (\lambda_0+\gamma)Lt)$ does not depend on L, and will therefore be denoted by $s^+_{\lambda_0,\gamma}(u)$. Now considering $s^+_{\lambda_0,\delta^*,\tau^*,L}(u)$, the u-quantile under (H_0) of the statistic $\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N)$ with $S_{\delta^*,\tau^*,\tau^*+\ell}(N)$ defined by (2.19), we have for all $L \geq 1$

$$\begin{cases} s^+_{\lambda_0,\delta^*,\tau^*,L}(u) & \leq & s^+_{\lambda_0,\delta^*/2}(u) \quad \textit{when } \delta^* > 0 \ , \\ s^+_{\lambda_0,\delta^*,\tau^*,L}(u) & \leq & \frac{\log(1/(1-u))}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))} \quad \textit{when } \delta^* \in (-\lambda_0^*, 0) \ . \end{cases} \tag{2.121}$$

Proof.
Equation (7) in [Pyk59] directly enables to state the first part of the result.
Under (H_0), since the processes $(N(\tau^*,\tau^*+\ell])_{\ell\in(0,1-\tau^*]}$ and $(N(0,\ell])_{\ell\in(0,1-\tau^*]}$ are left-continuous and have the same finite dimensional laws, we get

$$\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) \stackrel{d}{=} \sup_{\ell\in(0,1-\tau^*)} \left(\mathrm{sgn}(\delta^*)(N(0,\ell] - \lambda_0 \ell L) - \frac{|\delta^*|}{2}\ell L\right).$$

Assume first that $\delta^* > 0$. We compute

$$\begin{aligned} P_{\lambda_0}&\Big(\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) > s^+_{\lambda_0,\delta^*/2}(u)\Big) \\ &\leq \mathbb{P}\left(\sup_{t\in[0,+\infty)} \left(N^{\lambda_0}(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right)Lt\right) > s^+_{\lambda_0,\delta^*/2}(u)\right) \\ &\leq 1-u \ , \end{aligned}$$

by definition of $s^+_{\lambda_0,\delta^*/2}(u)$. This allows to conclude that the first part of (2.121) holds.
Assume then that δ^* belongs to $(-\lambda_0, 0)$. For all $x > 0$,

$$P_{\lambda_0}\left(\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) > x\right) = P_{\lambda_0}\left(\sup_{\ell\in(0,1-\tau^*)} \left(\left(\lambda_0 - \frac{|\delta^*|}{2}\right)\ell L - N(0,\ell]\right) > x\right) \ .$$

Theorem 3 in [Pyk59] then entails

$$P_{\lambda_0}\left(\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) > x\right) \leq \exp(-\omega x) \ ,$$

where ω is the largest real root of the equation $\lambda_0(1-e^{-\omega}) = \omega(\lambda_0 - |\delta^*|/2)$. Notice that $\omega > \log(\lambda_0/(\lambda_0 - |\delta^*|/2))$.
Then, correctly choosing x in the above exponential inequality leads to

$$P_{\lambda_0}\left(\sup_{\ell\in(0,1-\tau^*)} S_{\delta^*,\tau^*,\tau^*+\ell}(N) > \frac{\log(1/(1-u))}{\log(\lambda_0/(\lambda_0 - |\delta^*|/2))}\right) \leq 1-u \ ,$$

which implies the second part of (2.121). □

Lemma 2.30 (Quantile bound for $\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N)$).
Let $L \geq 1$. With the same notation as in Lemma 2.29 and $S_{\delta^*,\tau,1}(N)$ defined by (2.19), the u-quantile $s^+_{\lambda_0,\delta^*,L}(u)$ of the statistic $\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N)$ under (H_0) satisfies

$$\begin{cases} s^+_{\lambda_0,\delta^*,L}(u) & \leq & s^+_{\lambda_0,\delta^*/2}(u) \quad \textit{when } \delta^* > 0 \ , \\ s^+_{\lambda_0,\delta^*,L}(u) & \leq & \frac{\log(1/(1-u))}{\log(\lambda_0/(\lambda_0-|\delta^*|/2))} \quad \textit{when } \delta^* \in (-\lambda_0^*, 0) \ . \end{cases} \tag{2.122}$$

Proof.
Under (H_0), since the processes $(N(\tau,1])_{\tau\in(0,1)}$ and $(N(0,1-\tau])_{\tau\in(0,1)}$ are left-continuous and have the same finite dimensional laws, we get

$$\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N) \overset{d}{=} \sup_{\tau\in(0,1)}\left(\mathrm{sgn}(\delta^*)(N(0,\tau]-\lambda_0\tau L)-\frac{|\delta^*|}{2}\tau L\right).$$

The result then follows from the same arguments as in the proof of Lemma 2.29, by noticing that when $\delta^*>0$,

$$\begin{aligned}P_{\lambda_0}&\left(\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N) > s^+_{\lambda_0,\delta^*/2}(u)\right)\\ &\leq \mathbb{P}\left(\sup_{t\in[0,+\infty)}\left(N^{\lambda_0}(0,t]-\left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) > s^+_{\lambda_0,\delta^*/2}(u)\right)\ ,\end{aligned}$$

and when δ^* belongs to $(-\lambda_0,0)$,

$$P_{\lambda_0}\left(\sup_{\tau\in(0,1)} S_{\delta^*,\tau,1}(N) > x\right) = P_{\lambda_0}\left(\sup_{t\in(0,1)}\left(\left(\lambda_0-\frac{|\delta^*|}{2}\right)tL-N(0,t]\right) > x\right)$$

for all $x>0$. $\square$

CHAPTER
THREE

MINIMAX AND ADAPTIVE TESTS FOR DETECTING ABRUPT AND POSSIBLY TRANSITORY CHANGES IN A POISSON PROCESS WITH AN UNKNOWN BASELINE INTENSITY

In this chapter, we turn to the problem of detecting an abrupt change in the intensity of the Poisson process N when its constant baseline is not assumed to be known anymore. This detection problem probably more largely fits applications, especially with epidemiological data for which it is often more realistic not to assume that the baseline intensity of the underlying Poisson process is known. In the present chapter, we therefore consider the null hypothesis expressed as (H_0) "$\lambda \in \mathcal{S}_0^u[R]$", where $\mathcal{S}_0^u[R]$ is the set of all possible constant intensities upper bounded by a given $R > 0$. As in Chapter 2, we consider various alternative hypotheses, that are defined according to the persistent or transitory nature of the change, and its height, location and length knowledge. All the possible sets of alternatives according to whether each parameter of the change (height, location and length) is known or not, including the special case where the change is non transitory (jump detection), are handled. For each sets of alternatives, lower bounds for minimax separation rates are provided, as a preliminary basis for corresponding upper bounds (when appropriate, that is when at least the height or the length is unknown). As explained above, these upper bounds are obtained by constructing minimax or minimax adaptive tests, which are mainly based on aggregation of either linear or quadratic statistics, coupled with adjusted critical values. A simulation study is presented in Chapter 3 Section VII, whose aim is to compare linear and quadratic type tests, and also to compare them with standard tests used to detect nonhomogeneity of Poisson processes in practice. Proofs of the core results are postponed to Chapter 3 Section VIII, and proofs of technical results mostly based on exponential inequalities and devoted to quantiles and critical values upper bounds are postponed to Chapter 3 Section IX. All along the chapter, we will introduce some positive constants denoted by $C(\alpha,\beta,\ldots)$ and $L_0(\alpha,\beta,\ldots)$, meaning that they depend on $(\alpha,\beta,\ldots)$. Though they are denoted in the same way, they may vary from one line to another. When they appear in the main results about lower and upper bounds, we do not intend to precisely evaluate them. However, some possible, probably pessimistic, explicit expressions for them are proposed in the proofs.

In order to further cover the full range of alternatives in a unified notation, we introduce for δ^* in $(-R,R)\setminus\{0\}$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$ the set $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$ of intensities with a change of height δ^*, location τ^* and length ℓ^* from an unknown λ_0 in $\mathcal{S}_0^u[R]$, and still upper bounded by R,

$$[\mathbf{Alt^u.1}] \quad \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (-\delta^* \vee 0, (R-\delta^*) \wedge R], \quad \forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t)\Big\}. \quad (3.1)$$

Though it is not as immediate as in Chapter 2, testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$" also falls within the theory of UMP tests, and an Uniformly Most Powerful Unbiased (UMPU) test can be constructed by using a conditioning trick (see details below). As above, when the question of adaptivity w.r.t. some unknown parameters is tackled, the unknown parameters are replaced by single, double or triple dots in the notation $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$.

Notice that for any intensity λ such that $\lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell]}(t)$ for δ in $(-R,R)\setminus\{0\}$, τ in $(0,1)$, ℓ in $(0,1-\tau]$ and λ_0 in $(-\delta \vee 0, (R-\delta)\wedge R]$,

$$d_2(\lambda, \mathcal{S}^u_0[R]) = |\delta|\sqrt{\ell(1-\ell)}\ .$$

Hence, as soon as an alternative intensity has known change height $\delta = \delta^*$ and length $\ell = \ell^*$, the distance $d_2(\lambda, \mathcal{S}^u_0[R])$ is fixed, equal to $|\delta^*|\sqrt{\ell^*(1-\ell^*)}$. Hence, the β-uniform separation rate of any level α test over $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$ or $\mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$ is either 0 or $+\infty$, and so is the (α,β)-minimax separation rate. In these cases, our tests are not studied from the minimax point of view. As in Chapter 2, we nevertheless establish conditions, expressed as a sufficient minimal distance $d_2(\lambda, \mathcal{S}^u_0[R])$, guaranteeing that their second kind error rate is controlled by β.

I Uniformly most powerful detection of a possibly transitory change with known location and length

Let us first focus on the problem of testing (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$" with $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$ defined by (3.1) for δ^* in $(-R,R)\setminus\{0\}$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. Assume here that λ belongs to $\{\lambda : [0,1] \to (0,R],\ \exists \delta \in (-R,R),\ \exists \lambda_0 \in (-\delta\vee 0, (R-\delta)\wedge R],\ \lambda = \lambda_0 + \delta \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}\} \supset \mathcal{S}^u_0[R] \cup \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$. In this model parametrised by (δ,λ_0) in $\{(\delta,\lambda_0),\ \delta \in (-R,R), \lambda_0 \in (-\delta \vee 0, (R-\delta)\wedge R]\}$, the distribution P_λ is dominated by P_1 (see Lemma 2.1), with a likelihood ratio given by

$$(dP_\lambda/dP_1)(N) = e^{(\log(1+\delta/\lambda_0)N(\tau^*,\tau^*+\ell^*]+\log(\lambda_0)N(0,1]-L(\lambda_0+\delta\ell^*-1))}\ .$$

Reparametrising the model by $\theta_1 = \log(1+\delta/\lambda_0)$ and $\theta_2 = \log(\lambda_0)$, this likelihood ratio becomes

$$(dP_\lambda/dP_1)(N) = e^{-L\left(e^{\theta_2}\left(1+(e^{\theta_1}-1)\ell^*\right)-1\right)} e^{\theta_1 N(\tau^*,\tau^*+\ell^*]+\theta_2 N(0,1]}\ . \tag{3.2}$$

Our testing problem can then be viewed as a problem of testing (H_0)"$\theta_1 = 0$" versus (H_1)"$\theta_1 < 0$" or (H_1)"$\theta_1 > 0$" (depending on the sign of δ^*) in an exponential model with natural parameters $\theta = (\theta_1,\theta_2)$ and sufficient statistics $(N(\tau^*,\tau^*+\ell^*], N(0,1])$, and where θ_2 can be interpreted as a nuisance parameter. From (3.2) and Lemma 2.7.2 of [LR05] we can deduce that given $N_1 = n$, the conditional distribution of $N(\tau^*,\tau^*+\ell^*]$ defines an exponential family with respect to some measure ν_n, with natural parameter θ_1, and is in particular free of θ_2. In this conditional framework, one knows that there exists an UMP test of (H_0) versus (H_1) of the Neyman-Pearson form. Recalling that given $N_1 = n$, $N(\tau^*,\tau^*+\ell^*]$ has the same distribution as a binomial random variable Y_{n,ℓ^*} with parameters (n,ℓ^*), such conditional Neyman-Pearson tests lead us to consider the unilateral tests defined by

$$\begin{cases} \phi^{u,-}_{1,\alpha}(N) &= \mathbb{1}_{N(\tau^*,\tau^*+\ell^*]<b_{N_1,\ell^*}(\alpha)} + \gamma^-_{(N_1,\ell^*)}(\alpha)\mathbb{1}_{N(\tau^*,\tau^*+\ell^*]=b_{N_1,\ell^*}(\alpha)} \text{ if } \delta^*<0\ , \\ \phi^{u,+}_{1,\alpha}(N) &= \mathbb{1}_{N(\tau^*,\tau^*+\ell^*]>b_{N_1,\ell^*}(1-\alpha)} + \gamma^+_{(N_1,\ell^*)}(1-\alpha)\mathbb{1}_{N(\tau^*,\tau^*+\ell^*]=b_{N_1,\ell^*}(1-\alpha)} \text{ if } \delta^*>0\ , \end{cases} \tag{3.3}$$

where for all n in $\mathbb{N}$, $b_{n,\ell^*}(u)$ denotes the u-quantile of the distribution of Y_{n,ℓ^*}, and

$$\gamma^-_{(n,\ell^*)}(u) = (u - \mathbb{P}(Y_{n,\ell^*} < b_{n,\ell^*}(u)))/\mathbb{P}(Y_{n,\ell^*} = b_{n,\ell^*}(u)),\ \gamma^+_{(n,\ell^*)}(u) = 1-\gamma^-_{(n,\ell^*)}(u)\ . \tag{3.4}$$

From Theorem 4.4.1 in [LR05] and the remark below its proof, we obtain the following result.

Proposition 3.1 (Uniformly Most Powerful Unbiased tests).
Let $L \geq 1$, α in $(0,1)$, δ^* in $(-R,R)\setminus\{0\}$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. For the problem of testing (H_0) v.s. (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$", let $\phi^u_{1,\alpha}$ be the test $\phi^{u,-}_{1,\alpha}$ if $\delta^* < 0$, $\phi^{u,+}_{1,\alpha}$ if $\delta^* > 0$ (see (3.3)). Then $E_\lambda\left[\phi^u_{1,\alpha}(N)\middle|N_1 = n\right] = \alpha$ for all λ in $\mathcal{S}^u_0[R]$ and $\phi^u_{1,\alpha}$ is an UMPU test.

In order to follow the same line as the minimax results obtained when regarding other alternative hypotheses with unknown height and/or length change, we further study which minimal distance $d_2(\lambda, \mathcal{S}_0^u[R])$ guarantees a second kind error rate control.

Proposition 3.2 (Second kind error rates control for [$\mathbf{Alt^u.1}$]).
Let $L \geq 1$, α in $(0,1)$, δ^* in $(-R,R)\setminus\{0\}$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. For the problem of testing (H_0) v.s. (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$", let $\phi^u_{1,\alpha}$ be the test $\phi^{u,-}_{1,\alpha}$ if $\delta^* < 0$, $\phi^{u,+}_{1,\alpha}$ if $\delta^* > 0$. Then there exists $C(\alpha,\beta,R,\ell^*) > 0$ such that $P_\lambda(\phi^u_{1,\alpha}(N) = 0) \leq \beta$ if λ belongs to $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$ with

$$d_2(\lambda, \mathcal{S}_0^u[R]) \geq \frac{C(\alpha,\beta,R,\ell^*)}{\sqrt{L}} \quad . \tag{3.5}$$

Comments.
Noticing that for any λ in $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$, $d_2\left(\lambda, \mathcal{S}_0^u[R]\right) = |\delta^*|\sqrt{\ell^*(1-\ell^*)}$, the above proposition implies that if $L \geq C^2(\alpha,\beta,R,\ell^*)/(\delta^{*2}\ell^*(1-\ell^*))$, then $P_\lambda(\phi^u_{1,\alpha}(N) = 0) \leq \beta$. Therefore, in this case, the β-uniform separation rate of $\phi^u_{1,\alpha}$ over $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$ is equal to 0, and so is the corresponding (α,β)-minimax separation rate $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R])$.

In order to address the question of adaptation to the change height, we consider the problem of testing (H_0) v.s. (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$", where for $R > 0$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$,

$$[\mathbf{Alt^u}.2] \quad \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (0,R],\ \exists \delta \in (-\lambda_0, R-\lambda_0]\setminus\{0\},$$
$$\forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t)\Big\} \ . \tag{3.6}$$

Unsurprisingly, with the Bayesian arguments already used to prove Proposition 2.3, one obtains a lower bound for the minimax separation rate over $\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$ of the parametric order $1/\sqrt{L}$.

Proposition 3.3 (Minimax lower bound for [$\mathbf{Alt^u}.2$]).
Let α and β in $(0,1)$, $R > 0$, τ^* in $(0,1)$ and ℓ^* in $(0,1-\tau^*]$. For all $L \geq (2\log C_{\alpha,\beta}/(R\ell^*))$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]\right) \geq \sqrt{\frac{R(1-\ell^*)\log C_{\alpha,\beta}}{2L}}, \ \textit{with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

In order to prove that this lower bound is sharp, we construct two minimax adaptive tests.
The first one is based on the linear statistic $N(\tau^*,\tau^*+\ell^*]$ and is very similar in spirit to the test $\phi^{(1)}_{2,\alpha}$ defined by (2.7), except that the associated critical values are based on the conditional distribution of $N(\tau^*,\tau^*+\ell^*]$ given $N_1 = n$ under (H_0) instead of the unconditional distribution (which is not free from the unknown constant baseline intensity under (H_0)). Let

$$\phi^{u(1)}_{2,\alpha}(N) = \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] > b_{N_1,\ell^*}(1-\alpha_1)} + \gamma^+_{(N_1,\ell^*)}(1-\alpha_1)\mathbb{1}_{N(\tau^*,\tau^*+\ell^*] = b_{N_1,\ell^*}(1-\alpha_1)}$$
$$+ \mathbb{1}_{N(\tau^*,\tau^*+\ell^*] < b_{N_1,\ell^*}(\alpha_2)} + \gamma^-_{(N_1,\ell^*)}(\alpha_2)\mathbb{1}_{N(\tau^*,\tau^*+\ell^*] = b_{N_1,\ell^*}(\alpha_2)} \ , \tag{3.7}$$

where $\gamma^+_{(n,\ell^*)}$ and $\gamma^-_{(n,\ell^*)}$ are defined by (3.4), $b_{n,\ell^*}(u)$ denotes the u-quantile of the binomial distribution with parameters (n,ℓ^*) for all n in $\mathbb{N}$, and α_1 and α_2 in $(0,1)$ are determined by

$$\begin{cases} \alpha_1 + \alpha_2 = \alpha \ , \\ E_\lambda\Big[N(\tau^*,\tau^*+\ell^*]\phi^{u(1)}_{2,\alpha}(N)\Big|N_1 = n\Big] = \alpha E_\lambda\Big[N(\tau^*,\tau^*+\ell^*]\Big|N_1 = n\Big] \ \forall \lambda \in \mathcal{S}_0^u[R] \,. \end{cases} \tag{3.8}$$

Theorem 4.4.1 of [LR05] again (but considering the bilateral test) shows that $\phi^{u(1)}_{2,\alpha}$ is UMPU.

The second one is based on a quadratic statistic deduced from an estimation of the $\mathbb{L}_2$-distance between λ in $\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$ and $\mathcal{S}^u_0[R]$. For $0 < \tau_1 < \tau_2 \leq 1$, we define $\psi_0 = \mathbb{1}_{[0,1]}$ and $\psi_{\tau_1,\tau_2} = (\mathbb{1}_{(\tau_1,\tau_2]} - (\tau_2 - \tau_1)\psi_0)/\sqrt{(\tau_2-\tau_1)(1-\tau_2+\tau_1)}$, so that $(\psi_0, \psi_{\tau_1,\tau_2})$ is an orthonormal family. Denoting by Π_{W_0} and $\Pi_{W_{\tau_1,\tau_2}}$ the orthogonal projections onto $W_0 = \mathrm{Vect}(\psi_0)$ and $W_{\tau_1,\tau_2} = \mathrm{Vect}\left(\psi_0, \psi_{\tau_1,\tau_2}\right)$ in $\mathbb{L}_2([0,1])$ respectively, the quadratic statistic

$$T'_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\Big((N(\tau_1,\tau_2] - (\tau_2-\tau_1)N(0,1])^2 + (\tau_2-\tau_1)(N(\tau_1,\tau_2] - (\tau_2-\tau_1)N(0,1]) \\ - (1-\tau_2+\tau_1)N(\tau_1,\tau_2]\Big) \ , \quad (3.9)$$

is an unbiased estimator of $\|\Pi_{W_{\tau_1,\tau_2}}(\lambda - \Pi_{W_0}(\lambda))\|_2^2$. We therefore consider the particular statistic $T'_{\tau^*,\tau^*+\ell^*}(N)$ which is an unbiased estimator of the squared $\mathbb{L}_2$-distance between λ in $\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$ and the set of constant intensities, leading to the test defined by

$$\phi^{u(2)}_{2,\alpha}(N) = \mathbb{1}_{T'_{\tau^*,\tau^*+\ell^*}(N) > t'_{N_1,\tau^*,\tau^*+\ell^*}(1-\alpha)} \ , \quad (3.10)$$

where $t'_{n,\tau_1,\tau_2}(u)$ is the u-quantile of the distribution of $T'_{\tau_1,\tau_2}(N)$ given $N_1 = n$ under (H_0).
Since this conditional distribution under (H_0) is the distribution of the renormalised U-statistic

$$\frac{1}{L^2}\sum_{i\neq j=1}^{n} \psi_{\tau_1,\tau_2}(U_i)\psi_{\tau_1,\tau_2}(U_j)$$

based on a n-sample $(U_1,\ldots,U_n)$ of i.i.d. uniform random variables, we use an exponential inequality for U-statistics of order 2 due to Reynaud-Bouret and Houdré [HRB03] to control the quantiles $t'_{n,\tau_1,\tau_2}(u)$ in theory (see Lemma 3.20), and Monte-Carlo methods to evaluate them in practice.

Proposition 3.4 (Minimax upper bound for [**Alt$^{\mathbf{u}}$.2**]).
Let $L \geq 1$, α, β in $(0,1)$, $R > 0$, τ^* in $(0,1)$ and ℓ^* in $(0, 1-\tau^*]$. Let $\phi^{u(1/2)}_{2,\alpha}$ be one of the tests $\phi^{u(1)}_{2,\alpha}$ and $\phi^{u(2)}_{2,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$" respectively defined by (3.7)-(3.8) and (3.10). Then $\phi^{u(1/2)}_{2,\alpha}$ is of level α, that is $\sup_{\lambda\in\mathcal{S}^u_0[R]} P_\lambda(\phi^{u(1/2)}_{2,\alpha}(N) = 1) \leq \alpha$ ($\phi^{u(1)}_{2,\alpha}$ is even of size α). Moreover, there exists $C(\alpha,\beta,R,\ell^*) > 0$ such that

$$\mathrm{SR}_\beta(\phi^{u(1/2)}_{2,\alpha}, \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]) \leq \frac{C(\alpha,\beta,R,\ell^*)}{\sqrt{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]) \leq C(\alpha,\beta,R,\ell^*)/\sqrt{L}$.

Comments.
This result proves that both tests $\phi^{u(1)}_{2,\alpha}$ and $\phi^{u(2)}_{2,\alpha}$ are therefore β-minimax (up to a possible multiplicative constant) over the set of alternatives $\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$, where the height of the change is unknown, with an optimal uniform separation rate of the parametric order $1/\sqrt{L}$, as expected regarding results for $\phi^{(1)}_{2,\alpha}$ and $\phi^{(2)}_{2,\alpha}$ in Chapter 2.

Notice that this study involves the particular non transitory change or jump detection problem, with a known change location, taking $\ell^* = 1 - \tau^*$. Following the same layout as Chapter 2, we investigate the jump detection problem with unknown location in Chapter 3 Section IV.1.

II Minimax detection of a transitory change with known length

In this section, we deal with the problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0^u[R]$" versus alternatives where the length of the change from the unknown baseline intensity is known, with adaptation to the change location, and with or without adaptation to the change height. We therefore introduce for ℓ^* in $(0,1)$ and δ^* in $(-R,R)\setminus\{0\}$ the two following sets:

$$[\mathbf{Alt^u.3}]\ \mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (-\delta^* \vee 0, (R-\delta^*)\wedge R],\ \exists \tau \in (0,1-\ell^*),\ \forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,\tau+\ell^*]}(t)\Big\}\ , \quad (3.11)$$

$$[\mathbf{Alt^u.4}]\ \mathcal{S}^u_{\cdot,\cdot,\ell^*}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (0,R],\ \exists \delta \in (-\lambda_0, R-\lambda_0]\setminus\{0\},\ \exists \tau \in (0,1-\ell^*),\ \forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell^*]}(t)\Big\}\ . \quad (3.12)$$

Adapting the ideas of Chapter 2 Section III, we handle the question of adaptation to the change location τ^* by introducing aggregated tests based on the same linear and quadratic statistics as those used for testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$" above. We thus set on the one hand the unilateral tests

$$\begin{cases} \phi^{u(1)-}_{3,\alpha}(N) = \mathbb{1}_{\min_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)] < b^-_{N_1,\ell^*\wedge(1/2)}(\alpha)}\ , \\ \phi^{u(1)+}_{3,\alpha}(N) = \mathbb{1}_{\max_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)] > b^+_{N_1,\ell^*\wedge(1/2)}(1-\alpha)}\ , \end{cases} \quad (3.13)$$

and the bilateral test

$$\phi^{u(1)}_{4,\alpha}(N) = \phi^{u(1)-}_{3,\alpha/2}(N) \vee \phi^{u(1)+}_{3,\alpha/2}(N)\ , \quad (3.14)$$

where $b^-_{n,\ell}(u)$ and $b^+_{n,\ell}(u)$ respectively denote the u-quantiles of the conditional distributions of $\min_{\tau\in[0,1-\ell]} N(\tau,\tau+\ell]$ and $\max_{\tau\in[0,1-\ell]} N(\tau,\tau+\ell]$ given $N_1 = n$ under (H_0), for all n in $\mathbb{N}$ and ℓ in $(0,1/2]$. Then, we introduce on the other hand the aggregated test

$$\phi^{u(2)}_{3/4,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}}\left(T'_{\frac{k}{M},\frac{k}{M}+\ell^*}(N) - t'_{N_1,\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right)>0\right\}}\ , \quad (3.15)$$

where $M = \lceil 2/(\ell^*(1-\ell^*))\rceil$, $u_\alpha = \alpha/\lceil(1-\ell^*)M\rceil$, $T'_{k/M,k/M+\ell^*}$ is defined by (3.9) and $t'_{n,k/M,k/M+\ell^*}(u)$ is the u-quantile of $T'_{k/M,k/M+\ell^*}(N)$ given $N_1 = n$ under (H_0).

Since the set $\mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$ of (3.11) is composed of alternatives with known change height δ^* and length ℓ^*, the distance between any of its elements and $\mathcal{S}_0^u[R]$ is fixed, equal to $|\delta^*|\sqrt{\ell^*(1-\ell^*)}$. Hence, for this set, we only provide sufficient conditions for the tests $\phi^{u(1)+}_{3,\alpha}$, $\phi^{u(1)-}_{3,\alpha}$ and $\phi^{u(2)}_{3/4,\alpha}$ to have a second kind error rate controlled by a prescribed level β when $\lambda \in \mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$. As in Proposition 2.6, the key points of the proofs of the following results for $\phi^{u(1)-}_{3,\alpha}$ and $\phi^{u(1)+}_{3,\alpha}$ are sharp lower or upper bounds for the involved quantiles $b^-_{n,\ell^*\wedge(1/2)}(\alpha)$ and $b^+_{n,\ell^*\wedge(1/2)}(\alpha)$, which are deduced from inequalities for oscillations of empirical processes found in [SW86] (see Lemma 3.21 for details). The result for $\phi^{u(2)}_{3/4,\alpha}$ relies on the control of the quantile $t'_{n,\tau_1,\tau_2}(u_\alpha)$ obtained in Lemma 3.20 via the exponential inequality for U-statistics of order 2 due to Reynaud-Bouret and Houdré [HRB03], as in Proposition 3.4.

Proposition 3.5 (Second kind error rate control for [**Alt$^{\mathbf{u}}$.3**]).
Let $L \geq 1$, α and β in $(0,1)$, ℓ^* in $(0,1)$ and δ^* in $(-R,R)\setminus\{0\}$, and consider the problem of testing (H_0) v.s. (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$". Let $\phi^{u(1/2)}_{3,\alpha}$ be one of the tests $\phi^{u(1)+}_{3,\alpha}$ or $\phi^{u(2)}_{3/4,\alpha}$ if $\delta^* > 0$, and one of the tests $\phi^{u(1)-}_{3,\alpha}$ or $\phi^{u(2)}_{3/4,\alpha}$ if $\delta^* < 0$ (see (3.13) and (3.15)). The test $\phi^{u(1/2)}_{3,\alpha}$ is of level α, that is $\sup_{\lambda_0\in\mathcal{S}_0^u[R]} P_{\lambda_0}\left(\phi^{u(1/2)}_{3,\alpha}(N) = 1\right) \leq \alpha$.

Moreover, there exists $C(\alpha,\beta,R,\delta^*,\ell^*) > 0$ such that $P_\lambda\Big(\phi^{u(1/2)}_{3,\alpha}(N) = 0\Big) \leq \beta$ as soon as λ belongs to $\mathcal{S}^u_{\delta^*,\cdot\cdot,\ell^*}[R]$ with

$$d_2\left(\lambda, \mathcal{S}^u_0[R]\right) \geq \frac{C(\alpha,\beta,R,\delta^*,\ell^*)}{\sqrt{L}} \ .$$

Comments.
Remarking that for λ in $\mathcal{S}^u_{\delta^*,\cdot\cdot,\ell^*}[\lambda_0]$, $d_2\left(\lambda, \mathcal{S}^u_0[R]\right) = |\delta^*|\sqrt{\ell^*(1-\ell^*)}$, Proposition 3.5 provides a sufficient value $L_0(\alpha,\beta,R,\delta^*,\ell^*)$ for L so that the second kind error rates of the three tests is controlled by β. If $L \geq L_0(\alpha,\beta,R,\delta^*,\ell^*)$, their β-uniform separation rates over $\mathcal{S}^u_{\delta^*,\cdot\cdot,\ell^*}[R]$ is equal to 0, as well as the (α,β)-minimax separation rate.

Now considering the alternative set $\mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]$, that is the change height adaptation issue, the following lower bound is directly deduced from the lower bound for $\mathrm{mSR}_{\alpha,\beta}\big(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]\big)$ and the monotonicity property of the minimax separation rate recalled in Lemma 1.7.

Corollary 3.6 (Minimax lower bound for [**Alt^u.4**]).
Let α and β in $(0,1)$, $R > 0$ and ℓ^* in $(0,1)$. For all $L \geq (2\log C_{\alpha,\beta}/(R\ell^*))$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]\right) \geq \sqrt{\frac{R(1-\ell^*)\log C_{\alpha,\beta}}{2L}}, \ \textit{with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

Proposition 3.7 (Minimax upper bounds for [**Alt^u.4**]).
Let $L \geq 1$, α,β in $(0,1)$, $R > 0$ and ℓ^* in $(0,1)$. Let $\phi^{u(1/2)}_{4,\alpha}$ be one of the tests $\phi^{u(1)}_{4,\alpha}$ and $\phi^{u(2)}_{3/4,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]$", defined by (3.14) and (3.15). $\phi^{u(1/2)}_{4,\alpha}$ is of level α, that is $\sup_{\lambda_0\in\mathcal{S}^u_0[R]} P_{\lambda_0}(\phi^{u(1/2)}_{4,\alpha}(N) = 1) \leq \alpha$, and there exists $C(\alpha,\beta,R,\ell^*) > 0$ such that

$$\mathrm{SR}_\beta\Big(\phi^{u(1/2)}_{4,\alpha}, \mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]\Big) \leq \frac{C(\alpha,\beta,R,\ell^*)}{\sqrt{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\Big(\mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]\Big) \leq C(\alpha,\beta,R,\ell^*)/\sqrt{L}$.

Comments.
Proposition 3.7 and Corollary 3.6 mean that the tests $\phi^{u(1)}_{4,\alpha}$ and $\phi^{u(2)}_{3/4,\alpha}$ are minimax. Together with the ones obtained for [**Alt^u.2**], the two above results finally mean that, as when the baseline intensity is known, adaptation with respect to the change location can be achieved with a minimax separation rate of the parametric order, that is without any additional price to pay (possibly except multiplicative constants) as soon as the only change length is known.

III Minimax detection of a transitory change with known location

We consider the problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" versus alternative hypotheses where the location of the change from the baseline intensity is known, with adaptation to the change length, and with or without adaptation to the height. As in Chapter 2 Section IV, we see that adaptation to the length can be done without any incidence on the minimax separation rate order, while adaptation to both height and length leads to a cost factor of order $\sqrt{\log\log L}$.

III.1 Known change height

Let us first investigate the problem of testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]$", where for $R > 0$, δ^* in $(-R,R)\setminus\{0\}$ and τ^* in $(0,1)$,

$$[\mathbf{Alt^u.5}]\ \mathcal{S}^u_{\delta^*,\tau^*,\dots}[R] = \{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (-\delta^* \vee 0, (R-\delta^*) \wedge R],\ \exists \ell \in (0,1-\tau^*),$$
$$\forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau^*,\tau^*+\ell]}(t)\} \ . \quad (3.16)$$

As in Chapter 2 Section IV, the most intricate point here is the construction of a test achieving the minimax separation rate over $\mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]$, which will be proved to be of the parametric order $1/\sqrt{L}$, and therefore necessarily taking the knowledge of the change height δ^* into account. The test we propose is largely inspired from the aggregated test $\phi_{5,\alpha}$ defined by (2.18), where the test statistic is slightly adapted to compensate for the lack of the baseline intensity knowledge. Since the critical value can not be taken as a quantile of the test statistic, whose distribution under the null hypothesis is not free from the unknown baseline intensity anymore, we use the same conditioning trick as in the above sections.

Proposition 3.8 (Minimax lower bound for [$\mathbf{Alt^u.5}$]).
Let α,β in $(0,1)$ with $\alpha+\beta < 1$, $R > 0$, δ^* in $(-R,R)\setminus\{0\}$, τ^* in $(0,1)$. For $L > ((R-\delta^*)\wedge R)\log C_{\alpha,\beta}/\left(2\delta^{*2}\tau^*(1-\tau^*)\right)$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]\right) \geq \sqrt{\frac{((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{2L}}, \quad \text{with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

Let us now introduce the test

$$\phi^u_{5,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N) > s'^{+}_{N_1,\delta^*,\tau^*,L}(1-\alpha)\right\}} \ , \quad (3.17)$$

where $S'_{\delta^*,\tau_1,\tau_2}(N)$ is the statistic defined for $0 \leq \tau_1 < \tau_2 \leq 1$ by

$$S'_{\delta^*,\tau_1,\tau_2}(N) = \mathrm{sgn}(\delta^*)\Big(N(\tau_1,\tau_2] - (\tau_2-\tau_1)N_1\Big) - |\delta^*|L(\tau_2-\tau_1)(1-\tau_2+\tau_1)/2 \ , \quad (3.18)$$

and $s'^{+}_{n,\delta^*,\tau^*,L}(u)$ is the u-quantile of $\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$ given $N_1 = n$ under (H_0).

The main argument of the following upper bound is a control of the conditional quantile $s'^{+}_{n,\delta^*,\tau^*,L}(1-\alpha)$, provided in Lemma 3.22, and which is deduced from a refined Bernstein inequality based on some chaining techniques.

Proposition 3.9 (Minimax upper bound for [$\mathbf{Alt^u.5}$]).
Let $L \geq 1$, α,β in $(0,1)$, $R > 0$, δ^* in $(-R,R)\setminus\{0\}$ and τ^* in $(0,1)$. Let $\phi^u_{5,\alpha}$ be the test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]$" defined by (3.17). $\phi^u_{5,\alpha}$ is of level α, that is $\sup_{\lambda_0\in\mathcal{S}^u_0[R]} P_{\lambda_0}\left(\phi^u_{5,\alpha}(N) = 1\right) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,R,\delta^*) > 0$ such that

$$\mathrm{SR}_\beta\left(\phi^u_{5,\alpha}, \mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]\right) \leq \frac{C(\alpha,\beta,R,\delta^*)}{\sqrt{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\delta^*,\tau^*,\dots}[R]\right) \leq C(\alpha,\beta,R,\delta^*)/\sqrt{L}$.

III.2 Unknown change height

Now addressing the question of adaptation to the change height and length together, we consider for $R > 0$ and τ^* in $(0,1)$ the alternative set

$$[\mathbf{Alt^u.6}]\ \mathcal{S}^u_{\cdot,\tau^*,\dots}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (0,R],\ \exists \delta \in (-\lambda_0, R-\lambda_0] \setminus \{0\},\ \exists \ell \in (0, 1-\tau^*), \\ \forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau^*, \tau^*+\ell]}(t)\Big\} \ . \quad (3.19)$$

For the problem of testing (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\dots}[R]$", we obtain the following lower bound.

Proposition 3.10 (Minimax lower bound for [$\mathbf{Alt^u.6}$]).
Let α, β in $(0,1)$ with $\alpha + \beta < 1/2$, $R > 0$ and τ^* in $(0,1)$. There exists $L_0(\alpha, \beta, R, \tau^*) > 0$ such that for $L \geq L_0(\alpha, \beta, R, \tau^*)$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\tau^*,\dots}[R]\right) \geq \sqrt{\frac{R\tau^* \log\log L}{2L}} \ .$$

Let us assume now that $L \geq 3$. In order to prove that the above lower bound is of sharp order (with respect to L), we construct two aggregated tests: a first one based on a linear statistic and a second one based on a quadratic statistic as in Chapter 3 Section II.
We thus consider the discrete subset of $(0, 1-\tau^*)$ of the dyadic form

$$\left\{\ell_{\tau^*,k} = (1-\tau^*)2^{-k},\ k \in \{1, \dots, \lfloor \log_2 L \rfloor\}\right\} \ ,$$

and $u_\alpha = \alpha / \lfloor \log_2(L) \rfloor$, which allows to define the two following tests:

$$\phi^{u(1)}_{6,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\dots,\lfloor \log_2 L\rfloor\}}\left(N(\tau^*, \tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k}N_1 - \bar{b}_{N_1,\ell_{\tau^*,k}}\left(1-\frac{u_\alpha}{2}\right)\right)>0\right\}} \\ \vee \mathbb{1}_{\left\{\max_{k\in\{1,\dots,\lfloor \log_2 L\rfloor\}}\left(\bar{b}_{N_1,\ell_{\tau^*,k}}\left(\frac{u_\alpha}{2}\right) - N(\tau^*, \tau^*+\ell_{\tau^*,k}] + \ell_{\tau^*,k}N_1\right)>0\right\}} \ , \quad (3.20)$$

where $\bar{b}_{n,\tau_2-\tau_1}(u)$ is the u-quantile of $N(\tau_1,\tau_2] - (\tau_2-\tau_1)N_1$ given $N_1 = n$ under (H_0), that is the u-quantile of a recentered binomial distribution with parameters $(n, (\tau_2-\tau_1))$, whose sharp bound is obtained via Bennett's inequality (see Lemma 3.23 for details), and

$$\phi^{u(2)}_{6,\alpha}(N) = \mathbb{1}_{\left\{\max_{k\in\{1,\dots,\lfloor \log_2 L\rfloor\}}\left(T'_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N) - t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right)>0\right\}} \ , \quad (3.21)$$

where $T'_{\tau_1,\tau_2}(N)$ is the quadratic statistic defined in (3.9) and $t'_{n,\tau_1,\tau_2}(u)$ still denotes the u-quantile of its conditional distribution given $N_1 = n$ under (H_0).

Proposition 3.11 (Minimax upper bound for [$\mathbf{Alt^u.6}$]).
Let α, β in $(0,1)$, $R > 0$, τ^* in $(0,1)$, and let $\phi^{u(1/2)}_{6,\alpha}$ be one of the tests $\phi^{u(1)}_{6,\alpha}$ and $\phi^{u(2)}_{6,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\dots}[R]$" respectively defined by (3.20) and (3.21). Then $\phi^{u(1/2)}_{6,\alpha}$ is of level α, that is $\sup_{\lambda_0 \in \mathcal{S}^u_0[R]} P_{\lambda_0}\left(\phi^{u(1/2)}_{6,\alpha}(N) = 1\right) \leq \alpha$. Moreover, there exists $C(\alpha, \beta, R, \tau^*) > 0$ such that

$$\mathrm{SR}_\beta\left(\phi^{u(1/2)}_{6,\alpha}, \mathcal{S}^u_{\cdot,\tau^*,\dots}[R]\right) \leq C(\alpha, \beta, R, \tau^*)\sqrt{\frac{\log\log L}{L}} \ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\tau^*,\dots}[R]\right) \leq C(\alpha, \beta, R, \tau^*)\sqrt{\log\log L / L}$.

IV Minimax detection of a possibly transitory change with unknown location and length

Let us discuss as final stage the problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0^u[R]$" versus alternatives where both location and length of the change from the unknown baseline intensity are not known, distinguishing as in Chapter 2 Section V the transitory change case from the non transitory change particular case.

From the minimax point of view, we will emphasize that regardless if the baseline intensity is known or not, adaptation to both location and length of the change has the same minimax separation rate cost of order $\sqrt{\log L}$ in the transitory change case, and of order $\sqrt{\log\log L}$ at most (possibly cancelled by the change height knowledge) in the non transitory change case.

Since the non transitory change or jump detection problem, that we here study first, can be viewed as perfectly symmetrical to the transitory change with known location detection problem, our study uses tools and arguments that are very similar to the ones used in Chapter 3 Section III.

IV.1 Non transitory change

In order to investigate the problem of detecting a non transitory change with unknown location, but known height, we introduce for $R>0$ and δ^* in $(-R,R)\setminus\{0\}$ the alternative set

$$[\mathbf{Alt^u}.7]\ \mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (-\delta^* \vee 0, (R-\delta^*)\wedge R],\ \exists \tau \in (0,1),\ \forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}(t)\Big\} . \quad (3.22)$$

Considering the problem of testing the null hypothesis (H_0) "$\lambda \in \mathcal{S}_0^u[R]$" versus the alternative hypothesis (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]$", we obtain the following lower bound.

Proposition 3.12 (Minimax lower bound for $[\mathbf{Alt^u}.7]$).
Let α,β in $(0,1)$ with $\alpha+\beta<1$, $R>0$ and δ^* in $(-R,R)\setminus\{0\}$. For all $L > 2((R-\delta^*)\wedge R)\log C_{\alpha,\beta}/\delta^{*2}$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]) \geq \sqrt{\frac{((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{2L}},\ \text{with}\ C_{\alpha,\beta} = 1+4(1-\alpha-\beta)^2 .$$

Following the study and the notation of Chapter 3 Section III, we define the test

$$\phi^u_{7,\alpha}(N) = \mathbb{1}_{\left\{\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N) > s'^{+}_{N_1,\delta^*,L}(1-\alpha)\right\}} , \quad (3.23)$$

where $S'_{\delta^*,\tau_1,\tau_2}(N)$ is the statistic defined for $0 \leq \tau_1 < \tau_2 \leq 1$ by (3.18) and $s'^{+}_{n,\delta^*,L}(u)$ is the u-quantile of the conditional distribution of $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$ given $N_1 = n$ under (H_0).

Notice that a control of this conditional quantile $s'^{+}_{n,\delta^*,L}(1-\alpha)$, provided in Lemma 3.24, and deduced from the same chaining trick combined with Bernstein's inequality as in the proof of Lemma 3.22, is the main argument of the following result.

Proposition 3.13 (Minimax upper bound for $[\mathbf{Alt^u}.7]$).
Let $L \geq 1$, α and β in $(0,1)$, $R>0$ and δ^* in $(-R,R)\setminus\{0\}$. Let $\phi^u_{7,\alpha}$ be the test of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]$" defined by (3.23). Then $\phi^u_{7,\alpha}$ is of level α, that is $\sup_{\lambda_0\in\mathcal{S}_0^u[R]} P_{\lambda_0}(\phi^u_{7,\alpha}(N)=1) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,R,\delta^*)>0$ such that

$$\mathrm{SR}_\beta\left(\phi^u_{7,\alpha}, \mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]\right) \leq \frac{C(\alpha,\beta,R,\delta^*)}{\sqrt{L}} ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]\right) \leq C(\alpha,\beta,R,\delta^*)/\sqrt{L}$.

To address the question of adaptation to the change height, we introduce the alternative set

$$[\mathbf{Alt^u.8}]\ \mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (0,R],\ \exists \delta \in (-\lambda_0, R-\lambda_0] \setminus \{0\},\ \exists \tau \in (0,1),$$
$$\forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\Big\}\ , \quad (3.24)$$

and we consider the problem of testing (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$".
As usual, we start with a lower bound for the corresponding minimax separation rate.

Proposition 3.14 (Minimax lower bound for [$\mathbf{Alt^u.8}$]).
Let α, β in $(0,1)$ with $\alpha + \beta < 1/2$ and $R > 0$. There exists $L_0(\alpha,\beta,R) > 0$ such that for all $L \geq L_0(\alpha,\beta,R)$,

$$\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]) \geq \sqrt{\frac{R\log\log L}{4L}}\ .$$

Let us assume now that $L \geq 3$. In order to prove that the above lower bound is of sharp order (with respect to L), we consider the discrete subset of $(0,1)$ of the dyadic form

$$\mathcal{D}_L = \Big\{2^{-k},\ k \in \{2,\ldots,\lfloor \log_2(L)\rfloor\}\Big\} \cup \Big\{1-2^{-k},\ k \in \{1,\ldots,\lfloor \log_2(L)\rfloor\}\Big\}\ ,$$

we set $u_\alpha = \alpha/(2\lfloor\log_2(L)\rfloor - 1)$ and we define the two following tests:

$$\phi^{u(1)}_{8,\alpha}(N) = \mathbb{1}_{\{\max_{\tau\in\mathcal{D}_L}(N(\tau,1]-(1-\tau)N_1-\bar{b}_{N_1,1-\tau}(1-u_\alpha/2))>0\}} \vee \mathbb{1}_{\{\max_{\tau\in\mathcal{D}_L}(\bar{b}_{N_1,1-\tau}(u_\alpha/2)-N(\tau,1]+(1-\tau)N_1)>0\}}\ , \quad (3.25)$$

where $\bar{b}_{n,1-\tau}(u)$ is still the u-quantile of $N(\tau,1]-(1-\tau)N_1$ given $N_1 = n$ under (H_0), that is the u-quantile of a recentered binomial distribution with parameters $(n, 1-\tau)$, and

$$\phi^{u(2)}_{8,\alpha}(N) = \mathbb{1}_{\{\max_{\tau\in\mathcal{D}_L}(T'_{\tau,1}(N)-t'_{N_1,\tau,1}(1-u_\alpha))>0\}}\ , \quad (3.26)$$

where $T'_{\tau_1,\tau_2}(N)$ is the quadratic statistic defined by (3.9), and $t'_{n,\tau_1,\tau_2}(u)$ denotes the u-quantile of its conditional distribution given $N_1 = n$ under (H_0).

Proposition 3.15 (Minimax upper bound for [$\mathbf{Alt^u.8}$]).
Let α, β in $(0,1)$, $R > 0$, and let $\phi^{u(1/2)}_{8,\alpha}$ be one of the tests $\phi^{u(1)}_{8,\alpha}$ and $\phi^{u(2)}_{8,\alpha}$ of (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$" defined by (3.25) and (3.26). Then $\phi^{u(1/2)}_{8,\alpha}$ is of level α, that is $\sup_{\lambda_0\in\mathcal{S}^u_0[R]} P_{\lambda_0}\left(\phi^{u(1/2)}_{8,\alpha}(N) = 1\right) \leq \alpha$. Moreover, there exists a constant $C(\alpha,\beta,R) > 0$ such that

$$\mathrm{SR}_\beta\left(\phi^{u(1/2)}_{8,\alpha}, \mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]\right) \leq C(\alpha,\beta,R)\sqrt{\frac{\log\log L}{L}}\ ,$$

which entails in particular $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]\right) \leq C(\alpha,\beta,R)\sqrt{\log\log L/L}$.

IV.2 Transitory change

Let us investigate now the transitory change detection problem, focusing on adaptation to unknown location and length. As in Chapter 2 Section V.2, we will prove that minimax adaptation to these two parameters together has a cost of order as large as $\sqrt{\log L}$, so that adaptation to the height will have no additional cost. We therefore treat the two corresponding alternative sets quasi-simultaneously. For $R > 0$, δ^* in $(-R,R)\setminus\{0\}$, let

$$[\mathbf{Alt^u.9}]\ \mathcal{S}^u_{\delta^*,\cdot\cdot,\cdots}[R] = \Big\{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (-\delta^* \vee 0, (R-\delta^*)\wedge R],\ \exists \tau \in (0,1),\ \exists \ell \in (0,1-\tau),$$
$$\forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,\tau+\ell]}(t)\Big\}\ , \tag{3.27}$$

$$[\mathbf{Alt^u.10}]\ \mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R] = \{\lambda : [0,1] \to (0,R],\ \exists \lambda_0 \in (0,R],\ \exists \delta \in (-\lambda_0, R-\lambda_0]\setminus\{0\},\ \exists \tau \in (0,1),\ \exists \ell \in (0,1-\tau),$$
$$\forall t \in [0,1] \quad \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell]}(t)\}\ . \tag{3.28}$$

As usual, we begin by giving lower bounds for the minimax separation rates over these two alternative sets, noticing that the case where the change height is known can be straightforwardly extended (as a simple corollary then) to the general one, where all three parameters, location, length and height of the change are unknown.

Proposition 3.16 (Minimax lower bound for [**Alt$^{\mathbf{u}}$.9**]).
Let α,β in $(0,1)$ with $\alpha+\beta<1$, $R>0$, δ^* in $(-R,R)\setminus\{0\}$. There exists $L_0(\alpha,\beta,R,\delta^*)>0$ such that for $L \geq L_0(\alpha,\beta,R,\delta^*)$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\delta^*,\cdot\cdot,\cdots}[R]\right) \geq \sqrt{\frac{((R-\delta^*)\wedge R)\log L}{4L}}\ .$$

Since $\mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]$ includes $\mathcal{S}^u_{R/2,\cdot\cdot,\cdots}[R]$, Proposition 3.16 directly leads to the following corollary, whose proof is omitted for simplicity.

Corollary 3.17 (Minimax lower bound for [**Alt$^{\mathbf{u}}$.10**]).
Let α,β in $(0,1)$ with $\alpha+\beta<1$ and $R>0$. There exists $L_0(\alpha,\beta,R)>0$ such that for all $L \geq L_0(\alpha,\beta,R)$,

$$\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]\right) \geq \sqrt{\frac{R\log L}{8L}}\ .$$

In order to prove that the above lower bounds are sharp, we then construct two minimax adaptive tests, based on an aggregation principle. In order to customise the tests developed in Chapter 2 Section V.2 to the lack of knowledge of the baseline intensity, we consider the linear statistic $N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1$ and the quadratic statistic $T'_{\tau_1,\tau_2}(N)$ defined by (3.9), combined with the conditional trick already used in the above studies through the u-quantiles $\bar{b}_{n,\tau_2-\tau_1}(u)$ and $t'_{n,\tau_1,\tau_2}(u)$ of the conditional distributions of these statistics given $N_1=n$ under (H_0). Introducing $M_L = \lceil L/\log L\rceil$,

$$\begin{aligned}
\mathcal{K}_L^{(1)} &= \{(k,k'),\ k \in \{0,\ldots,\lceil L\rceil-1\}, k' \in \{1,\ldots,\lceil L\rceil - k\}\}\ ,\\
\mathcal{K}_L^{(2)} &= \{(k,k'),\ k \in \{0,\ldots,M_L-1\}, k' \in \{1,\ldots,M_L - k\}\}\setminus\{(0,M_L)\}\ ,
\end{aligned}$$

and the corrected levels $u_\alpha^{(1)} = 2\alpha/(\lceil L\rceil(\lceil L\rceil+1))$ and $u_\alpha^{(2)} = 2\alpha/(M_L(M_L+1)-2)$, we can thus propose the two following tests:

$$\phi^{u(1)}_{9/10,\alpha}(N) = \mathbb{1}_{\left\{\max_{(k,k')\in\mathcal{K}_L^{(1)}}\left(N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]-\frac{k'}{\lceil L\rceil}N_1-\bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\left(1-u_\alpha^{(1)}/2\right)\right)>0\right\}}$$
$$\vee\ \mathbb{1}_{\left\{\max_{(k,k')\in\mathcal{K}_L^{(1)}}\left(\bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\left(u_\alpha^{(1)}/2\right)-N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]+\frac{k'}{\lceil L\rceil}N_1\right)>0\right\}}\ , \tag{3.29}$$

and

$$\phi^{u(2)}_{9/10,\alpha}(N) = \mathbb{1}_{\left\{\max_{(k,k')\in\mathcal{K}_L^{(2)}}\left(T'_{\frac{k}{M_L},\frac{k+k'}{M_L}}(N)-t'_{N_1,\frac{k}{M_L},\frac{k+k'}{M_L}}\left(1-u_\alpha^{(2)}\right)\right)>0\right\}}\ . \tag{3.30}$$

Proposition 3.18 (Minimax upper bound for [**Alt$^{\mathbf{u}}$.9**] and [**Alt$^{\mathbf{u}}$.10**]).
Let α, β in $(0,1)$, $R > 0$ and δ^* in $(-R,R)\setminus\{0\}$. Let $\phi^{u(1/2)}_{9/10,\alpha}$ be one of the tests $\phi^{u(1)}_{9/10,\alpha}$ and $\phi^{u(2)}_{9/10,\alpha}$ defined by (3.29) and (3.30). Then $\phi^{u(1/2)}_{9/10,\alpha}$ is of level α for the problems of testing (H_0) versus (H_1) "$\lambda \in \mathcal{S}^u_{\delta^*,..,...}[R]$ or (H_1) "$\lambda \in \mathcal{S}^u_{.,..,...}[R]$", that is $\sup_{\lambda_0\in\mathcal{S}^u_0[R]} P_{\lambda_0}\left(\phi^{u(1/2)}_{9/10,\alpha}(N)=1\right)\le\alpha$. Moreover, there exist $C(\alpha,\beta,R,\delta^*)>0$ and $C(\alpha,\beta,R)>0$ such that

$$\begin{aligned}\mathrm{SR}_\beta\left(\phi^{u(1/2)}_{9/10,\alpha},\mathcal{S}^u_{\delta^*,..,...}[R]\right) &\le C(\alpha,\beta,R,\delta^*)\sqrt{\frac{\log L}{L}}\ ,\\ \mathrm{SR}_\beta\left(\phi^{u(1/2)}_{9/10,\alpha},\mathcal{S}^u_{.,..,...}[R]\right) &\le C(\alpha,\beta,R)\sqrt{\frac{\log L}{L}}\ .\end{aligned}$$

These upper bounds entail both $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\delta^*,..,...}[R]\right)\le C(\alpha,\beta,R,\delta^*)\sqrt{\log L/L}$ and $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{.,..,...}[R]\right)\le C(\alpha,\beta,R)\sqrt{\log L/L}$.

V Adjustment of individual levels for aggregated tests

As in Chapter 2 Section VI, we discuss here the possibility of adjusting the individual levels of the single tests involved in our aggregated tests to make them more powerful.

Most of the tests introduced in the present Chapter are based on aggregation principles coupled with conditional tricks. Among them, we can again distinguish aggregated tests of the form

$$\phi^u_{agg1,\alpha}(N)=\mathbb{1}_{\left\{\sup_{\theta\in\Theta}S_\theta(N)>s^+_{N_1}(1-\alpha)\right\}}\ \text{ or }\ \mathbb{1}_{\left\{\sup_{\theta\in\Theta}S_\theta(N)>s^+_{N_1}(1-\alpha/2)\right\}}\vee\mathbb{1}_{\left\{\inf_{\theta\in\Theta}S_\theta(N)<s^-_{N_1}(\alpha/2)\right\}}\ ,$$

where $s^+_n(u)$ and $s^-_n(u)$ respectively denote the u-conditional quantiles of $\sup_{\theta\in\Theta}S_\theta(N)$ and $\inf_{\theta\in\Theta}S_\theta(N)$ given $N_1=n$ under (H_0) (concerning $\phi^{u(1)-}_{3,\alpha}$, $\phi^{u(1)+}_{3,\alpha}$, $\phi^{u(1)}_{4,\alpha}$, $\phi^u_{5,\alpha}$ and $\phi^u_{7,\alpha}$), from aggregated tests of the form

$$\phi^u_{agg2,\alpha}(N)=\mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(S_\theta(N)-s_{N_1,\theta}(1-u_\alpha)\right)>0\right\}}\ \text{ or }\ \mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(S_\theta(N)-s_{N_1,\theta}(1-u_\alpha/2)\right)>0\right\}}\vee\mathbb{1}_{\left\{\sup_{\theta\in\Theta}\left(s_{N_1,\theta}(u_\alpha/2)-S_\theta(N)\right)>0\right\}}\ ,$$

$s_{n,\theta}(u)$ being the u-conditional quantile of $S_\theta(N)$ given $N_1=n$ under (H_0) and $u_\alpha=\alpha/|\Theta|$ (as $\phi^{u(2)}_{3/4,\alpha}$, $\phi^{u(1)}_{6,\alpha}$, $\phi^{u(2)}_{6,\alpha}$, $\phi^{u(1)}_{8,\alpha}$, $\phi^{u(2)}_{8,\alpha}$, $\phi^{u(1)}_{9/10,\alpha}$, $\phi^{u(2)}_{9/10,\alpha}$).

Notice that for any λ_0 in $\mathcal{S}^u_0[R]$, when $N\sim P_{\lambda_0}$, the distribution of $S_\theta(N)$ given $N_1=n$ is free from λ_0. Moreover, similarly to (2.35), a better choice than u_α can be made for levels of the single tests involved in aggregated tests of the form $\phi_{agg2,\alpha}(N)$, namely $u'_{n,\alpha}$ equal to

$$\left\{\begin{array}{l}\sup\left\{u\in(0,1),\ \sup_{\lambda_0\in\mathcal{S}^u_0[R]}P_{\lambda_0}\left(\sup_{\theta\in\Theta}\left(S_\theta(N)-s_{n,\theta}(1-u)\right)>0\ \middle|N_1=n\right)\le\alpha\right\}\\ \text{or }\sup\left\{u\in(0,1),\ \sup_{\lambda_0\in\mathcal{S}^u_0[R]}P_{\lambda_0}\left(\sup_{\theta\in\Theta}\left(S_\theta(N)-s_{n,\theta}(1-u/2)\right)\vee\left(s_{n,\theta}(u/2)-S_\theta(N)\right)>0\ \middle|N_1=n\right)\le\alpha\right\}\end{array}\right.\ . \tag{3.31}$$

Since for all n in $\mathbb{N}\setminus\{0\}$ $u_\alpha\le u'_{n,\alpha}$, by definition, $s_{n,\theta}(1-u'_{n,\alpha})\le s_{n,\theta}(1-u_\alpha)$, $s_{n,\theta}(1-u'_{n,\alpha}/2)\le s_{n,\theta}(1-u_\alpha/2)$ and $s_{n,\theta}(u_\alpha/2)\le s_{n,\theta}(u'_{n,\alpha}/2)$ therefore all the above tests of type $\phi^u_{agg2,\alpha}$ but with u_α replaced by $u'_{N_1,\alpha}$, that we can denote by $\phi'^u_{agg2,\alpha}$, satisfy the same minimax properties as $\phi^u_{agg2,\alpha}$. Our simulation study presented in Chapter 3 Section VII focuses on the practical performances of these adjusted aggregated tests $\phi'^u_{agg2,\alpha}$.

VI Summary and discussion

As in Chapter 2 Section VII, we present a summary of the results stated above. Recall (*c.f.* (3.9) and (3.18)) that for $0 \le \tau_1 < \tau_2 \le 1$,

$$T'_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\Big((N(\tau_1,\tau_2]-(\tau_2-\tau_1)N(0,1])^2+(\tau_2-\tau_1)(N(\tau_1,\tau_2]-(\tau_2-\tau_1)N(0,1])-(1-\tau_2+\tau_1)N(\tau_1,\tau_2]\Big),$$

$S'_{\delta^*,\tau_1,\tau_2}(N) = \mathrm{sgn}(\delta^*)\Big(N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1\Big)-|\delta^*|L(\tau_2-\tau_1)(1-\tau_2+\tau_1)/2$, and that $\bar{b}_{n,\tau_2-\tau_1}(u)$ and $t'_{n,\tau_1,\tau_2}(u)$ respectively stand for the u-quantiles of $N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1$ and $T'_{\tau_1,\tau_2}(N)$ given $N_1 = n$ under (H_0).

Non transitory change or jump detection		
Alternative set	$\mathrm{mSR}_{\alpha,\beta}$	Test statistics
$\mathcal{S}^u_{\delta^*,\tau^*,1-\tau^*}[R]$	-	$N(\tau^*,1]$
$\mathcal{S}^u_{\cdot,\tau^*,1-\tau^*}[R]$	$L^{-1/2}$	$N(\tau^*,1]$ $T'_{\tau^*,1}(N)$
$\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]$	$L^{-1/2}$	$\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$
$\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$	$\sqrt{\frac{\log\log L}{L}}$	$\max_{\tau\in\mathcal{D}_L}\Big(N(\tau,1]-(1-\tau)N_1-\bar{b}_{N_1,1-\tau}(1-u_\alpha/2)\Big)$ $\vee\Big(\bar{b}_{N_1,1-\tau}(u_\alpha/2)-N(\tau,1]+(1-\tau)N_1\Big)$ $\max_{\tau\in\mathcal{D}_L}\Big(T'_{\tau,1}(N)-t'_{N_1,\tau,1}(1-u_\alpha)\Big)$ $\mathcal{D}_L = \Big\{2^{-k},\ k\in\{2,\dots,\lfloor\log_2(L)\rfloor\}\Big\}\cup\Big\{1-2^{-k},\ k\in\{1,\dots,\lfloor\log_2(L)\rfloor\}\Big\}$
Transitory change or bump detection		
Alternative set	$\mathrm{mSR}_{\alpha,\beta}$	Test statistics
$\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$	-	$N(\tau^*,\tau^*+\ell^*]$
$\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$	$L^{-1/2}$	$N(\tau^*,\tau^*+\ell^*]$ $T'_{\tau^*,\tau^*+\ell^*}(N)$
$\mathcal{S}^u_{\delta^*,\cdot\cdot,\ell^*}[R]$	-	$\max_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)]$, $\min_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)]$ $\max_{k\in\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}}\Big(T'_{\frac{k}{M},\frac{k}{M}+\ell^*}(N)-t'_{N_1,\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\Big)$ $M = \lceil 2/(\ell^*(1-\ell^*))\rceil$
$\mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]$	$L^{-1/2}$	$\max_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)]$, $\min_{\tau\in[0,1-\ell^*\wedge(1/2)]} N(\tau,\tau+\ell^*\wedge(1/2)]$ $\max_{k\in\{0,\dots,\lceil(1-\ell^*)M\rceil-1\}}\Big(T'_{\frac{k}{M},\frac{k}{M}+\ell^*}(N)-t'_{N_1,\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\Big)$
$\mathcal{S}^u_{\delta^*,\tau^*,\cdots}[R]$	$L^{-1/2}$	$\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$
$\mathcal{S}^u_{\cdot,\tau^*,\cdots}[R]$	$\sqrt{\frac{\log\log L}{L}}$	$\max_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}}\Big(N\Big(\tau^*,\tau^*+\frac{1-\tau^*}{2^k}\Big]-\frac{1-\tau^*}{2^k}N_1-\bar{b}_{N_1,\frac{1-\tau^*}{2^k}}(1-u_\alpha/2)\Big)$ $\vee\Big(\bar{b}_{N_1,\frac{1-\tau^*}{2^k}}(u_\alpha/2)-N\Big(\tau^*,\tau^*+\frac{1-\tau^*}{2^k}\Big]+\frac{1-\tau^*}{2^k}N_1\Big)$ $\max_{k\in\{1,\dots,\lfloor\log_2 L\rfloor\}}\Big(T'_{\tau^*,\tau^*+\frac{1-\tau^*}{2^k}}(N)-t'_{N_1,\tau^*,\tau^*+\frac{1-\tau^*}{2^k}}(1-u_\alpha)\Big)$
$\mathcal{S}^u_{\delta^*,\cdot\cdot,\cdots}[R]$ $\mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]$	$\sqrt{\frac{\log L}{L}}$	$\max_{k\in\{0,\dots,\lceil L\rceil-1\},k'\in\{1,\dots,\lceil L\rceil-k\}}\Big(N\Big(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\Big]-\frac{k'}{\lceil L\rceil}N_1-\bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\Big(1-\frac{u_\alpha^{(1)}}{2}\Big)\Big)$ $\vee\Big(\bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\Big(\frac{u_\alpha^{(1)}}{2}\Big)-N\Big(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\Big]+\frac{k'}{\lceil L\rceil}N_1\Big)$ $\max_{\substack{k\in\{0,\dots,M_L-1\},k'\in\{1,\dots,M_L-k\}\\(k,k')\neq(0,M_L)}}\Big(T'_{\frac{k}{M_L},\frac{k+k'}{M_L}}(N)-t'_{N_1,\frac{k}{M_L},\frac{k+k'}{M_L}}\Big(1-u_\alpha^{(2)}\Big)\Big)$ $M_L = \lceil L/\log L\rceil$

As compared with the above overview in Chapter 2 Section VII, this one enables to see that the minimax separation rates do not suffer from the lack of knowledge of the baseline distribution: they indeed remain of the same order as in the problem of detecting a change from a given intensity, with the same phase transitions. Of course, these results are obtained at the price of a more important complexity of the test statistics, whether they are of linear or quadratic nature. This, combined with the need to use conditional quantiles instead of direct quantiles as critical values, brought in more technical arguments in the proofs. It can be furthermore noticed that up to our knowledge, except in the work of Verzelen et al. [VFLRB21] for the jump detection problem, this specific case of an unknown baseline distribution is in general not treated in the basic Gaussian model, where the only presence of a signal (that is a bump or jump from zero-mean) is tested.

VII Simulation study

We study in this section the performance of our minimax adaptive tests from an experimental point of view, by giving estimations of their size and their power for various distributions of the observed Poisson process, characterised by a jump or a bump in its intensity. Motivated by some applications in epidemiology and in cybersecurity, we check the feasibility of our new change-points detection procedures in practice and compare them with existing procedures.

As for the simulation study of Chapter 2, we focus here on the most general problems investigated in this chapter of detecting a change (a jump or a bump) in the intensity λ when the change location and height are unknown. The baseline intensity of λ, denoted by λ_0, is taken equal to 1 on $[0,1]$ in all the sequel. For several piecewise constant intensities λ with respect to the measure $d\Lambda(t) = Ldt$, where we have chosen $L = 50$, we take a level of test $\alpha = 0,05$.

We compare the estimated powers of our procedures with the Laplace test and the Z test recalled respectively in (2.36) and (2.37). The minimax adaptive tests we introduced to detect a change with unknown parameters from such an unknown intensity are still based on two kinds of statistics. The first statistic, of linear nature, is $N(\tau_1,\tau_2] - (\tau_2 - \tau_1)N(0,1]$, while the second statistic, of quadratic nature, is defined by

$$T'_{\tau_1,\tau_2}(N) = \frac{1}{L^2(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\Big((N(\tau_1,\tau_2]-(\tau_2-\tau_1)N(0,1])^2+(\tau_2-\tau_1)(N(\tau_1,\tau_2]-(\tau_2-\tau_1)N(0,1]) \\ -(1-\tau_2+\tau_1)N(\tau_1,\tau_2]\Big)\ .$$

VII.1 Detection of a non transitory change or jump

The aggregation approach we used to construct our new tests to detect a jump from an unknown baseline intensity consists in scanning these linear and quadratic statistics over a discrete subset of possible values for the change location on $(0,1)$. The subset introduced in Chapter 3 Section IV.1 is of the dyadic form

$$\Theta_d^u = \left\{2^{-k},\ k\in\{2,\ldots,5\}\right\}\cup\left\{1-2^{-k},\ k\in\{1,\ldots,5\}\right\}\ .$$

Considering the alternative [$\mathbf{Alt^u.8}$], the test statistic of our first procedure denoted by $(\mathtt{CP1}^u(\Theta_d^u))$ is thus

$$\mathcal{T}^{(1)}_{\cdot,\alpha}(N) = \max_{\tau\in\Theta_d^u}\Bigg(\Big(N(\tau,1]-(1-\tau)N(0,1]-\bar{b}_{N_1,1-\tau}\Big(1-u^{(1)}_{N_1,\alpha}/2\Big)\Big) \\ \vee\Big(\bar{b}_{N_1,1-\tau}\Big(u^{(1)}_{N_1,\alpha}/2\Big)-N(\tau,1]+(1-\tau)N(0,1]\Big)\Bigg)\ ,$$

where $\bar{b}_{n,1-\tau}(u)$ is the u-quantile of a recentered binomial distribution with parameters $(n, 1-\tau)$ and $u^{(1)}_{n,\alpha}$ is defined for all n in $\mathbb{N}$ as in (3.31) by

$$u^{(1)}_{n,\alpha} = \sup\Big\{u \in (0,1),\ \sup_{\lambda_0 \in \mathcal{S}^u_0[R]} P_{\lambda_0}\Big(\max_{\tau\in\Theta^u_d}\Big(\Big(N(\tau,1] - (1-\tau)N(0,1] - \bar{b}_{n,1-\tau}(1-u/2)\Big) \vee \Big(\bar{b}_{n,1-\tau}(u/2) - N(\tau,1] + (1-\tau)N(0,1]\Big)\Big) > 0 \,\Big|\, N_1 = n\Big) \le \alpha\Big\} \ . \tag{3.32}$$

The test statistic of our second procedure denoted by $(\mathtt{CP2}^u(\Theta^u_d))$ is

$$\mathcal{T}^{(2)}_{\cdot,\alpha}(N) = \max_{\tau\in\Theta^u_d}\Big(T'_{\tau,1}(N) - t'_{N_1,\tau,1}\Big(1 - u^{(2)}_{N_1,\alpha}\Big)\Big) \ ,$$

where $t'_{n,\tau_1,\tau_2}(u)$ is the u-quantile of the conditional distribution of $T'_{\tau_1,\tau_2}(N)$ given $N_1 = n$ under (H_0) and $u^{(2)}_{n,\alpha}$ is defined for all n in $\mathbb{N}$ by

$$u^{(2)}_{n,\alpha} = \sup\left\{u \in (0,1),\ \sup_{\lambda_0 \in \mathcal{S}^u_0[R]} P_{\lambda_0}\left(\max_{\tau\in\Theta^u_d}\Big(T'_{\tau,1}(N) - t'_{N_1,\tau,1}(1-u)\Big) > 0 \,\Big|\, N_1 = n\right) \le \alpha\right\} \ . \tag{3.33}$$

Then, the null hypothesis (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" is rejected when $\mathcal{T}^{(1)}_{\cdot,\alpha}(N) > 0$ for $(\mathtt{CP1}^u(\Theta^u_d))$, and when $\mathcal{T}^{(2)}_{\cdot,\alpha}(N) > 0$ for $(\mathtt{CP2}^u(\Theta^u_d))$.

As in the known baseline case (see Chapter 2 Section VIII), we have also considered the same tests, but replacing the dyadic set Θ^u_d by the regular set

$$\Theta^u_r = \left\{\frac{k}{10},\ k \in \{1,\dots,9\}\right\} \ .$$

The corresponding testing procedures are then denoted by $(\mathtt{CP1}^u(\Theta^u_r))$ and $(\mathtt{CP2}^u(\Theta^u_r))$

The quantities $u^{(1)}_{n,\alpha}$, $u^{(2)}_{n,\alpha}$ and $t'_{n,\tau,1}(1-u^{(2)}_{n,\alpha})$ have been estimated by Monte Carlo methods based on the simulation of 200 000 samples of n independent copies of a recentered binomial random variable with parameters $(n, 1-\tau)$ and of a uniform random variable on $[0,1]$. These samples were used to approximate the conditional probabilities occurring in (3.32) and (3.33). The approximations of $u^{(1)}_{n,\alpha}$ and $u^{(2)}_{n,\alpha}$ were obtained by dichotomy.

VII.2 Detection of a transitory change or bump

Let us consider the discrete sets

$$\Theta_1 = \left\{\left(\frac{k}{50}, \frac{k+k'}{50}\right),\ k \in \{0,\dots,49\},\ k' \in \{1,\dots,50-k\}\right\} \ ,$$

and

$$\Theta_2 = \left\{\left(\frac{k}{13}, \frac{k+k'}{13}\right),\ k \in \{0,\dots,12\},\ k' \in \{1,\dots,13-k\},\ (k,k') \neq (0,13)\right\} \ .$$

Considering the alternative [**Alt$^{\mathbf{u}}$.10**], the test statistic of our first procedure denoted by $(\mathtt{TC1}^u)$ is

$$\mathcal{T}^{(3)}_{\cdot,\alpha}(N) = \max_{(\tau,\tau')\in\Theta_1}\Big(\Big(N(\tau,\tau'] - (\tau'-\tau)N(0,1] - \bar{b}_{N_1,\tau'-\tau}\Big(1 - u^{(3)}_{N_1,\alpha}/2\Big)\Big) \vee \Big(\bar{b}_{N_1,\tau'-\tau}\Big(u^{(3)}_{N_1,\alpha}/2\Big) - N(\tau,\tau'] + (\tau'-\tau)N(0,1]\Big)\Big) \ ,$$

where $\bar{b}_{n,\tau'-\tau}(u)$ is the u-quantile of a recentered binomial distribution with parameters $(n, \tau' - \tau)$ and $u^{(3)}_{n,\alpha}$ is defined for all n in $\mathbb{N}$ by

$$u^{(3)}_{n,\alpha} = \sup\Bigg\{u \in (0,1),\ \sup_{\lambda_0 \in \mathcal{S}^u_0[R]} P_{\lambda_0}\Bigg(\max_{(\tau,\tau')\in\Theta_1} \Big(\Big(N(\tau,\tau'] - (\tau'-\tau)N(0,1] - \bar{b}_{n,\tau'-\tau}(1-u/2) \Big) \vee \Big(\bar{b}_{n,\tau'-\tau}(u/2) - N(\tau,\tau'] + (\tau'-\tau)N(0,1] \Big) \Big) > 0 \,\Big|\, N_1 = n \Bigg) \leq \alpha \Bigg\}\ . \tag{3.34}$$

while the test statistic of our second procedure denoted by (TC2^u) is

$$\mathcal{T}^{(4)}_{\cdot,\alpha}(N) = \max_{(\tau,\tau')\in\Theta_2} \Big(T'_{\tau,\tau'}(N) - t'_{N_1,\tau,\tau'}\Big(1 - u^{(4)}_{N_1,\alpha}\Big)\Big)\ ,$$

where $u^{(4)}_{n,\alpha}$ is defined for all n in $\mathbb{N}$ by

$$u^{(4)}_{n,\alpha} = \sup\left\{u \in (0,1),\ \sup_{\lambda_0 \in \mathcal{S}^u_0[R]} P_{\lambda_0}\left(\max_{(\tau,\tau')\in\Theta_2} \left(T'_{\tau,\tau'}(N) - t'_{N_1,\tau,\tau'}(1-u)\right) > 0 \,\Big|\, N_1 = n \right) \leq \alpha \right\}\ . \tag{3.35}$$

The null hypothesis (H_0) "$\lambda \in \mathcal{S}^u_0[R]$" is rejected when $\mathcal{T}^{(3)}_{\cdot,\alpha}(N) > 0$ for (TC1^u), and when $\mathcal{T}^{(4)}_{\cdot,\alpha}(N) > 0$ for (TC2^u).

The quantities $u^{(3)}_{n,\alpha}$, $u^{(4)}_{n,\alpha}$ and $t'_{n,\tau,\tau'}(1 - u^{(4)}_{n,\alpha})$ have been estimated by Monte Carlo methods based on the simulation of 200 000 independent copies of a recentered binomial random variable with parameter $(n, \tau' - \tau)$ and 200 000 independent copies of $T'_{\tau,\tau'}(N)$ given $N_1 = n$ under (H_0) obtained from the simulation of 200 000 samples of n i.i.d. random variables uniformly distributed on $[0,1]$. These samples have been used to approximate the conditional probabilities occurring in (3.34) and (3.35). The approximations of $u^{(3)}_{n,\alpha}$ and $u^{(4)}_{n,\alpha}$ have been obtained by dichotomy.

VII.3 Simulation results

We compare the tests (La) and (Z) with (CP1$^u(\Theta^u_d)$), (CP2$^u(\Theta^u_d)$), (CP1$^u(\Theta^u_r)$), and (CP2$^u(\Theta^u_r)$) when addressing the jump detection problem (described in VII.1), and with (TC1^u) and (TC2^u) when addressing the bump detection problem (described in VII.2).

Estimated sizes
We first study the size of each test by simulating 10 000 independent homogeneous Poisson processes of intensity $\lambda_0 = 1$ w.r.t. Λ on $[0,1]$. The probabilities of first kind error of all the considered tests have been estimated by the number of rejections divided by 10 000. The results are given in Table 3.1.

Table 3.1 – Estimated sizes

La	Z
0.050	0.049
(CP1$^u(\Theta^u_d)$)	(CP2$^u(\Theta^u_d)$)
0.047	0.046
(CP1$^u(\Theta^u_r)$)	(CP2$^u(\Theta^u_r)$)
0.046	0.047
TC1^u	TC2^u
0.049	0.049

Notice that the estimated sizes of our tests always remain below the target 0.05, as expected from the definitions of $u_{n,\alpha}^{(1)}$, $u_{n,\alpha}^{(2)}$, $u_{n,\alpha}^{(3)}$ and $u_{n,\alpha}^{(4)}$: the Monte Carlo estimation procedure does not affect this first kind error rate control property.

Estimated powers

For both testing problems, we study the estimated power of each test under various alternatives.

For the jump detection problem, we consider the same alternative intensities $\lambda_{\tau,\delta}$ as in the known baseline intensity case (see Chapter 2 Section VIII equation (2.42)), defined for all t in $[0,1]$ by

$$\lambda_{\tau,\delta}(t) = 1 + \delta \mathbb{1}_{(\tau,1]}(t) \ ,$$

where $\delta \in \{-0.8, -0.6, -0.4, -0.2, 0.4, 0.8, 1.2, 1.6, 2\}$ but with τ varying in $\{0.05, 0.1, 0.5, 0.9, 0.95\}$.
For each alternative, we have simulated 1 000 independent inhomogeneous Poisson processes with intensity $\lambda_{\tau,\delta}$ w.r.t. Λ on $[0,1]$, and the powers have been estimated for each test by the mean number of rejections. The results are gathered in Tables 3.2-3.6.

Concerning the bump detection problem, we have considered the same alternative intensities $\lambda_{0.2,\ell,\delta}$ as in the known baseline intensity case (see Chapter 2 Section VIII equation (2.43)) defined for all t in $[0,1]$ by

$$\lambda_{0.2,\ell,\delta}(t) = 1 + \delta \mathbb{1}_{(0.2,0.2+\ell]}(t) \ ,$$

where $\delta \in \{-0.8, -0.6, -0.4, -0.2, 0.4, 0.8, 1.2, 1.6, 2\}$ and ℓ varying in $\{0.1, 0.2, 0.5\}$. For each alternative, we have simulated 1 000 independent inhomogeneous Poisson processes with intensity $\lambda_{0.2,\ell,\delta}$ w.r.t. Λ on $[0,1]$. The powers have been estimated for each test by the mean number of rejections, giving the results presented in Tables 3.7-3.9

Table 3.2 – Estimated probability of detecting a jump with $\tau = 0.05$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.19	0.10	0.06	0.05	0.05	0.06	0.07	0.09	0.10
Z	0.36	0.18	0.09	0.06	0.05	0.09	0.13	0.17	0.24
$\mathrm{CP1}^u(\Theta_d^u)$	0.34	0.16	0.09	0.06	0.05	0.08	0.09	0.14	0.20
$\mathrm{CP2}^u(\Theta_d^u)$	0.34	0.17	0.10	0.06	0.05	0.06	0.07	0.11	0.16
$\mathrm{CP1}^u(\Theta_r^u)$	0.22	0.10	0.06	0.05	0.06	0.07	0.07	0.10	0.14
$\mathrm{CP2}^u(\Theta_r^u)$	0.23	0.11	0.07	0.05	0.06	0.06	0.07	0.09	0.13

Table 3.3 – Estimated probability of detecting a jump with $\tau = 0.1$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.41	0.17	0.08	0.06	0.07	0.09	0.13	0.19	0.24
Z	0.54	0.26	0.12	0.07	0.08	0.14	0.27	0.39	0.54
$\mathrm{CP1}^u(\Theta_d^u)$	0.54	0.25	0.11	0.07	0.06	0.11	0.20	0.35	0.49
$\mathrm{CP2}^u(\Theta_d^u)$	0.55	0.27	0.12	0.07	0.05	0.09	0.17	0.28	0.43
$\mathrm{CP1}^u(\Theta_r^u)$	0.53	0.24	0.10	0.06	0.07	0.13	0.24	0.42	0.60
$\mathrm{CP2}^u(\Theta_r^u)$	0.54	0.27	0.11	0.07	0.06	0.11	0.21	0.39	0.56

Table 3.4 – Estimated probability of detecting a jump with $\tau = 0.5$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.91	0.60	0.28	0.09	0.19	0.55	0.85	0.96	0.99
Z	0.62	0.35	0.17	0.08	0.16	0.45	0.72	0.88	0.97
$\mathrm{CP1}^u(\Theta_d^u)$	0.92	0.55	0.22	0.07	0.15	0.49	0.83	0.96	1
$\mathrm{CP2}^u(\Theta_d^u)$	0.90	0.51	0.20	0.07	0.14	0.46	0.81	0.95	0.99
$\mathrm{CP1}^u(\Theta_r^u)$	0.93	0.58	0.25	0.09	0.16	0.53	0.85	0.97	1
$\mathrm{CP2}^u(\Theta_r^u)$	0.91	0.56	0.23	0.08	0.16	0.51	0.84	0.96	1

Table 3.5 – Estimated probability of detecting a jump with $\tau = 0.9$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.13	0.10	0.07	0.05	0.08	0.15	0.24	0.37	0.53
Z	0.08	0.08	0.06	0.04	0.06	0.10	0.13	0.20	0.27
$\mathrm{CP1}^u(\Theta_d^u)$	0.17	0.10	0.07	0.05	0.09	0.22	0.36	0.57	0.73
$\mathrm{CP2}^u(\Theta_d^u)$	0.14	0.09	0.05	0.04	0.10	0.24	0.39	0.60	0.75
$\mathrm{CP1}^u(\Theta_r^u)$	0.22	0.13	0.07	0.05	0.08	0.20	0.36	0.58	0.75
$\mathrm{CP2}^u(\Theta_r^u)$	0.15	0.09	0.05	0.04	0.09	0.22	0.40	0.61	0.79

Table 3.6 – Estimated probability of detecting a jump with $\tau = 0.95$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.07	0.06	0.06	0.06	0.06	0.08	0.12	0.16	0.23
Z	0.05	0.05	0.06	0.05	0.06	0.06	0.08	0.10	0.12
$\mathrm{CP1}^u(\Theta_d^u)$	0.06	0.05	0.05	0.04	0.09	0.14	0.22	0.31	0.46
$\mathrm{CP2}^u(\Theta_d^u)$	0.05	0.04	0.05	0.04	0.10	0.15	0.25	0.35	0.50
$\mathrm{CP1}^u(\Theta_r^u)$	0.08	0.05	0.07	0.05	0.06	0.08	0.13	0.20	0.29
$\mathrm{CP2}^u(\Theta_r^u)$	0.06	0.05	0.06	0.05	0.07	0.09	0.15	0.22	0.32

Table 3.7 – Estimated probability of detecting a bump with $\ell = 0.1$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.08	0.06	0.07	0.05	0.06	0.08	0.10	0.13	0.19
Z	0.08	0.06	0.07	0.06	0.05	0.04	0.04	0.05	0.06
$\mathrm{TC1}^u$	0.12	0.09	0.07	0.05	0.06	0.14	0.26	0.37	0.59
$\mathrm{TC2}^u$	0.11	0.07	0.07	0.05	0.06	0.13	0.25	0.35	0.56

Table 3.8 – Estimated probability of detecting a bump with $\ell = 0.2$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.14	0.10	0.08	0.06	0.07	0.11	0.18	0.29	0.39
Z	0.09	0.09	0.07	0.05	0.05	0.04	0.04	0.05	0.05
$\mathtt{TC1}^u$	0.42	0.19	0.09	0.06	0.09	0.24	0.47	0.71	0.87
$\mathtt{TC2}^u$	0.29	0.17	0.08	0.06	0.10	0.24	0.46	0.70	0.86

Table 3.9 – Estimated probability of detecting a bump with $\ell = 0.5$

$\delta =$	-0.8	-0.6	-0.4	-0.2	0.4	0.8	1.2	1.6	2
La	0.15	0.12	0.07	0.06	0.04	0.04	0.05	0.07	0.09
Z	0.15	0.11	0.07	0.06	0.04	0.05	0.04	0.04	0.05
$\mathtt{TC1}^u$	0.83	0.45	0.17	0.08	0.12	0.37	0.67	0.89	0.98
$\mathtt{TC2}^u$	0.83	0.44	0.17	0.08	0.13	0.39	0.68	0.89	0.98

Comments

1. Considering the single change-point or jump detection problem, it first arises that among the (La) and (Z) procedures, neither is preferable to use: the Laplace and Z tests can have very low powers depending on when the change occurs. One can notice that their performances are significantly smaller than the ones of our procedures ($\mathtt{CP1}^u$) and ($\mathtt{CP2}^u$) when the jump occurs near to one, while the estimated powers remain comparable in the other cases. Moreover, it is worthwhile to note again that the jump detection problem in a Poisson process is not a symmetric problem. Indeed, it is easier to detect large negative jumps occurring close to zero than close to one, and easier to detect large positive jumps occurring close to one than close to zero.
2. Considering the transitory change or bump detection problem, our procedures have estimated powers significantly larger in all cases than the Laplace and Z tests. Moreover, we have to mention that complementary experiments (omitted in this study) showed that the estimated powers of ($\mathtt{TC1}^u$) and ($\mathtt{TC2}^u$) are equivalent for a same value of change length whatever the change location, which is not true for the procedures (La) and (Z).
3. Among our testing procedures, the estimated powers are quite similar in most cases. Nevertheless, it is to note that the procedures ($\mathtt{CP1}^u$) based on the linear statistics, are slightly more powerful than the procedures ($\mathtt{CP2}^u$) based on the quadratic statistics, for some positive jumps occurring near 0 and for some negative jumps occurring near 1, whereas the procedures ($\mathtt{CP2}^u$) are slightly more powerful for some positive jumps occurring near 1. As expected, the aggregated tests based on dyadic sets are significantly more efficient that the ones based on regular sets when the change occurs near 0 or 1. For the bump detection problem, the main difference in the estimated powers concerns the case $\ell = 0.2$ where the better performance of the linear statistics based test ($\mathtt{TC1}^u$) as compared with the quadratic statistics based test ($\mathtt{TC2}^u$) to detect large negative jumps is pronounced.
4. The comparison of the simulated powers of the different testing procedures confirm again the intuition that detecting a bump is harder than detecting a jump. The simulation study also highlights that it is substantially easier to detect a jump or a bump with negative change height than with positive change height probably for the same reasons as in the above known baseline intensity case (see Chapter 2 Section VIII).

VIII Proofs of the main results

Proof of Proposition 3.2

By definition of $b_{n,\ell^*}(u)$ as the u-quantile of a binomial distribution with parameters (n,ℓ^*), the Bienayme-Chebyshev inequality easily gives

$$b_{n,\ell^*}(1-\alpha) \le n\ell^* + \sqrt{n\ell^*(1-\ell^*)/\alpha}\ ,$$

for all n in $\mathbb{N}$. It also gives for every n in $\mathbb{N}$ and every $\varepsilon > 0$, $b_{n,\ell^*}(\alpha) > n\ell^* - \sqrt{n\ell^*(1-\ell^*)/(\alpha-\varepsilon)}$. Therefore, letting ε tending to 0,

$$b_{n,\ell^*}(\alpha) \ge n\ell^* - \sqrt{n\ell^*(1-\ell^*)/\alpha}\ .$$

(i) Assume first that $0 < \delta^* < R$ and let λ in $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$.
Setting $I(\lambda) = \int_0^1 \lambda(t)dt \le R$, the assumption (3.5) leads to

$$\delta^*\sqrt{\ell^*(1-\ell^*)} \ge \frac{1}{\sqrt{L}}\left(\sqrt{\frac{2(\lambda_0+\delta^*)}{\beta(1-\ell^*)}} + 2\sqrt{\frac{I(\lambda)\ell^*}{\beta(1-\ell^*)}} + \sqrt{\frac{1}{\alpha}\left(I(\lambda)+2\sqrt{\frac{I(\lambda)}{\beta L}}\right)}\right)\ .$$

Hence

$$\delta^*\ell^*(1-\ell^*)L \ge \sqrt{\frac{2(\lambda_0+\delta^*)\ell^*L}{\beta}} + 2\ell^*\sqrt{\frac{I(\lambda)L}{\beta}} + \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)}\ ,$$

and since $I(\lambda) = \lambda_0 + \delta^*\ell^*$,

$$\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)\ell^* + \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)} - (\lambda_0+\delta^*)\ell^*L \le -\sqrt{\frac{2(\lambda_0+\delta^*)\ell^*L}{\beta}}\ . \tag{3.36}$$

We get then the following inequalities

$$\begin{aligned}
&P_\lambda\left(\phi^{u,+}_{1,\alpha}(N)=0\right)\\
&\quad= P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < b_{N_1,\ell^*}(1-\alpha)\right)\\
&\quad= P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < b_{N_1,\ell^*}(1-\alpha)\ ,\ |N_1-I(\lambda)L| \le 2\sqrt{I(\lambda)L/\beta}\right)\\
&\qquad+ P_\lambda\left(N_1 < I(\lambda)L - 2\sqrt{I(\lambda)L/\beta}\right) + P_\lambda\left(N_1 > I(\lambda)L+2\sqrt{I(\lambda)L/\beta}\right)\\
&\quad\le P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < b_{N_1,\ell^*}(1-\alpha)\ ,\ |N_1-I(\lambda)L| \le 2\sqrt{I(\lambda)L/\beta}\right) + \frac{\beta}{2}\\
&\quad\le P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < N_1\ell^* + \sqrt{\frac{N_1\ell^*(1-\ell^*)}{\alpha}}\ ,\ |N_1-I(\lambda)L| \le 2\sqrt{I(\lambda)L/\beta}\right) + \frac{\beta}{2}\\
&\quad\le P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < \left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)\ell^* + \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)}\right) + \frac{\beta}{2}\\
&\quad\le P_\lambda\left(N(\tau^*,\tau^*+\ell^*] - (\lambda_0+\delta^*)\ell^*L < -\sqrt{\frac{2(\lambda_0+\delta^*)\ell^*L}{\beta}}\right) + \frac{\beta}{2}\ \text{(thanks to (3.36))}\\
&\quad\le \beta\ \text{(Bienayme-Chebyshev)}\ .
\end{aligned}$$

This concludes the proof of (i).

(ii) Assume now that $-R < \delta^* < 0$ and let λ in $\mathcal{S}^u_{\delta^*,\tau^*,\ell^*}[R]$. The assumption (3.5) entails

$$|\delta^*|\sqrt{\ell^*(1-\ell^*)} \geq \frac{1}{\sqrt{L}}\left(\sqrt{\frac{2(\lambda_0+\delta^*)}{\beta(1-\ell^*)}} + 2\sqrt{\frac{I(\lambda)\ell^*}{\beta(1-\ell^*)}} + \sqrt{\frac{1}{\alpha}\left(I(\lambda)+2\sqrt{\frac{I(\lambda)}{\beta L}}\right)}\right) .$$

Hence

$$-\delta^*\ell^*(1-\ell^*)L \geq \sqrt{\frac{2(\lambda_0+\delta^*)\ell^* L}{\beta}} + 2\ell^*\sqrt{\frac{I(\lambda)L}{\beta}} + \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)} ,$$

and then

$$\left(I(\lambda)L-2\sqrt{\frac{I(\lambda)L}{\beta}}\right)\ell^* - \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)} - (\lambda_0+\delta^*)\ell^* L \geq \sqrt{\frac{2(\lambda_0+\delta^*)\ell^* L}{\beta}} . \qquad (3.37)$$

We get then

$$\begin{aligned}
&P_\lambda\left(\phi^{u,-}_{1,\alpha}(N)=0\right)\\
&= P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > b_{N_1,\ell^*}(\alpha)\right)\\
&= P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > b_{N_1,\ell^*}(\alpha)\,,\ |N_1-I(\lambda)L| \leq 2\sqrt{I(\lambda)L/\beta}\right)\\
&\quad + P_\lambda\left(N_1 < I(\lambda)L-2\sqrt{I(\lambda)L/\beta}\right) + P_\lambda\left(N_1 > I(\lambda)L+2\sqrt{I(\lambda)L/\beta}\right)\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > b_{N_1,\ell^*}(\alpha)\,,\ |N_1-I(\lambda)L| \leq 2\sqrt{I(\lambda)L/\beta}\right) + \frac{\beta}{2}\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > N_1\ell^* - \sqrt{\frac{N_1\ell^*(1-\ell^*)}{\alpha}},\ |N_1-I(\lambda)L| \leq 2\sqrt{I(\lambda)L/\beta}\right) + \frac{\beta}{2}\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > \left(I(\lambda)L-2\sqrt{\frac{I(\lambda)L}{\beta}}\right)\ell^* - \sqrt{\frac{\ell^*(1-\ell^*)}{\alpha}\left(I(\lambda)L+2\sqrt{\frac{I(\lambda)L}{\beta}}\right)}\right) + \frac{\beta}{2}\\
&\leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] - (\lambda_0+\delta^*)\ell^* L > \sqrt{\frac{2(\lambda_0+\delta^*)\ell^* L}{\beta}}\right) + \frac{\beta}{2} \text{ (thanks to (3.37))}\\
&\leq \beta \text{ (Bienayme-Chebyshev)} .
\end{aligned}$$

This concludes the proof of (ii).

Proof of Proposition 3.3

Let us set $\lambda_0 = R/2$ and introduce for $r > 0$ the Poisson intensity λ_r defined for all t in $[0,1]$ by

$$\lambda_r(t) = \lambda_0 + \frac{r}{\sqrt{\ell^*(1-\ell^*)}}\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t) .$$

Notice that when $0 < r/\sqrt{\ell^*(1-\ell^*)} \leq R/2$, λ_r belongs to

$$(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R])_r = \{\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R],\ d_2(\lambda,\mathcal{S}_0[R]) \geq r\} ,$$

as defined in Lemma 1.7. We get from Lemma 2.1 and Lemma 2.24 that

$$E_{\lambda_0}\left[\left(\frac{dP_\lambda}{dP_{\lambda_0}}\right)^2(N)\right] = \exp\left(\frac{r^2L}{\lambda_0(1-\ell^*)}\right) .$$

Choosing $r = (\lambda_0(1-\ell^*)\log C_{\alpha,\beta}/L)^{1/2}$ then leads to $E_{\lambda_0}\left[(dP_\lambda/dP_{\lambda_0})^2(N)\right] = C_{\alpha,\beta}$.
For $L \geq 1$ such that $2\log C_{\alpha,\beta}/(\ell^* L) \leq R$, we obtain $0 < r/\sqrt{\ell^*(1-\ell^*)} \leq R/2$ whereby λ_r belongs to $(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R])_r$ and Lemma 1.8 allows us to conclude that $\rho\left(\left(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]\right)_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}\left(\mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]\right) \geq r$.

Proof of Proposition 3.4

The first statement of Proposition 3.4 is straightforward, just noticing that for every λ_0 in $\mathcal{S}^u_0[R]$

$$E_{\lambda_0}\left[\phi^{u(1/2)}_{2,\alpha}(N)\right] = E_{\lambda_0}\left[E_{\lambda_0}\left[\phi^{u(1/2)}_{2,\alpha}(N)\middle|N_1\right]\right] \leq \alpha .$$

Let us assume that $\lambda \in \mathcal{S}^u_{\cdot,\tau^*,\ell^*}[R]$, that is there exists λ_0 in $(0,R)$ and δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ satisfying $\lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell^*]}(t)$ for all t in $[0,1]$.

Let us first consider the test $\phi^{u(1)}_{2,\alpha}(N)$ and assume

$$d_2(\lambda,\mathcal{S}^u_0[R]) \geq \frac{1}{\sqrt{L}}\left(\sqrt{\frac{2R}{\beta(1-\ell^*)}} + 2\sqrt{\frac{R\ell^*}{\beta(1-\ell^*)}} + \sqrt{\frac{1}{\alpha_1\wedge\alpha_2}\left(R+2\sqrt{\frac{R}{\beta L}}\right)}\right) . \tag{3.38}$$

We may write $\phi^{u(1)}_{2,\alpha}(N) = \phi^{u,-}_{1,\alpha_2}(N)\vee\phi^{u,+}_{1,\alpha_1}(N)$ by definition of the tests $\phi^{u,-}_{1,\alpha}$ and $\phi^{u,+}_{1,\alpha}$ in (3.3). We therefore obtain $P_\lambda\left(\phi^{u(1)}_{2,\alpha}(N) = 0\right) = P_\lambda\left(\phi^{u,-}_{1,\alpha_2}(N) = 0,\ \phi^{u,+}_{1,\alpha_1}(N) = 0\right)$.
From the assumption (3.38) and the same computations as in the proof of Proposition 3.2, we get

$$P_\lambda\left(\phi^{u,-}_{1,\alpha_2}(N) = 0\right) \leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] > b_{N_1,\ell^*}(\alpha_2)\right) \leq \beta ,$$

when $-\lambda_0 < \delta < 0$ and

$$P_\lambda\left(\phi^{u,+}_{1,\alpha_1}(N) = 0\right) \leq P_\lambda\left(N(\tau^*,\tau^*+\ell^*] < b_{N_1,\ell^*}(1-\alpha_1)\right) \leq \beta ,$$

when $0 < \delta \leq R-\lambda_0$.
The result of Proposition 3.4 for the test $\phi^{u(1)}_{2,\alpha}(N)$ follows with

$$C(\alpha,\beta,R,\ell^*) = \sqrt{\frac{2R}{\beta(1-\ell^*)}} + 2\sqrt{\frac{R\ell^*}{\beta(1-\ell^*)}} + \sqrt{\frac{1}{\alpha_1\wedge\alpha_2}\left(R+2\sqrt{\frac{R}{\beta L}}\right)} .$$

Let us consider then the test $\phi^{u(2)}_{2,\alpha}(N)$. We get from Lemma 3.20

$$t'_{n,\tau^*,\tau^*+\ell^*}(1-\alpha) \leq \frac{C}{L^2}\left(5n\log\left(\frac{2.77}{\alpha}\right) + 3\max\left(\frac{1-\ell^*}{\ell^*},\frac{\ell^*}{1-\ell^*}\right)\log^2\left(\frac{2.77}{\alpha}\right)\right) .$$

Now, the Bienayme-Chebyshev inequality and the bound $\int_0^1 \lambda(x)L dx \leq RL$ give

$$P_\lambda\left(N_1 \geq RL + \sqrt{\frac{2RL}{\beta}}\right) \leq \frac{\beta}{2} .$$

This yields $P_\lambda\left(t'_{N_1,\tau^*,\tau^*+\ell^*}(1-\alpha)\geq C'(\alpha,\beta,R,\ell^*,L)\right)\leq\beta/2$ with

$$C'(\alpha,\beta,R,\ell^*,L)=C\left(5R\frac{\log\left(\frac{2.77}{\alpha}\right)}{L}+5\sqrt{\frac{2R}{\beta}}\frac{\log\left(\frac{2.77}{\alpha}\right)}{L^{3/2}}+3\max\left(\frac{1-\ell^*}{\ell^*},\frac{\ell^*}{1-\ell^*}\right)\frac{\log^2\left(\frac{2.77}{\alpha}\right)}{L^2}\right).$$

Noticing that

$$P_\lambda\left(\phi^{u(2)}_{2,\alpha}(N)=0\right)\leq P_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)\leq C'(\alpha,\beta,R,\ell^*,L)\right)+P_\lambda\left(t'_{N_1,\tau^*,\tau^*+\ell^*}(1-\alpha)>C'(\alpha,\beta,R,\ell^*,L)\right),$$

this enables to write

$$P_\lambda\left(\phi^{u(2)}_{2,\alpha}(N)=0\right)\leq P_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)\leq C'(\alpha,\beta,R,\ell^*,L)\right)+\frac{\beta}{2}. \tag{3.39}$$

Assume now

$$d_2(\lambda,\mathcal{S}^u_0[R])\geq\frac{1}{\sqrt{L}}\max\left(4\sqrt{\frac{2R}{\beta}},\left(\frac{4R}{\sqrt{\beta}}+2LC'(\alpha,\beta,R,\ell^*,L)\right)^{1/2}\right), \tag{3.40}$$

which ensures

$$|\delta|\sqrt{\ell^*(1-\ell^*)}\geq\frac{1}{\sqrt{L}}\max\left(4\sqrt{\frac{2(\lambda_0+\delta(1-\ell^*))}{\beta}},\left(\frac{4(\lambda_0+\delta(1-\ell^*))}{\sqrt{\beta}}+2LC'(\alpha,\beta,R,\ell^*,L)\right)^{1/2}\right).$$

We get then

$$\delta^2\ell^*(1-\ell^*)\geq 2\max\left(2\sqrt{\frac{2(\lambda_0+\delta(1-\ell^*))}{\beta L}}|\delta|\sqrt{\ell^*(1-\ell^*)},\frac{2(\lambda_0+\delta(1-\ell^*))}{\sqrt{\beta}L}+C'(\alpha,\beta,R,\ell^*,L)\right),$$

and using the simple facts that $a+b\leq 2\max(a,b)$ and $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for all $a,b\geq 0$,

$$\delta^2\ell^*(1-\ell^*)\geq\sqrt{\frac{2}{\beta}\left(\frac{4(\lambda_0+\delta(1-\ell^*))}{L}\delta^2\ell^*(1-\ell^*)+\frac{2(\lambda_0+\delta(1-\ell^*))^2}{L^2}\right)}+C'(\alpha,\beta,R,\ell^*,L). \tag{3.41}$$

Furthermore, Lemma 3.19 gives $E_\lambda[T'_{\tau^*,\tau^*+\ell^*}(N)]=\delta^2\ell^*(1-\ell^*)$ and

$$\mathrm{Var}_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)\right)=\frac{2(\lambda_0+\delta(1-\ell^*))^2}{L^2}+\frac{4(\lambda_0+\delta(1-\ell^*))}{L}\delta^2\ell^*(1-\ell^*),$$

so (3.41) leads to

$$E_\lambda[T'_{\tau^*,\tau^*+\ell^*}(N)]\geq\sqrt{2\mathrm{Var}_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)\right)/\beta}+C'(\alpha,\beta,R,\ell^*,L).$$

Combined with (3.39), this inequality entails

$$P_\lambda\left(\phi^{u(2)}_{2,\alpha}(N)=0\right)\leq P_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)-E_\lambda[T'_{\tau^*,\tau^*+\ell^*}(N)]\leq\sqrt{2\mathrm{Var}_\lambda\left(T'_{\tau^*,\tau^*+\ell^*}(N)\right)/\beta}\right)+\frac{\beta}{2},$$

and the proof ends with the Bienayme-Chebyshev inequality, thus giving

$$P_\lambda\left(\phi^{u(2)}_{2,\alpha}(N)=0\right)\leq\beta.$$

The result of Proposition 3.4 for the test $\phi^{u(2)}_{2,\alpha}(N)$ then follows with

$$C(\alpha,\beta,R,\ell^*) = \max\left(4\sqrt{\frac{2R}{\beta}}, \left(\frac{4R}{\sqrt{\beta}} + 2C\left(5R\log\left(\frac{2.77}{\alpha}\right) + 5\sqrt{\frac{2R}{\beta}}\log\left(\frac{2.77}{\alpha}\right) + 3\max\left(\frac{\ell^*}{1-\ell^*}, \frac{1-\ell^*}{\ell^*}\right)\log^2\left(\frac{2.77}{\alpha}\right)\right)\right)^{1/2}\right),$$

where the constant C is defined in Lemma 3.20.

Proof of Proposition 3.5

Start by remarking that the control of the first kind error rates of the three tests $\phi^{u(1)+}_{3,\alpha}$, $\phi^{u(1)-}_{3,\alpha}$ and $\phi^{u(2)}_{3/4,\alpha}$ is straightforward, considering the same conditioning trick as in the beginning of the proof of Proposition 3.4 above.

Let us first address the statement of Proposition 3.5 for $\phi^{u(1)+}_{3,\alpha}$.
Let $L \geq 1$ and let us consider $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau,\tau+\ell^*]}$ in $\mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$ with $\delta^* > 0$. Notice that

$$\begin{aligned} P_\lambda\left(\phi^{u(1)+}_{3,\alpha}(N) = 0\right) &= P_\lambda\left(\max_{t\in[0,1-\ell^*\wedge(1/2)]} N(t, t+\ell^*\wedge(1/2)] \leq b^+_{N_1,\ell^*\wedge(1/2)}(1-\alpha)\right) \\ &\leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \leq b^+_{N_1,\ell^*\wedge(1/2)}(1-\alpha)\right) . \end{aligned}$$

One deduces from Lemma 3.21 that for every n in $\mathbb{N}\setminus\{0\}$,

$$b^+_{n,\ell^*\wedge(1/2)}(1-\alpha) \leq (\ell^*\wedge(1/2))n + \frac{n}{2}g^{-1}\left(\frac{32}{n}\log\left(\frac{320}{\alpha}\right)\right) ,$$

with g defined by (2.44), and then from the inequality $g^{-1}(x) \leq 2x/3 + \sqrt{2x}$ for all $x > 0$ (see (2.45)),

$$b^+_{n,\ell^*\wedge(1/2)}(1-\alpha) \leq (\ell^*\wedge(1/2))n + 4\sqrt{n\log\left(\frac{320}{\alpha}\right)} + \frac{32}{3}\log\left(\frac{320}{\alpha}\right) .$$

Since $b^+_{0,\ell^*\wedge(1/2)}(1-\alpha) = 0$, the above control holds in fact for every n in $\mathbb{N}$, whereby

$$b^+_{N_1,\ell^*\wedge(1/2)}(1-\alpha) \leq (\ell^*\wedge(1/2))N_1 + 4\sqrt{N_1\log\left(\frac{320}{\alpha}\right)} + \frac{32}{3}\log\left(\frac{320}{\alpha}\right) .$$

Setting $I_{\lambda,L}(\beta) = I(\lambda)L + \sqrt{\frac{2I(\lambda)L}{\beta}}$ with $I(\lambda) = \int_0^1 \lambda(t)dt$, and

$$Q(\alpha,\beta,\ell,L) = \ell I_{\lambda,L}(\beta) + 4\sqrt{I_{\lambda,L}(\beta)\log\left(\frac{320}{\alpha}\right)} + \frac{32}{3}\log\left(\frac{320}{\alpha}\right) , \tag{3.42}$$

we obtain for ℓ in $(0,1/2]$

$$P_\lambda\left(\phi^{u(1)+}_{3,\alpha}(N) = 0\right) \leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \leq Q(\alpha,\beta,\ell^*\wedge(1/2),L)\right) + P_\lambda\left(N_1 > I_{\lambda,L}(\beta)\right) .$$

Therefore

$$P_\lambda\left(\phi^{u(1)+}_{3,\alpha}(N) = 0\right) \leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \leq Q(\alpha,\beta,\ell^*\wedge(1/2),L)\right) + \frac{\beta}{2} , \tag{3.43}$$

with

$$Q(\alpha,\beta,\ell^*\wedge(1/2),L)\le\left(\ell^*\wedge\frac{1}{2}\right)(\lambda_0+\delta^*\ell^*)L+\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{\frac{2RL}{\beta}}+4\sqrt{\left(RL+\sqrt{\frac{2RL}{\beta}}\right)\log\left(\frac{320}{\alpha}\right)}$$
$$+\frac{32}{3}\log\left(\frac{320}{\alpha}\right), \quad (3.44)$$

since $I(\lambda)=\lambda_0+\delta^*\ell^*\le R$. Let us now assume that

$$d_2\left(\lambda,\mathcal{S}_0^u[R]\right)=\delta^*\sqrt{\ell^*(1-\ell^*)}\ge\sqrt{\frac{\ell^*}{(1-\ell^*)L}}\left(\sqrt{\frac{2R}{\beta}}+\sqrt{\frac{2R}{\beta\left(\ell^*\wedge\frac{1}{2}\right)}}+\frac{4}{\ell^*\wedge\frac{1}{2}}\sqrt{\left(R+\sqrt{\frac{2R}{\beta L}}\right)\log\left(\frac{320}{\alpha}\right)}\right.$$
$$\left.+\frac{32\log(320/\alpha)}{3\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{L}}\right). \quad (3.45)$$

Then

$$\delta^*(1-\ell^*)\left(\ell^*\wedge\frac{1}{2}\right)L\ge\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{\frac{2RL}{\beta}}+\sqrt{\frac{2RL}{\beta}\left(\ell^*\wedge\frac{1}{2}\right)}+4\sqrt{\left(RL+\sqrt{\frac{2RL}{\beta}}\right)\log\left(\frac{320}{\alpha}\right)}+\frac{32}{3}\log\left(\frac{320}{\alpha}\right).$$

With (3.44) and using $R\ge\lambda_0+\delta^*$, this implies

$$\delta^*(1-\ell^*)\left(\ell^*\wedge\frac{1}{2}\right)L\ge Q(\alpha,\beta,\ell^*\wedge(1/2),L)+\sqrt{\frac{2(\lambda_0+\delta^*)L}{\beta}\left(\ell^*\wedge\frac{1}{2}\right)}-\left(\ell^*\wedge\frac{1}{2}\right)(\lambda_0+\delta^*\ell^*)L\ ,$$

and

$$Q(\alpha,\beta,\ell^*\wedge(1/2),L)\le\left(\ell^*\wedge\frac{1}{2}\right)(\lambda_0+\delta^*)L-\sqrt{\frac{2(\lambda_0+\delta^*)L}{\beta}\left(\ell^*\wedge\frac{1}{2}\right)}\ .$$

By simply using the exact computation of $E_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]]$ and $\mathrm{Var}_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]]$ which both equal $(\lambda_0+\delta^*)(\ell^*\wedge(1/2))L$, one can notice that this is equivalent to

$$Q(\alpha,\beta,\ell^*\wedge(1/2),L)\le E_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]]-\sqrt{\frac{2\mathrm{Var}_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]]}{\beta}}\ .$$

Coming back to (3.43), one finally deduces from the Bienayme-Chebyshev inequality that

$$P_\lambda\left(\phi_{3,\alpha}^{u(1)+}(N)=0\right)\le P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)]\le Q(\alpha,\beta,\ell^*\wedge(1/2),L)\right)+\frac{\beta}{2}\le\beta\ .$$

Let us now address the statement of Proposition 3.5 for $\phi_{3,\alpha}^{u(1)-}$.
Let $L\ge1$ and let us consider again $\lambda=\lambda_0+\delta^*\mathbb{1}_{(\tau,\tau+\ell^*]}$ in $\mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$, but with $-\lambda_0<\delta^*<0$ here. Notice first that

$$P_\lambda\left(\phi_{3,\alpha}^{u(1)-}(N)=0\right)=P_\lambda\left(\min_{t\in[0,1-\ell^*\wedge(1/2)]}N(t,t+\ell^*\wedge(1/2)]\ge b^-_{N_1,\ell^*\wedge(1/2)}(\alpha)\right)$$
$$\le P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)]\ge b^-_{N_1,\ell^*\wedge(1/2)}(\alpha)\right)\ .$$

From Lemma 3.21, one deduces that

$$b^-_{N_1,\ell^*\wedge(1/2)}(\alpha)\ge N_1\left(\ell^*\wedge(1/2)\right)-4\sqrt{2N_1\log\left(\frac{320}{\alpha}\right)}\ .$$

Setting for ℓ in $(0,1/2]$,

$$Q'(\alpha,\beta,\ell,L) = \ell I(\lambda)L - 2\ell\sqrt{\frac{I(\lambda)L}{\beta}} - 4\sqrt{I(\lambda)L + 2\sqrt{\frac{I(\lambda)L}{\beta}}}\sqrt{2\log\left(\frac{320}{\alpha}\right)}\ , \tag{3.46}$$

with $I(\lambda) = \int_0^1 \lambda(t)dt$ as above, this entails

$$P_\lambda\left(\phi_{3,\alpha}^{u(1)-}(N) = 0\right) \leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \geq Q'(\alpha,\beta,\ell^*\wedge(1/2),L)\right) \\ + P_\lambda\left(N_1 \notin \left[I(\lambda)L - 2\sqrt{I(\lambda)L/\beta}; I(\lambda)L + 2\sqrt{I(\lambda)L/\beta}\right]\right)\ .$$

The Bienayme-Chebyshev inequality therefore leads to

$$P_\lambda\left(\phi_{3,\alpha}^{u(1)-}(N) = 0\right) \leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \geq Q'(\alpha,\beta,\ell^*\wedge(1/2),L)\right) + \frac{\beta}{2}\ , \tag{3.47}$$

with

$$Q'(\alpha,\beta,\ell^*\wedge(1/2),L) \geq \left(\ell^*\wedge\frac{1}{2}\right)(\lambda_0+\delta^*\ell^*)L - 2\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{\frac{RL}{\beta}} - 4\sqrt{RL + 2\sqrt{\frac{RL}{\beta}}}\sqrt{2\log\left(\frac{320}{\alpha}\right)}\ . \tag{3.48}$$

since $I(\lambda) = \lambda_0 + \delta^*\ell^*$ belongs to $(0,R]$.
Let us furthermore assume that

$$d_2\left(\lambda, \mathcal{S}_0^u[R]\right) \geq \sqrt{\frac{\ell^*}{(1-\ell^*)L}}\left(2\sqrt{\frac{R}{\beta}} + \sqrt{\frac{2R}{\beta\left(\ell^*\wedge\frac{1}{2}\right)}} + \frac{4}{\ell^*\wedge\frac{1}{2}}\sqrt{R + 2\sqrt{\frac{R}{\beta L}}}\sqrt{2\log\left(\frac{320}{\alpha}\right)}\right)\ . \tag{3.49}$$

Then

$$|\delta^*|(1-\ell^*)\left(\ell^*\wedge\frac{1}{2}\right)L \geq \sqrt{\frac{2R(\ell^*\wedge(1/2))L}{\beta}} + 2\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{\frac{RL}{\beta}} + 4\sqrt{RL + 2\sqrt{\frac{RL}{\beta}}}\sqrt{2\log\left(\frac{320}{\alpha}\right)}\ .$$

With (3.48) and the bound $R \geq \lambda_0 + \delta^*$, this yields

$$|\delta^*|(1-\ell^*)\left(\ell^*\wedge\frac{1}{2}\right)L \geq \sqrt{\frac{2(\lambda_0+\delta^*)(\ell^*\wedge(1/2))L}{\beta}} + \left(\ell^*\wedge\frac{1}{2}\right)(\lambda_0+\delta^*\ell^*)L - Q'(\alpha,\beta,\ell^*\wedge(1/2),L)\ .$$

From $E_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]] = \mathrm{Var}_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]] = (\lambda_0+\delta^*)(\ell^*\wedge(1/2))L$, we then deduce that

$$Q'(\alpha,\beta,\ell^*\wedge(1/2),L) \geq E_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]] + \sqrt{\frac{2\mathrm{Var}_\lambda[N(\tau,\tau+\ell^*\wedge(1/2)]]}{\beta}}\ .$$

Inserting this inequality in (3.47) and using the Bienayme-Chebyshev inequality again, we finally obtain

$$P_\lambda\left(\phi_{3,\alpha}^{u(1)-}(N) = 0\right) \leq P_\lambda\left(N(\tau,\tau+\ell^*\wedge(1/2)] \geq Q'(\alpha,\beta,\ell^*\wedge(1/2),L)\right) + \frac{\beta}{2} \leq \beta\ .$$

This concludes the proof for the test $\phi_{3,\alpha}^{u(1)-}$.

Let us finally turn to the test $\phi_{3/4,\alpha}^{u(2)}$.

Let $L \geq 1$ and let us consider $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau, \tau+\ell^*]}$ in $\mathcal{S}^u_{\delta^*,\cdot,\ell^*}[R]$ satisfying

$$d_2(\lambda, \mathcal{S}_0^u[R]) \geq 2\max\left(\frac{1}{\sqrt{L}}\left(\sqrt{10CR\log\left(\frac{2.77}{u_\alpha}\right)} + 2\sqrt{\frac{R}{\sqrt{\beta}}}\right) + \frac{1}{L^{3/4}}\sqrt{10C\sqrt{\frac{2R}{\beta}}\log\left(\frac{2.77}{u_\alpha}\right)}\right.$$
$$\left. + \frac{1}{L}\sqrt{6C\max\left(\frac{\ell^*}{1-\ell^*}, \frac{1-\ell^*}{\ell^*}\right)\log\left(\frac{2.77}{u_\alpha}\right)}, 8\sqrt{\frac{2R}{\beta L}}\right), \quad (3.50)$$

C being the constant defined in Lemma 3.20.
In order to prove $P_\lambda(\phi^{u(2)}_{3/4,\alpha}(N) = 0) \leq \beta$, noticing first that

$$P_\lambda(\phi^{u(2)}_{3/4,\alpha}(N) = 0) \leq \inf_{k \in \{0,\ldots,\lceil (1-\ell^*)M \rceil - 1\}} P_\lambda\left(T'_{\frac{k}{M},\frac{k}{M}+\ell^*}(N) \leq t'_{N_1,\frac{k}{M},\frac{k}{M}+\ell^*}(1-u_\alpha)\right),$$

we only need to exhibit some k_τ in $\{0,\ldots,\lceil (1-\ell^*)M \rceil - 1\}$ such that

$$P_\lambda\left(T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) \leq t'_{N_1\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha)\right) \leq \beta \ .$$

Let

$$C'(u_\alpha,\beta,R,\ell^*,L) = C\left(5R\frac{\log\left(\frac{2.77}{u_\alpha}\right)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log\left(\frac{2.77}{u_\alpha}\right)}{L^{3/2}} + 3\max\left(\frac{1-\ell^*}{\ell^*}, \frac{\ell^*}{1-\ell^*}\right)\frac{\log^2\left(\frac{2.77}{u_\alpha}\right)}{L^2}\right).$$

Since Lemma 3.20 with the Bienayme-Chebyshev inequality together entail

$$P_\lambda\left(t'_{N_1\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(1-u_\alpha) \geq C'(u_\alpha,\beta,R,\ell^*,L)\right) \leq \frac{\beta}{2},$$

k_τ only needs to satisfy

$$P_\lambda\left(T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N) \leq C'(u_\alpha,\beta,R,\ell^*,L)\right) \leq \frac{\beta}{2} \ . \quad (3.51)$$

Using now the simple facts that $(a+b)^2 \geq a^2 + b^2$ and $a + b \leq 2\max(a,b)$ for all $a,b \geq 0$, (3.50) entails

$$d_2^2(\lambda, \mathcal{S}_0^u[R]) \geq 4C\left(5R\frac{\log\left(\frac{2.77}{u_\alpha}\right)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log\left(\frac{2.77}{u_\alpha}\right)}{L^{3/2}} + 3\max\left(\frac{\ell^*}{1-\ell^*}, \frac{1-\ell^*}{\ell^*}\right)\frac{\log^2\left(\frac{2.77}{u_\alpha}\right)}{L^2}\right) + \frac{8R}{L\sqrt{\beta}}$$
$$+ \frac{8\sqrt{2R}}{\sqrt{L\beta}}|\delta^*|\sqrt{\ell^*(1-\ell^*)} \ .$$

Further using $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for all $a,b \geq 0$, by definition of $C'(u_\alpha,\beta,R,\ell^*,L)$, this yields

$$\frac{\delta^{*2}\ell^*(1-\ell^*)}{4} \geq C'(u_\alpha,\beta,R,\ell^*,L) + \sqrt{\frac{4R^2}{L^2\beta} + \frac{8R\delta^{*2}\ell^*(1-\ell^*)}{L\beta}} \ . \quad (3.52)$$

Let us set $k_\tau = \lfloor \tau M \rfloor$. Since $0 < \tau < 1 - \ell^*$, k_τ actually belongs to $\{0,\ldots,\lceil (1-\ell^*)M \rceil - 1\}$, and since $M = \lceil 2/(\ell^*(1-\ell^*)) \rceil$, $k_\tau/M \leq \tau < k_\tau/M + (\ell^*(1-\ell^*)/2$. Therefore, since we get in particular $k_\tau/M \leq \tau < k_\tau/M + \ell^*$, using Lemma 3.19 (equations (3.165) and (3.166)) with $x = \lambda_0 k_\tau/M$, $y = \lambda_0\ell^* + \delta(k_\tau/M + \ell^* - \tau)$ and $z = \lambda_0(1 - \ell^* - k_\tau/M) + \delta(\tau - k_\tau/M)$, we get on the one hand

$$E_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right] = \delta^{*2}\frac{(\ell^*(1-\ell^*) + k_\tau/M - \tau)^2}{\ell^*(1-\ell^*)} \geq \frac{\delta^{*2}\ell^*(1-\ell^*)}{4},$$

and on the other hand

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]=\frac{2}{L^2}\left(\lambda_0+\delta^*\left(1-\ell^*+\left(\tau-\frac{k_\tau}{M}\right)\frac{2\ell^*-1}{\ell^*(1-\ell^*)}\right)\right)^2$$
$$+\frac{4}{L}\delta^{*2}\frac{1-\ell^*}{\ell^*}\left(\ell^*-\frac{\tau-k_\tau/M}{1-\ell^*}\right)^2\left(\lambda_0+\delta^*\left(1-\ell^*+\left(\tau-\frac{k_\tau}{M}\right)\frac{2\ell^*-1}{\ell^*(1-\ell^*)}\right)\right)\ .$$

Using again the fact that $\tau-k_\tau/M<(\ell^*(1-\ell^*)/2$, we obtain

$$0\leq 1-\ell^*+\left(\tau-\frac{k_\tau}{M}\right)\frac{2\ell^*-1}{\ell^*(1-\ell^*)}\leq 1$$

for all ℓ^* in $(0,1)$, leading to

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]\leq\left(\frac{2}{L^2}(\lambda_0+\delta^*)^2+\frac{4}{L}(\lambda_0+\delta^*)\delta^{*2}\ell^*(1-\ell^*)\right)\mathbb{1}_{\delta^*>0}+\left(\frac{2}{L^2}\lambda_0^2+\frac{4}{L}\lambda_0\delta^{*2}\ell^*(1-\ell^*)\right)\mathbb{1}_{\delta^*<0}\ ,$$

whereby

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]\leq\frac{2R^2}{L^2}+\frac{4R}{L}\delta^{*2}\ell^*(1-\ell^*)\ .$$

Combined with these computations, (3.52) leads to

$$E_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]\geq C'(u_\alpha,\beta,R,\ell^*,L)+\sqrt{2\mathrm{Var}_\lambda\left[T'_{\frac{k_\tau}{M},\frac{k_\tau}{M}+\ell^*}(N)\right]/\beta}\ .$$

The Bienayme-Chebyshev inequality finally allows to obtain (3.51), which ends the proof.

Proof of Proposition 3.7

As for the other Bonferroni type aggregated tests, the control of the first kind error rates of the two tests $\phi^{u(1)}_{4,\alpha}$ and $\phi^{u(2)}_{3/4,\alpha}$ is straightforward using simple union bounds.

(i) *Control of the second kind error rate of* $\phi^{u(1)}_{4,\alpha}$.
Let us first set λ in $\mathcal{S}^u_{\cdot,\cdot,\ell^*}[R]$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell^*]}$ with τ in $(0,1-\ell^*)$, λ_0 in $(0,R]$, δ in $(0,R-\lambda_0]$ or δ in $(-\lambda_0,0)$, and

$$d_2(\lambda,\mathcal{S}^u_0[R])\geq\sqrt{\frac{\ell^*}{(1-\ell^*)L}}\left(2\sqrt{\frac{R}{\beta}}+\sqrt{\frac{2R}{\beta\left(\ell^*\wedge\frac{1}{2}\right)}}+\frac{4}{\ell^*\wedge\frac{1}{2}}\sqrt{2\left(R+2\sqrt{\frac{R}{\beta L}}\right)\log\left(\frac{640}{\alpha}\right)}+\frac{32\log(640/\alpha)}{3\left(\ell^*\wedge\frac{1}{2}\right)\sqrt{L}}\right)\ .$$

This condition ensures that (3.45) and (3.49) both hold, but with α replaced by $\alpha/2$. Then, it suffices to notice that if $\delta\in(0,R-\lambda_0]$,

$$P_\lambda\left(\phi^{u(1)}_{4,\alpha}(N)=0\right)\leq P_\lambda\left(\phi^{(1)+}_{3,\alpha/2}(N)=0\right)\ ,$$

and if δ in $(-\lambda_0,0)$,

$$P_\lambda\left(\phi^{u(1)}_{4,\alpha}(N)=0\right)\leq P_\lambda\left(\phi^{(1)-}_{3,\alpha/2}(N)=0\right)\ .$$

Since (3.45) and (3.49) hold with $\alpha/2$ instead of α, by using exactly the same arguments as in the proof of Proposition 3.5, we obtain $P_\lambda\left(\phi^{(1)+}_{3,\alpha/2}(N)=0\right)\leq\beta$ when $\delta\in(0,R-\lambda_0]$ on the one hand, and $P_\lambda\left(\phi^{(1)-}_{3,\alpha/2}(N)=0\right)\leq\beta$ when δ in $(-\lambda_0,0)$ on the other hand.

In any case, whatever the value of δ in $(-\lambda_0, R-\lambda_0] \setminus \{0\}$, one has

$$P_\lambda\left(\phi^{u(1)}_{4,\alpha}(N)=0\right) \leq \beta \ .$$

(ii) Control of the second kind error rate of $\phi^{u(2)}_{3/4,\alpha}$.
Let us set now λ in $\mathcal{S}^u_{\cdot,\cdot\cdot,\ell^*}[R]$ such that $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,\tau+\ell^*]}$ with τ in $(0,1-\ell^*)$, λ_0 in $(0,R]$, δ in $(-\lambda_0, R-\lambda_0] \setminus \{0\}$ as above, but with

$$d_2(\lambda, \mathcal{S}^u_0[R]) \geq 2\max\left(\frac{1}{\sqrt{L}}\left(\sqrt{10CR\log\left(\frac{2.77}{u_\alpha}\right)}+2\sqrt{\frac{R}{\sqrt{\beta}}}\right)+\frac{1}{L^{3/4}}\sqrt{10C\sqrt{\frac{2R}{\beta}}\log\left(\frac{2.77}{u_\alpha}\right)}\right.$$
$$\left.+\frac{1}{L}\sqrt{6C\max\left(\frac{\ell^*}{1-\ell^*},\frac{1-\ell^*}{\ell^*}\right)\log\left(\frac{2.77}{u_\alpha}\right)}\ ,\ 8\sqrt{\frac{2R}{\beta L}}\right)\ ,$$

so that (3.50) holds.
Following the same arguments as in the proof of Proposition 3.5 (with the only change of δ instead of δ^*), we prove that

$$P_\lambda\left(\phi^{u(2)}_{3/4,\alpha}(N)=0\right) \leq \beta \ .$$

This ends the proof, just taking for instance $C(\alpha,\beta,R,\ell^*)$ as the maximum between

$$\sqrt{\frac{\ell^*}{(1-\ell^*)}}\left(2\sqrt{\frac{R}{\beta}}+\sqrt{\frac{2R}{\beta\left(\ell^*\wedge\frac{1}{2}\right)}}+\frac{4}{\ell^*\wedge\frac{1}{2}}\sqrt{2\left(R+2\sqrt{\frac{R}{\beta}}\right)\log\left(\frac{640}{\alpha}\right)}+\frac{32\log(640/\alpha)}{3\left(\ell^*\wedge\frac{1}{2}\right)}\right)\ ,$$

and

$$2\max\left(\left(\sqrt{10CR\log\left(\frac{2.77}{u_\alpha}\right)}+2\sqrt{\frac{R}{\sqrt{\beta}}}\right)+\sqrt{10C\sqrt{\frac{2R}{\beta}}\log\left(\frac{2.77}{u_\alpha}\right)}+\sqrt{6C\max\left(\frac{\ell^*}{1-\ell^*},\frac{1-\ell^*}{\ell^*}\right)\log\left(\frac{2.77}{u_\alpha}\right)}\ ,\ 8\sqrt{\frac{2R}{\beta}}\right)\ .$$

Proof of Proposition 3.8

Assume that

$$L > \frac{((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{2\delta^{*2}\tau^*(1-\tau^*)}\ , \tag{3.53}$$

and set

$$r = \sqrt{\frac{((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{2L}}\ . \tag{3.54}$$

The assumption (3.53) ensures

$$r^2 < \delta^{*2}\tau^*(1-\tau^*) \leq \delta^{*2}/4\ , \tag{3.55}$$

which enables us to define λ_r for t in $(0,1)$ by

$$\lambda_r(t) = \lambda_0 + \delta^*\mathbb{1}_{(\tau^*,\tau^*+\ell_r]}(t) \text{ with } \lambda_0 = ((R-\delta^*)\wedge R) \text{ and } \ell_r = \frac{1}{2}\left(1-\frac{\sqrt{\delta^{*2}-4r^2}}{|\delta^*|}\right)\ .$$

First, ℓ_r belongs to $(0,1-\tau^*)$ for all τ^* in $(0,1)$. Indeed, if $\tau^* \leq 1/2$ the result is straightforward by definition of ℓ_r and if $\tau^* > 1/2$ the result follows from (3.55).

Moreover, the definition of ℓ_r implies $\delta^{*2}\ell_r(1-\ell_r) = r^2$ and ensures that λ_r belongs to $(\mathcal{S}^u_{\delta^*,\tau^*,\ldots}[R])_r$. Furthermore, we get from (3.53) that $L > (2\lambda_0 \log C_{\alpha,\beta})/\delta^{*2}$ and then $1 - \lambda_0 \log C_{\alpha,\beta}/(L\delta^{*2}) > 1/2$, which leads to

$$r^2 < \frac{\lambda_0 \log C_{\alpha,\beta}}{L}\left(1 - \frac{\lambda_0 \log C_{\alpha,\beta}}{L\delta^{*2}}\right) = \frac{\delta^{*2}}{4}\left(1 - \left(1 - \frac{2\lambda_0 \log C_{\alpha,\beta}}{L\delta^{*2}}\right)^2\right) ,$$

hence $\delta^{*2}\ell_r < \lambda_0 \log C_{\alpha,\beta}/L$. We then obtain from Lemma 2.1 and Lemma 2.24

$$E_{\lambda_0}\left[\left(\frac{dP_{\lambda_r}}{dP_{\lambda_0}}\right)^2(N)\right] = \exp\left(\frac{L\delta^{*2}\ell_r}{\lambda_0}\right) < C_{\alpha,\beta} \ .$$

Lemmas 1.7 and 1.8 then entail $\rho\left((\mathcal{S}^u_{\delta^*,\tau^*,\ldots}[R])_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\delta^*,\tau^*,\ldots}[R]) \geq r$.

Proof of Proposition 3.9

The control of the first kind error rate is straightforward, and even more strong by using the same conditioning trick as in the proof of Proposition 3.4: in fact, for every λ_0 in $\mathcal{S}^u_0[R]$ and n in $\mathbb{N}$, $E_{\lambda_0}\left[\phi^u_{5,\alpha}(N)\middle|N_1 = n\right] = P_{\lambda_0}\left(\phi^u_{5,\alpha}(N) = 1\middle|N_1 = n\right) \leq \alpha$, so

$$\forall \lambda_0 \in \mathcal{S}^u_0[R],\ P_{\lambda_0}\left(\phi^u_{5,\alpha}(N) = 1\right) = E_{\lambda_0}\left[P_{\lambda_0}\left(\phi^u_{5,\alpha}(N)\middle|N_1\right)\right] \leq \alpha \ .$$

Let us turn to the control of the second kind error rate of $\phi^u_{5,\alpha}$.
Set λ in $\mathcal{S}^u_{\delta^*,\tau^*,\ldots}[R]$ such that $\lambda = \lambda_0 + \delta^*\mathbb{1}_{(\tau^*,\tau^*+\ell]}$ with λ_0 in $(-\delta^* \vee 0, (R-\delta^*) \wedge R]$ and ℓ in $(0, 1-\tau^*)$, and satisfying

$$d_2(\lambda, \mathcal{S}_0[R]) \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{|\delta^*|Q(2R,\delta^*,\alpha)},\ 2\sqrt{\frac{2R}{\beta}},\ \frac{|\delta^*|}{2\sqrt{2\beta R}}\right) , \tag{3.56}$$

where $Q(2R,\delta^*,\alpha)$ is the quantile upper bound defined by Lemma 3.22, and which does not depend on L. The condition (3.56) ensures that $|\delta^*|\sqrt{\ell(1-\ell)} \geq |\delta^*|/\sqrt{2\beta RL}$, that is, using the fact that $\ell(1-\ell) \leq 1/4$,

$$L \geq \frac{2}{\beta R} \ . \tag{3.57}$$

Setting $I(\lambda) = \int_0^1 \lambda(t)dt$ as in the proof of Proposition 3.5, with $I(\lambda) \leq R$ (and therefore obviously $2R - I(\lambda) \geq R$), since (3.57) also entails $L \geq 2R/(\beta R^2)$ we obtain

$$L \geq \frac{2I(\lambda)}{(2R - I(\lambda))^2\beta} \ . \tag{3.58}$$

Notice now that (3.58) and the Bienayme-Chebyshev inequality yield

$$\begin{aligned} P_\lambda(N_1 > 2RL) &= P_\lambda\big(N_1 - I(\lambda)L > L(2R - I(\lambda))\big) \\ &\leq P_\lambda\left(N_1 - I(\lambda)L > \sqrt{\frac{2I(\lambda)L}{\beta}}\right) \leq \frac{\beta}{2} \ . \end{aligned}$$

This leads to

$$P_\lambda\left(\phi^u_{5,\alpha}(N) = 0\right) \leq P_\lambda\left(\sup_{\ell' \in (0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell'}(N) \leq s'^{+}_{N_1,\delta^*,\tau^*,L}(1-\alpha),\ N_1 \leq 2RL\right) + \frac{\beta}{2} ,$$

and Lemma 3.22 allows to write that

$$P_\lambda\left(\phi^u_{5,\alpha}(N)=0\right)\le P_\lambda\left(\sup_{\ell'\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell'}(N)\le Q(2R,\delta^*,\alpha)\right)+\frac{\beta}{2}$$
$$\le P_\lambda\left(S'_{\delta^*,\tau^*,\tau^*+\ell}(N)\le Q(2R,\delta^*,\alpha)\right)+\frac{\beta}{2}\ .$$

Moreover, the assumption (3.56) implies

$$|\delta^*|\sqrt{\ell(1-\ell)}\ge\frac{2}{\sqrt{L}}\max\left(\sqrt{|\delta^*|Q(2R,\delta^*,\alpha)},\ 2\sqrt{\frac{2R}{\beta}}\right)\ ,$$

which entails

$$\frac{|\delta^*|}{2}\ell(1-\ell)L\ge 2\max\left(Q(2R,\delta^*,\alpha),\ \sqrt{\frac{2R\ell(1-\ell)L}{\beta}}\right)\ ,$$

hence

$$\frac{|\delta^*|}{2}\ell(1-\ell)L\ge 2\max\left(Q(2R,\delta^*,\alpha),\ \sqrt{\frac{2(\lambda_0+\delta^*(1-\ell))\ell(1-\ell)L}{\beta}}\right)\ .$$

Noticing that $E_\lambda[S'_{\delta^*,\tau^*,\tau^*+\ell}(N)]=|\delta^*|\ell(1-\ell)L/2$ and $\mathrm{Var}_\lambda(S'_{\delta^*,\tau^*,\tau^*+\ell}(N))=(\lambda_0+\delta^*(1-\ell))\ell(1-\ell)L$, we get

$$E_\lambda[S'_{\delta^*,\tau^*,\tau^*+\ell}(N)]\ge Q(2R,\delta^*,\alpha)+\sqrt{\frac{2\mathrm{Var}(S'_{\delta^*,\tau^*,\tau^*+\ell}(N))}{\beta}}\ . \tag{3.59}$$

Therefore,

$$P_\lambda\left(\phi^u_{5,\alpha}(N)=0\right)\le P_\lambda\left(S'_{\delta^*,\tau^*,\tau^*+\ell}(N)\le Q(2R,\delta^*,\alpha)\right)+\frac{\beta}{2}$$
$$\le P_\lambda\left(S'_{\delta^*,\tau^*,\tau^*+\ell}(N)-E_\lambda[S'_{\delta^*,\tau^*,\tau^*+\ell}(N)]\le-\sqrt{\frac{2\mathrm{Var}(S'_{\delta^*,\tau^*,\tau^*+\ell}(N))}{\beta}}\right)+\frac{\beta}{2}$$
$$\le\beta\ ,$$

with a last line simply following from the Bienayme-Chebyshev inequality.

Proof of Proposition 3.10

Assume that $L\ge 3$ and $\alpha+\beta<1/2$, and set $\lambda_0=R/2$. As in the proof of Proposition 2.12, we consider $C'_{\alpha,\beta}=4(1-\alpha-\beta)^2$, $K_{\alpha,\beta,L}=\lceil(\log_2 L)/C'_{\alpha,\beta}\rceil$, and for k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $\lambda_k=\lambda_0+\delta_k\mathbb{1}_{(\tau^*,\tau^*+\ell_k]}$ with $\ell_k=(1-\tau^*)/2^k$ and $\delta_k=(\lambda_0\log\log L/(\ell_k L))^{1/2}$. Then, for all k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, noticing that $\ell_k<1-\tau^*$, we get $d_2(\lambda_k,\mathcal{S}_0[R])>\sqrt{\lambda_0\tau^*\log\log L/L}$. Furthermore, assuming that

$$\frac{\log\log L}{L^{1+1/C'_{\alpha,\beta}}}\le\frac{R(1-\tau^*)}{4}\ , \tag{3.60}$$

one obtains that λ_k belongs to $\mathcal{S}^u_{\cdot,\tau^*,\ldots}[R]$.
Recall also that for all k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, P_{λ_k} denotes the distribution of a Poisson process with intensity λ_k with respect to the measure Λ, and consider κ, a random variable with uniform distribution on $\{1,\ldots,K_{\alpha,\beta,L}\}$, which allows to define the probability distribution μ of λ_κ. From Lemma 1.8, we know that it is enough to prove $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2]\le 1+C'_{\alpha,\beta}$ to conclude that $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\cdot,\tau^*,\ldots}[R])\ge\sqrt{R\tau^*\log\log L/(2L)}$.

The same calculation as in the proof of Proposition 2.12 (see (2.73)) gives for η such that $0 < \eta < 1 - 1/\sqrt{2}$,

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] \leq C'_{\alpha,\beta}\log 2 + \frac{2C'_{\alpha,\beta}\log 2}{\log L}(\log L)^{\eta+\frac{1}{\sqrt{2}}} + \exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) .$$

If we assume now that

$$\exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) + \frac{2C'_{\alpha,\beta}\log 2}{(\log L)^{1-\eta-\frac{1}{\sqrt{2}}}} \leq 1 + (1-\log 2)C'_{\alpha,\beta} \ , \tag{3.61}$$

we finally obtain the expected result

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] \leq 1 + C'_{\alpha,\beta} \ .$$

To end the proof, it remains to notice that there exists $L_0(\alpha,\beta,R,\tau^*) \geq 3$ such that for all $L \geq L_0(\alpha,\beta,R,\tau^*)$, both assumptions (3.60) and (3.61) hold.

Proof of Proposition 3.11

As for all our Bonferroni type aggregated tests, the control of the first kind error rates of the two tests $\phi^{u(1)}_{6,\alpha}$ and $\phi^{u(2)}_{6,\alpha}$ is straightforward using simple union bounds and the conditioning trick of the above proofs for upper bounds.

(i) Control of the second kind error rate of $\phi^{u(1)}_{6,\alpha}$.
Let λ in $\mathcal{S}^u_{\cdot,\tau^*,\ldots}[R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}$ with λ_0 in $(0,R]$, δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and ℓ in $(0,1-\tau^*)$ and assume that

$$d_2(\lambda,\mathcal{S}^u_0[R]) \geq 2\max\left(\sqrt{\frac{R}{3}}\sqrt{\frac{1+\tau^*}{\tau^*}}\sqrt{\frac{\log(2/u_\alpha)}{L}}\ ,\ \frac{1+\tau^*}{\tau^*}\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}}+\sqrt{\frac{2R}{\beta L}}\right)\ ,\ \frac{\sqrt{1-\tau^*}R}{\sqrt{2L}}\right) . \tag{3.62}$$

Let us prove the inequality $P_\lambda(\phi^{u(1)}_{6,\alpha}(N)=0) \leq \beta$. Applying Lemma 2.24, we get for all k in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$ and $\ell_{\tau^*,k} = (1-\tau^*)2^{-k}$

$$E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k}N_1] = \delta(\ell_{\tau^*,k}\wedge\ell)(1-\ell_{\tau^*,k}\vee\ell)L \ , \tag{3.63}$$

and

$$\mathrm{Var}_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k}N_1] = \begin{cases}(\lambda_0(1-\ell_{\tau^*,k}) + \delta(1-2\ell_{\tau^*,k}+\ell_{\tau^*,k}\ell))\ell_{\tau^*,k}L & \text{if } \ell_{\tau^*,k}\leq\ell \ ,\\ (\lambda_0\ell_{\tau^*,k}+\delta\ell(1-\ell_{\tau^*,k}))(1-\ell_{\tau^*,k})L & \text{if } \ell_{\tau^*,k}\geq\ell \ .\end{cases} \tag{3.64}$$

Assume first that δ belongs to $(0, R-\lambda_0]$. Noticing that

$$P_\lambda(\phi^{u(1)}_{6,\alpha}(N)=0) \leq \inf_{k\in\{1,\ldots,\lfloor\log_2 L\rfloor\}} P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k}N_1 \leq \bar{b}_{N_1,\ell_{\tau^*,k}}(1-u_\alpha/2)\right) ,$$

one can see it is enough to exhibit some k in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k}N_1 \leq \bar{b}_{N_1,\ell_{\tau^*,k}}(1-u_\alpha/2)\right) \leq \beta \ .$$

Under the condition (3.62), we have $d_2^2(\lambda, \mathcal{S}_0^u[R]) \geq 2R^2(1-\tau^*)/L$ which ensures the inequality $\ell(1-\ell) > 2(1-\tau^*)/L > (1-\tau^*)2^{-\lfloor \log_2 L \rfloor}$ and then

$$1-\tau^* > \ell > \frac{1-\tau^*}{2^{\lfloor \log_2 L \rfloor}} \ . \tag{3.65}$$

From (3.65), we deduce the existence of k_ℓ in $\{1, \ldots, \lfloor \log_2 L \rfloor\}$ satisfying $(1-\tau^*)2^{-k_\ell} \leq \ell < (1-\tau^*)2^{-k_\ell+1}$, that is

$$\ell_{\tau^*,k_\ell} \leq \ell < \ell_{\tau^*,k_\ell-1} \ . \tag{3.66}$$

Then

$$\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}} \leq \frac{\ell}{1-\ell} < \frac{\ell_{\tau^*,k_\ell-1}}{1-\ell_{\tau^*,k_\ell-1}} \ ,$$

and

$$\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}} = \underbrace{\frac{\ell_{\tau^*,k_\ell-1}}{1-\ell_{\tau^*,k_\ell-1}}}_{>\frac{\ell}{1-\ell}} \frac{1-\ell_{\tau^*,k_\ell-1}}{1-\ell_{\tau^*,k_\ell}} \underbrace{\frac{\ell_{\tau^*,k_\ell}}{\ell_{\tau^*,k_\ell-1}}}_{=\frac{1}{2}} > \frac{\ell}{2(1-\ell)} \frac{1-\ell_{\tau^*,k_\ell-1}}{1-\ell_{\tau^*,k_\ell}} \ .$$

But

$$\frac{1-\ell_{\tau^*,k_\ell-1}}{1-\ell_{\tau^*,k_\ell}} = 1 - \frac{1-\tau^*}{2^{k_\ell}-(1-\tau^*)} \geq 2\frac{\tau^*}{1+\tau^*} \ ,$$

so we finally obtain

$$\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}} > \frac{\tau^*}{1+\tau^*} \frac{\ell}{1-\ell} \ . \tag{3.67}$$

The condition (3.62) then gives on the one hand

$$\delta\sqrt{\ell(1-\ell)} \geq \frac{2}{\sqrt{3}}\sqrt{R}\sqrt{\frac{1+\tau^*}{\tau^*}}\sqrt{\frac{\log(2/u_\alpha)}{L}} \ ,$$

and using the fact that $\delta < R$

$$\delta\ell(1-\ell)\frac{\tau^*}{1+\tau^*} \geq \frac{4\log(2/u_\alpha)}{3L} \ ,$$

which entails with (3.66) and (3.67)

$$\delta\ell_{\tau^*,k_\ell}(1-\ell) \geq \frac{4\log(2/u_\alpha)}{3L} \ . \tag{3.68}$$

On the other hand, (3.62) yields

$$\delta\sqrt{\ell(1-\ell)} \geq 2\frac{1+\tau^*}{\tau^*}\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}} + \sqrt{\frac{2R}{\beta L}}\right) \ ,$$

which entails with (3.66) and (3.67)

$$\delta(1-\ell)\sqrt{\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}}} \geq 2\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}} + \sqrt{\frac{2R}{\beta L}}\right) \ ,$$

whereby

$$\delta\ell_{\tau^*,k_\ell}(1-\ell) \geq 2\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}} + \sqrt{\frac{2R}{\beta L}}\right) \ . \tag{3.69}$$

Thus, with (3.68) and (3.69), the condition (3.62) leads to

$$\delta\ell_{\tau^*,k_\ell}(1-\ell) \geq \max\left(\frac{4\log(2/u_\alpha)}{3L}\ ,\ 2\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}}+\sqrt{\frac{2R}{\beta L}}\right)\right)\ .$$

Using the fact that $2\max(a,b) \geq a+b$ for all $a,b \geq 0$, we get

$$\delta\ell_{\tau^*,k_\ell}(1-\ell)L \geq \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)+\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\left(\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}}+\sqrt{\frac{2RL}{\beta}}\right)\ , \tag{3.70}$$

where we recall that $I(\lambda)$ stands for $\int_0^1 \lambda(t)dt$.
Moreover, with (3.66), the equations (3.63) and (3.64) lead to $E_\lambda\left[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1\right] = \delta\ell_{\tau^*,k_\ell}(1-\ell)L$ and $\mathrm{Var}_\lambda\left[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1\right] = (\lambda_0(1-\ell_{\tau^*,k})+\delta(1-2\ell_{\tau^*,k}+\ell_{\tau^*,k}\ell))\ell_{\tau^*,k}L \leq \ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})RL$. So (3.70) entails

$$E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1] \geq \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)+\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}}$$
$$+\sqrt{2\mathrm{Var}_\lambda\left[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1\right]/\beta}\ . \tag{3.71}$$

The total probability formula then ensures that

$$\begin{aligned}
&P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1 \leq \bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(1-u_\alpha/2)\right)\\
\leq &P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1 \leq E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1]\right.\\
&\left.-\sqrt{2\mathrm{Var}_\lambda\left[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1\right]/\beta}\right)+P_\lambda\left(\bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(1-u_\alpha/2) > \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)\right.\\
&\left.+\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}}\right)\ .
\end{aligned}$$

From the Bienayme-Chebyshev inequality, we deduce on the one hand that the first right hand side term is upper bounded by $\beta/2$, and on the other hand that

$$P_\lambda\left(N_1 \geq LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}\right) \leq \frac{\beta}{2}\ .$$

This last inequality, combined with Lemma 3.23, which follows from a simple application of Bennett's inequality, leads to

$$P_\lambda\left(\bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(1-u_\alpha/2) > \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)+\sqrt{\ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})}\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}}\right) \leq \frac{\beta}{2}\ .$$

We therefore conclude that

$$P_\lambda\left(N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1 \leq \bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(1-u_\alpha/2)\right) \leq \beta\ ,$$

so, as expected, $P_\lambda(\phi^{u(1)}_{6,\alpha}(N)=0) \le \beta$.
Assume now that δ belongs to $(-\lambda_0, 0)$ and notice that

$$P_\lambda(\phi^{u(1)}_{6,\alpha}(N)=0) \le \inf_{k\in\{1,\ldots,\lfloor \log_2 L\rfloor\}} P_\lambda\left(N(\tau^*, \tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k} N_1 \ge \bar{b}_{N_1,\ell_{\tau^*,k}}(u_\alpha/2)\right) \ .$$

One can see it is enough to exhibit some k in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(N(\tau^*, \tau^*+\ell_{\tau^*,k}] - \ell_{\tau^*,k} N_1 \ge \bar{b}_{N_1,\ell_{\tau^*,k}}(u_\alpha/2)\right) \le \beta \ .$$

The same choice of k_ℓ as above leads, with (3.63) and (3.64), to $E_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1] = -|\delta|\ell_{\tau^*,k_\ell}(1-\ell)L$ and $\mathrm{Var}_\lambda[N(\tau^*,\tau^*+\ell_{\tau^*,k_\ell}]-\ell_{\tau^*,k_\ell}N_1] \le \ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})\lambda_0 L \le \ell_{\tau^*,k_\ell}(1-\ell_{\tau^*,k_\ell})RL$. With very similar arguments and calculations (mainly replacing δ by $|\delta|$ and using the lower bound for $\bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(u_\alpha/2)$ of Lemma 3.23) as above, we also conclude that condition (3.62) implies

$$P_\lambda\left(N(\tau^*, \tau^*+\ell_{\tau^*,k_\ell}] - \ell_{\tau^*,k_\ell} N_1 \ge \bar{b}_{N_1,\ell_{\tau^*,k_\ell}}(u_\alpha/2)\right) \le \beta \ ,$$

and $P_\lambda(\phi^{u(1)}_{6,\alpha}(N)=0) \le \beta$.
Since $\log(2/u_\alpha) = \log(2\lfloor \log_2 L\rfloor/\alpha)$ and $L \ge 3$, there exists $C(\alpha,\beta,R,\tau^*) > 0$ such that

$$2\max\left(\sqrt{\frac{R}{3}}\sqrt{\frac{1+\tau^*}{\tau^*}}\sqrt{\frac{\log(2/u_\alpha)}{L}}, \frac{1+\tau^*}{\tau^*}\left(\sqrt{\frac{2\log(2/u_\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}}+\sqrt{\frac{2R}{\beta L}}\right), \frac{\sqrt{1-\tau^*}R}{\sqrt{2L}}\right)$$
$$\le C(\alpha,\beta,R,\tau^*)\sqrt{\frac{\log\log L}{L}} \ ,$$

which allows to conclude the proof.

(ii) Control of the second kind error rate of $\phi^{u(2)}_{6,\alpha}$.

Let λ in $\mathcal{S}^u_{\cdot,\tau^*,\ldots}[R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau^*,\tau^*+\ell]}$ with λ_0 in $(0,R]$, δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and ℓ in $(0,1-\tau^*)$ and assume that

$$d_2(\lambda,\mathcal{S}^u_0[R]) \ge \sqrt{\frac{1+\tau^*}{\tau^*}}\max\left(\sqrt{2C}\left(\sqrt{5R}\sqrt{\frac{\log(2.77/u_\alpha)}{L}}+\sqrt{5}\left(\frac{2R}{\beta}\right)^{1/4}\frac{\sqrt{\log(2.77/u_\alpha)}}{L^{3/4}}+\frac{\sqrt{3}\log(2.77/u_\alpha)}{2\sqrt{2L\log\log L}}\right.\right.$$
$$\left.\left.+\frac{2\sqrt{R}}{\sqrt{L\sqrt{\beta}}}\right), 4\sqrt{\frac{1+\tau^*}{\tau^*}}\sqrt{\frac{2R}{L\beta}}, 4R\sqrt{\frac{\tau^*}{1+\tau^*}}\sqrt{\frac{\log\log L}{L}}\right) , \quad (3.72)$$

where C is the constant defined in Lemma 3.20, and let us prove $P_\lambda(\phi^{u(2)}_{6,\alpha}(N)=0) \le \beta$.
Noticing that

$$P_\lambda(\phi^{u(2)}_{6,\alpha}(N)=0) \le \inf_{k\in\{1,\ldots,\lfloor \log_2 L\rfloor\}} P_\lambda\left(T'_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N) \le t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right) \ ,$$

one only needs to exhibit some k in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$ satisfying

$$P_\lambda\left(T'_{\tau^*,\tau^*+\ell_{\tau^*,k}}(N) \le t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k}}(1-u_\alpha)\right) \le \beta \ ,$$

in order to prove the result.
Under the condition (3.72), we first obtain $d_2^2(\lambda,\mathcal{S}^u_0[R]) \ge 16R^2(\log\log L)/L$ which ensures $\ell(1-\ell) > 16(\log\log L)/L$ as well as

$$\ell > \frac{16\log\log L}{L} \text{ and } 1-\ell > \frac{16\log\log L}{L} \ . \quad (3.73)$$

Moreover, since $16(\log\log L)/L > 2^{-\lfloor \log_2 L\rfloor} > (1-\tau^*)2^{-\lfloor \log_2 L\rfloor}$, we obtain

$$1-\tau^* > \ell > \frac{1-\tau^*}{2^{\lfloor \log_2 L\rfloor}} \ . \tag{3.74}$$

From (3.74), as in the above part (i), we deduce that there exists k_ℓ in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$ satisfying $(1-\tau^*)2^{-k_\ell} \leq \ell < (1-\tau^*)2^{-k_\ell+1}$, that is

$$\ell_{\tau^*,k_\ell} \leq \ell < \ell_{\tau^*,k_\ell-1} \ . \tag{3.75}$$

As above again, this leads to the same inequality as (3.67), i.e.

$$\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}} > \frac{\tau^*}{1+\tau^*}\frac{\ell}{1-\ell} \ . \tag{3.76}$$

Applying Lemma 3.19, we get with (3.75)

$$E_\lambda[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)] = \delta^2(1-\ell)^2\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}} \ , \tag{3.77}$$

and

$$\mathrm{Var}_\lambda\left[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)\right] = \frac{2}{L^2}\left(\lambda_0+\delta\left(1-(1-\ell)\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}}\right)\right)^2 + \frac{4}{L}\delta^2(1-\ell)^2\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}}\left(\lambda_0+\delta\left(1-(1-\ell)\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}}\right)\right) \ . \tag{3.78}$$

Using (3.76),

$$E_\lambda[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)] > \delta^2\ell(1-\ell)\frac{\tau^*}{1+\tau^*} \ , \tag{3.79}$$

and with (3.75) we obtain

$$\mathrm{Var}_\lambda\left[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)\right] \leq \frac{2R^2}{L^2} + \frac{4R}{L}\delta^2\ell(1-\ell) \ . \tag{3.80}$$

On the one hand, using $a^2+b^2 \leq (a+b)^2$ for $a,b \geq 0$, the condition (3.72) ensures that

$$\delta^2\ell(1-\ell) \geq 2\frac{1+\tau^*}{\tau^*}\left(C\left(5R\frac{\log(2.77/u_\alpha)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}} + \frac{3\log^2(2.77/u_\alpha)}{8L\log\log L}\right) + \frac{2R}{L\sqrt{\beta}}\right) \ ,$$

and on the other hand

$$\delta^2\ell(1-\ell) \geq 2\frac{1+\tau^*}{\tau^*}\left(2\sqrt{\frac{2R}{L\beta}}|\delta|\sqrt{\ell(1-\ell)}\right) \ ,$$

hence

$$\delta^2\ell(1-\ell) \geq 2\frac{1+\tau^*}{\tau^*}\max\left(C\left(5R\frac{\log(2.77/u_\alpha)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}} + \frac{3\log^2(2.77/u_\alpha)}{8L\log\log L}\right) + \frac{2R}{L\sqrt{\beta}} \ , \ 2\sqrt{\frac{2R}{L\beta}}|\delta|\sqrt{\ell(1-\ell)}\right) \ .$$

Finally, with the inequality $a+b \leq 2\max(a,b)$, we get

$$\delta^2\ell(1-\ell)\frac{\tau^*}{1+\tau^*} \geq C\left(5R\frac{\log(2.77/u_\alpha)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}} + \frac{3\log^2(2.77/u_\alpha)}{8L\log\log L}\right) + \frac{2R}{L\sqrt{\beta}} + 2\sqrt{\frac{2R}{L\beta}}|\delta|\sqrt{\ell(1-\ell)} \ ,$$

that is, with (3.79), (3.80) and using $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$,

$$E_\lambda[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)] \ge C\left(5R\frac{\log(2.77/u_\alpha)}{L}+5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}}+\frac{3\log^2(2.77/u_\alpha)}{8L\log\log L}\right) + \sqrt{2\mathrm{Var}_\lambda\left[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)\right]/\beta} \ . \quad (3.81)$$

Furthermore, (3.73) and (3.75) lead to

$$\frac{1}{\ell_{\tau^*,k_\ell}} < \frac{L}{8\log\log L} \text{ and } \frac{1}{1-\ell_{\tau^*,k_\ell}} < \frac{L}{16\log\log L} \ ,$$

and therefore

$$\max\left(\frac{1-\ell_{\tau^*,k_\ell}}{\ell_{\tau^*,k_\ell}}, \frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}}\right) < \frac{L}{8\log\log L} \ . \quad (3.82)$$

We set now

$$Q(\alpha,\beta,L,R,\ell_{\tau^*,k_\ell}) = C\left(5R\frac{\log(2.77/u_\alpha)}{L}+5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}}+3\max\left(\frac{\ell_{\tau^*,k_\ell}}{1-\ell_{\tau^*,k_\ell}},\frac{1-\ell_{\tau^*,k_\ell}}{\ell_{\tau^*,k_\ell}}\right)\frac{\log^2(2.77/u_\alpha)}{L^2}\right) ,$$

and with (3.82), the condition (3.81) ensures that

$$E_\lambda[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)] \ge Q(\alpha,\beta,L,R,k_\ell) + \sqrt{2\mathrm{Var}_\lambda\left[T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N)\right]/\beta} \ . \quad (3.83)$$

The Bienayme-Chebyshev inequality leads to

$$P_\lambda\left(N_1 \ge LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}\right) \le \frac{\beta}{2} \ ,$$

and combined with Lemma 3.20 and the fact that $I(\lambda) \le R$, we obtain

$$P_\lambda\left(t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(1-u_\alpha) \ge Q(\alpha,\beta,L,R,\ell_{\tau^*,k_\ell})\right) \le \frac{\beta}{2} \ . \quad (3.84)$$

As a consequence,

$$P_\lambda(T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N) \le t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(1-u_\alpha)) \le P_\lambda\left(T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N) < Q(\alpha,\beta,L,R,\ell_{\tau^*,k_\ell})\right)+\frac{\beta}{2} \ ,$$

whereby we finally obtain with (3.83) and the Bienayme-Chebyshev inequality

$$P_\lambda(T'_{\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(N) \le t'_{N_1,\tau^*,\tau^*+\ell_{\tau^*,k_\ell}}(1-u_\alpha)) \le \beta \ .$$

The proof is ended by noticing that there exists $C(\alpha,\beta,R,\tau^*)>0$ such that

$$\sqrt{\frac{1+\tau^*}{\tau^*}}\max\left(\sqrt{2C}\left(\sqrt{5R}\sqrt{\frac{\log(2.77/u_\alpha)}{L}}+\sqrt{5}\left(\frac{2R}{\beta}\right)^{1/4}\frac{\sqrt{\log(2.77/u_\alpha)}}{L^{3/4}}+\frac{\sqrt{3}\log(2.77/u_\alpha)}{2\sqrt{2L\log\log L}}\right)+\frac{2\sqrt{R}}{\beta^{1/4}}\frac{1}{\sqrt{L}}, 4\sqrt{\frac{1+\tau^*}{\tau^*}}\sqrt{\frac{2R}{\beta}}\frac{1}{\sqrt{L}}, 4R\sqrt{\frac{\tau^*}{1+\tau^*}}\sqrt{\frac{\log\log L}{L}}\right) \le C(\alpha,\beta,R,\tau^*)\sqrt{\frac{\log\log L}{L}} \ ,$$

as $\log(2.77/u_\alpha) = \log(2.77\lfloor\log_2 L\rfloor/\alpha)$ and $L \ge 3$.

Proof of Proposition 3.12

Assume that

$$L > \frac{2((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{\delta^{*2}} \ , \tag{3.85}$$

and set

$$r = \sqrt{\frac{((R-\delta^*)\wedge R)\log C_{\alpha,\beta}}{2L}} \ . \tag{3.86}$$

The assumption (3.85) entails

$$r^2 < \frac{\delta^{*2}}{4} \ , \tag{3.87}$$

which enables us to define λ_r by $\lambda_r(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau_r,1]}(t)$ for t in $(0,1)$, with $\lambda_0 = (R-\delta^*)\wedge R$ and

$$\tau_r = \frac{1}{2}\left(1 + \frac{\sqrt{\delta^{*2} - 4r^2}}{|\delta^*|}\right) \ .$$

Thanks to (3.87), τ_r belongs to $(0,1)$ and satisfies $\delta^{*2}\tau_r(1-\tau_r) = r^2$, that is λ_r belongs to $\left(\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]\right)_r$. Moreover, the inequality $1 - \lambda_0 \log C_{\alpha,\beta}/(L\delta^{*2}) > 1/2$ comes from (3.85) and yields

$$\begin{aligned} r^2 &< \frac{\lambda_0 \log C_{\alpha,\beta}}{L}\left(1 - \frac{\lambda_0 \log C_{\alpha,\beta}}{L\delta^{*2}}\right) \\ &= \frac{\delta^{*2}}{4}\left(1 - \left(1 - \frac{2\lambda_0 \log C_{\alpha,\beta}}{L\delta^{*2}}\right)^2\right) \ , \end{aligned}$$

hence $\delta^{*2}(1-\tau_r) < \lambda_0 \log C_{\alpha,\beta}/L$. We get now from Lemma 2.1 and Lemma 2.24

$$E_{\lambda_0}\left[\left(\frac{dP_\lambda}{dP_{\lambda_0}}\right)^2(N)\right] = \exp\left(\frac{L(1-\tau_r)\delta^{*2}}{\lambda_0}\right) < C_{\alpha,\beta} \ .$$

Lemmas 1.7 and 1.8 then entail $\rho\left((\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R])_r\right) \geq \beta$ and $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]) \geq r$.

Proof of Proposition 3.13

The control of the first kind error rate is straightforward, and even more strong by using the same conditioning trick as in the proof of Proposition 3.9: in fact, for every λ_0 in $\mathcal{S}^u_0[R]$ and n in $\mathbb{N}$, $E_{\lambda_0}\left[\phi^u_{7,\alpha}(N)\middle|N_1 = n\right] = P_{\lambda_0}\left(\phi^u_{7,\alpha}(N) = 1\middle|N_1 = n\right) \leq \alpha$.

Let us turn to the control of the second kind error rate of $\phi^u_{7,\alpha}$.

Set λ in $\mathcal{S}^u_{\delta^*,\cdot\cdot,1-\cdot\cdot}[R]$ such that $\lambda = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}$ with λ_0 in $(-\delta^* \vee 0, (R-\delta^*)\wedge R]$ and τ in $(0,1)$, and satisfying

$$d_2(\lambda, \mathcal{S}^u_0[R]) \geq \frac{2}{\sqrt{L}}\max\left(\sqrt{|\delta^*|Q(2R,\delta^*,\alpha)},\ 2\sqrt{\frac{2R}{\beta}},\ \frac{|\delta^*|}{2\sqrt{2\beta R}}\right) \ , \tag{3.88}$$

where $Q(2R,\delta^*,\alpha)$ is the constant, not depending on L, defined in Lemma 3.22 and used in Lemma 3.24. The condition (3.88) ensures that $|\delta^*|\sqrt{\tau(1-\tau)} \geq |\delta^*|/\sqrt{2\beta RL}$, and therefore, using the fact that $\tau(1-\tau) \leq 1/4$,

$$L \geq \frac{2}{\beta R} \ . \tag{3.89}$$

Setting $I(\lambda) = \int_0^1 \lambda(t)dt$, we obtain with $I(\lambda) \le R$ (and therefore obviously $2R - I(\lambda) \ge R$) and the fact that (3.89) entails $L \ge 2R/(\beta R^2)$

$$L \ge \frac{2I(\lambda)}{(2R - I(\lambda))^2 \beta} \quad . \tag{3.90}$$

Notice now that (3.90) and the Bienayme-Chebyshev inequality yield

$$\begin{aligned} P_\lambda(N_1 > 2RL) &= P_\lambda\Big(N_1 - I(\lambda)L > L(2R - I(\lambda))\Big) \\ &\le P_\lambda\left(N_1 - I(\lambda)L > \sqrt{\frac{2I(\lambda)L}{\beta}}\right) \le \frac{\beta}{2} \quad . \end{aligned}$$

This leads to

$$\begin{aligned} P_\lambda\left(\phi^u_{7,\alpha}(N) = 0\right) &\le P_\lambda\left(\sup_{\tau' \in (0,1)} S'_{\delta^*,\tau',1}(N) \le s'^{+}_{N_1,\delta^*,L}(1-\alpha),\ N_1 \le 2RL\right) + \frac{\beta}{2} \\ &\le P_\lambda\left(\sup_{\tau' \in (0,1)} S'_{\delta^*,\tau',1}(N) \le Q(2R,\delta^*,\alpha)\right) + \frac{\beta}{2} \quad \text{with Lemma 3.24} \\ &\le P_\lambda\left(S'_{\delta^*,\tau,1}(N) \le Q(2R,\delta^*,\alpha)\right) + \frac{\beta}{2} \quad . \end{aligned}$$

Moreover, the assumption (3.88) implies

$$|\delta^*|\sqrt{\tau(1-\tau)} \ge \frac{2}{\sqrt{L}} \max\left(\sqrt{|\delta^*| Q(2R,\delta^*,\alpha)},\ 2\sqrt{\frac{2R}{\beta}}\right) \quad ,$$

which entails

$$\frac{|\delta^*|}{2}\tau(1-\tau)L \ge 2\max\left(Q(2R,\delta^*,\alpha),\ \sqrt{\frac{2R\tau(1-\tau)L}{\beta}}\right) \quad ,$$

whereby

$$\frac{|\delta^*|}{2}\tau(1-\tau)L \ge 2\max\left(Q(2R,\delta^*,\alpha),\ \sqrt{\frac{2(\lambda_0 + \delta^*\tau)\tau(1-\tau)L}{\beta}}\right) \quad .$$

Noticing that $E_\lambda[S'_{\delta^*,\tau,1}(N)] = |\delta^*|\tau(1-\tau)L/2$ and $\mathrm{Var}(S'_{\delta^*,\tau,1}(N)) = (\lambda_0 + \delta^*\tau)\tau(1-\tau)L$, this leads to

$$E_\lambda[S'_{\delta^*,\tau,1}(N)] \ge Q(2R,\delta^*,\alpha) + \sqrt{\frac{2\mathrm{Var}(S'_{\delta^*,\tau,1}(N))}{\beta}} \quad . \tag{3.91}$$

Therefore,

$$\begin{aligned} P_\lambda\left(\phi^u_{7,\alpha}(N) = 0\right) &\le P_\lambda\left(S'_{\delta^*,\tau,1}(N) - E_\lambda[S'_{\delta^*,\tau,1}(N)] \le -\sqrt{\frac{2\mathrm{Var}(S'_{\delta^*,\tau,1}(N))}{\beta}}\right) + \frac{\beta}{2} \quad \text{with (3.91)} \\ &\le \beta \quad , \end{aligned}$$

with a last line simply deduced from the Bienayme-Chebyshev inequality.
Setting

$$C(\alpha,\beta,R,\delta^*) = 2\max\left(\sqrt{|\delta^*| Q(2R,\delta^*,\alpha)},\ 2\sqrt{\frac{2R}{\beta}},\ \frac{|\delta^*|}{2\sqrt{2\beta R}}\right) \quad ,$$

allows to conclude the proof.

Proof of Proposition 3.14

Assume that $L \geq 3$ and $\alpha + \beta < 1/2$. We consider $C'_{\alpha,\beta} = 4(1-\alpha-\beta)^2$, $K_{\alpha,\beta,L} = \lceil (\log_2 L)/C'_{\alpha,\beta} \rceil$, $\lambda_0 = R/2$ and for k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $\lambda_k = \lambda_0 + \delta_k \mathbb{1}_{(\tau_k,1]}$, with $\tau_k = 1 - 2^{-k}$ and $\delta_k = (2^k \lambda_0 \log\log L/L)^{1/2}$. Then, for every k in $\{1,\ldots,K_{\alpha,\beta,L}\}$, $d_2\left(\lambda_k, \mathcal{S}_0^u[R]\right) \geq \sqrt{R \log\log L/(4L)}$, and λ_k belongs to $\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$ assuming

$$\frac{\log\log L}{L^{1-1/C'_{\alpha,\beta}}} \leq \frac{R}{4} \ . \tag{3.92}$$

The proof then essentially follows the same arguments as the proof of Proposition 2.17. Considering a random variable κ with uniform distribution on $\{1,\ldots,K_{\alpha,\beta,L}\}$ and the probability distribution μ of λ_κ, we aim at proving that $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2] \leq 1 + C'_{\alpha,\beta}$, with P_μ defined as in Lemma 1.8, in order to conclude that $\mathrm{mSR}_{\alpha,\beta}(\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]) \geq \sqrt{R\log\log L/(4L)}$.
The same calculation as in the proof of Proposition 2.12 and Proposition 2.17 gives for η such that $0 < \eta < 1 - 1/\sqrt{2}$,

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] \leq C'_{\alpha,\beta}\log 2 + \frac{2C'_{\alpha,\beta}\log 2}{\log L}(\log L)^{\eta+\frac{1}{\sqrt{2}}} + \exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) \ .$$

If we assume now that

$$\exp\left(\frac{\log\log L}{2^{(\log L)^\eta/2}}\right) + \frac{2C'_{\alpha,\beta}\log 2}{(\log L)^{1-\eta-\frac{1}{\sqrt{2}}}} \leq 1 + (1-\log 2)C'_{\alpha,\beta} \ , \tag{3.93}$$

we finally obtain the expected result, that is

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2\right] \leq 1 + C'_{\alpha,\beta} \ .$$

To end the proof, it remains to notice that there exists $L_0(\alpha,\beta,R) \geq 3$ such that for all $L \geq L_0(\alpha,\beta,R)$, both assumptions (3.92) and (3.93) hold.

Proof of Proposition 3.15

As for all our Bonferroni type aggregated tests, the control of the first kind error rates of the two tests $\phi^{u(1)}_{8,\alpha}$ and $\phi^{u(2)}_{8,\alpha}$ is straightforward using union bounds and the conditioning trick of the above proofs for upper bounds. Let us now turn to the second kind error rates.

(i) Control of the second kind error rate of $\phi^{u(1)}_{8,\alpha}$.
Let λ in $\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$, with λ_0 in $(0,R]$, τ in $(0,1)$, δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$, and assume that

$$d_2(\lambda,\mathcal{S}_0^u[R]) \geq \max\left(\sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{R}\ ,\ \sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{6}\sqrt{R + \sqrt{\frac{2R}{\beta L}} + \frac{2}{\sqrt{L}}\sqrt{\frac{6R}{\beta}}}\ ,\ \sqrt{\frac{2}{L}R}\right) \ . \tag{3.94}$$

Let us prove that under this assumption, $P_\lambda(\phi^{u(1)}_{8,\alpha}(N) = 0) \leq \beta$.
Applying Lemma 2.24, we get for all τ' in $\mathcal{D}_L$

$$E_\lambda[N(\tau',1] - (1-\tau')N_1] = \delta(\tau'\wedge\tau)(1-\tau'\vee\tau)L \ , \tag{3.95}$$

and

$$\mathrm{Var}_\lambda\left[N(\tau',1]-(1-\tau')N_1\right]=\begin{cases}(\lambda_0(1-\tau')+\delta\tau'(1-\tau))\tau' L & \text{if } \tau'\le\tau\\ (\lambda_0\tau'+\delta(\tau'-\tau+\tau'\tau))(1-\tau')L & \text{if } \tau'\ge\tau\end{cases}\ . \tag{3.96}$$

Assume first that δ belongs to $(0,R-\lambda_0]$.
Noticing that

$$P_\lambda(\phi^{u(1)}_{8,\alpha}(N)=0)\le\inf_{\tau'\in\mathcal{D}_L}P_\lambda\left(N(\tau',1]-(1-\tau')N_1\le\bar{b}_{N_1,1-\tau'}(1-u_\alpha/2)\right)\ ,$$

one can see that it is enough to exhibit some τ' in $\mathcal{D}_L$ such that

$$P_\lambda\left(N(\tau',1]-(1-\tau')N_1\le\bar{b}_{N_1,1-\tau'}(1-u_\alpha/2)\right)\le\beta\ .$$

Under the condition (3.94), we have $d_2^2(\lambda,\mathcal{S}_0^u[R])\ge 2R^2/L$ which entails

$$\tau(1-\tau)>\frac{2}{L}\ . \tag{3.97}$$

On the one hand, if τ belongs to $(0,1/2)$, the condition (3.97) implies the inequalities $2^{-1}>\tau>2/L>2^{-\lfloor\log_2 L\rfloor}$ and the existence of k_τ in $\{2,\ldots,\lfloor\log_2 L\rfloor\}$ satisfying $2^{-k_\tau}\le\tau<2^{-k_\tau+1}$. We set $\tau_{k_\tau}=2^{-k_\tau}$ and then

$$\tau_{k_\tau}\le\tau<\tau_{k_\tau-1}\ , \tag{3.98}$$

whereby

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}\le\frac{\tau}{1-\tau}<\frac{\tau_{k_\tau-1}}{1-\tau_{k_\tau-1}}\ ,$$

and therefore

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}=\underbrace{\frac{\tau_{k_\tau-1}}{1-\tau_{k_\tau-1}}}_{>\frac{\tau}{1-\tau}}\frac{1-\tau_{k_\tau-1}}{1-\tau_{k_\tau}}\underbrace{\frac{\tau_{k_\tau}}{\tau_{k_\tau-1}}}_{=\frac{1}{2}}>\frac{1}{2}\frac{\tau}{1-\tau}\frac{1-\tau_{k_\tau-1}}{1-\tau_{k_\tau}}\ .$$

But

$$\frac{1-\tau_{k_\tau-1}}{1-\tau_{k_\tau}}=1-\frac{1}{2^{k_\tau}-1}\ge\frac{2}{3}\ ,$$

so we finally obtain

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}>\frac{\tau}{3(1-\tau)}\ . \tag{3.99}$$

On the other hand, if τ belongs to $[1/2,1)$, the condition (3.97) implies the inequalities $1-2^{-1}\le\tau<1-2/L<1-2^{-\lfloor\log_2 L\rfloor}$ and the existence of k_τ in $\{1,\ldots,\lfloor\log_2 L\rfloor-1\}$ satisfying $1-2^{-k_\tau}\le\tau<1-2^{-k_\tau-1}$. We set $\tau_{k_\tau}=1-2^{-k_\tau}$ and we obtain

$$\tau_{k_\tau}\le\tau<\tau_{k_\tau+1}\ , \tag{3.100}$$

whereby

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}\le\frac{\tau}{1-\tau}<\frac{\tau_{k_\tau+1}}{1-\tau_{k_\tau+1}}\ ,$$

and therefore

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}=\underbrace{\frac{\tau_{k_\tau+1}}{1-\tau_{k_\tau+1}}}_{>\frac{\tau}{1-\tau}}\underbrace{\frac{1-\tau_{k_\tau+1}}{1-\tau_{k_\tau}}}_{=\frac{1}{2}}\frac{\tau_{k_\tau}}{\tau_{k_\tau+1}}>\frac{1}{2}\frac{\tau}{1-\tau}\frac{\tau_{k_\tau}}{\tau_{k_\tau+1}}\ .$$

But

$$\frac{\tau_{k_\tau}}{\tau_{k_\tau+1}} = 1 - \frac{1}{2^{k_\tau+1}-1} \geq \frac{2}{3} \ ,$$

so we finally get

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}} > \frac{\tau}{3(1-\tau)} \ . \tag{3.101}$$

Recall that $I(\lambda) = \int_0^1 \lambda(t)dt \leq R$ and notice that the assumption (3.94) leads to

$$\delta\sqrt{\tau(1-\tau)} \geq \max\left(\sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{R} \ , \ \sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{6}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+\frac{2}{\sqrt{L}}\sqrt{\frac{6(\lambda_0+\delta\tau_{k_\tau})}{\beta}}}\right) \ . \tag{3.102}$$

In particular, (3.102) gives on the one hand

$$\delta^2\tau(1-\tau) \geq 4R\frac{\log(2/u_\alpha)}{L} \ ,$$

which entails, using (3.98), (3.100) and the inequality $\delta < R$

$$\delta\tau(1-\tau_{k_\tau}) \geq 4\frac{\log(2/u_\alpha)}{L} \ .$$

Then, (3.99) and (3.101) ensure that $\tau(1-\tau_{k_\tau})/(3(1-\tau)) < \tau_{k_\tau}$ and thus

$$\delta\tau_{k_\tau}(1-\tau) \geq \frac{4}{3}\frac{\log(2/u_\alpha)}{L} \ . \tag{3.103}$$

On the other hand, it follows from (3.102) that

$$\delta\sqrt{\tau(1-\tau)} \geq \sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{6}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+\frac{2}{\sqrt{L}}\sqrt{\frac{6(\lambda_0+\delta\tau_{k_\tau})}{\beta}}} \ ,$$

which entails with (3.99) and (3.101)

$$\delta(1-\tau)\sqrt{\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}} \geq \sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{2}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+\frac{2}{\sqrt{L}}\sqrt{\frac{2(\lambda_0+\delta\tau_{k_\tau})}{\beta}}} \ ,$$

that is

$$\delta\tau_{k_\tau}(1-\tau)L \geq 2\sqrt{2\tau_{k_\tau}(1-\tau_{k_\tau})}\sqrt{L\log(2/u_\alpha)}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+2\sqrt{\frac{2\tau_{k_\tau}(1-\tau_{k_\tau})(\lambda_0+\delta\tau_{k_\tau})L}{\beta}}} \ . \tag{3.104}$$

Thereby, with (3.103) and (3.104),

$$\delta\tau_{k_\tau}(1-\tau)L \geq 2\max\left(\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right), \ \sqrt{\tau_{k_\tau}(1-\tau_{k_\tau})}\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}+\sqrt{\frac{2\tau_{k_\tau}(1-\tau_{k_\tau})(\lambda_0+\delta\tau_{k_\tau})L}{\beta}}}\right) ,$$

hence

$$\delta\tau_{k_\tau}(1-\tau)L \geq Q(\alpha,\beta,L,\tau_{k_\tau}) + \sqrt{\frac{2\tau_{k_\tau}(1-\tau_{k_\tau})(\lambda_0+\delta\tau_{k_\tau})L}{\beta}} \ , \tag{3.105}$$

with

$$Q(\alpha,\beta,L,\tau_{k_\tau}) = \frac{2}{3}\log\left(\frac{2}{u_\alpha}\right) + \sqrt{\tau_{k_\tau}(1-\tau_{k_\tau})}\sqrt{2\log\left(\frac{2}{u_\alpha}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}}\ .$$

Furthermore, with (3.98) and (3.100), the expressions of $E_\lambda[N(\tau',1]-(1-\tau')N_1]$ and $\mathrm{Var}_\lambda[N(\tau',1]-(1-\tau')N_1]$ given in (3.95) and (3.96) entail

$$\begin{cases} E_\lambda[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1] = \delta\tau_{k_\tau}(1-\tau)L\ , \\ \mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1\right] \le (\lambda_0+\delta\tau_{k_\tau})\tau_{k_\tau}(1-\tau_{k_\tau})L\ . \end{cases} \tag{3.106}$$

The Bienayme-Chebyshev inequality leads to

$$P_\lambda\left(N_1 \ge LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}\right) \le \frac{\beta}{2}\ ,$$

and combined with Lemma 3.23, we get that

$$P_\lambda\left(\bar{b}_{N_1,1-\tau_{k_\tau}}(1-u_\alpha/2) \ge Q(\alpha,\beta,L,\tau_{k_\tau})\right) \le \frac{\beta}{2}\ . \tag{3.107}$$

We conclude with the following inequalities:

$$\begin{aligned}
&P_\lambda(N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1 \le s'_{N_1,\tau_{k_\tau},1}(1-u_\alpha)) \\
&\quad\le P_\lambda\left(N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1 < Q(\alpha,\beta,L,\tau_{k_\tau})\right)+\frac{\beta}{2} \text{ with (3.107)} \\
&\quad\le P_\lambda\Bigg(N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1 - E_\lambda[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1] \\
&\qquad\qquad < -\sqrt{\frac{2\mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1\right]}{\beta}}\Bigg)+\frac{\beta}{2} \text{ with (3.105) and (3.106)} \\
&\quad\le \beta \text{ with the Bienayme-Chebyshev inequality again.}
\end{aligned}$$

Assume now that δ belongs to $(-\lambda_0,0)$ and notice that we have also

$$P_\lambda(\phi^{u(1)}_{8,\alpha}(N)=0) \le \inf_{\tau'\in\mathcal{D}_L} P_\lambda\left(N(\tau',1]-(1-\tau')N_1 \ge \bar{b}_{N_1,1-\tau'}(u_\alpha/2)\right)\ .$$

The same choice of τ_{k_τ} as above leads, with (3.95) and (3.96), to $E_\lambda[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1] = -|\delta|\tau_{k_\tau}(1-\tau)L$ and $\mathrm{Var}_\lambda\left[N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1\right] \le \lambda_0\tau_{k_\tau}(1-\tau_{k_\tau})L$ and we obtain, with very similar arguments and calculations (mainly replacing δ by $|\delta|$ and using the lower bound for $\bar{b}_{N_1,1-\tau_{k_\tau}}(u_\alpha/2)$ deduced from Lemma 3.23), that the assumption (3.94) implies

$$P_\lambda\left(N(\tau_{k_\tau},1]-(1-\tau_{k_\tau})N_1 \ge \bar{b}_{N_1,1-\tau_{k_\tau}}(u_\alpha/2)\right) \le \beta\ ,$$

so $P_\lambda(\phi^{u(1)}_{8,\alpha}(N)=0) \le \beta$.
Since $\log(2/u_\alpha) = \log(2(2\lfloor\log_2 L\rfloor-1)/\alpha)$ and $L \ge 3$, there exists $C(\alpha,\beta,R) > 0$ such that

$$\max\left(\sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{R},\ \sqrt{\frac{\log(2/u_\alpha)}{L}}2\sqrt{6}\sqrt{R+\sqrt{\frac{2R}{\beta L}}+\frac{2}{\sqrt{L}}\sqrt{\frac{6R}{\beta}}},\ \sqrt{\frac{2}{L}}R\right) \le C(\alpha,\beta,R)\sqrt{\frac{\log\log L}{L}}\ , \tag{3.108}$$

which allows to conclude the proof.

(ii) Control of the second kind error rate of $\phi^{u(2)}_{8,\alpha}$.

Let λ in $\mathcal{S}^u_{\cdot,\cdot\cdot,1-\cdot\cdot}[R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$ with λ_0 in $(0,R]$, τ in $(0,1)$ and δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and assume that

$$d_2(\lambda,\mathcal{S}^u_0[R]) \geq \max\left(\sqrt{\frac{30CR\log(2.77/u_\alpha)}{L}} + \frac{\sqrt{30C\log(2.77/u_\alpha)}}{L^{3/4}}\left(\frac{2R}{\beta}\right)^{1/4} + \frac{3\sqrt{C}\log(2.77/u_\alpha)}{2\sqrt{L\log\log L}} + 2\sqrt{\frac{3R}{L\sqrt{\beta}}}\,,\right.$$
$$\left. 12\sqrt{\frac{2R}{L\beta}}\,,\, 4R\sqrt{\frac{\log\log L}{L}}\right)\ , \quad (3.109)$$

where C is the constant defined in Lemma 3.20.

Let us prove the inequality $P_\lambda(\phi^{u(2)}_{8,\alpha}(N)=0) \leq \beta$. Notice first that

$$P_\lambda(\phi^{u(2)}_{8,\alpha}(N)=0) \leq \inf_{\tau'\in\mathcal{D}_L} P_\lambda\left(T'_{\tau',1}(N) \leq t'_{N_1,\tau',1}(1-u_\alpha)\right)\ ,$$

so one only needs to exhibit some τ' in $\mathcal{D}_L$ satisfying

$$P_\lambda\left(T'_{\tau',1}(N) \leq t'_{N_1,\tau',1}(1-u_\alpha)\right) \leq \beta\ ,$$

to obtain the expected result.

Under the assumption (3.109), we get $d_2^2(\lambda,\mathcal{S}^u_0[R]) \geq 16R^2(\log\log L)/L$ which entails

$$\tau(1-\tau) > \frac{16\log\log L}{L}\ . \quad (3.110)$$

Assume first that τ belongs to $(0,1/2)$. The condition (3.110) leads to $\tau > 16(\log\log L)/L$ and since $L\geq 3$, we get the inequality $16(\log\log L)/L > 2^{-\lfloor\log_2 L\rfloor}$ and the existence of k_τ in $\{2,\ldots,\lfloor\log_2 L\rfloor\}$ satisfying $2^{-k_\tau} \leq \tau < 2^{-k_\tau+1}$. Setting $\tau_{k_\tau} = 2^{-k_\tau}$ we can prove as in the above case (i) that

$$\tau_{k_\tau} \leq \tau < \tau_{k_\tau-1}\ , \quad (3.111)$$

and

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}} > \frac{\tau}{3(1-\tau)}\ . \quad (3.112)$$

Moreover, since $\tau_{k_\tau} < 1/2$ and by definition of k_τ ,

$$\max\left(\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}, \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}}\right) = \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}} < \frac{2-\tau}{\tau}\ .$$

Since (3.110) implies $\tau > 16(\log\log L)/L$, we get

$$\max\left(\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}, \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}}\right) < \frac{L}{8\log\log L}\ . \quad (3.113)$$

Assume now that τ belongs to $[1/2,1)$. The condition (3.110) entails $\tau < 1-16(\log\log L)/L$ and since $L\geq 3$, $1-2^{-1} \leq \tau < 1-2^{-\lfloor\log_2 L\rfloor}$. Hence, there exists k_τ in $\{1,\ldots,\lfloor\log_2 L\rfloor-1\}$ satisfying $1-2^{-k_\tau} \leq \tau < 1-2^{-k_\tau-1}$. We set $\tau_{k_\tau} = 1-2^{-k_\tau}$ and we obtain as in the above case (i)

$$\tau_{k_\tau} \leq \tau < \tau_{k_\tau+1}\ , \quad (3.114)$$

and

$$\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}} > \frac{\tau}{3(1-\tau)}\ . \quad (3.115)$$

Moreover, since τ_{k_τ} belongs to $[1/2,1)$, we get using (3.114)

$$\max\left(\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}, \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}}\right) = \frac{\tau_{k_\tau}}{1-\tau_{k_\tau}} \leq \frac{\tau}{1-\tau} \quad .$$

Since (3.110) yields $1-\tau > 16(\log\log L)/L$, we get also

$$\max\left(\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}, \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}}\right) < \frac{L}{16\log\log L} \quad . \tag{3.116}$$

Applying Lemma 3.19, we obtain with (3.111) and (3.114)

$$E_\lambda[T'_{\tau_{k_\tau},1}(N)] = \delta^2(1-\tau)^2\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}} \quad , \tag{3.117}$$

and

$$\mathrm{Var}_\lambda\left[T'_{\tau_{k_\tau},1}(N)\right] = \frac{4}{L}\delta^2(1-\tau)^2\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}\left(\lambda_0+\delta(1-\tau)\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}\right) + \frac{2}{L^2}\left(\lambda_0+\delta(1-\tau)\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}\right)^2 \quad . \tag{3.118}$$

On the one hand, we finally get with (3.112) and (3.115),

$$E_\lambda[T'_{\tau_{k_\tau},1}(N)] > \frac{\delta^2\tau(1-\tau)}{3} \quad , \tag{3.119}$$

and on the other hand, using (3.111) and (3.114),

$$\mathrm{Var}_\lambda\left[T'_{\tau_{k_\tau},1}(N)\right] \leq \frac{4\delta^2\tau(1-\tau)R}{L} + \frac{2R^2}{L^2} \quad . \tag{3.120}$$

Notice that the assumption (3.109) entails

$$\delta^2\tau(1-\tau) \geq \max\left(\frac{30CR\log(2.77/u_\alpha)}{L} + \frac{30C\log(2.77/u_\alpha)}{L^{3/2}}\sqrt{\frac{2R}{\beta}} + \frac{9C\log^2(2.77/u_\alpha)}{4L\log\log L} + \frac{12R}{L\sqrt{\beta}}, \right.$$
$$\left. 12|\delta|\sqrt{\tau(1-\tau)}\sqrt{\frac{2R}{L\beta}}\right) \quad ,$$

hence

$$\delta^2\tau(1-\tau) \geq \frac{15CR\log(2.77/u_\alpha)}{L} + \frac{15C\log(2.77/u_\alpha)}{L^{3/2}}\sqrt{\frac{2R}{\beta}} + \frac{9C\log^2(2.77/u_\alpha)}{8L\log\log L} + \frac{6R}{L\sqrt{\beta}} + 6|\delta|\sqrt{\tau(1-\tau)}\sqrt{\frac{2R}{L\beta}} \quad . \tag{3.121}$$

Using (3.119) and (3.120), the condition (3.121) then leads to

$$E_\lambda[T'_{\tau_{k_\tau},1}(N)] - \sqrt{2\mathrm{Var}_\lambda\left[T'_{\tau_{k_\tau},1}(N)\right]/\beta} \geq C\left(5R\frac{\log(2.77/u_\alpha)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}} + \frac{3}{8}\frac{\log^2(2.77/u_\alpha)}{L\log\log L}\right) \quad . \tag{3.122}$$

Now, if we set

$$Q(\alpha,\beta,L,R,\tau_{k_\tau}) = C\left(5R\frac{\log(2.77/u_\alpha)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log(2.77/u_\alpha)}{L^{3/2}} + 3\max\left(\frac{\tau_{k_\tau}}{1-\tau_{k_\tau}}, \frac{1-\tau_{k_\tau}}{\tau_{k_\tau}}\right)\frac{\log^2(2.77/u_\alpha)}{L^2}\right) \quad , \tag{3.123}$$

combined with (3.113) and (3.116), the condition (3.122) yields

$$E_\lambda[T'_{\tau_{k_\tau},1}(N)] - \sqrt{2\mathrm{Var}_\lambda\left[T'_{\tau_{k_\tau},1}(N))\right]/\beta} \geq Q(\alpha,\beta,L,R,\tau_{k_\tau}) \ . \tag{3.124}$$

Furthermore, from the inequality

$$P_\lambda\left(N_1 \geq LI(\lambda) + \sqrt{\frac{2LI(\lambda)}{\beta}}\right) \leq \frac{\beta}{2} \ ,$$

Lemma 3.20 and the fact that $I(\lambda) \leq R$, we deduce that

$$P_\lambda\left(t'_{N_1,\tau_{k_\tau},1}(1-u_\alpha) \geq Q(\alpha,\beta,L,R,\tau_{k_\tau})\right) \leq \frac{\beta}{2} \ . \tag{3.125}$$

We finally obtain using successively (3.125), (3.124) and the Bienayme-Chebyshev inequality

$$\begin{aligned} P_\lambda(T'_{\tau_{k_\tau},1}(N) \leq t'_{N_1,\tau_{k_\tau},1}(1-u_\alpha)) &\leq P_\lambda\left(T'_{\tau_{k_\tau},1}(N) < Q(\alpha,\beta,L,R,\tau_{k_\tau})\right) + \frac{\beta}{2} \\ &\leq P_\lambda\left(T'_{\tau_{k_\tau},1}(N) - E_\lambda[T'_{\tau_{k_\tau},1}(N)] < -\sqrt{2\mathrm{Var}_\lambda\left[T'_{\tau_{k_\tau},1}(N))\right]/\beta}\right) + \frac{\beta}{2} \\ &\leq \beta \ . \end{aligned}$$

Since $\log(2/u_\alpha) = \log((4\lfloor\log_2 L\rfloor - 2)/\alpha)$ and $L \geq 3$, there exists $C(\alpha,\beta,R) > 0$ such that (3.109) is implied by

$$d_2(\lambda, \mathcal{S}_0^u[R]) \geq C(\alpha,\beta,R)\sqrt{\frac{\log\log L}{L}} \ ,$$

which concludes the proof.

Proof of Proposition 3.16

Let $L \geq 2$ and set $\lambda_0 = (R-\delta^*) \wedge R$.
For all k in $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$, let us define $\lambda_k(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau_k,\tau_k+\ell]}(t)$ with $\tau_k = k/L$ and $\ell = \lambda_0 \log L/(2\delta^{*2}L)$. Then, as soon as

$$\frac{\lceil L^{3/4}\rceil}{L} + \frac{((R-\delta^*)\wedge R)\log L}{2\delta^{*2}L} < 1 \ , \tag{3.126}$$

λ_k belongs to $\mathcal{S}^u_{\delta^*,..,...}[R]$ for any k in $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$. If in addition

$$\frac{\log L}{L} \leq \frac{\delta^{*2}}{(R-\delta^*)\wedge R} \ , \tag{3.127}$$

λ_k satisfies for all k in $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$,

$$d_2^2(\lambda_k, \mathcal{S}_0^u[R]) \geq \frac{((R-\delta^*)\wedge R)\log L}{4L} \ .$$

Using Lemma 1.8 and considering a random variable J uniformly distributed on $\left\{1,\ldots,\lceil L^{3/4}\rceil\right\}$ and the distribution μ of λ_J, one can see that it is enough to prove that $E_{\lambda_0}[\left(dP_\mu/dP_{\lambda_0}\right)^2] \leq 1 + 4(1-\alpha-\beta)^2$ to obtain the expected lower bound. The same calculation as in the proof of Proposition 2.19 leads to

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N)\right] \leq 1 + \frac{\sqrt{L}}{\lceil L^{3/4}\rceil}\frac{e^{\delta^{*2}/\lambda_0}+1}{e^{\delta^{*2}/\lambda_0}-1} \ .$$

Therefore, assuming that

$$\frac{\sqrt{L}}{\lceil L^{3/4}\rceil}\frac{e^{\delta^{*2}/(R-\delta^*)\wedge R}+1}{e^{\delta^{*2}/(R-\delta^*)\wedge R}-1}\leq 4(1-\alpha-\beta)^2\ , \tag{3.128}$$

we get the expected result, that is

$$E_{\lambda_0}\left[\left(\frac{dP_\mu}{dP_{\lambda_0}}\right)^2(N)\right]\leq 1+4(1-\alpha-\beta)^2\ .$$

We then conclude the proof noticing that there exists $L_0(\alpha,\beta,\delta^*,R)\geq 2$ such that the assumptions (3.126), (3.127) and (3.128) hold for all $L\geq L_0(\alpha,\beta,\delta^*,R)$.

Proof of Proposition 3.18

As for all our Bonferroni type aggregated tests of Chapter 3 dedicated to change detection from an unknown baseline intensity, the control of the first kind error rates of the two tests $\phi^{u(1)}_{9/10,\alpha}$ and $\phi^{u(2)}_{9/10,\alpha}$ is easily deduced from union bounds and the conditioning trick of the above proofs for upper bounds. We therefore focus here on the second kind error rates.

(i) Control of the second kind error rate of $\phi^{u(1)}_{9/10,\alpha}$.

Let λ in $\mathcal{S}^u_{\cdot,\cdot,\cdot,\cdot\cdot\cdot}[R]$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,\tau+\ell]}$, with λ_0 in $(0,R]$, τ in $(0,1)$, δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$ and ℓ in $(0,1-\tau)$. By now, we aim at proving that $P_\lambda\left(\phi^{u(1)}_{9/10,\alpha}(N)=0\right)\leq\beta$ as soon as we assume that

$$d_2(\lambda,\mathcal{S}^u_0[R])\geq\max\left(2\sqrt{\frac{R\log\left(2/u^{(1)}_\alpha\right)}{L}}\ ,\ 6\sqrt{\frac{2\log\left(2/u^{(1)}_\alpha\right)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}+6\sqrt{\frac{2R}{\beta L}}}\ ,\ \frac{\sqrt{3}R}{\sqrt{L}}\right)\ . \tag{3.129}$$

Let us first consider the case where δ belongs to $(0,R-\lambda_0]$.
Noticing that

$$P_\lambda\left(\phi^{u(1)}_{9/10,\alpha}(N)=0\right)\leq\inf_{k\in\{0,\ldots,\lceil L\rceil-1\}}\inf_{k'\in\{1,\ldots,\lceil L\rceil-k\}}P_\lambda\left(N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]-\frac{k'}{\lceil L\rceil}N_1\leq\bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\left(1-\frac{u^{(1)}_\alpha}{2}\right)\right)\ ,$$

one can see that it is enough to exhibit some k_0 in $\{0,\ldots,\lceil L\rceil-1\}$ and k_0' in $\{1,\ldots,\lceil L\rceil-k_0\}$ such that

$$P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\leq\bar{b}_{N_1,\frac{k_0'}{\lceil L\rceil}}\left(1-\frac{u^{(1)}_\alpha}{2}\right)\right)\leq\beta\ .$$

We get from (3.129) that $d_2^2(\lambda,\mathcal{S}^u_0[R])\geq 3R^2/L\geq 3R^2/\lceil L\rceil$ which entails

$$\ell(1-\ell)>3/\lceil L\rceil\ . \tag{3.130}$$

Assume first that $\ell\leq 1/2$. The condition (3.130) leads to

$$\ell>3/\lceil L\rceil\text{ and }\tau<1-3/\lceil L\rceil\ . \tag{3.131}$$

Therefore, we can define $k_0=\min(k\in\{0,\ldots,\lceil L\rceil-1\},\ \tau\leq k/\lceil L\rceil)$ and $k_0'=\max(k'\in\{1,\ldots,\lceil L\rceil-k_0\},\ (k_0+k')/\lceil L\rceil\leq\tau+\ell)$, so that $\tau\leq k_0/\lceil L\rceil<(k_0+k_0')/\lceil L\rceil\leq\tau+\ell$. Since by definition $k_0/\lceil L\rceil-\tau<1/\lceil L\rceil$ and $\tau+\ell-(k_0+k_0')/\lceil L\rceil<1/\lceil L\rceil$, we get that

$$\frac{k_0'}{\lceil L\rceil}=\ell-\left(\left(\frac{k_0}{\lceil L\rceil}-\tau\right)+\left(\tau+\ell-\frac{k_0+k_0'}{\lceil L\rceil}\right)\right)>\ell-\frac{2}{\lceil L\rceil},$$

and then, combining with (3.131),

$$\frac{k_0'}{\lceil L\rceil} > \frac{\ell}{3} \ . \tag{3.132}$$

Lemma 2.24 leads to $E_\lambda\left[N\left(k_0/\lceil L\rceil,(k_0+k_0')/\lceil L\rceil\right]-(k_0'/\lceil L\rceil)N_1\right]=\delta(1-\ell)Lk_0'/\lceil L\rceil$ and

$$\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\right]=\left(\lambda_0\left(1-\frac{k_0'}{\lceil L\rceil}\right)+\delta\left(1-(2-\ell)\frac{k_0'}{\lceil L\rceil}\right)\right)\frac{k_0'}{\lceil L\rceil}L \ .$$

With (3.132), we obtain on the one hand

$$E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\right]>\frac{\delta}{3}\ell(1-\ell)L \ . \tag{3.133}$$

On the other hand, $k_0'/\lceil L\rceil\le\ell<1$ yields $0<1-(2-\ell)k_0'/\lceil L\rceil<1-k_0'/\lceil L\rceil$. Moreover, using $k_0'/\lceil L\rceil\le\ell$ again and the fact that $\ell\le 1/2$ one obtains

$$\frac{k_0'}{\lceil L\rceil}\left(1-\frac{k_0'}{\lceil L\rceil}\right)\le\ell(1-\ell) \ , \tag{3.134}$$

and we finally get

$$\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\right]\le(\lambda_0+\delta)\ell(1-\ell)L \ . \tag{3.135}$$

Recall that $I(\lambda)=\int_0^1\lambda(t)dt$ and notice that (3.129) entails

$$\delta\sqrt{\ell(1-\ell)}\ge\max\left(2\sqrt{\frac{\delta\log\left(2/u_\alpha^{(1)}\right)}{L}}\ ,\ 6\sqrt{\frac{2\log\left(2/u_\alpha^{(1)}\right)}{L}}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+6\sqrt{\frac{2(\lambda_0+\delta)}{\beta L}}}\right) \ ,$$

thereby

$$\delta\ell(1-\ell)\ge 2\max\left(\frac{2\log\left(2/u_\alpha^{(1)}\right)}{L}\ ,\ 3\sqrt{\ell(1-\ell)}\left(\sqrt{\frac{2\log\left(2/u_\alpha^{(1)}\right)}{L}}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+\sqrt{\frac{2(\lambda_0+\delta)}{\beta L}}}\right)\right) \ ,$$

and then

$$\frac{\delta}{3}\ell(1-\ell)L\ge\frac{2}{3}\log\left(\frac{2}{u_\alpha^{(1)}}\right)+\sqrt{\ell(1-\ell)}\sqrt{2L\log\left(\frac{2}{u_\alpha^{(1)}}\right)}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}+\sqrt{\frac{2\ell(1-\ell)L(\lambda_0+\delta)}{\beta}}} \ . \tag{3.136}$$

Thus, with (3.133) and (3.135), the inequality (3.136) ensures that

$$E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\right]\ge\frac{2}{3}\log\left(\frac{2}{u_\alpha^{(1)}}\right)+\sqrt{\ell(1-\ell)}\sqrt{2L\log\left(\frac{2}{u_\alpha^{(1)}}\right)}\sqrt{I(\lambda)+\sqrt{\frac{2I(\lambda)}{\beta L}}}$$
$$+\sqrt{\frac{2\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right]-\frac{k_0'}{\lceil L\rceil}N_1\right]}{\beta}} \ . \tag{3.137}$$

Furthermore, we deduce from the inequality

$$P_\lambda\left(N_1 \geq LI(\lambda) + \sqrt{\frac{2LI(\lambda)}{\beta}}\right) \leq \frac{\beta}{2} \ ,$$

combined with Lemma 3.23 that

$$P_\lambda\left(\bar{b}_{N_1,\frac{k_0'}{\lceil L\rceil}}\left(1-\frac{u_\alpha^{(1)}}{2}\right) \geq Q(\alpha,\beta,L,k_0')\right) \leq \frac{\beta}{2} \ , \tag{3.138}$$

with

$$Q(\alpha,\beta,L,k_0') = \frac{2}{3}\log\left(\frac{2}{u_\alpha^{(1)}}\right) + \sqrt{\frac{k_0'}{\lceil L\rceil}\left(1-\frac{k_0'}{\lceil L\rceil}\right)}\sqrt{2\log\left(\frac{2}{u_\alpha^{(1)}}\right)}\sqrt{LI(\lambda)+\sqrt{\frac{2LI(\lambda)}{\beta}}} \ . \tag{3.139}$$

Using (3.134), the inequality (3.137) leads to

$$E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1\right] \geq Q(\alpha,\beta,L,k_0') + \sqrt{2\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1\right]/\beta} \ , \tag{3.140}$$

and we conclude with the following inequalities

$$\begin{aligned}
&P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1 \leq \bar{b}_{N_1,\frac{k_0'}{\lceil L\rceil}}\left(1-\frac{u_\alpha^{(1)}}{2}\right)\right)\\
&\leq P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1 < Q(\alpha,\beta,L,k_0')\right) + P_\lambda\left(\bar{b}_{N_1,\frac{k_0'}{\lceil L\rceil}}\left(1-\frac{u_\alpha^{(1)}}{2}\right) \geq Q(\alpha,\beta,L,k_0')\right)\\
&\leq P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1 < Q(\alpha,\beta,L,k_0')\right) + \frac{\beta}{2} \quad \text{with (3.138)}\\
&\leq P_\lambda\Bigg(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1 - E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1\right]\\
&\qquad\qquad < -\sqrt{2\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k_0'}{\lceil L\rceil}\right] - \frac{k_0'}{\lceil L\rceil}N_1\right]/\beta}\Bigg) + \frac{\beta}{2} \quad \text{with (3.140)}\\
&\leq \beta \quad \text{with the Bienayme-Chebyshev inequality} \ .
\end{aligned}$$

Assume now that $\ell > 1/2$. We define $k_0 = \max(k \in \{0,\ldots,\lceil L\rceil - 1\},\ \tau \geq k/\lceil L\rceil)$ and $k_0' = \min(k' \in \{1,\ldots,\lceil L\rceil - k_0\},\ (k_0+k')/\lceil L\rceil \geq \tau+\ell)$ so that $k_0/\lceil L\rceil \leq \tau < \tau+\ell \leq (k_0+k_0')/\lceil L\rceil$. Notice that the condition (3.130) entails

$$1-\ell > 3/\lceil L\rceil \ . \tag{3.141}$$

Since by definition of k_0 and k_0', $\tau - k_0/\lceil L\rceil < 1/\lceil L\rceil$ and $(k_0+k_0')/\lceil L\rceil - (\tau+\ell) < 1/\lceil L\rceil$, then, with (3.141),

$$\frac{k_0'}{\lceil L\rceil} = \frac{k_0+k_0'}{\lceil L\rceil} - (\tau+\ell) + \ell + \tau - \frac{k_0}{\lceil L\rceil} < 1/\lceil L\rceil + (1-3/\lceil L\rceil) + 1/\lceil L\rceil \leq 1 - 1/\lceil L\rceil < 1 \ .$$

In the same way, we can also obtain with (3.141) that

$$1-\frac{k_0'}{\lceil L\rceil} = 1-\ell-\left(\left(\tau-\frac{k_0}{\lceil L\rceil}\right)+\left(\frac{k_0+k_0'}{\lceil L\rceil}-(\tau+\ell)\right)\right) > 1-\ell-\frac{2}{\lceil L\rceil} > \frac{1-\ell}{3} \ . \tag{3.142}$$

Furthermore, by definition of k_0 and k'_0, one has $k'_0/\lceil L\rceil \geq \ell$ and since $\ell > 1/2$,

$$\frac{k'_0}{\lceil L\rceil}\left(1-\frac{k'_0}{\lceil L\rceil}\right) \leq \ell(1-\ell) \ . \tag{3.143}$$

From Lemma 2.24, we therefore deduce

$$\begin{aligned} E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right] &= \delta\ell\left(1-\frac{k'_0}{\lceil L\rceil}\right)L \\ &> \frac{\delta}{3}\ell(1-\ell)L \ \text{ with (3.142)} \ , \end{aligned}$$

and since $k'_0/\lceil L\rceil \geq \ell$,

$$\begin{aligned} \mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right] &= \left(\lambda_0\frac{k'_0}{\lceil L\rceil}+\delta\ell\left(1-\frac{k'_0}{\lceil L\rceil}\right)\right)\left(1-\frac{k'_0}{\lceil L\rceil}\right)L \\ &\leq \left(\lambda_0+\delta\left(1-\frac{k'_0}{L}\right)\right)\left(1-\frac{k'_0}{L}\right)\frac{k'_0}{\lceil L\rceil}L \\ &\leq (\lambda_0+\delta)\ell(1-\ell)L \ \text{ with (3.143)} \ . \end{aligned}$$

Thus, the condition (3.136) also yields

$$E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right] \geq Q(\alpha,\beta,L,k'_0)+\sqrt{2\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right]/\beta} \ ,$$

where $Q(\alpha,\beta,L,k'_0)$ is defined by (3.139) and we conclude that

$$P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1 \leq \bar{b}_{N_1,\frac{k'_0}{\lceil L\rceil}}\left(1-\frac{u_\alpha^{(1)}}{2}\right)\right) \leq \beta \ ,$$

with the same arguments as above.

Let us then treat the case where δ belongs to $(-\lambda_0, 0)$. We start by noticing that

$$P_\lambda\left(\phi^{(1)}_{9/10,\alpha}(N)=0\right) \leq \inf_{k\in\{0,\ldots,\lceil L\rceil-1\}}\ \inf_{k'\in\{1,\ldots,\lceil L\rceil-k\}} P_\lambda\left(N\left(\frac{k}{\lceil L\rceil},\frac{k+k'}{\lceil L\rceil}\right]-\frac{k'}{\lceil L\rceil}N_1 \geq \bar{b}_{N_1,\frac{k'}{\lceil L\rceil}}\left(\frac{u_\alpha^{(1)}}{2}\right)\right) \ .$$

The same choice of k_0 and k'_0 as in the case where δ belongs to $(0, R-\lambda_0]$ leads to

$$E_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right] < -\frac{|\delta|\ell(1-\ell)L}{3}$$

and

$$\mathrm{Var}_\lambda\left[N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1\right] \leq \lambda_0\frac{k'_0}{\lceil L\rceil}\left(1-\frac{k'_0}{\lceil L\rceil}\right)L \leq R\frac{k'_0}{\lceil L\rceil}\left(1-\frac{k'_0}{\lceil L\rceil}\right)L \ ,$$

and we obtain

$$P_\lambda\left(N\left(\frac{k_0}{\lceil L\rceil},\frac{k_0+k'_0}{\lceil L\rceil}\right]-\frac{k'_0}{\lceil L\rceil}N_1 \geq \bar{b}_{N_1,\frac{k'_0}{\lceil L\rceil}}\left(\frac{u_\alpha^{(1)}}{2}\right)\right) \leq \beta \ ,$$

following the same lines of proof as above, but with δ replaced by $|\delta|$ except when it is involved in $\lambda_0+\delta$, and using the lower bound for $\bar{b}_{N_1,\frac{k'_0}{\lceil L\rceil}}\left(u_\alpha^{(1)}/2\right)$ deduced from Lemma 3.23.

Coming back to the assumption (3.129) and the definition of $u_\alpha^{(1)}$, one can finally claim that

$$\mathrm{SR}_\beta\left(\phi^{u(1)}_{9/10,\alpha}, \mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]\right) \le \max\left(2\sqrt{\frac{R\log(\lceil L\rceil(\lceil L\rceil+1)/\alpha)}{L}}\, ,\, 6\sqrt{\frac{2\log(\lceil L\rceil(\lceil L\rceil+1)/\alpha)}{L}}\sqrt{R+\sqrt{\frac{2R}{\beta L}}}+6\sqrt{\frac{2R}{\beta L}}\, ,\, \frac{\sqrt{3}R}{\sqrt{L}}\right)\ , \tag{3.144}$$

which leads to the result stated in Proposition 3.18 for $\phi^{u(1)}_{9/10,\alpha}$ and the set $\mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]$ of the alternative [$\mathbf{Alt^u.10}$]. Since $\mathcal{S}^u_{\delta^*,\cdot\cdot,\cdots}[R] \subset \mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]$ for any δ^* in $(-R,R)\setminus\{0\}$, the same result holds for $\phi^{u(1)}_{9/10,\alpha}$ and the set $\mathcal{S}^u_{\delta^*,\cdot\cdot,\cdots}[R]$ of the alternative [$\mathbf{Alt^u.9}$]. Notice that in this case, the constant $C(\alpha,\beta,R)$ involved in the upper bound can be refined, benefiting from the knowledge of δ^*, which explains the formulation with $C(\alpha,\beta,R,\delta^*)$ instead of $C(\alpha,\beta,R)$ in Proposition 3.18.

(ii) Control of the second kind error rate of $\phi^{u(2)}_{9/10,\alpha}$.
Let λ in $\mathcal{S}^u_{\cdot,\cdot\cdot,\cdots}[R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,\tau+\ell]}$, with λ_0 in $(0,R]$, τ in $(0,1)$, δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and ℓ in $(0,1-\tau)$. We assume by now that

$$d_2(\lambda,\mathcal{S}^u_0[R]) \ge \max\left(\sqrt{\frac{60CR\log\left(2.77/u_\alpha^{(2)}\right)}{L}}+\left(\frac{2R}{\beta}\right)^{1/4}\frac{\sqrt{60C\log\left(2.77/u_\alpha^{(2)}\right)}}{L^{3/4}}+6\sqrt{C}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{\sqrt{L\log L}}+2\sqrt{\frac{6R}{L\sqrt{\beta}}}\, ,\, 24\sqrt{\frac{2R}{\beta L}}\, ,\, R\sqrt{\frac{3\log L}{L}}\right)\ , \tag{3.145}$$

where C is the constant defined in Lemma 3.20.
Let us prove that this assumption implies $P_\lambda\left(\phi^{u(2)}_{9/10,\alpha}(N)=0\right)\le\beta$.
As above, we invoke that

$$P_\lambda\left(\phi^{u(2)}_{9/10,\alpha}(N)=0\right) \le \inf_{k\in\{0,\dots,M_L-1\}}\ \inf_{\substack{k'\in\{1,\dots,M_L-k\}\\(k,k')\ne(0,M_L)}} P_\lambda\left(T'_{\frac{k}{M_L},\frac{k+k'}{M_L}}(N)\le t'_{N_1,\frac{k}{M_L},\frac{k+k'}{M_L}}\left(1-u_\alpha^{(2)}\right)\right)\ ,$$

to argue that we only need to exhibit some k_0 in $\{0,\dots,M_L-1\}$ and k_0' in $\{1,\dots,M_L-k_0\}$ such that $(k_0,k_0')\ne(0,M_L)$ satisfying

$$P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\le t'_{N_1,\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}\left(1-u_\alpha^{(2)}\right)\right)\le\beta\ .$$

Recalling that $M_L=\lceil L/\log L\rceil$, we get from (3.145) that $d_2^2(\lambda,\mathcal{S}^u_0[R])\ge 3R^2\log L/L\ge 3R^2/M_L$ which entails

$$\ell(1-\ell) > 3/M_L\ . \tag{3.146}$$

Let us first consider the case where $\ell\le 1/2$.
The condition (3.146) leads to

$$\ell > 3/M_L\ , \tag{3.147}$$

and therefore, we can define $k_0=\min(k\in\{1,\dots,M_L-1\},\ \tau\le k/M_L)$ and $k_0'=\max(k'\in\{1,\dots,M_L-k_0\},\ (k_0+k')/M_L\le\tau+\ell)$, so that $\tau\le k_0/M_L<(k_0+k_0')/M_L\le\tau+\ell$. Since by definition $k_0/M_L-\tau<1/M_L$ and $\tau+\ell-(k_0+k_0')/M_L<1/M_L$, notice that

$$\frac{k_0'}{M_L}=\ell-\left(\left(\frac{k_0}{M_L}-\tau\right)+\left(\tau+\ell-\frac{k_0+k_0'}{M_L}\right)\right)>\ell-\frac{2}{M_L}\ ,$$

and then, combined with (3.147),

$$\frac{k_0'}{M_L} > \frac{\ell}{3} \ . \tag{3.148}$$

Lemma 3.19 gives $E_\lambda\left[T'_{k_0/M_L,(k_0+k_0')/M_L}(N)\right] = \delta^2(1-\ell)^2(k_0'/M_L)/(1-k_0'/M_L)$, and with (3.148) and the facts that $1-\ell \geq 1/2$ and $1/(1-k_0'/M_L) > 1$, we get

$$E_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] > \frac{\delta^2}{6}\ell(1-\ell) \ . \tag{3.149}$$

Lemma 3.19 also gives

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] = \frac{2}{L^2}\left(\lambda_0+\delta\frac{1-(2-\ell)k_0'/M_L}{1-k_0'/M_L}\right)^2 + \frac{4}{L}\delta^2(1-\ell)^2\frac{k_0'/M_L}{1-k_0'/M_L}\left(\lambda_0+\delta\frac{1-(2-\ell)k_0'/M_L}{1-k_0'/M_L}\right) , \tag{3.150}$$

and since $k_0'/M_L \leq \ell < 1$, notice that $0 < 1-(2-\ell)k_0'/M_L < 1-k_0'/M_L$. Moreover using $k_0'/M_L \leq \ell$ again, one obtains

$$\frac{k_0'/M_L}{1-k_0'/M_L} \leq \frac{\ell}{1-\ell} \ . \tag{3.151}$$

Therefore, using (3.151),

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \leq \left(\frac{2}{L^2}(\lambda_0+\delta)^2+\frac{4}{L}(\lambda_0+\delta)\delta^2\ell(1-\ell)\right)\mathbb{1}_{\delta>0}+\left(\frac{2}{L^2}\lambda_0^2+\frac{4}{L}\lambda_0\delta^2\ell(1-\ell)\right)\mathbb{1}_{\delta<0} \ ,$$

hence

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \leq \frac{2R^2}{L^2}+\frac{4R}{L}\delta^2\ell(1-\ell) \ . \tag{3.152}$$

Now, on the one hand, notice that the assumption (3.145) ensures that

$$|\delta|\sqrt{\ell(1-\ell)} \geq \sqrt{\frac{60CR\log\left(2.77/u_\alpha^{(2)}\right)}{L}+\left(\frac{2R}{\beta}\right)^{1/4}\frac{\sqrt{60C\log\left(2.77/u_\alpha^{(2)}\right)}}{L^{3/4}}+6\sqrt{C(M_L-1)}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L}+2\sqrt{\frac{6R}{L\sqrt{\beta}}}} \ ,$$

which entails

$$\delta^2\ell(1-\ell) \geq 60CR\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L}+60C\sqrt{\frac{2R}{\beta}}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L^{3/2}}+36C(M_L-1)\frac{\log^2\left(2.77/u_\alpha^{(2)}\right)}{L^2}+\frac{24R}{\sqrt{\beta}L} \ . \tag{3.153}$$

On the other hand, the same assumption (3.145) ensures that $|\delta|\sqrt{\ell(1-\ell)} \geq 24\sqrt{2R/(\beta L)}$, which leads to

$$\delta^2\ell(1-\ell) \geq 24\sqrt{\frac{2R}{\beta L}}|\delta|\sqrt{\ell(1-\ell)} \ . \tag{3.154}$$

Therefore, (3.153) and (3.154) imply

$$\delta^2\ell(1-\ell) \geq 2\max\left(C\left(30R\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L}+30\sqrt{\frac{2R}{\beta}}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L^{3/2}}+18(M_L-1)\frac{\log^2\left(2.77/u_\alpha^{(2)}\right)}{L^2}\right)+\frac{12R}{\sqrt{\beta}L}, \right.$$
$$\left. 12\sqrt{\frac{2R}{\beta L}}|\delta|\sqrt{\ell(1-\ell)}\right) ,$$

so

$$\frac{\delta^2}{6}\ell(1-\ell) \geq C\left(5R\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L^{3/2}} + 3(M_L-1)\frac{\log^2\left(2.77/u_\alpha^{(2)}\right)}{L^2}\right) + \frac{2R}{\sqrt{\beta L}} + 2\sqrt{\frac{2R}{\beta L}}|\delta|\sqrt{\ell(1-\ell)} \ . \quad (3.155)$$

Thus, with (3.149) and (3.152), the inequality (3.155) ensures that

$$E_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \geq C\left(5R\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L^{3/2}} + 3(M_L-1)\frac{\log^2\left(2.77/u_\alpha^{(2)}\right)}{L^2}\right) + \sqrt{2\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right]/\beta} \ . \quad (3.156)$$

Furthermore, the inequality

$$P_\lambda\left(N_1 \geq LI(\lambda) + \sqrt{\frac{2LI(\lambda)}{\beta}}\right) \leq \frac{\beta}{2} \ ,$$

combined with Lemma 3.20 and the upper bound $I(\lambda) \leq R$, leads to

$$P_\lambda\left(t'_{N_1,\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}\left(1-u_\alpha^{(2)}\right) \geq Q(\alpha,\beta,L,k_0')\right) \leq \frac{\beta}{2} \ , \quad (3.157)$$

with

$$Q(\alpha,\beta,L,k_0') = C\left(5R\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L} + 5\sqrt{\frac{2R}{\beta}}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{L^{3/2}} + 3\max\left(\frac{k_0'/M_L}{1-k_0'/M_L}, \frac{1-k_0'/M_L}{k_0'/M_L}\right)\frac{\log^2\left(2.77/u_\alpha^{(2)}\right)}{L^2}\right) \ . \quad (3.158)$$

Since $k_0'/M_L \leq \ell \leq 1/2$, one has

$$\max\left(\frac{k_0'/M_L}{1-k_0'/M_L}, \frac{1-k_0'/M_L}{k_0'/M_L}\right) = \frac{1-k_0'/M_L}{k_0'/M_L} = \frac{M_L}{k_0'} - 1 \leq M_L - 1 \ ,$$

and the inequality (3.156) leads to

$$E_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \geq Q(\alpha,\beta,L,k_0') + \sqrt{2\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right]/\beta} \ . \quad (3.159)$$

We conclude with the following inequalities

$$\begin{aligned}
&P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N) \leq t'_{N_1,\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}\left(1-u_\alpha^{(2)}\right)\right) \\
&\leq P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N) < Q(\alpha,\beta,L,k_0')\right) + P_\lambda\left(t'_{N_1,\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}\left(1-u_\alpha^{(2)}\right) \geq Q(\alpha,\beta,L,k_0')\right) \\
&\leq P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N) < Q(\alpha,\beta,L,k_0')\right) + \frac{\beta}{2} \quad \text{with (3.157)} \\
&\leq P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N) - E_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] < -\sqrt{2\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right]/\beta}\right) + \frac{\beta}{2} \quad \text{with (3.159)} \\
&\leq \beta \quad \text{with the Bienayme-Chebyshev inequality} \ .
\end{aligned}$$

Let us then consider the case where $\ell > 1/2$.
We define $k_0 = \max(k \in \{0,\dots,M_L-1\},\ \tau \geq k/M_L)$ and $k_0' = \min(k' \in \{1,\dots,M_L-k_0\},\ (k_0+k')/M_L \geq \tau+\ell)$ so that $k_0/M_L \leq \tau < \tau+\ell \leq (k_0+k_0')/M_L$.
Notice that the condition (3.146) entails

$$1-\ell > \frac{3}{M_L} \quad . \tag{3.160}$$

Since by definition of k_0 and k_0', $\tau - k_0/M_L < 1/M_L$ and $(k_0+k_0')/M_L - (\tau+\ell) < 1/M_L$, then, with (3.160),

$$\frac{k_0'}{M_L} = \frac{k_0+k_0'}{M_L} - (\tau+\ell) + \ell + \tau - \frac{k_0}{M_L} < 1/M_L + (1-3/M_L) + 1/M_L < 1 \quad . \tag{3.161}$$

In the same way, we can also get with (3.160)

$$1 - \frac{k_0'}{M_L} = 1-\ell - \left(\left(\tau - \frac{k_0}{M_L}\right) + \left(\frac{k_0+k_0'}{M_L} - (\tau+\ell)\right)\right) > 1-\ell-\frac{2}{M_L} > \frac{1-\ell}{3} \quad . \tag{3.162}$$

Furthermore, by definition of k_0 and k_0' again, one has $k_0'/M_L \geq \ell$, so

$$\frac{1-k_0'/M_L}{k_0'/M_L} \leq \frac{1-\ell}{\ell} \quad . \tag{3.163}$$

From Lemma 3.19, we deduce the expectation

$$E_\lambda\left[T'_{k_0/M_L,(k_0+k_0')/M_L}(N)\right] = \delta^2\ell^2(1-k_0'/M_L)/(k_0'/M_L) \quad ,$$

which satisfies with (3.162) and $\ell > 1/2$

$$E_\lambda\left[T'_{k_0/M_L,(k_0+k_0')/M_L}(N)\right] > \frac{\delta^2}{6}\ell(1-\ell) \quad .$$

Lemma 3.19 also gives the variance

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] = \frac{2}{L^2}\left(\lambda_0 + \delta\ell\frac{1-k_0'/M_L}{k_0'/M_L}\right)^2 + \frac{4}{L}\delta^2\ell^2\frac{1-k_0'/M_L}{k_0'/M_L}\left(\lambda_0+\delta\ell\frac{1-k_0'/M_L}{k_0'/M_L}\right) \quad ,$$

which satisfies with (3.163)

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \leq \left(\frac{2}{L^2}(\lambda_0+\delta)^2 + \frac{4}{L}\delta^2\ell(1-\ell)(\lambda_0+\delta)\right)\mathbb{1}_{\delta>0} + \left(\frac{2}{L^2}\lambda_0^2 + \frac{4}{L}\lambda_0\delta^2\ell(1-\ell)\right)\mathbb{1}_{\delta<0} \quad ,$$

thereby

$$\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \leq \frac{2R^2}{L^2} + \frac{4R}{L}\delta^2\ell(1-\ell) \quad .$$

Finally, since $k_0'/M_L \geq \ell > 1/2$, (3.161) leads to

$$\max\left(\frac{k_0'/M_L}{1-k_0'/M_L}, \frac{1-k_0'/M_L}{k_0'/M_L}\right) = \frac{k_0'/M_L}{1-k_0'/M_L} = \frac{k_0'}{M_L-k_0'} \leq M_L - 1 \quad .$$

As in the above case where $\ell \leq 1/2$, we can therefore prove that the assumption (3.145) leads to (3.159), that is

$$E_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right] \geq Q(\alpha,\beta,L,k_0') + \sqrt{2\mathrm{Var}_\lambda\left[T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N)\right]/\beta} \quad ,$$

with $Q(\alpha,\beta,L,k_0')$ is defined by (3.158). We then use the same final arguments as in this case, to conclude that

$$P_\lambda\left(T'_{\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}(N) \le t'_{N_1,\frac{k_0}{M_L},\frac{k_0+k_0'}{M_L}}\left(1-u_\alpha^{(2)}\right)\right) \le \beta \ .$$

Coming back to the assumption (3.145), we can thus affirm that

$$\mathrm{SR}_\beta\left(\phi^{u(2)}_{9/10,\alpha}, \mathcal{S}^u_{\cdot,\cdot\cdot,\ldots}[R]\right) \le \max\left(\sqrt{\frac{60CR\log\left(2.77/u_\alpha^{(2)}\right)}{L}} + \left(\frac{2R}{\beta}\right)^{1/4}\frac{\sqrt{60C\log\left(2.77/u_\alpha^{(2)}\right)}}{L^{3/4}} + 6\sqrt{C}\frac{\log\left(2.77/u_\alpha^{(2)}\right)}{\sqrt{L\log L}} \right.$$
$$\left. + 2\sqrt{\frac{6R}{L\sqrt{\beta}}},\ 24\sqrt{\frac{2R}{\beta L}},\ R\sqrt{\frac{3\log L}{L}}\right), \tag{3.164}$$

which, since $M_L = \lceil L/\log L\rceil$ and $u_\alpha^{(2)} = 2\alpha/(M_L(M_L+1)-2)$, leads to the result stated in Proposition 3.18 for $\phi^{u(2)}_{9/10,\alpha}$ and the set $\mathcal{S}^u_{\cdot,\cdot\cdot,\ldots}[R]$ of the alternative [$\mathbf{Alt^u.10}$].

Since $\mathcal{S}^u_{\delta^*,\cdot\cdot,\ldots}[R] \subset \mathcal{S}^u_{\cdot,\cdot\cdot,\ldots}[R]$ for any δ^* in $(-R,R)\setminus\{0\}$, the same result holds for $\phi^{u(2)}_{9/10,\alpha}$ and the set $\mathcal{S}^u_{\delta^*,\cdot\cdot,\ldots}[R]$ of the alternative [$\mathbf{Alt^u.9}$]. Again, notice that in this case, the constant $C(\alpha,\beta,R)$ could be refined thanks to the knowledge of δ^*, which justifies the formulation with $C(\alpha,\beta,R,\delta^*)$ instead of $C(\alpha,\beta,R)$ in Proposition 3.18.

IX Technical results for minimax separation rates upper bounds

Lemma 3.19 (Expectation and variance of $T'_{\tau_1,\tau_2}(N)$)**.**
Let τ_1 and τ_2 in $(0,1]$ such that $0<\tau_1<\tau_2\le 1$ and let N be a Poisson process such that $N(0,\tau_1]$, $N(\tau_1,\tau_2]$, and $N(\tau_2,1]$ follow a Poisson distribution with respective parameters $Lx>0$, $Ly>0$ and $Lz>0$. Considering the statistic T'_{τ_1,τ_2} defined by (3.9), one has

$$\mathbb{E}[T'_{\tau_1,\tau_2}(N)] = \left(\sqrt{\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}}(x+z) - \sqrt{\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}}y\right)^2 , \tag{3.165}$$

and

$$\mathrm{Var}(T'_{\tau_1,\tau_2}(N)) = \frac{2}{L^2}\left(\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(x+z) + \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}y\right)^2$$
$$+ \frac{4}{L}\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\right)^2\left(y - \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(x+z)\right)^2\left(\left(\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)^2(x+z)+y\right) . \tag{3.166}$$

Proof.
Notice that the statistic T' can be written as

$$T'_{\tau_1,\tau_2}(N) = \frac{1}{L^2}\left[\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\left((N(0,\tau_1]+N(\tau_2,1])^2 - (N(0,\tau_1]+N(\tau_2,1])\right)\right.$$
$$\left. + \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\left(N(\tau_1,\tau_2]^2 - N(\tau_1,\tau_2]\right) - 2N(\tau_1,\tau_2](N(0,\tau_1]+N(\tau_2,1])\right] .$$

Set, as in the proof of Lemma 2.26, m_i and $\bar{m}_i$ the moments of order i of $N(\tau_1,\tau_2]$ and $(N(0,\tau_1]+N(\tau_2,1])$ respectively, and c_i and $\bar{c}_i$ the corresponding centered moments of order i. Then, by independence of $N(0,\tau_1]$, $N(\tau_1,\tau_2]$ and $N(\tau_2,1]$, and since $m_2 = m_1^2 + m_1$, $\bar{m}_2 = \bar{m}_1^2 + \bar{m}_1$ with $m_1 = Ly$ and $\bar{m}_1 = L(x+z)$,

$$\begin{aligned}\mathbb{E}[T'_{\tau_1,\tau_2}(N)] &= \frac{1}{L^2}\left[\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(\bar{m}_2-\bar{m}_1) + \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}(m_2-m_1) - 2m_1\bar{m}_1\right] \\ &= \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(x+z)^2 + \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}y^2 - 2y(x+z)\ ,\end{aligned}$$

which gives (3.165). Moreover,

$$\begin{aligned}T'_{\tau_1,\tau_2}(N) = \frac{1}{L^2}\Bigg[&\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\Big((N(0,\tau_1]+N(\tau_2,1])^2 - (N(0,\tau_1]+N(\tau_2,1])\Big) \\ &+ \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\Big(N(\tau_1,\tau_2]^2 - N(\tau_1,\tau_2]\Big) - 2N(\tau_1,\tau_2](N(0,\tau_1]+N(\tau_2,1])\Bigg]\ ,\end{aligned}$$

and then

$$\begin{aligned}T'_{\tau_1,\tau_2}(N) - \mathbb{E}[T'_{\tau_1,\tau_2}(N)] = \frac{1}{L^2}\Bigg[&\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\Big((N(0,\tau_1]+N(\tau_2,1]-\bar{m}_1)^2 + (2\bar{m}_1-1)(N(0,\tau_1]+N(\tau_2,1]-\bar{m}_1) - \bar{m}_1\Big) \\ &+ \frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\Big((N(\tau_1,\tau_2]-m_1)^2 + (2m_1-1)(N(\tau_1,\tau_2]-m_1) - m_1\Big) \\ &- 2(N(\tau_1,\tau_2]-m_1)\Big((N(0,\tau_1]+N(\tau_2,1]-\bar{m}_1)+\bar{m}_1\Big) - 2m_1\,(N(0,\tau_1]+N(\tau_2,1]-\bar{m}_1)\Bigg]\ .\end{aligned}$$

This entails

$$\begin{aligned}\mathrm{Var}(T'_{\tau_1,\tau_2}(N)) = \frac{1}{L^4}\Bigg[&\left(\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)^2\Big(\bar{c}_4 + (2\bar{m}_1-1)^2\bar{c}_2 + \bar{m}_1^2 + 2(2\bar{m}_1-1)\bar{c}_3 - 2\bar{m}_1\bar{c}_2\Big) \\ &+ \left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\right)^2\Big(c_4 + (2m_1-1)^2c_2 + m_1^2 + 2(2m_1-1)c_3 - 2m_1c_2\Big) \\ &+ 4\Big(c_2\bar{m}_2 + m_1^2\bar{c}_2\Big) + 2(c_2-m_1)(\bar{c}_2-\bar{m}_1) - 4m_1\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(\bar{c}_3 + (2\bar{m}_1-1)\bar{c}_2) \\ &- 4\bar{m}_1\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}(c_3 + (2m_1-1)c_2)\Bigg]\ .\end{aligned}$$

This leads using Lemma 2.24 to

$$\begin{aligned}\mathrm{Var}(T'_{\tau_1,\tau_2}(N)) = \frac{1}{L^4}\Bigg[&\left(\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)^2\Big(4L^3(x+z)^3 + 2L^2(x+z)^2\Big) + \left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\right)^2\Big(4L^3y^3 + 2L^2y^2\Big) \\ &+ 4\Big(Ly(L^2(x+z)^2 + L(x+z)) + L^2y^2L(x+z)\Big) - 4Ly\frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}(L(x+z) + (2L(x+z)-1)L(x+z)) \\ &- 4L(x+z)\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}(Ly + (2Ly-1)Ly)\Bigg]\ .\end{aligned}$$

The second statement of Lemma 3.19 given in (3.166) finally follows from direct computations. □

Lemma 3.20 (Conditional quantile bound for $T'_{\tau_1,\tau_2}(N)$).
Assume that N is a homogeneous Poisson process with a constant intensity λ_0 with respect to the measure $\Lambda = Ldt$. For τ_1 and τ_2 in $(0,1)$ such that $0 < \tau_1 < \tau_2 \leq 1$, u in $(0,1)$ and n in $\mathbb{N}$, let $t'_{n,\tau_1,\tau_2}(1-u)$ the $(1-u)$-quantile of the conditional distribution of $T'_{\tau_1,\tau_2}(N)$ defined by (3.9) given the event $\{N_1 = n\}$. Then

$$t'_{n,\tau_1,\tau_2}(1-u) \leq \frac{C}{L^2}\left(5n\log\left(\frac{2.77}{u}\right) + 3\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}, \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)\log^2\left(\frac{2.77}{u}\right)\right) ,$$

with $C = \min_{\varepsilon>0}\left(e(1+\varepsilon^{-1})^2(2.5+32\varepsilon^{-1}\right) + \left((2\sqrt{2}(2+\varepsilon+\varepsilon^{-1})) \vee ((1+\varepsilon)^2)/\sqrt{2}\right)$.

Proof.
For $n \geq 2$ and conditionally on the event $\{N_1 = n\}$, the points of the process N obey the same law as a n-sample $(U_1, \ldots, U_n)$ of i.i.d. random variables uniformly distributed on $(0,1)$. $t'_{n,\tau_1,\tau_2}(1-u)$ is thus equal to the $(1-u)$-quantile of the following U-statistic of order 2

$$T'_{n,L,\tau^1,\tau^2} = \frac{1}{L^2}\sum_{i\neq j=1}^{n} \psi_{\tau_1,\tau_2}(U_i)\psi_{\tau_1,\tau_2}(U_j) = \sum_{i=2}^{n}\sum_{j=1}^{i-1} H_{L,\tau_1,\tau_2}(U_i,U_j) ,$$

where $H_{L,\tau_1,\tau_2}(x,y) = 2\psi_{\tau_1,\tau_2}(x)\psi_{\tau_1,\tau_2}(y)/L^2$ for any x and y in $[0,1]$.
Since for all $0 \leq \tau_1 < \tau_2 \leq 1$, ψ_{τ_1,τ_2} is orthogonal to ψ_0 (in $\mathbb{L}_2([0,1])$), the variables $\psi_{\tau_1,\tau_2}(U_i)$ are centered and we can apply Theorem 3.4 in [HRB03]. We obtain that there exists some absolute constant $C > 0$ such that for all $x > 0$ and for all $n \geq 2$

$$\mathbb{P}\left(T'_{n,L,\tau^1,\tau^2} \geq C(A_1\sqrt{x} + A_2 x + A_3 x^{3/2} + A_4 x^2)\right) \leq 2.77e^{-x} ,$$

where

$$\begin{aligned}
A_1^2 &= n^2\mathbb{E}[H_{L,\tau_1,\tau_2}(U_1,U_2)^2] , \\
A_2 &= \sup\left(\left|\mathbb{E}\left[\sum_{i=2}^{n}\sum_{j=1}^{i-1} H_{L,\tau_1,\tau_2}(U_i,U_j)f_i(U_i)g_j(U_j)\right]\right| ,\right. \\
&\qquad \left.\mathbb{E}\left[\sum_{i=2}^{n} f_i^2(U_i)\right] \leq 1,\ \mathbb{E}\left[\sum_{j=1}^{n-1} g_j^2(U_j)\right] \leq 1,\ f_i \text{ and } g_i \text{ Borel measurable functions}\right) , \\
A_3^2 &= n \sup_{y\in[0,1]} \int_0^1 H_{L,\tau_1,\tau_2}^2(x,y)dx , \\
A_4 &= \sup_{x,y\in[0,1]} |H_{L,\tau_1,\tau_2}(x,y)| .
\end{aligned}$$

From Theorem 3.4 in [HRB03], notice that the constant C can be taken equal to

$$C = \min_{\varepsilon>0}\left(e(1+\varepsilon^{-1})^2(2.5+32\varepsilon^{-1}\right) + \left((2\sqrt{2}(2+\varepsilon+\varepsilon^{-1})) \vee ((1+\varepsilon)^2/\sqrt{2})\right) .$$

Let us now evaluate A_1, A_2, A_3 and A_4.
Since the function ψ_{τ_1,τ_2} has a $\mathbb{L}_2$-norm equal to 1 and the U_i's are independent, we get

$$A_1^2 = \frac{4n^2}{L^4}\mathbb{E}\left[\psi_{\tau_1,\tau_2}^2(U_1)\right]^2 = \frac{4n^2}{L^4} .$$

The independence of the U_i's and the Cauchy-Schwarz inequality applied twice also yields

$$\begin{aligned}
&\left|\mathbb{E}\left[\sum_{i=2}^{n}\sum_{j=1}^{i-1} H_{L,\tau_1,\tau_2}(U_i,U_j)f_i(U_i)g_j(U_j)\right]\right| \\
&\quad \le \frac{2}{L^2}\sum_{i=2}^{n}\sum_{j=1}^{i-1}\left|\int_0^1 \psi_{\tau_1,\tau_2}(x)f_i(x)dx\right|\left|\int_0^1 \psi_{\tau_1,\tau_2}(y)g_j(y)dy\right| \\
&\quad \le \frac{2}{L^2}\sum_{i=2}^{n}\sqrt{\int_0^1 f_i^2(x)dx}\sum_{j=1}^{n-1}\sqrt{\int_0^1 g_j^2(y)dy} \\
&\quad \le \frac{2(n-1)}{L^2}\sqrt{\sum_{i=2}^{n}\underbrace{\int_0^1 f_i^2(x)dx}_{=\mathbb{E}[f_i^2(U_i)]}}\sqrt{\sum_{j=1}^{n-1}\underbrace{\int_0^1 g_j(y)^2dy}_{=\mathbb{E}[g_j^2(U_j)]}} \ ,
\end{aligned}$$

hence $A_2 \le 2n/L^2$. Moreover,

$$\begin{aligned}
A_3^2 &= \frac{4n}{L^4}\sup_{y\in[0,1]}\int_0^1 \psi^2_{\tau_1,\tau_2}(x)\psi^2_{\tau_1,\tau_2}(y)dx \\
&= \frac{4n}{L^4}\sup_{y\in[0,1]}\psi^2_{\tau_1,\tau_2}(y) \\
&= \frac{4n}{L^4}\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right) \ .
\end{aligned}$$

To conclude,

$$A_4 = \frac{2}{L^2}\left(\sup_{x\in[0,1]}|\psi_{\tau_1,\tau_2}(x)|\right)^2 = \frac{2}{L^2}\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right) \ .$$

We finally obtain for all $x>0$ and for all $n\ge 2$

$$\begin{aligned}
\mathbb{P}\Bigg(T'_{n,L,\tau^1,\tau^2} \ge \frac{C}{L^2}\Bigg(2n\sqrt{x} + 2nx + 2\sqrt{n\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)}x^{3/2} \\
+ 2\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)x^2\Bigg)\Bigg) \le 2.77e^{-x} \ .
\end{aligned}$$

Then notice that for all $x>0$

$$2\sqrt{n\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)}x^{3/2} \le nx + \max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)x^2,$$

whereby

$$\mathbb{P}\left(T'_{n,L,\tau^1,\tau^2} \ge \frac{C}{L^2}\left(2n\sqrt{x} + 3nx + 3\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}\ ,\ \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)x^2\right)\right) \le 2.77e^{-x} \ . \tag{3.167}$$

By convention, (3.167) also holds for n in $\{0,1\}$ such that $T'_{n,L,\tau^1,\tau^2}=0$. We then obtain for all n in $\mathbb{N}$ and $x=\log(2.77/u)$ (which satisfies $x\ge 1$ for all u in $(0,1)$)

$$\mathbb{P}\left(T'_{n,L,\tau^1,\tau^2} \ge \frac{C}{L^2}\left(5n\log\left(\frac{2.77}{u}\right) + 3\max\left(\frac{1-\tau_2+\tau_1}{\tau_2-\tau_1}, \frac{\tau_2-\tau_1}{1-\tau_2+\tau_1}\right)\log^2\left(\frac{2.77}{u}\right)\right)\right) \le u \ .$$

This allows to end the proof. □

Lemma 3.21 (Conditional quantile bound for $\max/\min_{\tau\in[0,1-\ell]} N(\tau,\tau+\ell]$).
Let $L \geq 1$ and $0<\ell<1$. For any n in $\mathbb{N}\setminus\{0\}$, the $(1-\alpha)$-quantile $b^+_{n,\ell}(1-\alpha)$ of the conditional distribution of $\max_{\tau\in[0,1-\ell]} N(\tau,\tau+\ell]$ given $N_1=n$ under (H_0) satisfies

$$b^+_{n,\ell}(1-\alpha) \leq \ell n + \frac{n}{2} g^{-1}\left(\frac{32}{n}\log\left(\frac{320}{\alpha}\right)\right) ,$$

where g is defined by (2.44). Moreover, $b^+_{0,\ell}(1-\alpha)=0$.
For any n in $\mathbb{N}$, the α-quantile $b^-_{n,\ell}(\alpha)$ of the conditional distribution of $\min_{\tau\in[0,1-\ell]} N(\tau,\tau+\ell]$ given $N_1=n$ under (H_0) satisfies

$$b^-_{n,\ell}(\alpha) \geq \ell n - 4\sqrt{2n\log\left(\frac{320}{\alpha}\right)} .$$

Proof.
Let n in $\mathbb{N}\setminus\{0\}$. If $U_1,\ldots,U_n$ denote independent uniform random variables on $(0,1)$, let us consider the empirical process $\mathbb{U}_n$ associated with these random variables, defined for $0\leq t\leq 1$ by

$$\mathbb{U}_n(t) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i\leq t} - t\right) .$$

Let ℓ in $(0,1/2]$ and $x>0$ satisfying

$$x \geq n\ell + \frac{n}{2} g^{-1}\left(\frac{32}{n}\log\left(\frac{320}{\alpha}\right)\right) . \tag{3.168}$$

Then, we may compute for all λ_0 in $(0,R)$

$$\begin{aligned}
P_{\lambda_0}\left(\max_{t\in[0,1-\ell]} N(t,t+\ell] > x \,\middle|\, N_1=n\right) &= P_{\lambda_0}\left(\max_{t\in[0,1-\ell]}\left(\sum_{i=1}^n \mathbb{1}_{X_i\leq t+\ell} - \sum_{i=1}^n \mathbb{1}_{X_i\leq t}\right) > x \,\middle|\, N_1=n\right)\\
&= P_{\lambda_0}\left(\max_{t\in[0,1-\ell]} \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i\leq t+\ell} - \frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i\leq t} - \ell\right) > \sqrt{n}\left(\frac{x}{n}-\ell\right)\middle| N_1=n\right)\\
&\leq P_{\lambda_0}\left(\max_{t\in[0,1-\ell]} |\mathbb{U}_n(t+\ell)-\mathbb{U}_n(t)| > \sqrt{n}\left(\frac{x}{n}-\ell\right)\middle| N_1=n\right)\\
&\leq P_{\lambda_0}\left(\max_{0\leq z\leq\frac{1}{2}}\max_{t\in[0,1-z]} |\mathbb{U}_n(t+z)-\mathbb{U}_n(t)| > \sqrt{n}\left(\frac{x}{n}-\ell\right)\middle| N_1=n\right) .
\end{aligned}$$

We shall use now an inequality for controlling the oscillations of the empirical process which is due to Mason, Shorack and Wellner, and which can be found in Chapter 14 of [SW86] page 545 under the name of Inequality 1. We get

$$\begin{aligned}
\sup_{\lambda_0\in(0,R)} P_{\lambda_0}\left(\max_{t\in[0,1-\ell]} N(t,t+\ell] > x \,\middle|\, N_1=n\right) &\leq 320\exp\left(-\frac{n}{32}g\left(2\left(\frac{x}{n}-\ell\right)\right)\right)\\
&\leq \alpha \text{ using (3.168)} .
\end{aligned}$$

This yields $b^+_{n,\ell}(1-\alpha) \leq n\ell + ng^{-1}(32\log(320/\alpha)/n)/2$ for every n in $\mathbb{N}\setminus\{0\}$, which is the first statement of Lemma 3.21. The fact that $b^+_{0,\ell}(1-\alpha)=0$ is obvious.
Now let again n in $\mathbb{N}\setminus\{0\}$, $\varepsilon>0$ and y in $\mathbb{R}$ satisfying

$$y \leq n\ell - 4\sqrt{2n\log\left(\frac{320}{\alpha-\varepsilon}\right)} . \tag{3.169}$$

We compute for all λ_0 in $(0,R)$

$$\begin{aligned}
P_{\lambda_0}\left(\min_{t\in[0,1-\ell]} N(t,t+\ell]\leq y \,\middle|\, N_1=n\right) &= P_{\lambda_0}\left(\min_{t\in[0,1-\ell]}(\mathbb{U}_n(t+\ell)-\mathbb{U}_n(t))\leq\sqrt{n}\left(\frac{y}{n}-\ell\right)\,\middle|\, N_1=n\right)\\
&= P_{\lambda_0}\left(\max_{t\in[0,1-\ell]}(\mathbb{U}_n(t)-\mathbb{U}_n(t+\ell))\geq\sqrt{n}\left(\ell-\frac{y}{n}\right)\,\middle|\, N_1=n\right)\\
&\leq P_{\lambda_0}\left(\max_{0\leq z\leq\frac{1}{2}}\max_{t\in[0,1-z]}(\mathbb{U}_n(t)-\mathbb{U}_n(t+z))\geq\sqrt{n}\left(\ell-\frac{y}{n}\right)\,\middle|\, N_1=n\right)\ .
\end{aligned}$$

Applying now the second part of Inequality 1 page 545 in [SW86], we obtain

$$\begin{aligned}
\sup_{\lambda_0\in(0,R)} P_{\lambda_0}\left(\min_{t\in[0,1-\ell]} N(t,t+\ell]\leq y \,\middle|\, N_1=n\right) &\leq 320\exp\left(-\frac{n}{32}\left(\ell-\frac{y}{n}\right)^2\right)\\
&\leq \alpha-\varepsilon<\alpha \ \text{ using (3.169)}\ .
\end{aligned}$$

This entails $b^-_{n,\ell}(\alpha) > n\ell - 4\sqrt{2n\log(320/(\alpha-\varepsilon))}$ for all $\varepsilon>0$. With ε tending to 0, we get $b^-_{n,\ell}(\alpha)\geq n\ell-4\sqrt{2n\log(320/\alpha)}$, which leads to the second statement of Lemma 3.21 with the obvious fact that $b^-_{0,\ell}(\alpha)=0$. □

Lemma 3.22 (Conditional quantile bound for $\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$).
Let $L\geq 1$ and $n_0\geq 1$. For all $0\leq n\leq n_0L$, the $(1-\alpha)$-quantile $s'^{+}_{n,\delta^*,\tau^*,L}(1-\alpha)$ of the conditional distribution of $\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$ given $N_1=n$ under (H_0) with $S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$ defined by (3.18) satisfies

$$s'^{+}_{n,\delta^*,\tau^*,L}(1-\alpha)\leq Q(n_0,\delta^*,\alpha)\ ,$$

where

$$Q(n_0,\delta^*,\alpha)=\log(2)\frac{(6n_0+|\delta^*|)(6n_0+2|\delta^*|/3)}{\delta^{*2}}+\log\left(\frac{\pi^2}{3\alpha}\right)\frac{9n_0+|\delta^*|}{3|\delta^*|}\ .$$

Proof.
First recall that (see (3.18))

$$S'_{\delta^*,\tau^*,\tau^*+\ell}(N)=\mathrm{sgn}(\delta^*)\Big(N(\tau^*,\tau^*+\ell]-\ell N_1\Big)-|\delta^*|\ell(1-\ell)L/2\ .$$

Since N is an homogeneous Poisson process, the processes $(N_1,N(\tau^*,\tau^*+\ell])_{\ell\in(0,1-\tau^*]}$ and $(N_1,N(0,\ell])_{\ell\in(0,1-\tau^*]}$ have the same finite dimensional law and since N is a right continuous process, we get that $\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$ is distributed as

$$\sup_{\ell\in(0,1-\tau^*)}\left(\mathrm{sgn}(\delta^*)(N(0,\ell]-\ell N_1)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)\ .$$

As seen above, for $n\geq 1$ and conditionally on the event $\{N_1=n\}$, the points of the process N obey the same law as a n-sample $(U_1,\ldots,U_n)$ of i.i.d. random variables uniformly distributed on $(0,1)$. Therefore, considering the empirical distribution function F_n associated with this sample defined for $0\leq t\leq 1$ by

$$F_n(t)=\frac{1}{n}\sum_{i=1}^n \mathbb{1}_{X_i\leq t}\ ,$$

conditionally on the event $\{N_1 = n\}$, $\sup_{\ell\in(0,1-\tau^*)} S'_{\delta^*,\tau^*,\tau^*+\ell}(N)$ is distributed as

$$\sup_{\ell\in(0,1-\tau^*)}\left(\mathrm{sgn}(\delta^*)\Big(nF_n(\ell)-n\ell\Big)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right) .$$

Notice first that for all $n \geq 1$ and for all $x > 0$

$$\mathbb{P}\left(\sup_{\ell\in(0,1-\tau^*)}\left(\mathrm{sgn}\,(\delta^*)\,(nF_n(\ell)-n\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)\leq\mathbb{P}\left(\sup_{\ell\in(0,1)}\left(\mathrm{sgn}\,(\delta^*)\,(nF_n(\ell)-n\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right) .$$

Moreover, since the process $\Big(-(nF_n(\ell)-n\ell)\Big)_{\ell\in(0,1)}$ has the same distribution as the process $\Big(nF_n(1-\ell)-n(1-\ell)\Big)_{\ell\in(0,1)}$, one has when $\delta^* < 0$

$$\begin{aligned}\mathbb{P}\left(\sup_{\ell\in(0,1)}\left(\mathrm{sgn}\,(\delta^*)\,(nF_n(\ell)-n\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)&=\mathbb{P}\left(\sup_{\ell\in(0,1)}\left(nF_n(1-\ell)-n(1-\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)\\&=\mathbb{P}\left(\sup_{\ell\in(0,1)}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right) .\end{aligned}$$

Therefore, whatever the sign of δ^*,

$$\mathbb{P}\left(\sup_{\ell\in(0,1-\tau^*)}\left(\mathrm{sgn}\,(\delta^*)\,(nF_n(\ell)-n\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)\leq\mathbb{P}\left(\sup_{\ell\in(0,1)}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right) ,$$

and

$$\begin{aligned}\mathbb{P}\left(\sup_{\ell\in(0,1-\tau^*)}\left(\mathrm{sgn}\,(\delta^*)\,(nF_n(\ell)-n\ell)-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)&\leq\mathbb{P}\left(\sup_{\ell\in(0,1/2]}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right)\\&+\mathbb{P}\left(\sup_{\ell\in(1/2,1)}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>x\right) .\end{aligned}\tag{3.170}$$

Let us now prove that on the one hand

$$\mathbb{P}\left(\sup_{\ell\in(0,1/2]}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>Q(n_0,\delta^*,\alpha)\right)\leq\frac{\alpha}{2} ,\tag{3.171}$$

and on the other hand

$$\mathbb{P}\left(\sup_{\ell\in(1/2,1)}\left(nF_n(\ell)-n\ell-\frac{|\delta^*|}{2}\ell(1-\ell)L\right)>Q(n_0,\delta^*,\alpha)\right)\leq\frac{\alpha}{2} .\tag{3.172}$$

Let $1 \leq n \leq n_0 L$. Set $C(\delta^*, n_0) = |\delta^*|/(6n_0)$ and $C'(\delta^*, n_0, n) = \left\lfloor\exp\left(\frac{3n}{2}\frac{C(\delta^*,n_0)^2}{3+2C(\delta^*,n_0)}\right)\right\rfloor$.
For all k in $\{0,\ldots,C'(\delta^*,n_0,n)-1\}$, we define

$$\ell_k=\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\frac{\log(k+1)}{n} ,$$

and

$$\ell_{C'(\delta^*,n_0,n)}=\left(\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\frac{\log(C'(\delta^*,n_0,n)+1)}{n}\right)\wedge 1 .$$

Notice that with such definitions, ℓ_k belongs to $[0,1/2]$ for all k in $\{0,\dots,C'(\delta^*,n_0,n)-1\}$ and $\ell_{C'(\delta^*,n_0,n)}$ belongs to $(1/2,1]$.
Applying Bernstein's inequality, as stated in equation (2.6) in [BDR15][page 12], one obtains for every $x>0$ and every k in $\{1,\dots,C'(\delta^*,n_0,n)\}$

$$\mathbb{P}\left(nF_n(\ell_k) > n\ell_k + \sqrt{2n\ell_k(1-\ell_k)(x+2\log k)} + \frac{x+2\log k}{3}\right) \le \frac{e^{-x}}{k^2} \ .$$

A union bound therefore gives for every $x>0$

$$\mathbb{P}\left(\forall k\in\{1,\dots,C'(\delta^*,n_0,n)\},\ nF_n(\ell_k) \le n\ell_k + \sqrt{2n\ell_k(1-\ell_k)(x+2\log k)} + \frac{x+2\log k}{3}\right) \ge 1-\frac{\pi^2}{6}e^{-x} \ .$$

Using the inequality $\sqrt{2ab} \le aC(\delta^*,n_0) + b/(2C(\delta^*,n_0))$ and the fact that $\ell \mapsto nF_n(\ell)$ is nondecreasing, we get

$$\mathbb{P}\left(\forall k\in\{1,\dots,C'(\delta^*,n_0,n)\},\ \forall \ell\in(\ell_{k-1},\ell_k],\ nF_n(\ell) \le n\ell_k + C(\delta^*,n_0)n\ell_k(1-\ell_k) + \frac{x+2\log k}{2C(\delta^*,n_0)} + \frac{x+2\log k}{3}\right)$$
$$\ge 1-\frac{\pi^2}{6}e^{-x} \ .$$

Therefore, with probability larger than $1-\pi^2 e^{-x}/6$, for all k in $\{1,\dots,C'(\delta^*,n_0,n)\}$ and all ℓ in $(\ell_{k-1},\ell_k]$,

$$\begin{aligned}
nF_n(\ell) \le\,& n\ell_k + C(\delta^*,n_0)n\ell_k(1-\ell_k) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + \log k\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)}\\
=\,& n\ell_k + C(\delta^*,n_0)n\ell_k(1-\ell_k) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + C(\delta^*,n_0)n\ell_{k-1}\\
<\,& n\ell_k + C(\delta^*,n_0)n\ell_k(1-\ell_k) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + C(\delta^*,n_0)n\ell\\
=\,& n\ell + C(\delta^*,n_0)n\ell(1-\ell) + C(\delta^*,n_0)n\ell + n(\ell_k-\ell) + C(\delta^*,n_0)n\left(\ell_k(1-\ell_k)-\ell(1-\ell)\right) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)}\\
\le\,& n\ell + C(\delta^*,n_0)n\ell(1-\ell) + C(\delta^*,n_0)n\ell + n(\ell_k-\ell) + C(\delta^*,n_0)n(1-\ell)(\ell_k-\ell) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)}\\
<\,& n\ell + C(\delta^*,n_0)n\ell(1-\ell) + C(\delta^*,n_0)n\ell + (1+C(\delta^*,n_0)(1-\ell))\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\log 2 + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} \ .
\end{aligned}$$

Finally, we have proved that with probability larger than $1-\pi^2e^{-x}/6$, for all ℓ in $(0,1/2]$,

$$nF_n(\ell) \le n\ell + 3C(\delta^*,n_0)n_0L\ell(1-\ell) + (1+C(\delta^*,n_0))\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\log 2 + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} \ ,$$

hence

$$\mathbb{P}\left(\sup_{\ell\in(0,1/2]}\left(nF_n(\ell) - n\ell - \frac{|\delta^*|}{2}\ell(1-\ell)L\right) \le Q(n_0,\delta^*,\alpha)\right) \ge 1-\frac{\alpha}{2} \ ,$$

that is (3.171).
Define now $\ell'_k = 1-\ell_k$ for all k in $\{0,\dots,C'(\delta^*,n_0,n)\}$ and notice that ℓ'_k belongs to $[1/2,1]$ for all k in $\{0,\dots,C'(\delta^*,n_0,n)-1\}$ and $\ell'_{C'(\delta^*,n_0,n)}$ belongs to $[0,1/2)$.
Applying Bernstein's inequality again, one obtains for every $x>0$

$$\mathbb{P}\left(\forall k\in\{1,\dots,C'(\delta^*,n_0,n)\},\ nF_n(\ell'_{k-1}) \le n\ell'_{k-1} + \sqrt{2n\ell'_{k-1}(1-\ell'_{k-1})(x+2\log k)} + \frac{x+2\log k}{3}\right) \ge 1-\frac{\pi^2}{6}e^{-x} \ .$$

With the same computations as in the above case, we get

$$\mathbb{P}\left(\forall k \in \{1,\ldots,C'(\delta^*,n_0,n)\}, \forall \ell \in [\ell'_k,\ell'_{k-1}),\ nF_n(\ell) \le n\ell'_{k-1} + C(\delta^*,n_0)n\ell'_{k-1}(1-\ell'_{k-1}) + \frac{x+2\log k}{2C(\delta^*,n_0)} + \frac{x+2\log k}{3} \right) \ge 1 - \frac{\pi^2}{6}e^{-x} \ .$$

Therefore, with probability larger than $1-\pi^2 e^{-x}/6$, for all k in $\{1,\ldots,C'(\delta^*,n_0,n)\}$ and all ℓ in $[\ell'_k,\ell'_{k-1})$,

$$\begin{aligned}
nF_n(\ell) \le & n\ell'_{k-1} + C(\delta^*,n_0)n\ell'_{k-1}(1-\ell'_{k-1}) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + \log k \frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)} \\
= & n\ell'_{k-1} + C(\delta^*,n_0)n\ell'_{k-1}(1-\ell'_{k-1}) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + C(\delta^*,n_0)n(1-\ell'_{k-1}) \\
< & n\ell'_{k-1} + C(\delta^*,n_0)n\ell'_{k-1}(1-\ell'_{k-1}) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} + C(\delta^*,n_0)n(1-\ell) \\
< & n\ell + C(\delta^*,n_0)n\ell(1-\ell) + C(\delta^*,n_0)n(1-\ell) + n(\ell'_{k-1}-\ell) + C(\delta^*,n_0)n(1-\ell)\left(\ell'_{k-1}-\ell\right) + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} \\
\le & n\ell + C(\delta^*,n_0)n\ell(1-\ell) + C(\delta^*,n_0)n(1-\ell) + (1+C(\delta^*,n_0)(1-\ell))\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\log 2 + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} \ .
\end{aligned}$$

Finally, we have proved that with probability larger than $1-\pi^2 e^{-x}/6$, for all ℓ in $[1/2,1)$,

$$nF_n(\ell) \le n\ell + 3C(\delta^*,n_0)n_0L\ell(1-\ell) + (1+C(\delta^*,n_0))\frac{3+2C(\delta^*,n_0)}{3C(\delta^*,n_0)^2}\log 2 + x\frac{3+2C(\delta^*,n_0)}{6C(\delta^*,n_0)} \ ,$$

hence

$$\mathbb{P}\left(\sup_{\ell\in[1/2,1)} \left(nF_n(\ell) - n\ell - \frac{|\delta^*|}{2}\ell(1-\ell)L \right) \le Q(n_0,\delta^*,\alpha) \right) \ge 1 - \frac{\alpha}{2} \ ,$$

that is (3.172). Combined with (3.170), the equations (3.171) and (3.172) conclude the proof using the obvious fact that $s'^{+}_{0,\delta^*,\tau^*,L}(1-\alpha)=0$. □

Lemma 3.23 (Conditional quantile bounds for $N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1$).
Let $L\ge 1$, u in $(0,1)$, τ_1 and τ_2 such that $0<\tau_1<\tau_2\le 1$ and n in $\mathbb{N}$. The u-quantile $\bar{b}_{n,\tau_2-\tau_1}(u)$ of the conditional distribution of $N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1$ given $N_1=n$ under (H_0) satisfies

$$-\frac{2}{3}\log(1/u) - \sqrt{n(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\sqrt{2\log(1/u)} \le \bar{b}_{n,\tau_2-\tau_1}(u) \le \frac{2}{3}\log(1/(1-u)) + \sqrt{n(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\sqrt{2\log(1/(1-u))} \ ,$$

Proof.
Let $n\ge 1$. Under (H_0) and conditionally on $N_1=n$, $N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1$ follows a recentered binomial distribution with parameters $(n,\tau_2-\tau_1)$. Applying Bennett's inequality as stated in Theorem 2.28 in [BDR15] to $\sum_{i=1}^n (X_i-(\tau_2-\tau_1))$ where $(X_1,\ldots,X_n)$ is a sample of i.i.d. random variables with a Bernoulli distribution with parameter $\tau_2-\tau_1$, we obtain for all $x>0$

$$\sup_{\lambda_0\in\mathcal{S}_0^u[R]} P_{\lambda_0}\left(N(\tau_1,\tau_2]-(\tau_2-\tau_1)N_1 > x\,|N_1=n\right) \le \exp\left(-n(\tau_2-\tau_1)(1-\tau_2+\tau_1)g\left(\frac{x}{n(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\right)\right) \ .$$

It directly follows that

$$\bar{b}_{n,\tau_2-\tau_1}(1-u) \leq n(\tau_2-\tau_1)(1-\tau_2+\tau_1)g^{-1}\left(\frac{\log(1/u)}{n(\tau_2-\tau_1)(1-\tau_2+\tau_1)}\right) .$$

The upper bound (2.45) for g^{-1} and the fact that $\bar{b}_{0,\tau_2-\tau_1}(1-u) = 0$ allow to conclude for the first part of Lemma 3.23. Then, following similar arguments, but applying Bennett's inequality to $\sum_{i=1}^{n}((\tau_2-\tau_1)-X_i)$ and using the continuity of g^{-1} as in Lemma 2.25, one obtains the second part of Lemma 3.23. □

Lemma 3.24 (Conditional quantile bound for $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$).
Let $L \geq 1$ and $n_0 \geq 1$. For all $0 \leq n \leq n_0 L$, the $(1-\alpha)$-quantile $s'^{+}_{n,\delta^*,L}(1-\alpha)$ of the conditional distribution of $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$ given $N_1 = n$ under (H_0) with $S'_{\delta^*,\tau,1}(N)$ defined by (3.18) satisfies

$$s'^{+}_{n,\delta^*,L}(1-\alpha) \leq Q(n_0,\delta^*,\alpha) ,$$

where $Q(n_0,\delta^*,\alpha)$ is defined in Lemma 3.22.

Proof.
Notice first that $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$ is distributed as

$$\sup_{\tau\in(0,1)}\left(\mathrm{sgn}(\delta^*)(N(0,1-\tau]-(1-\tau)N_1) - \frac{|\delta^*|}{2}\tau(1-\tau)L\right) .$$

Now, recall that for $n \geq 1$ and conditionally on the event $\{N_1 = n\}$, the points of the process N obey the same law as a n-sample $(U_1,\ldots,U_n)$ of i.i.d. random variables uniformly distributed on $(0,1)$ and consider the empirical distribution function F_n associated with this sample defined for $0 \leq t \leq 1$ as in the proof of Lemma 3.22. Then, conditionally on the event $\{N_1 = n\}$, $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$ is distributed as

$$\sup_{\tau\in(0,1)}\left(\mathrm{sgn}(\delta^*)\Big(nF_n(1-\tau)-n(1-\tau)\Big) - \frac{|\delta^*|}{2}\tau(1-\tau)L\right) .$$

Since the process $(-(nF_n(1-\tau)-n(1-\tau)))_{\tau\in(0,1)}$ has the same distribution as the process $(nF_n(\tau)-n\tau)_{\tau\in(0,1)}$, whatever the sign of δ^*, $\sup_{\tau\in(0,1)} S'_{\delta^*,\tau,1}(N)$ is distributed as

$$\sup_{\tau\in(0,1)}\left(nF_n(\tau)-n\tau - \frac{|\delta^*|}{2}\tau(1-\tau)L\right) .$$

The proof now follows the same line as the proof of Lemma 3.22, just replacing ℓ by τ and using the fact that $s'^{+}_{0,\delta^*,L}(1-\alpha) = 0$. □

CHAPTER
FOUR

MINIMAX AND ADAPTIVE MULTIPLE TESTS FOR DETECTING AND LOCALISING AN ABRUPT CHANGE IN A POISSON PROCESS WITH A KNOWN BASELINE INTENSITY

In this chapter, we focus on the question of simultaneously detecting and localizing a single change point in the intensity of the Poisson process N when its baseline is assumed to be known, equal to a positive constant function λ_0 on $[0,1]$, and on the construction of change point estimation procedures. The results of this chapter go further than the minimax study done in Chapter 2 since it also consider the change point localisation in addition to its detection. We formulate this change point localization problem as a multiple testing problem, and we present a nonasymptotic minimax study.

As in the previous chapters, we consider various sets of intensities that are defined according to the change height knowledge. For each sets, lower bounds for minimax family-wise separation rates are provided, as a preliminary basis for corresponding upper bounds. As explained in Chapter 1, these upper bounds are obtained by constructing minimax or minimax adaptive multiple tests, which are based on aggregation of either linear or quadratic statistics, coupled with adjusted critical values. In Chapter 4 Section III, we draw a parallel between minimax multiple testing procedures and confidence intervals for the jump localisation and when appropriate, it is shown that the confidence intervals are of minimal length. Proofs of the results are postponed to Chapter 4 Section V. All along the chapter, we will introduce some positive constants denoted by $C(\alpha,\beta,\ldots)$ and $L_0(\alpha,\beta,\ldots)$, meaning that they depend on $(\alpha,\beta,\ldots)$. Though they are denoted in the same way, they may vary from one line to another. When they appear in the main results about lower and upper bounds, we do not intend to precisely evaluate them. However, some possible, probably pessimistic, explicit expressions for them are proposed in the proofs.

I Minimax multiple test for detecting and localising a jump with a known change height

In this section, we focus on the case where the height of the possible jump from λ_0 in the intensity λ of the Poisson process N is known, equal to δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. Hence, for $\lambda_0>0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$, we introduce the set $\mathcal{S}[\lambda_0,\delta^*]$ of intensities with one jump of height δ^* from λ_0 at location τ, that is

$$\mathcal{S}[\lambda_0,\delta^*]=\{\lambda:[0,1]\to(0,+\infty),\ \exists\tau\in(0,1),\ \forall t\in[0,1],\ \lambda(t)=\lambda_0+\delta^*\mathbb{1}_{(\tau,1]}(t)\}\ , \tag{4.1}$$

and the set $\overline{\mathcal{S}}[\lambda_0,\delta^*]=\mathcal{S}[\lambda_0,\delta^*]\cup\{\lambda_0\}$ of intensities with at most one jump of height δ^* from λ_0 at location τ. By convention, we say that the intensity λ in $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ is constant and equal to λ_0 when its jump location τ is equal to 1.

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,\delta^*} = \{H_k[\lambda_0, \delta^*],\ k \in \{1,\dots,M\}\}$ where for all $k \in \{1,\dots,M\}$,

$$H_k[\lambda_0, \delta^*] = \{\lambda : [0,1] \to (0,+\infty),\ \exists \tau \in [k/M, 1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta^* \mathbb{1}_{(\tau,1]}(t)\} \ .$$

Notice that for all k in $\{1,\dots,M\}$, the hypothesis $H_k[\lambda_0,\delta^*]$ is included in $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ and that $H_M[\lambda_0,\delta^*] = \{\lambda_0\}$. Moreover, since the hypotheses are nested in the sense that

$$H_M[\lambda_0,\delta^*] \subset H_{M-1}[\lambda_0,\delta^*] \subset \ldots \subset H_1[\lambda_0,\delta^*] \ ,$$

the collection $\mathcal{H}_{M,\delta^*}$ is closed under intersection (that is any intersection $H \cap H'$ of two hypotheses H and H' in $\mathcal{H}_{M,\delta^*}$ also belongs to $\mathcal{H}_{M,\delta^*}$). In particular $\bigcap \mathcal{H}_{M,\delta^*} = H_M[\lambda_0,\delta^*] = \{\lambda_0\}$ and then, for the considered multiple testing problem, given prescribed levels α and β in $(0,1)$, a lower bound for the (α,β)-*minimax Family-Wise Separation Rate* over $\mathcal{S}[\lambda_0,\delta^*]$ can be deduced from Lemma 1.10:

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*]) \geq \mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0,\delta^*]) \ ,$$

where $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0,\delta^*])$ is the (α,β)-*minimax Separation Rate* over $\mathcal{S}[\lambda_0,\delta^*]$ for the problem of testing the null hypothesis "$\lambda = \lambda_0$" versus "$\lambda \neq \lambda_0$", defined by (1.3). Since $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}[\lambda_0,\delta^*])$ corresponds to the minimax separation rate of the jump detection problem with known baseline and jump height studied in Chapter 2 Section V.1, the Proposition 2.14 finally leads to the following lower bound.

Proposition 4.1 (Minimax lower bound).
Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1$, $\lambda_0>0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. For all M in $\mathbb{N}^*$ and for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \ \textit{where} \ \ C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

In order to prove that the above lower bound is sharp (possibly up to a constant), we secondly construct a minimax multiple testing procedure. Let us begin with a preliminary remark which provides a rough upper bound for the mFWSR.

Remark 4.1.
Let $r > 0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}[\lambda_0,\delta^*]$, $\mathcal{F}_r(\lambda) = \{H_k[\lambda_0,\delta^*] \in \mathcal{H}_{M,\delta^*},\ d_2(\lambda, H_k[\lambda_0,\delta^*]) \geq r\}$ with $d_2(\lambda, H_k[\lambda_0,\delta^*]) = |\delta^*|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1,\dots,M\}$. One has the following straightforward assertion

$$\forall \lambda \in \mathcal{S}[\lambda_0,\delta^*],\ \mathcal{F}_r(\lambda) = \varnothing \iff r \geq |\delta^*| \ . \tag{4.2}$$

In particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0,\delta^*]) \leq |\delta^*| \ . \tag{4.3}$$

Therefore, with the lower bound established in Proposition 4.1, we get that for all M in $\mathbb{N}^*$ and for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\delta^{*2}$,

$$\sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}} \leq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*]) \leq |\delta^*| \ \text{ with } C_{\alpha,\beta} = 1 + 4(1-\alpha-\beta)^2 \ .$$

A noteworthy fact highlights by the previous inequality is that $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*])$ is bounded by quantities which does not depend on the choice of M.

To construct a multiple test whose *β-Family-Wise separation rate* over $\mathcal{S}[\lambda_0,\delta^*]$ achieves, possibly up to a multiplicative constant, the minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*])$ established in Proposition 4.1, we use the closure method of Marcus et al. [MPG76] applied to single tests of the null hypotheses $H_k[\lambda_0,\delta^*]$ in $\mathcal{H}_{M,\delta^*}$ versus the alternatives $\mathcal{S}[\lambda_0,\delta^*]\setminus H_k[\lambda_0,\delta^*]$ respectively. For k in $\{1,\ldots,M\}$, let $\phi_{1,k}$ be the test defined by

$$\phi_{1,k}(N)=\mathbb{1}_{S_{\delta^*,k}(N)>s_{\delta^*,k}(1-\alpha)}\ , \tag{4.4}$$

where $S_{\delta^*,k}$ is defined by

$$S_{\delta^*,k}(N)=\sup_{t\in(0,k/M)}\left(\mathrm{sgn}(\delta^*)\left(N\left(t,\frac{k}{M}\right]-\lambda_0L\left(\frac{k}{M}-t\right)\right)-\frac{|\delta^*|}{2}L\left(\frac{k}{M}-t\right)\right)\ , \tag{4.5}$$

and $s_{\delta^*,k}(u)$ is the u-quantile of $S_{\delta^*,k}$ under $H_k[\lambda_0,\delta^*]$.
These simple tests, which take the knowledge of the change height δ^* into account, are inspired of the one defined in (2.19) and considered in Chapter 2 Section V.1, which achieves the minimax separation rate of the simple testing problem of detecting a single change point in the intensity of a Poisson process when the jump height is known.

We then define $\hat{k}_1=\sup\{k'\in\{1,\ldots,M\},\ \phi_{1,k'}=0\}\vee 0$, leading to the definition of our multiple testing procedure $\mathcal{R}_1$:

$$\mathcal{R}_1=\{H_k[\lambda_0,\delta^*]:\ k\geq\hat{k}_1+1\}\ , \tag{4.6}$$

where we recall that $\sup\varnothing=-\infty$.

Studying the FWSR of the proposed multiple test from a nonasymptotic point of view necessarily involves to quantify or at least to derive a sharp upper bound of the quantiles $s_{\delta^*,k}(u)$.

The following lemma, which generalises the result of Lemma 2.30 and which is deduced from an early result of Pyke [Pyk59], allows in particular to see that the quantile of the supremum of a homogeneous Poisson process with intensity $\xi L>0$ (with respect to the Lebesgue measure on $[0,+\infty)$) and with some well chosen drift can be upper bounded by a positive constant not depending on L. This will be a key point of the proof of Theorem 4.3 below, providing an upper bound for the FWSR of our multiple test $\mathcal{R}_1$ over $\mathcal{S}[\lambda_0,\delta^*]$.

Lemma 4.2.
Let $L\geq 1$, $\xi>0$ and $\sigma>0$. Let $(N_t^\xi)_{t\geq 0}$ be an homogeneous Poisson process with a constant intensity $\xi L>0$ defined with respect to the Lebesgue measure. Then for all u in $(0,1)$, the u-quantile of $\sup_{t\geq 0}(N_t^\xi-(\xi+\sigma)Lt)$, denoted by $q_\xi(u,\sigma)$, is a positive constant which does not depend on L.

Comment.
Notice that when t tends to infinity $(N_t^\xi-(\xi+\sigma)Lt)$ tends to $-\infty$ almost surely which ensures that $\sup_{t\geq 0}(N_t^\xi-(\xi+\sigma)Lt)$ is finite almost surely. Therefore, $\mathbb{P}(\sup_{t\geq 0}(N_t^\xi-(\xi+\sigma)Lt)>x)$ tends to 0 when x tends to $+\infty$. Gathering this remark with the equation (7) in [Pyk59] then straightforwardly leads to the above result.

We can now state the main result of this section, showing that our multiple test $\mathcal{R}_1$ has a *Family-Wise error rate* over $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ controlled by α, and is minimax over $\mathcal{S}[\lambda_0,\delta^*]$.

Theorem 4.3 (Minimax upper bound)**.**
Let $L\geq 1$, α and β be fixed levels in $(0,1)$, $\lambda_0>0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,\delta^*)>0$ such that the multiple testing procedure $\mathcal{R}_1$ defined by (4.6) satisfies for all M in $\mathbb{N}^*$,

$$\mathrm{FWER}(\mathcal{R}_1)\leq\alpha,\ \ \textit{and}\ \ \mathrm{FWSR}_\beta(\mathcal{R}_1,\mathcal{S}[\lambda_0,\delta^*])\leq\min\left(|\delta^*|\,,\ \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}\right)\ .$$

In particular, for all M in $\mathbb{N}^*$ and for all $L \geq (C(\alpha,\beta,\lambda_0,\delta^*)/\delta^*)^2$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,\delta^*])) \leq \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}} \ .$$

Comment.
This result, combined with its corresponding lower bound, shows that the minimax family-wise separation rate over the alternative set $\mathcal{S}[\lambda_0,\delta^*]$ is of parametric order $L^{-1/2}$. For $M = 1$, recall that the minimax family-wise separation rate is equal to the minimax separation rate since only one hypothesis is tested. In this case, we recover in particular the minimax separation rate established in Chapter 2 Section V.1 dealing with the change-point detection problem with known baseline and jump height. By the way, the proof of Theorem 4.3 needs the condition on the distance between λ in $\mathcal{S}[\lambda_0,\delta^*]$ and $H_M[\lambda_0,\delta^*] = \{\lambda_0\}$ established in Proposition 2.15 to prove the upper bound for the minimax separation rate of jump detection problem. It is worth noting that the multiplicity of the hypotheses in our multiple testing framework does not affect the rate of the mFWSR which remains in the parametric order whatever the value of M.

II Minimax multiple test for detecting and localising a jump with an unknown change height

In this section, we tackle the question of adaptation with respect to the change height and the jump location, and therefore introduce to this end a preliminary set for $\lambda_0 > 0$,

$$\mathcal{S}[\lambda_0] = \{\lambda : [0,1] \to (0,+\infty),\ \exists(\delta,\tau) \in \{(-\lambda_0,+\infty) \setminus \{0\}\} \times (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\} \ . \quad (4.7)$$

As in Chapter 4 Section I, a lower bound for the (α,β)-*minimax Family-Wise separation rate* over $\mathcal{S}[\lambda_0]$ is determined using Lemma 1.10 and the (α,β)-*minimax Separation Rate* dealing with simple testing problem for the null hypothesis (H_0) "$\lambda = \{\lambda_0\}$" versus the alternative (H_1) "$\lambda \in \mathcal{S}[\lambda_0]$". However, recall that the Lemma 2.16 underlines that the (α,β)-*minimax Separation Rate* over $\mathcal{S}[\lambda_0]$ is infinite.

We therefore consider, for $R > \lambda_0$, the more suitable set of alternatives bounded by R, defined by:

$$\mathcal{S}[\lambda_0,R] = \{\lambda : [0,1] \to (0,R],\ \exists(\delta,\tau) \in \{(-\lambda_0,R-\lambda_0] \setminus \{0\}\} \times (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\} \ .$$

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,R} = \{H_k[\lambda_0,R],\ k \in \{1,\ldots,M\}\}$ where, for all k in $\{1,\ldots,M\}$,

$$H_k[\lambda_0,R] = \{\lambda :\ \exists(\delta,\tau) \in \{(-\lambda_0,R-\lambda_0] \setminus \{0\}\} \times [k/M,1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\} \ .$$

In particular, for all k in $\{1,\ldots,M\}$, the single hypothesis $H_k[\lambda_0,R]$ is included in the set $\overline{\mathcal{S}}[\lambda_0,R] = \mathcal{S}[\lambda_0,R] \cup \{\lambda_0\}$ of intensities bounded by R with at most one jump. As in Chapter 4 Section I, we say that the intensity λ in $\overline{\mathcal{S}}[\lambda_0,R]$ is constant and equal to λ_0 when its jump location τ is equal to 1 and we notice that $H_M[\lambda_0,R] = \{\lambda_0\}$.

The collection $\mathcal{H}_{M,R}$ is closed under intersection because the hypotheses are nested: $H_M[\lambda_0,R] \subset H_{M-1}[\lambda_0,R] \subset \ldots \subset H_1[\lambda_0,R]$, and in particular $\bigcap \mathcal{H}_{M,R} = H_M[\lambda_0,R] = \{\lambda_0\}$. Then we get by Lemma 1.10 that for α and β in $(0,1)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \geq \mathrm{mSR}^{\{\lambda_0\}}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \ ,$$

where $\mathrm{mSR}^{\{\lambda_0\}}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])$ is the (α,β)-*minimax Separation Rate* over $\mathcal{S}[\lambda_0,R]$ which corresponds to the minimax separation rate of the jump detection problem with known baseline studied in Chapter 2 Section V.1. Therefore, Proposition 2.17 ensures the following lower bound.

Proposition 4.4 (Minimax lower bound).
Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta<1/2$, $\lambda_0>0$ and $R>\lambda_0$. There exists $L_0(\alpha,\beta,\lambda_0,R)>0$ such that for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \geq \sqrt{\frac{\lambda_0 \log\log L}{L}} \ .$$

Let us now construct a multiple test whose *β-Family-Wise separation rate* over $\mathcal{S}[\lambda_0,R]$ achieves, possibly up to a multiplicative constant, the above minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])$.
We begin with a preliminary remark which provides a rough upper bound for the mFWSR.

Remark 4.2.
Let $r>0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{F}_r(\lambda)=\{H_k[\lambda_0,R]\in\mathcal{H}_{M,R},\ d_2(\lambda,H_k[\lambda_0,R])\geq r\}$ with $d_2(\lambda,H_k[\lambda_0,R])=|\delta|\sqrt{k/M-\tau}\mathbb{1}_{\tau\leq k/M}$ for all k in $\{1,\ldots,M\}$. One has the following straightforward assertion

$$\forall\lambda\in\mathcal{S}[\lambda_0,R],\ \mathcal{F}_r(\lambda)=\varnothing \iff r\geq\lambda_0\vee(R-\lambda_0)\ . \tag{4.8}$$

In particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R},\mathcal{S}[\lambda_0,R])\leq\lambda_0\vee(R-\lambda_0)\ . \tag{4.9}$$

Indeed, let $r>0$ be such that $r\geq\lambda_0\vee(R-\lambda_0)$. Then for all λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{F}_r(\lambda)=\varnothing$ and then $P_\lambda(\mathcal{F}_r(\lambda)\subset\mathcal{R})=1\geq1-\beta$ for all multiple testing procedure $\mathcal{R}$ and for all λ in $\mathcal{S}[\lambda_0,R]$.
Therefore, with the lower bound established in Proposition 4.4, we get that for all M in $\mathbb{N}^*$ and for L large enough

$$\sqrt{\frac{\lambda_0\log\log L}{L}}\leq\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])\leq\lambda_0\vee(R-\lambda_0)\ .$$

Again, it is noteworthy that $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])$ is bounded by quantities which does not depend on the choice of M.

Following the discussion of Chapter 2 Section II, we define two different minimax multiple tests: the first one is based on counting statistics and the second one is based on quadratic statistics. Both statistics are inspired of the ones considered in Chapter 2 Section V.1 which achieve the minimax separation rate of the jump detection problem. Both multiple testing procedures are defined according to the closure method of Marcus et al. [MPG76] applied to single tests of the null hypotheses $H_k[\lambda_0,R]$ in $\mathcal{H}_{M,R}$ versus the alternatives $\mathcal{S}[\lambda_0,R]\setminus H_k[\lambda_0,R]$ respectively.
We consider the corrected level $u_\alpha=\alpha/\lfloor\log_2 L\rfloor$ which allows to define the two following multiple tests.

Let us begin with the minimax multiple testing procedure based on counting statistics and let k in $\{1,\ldots,M\}$. We then consider a simple test $\phi^{(1)}_{2,k}$ for the null hypothesis $H_k[\lambda_0,R]$ versus the alternative $\mathcal{S}[\lambda_0,R]\setminus H_k[\lambda_0,R]$ defined by

$$\begin{aligned}\phi^{(1)}_{2,k}(N)=\mathbb{1}&\left\{\max_{j\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(N\left(\tfrac{k}{M}\left(1-2^{-j}\right),\tfrac{k}{M}\right]-p_{\frac{\lambda_0 kL}{M2^j}}\left(1-\tfrac{u_\alpha}{2}\right)\right)>0\right\}\\ &\vee\mathbb{1}\left\{\max_{j\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(p_{\frac{\lambda_0 kL}{M2^j}}\left(\tfrac{u_\alpha}{2}\right)-N\left(\tfrac{k}{M}\left(1-2^{-j}\right),\tfrac{k}{M}\right]\right)>0\right\}\ ,\end{aligned} \tag{4.10}$$

where $p_\xi(u)$ stands for the u-quantile of the Poisson distribution of parameter ξ.

We consider $\hat{k}^{(1)}_2=\sup\{k'\in\{1,\ldots,M\},\ \phi^{(1)}_{2,k'}=0\}\vee0$ and we define our multiple testing procedure $\mathcal{R}^{(1)}_2$ by

$$\mathcal{R}^{(1)}_2=\{H_k[\lambda_0,R]:\ k\geq\hat{k}^{(1)}_2+1\}\ . \tag{4.11}$$

We establish in Theorem 4.5 that our multiple test $\mathcal{R}_2^{(1)}$ has a FWER over $\overline{\mathcal{S}}[\lambda_0, R]$ controlled by α and is minimax over $\mathcal{S}[\lambda_0, R]$.

Theorem 4.5 (Minimax upper bound)**.**
Let $L \geq 3$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,R) > 0$ such that the multiple testing procedure $\mathcal{R}_2^{(1)}$ defined by (4.11) satisfies for all M in $\mathbb{N}^*$,

$$\mathrm{FWER}(\mathcal{R}_2^{(1)}) \leq \alpha, \ \textit{and} \ \mathrm{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R]) \leq \min\left(\max(\lambda_0, R-\lambda_0)\, ,\, C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} \right) .$$

In particular, there exists $L_0(\alpha,\beta,\lambda_0,R) > 0$ such that, for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} .$$

Now, let us turn to the second minimax multiple testing procedure, based on quadratic statistics. For k in $\{1,\ldots,M\}$, we consider a simple test $\phi_{2,k}^{(2)}$ for the null hypothesis $H_k[\lambda_0,R]$ versus the alternative $\mathcal{S}[\lambda_0,R] \setminus H_k[\lambda_0,R]$ defined by

$$\phi_{2,k}^{(2)}(N) = \mathbb{1}_{\max_{j\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}(T_{j,k}(N) - t_{j,k}(1-u_\alpha))>0} , \tag{4.12}$$

where for all j in $\{1,\ldots,\lfloor\log_2 L\rfloor\}$,

$$T_{j,k}(N) = \frac{2^j M}{L^2 k}\left(N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right), \frac{k}{M}\right]^2 - N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right), \frac{k}{M}\right]\right) - \frac{2\lambda_0}{L} N\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right), \frac{k}{M}\right] + \frac{\lambda_0^2 k}{2^j M} , \tag{4.13}$$

and $t_{j,k}(u)$ is the u-quantile of $T_{j,k}$ under $H_k[\lambda_0,R]$.

We set $\hat{k}_2^{(2)} = \sup\{k' \in \{1,\ldots,M\},\ \phi_{2,k'}^{(2)} = 0\} \vee 0$ and we define our multiple testing procedure $\mathcal{R}_2^{(2)}$ by

$$\mathcal{R}_2^{(2)} = \{H_k[\lambda_0,R] :\ k \geq \hat{k}_2^{(2)} + 1\} . \tag{4.14}$$

We establish in Theorem 4.6 an alternate minimax upper bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R])$ using the multiple testing procedure $\mathcal{R}_2^{(2)}$.

Theorem 4.6 (Alternate minimax upper bound)**.**
Let $L \geq 3$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,R) > 0$ such that the multiple testing procedure $\mathcal{R}_2^{(2)}$ defined by (4.14) satisfies for all M in $\mathbb{N}^*$,

$$\mathrm{FWER}(\mathcal{R}_2^{(2)}) \leq \alpha, \ \textit{and} \ \mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R]) \leq \min\left(\max(\lambda_0, R-\lambda_0)\, ,\, C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} \right) .$$

In particular, there exists $L_0(\alpha,\beta,\lambda_0,R) > 0$ such that, for all M in $\mathbb{N}^*$ and for all $L \geq L_0(\alpha,\beta,\lambda_0,R)$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0,R]) \leq C(\alpha,\beta,\lambda_0,R)\sqrt{\frac{\log\log L}{L}} .$$

Comment.
This result, combined with its corresponding lower bound, brings out a phase transition in the minimax family-wise separation rate orders, from the parametric order $1/\sqrt{L}$ (see Chapter 4 Section I) to $\sqrt{\log\log L/L}$. This means that adaptation with respect to both location and height of the abrupt change has an unavoidable logarithmic cost, while adaptation to only the change height does not cause any additional price. A comparable phase transition has already been observed in the particular case where $M = 1$ dealing with the change-point detection problem investigated in Chapter 2 Section V.1. By the way, the proofs of Theorem 4.5 and Theorem 4.6 need the conditions on the distance between λ in $\mathcal{S}[\lambda_0, R]$ and $H_M[\lambda_0, R] = \{\lambda_0\}$ established in Proposition 2.18 to prove the upper bound for the minimax separation rate of the simple testing problem of detecting an abrupt change in the intensity of a Poisson process. It is worth noting again that the multiplicity of the hypotheses in our multiple testing framework does not affect the rate of the mFWSR which remains in the order $\sqrt{\log\log L/L}$ whatever the value of M.

III Links between minimax multiple testing procedures and minimal length of confidence intervals for the jump localisation

In the sequel, the subset $\overline{\mathcal{S}}$ stands for $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ or $\overline{\mathcal{S}}[\lambda_0, R]$, $\mathcal{H}$ to the related collection of hypotheses $\mathcal{H}_{M,\delta^*}$ or $\mathcal{H}_{M,R}$ and H_k to the corresponding hypothesis $H_k[\lambda_0, \delta^*]$ or $H_k[\lambda_0, R]$ for all k in $\{1, \ldots, M\}$ with M a non-zero integer. Given the observation of an inhomogeneous Poisson process $N = (N_t)_{t\in[0,1]}$ with an unknown intensity λ in $\overline{\mathcal{S}}$ defined with respect to the measure Λ, a $(1-\varepsilon)$-confidence interval for the rupture location τ on $\overline{\mathcal{S}}$ is a real interval I_ϵ not depending on the unknown parameters such that

$$\inf_{\lambda\in\overline{\mathcal{S}}} P_\lambda(\tau \in I_\epsilon) \geq 1 - \varepsilon \ .$$

Since λ belongs to $\overline{S}$, recall that we set $\tau = 1$ by convention if $\lambda = \lambda_0$ on $[0, 1]$. Notice that the interval I_ε can be defined from an estimator $\hat{\tau}$ of τ and could depend on known parameters of the problem. The relationships between confidence intervals for the rupture localisation τ on $\overline{\mathcal{S}}$ and multiple testing procedures are summarized in the following lemmas.

First, it is well known that an α-level test for a single null hypothesis is intrinsically related to a $(1-\alpha)$-confidence region, and that a similar correspondence can be made between a FWER controlling procedure and a confidence interval (see for example [ABR10]).

Lemma 4.7.
Let M in $\mathbb{N}^*$. From any multiple procedure $\mathcal{R}$ on $\mathcal{H}$ such that $\mathrm{FWER}(\mathcal{R}) \leq \alpha$*, one can build a $(1-\alpha)$–confidence interval I_α for τ on $\overline{\mathcal{S}}$ defined by $I_\alpha = \{x \in [0, 1] : \ x \leq ((\sup\{k \in \{1, \ldots, M\} : \ H_k \notin \mathcal{R}\} + 1)/M) \wedge 1\}$. Conversely, from any $(1-\alpha)$-confidence region I_α for τ on $\overline{\mathcal{S}}$, one can build a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}$ such that* $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ *which is defined by $\mathcal{R} = \{H_k \in \mathcal{H},\ k/M > \sup\{x \in [0, 1],\ x \in I_\alpha\}\}$.*

We are then interested in finding the smallest confidence interval for the estimation of the jump location of the intensity. We consider $\mathcal{D}[0, 1]$ the space of càdlàg functions on $[0, 1]$ and $\mathcal{MD}$ the set of all the measurable real functions defined on $\mathcal{D}[0, 1]$ taking values in $[0, 1]$. We then focus on confidence intervals of the form $(\phi(N) - a, \phi(N) + b]$ where $a, b \geq 0$ and ϕ in $\mathcal{MD}$. One may think about $\phi(N)$ as an estimator of τ. We define the minimal length of $(1-\varepsilon)$-confidence intervals for the estimator $\phi(N)$ of τ with λ in $\overline{\mathcal{S}}$ by

$$\mathcal{L}_\varepsilon(\phi, \overline{\mathcal{S}}) = \inf\{a + b : a, b > 0, \inf_{\lambda\in\overline{\mathcal{S}}} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \varepsilon\} \ ,$$

and the minimal length of $(1-\varepsilon)$-confidence intervals for τ with λ in $\overline{\mathcal{S}}$ by

$$\mathcal{L}_\varepsilon(\overline{\mathcal{S}}) = \inf\{\mathcal{L}_\varepsilon(\phi, \overline{\mathcal{S}}) : \phi \in \mathcal{MD}\} \ .$$

III.1 Confidence interval for the jump localisation when the change height is known

The following lemma gives bounds for the minimal length of confidence intervals for the estimation problem of τ when the change height is known.

Lemma 4.8.
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ and M in $\mathbb{N}^*$. Consider first $a, b > 0$ and ϕ in $\mathcal{MD}$ such that

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, \delta^*]} P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \geq 1 - \alpha \ . \tag{4.15}$$

The multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\delta^*}$ defined by $\mathcal{R} = \{H_k \in \mathcal{H}_{M,\delta^*},\ k/M > \phi(N) + b\}$ is satisfying

$$\mathrm{FWER}(\mathcal{R}) \leq \alpha, \ \textit{and} \ \mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b} \ .$$

In particular, we get for all M in $\mathbb{N}^*$,

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}} \ .$$

Conversely, consider $r > 0$ be such that $r \geq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])$ and a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\delta^*}$ satisfying the two inequalities $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq r$. Set $\hat{k} = \sup\{k \in \{1, \ldots, M\},\ H_k \notin \mathcal{R}\} \vee 0$ and $\hat{\tau} = \hat{k}/M$. The interval $I_{r,\delta^*,M,\mathcal{R}}$ defined by $I_{r,\delta^*,M,\mathcal{R}} = (\hat{\tau} - r^2/\delta^{*2}, \hat{\tau} + 1/M]$ then satisfies

$$\inf_{\lambda \in \overline{\mathcal{S}}[\lambda_0, \delta^*]} P_\lambda(\tau \in I_{r,\delta^*,M,\mathcal{R}}) \geq 1 - \alpha - \beta \ .$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_{\alpha+\beta}(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \leq \frac{1}{M} + \frac{\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}} \ .$$

This lemma, combined with Proposition 4.1 and Theorem 4.3, allows us to construct a minimal confidence interval for the jump localisation from a multiple testing procedure. Assume that we observe a Poisson process $N = (N_t)_{t \in [0,1]}$ on the interval $[0,1]$, with intensity λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ with respect to some measure Λ on $[0,1]$, where λ_0, δ^* and L are known parameters. We shall estimate the jump location τ using a multiple testing procedure.

For α in $(0, 1/2)$, if we assume that $L \geq \max(\lambda_0 \log C_{\alpha,\beta}, C(\alpha/2, \alpha/2, \lambda_0, \delta^*)^2)/\delta^{*2}$, Lemma 4.8, Proposition 4.1 and Theorem 4.3 ensure that for all M in $\mathbb{N}^*$,

$$\frac{\lambda_0 \log C_\alpha}{L\delta^{*2}} \leq \mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \leq \frac{1}{M} + \frac{C(\alpha/2, \alpha/2, \lambda_0, \delta^*)^2}{L\delta^{*2}}$$

where $C_\alpha = 1 + 4(1 - 2\alpha)^2$ and $C(\alpha/2, \alpha/2, \lambda_0, \delta^*)$ is defined in Theorem 4.3.

We then aim at providing a confidence interval for τ of minimal length. To this end, we consider for $L \geq (C(\alpha/2, \alpha/2, \lambda_0, \delta^*)/\delta^*)^2$, the collection of hypotheses $\mathcal{H}_{M,\delta^*} = \{H_k[\lambda_0, \delta^*],\ k \in \{1, \ldots, M\}\}$ with

$$M = \left\lfloor \frac{L\delta^{*2}}{C(\alpha/2, \alpha/2, \lambda_0, \delta^*)^2} \right\rfloor \ . \tag{4.16}$$

We then apply our multiple testing procedure $\mathcal{R}_1$ introduced in Chapter 4 Section I. Recall that we define

$$\hat{k}_1 = \sup\{k' \in \{1,\ldots,M\},\ \phi_{1,k'} = 0\} \vee 0\ ,$$

with $\phi_{1,k}$ defined by (4.4), leading to the definition of our multiple testing procedure $\mathcal{R}_1$:

$$\mathcal{R}_1 = \{H_k[\lambda_0,\delta^*]:\ k \geq \hat{k}_1 + 1\}\ .$$

We introduce an estimator of the location of the intensity jump by

$$\hat{\tau} = \frac{\hat{k}_1}{M}\ . \tag{4.17}$$

Corollary 4.9.
Let α in $(0,1/2)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0,+\infty)\setminus\{0\}$. For $L \geq (C(\alpha/2,\alpha/2,\lambda_0,\delta^*)/\delta^*)^2$ and M defined in (4.16), the interval

$$\left(\hat{\tau} - \frac{1}{M},\ \hat{\tau} + \frac{1}{M}\right]$$

is a $(1-\alpha)$-confidence interval for the jump localisation on $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ which achieves, up to a constant, the minimal length of such confidence intervals.

III.2 Confidence interval for the jump localisation when the change height is unknown

When the change height δ^* is an unknown parameter, the notion of minimal confidence interval is quite irrelevant. Indeed, the minimal length for confidence intervals on the whole space $\overline{\mathcal{S}}[\lambda_0,R]$ is almost always equal to 1 and then, the interval $[0,1]$ is a confidence interval of minimal length.

Lemma 4.10.
Let α in $(0,1/2)$, $\lambda_0 > 0$ and $R > \lambda_0$. Then for all $L \geq 1$,

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0,R]) = 1\ .$$

The Lemma 4.10 invites us to consider a smaller set for the intensities λ of the Poisson process. Therefore, we focus on intensities whose jumps are large enough. For $\lambda_0 > 0$ and Δ in $(0,\lambda_0)$ we introduce the set $\mathcal{S}_{\geq\Delta}[\lambda_0]$ of intensities whose jumps are at least of height Δ:

$$\mathcal{S}_{\geq\Delta}[\lambda_0] = \{\lambda : [0,1] \to (0,+\infty),\ \exists\delta \in \{(-\lambda_0,-\Delta] \cup [\Delta,+\infty)\},\ \exists\tau \in (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}(t)\}\ . \tag{4.18}$$

However, the following lemma underlines that the (α,β)-*minimax Separation Rate* over this preliminary alternative set is infinite and then, the arguments used with Lemma 1.10 provide an infinite (α,β)-*minimax Family-Wise separation rate* over $\mathcal{S}_{\geq\Delta}[\lambda_0]$.

Lemma 4.11.
Let α and β be fixed levels in $(0,1)$ such that $\alpha+\beta < 1$. For $\lambda_0 > 0$ and Δ in $(0,\lambda_0)$, considering the testing problem (H_0) "$\lambda = \lambda_0$" versus (H_1) "$\lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0]$" with $\mathcal{S}_{\geq\Delta}[\lambda_0]$ defined by (4.18), one has

$$\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) = +\infty\ .$$

We therefore introduce, for $\lambda_0 > 0$, $R > \lambda_0$ and Δ in $(0,\lambda_0 \wedge (R-\lambda_0))$, the more suitable set $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ of intensities bounded by R whose jumps are at least of height Δ from λ_0:

$$\mathcal{S}_{\geq\Delta}[\lambda_0,R] = \{\lambda : [0,1] \to (0,R],\ \exists\delta \in \{(-\lambda_0,-\Delta] \cup [\Delta,R-\lambda_0]\},\ \exists\tau \in (0,1),\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}(t)\}\ .$$

In the aim of localising the jump location, we consider for M in $\mathbb{N}^*$, the collection of hypotheses $\mathcal{H}_{M,\Delta,R} = \{H_k[\lambda_0, \Delta, R],\ k \in \{1, \ldots, M\}\}$ where for all k in $\{1, \ldots, M\}$,

$$H_k[\lambda_0, \Delta, R] = \{\lambda : \exists \delta \in \{(-\lambda_0, -\Delta] \cup [\Delta, R - \lambda_0]\},\ \exists \tau \in [k/M, 1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\}\ .$$

For all k in $\{1, \ldots, M\}$, notice that the hypothesis $H_k[\lambda_0, \Delta, R]$ is included in the set $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R] = \mathcal{S}_{\geq\Delta}[\lambda_0, R] \cup \{\lambda_0\}$ of intensities bounded by R and with at most a change whose height is lower bounded by Δ.

Since the set $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$ includes $\mathcal{S}[\lambda_0, \Delta]$ defined in (4.1) for all Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$, the Lemma 1.11 combined with Proposition 4.1 leads to

Proposition 4.12 (Minimax lower bound)**.**
Let α and β be fixed levels in $(0,1)$ such that $\alpha + \beta < 1$, $\lambda_0 > 0$, $R > \lambda_0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$. For all M in $\mathbb{N}^*$ and for all $L \geq \lambda_0 \log C_{\alpha,\beta}/\Delta^2$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0, R]) \geq \sqrt{\frac{\lambda_0 \log C_{\alpha,\beta}}{L}}, \ \textit{where}\ \ C_{\alpha,\beta} = 1 + 4(1 - \alpha - \beta)^2\ .$$

To define a multiple testing procedure whose *β-Family-Wise separation rate* over $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$ achieves, possibly up to a multiplicative constant, the above minimax lower bound for $\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0, R])$, we construct for k in $\{1, \ldots, M\}$ a simple test for the null hypothesis $H_k[\lambda_0, \Delta, R]$ versus the alternative $\mathcal{S}_{\geq\Delta}[\lambda_0, R] \setminus H_k[\lambda_0, \Delta, R]$. More precisely, we define the test $\phi_{3,k}$ for all k in $\{1, \ldots, M\}$ by

$$\phi_{3,k}(N) = \mathbb{1}_{S_{\Delta,k}(N) > s_{\Delta,k}(1-\alpha/2)} \vee \mathbb{1}_{S_{-\Delta,k}(N) > s_{-\Delta,k}(1-\alpha/2)}\ , \tag{4.19}$$

where for all δ in $\mathbb{R}^*$, $S_{\delta,k}(N)$ is defined in (4.5) by

$$S_{\delta,k}(N) = \sup_{t \in (0, k/M)} \left(\mathrm{sgn}(\delta) \left(N\left(t, \frac{k}{M}\right] - \lambda_0 L \left(\frac{k}{M} - t \right) \right) - \frac{|\delta|}{2} L \left(\frac{k}{M} - t \right) \right)\ ,$$

and $s_{\Delta,k}(u)$ (respectively $s_{-\Delta,k}(u)$) is the u-quantile of $S_{\Delta,k}$ (respectively the u-quantile of $S_{-\Delta,k}$) under $H_k[\lambda_0, \Delta, R]$. These simple tests, which take the knowledge of the minimal value Δ of the change height into account, are closed to the ones considered in Chapter 4 Section I.

We then define $\hat{k}_3 = \sup\{k' \in \{1, \ldots, M\},\ \phi_{3,k'}(N) = 0\} \vee 0$, leading to the definition of our multiple testing procedure $\mathcal{R}_3$:

$$\mathcal{R}_3 = \{H_k[\lambda_0, \Delta, R] :\ k \geq \hat{k}_3 + 1\}\ . \tag{4.20}$$

The following lemma, deduced from a result of Loader [Loa90], ensures that the quantile of a homogeneous Poisson process with intensity $\xi L > 0$ and with a well chosen drift is an increasing function of ξ. This will be a key point of the proof of Theorem 4.14 below, providing an upper bound for the FWSR of our multiple test $\mathcal{R}_3$ over $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$.

Lemma 4.13.
Let $\xi > 0$ and $\sigma > 0$. For all u in $(0,1)$, the function $\xi \mapsto q_\xi(u, \sigma)$ is increasing, where $q_\xi(u, \sigma)$ is defined in Lemma 4.2.

Comment.
Let $(N_t^\xi)_{t \geq 0}$ be a homogeneous Poisson process with a constant intensity $\xi L > 0$. For all $x \geq 0$, an Abel transform on the exact expression of $\mathbb{P}(\sup_{t\geq 0}(N_t^\xi - (\xi + \sigma)Lt) > x)$ given in [Loa90, Theorem 2.2] shows that $\lambda \mapsto \mathbb{P}(\sup_{t \geq 0}(N_t^\xi - (\xi + \sigma)Lt) > x)$ is increasing. The above result then follows easily.

The following theorem shows that the multiple test $\mathcal{R}_3$ is minimax over $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$ and its proof follows essentially the same ideas of the proof of Theorem 4.3 with the argument given in Lemma 4.13. For the sake of clarity and completeness, the proof of Theorem 4.14 is detailed in Chapter 4 Section V.4.

Theorem 4.14 (Minimax upper bound).
Let $L \geq 1$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$ and Δ in $(0, \lambda_0 \wedge (R-\lambda_0))$. Then, there exists a constant $C(\alpha,\beta,\lambda_0,\Delta,R) > 0$ such that the multiple testing procedure $\mathcal{R}_3$ defined by (4.20) satisfies for all M in $\mathbb{N}^*$,

$$\mathrm{FWER}(\mathcal{R}_3) \leq \alpha, \ \textit{and} \ \mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq \min\left(\lambda_0 \vee (R-\lambda_0)\,,\ \frac{C(\alpha,\beta,\lambda_0,\Delta,R)}{\sqrt{L}}\right) .$$

In particular, for all M in $\mathbb{N}^*$ and for all $L \geq (C(\alpha,\beta,\lambda_0,\Delta,R)/(\lambda_0 \vee (R-\lambda_0)))^2$,

$$\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])) \leq \frac{C(\alpha,\beta,\lambda_0,\Delta,R)}{\sqrt{L}} .$$

Comment.
This result, combined with its corresponding lower bound, shows that the minimax family-wise separation rate over the alternative $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ has the same parametric order $L^{-1/2}$ as the case where the height of the jump is known (see Chapter 4 Section I) whatever the value of M. Then, when the change-point location is unknown, it is the possibility for the jump height to be close to zero (and not the fact that it is unknown) which deteriorates the mFWSR with a logarithmic factor (see Chapter 4 Section II). For $M = 1$, the lower bound and the upper bound established in Proposition 4.12 and Theorem 4.14 complete the minimax study for the detection of an abrupt change in the intensity of a Poisson process from a known constant baseline considered in Chapter 2 Section V.1: when both location and height of the change are unknown but with a minimal known value for the jump height, the minimax separation rate of the simple testing problem (H_0)"$\lambda = \lambda_0$" versus (H_1)"$\lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0,R]$" is of parametric order $\sqrt{1/L}$.

Now, let us turn back to the construction of a confidence interval of minimal length for the jump location. The following lemma gives bounds for the minimal length of confidence intervals for the estimation problem of τ when the change height is lower bounded by Δ.

Lemma 4.15.
Let $L \geq 1$, α and β in $(0,1)$, $\lambda_0 > 0$, $R > \lambda_0$, Δ in $(0,\lambda_0 \wedge (R-\lambda_0))$ and M in $\mathbb{N}^*$. Consider first $a, b > 0$ and ϕ in $\mathcal{MD}$ such that

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]} P_\lambda(\tau \in (\phi(N)-a, \phi(N)+b]) \geq 1-\alpha . \tag{4.21}$$

The multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\Delta,R}$ defined by $\mathcal{R} = \{H_k \in \mathcal{H}_{M,\Delta,R},\ k/M > \phi(N)+b\}$ is satisfying

$$\mathrm{FWER}(\mathcal{R}) \leq \alpha, \ \textit{and} \ \mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq (\lambda_0 \wedge (R-\lambda_0))\sqrt{a+b} .$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])^2}{(\lambda_0 \wedge (R-\lambda_0))^2} .$$

Conversely, consider $r > 0$ be such that $r \geq \mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])$ and a multiple testing procedure $\mathcal{R}$ on $\mathcal{H}_{M,\Delta,R}$ satisfying the two inequalities $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq r$. Set $\hat{k} = \sup\{k \in \{1,\ldots,M\},\ H_k \notin \mathcal{R}\}$ and $\hat{\tau} = \hat{k}/M$. The interval $I_{r,\Delta,M,\mathcal{R}}$ defined by $I_{r,\Delta,M,\mathcal{R}} = (\hat{\tau} - r^2/\Delta^2, \hat{\tau} + 1/M]$ then satisfies

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]} P_\lambda(\tau \in I_{r,\Delta,M,\mathcal{R}}) \geq 1-\alpha-\beta .$$

In particular, we get that for all M in $\mathbb{N}^*$,

$$\mathcal{L}_{\alpha+\beta}(\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]) \leq \frac{1}{M} + \frac{\mathrm{mFWSR}_{\alpha,\beta}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])^2}{\Delta^2} .$$

This lemma, combined with Proposition 4.12 and Theorem 4.14, allows us to construct a minimal confidence interval for the jump localisation from a multiple testing procedure. Assume that we observe a Poisson process $N = (N_t)_{t\in[0,1]}$ on the interval $[0,1]$, with intensity λ in $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$ with respect to some measure Λ on $[0,1]$, where λ_0, Δ, R and L are known parameters. We shall estimate the jump location τ using a multiple testing procedure.

For α in $(0,1/2)$, if we assume that $L \geq \max(\lambda_0 \log C_{\alpha,\beta}/\Delta^2, C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)^2/(\lambda_0^2 \vee (R-\lambda_0)^2)$, Lemma 4.15, Proposition 4.12 and Theorem 4.14 ensure that for all M in $\mathbb{N}^*$,

$$\frac{\lambda_0 \log C_\alpha}{L(\lambda_0^2 \wedge (R-\lambda_0)^2)} \leq \mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]) \leq \frac{1}{M} + \frac{C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)^2}{L\Delta^2} ,$$

where $C_\alpha = 1 + 4(1-2\alpha)^2$ and $C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)$ is defined in Theorem 4.14.
We then aim at providing a confidence interval for τ of minimal length. To this end, we consider for $L \geq (C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)/\Delta)^2$, the collection of hypotheses $\mathcal{H}_{M,\Delta,R} = \{H_k[\lambda_0,\Delta,R],\ k \in \{1,\dots,M\}\}$ with

$$M = \left\lfloor \frac{L\Delta^2}{C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)^2} \right\rfloor . \tag{4.22}$$

We then apply our multiple testing procedure $\mathcal{R}_3$ defined in (4.20) and we introduce an estimator of the location of the intensity jump by

$$\hat{\tau} = \frac{\hat{k}_3}{M} . \tag{4.23}$$

Corollary 4.16.
Let α in $(0,1/2)$, $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R-\lambda_0))$. For $L \geq (C(\alpha/2,\alpha/2,\lambda_0,\Delta,R)/\Delta)^2$ and M defined in (4.22), the interval

$$\left(\hat{\tau} - \frac{1}{M},\ \hat{\tau} + \frac{1}{M}\right]$$

is a $(1-\alpha)$-confidence interval for the jump localisation on $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]$ which achieves, up to a constant, the minimal length of such confidence intervals.

IV Simulation study

From an experimental point of view, we study in this section the performance of the change location estimators build from the minimax adaptive multiple testing procedures, by giving estimations of their risk for various distributions of the observed Poisson process, characterised by a jump in its intensity. Motivated by some applications in epidemiology and in cybersecurity, we illustrate the feasibility of our new estimation approach in practice.

As for the simulation studies of Chapter 2 and Chapter 3, we focus here on the most general problem investigated in this chapter of localising a single change in the intensity λ when the change height is unknown. The known baseline intensity of λ, denoted by λ_0, is taken equal to 1 on $[0,1]$ in all the sequel. For several piecewise constant intensity λ with respect to the measure $d\Lambda(t) = Ldt$, where we have chosen $L = 50$, we simultaneously test the $M = 25$ hypotheses of the collection $\mathcal{H}_{25,R} = \{H_k[\lambda_0,R],\ k \in \{1,\dots,25\}\}$ where, for all k in $\{1,\dots,25\}$,

$$H_k[\lambda_0,R] = \{\lambda :\ \exists(\delta,\tau) \in \{(-\lambda_0, R-\lambda_0] \setminus \{0\}\} \times [k/25,1],\ \forall t \in [0,1],\ \lambda(t) = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}(t)\} ,$$

and we consider multiple tests associated to the collection $\mathcal{H}_{25,R}$ that control the FWER by $\alpha = 0.05$.

The minimax adaptive multiple tests we introduced in Chapter 4 Section II to localise the change with unknown heigh from such a known intensity are based on two kinds of statistics: a linear one and a quadratic one. In order to define an estimator for the rupture location from the linear statistics based multiple tests (see (4.11)), recall that we consider

$$\hat{k}^{(1)} = \sup\{k' \in \{1,\ldots,25\},\ \phi_{k'}^{(1)} = 0\} \vee 0 \ ,$$

where $\phi_k^{(1)}$ is the simple test defined for k in $\{1,\ldots,25\}$ by

$$\phi_k^{(1)}(N) = \mathbb{1}_{\left\{\max_{j\in\{1,\ldots,5\}}\left(N\left(\frac{k}{25}\left(1-2^{-j}\right),\frac{k}{25}\right]-p_{\frac{\lambda_0 k}{2^{j-1}}}\left(1-u_{\alpha,k}^{(1)}/2\right)\right)>0\right\}} \vee \mathbb{1}_{\left\{\max_{j\in\{1,\ldots,5\}}\left(p_{\frac{\lambda_0 k}{2^{j-1}}}\left(u_{\alpha,k}^{(1)}/2\right)-N\left(\frac{k}{25}\left(1-2^{-j}\right),\frac{k}{25}\right]\right)>0\right\}} \ ,$$

where $p_\xi(u)$ stands for the u-quantile of the Poisson distribution of parameter ξ. Following the same idea of Chapter 2 Section VI, $u_{\alpha,k}^{(1)}$ stands for an adjusted individual level for the test $\phi_k^{(1)}$ defined by

$$\begin{aligned} u_{\alpha,k}^{(1)} = \sup\Bigg\{u \in (0,1),\ P_{\lambda_0}\Bigg(\max_{j\in\{1,\ldots,5\}}\Bigg(\left(N\left(\frac{k}{25}\left(1-2^{-j}\right),\frac{k}{25}\right] - p_{\frac{\lambda_0 k}{2^{j-1}}}\left(1-u/2\right)\right) \\ \vee \left(p_{\frac{\lambda_0 k}{2^{j-1}}}\left(u/2\right) - N\left(\frac{k}{25}\left(1-2^{-j}\right),\frac{k}{25}\right]\right)\Bigg) > 0\Bigg) \le \alpha\Bigg\} \ . \end{aligned} \tag{4.24}$$

Finally, the first estimator for the rupture location, denoted by $\widehat{\tau_1}$, is then defined by

$$\widehat{\tau_1} = \frac{\hat{k}^{(1)}}{25} \ . \tag{4.25}$$

Now, let us turn to the minimax multiple testing procedure based on quadratic statistics (see (4.14)), which provide a second estimator for the jump location. Recall that we consider

$$\hat{k}^{(2)} = \sup\{k' \in \{1,\ldots,25\},\ \phi_{k'}^{(2)} = 0\} \vee 0 \ ,$$

where $\phi_k^{(2)}$ is the simple test defined for k in $\{1,\ldots,25\}$ by

$$\phi_k^{(2)}(N) = \mathbb{1}_{\max_{j\in\{1,\ldots,5\}}\left(T_{j,k}(N)-t_{j,k}\left(1-u_{\alpha,k}^{(2)}\right)\right)>0} \ ,$$

where for all j in $\{1,\ldots,5\}$,

$$T_{j,k}(N) = \frac{1}{25}\left(\frac{2^{j-1}}{2k}\left(N\left(\frac{k}{25}\left(1-\frac{1}{2^j}\right),\frac{k}{25}\right]^2 - N\left(\frac{k}{25}\left(1-\frac{1}{2^j}\right),\frac{k}{25}\right]\right) - \lambda_0 N\left(\frac{k}{25}\left(1-\frac{1}{2^j}\right),\frac{k}{25}\right] + \frac{\lambda_0^2 k}{2^j}\right) \ ,$$

$t_{j,k}(u)$ is the u-quantile of $T_{j,k}$ under $H_k[\lambda_0, R]$, and $u_{\alpha,k}^{(2)}$ is an adjusted individual level for the test $\phi_k^{(2)}$ defined like in the discussion of Chapter 2 Section VI by

$$u_{\alpha,k}^{(2)} = \sup\left\{u \in (0,1),\ P_{\lambda_0}\left(\max_{j\in\{1,\ldots,5\}}\left(T_{j,k}(N) - t_{j,k}(1-u)\right) > 0\right) \le \alpha\right\} \ . \tag{4.26}$$

Finally, the second estimator for the jump location, denoted by $\widehat{\tau_2}$, is then defined by

$$\widehat{\tau_2} = \frac{\hat{k}^{(2)}}{25} \ . \tag{4.27}$$

For all k in $\{1,\dots,25\}$ and all j in $\{1,\dots,5\}$, we have estimated the quantities $u^{(1)}_{\alpha,k}$, $u^{(2)}_{\alpha,k}$ and $t_{j,k}\left(1-u^{(2)}_{\alpha,k}\right)$ by classical Monte Carlo methods based on the simulation of 200 000 independent copies of a Poisson process random variable with parameter $\lambda_0 k/2^{j-1}$ or $T_{j,k}$ under $H_k[\lambda_0, R]$. The approximations of $u^{(1)}_{\alpha,k}$ and $u^{(2)}_{\alpha,k}$ were obtained by dichotomy, such that the probabilities or estimated probabilities occurring in (4.24) and (4.26) are less than α, but as close to α as possible.

We evaluate the performance of both estimators for different intensities of the Poisson process. Let us consider intensities $\lambda_{\tau,\delta}$ defined for all t in $[0,1]$ by

$$\lambda_{\tau,\delta}(t) = 1 + \delta \mathbb{1}_{(\tau,1]}(t) \ ,$$

where $\delta \in \{-0.8, -0.5, 0.5, 1, 2\}$, and $\tau = 0.2$ (Table 4.1), $\tau = 0.5$ (Table 4.2), $\tau = 0.8$ (Table 4.3) and $\tau = 0.9$ (Table 4.4), or when $(\delta,\tau) = (0,1)$ (Table 4.5). For each intensities $\lambda_{\tau,\delta}$ described above, we compute the risk of the corresponding estimators $\widehat{\tau}_1$ and $\widehat{\tau}_2$ defined respectively by (4.25) and (4.27), denoted by

$$r(\widehat{\tau}_1,\tau) = E_{\lambda_{\tau,\delta}}[|\widehat{\tau}_1 - \tau|], \ \text{ and } \ r(\widehat{\tau}_2,\tau) = E_{\lambda_{\tau,\delta}}[|\widehat{\tau}_2 - \tau|] \ .$$

For each considered intensities, 1 000 independent Poisson process with intensity $\lambda_{\tau,\delta}$ w.r.t. Λ on $[0,1]$ have been simulated and the risks $r(\widehat{\tau}_1,\tau)$ and $r(\widehat{\tau}_2,\tau)$ have been estimated by classical Monte Carlo method.

Table 4.1 – Estimated risk for the jump location estimators with $\tau = 0.2$

$\delta =$	-0.8	-0.5	0.5	1	2
$r(\widehat{\tau}_1,\tau)$	0.198	0.721	0.756	0.429	0.048
$r(\widehat{\tau}_2,\tau)$	0.242	0.743	0.753	0.429	0.045

Table 4.2 – Estimated risk for the jump location estimators with $\tau = 0.5$

$\delta =$	-0.8	-0.5	0.5	1	2
$r(\widehat{\tau}_1,\tau)$	0.194	0.430	0.458	0.268	0.053
$r(\widehat{\tau}_2,\tau)$	0.220	0.449	0.456	0.263	0.053

Table 4.3 – Estimated risk for the jump location estimators with $\tau = 0.8$

$\delta =$	-0.8	-0.5	0.5	1	2
$r(\widehat{\tau}_1,\tau)$	0.154	0.190	0.186	0.145	0.048
$r(\widehat{\tau}_2,\tau)$	0.160	0.192	0.185	0.144	0.047

Table 4.4 – Estimated risk for the jump location estimators with $\tau = 0.9$

$\delta =$	-0.8	-0.5	0.5	1	2
$r(\widehat{\tau}_1,\tau)$	0.091	0.097	0.092	0.078	0.041
$r(\widehat{\tau}_2,\tau)$	0.093	0.097	0.091	0.076	0.040

Table 4.5 – Estimated risk for the jump location estimators with $\tau = 1$ and $\delta = 0$

$r(\widehat{\tau}_1, \tau)$	$r(\widehat{\tau}_2, \tau)$
0.002	0.002

Comments

1. It first arises that our jump location estimators are close to the target τ for large positive jumps and when the change point occurs near to 1. Nevertheless, the estimated risks of the estimators build from minimax and adaptive multiple tests are very large in other cases: when the jump does not occur near to 1, the presence of false negative hypotheses deteriorates the estimation of the change location, especially for small change heights (that is to say when the change-point is more difficult to detect). Therefore, the simulations study confirms that the tricky point in practice is the choice of the number of hypotheses M that are simultaneously tested by the multiple testing procedures.
2. Moreover, among our two estimators, it is to note that $\widehat{\tau}_1$, defined from multiple tests based on linear statistics, offers better performances than $\widehat{\tau}_2$, defined from multiple tests based on quadratic statistics, for negative jump heights when the jump location does not occur near to 1. Finally, the simulations study highlights that it seems to be more difficult to localise the change for large negative jump heights than for large positive jump heights. Notice that this fact was also observed for the detection problem (see Chapter 2 Section VIII and Chapter 3 Section VII).

V Proofs of the main results

V.1 Proof of Theorem 4.3

By now, for $\lambda_0 > 0$, δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$ and M in $\mathbb{N}^*$, the simple hypothesis $H_k[\lambda_0, \delta^*]$ is simply written H_k for short. We begin with a lemma which gives an upper bound for the quantile $s_{\delta^*,k}(1-\alpha)$ of $S_{\delta^*,k}$ under H_k. It highlights in particular that we can bound this quantile by some constants which do not depend on k, L and M. The proof of Lemma 4.17 follows the same ideas of the one of Lemma 2.30.

Lemma 4.17 (Control of the quantiles).
Let α in $(0,1)$, $\lambda_0 > 0$ and δ^* in $(-\lambda_0, +\infty) \setminus \{0\}$. For all M in $\mathbb{N}^*$, for all $L \geq 1$ and for all $k \in \{1, \ldots, M\}$, one has

$$\begin{cases} s_{\delta^*,k}(1-\alpha) \leq q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right) & \text{if } \delta^* > 0 \ , \\ s_{\delta^*,k}(1-\alpha) \leq \dfrac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)} & \text{if } -\lambda_0 < \delta^* < 0 \ , \end{cases} \tag{4.28}$$

where $q_{\lambda_0}(1-\alpha, \delta^*/2)$ is defined in Lemma 4.2.

Proof of Lemma 4.17.
Let k in $\{1, \ldots, M\}$. Under (H_k), N is a homogeneous Poisson process on $[0, k/M]$ of intensity λ_0 with respect to the measure Λ, and since the processes $(N(t, k/M])_{t\in(0,k/M)}$ and $(N(0, k/M - t])_{t\in(0,k/M)}$ are left continuous and have the same finite dimensional laws, one obtains

$$S_{\delta^*,k} \overset{d}{=} \sup_{t\in(0,k/M)} \left(\operatorname{sgn}(\delta^*)(N(0,t] - \lambda_0 Lt) - \frac{|\delta^*|}{2} Lt \right) . \tag{4.29}$$

Assume first that $\delta^* > 0$. The equality (4.29) simply reads in this case

$$S_{\delta^*,k} \overset{d}{=} \sup_{t\in(0,k/M)} \left(N(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right) Lt \right) ,$$

and we get for all λ in H_k,

$$
\begin{aligned}
P_\lambda\left(S_{\delta^*,k} > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right) &= \mathbb{P}\left(\sup_{t\in(0,k/M)}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right)Lt\right) > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right) \\
&\leq \mathbb{P}\left(\sup_{t\in[0,+\infty)}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 + \frac{\delta^*}{2}\right)Lt\right) > q_{\lambda_0}\left(1-\alpha, \frac{\delta^*}{2}\right)\right) \ ,
\end{aligned}
$$

where $(N_t^{\lambda_0})_{t\geq 0}$ is a homogeneous Poisson process of intensity $\lambda_0 L$ with respect to the Lebesgue measure on $\mathbb{R}^+$. By definition of $q_{\lambda_0}(1-\alpha, \delta^*/2)$ (see Lemma 4.2), this leads to $\sup_{\lambda\in H_k} P_\lambda\left(S_{\delta^*,k} > q_{\lambda_0}(1-\alpha,\delta^*/2)\right) \leq \alpha$ and the first part in (4.28) holds by definition of the quantile $s_{\delta^*,k}(1-\alpha)$.
Assume now that δ^* belongs to $(-\lambda_0, 0)$. For all $x > 0$ and for all λ in H_k,

$$
\begin{aligned}
P_\lambda\left(S_{\delta^*,k} > x\right) &= P_\lambda\left(\sup_{t\in(0,k/M)}\left(\left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) - N(0,t] > x\right) \\
&= \mathbb{P}\left(\sup_{t\in(0,k/M)}\left(\left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) - N^{\lambda_0}(0,t] > x\right) \\
&\leq \mathbb{P}\left(\inf_{t\geq 0}\left(N^{\lambda_0}(0,t] - \left(\lambda_0 - \frac{|\delta^*|}{2}\right)Lt\right) < -x\right) \ .
\end{aligned}
$$

Theorem 3 and equation (15) in [Pyk59] yield for all λ in H_k, $P_\lambda\left(S_{\delta^*,k} > x\right) \leq \exp(-\omega x)$ where ω is the largest real root of the equation $\lambda_0(1-e^{-\omega}) = \omega(\lambda_0 - |\delta^*|/2)$. A straightforward study of the function $x \mapsto \lambda_0(1-e^{-x}) - x(\lambda_0 - |\delta^*|/2)$ on $\mathbb{R}$ therefore ensures that ω satisfies $\omega > \log(\lambda_0/(\lambda_0 - |\delta^*|/2))$. If we assume that $x \geq -\log\alpha/\log(\lambda_0/(\lambda_0 - |\delta^*|/2))$, then $\sup_{\lambda\in H_k} P_\lambda\left(S_{\delta^*,k} > x\right) \leq \alpha$ and the second part of (4.28) holds by definition of the quantile $s_{\delta^*,k}(1-\alpha)$. □

Let us turn back to the proof of Theorem 4.3 and first recall that for λ in $\mathcal{S}[\lambda_0,\delta^*]$, $\mathcal{T}(\lambda) = \{H_k \in \mathcal{H}_{M,\delta^*},\ \lambda \in H_k\}$ is the set of true hypotheses.

Proof of Theorem 4.3.
For all k in $\{1,\ldots,M\}$, recall that H_k stands for $H_k[\lambda_0,\delta^*]$. We start with the control of FWER($\mathcal{R}_1$) over $\overline{\mathcal{S}}[\lambda_0,\delta^*]$, and for λ in $\overline{\mathcal{S}}[\lambda_0,\delta^*]$ we compute to this end

$$
P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = P_\lambda\left(\exists k \in \{1,\ldots,\lfloor\tau M\rfloor\},\ k \geq \hat{k}_1 + 1,\ \lambda \in H_k\right)
$$

because λ belongs to $H_{\lfloor\tau M\rfloor}$ and not to $H_{\lfloor\tau M\rfloor+1}$. If $\tau < 1/M$ then $P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = 0$, and if $\tau \geq 1/M$ one has

$$
\begin{aligned}
P_\lambda\left(\mathcal{R}_1 \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda(\hat{k}_1 + 1 \leq \lfloor\tau M\rfloor) \\
&= P_\lambda\left(\phi_{1,\lfloor\tau M\rfloor} = 1\right) \\
&= P_\lambda\left(S_{\delta^*,\lfloor\tau M\rfloor}(\delta^*) > s_{\delta^*,\lfloor\tau M\rfloor}(1-\alpha)\right) \\
&\leq \alpha \ ,
\end{aligned}
$$

that is FWER($\mathcal{R}_1$) is bounded by α.

Let us compute now an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_1, \mathcal{S}[\lambda_0, \delta^*])$. We shall consider in our results the following constant

$$C(\alpha,\beta,\lambda_0,\delta^*) = 2\max\left(\sqrt{|\delta^*|}\left(\sqrt{q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)+\frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{\delta^*>0}\right.\right.$$
$$\left.\left.+\sqrt{q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right)+\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{-\lambda_0<\delta^*<0}\right),\ 2\sqrt{\frac{3(\lambda_0+\delta^*)}{\beta}}\right), \tag{4.30}$$

where $q_{\lambda_0}(1-\alpha,\delta^*/2)$ for $\delta^*>0$ and $q_{\lambda_0+\delta^*}(1-\beta/3,|\delta^*|/2)$ for $-\lambda_0<\delta^*<0$ are two positive constants defined in Lemma 4.2. Recall that (4.3) leads to $\mathrm{FWSR}_\beta(\mathcal{R}_1,\mathcal{S}[\lambda_0,\delta^*]) \le |\delta^*|$. Now, assume that $L > (C(\alpha,\beta,\lambda_0,\delta^*)/\delta^*)^2$ and let $r>0$ be such that

$$|\delta^*| > r \ge \frac{C(\alpha,\beta,\lambda_0,\delta^*)}{\sqrt{L}}, \tag{4.31}$$

where $C(\alpha,\beta,\lambda_0,\delta^*)$ is defined by (4.30).
Recall that for λ in $\mathcal{S}[\lambda_0,\delta^*]$, $\mathcal{F}_r(\lambda)=\{H_k \in \mathcal{H}_{M,\delta^*},\ d_2(\lambda,H_k)\ge r\}$ with $d_2(\lambda,H_k)=|\delta^*|\sqrt{k/M-\tau}\mathbb{1}_{\tau\le k/M}$, and that to bound $\mathrm{FWSR}_\beta(\mathcal{R}_1,\mathcal{S}[\lambda_0,\delta^*])$ by r, it is sufficient to obtain $P_\lambda(\mathcal{F}_r(\lambda)\subset\mathcal{R}_1)\ge 1-\beta$ for all λ in $\mathcal{S}[\lambda_0,\delta^*]$.
We consider λ in $\mathcal{S}[\lambda_0,\delta^*]$ of the form $\lambda=\lambda_0+\delta^*\mathbb{1}_{(\tau,1]}$ with τ in $(0,1)$. If $\mathcal{F}_r(\lambda)=\varnothing$, we easily get $P_\lambda(\mathcal{F}_r(\lambda)\subset\mathcal{R}_1)=1\ge 1-\beta$. We therefore assume by now that λ is satisfying $\mathcal{F}_r(\lambda)\neq\varnothing$, and we define

$$k_r = \min\{\tau M < k' \le M,\ \delta^{*2}(k'/M-\tau)\ge r^2\}. \tag{4.32}$$

By virtue of $\{\mathcal{F}_r(\lambda)\subset\mathcal{R}_1\}=\{k_r\ge\hat{k}_1+1\}$, we want to prove the following inequality

$$P_\lambda\left(\hat{k}_1\ge k_r\right)\le\beta$$

to obtain the expected result.
First, if $k_r=M$ then

$$\begin{aligned}
P_\lambda\left(\hat{k}_1\ge k_r\right) &= P_\lambda\left(\hat{k}_1=M\right)\\
&= P_\lambda(\phi_{1,M}=0)\\
&= P_\lambda(S_{\delta^*,M}\le s_{\delta^*,M}(1-\alpha))\\
&= P_\lambda\left(\sup_{t\in(0,1)}\left(\mathrm{sgn}(\delta^*)(N(t,1]-\lambda_0L(1-t))-\frac{|\delta^*|}{2}L(1-t)\right)\le s_{\delta^*,M}(1-\alpha)\right).
\end{aligned}$$

By definition of k_r, since the condition (4.31) ensures in particular that

$$|\delta^*|\sqrt{1-\tau}\ge\frac{2}{\sqrt{L}}\max\left(\sqrt{\delta^* q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)}\mathbb{1}_{\delta^*>0}+\sqrt{\frac{|\delta^*|\log(1/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0+\delta^*/2}\right)}}\mathbb{1}_{-\lambda_0<\delta^*<0},\ 2\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\right),$$

we immediately obtain that

$$P_\lambda\left(\sup_{t\in(0,1)}\left(\mathrm{sgn}(\delta^*)(N(t,1]-\lambda_0L(1-t))-\frac{|\delta^*|}{2}L(1-t)\right)\le s_{\delta^*,M}(1-\alpha)\right)\le\beta,$$

according to the minimax study of the change-point detection done in Proposition 2.15 which involves the statistic $\sup_{t\in(0,1)}(\mathrm{sgn}(\delta^*)(N(t,1]-\lambda_0L(1-t))-|\delta^*|L(1-t)/2)$.

Assume by now that $k_r \le M-1$, and we compute

$$\begin{aligned}
P_\lambda\left(\hat{k}_1 \ge k_r\right) &= P_\lambda\left(\exists k \ge k_r, \phi_{1,k}=0\right) \\
&= P_\lambda\left(\exists k \ge k_r,\ S_{\delta^*,k} \le s_{\delta^*,k}(1-\alpha)\right) \\
&= P_\lambda\left(\exists k \ge k_r,\ \sup_{t\in(0,k/M)}\left(\operatorname{sgn}(\delta^*)\left(N\left(t,\frac{k}{M}\right]-\lambda_0 L\left(\frac{k}{M}-t\right)\right)-\frac{|\delta^*|}{2}L\left(\frac{k}{M}-t\right)\right)\le s_{\delta^*,k}(1-\alpha)\right) \\
&\le P_\lambda\left(\exists k \ge k_r,\ \operatorname{sgn}(\delta^*)\left(N\left(\tau,\frac{k}{M}\right]-\lambda_0 L\left(\frac{k}{M}-\tau\right)\right)-\frac{|\delta^*|}{2}L\left(\frac{k}{M}-\tau\right)\le s_{\delta^*,k}(1-\alpha)\right)\ .
\end{aligned} \tag{4.33}$$

Assume first that $\delta^* > 0$.
We use (4.28) in Lemma 4.17 to get

$$\begin{aligned}
P_\lambda\left(\hat{k}_1 \ge k_r\right) &\le P_\lambda\left(\exists k \ge k_r,\ N\left(\tau,\frac{k}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k}{M}-\tau\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right) \\
&= P_\lambda\left(\inf_{k\in[k_r,M]}\left(N\left(\tau,\frac{k}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k}{M}-\tau\right)\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right) \\
&\le P_\lambda\left(\inf_{s\in[k_r/M,1]}\left(N(\tau,s]-\left(\lambda_0+\frac{\delta^*}{2}\right)L(s-\tau)\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right) \\
&= P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0+\frac{\delta^*}{2}\right)L\left(\frac{k_r}{M}-\tau\right)+\inf_{s\in(k_r/M,1]}Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)\right) \\
&= P_\lambda\left(\inf_{s\in(k_r/M,1]}Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-N\left(\tau,\frac{k_r}{M}\right]+\left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right)\ .
\end{aligned}$$

where $Z_t = N(k_r/M,t]-(\lambda_0+\delta^*/2)(t-k_r/M)L$ for t in $(k_r/M,1]$. Let us write J for the interval

$$J=\left[(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}};\ (\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right]\ . \tag{4.34}$$

Using the total probability formula, we get

$$\begin{aligned}
P_\lambda\left(\hat{k}_1 \ge k_r\right) &\le P_\lambda\left(\inf_{s\in(k_r/M,1]}Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-N\left(\tau,\frac{k_r}{M}\right]+\left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right) \\
&\quad + P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]<(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right) \\
&\quad + P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]>(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right) \\
&\le P_\lambda\left(\inf_{s\in(k_r/M,1]}Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-N\left(\tau,\frac{k_r}{M}\right]+\left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)+\frac{2\beta}{3}
\end{aligned} \tag{4.35}$$

with the Bienayme-Chebyshev inequality. To compute this last probability, we consider a simple Poisson process $(N_t^{\lambda_0+\delta^*})_{t\ge 0}$ of intensity $(\lambda_0+\delta^*)L$ with respect to the Lebesgue measure on $\mathbb{R}^+$, which is the distribution of N_t for t greater than k_r/M. We then obtain

$$\begin{aligned}
&P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\delta^*}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\,,\; N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\
&\quad\le P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\
&\quad= \mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L\right.\\
&\qquad\qquad\left. + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\,.
\end{aligned}$$

By definition of k_r in (4.32), the condition (4.31) gives with (4.30)

$$\delta^*\sqrt{\frac{k_r}{M}-\tau} \ge \frac{2}{\sqrt{L}}\max\left(\sqrt{\delta^*}\sqrt{q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}}\,,\; 2\sqrt{3}\sqrt{\frac{\lambda_0+\delta^*}{\beta}}\right)\,. \tag{4.36}$$

On the one hand, (4.36) leads to $\delta^*\sqrt{k_r/M-\tau} \ge 2\sqrt{\delta^*/L}\sqrt{q_{\lambda_0}(1-\alpha,\delta^*/2)+\log(3/\beta)/\log((\lambda_0+\delta^*)/(\lambda_0+\delta^*/2))}$, and then $\delta^*(k_r/M-\tau)L \ge 4\left(q_{\lambda_0}(1-\alpha,\delta^*/2)+\log(3/\beta)/\log((\lambda_0+\delta^*)/(\lambda_0+\delta^*/2))\right)$. On the other hand, (4.36) yields $\delta^*\sqrt{k_r/M-\tau} \ge 4\sqrt{3(\lambda_0+\delta^*)/(\beta L)}$ and then $\delta^*(k_r/M-\tau)L \ge 4\sqrt{3(\lambda_0+\delta^*)(k_r/M-\tau)L/\beta}$. This leads to

$$\delta^*\left(\frac{k_r}{M}-\tau\right)L \ge 4\max\left(q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}\,,\; \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\,,$$

and using the fact that $a+b\le 2\max(a,b)$ for all $a,b\ge 0$, one obtains

$$q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}} \le \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)}\,, \tag{4.37}$$

which is equivalent to

$$\exp\left(\left(q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)\right) \le \frac{\beta}{3}\,. \tag{4.38}$$

Notice that since $\beta<1$, (4.37) ensures

$$q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}} \le 0\,. \tag{4.39}$$

We therefore may apply Theorem 3 and equation (15) in [Pyk59] to obtain

$$\begin{aligned}
&\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t] - \left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right) \le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\
&\qquad\le \exp\left(\left(q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right) - \frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\omega\right)\,,
\end{aligned}$$

where ω is the largest real root of the equation $(\lambda_0+\delta^*)(1-e^{-\omega}) = \omega(\lambda_0+\delta^*/2)$. The root ω satisfies $\omega > \log((\lambda_0+\delta^*)/(\lambda_0+\delta^*/2))$, and then

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t]-\left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-\frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)$$
$$\le \exp\left(\left(q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-\frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta^*}{\lambda_0+\delta^*/2}\right)\right),$$

which entails with (4.38)

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta^*}(0,t]-\left(\lambda_0+\frac{\delta^*}{2}\right)Lt\right)\le q_{\lambda_0}\left(1-\alpha,\frac{\delta^*}{2}\right)-\frac{\delta^*}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\le\frac{\beta}{3}.$$

Gathering this inequality with (4.35) leads finally to $P_\lambda\left(\hat{k}_1\ge k_r\right)\le\beta$.

Now, assume that $-\lambda_0<\delta^*<0$.

We proceed as in the case $\delta^*>0$. Let us define to this end $X_t = N(k_r/M,t]-(\lambda_0-|\delta^*|/2)(t-k_r/M)L$ for all t in $(k_r/M,1]$. Applying the inequalities (4.28) and (4.33), we get

$$P_\lambda\left(\hat{k}_1\ge k_r\right)\le P_\lambda\left(\exists k\ge k_r,\ \left(\lambda_0-\frac{|\delta^*|}{2}\right)L\left(\frac{k}{M}-\tau\right)-N\left(\tau,\frac{k}{M}\right]\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}\right)$$
$$\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right).$$

Using the interval J defined by (4.34), we obtain using the total probability formula and the Bienayme-Chebyshev inequality

$$\begin{aligned}P_\lambda\left(\hat{k}_1\ge k_r\right)&\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M}-\tau\right)L,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\&\quad+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]<(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\&\quad+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]>(\lambda_0+\delta^*)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\&\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M}-\tau\right)L,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)+\frac{2\beta}{3}.\end{aligned}\tag{4.40}$$

We conclude the proof giving an upper bound for this last probability. We compute

$$\begin{aligned}
&P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\,,\,N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\
&\quad\le P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\le\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}-\frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\
&\quad=P_\lambda\left(\sup_{s\in(k_r/M,1]}X_s\ge\frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+\frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\
&\quad\le\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right)\ge\frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+\frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right),
\end{aligned}\tag{4.41}$$

By definition of k_r in (4.32), the condition (4.31) gives with (4.30),

$$|\delta^*|\sqrt{\frac{k_r}{M}-\tau}\ge\frac{2}{\sqrt{L}}\max\left(\sqrt{|\delta^*|}\sqrt{q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right)+\frac{-\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}}\,,\,2\sqrt{\frac{3(\lambda_0+\delta^*)}{\beta}}\right),$$

which ensures, as in the case $\delta^*>0$, that

$$\frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+\frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\ge q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right).$$

We therefore obtain with (4.41)

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right)\ge\frac{\log\alpha}{\log\left(\frac{\lambda_0}{\lambda_0-|\delta^*|/2}\right)}+\frac{|\delta^*|}{2}\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta^*)(k_r/M-\tau)L}{\beta}}\right)\\
&\quad\le\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta^*}(0,t]-\left(\lambda_0-\frac{|\delta^*|}{2}\right)Lt\right)\ge q_{\lambda_0+\delta^*}\left(1-\frac{\beta}{3},\frac{|\delta^*|}{2}\right)\right)\\
&\quad\le\frac{\beta}{3}
\end{aligned}$$

by definition of $q_{\lambda_0+\delta^*}(1-\beta/3,|\delta^*|/2)$ in Lemma 4.2. The proof is then complete using (4.40).

□

V.2 Proof of Theorem 4.5

For $\lambda_0>0$ and $R>\lambda_0$, the simple hypothesis $H_k[\lambda_0,R]$ is denoted H_k for short and recall that u_α stands for $\alpha/\lfloor\log_2 L\rfloor$. Recall that an upper bound for the u-quantile $p_\xi(u)$ of the Poisson distribution of parameter $\xi>0$ is given by Lemma 2.25 (equation (2.116)).

For τ in $(0,1)$, let us define now $k_\tau=\min\{k'\in\{1,\ldots,M\},\ k'>\tau M\}$ and

$$j_\tau(k)=\left\lceil-\log_2\left(1-\frac{\tau M}{k}\right)\right\rceil\wedge\lfloor\log_2(L)\rfloor\tag{4.42}$$

for $k \geq k_\tau$, in order to get $j_\tau(k)$ in $\{1,\ldots,\lfloor \log_2(L)\rfloor\}$ as well as the inequalities

$$\frac{k}{M}\left(1-\frac{1}{2^{j_\tau(k)-1}}\right) < \tau \leq \frac{k}{M}\left(1-\frac{1}{2^{j_\tau(k)}}\right) \tag{4.43}$$

under the following condition

$$\left\lceil -\log_2\left(1-\frac{\tau M}{k}\right)\right\rceil \leq \lfloor \log_2(L)\rfloor \ . \tag{4.44}$$

We shall consider also a cover of the interval $[k_\tau, M]$, denoted $\mathcal{P} = \cup_{i=0}^{\Psi-1}[x_i, x_{i+1})$ where Ψ in $\mathbb{N}^*$ is the cardinal of the cover $\mathcal{P}$ and the real $x_0 < \ldots < x_\Psi$ satisfy $x_0 = k_\tau$ and $x_\Psi > M$. Note that the two real x_i and x_{i+1} may depend on λ_0, τ, L, M and k_τ. We will write $\sum_{[c,d)\in\mathcal{P}}$ for the sum over each disjoint interval $[c,d)$ of the cover $\mathcal{P}$. Assuming (4.44), the key argument in the proof of Theorem 4.5 will consist on giving an upper bound for the probabilities $P_\lambda(\sup_{k\in\mathcal{P}}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\in\mathcal{P}}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) \geq 0)$, instead of the probabilities $P_\lambda(\sup_{k\geq k_\tau}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\geq k_\tau}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) \geq 0)$ in order to get more refined bounds. The following technical lemmas allow us to bound $P_\lambda(\sup_{k\in[c,d)}(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]) \geq 0)$ and $P_\lambda(\sup_{k\in[c,d)}(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)) \geq 0)$ for each interval $[c,d)$ of the cover $\mathcal{P}$, using an exponential inequality related to the oscillation modulus of some martingales developed in [LG21, Theorem 8].

Lemma 4.18.
Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume (4.44) and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:

$$\inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(|\delta| - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right) \geq 2\left(\frac{d}{M}-\tau\right)L(\lambda_0+\delta)g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)}\right) \ , \tag{4.45}$$

where the function g is defined by (2.44). Then, when $\delta > 0$

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M]\right) \geq 0\right) \leq \beta \ ,$$

and when $\delta < 0$

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2)\right) \geq 0\right) \leq \beta \ .$$

Proof of Lemma 4.18.
Let λ in $\mathcal{S}[\lambda_0, R]$ be such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$, where δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$. Notice that for all k in $\{1,\ldots,M\}$ and j in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$, the counting statistic $N(k(1-2^{-j})/M, k/M]$ can be written

$$N(k(1-2^{-j})/M, k/M] = M_{k(1-2^{-j})/M}^{k/M} + B_{k(1-2^{-j})/M, k/M} \ , \tag{4.46}$$

where for all $0 \leq s < t$, $M_s^t = \int_s^t (dN_x - \lambda(x)Ldx)$ and $B_{s,t} = \int_s^t \lambda(x)Ldx$ is the bias of $N(s,t]$.
Assume first that $0 < \delta \leq R - \lambda_0$. For $[c,d)$ in $\mathcal{P}$ we define

$$\rho_{c,d}^{(1)} = \inf_{k\in[c,d)}\left(B_{k(1-2^{-j_\tau(k)})/M, k/M} - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right) \ ,$$

and we prove that

$$\varrho_{c,d}^{(1)} \geq \inf_{k\in[c,d)} \left(\frac{Lk}{M2^{j_\tau(k)}} \left(\delta - \lambda_0 g^{-1} \left(\frac{\log(2/u_\alpha) M 2^{j_\tau(k)}}{\lambda_0 k L} \right) \right) \right) \ . \tag{4.47}$$

Let k in $[c,d)$ and we compute

$$\begin{aligned} B_{k(1-2^{-j_\tau(k)})/M, k/M} - p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) &= (\lambda_0+\delta)L\frac{k}{M2^{j_\tau(k)}} - p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) \text{ with (4.43)} \\ &\geq \frac{Lk}{M2^{j_\tau(k)}} \left(\delta - \lambda_0 g^{-1} \left(\frac{\log(2/u_\alpha) M 2^{M2^{j_\tau(k)}}}{\lambda_0 k L} \right) \right) \text{ with (2.116)} \ , \end{aligned}$$

that is (4.47). Notice in particular that the condition (4.45) ensures that for all k in $[c,d)$,

$$B_{k(1-2^{-j_\tau(k)})/M, k/M} - p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) \geq 0 \ . \tag{4.48}$$

Using the condition (4.45) again we obtain

$$\varrho_{c,d}^{(1)} \geq 2\left(\frac{d}{M} - \tau\right) L(\lambda_0+\delta) g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)} \right) \ ,$$

which is equivalent to

$$2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta) L g\left(\frac{\varrho_{c,d}^{(1)}}{2(d/M-\tau)L(\lambda_0+\delta)} \right) \right) \leq \frac{\beta}{\Psi} \ . \tag{4.49}$$

Finally, we get the following inequalities

$$\begin{aligned} &P_\lambda\left(\sup_{k\in[c,d)} \left(p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M] \right) \geq 0 \right) \\ &= P_\lambda\left(\sup_{k\in[c,d)} \left(p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M, k/M} - M^{k/M}_{k(1-2^{-j_\tau(k)})/M} \right) \geq 0 \right) \text{ with (4.46)} \\ &\leq P_\lambda\left(\exists k \in [c,d), \ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}| \geq B_{k(1-2^{-j_\tau(k)})/M, k/M} - p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) \right) \text{ with (4.48)} \\ &\leq P_\lambda\left(\exists k \in [c,d), \ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}| \geq \varrho_{c,d}^{(1)} \right) \\ &\leq P_\lambda\left(\sup_{s,t\in[\tau,d/M]} |M_s^t| \geq \varrho_{c,d}^{(1)} \right) \text{ with (4.43)} \ , \end{aligned}$$

and Theorem 8 in [LG21] leads to

$$P_\lambda\left(\sup_{s,t\in[\tau,d/M]} |M_s^t| \geq \varrho_{c,d}^{(1)} \right) \leq 2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta) L g\left(\frac{\varrho_{c,d}^{(1)}}{2(d/M-\tau)L(\lambda_0+\delta)} \right) \right) \ .$$

Therefore (4.49) yields

$$P_\lambda\left(\sup_{k\in[c,d)} \left(p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2) - N(k(1-2^{-j_\tau(k)})/M, k/M] \right) \geq 0 \right) \leq \frac{\beta}{\Psi} \ ,$$

and the result follows summing over all the interval of the cover $\mathcal{P}$.

Assume now that $-\lambda_0 < \delta < 0$ and define for $[c,d)$ in $\mathcal{P}$

$$\rho^{(2)}_{c,d} = \inf_{k\in[c,d)} \left(p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M,k/M} \right) \ .$$

As for the case where $\delta > 0$, the equations (2.116) and (4.43) entails that for all k in $[c,d)$

$$p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M,k/M} \geq \frac{Lk}{M2^{j_\tau(k)}} \left(|\delta| - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL} \right)\right) \ ,$$

and we get

$$\rho^{(2)}_{c,d} \geq \inf_{k\in[c,d)} \left(\frac{Lk}{M2^{j_\tau(k)}} \left(|\delta| - \lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL} \right)\right)\right) \ . \tag{4.50}$$

The condition (4.45) ensures on the one hand that for all k in $[c,d)$

$$p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M,k/M} \geq 0 \ , \tag{4.51}$$

and on the other hand that

$$2\exp\left(-\left(\frac{d}{M} - \tau\right)(\lambda_0+\delta)Lg\left(\frac{\rho^{(2)}_{c,d}}{2(d/M-\tau)L(\lambda_0+\delta)} \right)\right) \leq \frac{\beta}{\Psi} \ . \tag{4.52}$$

Finally, we get the following inequalities

$$\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)} \left(N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) \right) \geq 0 \right) \\
&\leq P_\lambda\left(\exists k \in [c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}| \geq p_{\lambda_0 kL2^{-j_\tau(k)}/M}(u_\alpha/2) - B_{k(1-2^{-j_\tau(k)})/M,k/M} \right) \text{ with (4.51)} \\
&\leq P_\lambda\left(\exists k \in [c,d),\ |M^{k/M}_{k(1-2^{-j_\tau(k)})/M}| \geq \rho^{(2)}_{c,d} \right) \\
&\leq P_\lambda\left(\sup_{s,t\in[\tau,d/M]} |M^t_s| \geq \rho^{(2)}_{c,d} \right) \text{ with (4.43)} \ ,
\end{aligned}$$

and we conclude with the same lines as above using the Theorem 8 in [LG21], the inequality (4.52) and summing over all the interval of the cover $\mathcal{P}$. □

Lemma 4.19.
Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume (4.44) and that for all interval $[c,d)$ of the cover $\mathcal{P}$ one has

$$\frac{d/M - \tau}{c/M - \tau} = 2 \ , \tag{4.53}$$

and

$$|\delta|\sqrt{\frac{c}{M} - \tau} \geq 2\max\left(\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(2/u_\alpha)}{L} + \frac{2\log(2\Psi/\beta)}{L}},\ 4\sqrt{\frac{\lambda_0\log(2/u_\alpha)}{L}} + 8\sqrt{\frac{R\log(2\Psi/\beta)}{L}} \right) \ . \tag{4.54}$$

Then the inequality (4.45) is satisfied.

Proof of Lemma 4.19.
Let λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0, R-\lambda_0] \setminus \{0\}$ and τ in $(0,1)$. Assume first that $0 < \delta \le R - \lambda_0$ and let $[c,d)$ in $\mathcal{P}$. Notice that (4.54) entails on the one hand

$$\delta \sqrt{\frac{c}{M} - \tau} \ge 8 \left(\sqrt{\frac{\lambda_0 \log(2/u_\alpha)}{L}} + 2 \sqrt{\frac{R \log(2\Psi/\beta)}{L}} \right) ,$$

and then, since $\lambda_0 + \delta \le R$,

$$\delta \left(\frac{c}{M} - \tau \right) L \ge 8 \sqrt{\left(\frac{c}{M} - \tau \right) L} \left(\sqrt{\lambda_0 \log\left(\frac{2}{u_\alpha} \right)} + 2 \sqrt{(\lambda_0 + \delta) \log\left(\frac{2\Psi}{\beta} \right)} \right) ,$$

hence, using (4.53),

$$\delta \left(\frac{c}{M} - \tau \right) L \ge 4 \sqrt{2 \frac{d/M - \tau}{c/M - \tau} \left(\frac{c}{M} - \tau \right) L} \left(\sqrt{\lambda_0 \log\left(\frac{2}{u_\alpha} \right)} + 2 \sqrt{(\lambda_0 + \delta) \log\left(\frac{2\Psi}{\beta} \right)} \right) . \tag{4.55}$$

On the other hand, (4.54) yields

$$\delta \sqrt{\frac{c}{M} - \tau} \ge 2 \sqrt{\frac{2R}{3}} \sqrt{\frac{\log(2/u_\alpha)}{L} + \frac{2\log(2\Psi/\beta)}{L}} ,$$

and then, since $\delta < R$,

$$\delta^2 \left(\frac{c}{M} - \tau \right) \ge \frac{8\delta}{3} \left(\frac{\log(2/u_\alpha)}{L} + \frac{2\log(2\Psi/\beta)}{L} \right) ,$$

hence

$$\delta \left(\frac{c}{M} - \tau \right) L \ge \frac{8}{3} \log\left(\frac{2}{u_\alpha} \right) + \frac{16}{3} \log\left(\frac{2\Psi}{\beta} \right) . \tag{4.56}$$

Gathered together, the conditions (4.55) and (4.56) give

$$\delta \left(\frac{c}{M} - \tau \right) L \ge 2 \max\left(\frac{4}{3} \log\left(\frac{2}{u_\alpha} \right) + \frac{8}{3} \log\left(\frac{2\Psi}{\beta} \right), 2\sqrt{2\left(\frac{d}{M} - \tau \right) L} \left(\sqrt{\lambda_0 \log\left(\frac{2}{u_\alpha} \right)} + 2 \sqrt{(\lambda_0 + \delta) \log\left(\frac{2\Psi}{\beta} \right)} \right) \right) ,$$

and since $2\max(a,b) \ge a + b$ for all $a, b \ge 0$, we get

$$\frac{1}{2} \delta \left(\frac{c}{M} - \tau \right) L \ \ge \ \frac{2}{3} \log\left(\frac{2}{u_\alpha} \right) + \frac{4}{3} \log\left(\frac{2\Psi}{\beta} \right) + \sqrt{2\left(\frac{d}{M} - \tau \right) L} \left(\sqrt{\lambda_0 \log\left(\frac{2}{u_\alpha} \right)} + 2 \sqrt{(\lambda_0 + \delta) \log\left(\frac{2\Psi}{\beta} \right)} \right) ,$$

hence

$$\frac{1}{2} \delta \left(\frac{c}{M} - \tau \right) L - \frac{2}{3} \log\left(\frac{2}{u_\alpha} \right) - \sqrt{2\lambda_0 L \log\left(\frac{2}{u_\alpha} \right) \left(\frac{d}{M} - \tau \right)} \ge \frac{4}{3} \log\left(\frac{2\Psi}{\beta} \right) + 2\sqrt{2(\lambda_0 + \delta) L \log\left(\frac{2\Psi}{\beta} \right) \left(\frac{d}{M} - \tau \right)} . \tag{4.57}$$

Moreover, the condition (4.43) ensures that for all k in $[c,d)$,

$$\frac{1}{2} \left(\frac{k}{M} - \tau \right) < \frac{k}{M 2^{j_\tau(k)}} \le \frac{k}{M} - \tau , \tag{4.58}$$

and we then obtain

$$\begin{aligned}\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-&\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)}\\ &\leq \frac{1}{2}\delta\left(\frac{k}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{k}{M}-\tau\right)}\\ &\leq \frac{\delta Lk}{M2^{j_\tau(k)}}-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0\frac{Lk}{M2^{j_\tau(k)}}\log\left(\frac{2}{u_\alpha}\right)} \text{ with (4.58)} .\end{aligned}$$

Using a lower bound for g^{-1} recalled in (2.45), we finally obtain for all k in $[c,d)$,

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)}\leq \frac{Lk}{M2^{j_\tau(k)}}\left(\delta-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right) ,$$

hence

$$\frac{1}{2}\delta\left(\frac{c}{M}-\tau\right)L-\frac{2}{3}\log\left(\frac{2}{u_\alpha}\right)-\sqrt{2\lambda_0 L\log\left(\frac{2}{u_\alpha}\right)\left(\frac{d}{M}-\tau\right)}\leq \inf_{k\in[c,d)}\left(\frac{Lk}{M2^{j_\tau(k)}}\left(\delta-\lambda_0 g^{-1}\left(\frac{\log(2/u_\alpha)M2^{j_\tau(k)}}{\lambda_0 kL}\right)\right)\right) . \tag{4.59}$$

Using (2.45) again, we get

$$\frac{4}{3}\log\left(\frac{2\Psi}{\beta}\right)+2\sqrt{2(\lambda_0+\delta)L\log\left(\frac{2\Psi}{\beta}\right)\left(\frac{d}{M}-\tau\right)}\geq 2\left(\frac{d}{M}-\tau\right)L(\lambda_0+\delta)g^{-1}\left(\frac{\log(2\Psi/\beta)}{(d/M-\tau)L(\lambda_0+\delta)}\right) . \tag{4.60}$$

Finally, (4.59) and (4.60) combined with (4.57) lead to the expected result.
If we assume that $-\lambda_0<\delta<0$, the proof follows the same lines as above just replacing δ by $|\delta|$ except when it is involved in $\lambda_0+\delta$. □

Let us turn back now to the proof of Theorem 4.5 and first recall that for all λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{T}(\lambda)=\{H_k\in\mathcal{H}_{M,R}\ ,\ \lambda\in H_k\}$ is the set of true hypotheses.

Proof of Theorem 4.5.
Recall that H_k stands for $H_k[\lambda_0,R]$. We begin with the control of FWER($\mathcal{R}_2^{(1)}$) over $\overline{\mathcal{S}}[\lambda_0,R]$. To this end, we compute for all λ in $\overline{\mathcal{S}}[\lambda_0,R]$

$$\begin{aligned}P_\lambda\left(\mathcal{R}_2^{(1)}\cap\mathcal{T}(\lambda)\neq\varnothing\right)&=P_\lambda\left(\exists k\in\{1,\dots,M\},\ H_k\in\mathcal{R}_2^{(1)},\ H_k\in\mathcal{T}(\lambda)\right)\\ &=P_\lambda\left(\exists k\in\{1,\dots,\lfloor\tau M\rfloor\},\ k\geq\hat{k}_2^{(1)}+1,\ \lambda\in H_k\right)\end{aligned}$$

because λ belongs to $H_{\lfloor\tau M\rfloor}$ and not to $H_{\lfloor\tau M\rfloor+1}$. If $\tau<1/M$ then $P_\lambda\left(\mathcal{R}_2^{(1)}\cap\mathcal{T}(\lambda)\neq\varnothing\right)=0\leq\alpha$, and if

$\tau \geq 1/M$ one has

$$
\begin{aligned}
P_\lambda\left(\mathcal{R}_2^{(1)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda\left(\hat{k}_2^{(1)} + 1 \leq \lfloor \tau M \rfloor\right) \\
&= P_\lambda\left(\phi^{(1)}_{2,\lfloor \tau M \rfloor} = 1\right) \\
&\leq P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, N(\lfloor \tau M \rfloor (1-2^{-j})/M, \lfloor \tau M \rfloor /M] > p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(1-u_\alpha/2)\right) \\
&\quad + P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, N(\lfloor \tau M \rfloor (1-2^{-j})/M, \lfloor \tau M \rfloor /M] < p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(u_\alpha/2)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(N(\lfloor \tau M \rfloor (1-2^{-j})/M, \lfloor \tau M \rfloor /M] > p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(1-u_\alpha/2)\right) \\
&\quad + \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(N(\lfloor \tau M \rfloor (1-2^{-j})/M, \lfloor \tau M \rfloor /M] < p_{\lambda_0 \lfloor \tau M \rfloor L 2^{-j}/M}(u_\alpha/2)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha/2 + \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha/2 \leq \alpha \quad ,
\end{aligned}
$$

which proves the control of $\mathrm{FWER}(\mathcal{R}_2^{(1)})$ by α.

Let us compute an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R])$. Recall that (4.9) leads to $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0)$. Now, assume that

$$
L > \left(\frac{C(\alpha, \beta, \lambda_0, R, L)}{\lambda_0 \vee (R - \lambda_0)}\right)^2 \quad ,
$$

where

$$
\begin{aligned}
C(\alpha, \beta, \lambda_0, R, L) = 2 \max\Bigg(& \sqrt{\frac{2R}{3}} \sqrt{\log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right) + 2\log\left(\frac{2\lceil \log_2 L \rceil}{\beta}\right)}, \\
& 4\sqrt{\lambda_0 \log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right)} + 8\sqrt{R \log\left(\frac{2\lceil \log_2 L \rceil}{\beta}\right)}, \\
& 2\sqrt{\lambda_0 \log\left(\frac{2\lfloor \log_2 L \rfloor}{\alpha}\right)} + \sqrt{\frac{2R}{\beta}}, \frac{R}{\sqrt{2}}, \frac{7R}{2}\sqrt{\log \log L} \Bigg) \quad ,
\end{aligned}
$$

and let $r > 0$ be such that

$$
\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha, \beta, \lambda_0, R, L)}{\sqrt{L}} \quad . \tag{4.61}
$$

Recall that for λ in $\mathcal{S}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H}_{M,R},\ d_2(\lambda, H_k) \geq r\}$ with $d_2(\lambda, H_k) = |\delta|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \ldots, M\}$. To bound $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(1)}, \mathcal{S}[\lambda_0, R])$ by r, it is sufficient to prove that for all λ in $\mathcal{S}[\lambda_0, R]$, $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\right) \geq 1 - \beta$.

Let us consider then λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau, 1]}$ where δ in $(-\lambda_0, R - \lambda_0] \setminus \{0\}$ and τ in $(0, 1)$. If $\mathcal{F}_r(\lambda) = \varnothing$, we get that $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\right) = 1 \geq 1 - \beta$, so by now we assume that λ is such that $\mathcal{F}_r(\lambda) \neq \varnothing$. We define

$$
k_r = \min\{\tau M < k' \leq M,\ \delta^2 (k'/M - \tau) \geq r^2\} \quad , \tag{4.62}
$$

and with the relationship $\{\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(1)}\} = \{k_r \geq \hat{k}_2^{(1)} + 1\}$, we have to prove the following inequality

$$P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) \leq \beta \tag{4.63}$$

to obtain the expected result.

First, if $k_r = M$ then

$$\begin{aligned}P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &= P_\lambda\left(\hat{k}_2^{(1)} = M\right)\\ &= P_\lambda\left(\phi_{2,M}^{(1)} = 0\right)\\ &= P_\lambda\left(\forall j \in \{1,\ldots,\lfloor \log_2 L\rfloor\},\ N((1-2^{-j}),1] \leq p_{\lambda_0 L 2^{-j}}(1-u_\alpha/2),\ N((1-2^{-j}),1] \geq p_{\lambda_0 L 2^{-j}}(u_\alpha/2)\right) \ .\end{aligned}$$

By definition of k_r, since the condition (4.61) ensures in particular that

$$|\delta|\sqrt{1-\tau} \geq 2\max\left(\sqrt{\frac{2R\log(2/u_\alpha)}{3L}}\ ,\ 2\sqrt{\frac{\lambda_0\log(2/u_\alpha)}{L}} + \sqrt{\frac{2R}{\beta L}}\ ,\ \frac{R}{\sqrt{2L}}\right)\ ,$$

we immediately obtain that

$$P_\lambda\left(\forall j \in \{1,\ldots,\lfloor \log_2 L\rfloor\},\ N((1-2^{-j}),1] \leq p_{\lambda_0 L 2^{-j}}(1-u_\alpha/2),\ N((1-2^{-j}),1] \geq p_{\lambda_0 L 2^{-j}}(u_\alpha/2)\right) \leq \beta\ ,$$

according to the minimax study of the change-point detection done in Proposition 2.18 which involves the statistic $\max_{j\in\{1,\ldots,\lfloor\log_2 L\rfloor\}}\left(\left(N((1-2^{-j}),1] - p_{\lambda_0 L 2^{-j}}(1-u_\alpha/2)\right) \vee \left(p_{\lambda_0 L 2^{-j}}(u_\alpha/2) - N((1-2^{-j}),1]\right)\right)$.

Assume by now that $k_r \leq M-1$, and we compute then

$$\begin{aligned}P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &= P_\lambda\left(\exists k \geq k_r, \phi_{2,k}^{(1)} = 0\right)\\ &= P_\lambda\Big(\exists k \geq k_r,\ \forall j \in \{1,\ldots,\lfloor\log_2 L\rfloor\};\ N(k(1-2^{-j})/M, k/M] \leq p_{\lambda_0 k L 2^{-j}/M}(1-u_\alpha/2),\\ &\qquad N(k(1-2^{-j})/M, k/M] \geq p_{\lambda_0 k L 2^{-j}/M}(u_\alpha/2)\Big)\ .\end{aligned} \tag{4.64}$$

Assume first that $0 < \delta \leq R - \lambda_0$. Then (4.64) ensures that

$$\begin{aligned}P_\lambda\left(\hat{k}_2^{(1)} \geq k_r\right) &\leq P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1,\ldots,\lfloor\log_2 L\rfloor\};\ N(k(1-2^{-j})/M, k/M] \leq p_{\lambda_0 k L 2^{-j}/M}(1-u_\alpha/2)\right)\\ &\leq P_\lambda\left(\exists k \geq k_r,\ N(k(1-2^{-j_\tau(k)})/M, k/M] \leq p_{\lambda_0 k L 2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right)\ ,\end{aligned}$$

where $j_\tau(k)$ is defined by (4.42). The assumption (4.61) entails that $r \geq 7R\sqrt{\log\log L/L}$ and then for all $k \geq k_r$

$$\delta^2\left(\frac{k}{M} - \tau\right) \geq 49R^2\frac{\log\log L}{L}\ . \tag{4.65}$$

Since $\delta < R$, we therefore get $k/M - \tau \geq 49\log\log L/L$, which leads to the condition (4.44) for $L \geq 3$. Indeed, for all $L \geq 3$, we have $49\log\log L/L \geq 2^{-\lfloor\log_2 L\rfloor+1}$ which entails $k(1-2^{-\lfloor\log_2 L\rfloor+1})/M \geq \tau$. As a consequence, we have $-\log_2(1-\tau M/k)+1 \leq \lfloor\log_2 L\rfloor$ which implies (4.44). Then for all $k \geq k_r$, $j_\tau(k)$ defined by (4.42) is such that $j_\tau(k)$ belongs to $\{1,\ldots,\lfloor\log_2 L\rfloor)\}$ and satisfies (4.43).
Consider now the following partition of cardinality $\Psi = \lceil\log_2(L)\rceil$:

$$\bigcup_{i=0}^{\lceil\log_2(L)\rceil-1} I_{\tau,M,i,k_r} = \bigcup_{i=0}^{\lceil\log_2(L)\rceil-1} [\tau M + 2^i(k_r - \tau M)\ ;\ \tau M + 2^{i+1}(k_r - \tau M))\ .$$

The condition (4.65) yields $1/(k_r/M-\tau) \le L \le 2^{\lceil \log_2(L)\rceil}$ and $\tau M + 2^{(\lceil \log_2(L)\rceil -1)+1}(k_r-\tau M) > M$, so that

$$[k_r;M] \subset \bigcup_{i=0}^{\lceil \log_2(L)\rceil -1} I_{\tau,M,i,k_r} \ .$$

As a consequence

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(1)} \ge k_r\right) &\le \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\exists k \in I_{\tau,M,i,k_r},\ N(k(1-2^{-j_\tau(k)})/M, k/M] - p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2) \le 0\right) \\ &= \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}} \left(-N(k(1-2^{-j_\tau(k)})/M, k/M] + p_{\lambda_0 kL2^{-j_\tau(k)}/M}(1-u_\alpha/2)\right) \ge 0\right) \ . \end{aligned}$$

The proof is then completed applying Lemma 4.18. Recall that (4.44) is satisfied and notice that

$$\frac{\frac{\tau M + 2^{i+1}(k_r-\tau M)}{M} - \tau}{\frac{\tau M + 2^{i}(k_r-\tau M)}{M} - \tau} = 2 \ , \tag{4.66}$$

that is (4.53). Then, by definition of k_r (see (4.62)), the condition (4.61) gives

$$\begin{aligned} |\delta|\sqrt{\frac{k_r}{M}-\tau} \ge 2\max\Bigg(&\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(2\lfloor \log_2 L\rfloor/\alpha)}{L} + \frac{2\log(2\lceil \log_2 L\rceil/\beta)}{L}}\ , \\ &4\sqrt{\frac{\lambda_0\log(2\lfloor \log_2 L\rfloor/\alpha)}{L}} + 8\sqrt{\frac{R\log(2\lceil \log_2 L\rceil/\beta)}{L}}\Bigg) \ , \end{aligned}$$

and since $\tau M + 2^i(k_r-\tau M) \ge k_r$ for all i in $\{0,\dots,\lceil \log_2(L)\rceil -1\}$, the inequality (4.54) is satisfied. As a consequence of Lemma 4.19, the condition (4.45) is satisfied by the cover $\bigcup_{i=0}^{\lceil \log_2(L)\rceil -1} I_{\tau,M,i,k_r}$ and we may apply Lemma 4.18 to conclude the proof.
Now, if we assume that $-\lambda_0 < \delta < 0$, then (4.64) ensures that

$$P_\lambda\left(\hat{k}_2^{(1)} \ge k_r\right) \le P_\lambda\left(\exists k \ge k_r,\ \forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ N(k(1-2^{-j})/M, k/M] \ge p_{\lambda_0 kL2^{-j}/M}(u_\alpha/2)\right) \ ,$$

where $j_\tau(k)$ is defined by (4.42) and the proof essentially follows the same line as above. □

V.3 Proof of Theorem 4.6

For $\lambda_0 > 0$ and $R > \lambda_0$, recall that the simple hypotheses $H_k[\lambda_0,R]$ are denoted H_k for short and that u_α stands for $\alpha/\lfloor \log_2 L\rfloor$. For k in $\{1,\dots,M\}$ and j in $\{1,\dots,\lfloor \log_2 L\rfloor\}$, we set for all $x>0$

$$\varphi_{j,k}(x) = \sqrt{\frac{2^j M}{k}}\mathbb{1}_{\left\{\left(\frac{k}{M}\left(1-\frac{1}{2^j}\right),\frac{k}{M}\right]\right\}}(x) \ .$$

$T_{j,k}$ which is defined by (4.13) can then be written

$$T_{j,k} = U_{j,k} + 2V_{j,k} + C_{j,k} \ , \tag{4.67}$$

where

$$U_{j,k} = \frac{1}{L^2}\left(\left(\int_0^1 \varphi_{j,k}(x)(dN_x - \lambda(x)Ldx)\right)^2 - \int_0^1 \varphi_{j,k}^2(x)dN_x\right) , \tag{4.68}$$

$$V_{j,k} = \frac{1}{L}\int_0^1 \varphi_{j,k}(x)(dN_x - \lambda(x)Ldx)\left(\int_0^1 \varphi_{j,k}(x)\lambda(x)dx - \lambda_0\sqrt{\frac{k}{2^jM}}\right) , \tag{4.69}$$

and $C_{j,k}$ is the squared bias of $T_{j,k}$

$$C_{j,k} = \left(\int_0^1 \varphi_{j,k}(x)\lambda(x)dx - \lambda_0\sqrt{\frac{k}{2^jM}}\right)^2 . \tag{4.70}$$

We begin by giving an upper bound for the quantile $t_{j,k}$ of $T_{j,k}$ under (H_k). The proof of Lemma 4.20 follows the same ideas of the one of Lemma 2.27: the key argument to obtain the following upper bound is the use of an exponential inequality established in [LG21, Theorem 6] for the square martingale $\left(\int_0^s (dN_x - \lambda_0 Ldx)\right)^2 - \int_0^s dN_x$.

Lemma 4.20 (Control of the quantiles)**.**
Let $L \geq 3$, α in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For all M in $\mathbb{N}^*$, for all j in $\{1,\ldots,\lfloor \log_2 L\rfloor\}$ and for all k in $\{1,\ldots,M\}$, we have the following inequality

$$t_{j,k}(1-u_\alpha) \leq \frac{2k\lambda_0^2}{2^jM}\left(g^{-1}\left(\frac{M2^j}{\lambda_0 Lk}\log\left(\frac{3\lfloor \log_2(L)\rfloor}{\alpha}\right)\right)\right)^2 , \tag{4.71}$$

where the function g is defined by (2.44).

Proof of Lemma 4.20.
Let k in $\{1,\ldots,M\}$ and j in $\{1,\ldots,\lfloor \log_2(L)\rfloor\}$.
Under the hypothesis (H_k), N is a simple Poisson process on the interval $[0,k/M]$, of intensity λ_0 with respect to the measure Ldt, so that the equality (4.67) reduces to $T_{j,k} = U_{j,k}$ where

$$\begin{aligned} U_{j,k} &= \frac{M2^j}{kL^2}\left(\left(\int_{\frac{k}{M}\left(1-\frac{1}{2^j}\right)}^{\frac{k}{M}} (dN_x - \lambda_0 Ldx)\right)^2 - \int_{\frac{k}{M}\left(1-\frac{1}{2^j}\right)}^{\frac{k}{M}} dN_x\right) \\ &\overset{d}{=} \frac{M2^j}{kL^2}\left(\left(\int_0^{\frac{k}{2^jM}} (dN_x - \lambda_0 Ldx)\right)^2 - \int_0^{\frac{k}{2^jM}} dN_x\right) . \end{aligned}$$

We then apply inequality (8) of Theorem 6 in [LG21] in order to get for all $x > 0$ and for all λ in H_k,

$$P_\lambda\left(\sup_{0\leq s\leq \frac{k}{2^jM}}\left(\left(\int_0^s (dN_x - \lambda_0 Ldx)\right)^2 - \int_0^s dN_x\right) > \frac{xkL^2}{M2^j}\right) \leq 3\exp\left(\frac{-\lambda_0 Lk}{M2^j}g\left(\frac{M2^j}{\lambda_0 Lk}\sqrt{\frac{xkL^2}{M2^{j+1}}}\right)\right) ,$$

where g is defined by (2.44). To conclude, we obtain for

$$x \geq \frac{2k\lambda_0^2}{2^jM}\left(g^{-1}\left(\frac{M2^j}{\lambda_0 Lk}\log\left(\frac{3}{u_\alpha}\right)\right)\right)^2 ,$$

that for all λ in H_k,

$$\begin{aligned}
P_\lambda\left(T_{j,k} > x\right) &= P_\lambda\left(U_{j,k} > x\right) \\
&\leq P_\lambda\left(\sup_{0\leq s\leq \frac{k}{2^j M}}\left(\left(\int_0^s (dN_x - \lambda_0 L dx)\right)^2 - \int_0^s dN_x\right) > \frac{xkL^2}{M2^j}\right) \\
&\leq 3\exp\left(\frac{-\lambda_0 Lk}{M2^j} g\left(\frac{M2^j}{\lambda_0 Lk}\sqrt{\frac{xkL^2}{M2^{j+1}}}\right)\right) \\
&\leq u_\alpha \ ,
\end{aligned}$$

and (4.71) holds by definition of the quantile. □

As for the proof of Theorem 4.5, let us define for τ in $(0,1)$, $k_\tau = \min\{k' \in \{1,\ldots,M\},\ k' > \tau M\}$ and

$$j_\tau(k) = \left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \wedge \lfloor \log_2(L)\rfloor \tag{4.72}$$

for $k \geq k_\tau$, in order to get $j_\tau(k)$ in $\{1,\ldots,\lfloor\log_2(L)\rfloor\}$ as well as the inequalities

$$\frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)-1}}\right) < \tau \leq \frac{k}{M}\left(1 - \frac{1}{2^{j_\tau(k)}}\right) \tag{4.73}$$

under the following condition

$$\left\lceil -\log_2\left(1 - \frac{\tau M}{k}\right)\right\rceil \leq \lfloor \log_2(L)\rfloor \ . \tag{4.74}$$

Following the same idea of the proof of Theorem 4.5, we shall consider also a cover of the interval $[k_\tau, M]$, denoted $\mathcal{P} = \cup_{i=0}^{\Psi-1}[x_i, x_{i+1})$ where Ψ in $\mathbb{N}^*$ is the cardinal of the cover $\mathcal{P}$ and the real $x_0 < \ldots < x_\Psi$ satisfy $x_0 = k_\tau$ and $x_\Psi > M$. Note that the two real x_i and x_{i+1} may depend on λ_0, τ, L, M and k_τ. We will write $\sum_{[c,d)\in\mathcal{P}}$ for the sum over each disjoint interval $[c,d)$ of the cover $\mathcal{P}$. Assuming (4.74), the key argument in the proof of Theorem 4.6 will consist on giving an upper bound for the probability $P_\lambda(\sup_{k\in\mathcal{P}}(t_{j_\tau(k),k}(1 - u_\alpha) - T_{j_\tau(k),k}) \geq 0)$ instead of the probability $P_\lambda(\sup_{k\geq k_\tau}(t_{j_\tau(k),k}(1 - u_\alpha) - T_{j_\tau(k),k}) \geq 0)$, in order to get a more refined bound. The following technical lemmas allow us to bound $P_\lambda(\sup_{k\in[c,d)}(t_{j_\tau(k),k}(1-u_\alpha) - T_{j_\tau(k),k}) \geq 0)$ for each interval $[c,d)$ of the cover $\mathcal{P}$, using exponential inequalities related to the oscillation modulus of martingales or square martingales developed in [LG21].

Lemma 4.21 (Control 1)**.**
Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, assume (4.74) and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:

$$\begin{aligned}
&\frac{L^2}{2}\left(\frac{c}{M} - \tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M} - \tau\right) - \sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right) \geq \\
&\qquad 8\left(\left(\frac{d}{M} - \tau\right)(\lambda_0 + \delta)Lg^{-1}\left(\frac{\log(20\Psi/\beta)}{(d/M - \tau)(\lambda_0 + \delta)L}\right)\right)^2 \ ,
\end{aligned} \tag{4.75}$$

where the function g is defined by (2.44). Then

$$\sum_{[c,d)\in\mathcal{P}} P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1 - u_\alpha) - C_{j_\tau(k),k}}{2}\right) \geq 0\right) \leq \frac{\beta}{2} \ .$$

Proof of Lemma 4.21.
Let λ in $\mathcal{S}[\lambda_0, R]$ be such that $\lambda = \lambda_0 + \delta \mathbb{1}_{(\tau,1]}$, where δ in $(-\lambda_0, R-\lambda_0] \setminus \{0\}$ and τ in $(0,1)$. Let $[c,d)$ in $\mathcal{P}$ and notice that for k in $[c,d)$, (4.70) and (4.73) ensure the inequality

$$C_{j_\tau(k),k} = \delta^2 \frac{k}{M} \frac{1}{2^{j_\tau(k)}} > \frac{1}{2}\delta^2 \left(\frac{k}{M} - \tau \right) , \tag{4.76}$$

which combined with (4.71) in Lemma 4.20 gives

$$C_{j_\tau(k),k} > t_{j_\tau(k),k}(1-u_\alpha) \ . \tag{4.77}$$

Let us define now $\tilde{M}_s^t$ by $\tilde{M}_s^t = \left(\int_s^t (dN_x - \lambda(x)Ldx) \right)^2 - \int_s^t dN_x$ for all $0 \le s < t$, in order to write with (4.68)

$$U_{j_\tau(k),k} = \frac{M 2^{j_\tau(k)}}{kL^2} \tilde{M}^{k/M}_{k/M\left(1-2^{-j_\tau(k)}\right)}.$$

We define also $\xi_{c,d}$ by

$$\xi_{c,d} = \inf_{k \in [c,d)} \frac{L^2 k}{M 2^{j_\tau(k)}} \frac{C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha)}{2} ,$$

and we prove now that

$$\xi_{c,d} \ge \frac{L^2}{2}\left(\frac{c}{M} - \tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)} \left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M} \left(g^{-1}\left(\frac{M 2^{j_\tau(k)}}{\lambda_0 L k} \log\left(\frac{3\lfloor \log_2(L) \rfloor}{\alpha} \right)\right)\right)^2 \right)\right) . \tag{4.78}$$

Notice first that (4.75) leads to

$$\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)} \left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M} \left(g^{-1}\left(\frac{M 2^{j_\tau(k)}}{\lambda_0 L k} \log\left(\frac{3\lfloor \log_2(L) \rfloor}{\alpha} \right)\right)\right)^2 \right) \ge 0 \ .$$

The inequality (4.73) then ensures that

$$\frac{L^2 k}{M 2^{j_\tau(k)}} > \frac{L^2}{2}\left(\frac{k}{M} - \tau\right) ,$$

and with (4.71) and (4.76), we get for all k in $[c,d)$

$$\begin{aligned} \frac{L^2 k}{M 2^{j_\tau(k)}} \frac{C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha)}{2} > \frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\Bigg(& \frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) \\ & - \sup_{k\in[c,d)} \left(\frac{k\lambda_0^2}{2^{j_\tau(k)}M} \left(g^{-1}\left(\frac{M 2^{j_\tau(k)}}{\lambda_0 L k} \log\left(\frac{3\lfloor \log_2(L) \rfloor}{\alpha} \right)\right)\right)^2 \right)\Bigg) \end{aligned}$$

which is (4.78). Using the condition (4.75) we then obtain

$$\xi_{c,d} \ge 8\left(\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L g^{-1}\left(\frac{\log(20\Psi/\beta)}{(d/M-\tau)(\lambda_0+\delta)L} \right)\right)^2$$

which is equivalent to

$$10\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{1}{(d/M-\tau)L(\lambda_0+\delta)} \sqrt{\frac{\xi_{c,d}}{8}} \right)\right) \le \frac{\beta}{2\Psi} \ . \tag{4.79}$$

Finally, we get the following inequalities:

$$\begin{aligned}
&P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k}+\frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2}\right)\geq 0\right)\\
&\leq P_\lambda\left(\exists k\in[c,d),\ |U_{j_\tau(k),k}|\geq \frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{2}\right)\quad \text{with (4.77)}\\
&\leq P_\lambda\left(\exists k\in[c,d),\ \left|\tilde{M}^{k/M}_{k/M(1-2^{-j_\tau(k)})}\right|\geq \xi_{c,d}\right)\\
&\leq P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]}|\tilde{M}^t_s|\geq \xi_{c,d}\right)\quad \text{with (4.73)}\quad ,
\end{aligned}$$

and applying inequality (15) of Theorem 8 in [LG21],

$$P_\lambda\left(\sup_{s,t\in\left[\tau,\frac{d}{M}\right]}|\tilde{M}^t_s|\geq \xi_{c,d}\right)\leq 10\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{1}{(d/M-\tau)L(\lambda_0+\delta)}\sqrt{\frac{\xi_{c,d}}{8}}\right)\right)\ .$$

To conclude, (4.79) yields

$$P_\lambda\left(\sup_{k\in[c,d)}\left(-U_{j_\tau(k),k}+\frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2}\right)\geq 0\right)\leq \frac{\beta}{2\Psi}\ ,$$

and the result follows summing over all the interval of the cover $\mathcal{P}$. □

Lemma 4.22 (Control 2)**.**
Let $L\geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0>0$ and $R>\lambda_0$. For λ in $\mathcal{S}[\lambda_0,R]$, assume (4.74) and that the following inequality holds for all interval $[c,d)$ of the cover $\mathcal{P}$:

$$\begin{aligned}
&\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right)-\sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right)\\
&\quad\geq 2\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(4\Psi/\beta)}{(d/M-\tau)(\lambda_0+\delta)L}\right)\ ,
\end{aligned}\tag{4.80}$$

where the function g is defined by (2.44). Then

$$\sum_{[c,d)\in\mathcal{P}}P_\lambda\left(\sup_{k\in[c,d)}\left(-2V_{j_\tau(k),k}+\frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2}\right)\geq 0\right)\leq\frac{\beta}{2}\ .$$

Proof of Lemma 4.22.
We proceed as in the proof of Lemma 4.21. Let λ in $\mathcal{S}[\lambda_0,R]$ and $[c,d)$ in $\mathcal{P}$. We may fix δ in $(-\lambda_0,R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$ such that $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$. For all $0\leq s<t$ we define M^t_s by $M^t_s=\int_s^t(dN_x-\lambda(x)Ldx)$ in order to write with (4.69)

$$V_{j_\tau(k),k}=\frac{1}{L}\sqrt{\frac{M2^{j_\tau(k)}}{k}}\left(\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right)M^{k/M}_{k/M(1-2^{-j_\tau(k)})}\ .$$

We define also $\zeta_{c,d}$ by

$$\zeta_{c,d}=L\inf_{k\in[c,d)}\sqrt{\frac{k}{M2^{j_\tau(k)}}}\left|\int_0^1\varphi_{j_\tau(k),k}(x)\lambda(x)dx-\lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1}\frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{4}\ ,$$

and we prove that

$$\zeta_{c,d} \geq \frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right) . \tag{4.81}$$

A straightforward calculation gives for all k in $[c,d)$

$$\frac{L\sqrt{k}}{\sqrt{M2^{j_\tau(k)}}}\left|\int_0^1 \varphi_{j_\tau(k),k}(x)\lambda(x)dx - \lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1} = \frac{L}{|\delta|} ,$$

and then with (4.71) and (4.76), we get

$$\begin{aligned}\frac{L\sqrt{k}}{\sqrt{M2^{j_\tau(k)}}}\left|\int_0^1 \varphi_{j_\tau(k),k}(x)\lambda(x)dx - \lambda_0\sqrt{\frac{k}{2^{j_\tau(k)}M}}\right|^{-1}\frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{4} > \\ \frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right) - \sup_{k\in[c,d)}\left(\frac{k\lambda_0^2}{2^{j_\tau(k)+1}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2\right)\right) ,\end{aligned}$$

which entails (4.81). Using the condition (4.80) we obtain

$$\zeta_{c,d} \geq 2\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(4\Psi/\beta)}{(d/M-\tau)(\lambda_0+\delta)L}\right) ,$$

which is equivalent to

$$2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\zeta_{c,d}}{2(d/M-\tau)L(\lambda_0+\delta)}\right)\right) \leq \frac{\beta}{2\Psi} . \tag{4.82}$$

Finally, we get the following inequalities

$$\begin{aligned}&P_\lambda\left(\sup_{k\in[c,d)} -2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2} \geq 0\right) \\ &\leq P_\lambda\left(\exists k\in[c,d),\ |V_{j_\tau(k),k}| \geq \frac{C_{j_\tau(k),k}-t_{j_\tau(k),k}(1-u_\alpha)}{4}\right) \quad \text{with (4.77)} \\ &\leq P_\lambda\left(\exists k\in[c,d),\ \left|M_{k/M(1-2^{-j_\tau(k)})}^{k/M}\right| \geq \zeta_{c,d}\right) \\ &\leq P_\lambda\left(\sup_{s,t\in[\tau,\frac{d}{M}]} |M_s^t| \geq \zeta_{c,d}\right) \quad \text{with (4.73)} ,\end{aligned}$$

and Theorem 8 in [LG21] leads to

$$P_\lambda\left(\sup_{s,t\in[\tau,\frac{d}{M}]} |M_s^t| \geq \zeta_{c,d}\right) \leq 2\exp\left(-\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg\left(\frac{\zeta_{c,d}}{2(d/M-\tau)L(\lambda_0+\delta)}\right)\right) .$$

Therefore (4.82) yields

$$P_\lambda\left(\sup_{k\in[c,d)} -2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha)-C_{j_\tau(k),k}}{2} \geq 0\right) \leq \frac{\beta}{2\Psi} ,$$

and the result follows summing over all the interval of the cover $\mathcal{P}$.

□

The following lemma gives some conditions on the cover $\mathcal{P}$ to ensure the inequalities (4.75) and (4.80) in order to apply Lemma 4.21 and Lemma 4.22.

Lemma 4.23.
Let $L \geq 3$, M in $\mathbb{N}^*$, α and β be fixed levels in $(0,1)$, $\lambda_0 > 0$ and $R > \lambda_0$. For λ in $\mathcal{S}[\lambda_0, R]$, we assume (4.74), that for all interval $[c,d)$ of the cover $\mathcal{P}$

$$\frac{d/M-\tau}{c/M-\tau} = 2 \ , \tag{4.83}$$

and that the following inequality holds

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(32\sqrt{3R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\ ,\ 8\sqrt{2\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ,\ 2\cdot 32^{1/3}\cdot\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ,\right.$$
$$\left. 2\cdot\left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\ ,\ 128\sqrt{R}\sqrt{\frac{\log(4\Psi/\beta)}{L}}\right) \ . \tag{4.84}$$

Then the inequalities (4.75) and (4.80) are satisfied.

Proof of Lemma 4.23.
Let λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0, R-\lambda_0]\setminus\{0\}$ and τ in $(0,1)$. Let $[c,d)$ in $\mathcal{P}$, k in $[c,d)$ and notice that (4.73) ensures that $2^{j_\tau(k)}M/k < 2/(c/M-\tau)$. Combined with (2.45), we deduce that

$$\sup_{k\in[c,d)}\frac{k\lambda_0^2}{2^{j_\tau(k)}M}\left(g^{-1}\left(\frac{M2^{j_\tau(k)}}{\lambda_0 Lk}\log\left(\frac{3\lfloor\log_2(L)\rfloor}{\alpha}\right)\right)\right)^2 \leq$$
$$2\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L} + \frac{8}{9}\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{c/M-\tau} + \frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}} \ . \tag{4.85}$$

Moreover, applying the inequality (2.45) again, we obtain for all u in $\mathbb{R}_*^+$

$$g^{-1}\left(\frac{\log(u)}{(d/M-\tau)(\lambda_0+\delta)L}\right) \leq \sqrt{\frac{2\log(u)}{(d/M-\tau)(\lambda_0+\delta)L}} + \frac{2\log(u)}{3(d/M-\tau)(\lambda_0+\delta)L} \ . \tag{4.86}$$

To get (4.75), notice first that (4.86) applied with $u = 20\Psi/\beta$ leads to

$$8\left(\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)Lg^{-1}\left(\frac{\log(20\Psi/\beta)}{\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L}\right)\right)^2 \leq \frac{32}{9}\log^2\left(\frac{20\Psi}{\beta}\right) + 16\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L\log\left(\frac{20\Psi}{\beta}\right) +$$
$$\frac{32\sqrt{2}}{3}\sqrt{\frac{d}{M}-\tau}\sqrt{\lambda_0+\delta}\sqrt{L}\log^{3/2}\left(\frac{20\Psi}{\beta}\right) \ . \tag{4.87}$$

Using (4.85) and (4.87), it is enough to prove that

$$\frac{L^2}{2}\left(\frac{c}{M}-\tau\right)\left(\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right) - \left(2\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L} + \frac{8}{9}\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{c/M-\tau} + \right.\right.$$
$$\left.\left.\frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\right)\right) \geq \frac{32}{9}\log^2\left(\frac{20\Psi}{\beta}\right) + 16\left(\frac{d}{M}-\tau\right)(\lambda_0+\delta)L\log\left(\frac{20\Psi}{\beta}\right) +$$
$$\frac{32\sqrt{2}}{3}\sqrt{\frac{d}{M}-\tau}\sqrt{\lambda_0+\delta}\sqrt{L}\log^{3/2}\left(\frac{20\Psi}{\beta}\right) \tag{4.88}$$

to get (4.75). Let us prove that (4.84) implies (4.88). From (4.84), we get in particular

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\Bigg(32\sqrt{3R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\,,\ 4\sqrt{6\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ 16\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\,,\\ 2\cdot 32^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ 2\cdot\left(\frac{256}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}}\,,\\ 2\cdot\left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\Bigg)\,,$$

and using the fact that $a+b \leq 2\max(a,b)$ for all a,b in $\mathbb{R}^+$, one obtains

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\Bigg(16\sqrt{3R}\sqrt{\frac{\log(20\Psi/\beta)}{L}} + 2\sqrt{6\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\\ 8\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}} + 32^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\\ \left(\frac{256}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(20\Psi/\beta)}{L}} + \left(\frac{32}{3}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\Bigg)\,. \tag{4.89}$$

For all a,b in $\mathbb{R}^+$ and s in $(0,1)$, since $(a+b)^s \leq a^s+b^s$, we deduce from (4.89) that

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\Bigg(\sqrt{768R\frac{\log(20\Psi/\beta)}{L} + 24\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\\ \left(512R^{3/2}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}} + 32\sqrt{\lambda_0}R\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right)^{1/3}\,,\\ \left(\frac{256}{3}R^2\frac{\log^2(20\Psi/\beta)}{L^2} + \frac{32}{3}R^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right)^{1/4}\Bigg)\,. \tag{4.90}$$

Then (4.90) entails on the one hand

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \left(512R^{3/2}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}} + 32\sqrt{\lambda_0}R\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right)^{1/3}$$

that is

$$|\delta|^3\left(\frac{c}{M}-\tau\right)^{3/2} \geq 512R^{3/2}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}} + 32\sqrt{\lambda_0}R\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\,,$$

which gives

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq \frac{1}{|\delta|\sqrt{c/M-\tau}}\left(512|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}} + 32\sqrt{\lambda_0}|\delta|\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right)\,.$$

On the other hand, we get that

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \left(\frac{256}{3}R^2\frac{\log^2(20\Psi/\beta)}{L^2} + \frac{32}{3}R^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right)^{1/4}$$

that is

$$\delta^4\left(\frac{c}{M}-\tau\right)^2 \geq \frac{256}{3}R^2\frac{\log^2(20\Psi/\beta)}{L^2} + \frac{32}{3}R^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\,,$$

and then

$$\delta^2\left(\frac{c}{M}-\tau\right)\geq \frac{1}{\delta^2(c/M-\tau)}\left(\frac{256}{3}\delta^2\frac{\log^2(20\Psi/\beta)}{L^2}+\frac{32}{3}\delta^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right) .$$

We therefore obtain

$$\begin{aligned}\delta^2\left(\frac{c}{M}-\tau\right)\geq 3\max\Bigg(&256(\lambda_0+\delta)\frac{\log(20\Psi/\beta)}{L}+8\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L},\\ &\frac{1}{|\delta|\sqrt{c/M-\tau}}\left(\frac{512}{3}|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right),\\ &\frac{1}{\delta^2(c/M-\tau)}\left(\frac{256}{9}\delta^2\frac{\log^2(20\Psi/\beta)}{L^2}+\frac{32}{9}\delta^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right)\Bigg) ,\end{aligned}$$

and using the fact that $a+b+c\leq 3\max(a,b,c)$ for all a,b,c in $\mathbb{R}^+$, it yields

$$\begin{aligned}\delta^2\left(\frac{c}{M}-\tau\right)\geq\ &256(\lambda_0+\delta)\frac{\log(20\Psi/\beta)}{L}+8\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}+\\ &\frac{1}{|\delta|\sqrt{c/M-\tau}}\left(\frac{512}{3}|\delta|\sqrt{\lambda_0+\delta}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right)+\\ &\frac{1}{\delta^2(c/M-\tau)}\left(\frac{256}{9}\delta^2\frac{\log^2(20\Psi/\beta)}{L^2}+\frac{32}{9}\delta^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right) .\end{aligned}\tag{4.91}$$

We use now (4.83) to get

$$\begin{aligned}\delta^2\left(\frac{c}{M}-\tau\right)\geq\ &128(\lambda_0+\delta)\frac{\log(20\Psi/\beta)}{L}\frac{d/M-\tau}{c/M-\tau}+8\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}+\\ &\frac{1}{\sqrt{c/M-\tau}}\left(\frac{256\sqrt{2}}{3}\sqrt{\frac{d/M-\tau}{c/M-\tau}}\sqrt{\lambda_0+\delta}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}}+\frac{32}{3}\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\right)+\\ &\frac{1}{c/M-\tau}\left(\frac{256}{9}\frac{\log^2(20\Psi/\beta)}{L^2}+\frac{32}{9}\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\right) ,\end{aligned}$$

which can be rewritten as

$$\begin{aligned}&\frac{1}{4}\delta^2\left(\frac{c}{M}-\tau\right)-\left(2\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}+\frac{8}{9}\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{c/M-\tau}+\frac{8}{3}\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\right)\\ &\geq\frac{64}{9}\frac{\log^2(20\Psi/\beta)}{L^2}\frac{1}{c/M-\tau}+32\frac{d/M-\tau}{c/M-\tau}(\lambda_0+\delta)\frac{\log(20\Psi/\beta)}{L}+\frac{64\sqrt{2}}{3}\sqrt{\frac{d/M-\tau}{c/M-\tau}}\frac{1}{\sqrt{c/M-\tau}}\sqrt{\lambda_0+\delta}\frac{\log^{3/2}(20\Psi/\beta)}{L^{3/2}} ,\end{aligned}$$

that is (4.88). Let us prove now that (4.80) is satisfied. Using (4.85) and (4.86) with $u=4\Psi/\beta$, it is enough to prove that

$$\begin{aligned}&\frac{L}{|\delta|}\left(\frac{1}{8}\delta^2\left(\frac{c}{M}-\tau\right)-\left(\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}+\frac{4}{9}\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{c/M-\tau}\right.\right.\\ &\left.\left.+\frac{4}{3}\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{\sqrt{c/M-\tau}}\right)\right)\geq\frac{4}{3}\log\left(\frac{4\Psi}{\beta}\right)+2\sqrt{2(\lambda_0+\delta)}\sqrt{\frac{d}{M}-\tau}\sqrt{L\log\left(\frac{4\Psi}{\beta}\right)} .\end{aligned}\tag{4.92}$$

Let's prove that (4.84) implies (4.92). From (4.84), we get in particular

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(16\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(4\Psi/\beta)}{L}}\,,\ 8\sqrt{2\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ 128\sqrt{R}\sqrt{\frac{\log(4\Psi/\beta)}{L}}\,,\right.$$
$$\left.\left(\frac{128}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ \left(\frac{128}{9}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\right)\,,$$

which leads, using the fact that $a+b\leq 2\max(a,b)$ for all a,b in $\mathbb{R}^+$, to

$$|\delta|\sqrt{\frac{c}{M}-\tau} \geq \max\left(8\sqrt{\frac{2R}{3}}\sqrt{\frac{\log(4\Psi/\beta)}{L}}+4\sqrt{2\lambda_0}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ 128\sqrt{R}\sqrt{\frac{\log(4\Psi/\beta)}{L}}\,,\right.$$
$$\left.\left(\frac{128}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\,,\ \left(\frac{128}{9}\right)^{1/4}\sqrt{R}\sqrt{\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}}\right)\,. \tag{4.93}$$

With the same computations as before, (4.93) yields

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq 4\max\left(\frac{32}{3}|\delta|\frac{\log(4\Psi/\beta)}{L}+8\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}\,,\ 32\sqrt{\lambda_0+\delta}\sqrt{\frac{\log(4\Psi/\beta)}{L}}|\delta|\sqrt{\frac{c}{M}-\tau}\,,\right.$$
$$\left.\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{|\delta|\sqrt{c/M-\tau}}\,,\ \frac{32}{9}\delta^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{\delta^2(c/M-\tau)}\right)\,. \tag{4.94}$$

Since $a+b+c+d\leq 4\max(a,b,c,d)$ for all a,b,c,d in $\mathbb{R}^+$, (4.94) ensures using (4.83)

$$\delta^2\left(\frac{c}{M}-\tau\right) \geq \frac{32}{3}|\delta|\frac{\log(4\Psi/\beta)}{L}+32\sqrt{\lambda_0+\delta}\sqrt{\frac{\log(4\Psi/\beta)}{L}}|\delta|\sqrt{\frac{c}{M}-\tau}+8\lambda_0\frac{\log(3\lfloor\log_2 L\rfloor/\alpha)}{L}$$
$$+\frac{32}{9}\delta^2\frac{\log^2(3\lfloor\log_2 L\rfloor/\alpha)}{L^2}\frac{1}{\delta^2(c/M-\tau)}+\frac{32}{3}|\delta|\sqrt{\lambda_0}\frac{\log^{3/2}(3\lfloor\log_2 L\rfloor/\alpha)}{L^{3/2}}\frac{1}{|\delta|\sqrt{c/M-\tau}}\,,$$

that is (4.92) . □

Let us turn back now to the proof of Theorem 4.6 and first recall that for all λ in $\mathcal{S}[\lambda_0,R]$, $\mathcal{T}(\lambda)=\{H_k\in\mathcal{H}_{M,R}\,,\ \lambda\in H_k\}$ is the set of true hypotheses.

Proof of Theorem 4.6.
Recall that H_k stands for $H_k[\lambda_0,R]$. We begin with the control of FWER($\mathcal{R}_2^{(2)}$) over $\overline{\mathcal{S}}[\lambda_0,R]$. To this end, we compute for all λ in $\overline{\mathcal{S}}[\lambda_0,R]$

$$P_\lambda\left(\mathcal{R}_2^{(2)}\cap\mathcal{T}(\lambda)\neq\varnothing\right)=P_\lambda\left(\exists k\in\{1,\ldots,M\},\ H_k\in\mathcal{R}_2^{(2)},\ H_k\in\mathcal{T}(\lambda)\right)$$
$$=P_\lambda\left(\exists k\in\{1,\ldots,\lfloor\tau M\rfloor\},\ k\geq\hat{k}_2^{(2)}+1,\ \lambda\in H_k\right)\,,$$

because λ belongs to $H_{\lfloor\tau M\rfloor}$ and not to $H_{\lfloor\tau M\rfloor+1}$. If $\tau<1/M$ then $P_\lambda\left(\mathcal{R}_2^{(2)}\cap\mathcal{T}(\lambda)\neq\varnothing\right)=0\leq\alpha$, and if

$\tau \geq 1/M$ one has

$$\begin{aligned}
P_\lambda\left(\mathcal{R}_2^{(2)} \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda(\hat{k}_2^{(2)} + 1 \leq \lfloor \tau M \rfloor) \\
&= P_\lambda\left(\phi^{(2)}_{2,\lfloor \tau M \rfloor} = 1\right) \\
&= P_\lambda\left(\exists j \in \{1, \ldots, \lfloor \log_2 L \rfloor\}, T_{j,\lfloor \tau M \rfloor} > t_{j,\lfloor \tau M \rfloor}(1-u_\alpha)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} P_\lambda\left(T_{j,\lfloor \tau M \rfloor} > t_{j,\lfloor \tau M \rfloor}(1-u_\alpha)\right) \\
&\leq \sum_{j=1}^{\lfloor \log_2 L \rfloor} u_\alpha \leq \alpha \quad ,
\end{aligned}$$

which proves the control of $\mathrm{FWER}(\mathcal{R}_2^{(2)})$ by α.
Let us compute an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R])$. Recall that (4.9) leads to $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0)$. Now, assume that

$$L > \left(\frac{C(\alpha, \beta, \lambda_0, R, L)}{\lambda_0 \vee (R - \lambda_0)}\right)^2 \quad ,$$

where

$$\begin{aligned}
C(\alpha, \beta, \lambda_0, R, L) = \max\Bigg(& 128\sqrt{R}\sqrt{\log\left(\frac{20\lceil \log_2 L \rceil}{\beta}\right)},\ 4\sqrt{2\lambda_0 \log\left(\frac{3\lfloor \log_2 L \rfloor}{\alpha}\right)} + 2\sqrt{2R\sqrt{\frac{2}{\beta}}},\ 16\sqrt{\frac{R}{\beta}}, \\
& \max\left(8\sqrt{2\lambda_0},\ 2 \cdot 32^{1/3} \cdot \lambda_0^{1/6} R^{1/3},\ 2 \cdot (32/3)^{1/4}\sqrt{R},\ 2R\right)\sqrt{\log\left(\frac{3\lfloor \log_2 L \rfloor}{\alpha}\right)}\Bigg) \quad ,
\end{aligned}$$

and let $r > 0$ be such that

$$\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha, \beta, \lambda_0, R, L)}{\sqrt{L}} \quad . \tag{4.95}$$

Recall that for λ in $\mathcal{S}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \{H_k \in \mathcal{H}_{M,R},\ d_2(\lambda, H_k) \geq r\}$ with $d_2(\lambda, H_k) = |\delta|\sqrt{k/M - \tau}\mathbb{1}_{\tau \leq k/M}$ for all k in $\{1, \ldots, M\}$. To bound $\mathrm{FWSR}_\beta(\mathcal{R}_2^{(2)}, \mathcal{S}[\lambda_0, R])$ by r, it is sufficient to prove that for all λ in $\mathcal{S}[\lambda_0, R]$, $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\right) \geq 1 - \beta$.
Let us consider then λ in $\mathcal{S}[\lambda_0, R]$ such that $\lambda = \lambda_0 + \delta\mathbb{1}_{(\tau,1]}$ where δ in $(-\lambda_0, R - \lambda_0] \setminus \{0\}$ and τ in $(0,1)$.
If $\mathcal{F}_r(\lambda) = \varnothing$, we get that $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\right) = 1 \geq 1 - \beta$, so by now we assume that λ is such that $\mathcal{F}_r(\lambda) \neq \varnothing$.
We define

$$k_r = \min\{\tau M < k' \leq M,\ \delta^2 (k'/M - \tau) \geq r^2\} \quad , \tag{4.96}$$

and with the relationship $\{\mathcal{F}_r(\lambda) \subset \mathcal{R}_2^{(2)}\} = \{k_r \geq \hat{k}_2^{(2)} + 1\}$, we have to prove the following inequality

$$P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) \leq \beta \tag{4.97}$$

to obtain the expected result.

First, if $k_r = M$ then

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) &= P_\lambda\left(\hat{k}_2^{(2)} = M\right) \\ &= P_\lambda\left(\phi_{2,M}^{(2)} = 0\right) \\ &= P_\lambda\left(\forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ T_{j,M} \leq t_{j,M}(1-u_\alpha)\right) \ , \end{aligned}$$

where for all j in $\{1,\dots,\lfloor \log_2 L\rfloor\}$,

$$T_{j,M}(N) = \frac{2^j}{L^2}\left(N\left(\left(1-\frac{1}{2^j}\right),1\right]^2 - N\left(\left(1-\frac{1}{2^j}\right),1\right]\right) - \frac{2\lambda_0}{L} N\left(\left(1-\frac{1}{2^j}\right),1\right] + \frac{\lambda_0^2}{2^j} \ .$$

By definition of k_r, since the condition (4.95) ensures in particular that

$$|\delta|\sqrt{1-\tau} \geq \max\left(4\sqrt{\frac{2\lambda_0 \log(3/u_\alpha)}{L}} + 2\sqrt{2\sqrt{\frac{2}{\beta}}\frac{R}{L}}\ ,\ 2\sqrt{\frac{2\sqrt{2}R\log(3/u_\alpha)}{3L}}\ ,\ 4\left(\frac{2}{3}\right)^{1/3}\lambda_0^{1/6}R^{1/3}\sqrt{\frac{\log(3/u_\alpha)}{L}}\ ,\right.$$
$$\left. 16\sqrt{\frac{R}{\beta L}}\ ,\ \frac{\sqrt{2}R}{\sqrt{L}}\right) \ ,$$

we immediately conclude that

$$P_\lambda\left(\forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ T_{j,M} \leq t_{j,M}(1-u_\alpha)\right) \leq \beta \ ,$$

according to the minimax study of the change-point detection done in Proposition 2.18 which involves the statistic $\max_{j\in\{1,\dots,\lfloor \log_2 L\rfloor\}}(T_{j,M} - t_{j,M})$.

Assume by now that $k_r \leq M-1$, and we compute then

$$\begin{aligned} P_\lambda\left(\hat{k}_2^{(2)} \geq k_r\right) &= P_\lambda\left(\exists k \geq k_r, \phi_{2,k}^{(2)} = 0\right) \\ &= P_\lambda\left(\exists k \geq k_r,\ \forall j \in \{1,\dots,\lfloor \log_2 L\rfloor\},\ T_{j,k} \leq t_{j,k}(1-u_\alpha)\right) \\ &\leq P_\lambda\left(\exists k \geq k_r,\ T_{j_\tau(k),k} \leq t_{j_\tau(k),k}(1-u_\alpha)\right) \\ &= P_\lambda\left(\exists k \geq k_r;\ U_{j_\tau(k),k} + 2V_{j_\tau(k),k} + C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha) \leq 0\right) \ , \end{aligned}$$

where $j_\tau(k)$ is defined by (4.72). The assumption (4.95) entails that $r \geq 2R\sqrt{\log(3\lfloor \log_2 L\rfloor/\alpha)/L}$ and then for all $k \geq k_r$

$$\delta^2\left(\frac{k}{M} - \tau\right) \geq 4R^2 \frac{\log(3\lfloor \log_2 L\rfloor/\alpha)}{L} \ . \tag{4.98}$$

We therefore get $k/M - \tau \geq 4\log(3\lfloor \log_2 L\rfloor/\alpha)/L$, which leads to the condition (4.74) for $L \geq 3$. Indeed, for all $L \geq 3$, we have $2\log(3\lfloor \log_2 L\rfloor/\alpha)/L \geq 2^{-\lfloor \log_2 L\rfloor}$ which entails $k/M(1 - 2^{-\lfloor \log_2 L\rfloor+1}) \geq \tau$. As a consequence, we have $-\log_2(1-\tau M/k) + 1 \leq \lfloor \log_2 L\rfloor$ which implies (4.74). Then for all $k \geq k_r$, $j_\tau(k)$ defined by (4.72) is such that $j_\tau(k)$ belongs to $\{1,\dots,\lfloor \log_2 L\rfloor)\}$ and satisfies (4.73).
Consider now the following partition of cardinality $\Psi = \lceil \log_2(L)\rceil$:

$$\bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} I_{\tau,M,i,k_r} = \bigcup_{i=0}^{\lceil \log_2(L)\rceil-1} [\tau M + 2^i(k_r - \tau M)\ ;\ \tau M + 2^{i+1}(k_r - \tau M)) \ .$$

The condition (4.98) implies $1/(k_r/M-\tau) \le L \le 2^{\lceil \log_2(L)\rceil}$ and $\tau M + 2^{(\lceil \log_2(L)\rceil -1)+1}(k_r-\tau M) > M$, so that

$$[k_r;M] \subset \bigcup_{i=0}^{\lceil \log_2(L)\rceil -1} I_{\tau,M,i,k_r} \ .$$

As a consequence

$$\begin{aligned}
P_\lambda\left(\hat{k}_2^{(2)} \ge k_r\right) &\le \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\exists k \in I_{\tau,M,i,k_r}, U_{j_\tau(k),k} + 2V_{j_\tau(k),k} + C_{j_\tau(k),k} - t_{j_\tau(k),k}(1-u_\alpha) \le 0\right)\\
&= \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}}\left(-U_{j_\tau(k),k} - 2V_{j_\tau(k),k} - C_{j_\tau(k),k} + t_{j_\tau(k),k}(1-u_\alpha)\right) \ge 0\right)\\
&\le \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}}\left(-U_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \ge 0\right)\\
&\quad + \sum_{i=0}^{\lceil \log_2(L)\rceil -1} P_\lambda\left(\sup_{k\in I_{\tau,M,i,k_r}}\left(-2V_{j_\tau(k),k} + \frac{t_{j_\tau(k),k}(1-u_\alpha) - C_{j_\tau(k),k}}{2}\right) \ge 0\right) \ .
\end{aligned}$$

The proof is then completed applying Lemma 4.21 and Lemma 4.22. Recall that (4.73) is satisfied and notice that

$$\frac{\frac{\tau M + 2^{i+1}(k_r-\tau M)}{M} - \tau}{\frac{\tau M + 2^{i}(k_r-\tau M)}{M} - \tau} = 2 \ , \tag{4.99}$$

that is (4.83). Then, by definition of k_r (4.96), the condition (4.95) gives

$$|\delta|\sqrt{\frac{k_r}{M} - \tau} \ge \max\left(128\sqrt{R}\sqrt{\frac{\log(20\lceil \log_2 L\rceil/\beta)}{L}} \ , \ C(\lambda_0,R)\sqrt{\frac{\log(3\lfloor \log_2 L\rfloor/\alpha)}{L}}\right) \ ,$$

where $C(\lambda_0,R) = \max\left(8\sqrt{2\lambda_0} \ , \ 2\cdot 32^{1/3}\cdot \lambda_0^{1/6}R^{1/3} \ , \ 2\cdot(32/3)^{1/4}\sqrt{R} \ , \ 2R\right)$, and since $\tau M + 2^i(k_r-\tau M) \ge k_r$ for all i in $\{0,\ldots,\lceil \log_2(L)\rceil -1\}$, the inequality (4.84) is satisfied. As a consequence of Lemma 4.23, the two conditions (4.75) and (4.80) are satisfied by the cover $\bigcup_{i=0}^{\lceil \log_2(L)\rceil -1} I_{\tau,M,i,k_r}$ and we may apply Lemma 4.21 and Lemma 4.22 to conclude the proof. □

V.4 Proofs of Section III

Proof of Lemma 4.7

Let $\mathcal{R}$ be a multiple testing procedure on $\mathcal{H}$ such that $\mathrm{FWER}(\mathcal{R}) \le \alpha$ and let λ in $\overline{\mathcal{S}}$. Recall that we define I_α by $I_\alpha = \{x \in [0,1] : \ x \le ((\sup\{k \in \{1,\ldots,M\}, \ H_k \notin \mathcal{R}\} + 1)/M) \wedge 1\}$. If $\tau < 1/M$, one obtains immediately that $P_\lambda(\tau \in I_\alpha) = 1 \ge 1-\alpha$, and if $\tau \ge 1/M$ we get

$$\begin{aligned}
\alpha \ge P_\lambda(\mathcal{R}\cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H}, \ H_k \in \mathcal{R}, \ \lambda \in H_k)\\
&= P_\lambda(\exists k \in \{1,\ldots,\lfloor \tau M\rfloor\}, \ H_k \in \mathcal{R}, \ \lambda \in H_k)
\end{aligned}$$

because λ belongs to $H_{\lfloor \tau M\rfloor}$ and not to $H_{\lfloor \tau M\rfloor+1}$. Then

$$P_\lambda(\exists k \in \{1,\ldots,\lfloor \tau M\rfloor\}, \ H_k \in \mathcal{R}, \ \lambda \in H_k) \ge P_\lambda(\tau \notin I_\alpha)$$

because on the event $\{\tau \notin I_\alpha\}$, one has $H_{\lfloor \tau M \rfloor}$ in $\mathcal{R}$. Therefore $P_\lambda(\tau \in I_\alpha) \geq 1-\alpha$ for all λ in $\overline{\mathcal{S}}$ which proves the first part of the lemma.
Conversely, let λ in $\overline{\mathcal{S}}$ and let I_α a $(1-\alpha)$-confidence region for τ on $\overline{\mathcal{S}}$. Set $\mathcal{R} = \{H_k \in \mathcal{H},\ k/M > \sup\{x \in [0,1],\ x \in I_\alpha\}\}$ and notice that if $\tau < 1/M$ then $P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) = 0 \leq \alpha$. If $\tau \geq 1/M$, we get

$$
\begin{aligned}
P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H},\ H_k \in \mathcal{R},\ \lambda \in H_k) \\
&= P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ k/M > \sup\{x \in [0,1],\ x \in I_\alpha\},\ \lambda \in H_k) \\
&= P_\lambda\left(\sup\{x \in [0,1],\ x \in I_\alpha\} < \frac{\lfloor \tau M \rfloor}{M}\right) \\
&= 1 - P_\lambda\left(\sup\{x \in [0,1],\ x \in I_\alpha\} \geq \frac{\lfloor \tau M \rfloor}{M}\right) .
\end{aligned}
$$

On the event $\{\tau \in I_\alpha\}$, $\tau \leq \sup\{x \in [0,1],\ x \in I_\alpha\}$ and since $\lfloor \tau M \rfloor / M \leq \tau$, we obtain that $\lfloor \tau M \rfloor / M \leq \sup\{x \in [0,1],\ x \in I_\alpha\}$. Then

$$
P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) \leq 1 - P_\lambda(\tau \in I_\alpha) ,
$$

and $P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) \leq \alpha$ for all λ in $\overline{\mathcal{S}}$ which is the expected result.

Proof of Lemma 4.8

The control of $\mathrm{FWER}(\mathcal{R})$ over $\overline{\mathcal{S}}[\lambda_0, \delta^*]$ by α is a consequence of the second part of Lemma 4.7.
Now, let λ in $\mathcal{S}[\lambda_0, \delta^*]$ and notice that on the event $\{\tau \in (\phi(N) - a, \phi(N) + b]\}$, if H_k belongs to $\{H_k \in \mathcal{H}_{M,\delta^*},\ k/M - \tau \geq a + b\}$, then $k/M \geq \tau + a + b$ and since $\phi(N) + b < \tau + a + b$, we obtain $k/M > \phi(N) + b$ that is H_k belongs to $\mathcal{R}$. Then (4.15) leads to

$$
1 - \alpha \leq P_\lambda(\tau \in (\phi(N) - a, \phi(N) + b]) \leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\delta^*},\ k/M - \tau \geq a + b\} \subset \mathcal{R}\right) .
$$

Therefore, for all λ in $\mathcal{S}[\lambda_0, \delta^*]$,

$$
\begin{aligned}
1 - \alpha &\leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\delta^*},\ \delta^{*2}\,(k/M - \tau) \geq \delta^{*2}(a+b)\} \subset \mathcal{R}\right) \\
&= P_\lambda\left(\mathcal{F}_{|\delta^*|\sqrt{a+b}}(\lambda) \subset \mathcal{R}\right) ,
\end{aligned}
$$

and then $\mathrm{FWSR}_\alpha(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b}$. In particular one has $\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*]) \leq |\delta^*|\sqrt{a+b}$ hence

$$
\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0, \delta^*]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}[\lambda_0, \delta^*])^2}{\delta^{*2}}
$$

which proves the first part of the Lemma.
Assume now that $\mathcal{R}$ is a multiple procedure on $\mathcal{H}_{M,\delta^*}$ satisfying $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}[\lambda_0, \delta^*]) \leq r$. Recall that we define an estimator of τ from $\mathcal{R}$ by $\hat{\tau} = \hat{k}/M$ where $\hat{k} = \sup\{k \in \{1,\dots,M\},\ H_k \notin \mathcal{R}\} \vee 0$. If $\tau < 1/M$ then $P_\lambda(\tau > \hat{\tau} + 1/M) = 0 \leq \alpha$ and if $\tau \geq 1/M$, we get for all λ in $\overline{\mathcal{S}}[\lambda_0, \delta^*]$,

$$
\begin{aligned}
\alpha \geq P_\lambda(\mathcal{R} \cap \mathcal{T}(\lambda) \neq \varnothing) &= P_\lambda(\exists H_k \in \mathcal{H}_{M,\delta^*},\ H_k \in \mathcal{R},\ H_k \in \mathcal{T}(\lambda)) \\
&= P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k)
\end{aligned}
$$

because λ belongs to $H_{\lfloor \tau M \rfloor}$ and not in $H_{\lfloor \tau M \rfloor + 1}$. On the event $\{\tau > \hat{\tau} + 1/M\}$, $\hat{k} < \lfloor \tau M \rfloor$ and then $H_{\lfloor \tau M \rfloor}$ belongs to $\mathcal{R}$. This leads to $P_\lambda(\exists k \in \{1,\dots,\lfloor \tau M \rfloor\},\ H_k \in \mathcal{R},\ \lambda \in H_k) \geq P_\lambda(\tau > \hat{\tau} + 1/M)$ and

$$
P_\lambda(\tau > \hat{\tau} + 1/M) \leq \alpha
$$

for all λ in $\overline{\mathcal{S}}[\lambda_0,\delta^*]$. Moreover, since $\mathrm{FWSR}_\beta(\mathcal{R},\mathcal{S}[\lambda_0,\delta^*]) \le r$, one has $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) \ge 1-\beta$ for all λ in $\mathcal{S}[\lambda_0,\delta^*]$. We get then

$$\begin{aligned}\beta \ge P_\lambda(\mathcal{F}_r(\lambda)\cap(\mathcal{H}_{M,\delta^*}\setminus\mathcal{R})\neq\varnothing) &= P_\lambda(\exists k\in\{1,\dots,M\},\ d_2(\lambda,H_k)\ge r,\ H_k\notin\mathcal{R})\\ &= P_\lambda(\exists k\in\{\lceil\tau M\rceil,\dots,\hat{k}\},\ \delta^{*2}(k/M-\tau)\ge r^2,\ H_k\notin\mathcal{R})\\ &\ge P_\lambda(\hat{k}/M-\tau\ge r^2/\delta^{*2})\ ,\end{aligned}$$

and one obtains $P_\lambda\left(\tau\le\hat{\tau}-r^2/\delta^{*2}\right)\le\beta$ for all λ in $\mathcal{S}[\lambda_0,\delta^*]$. This inequality remains true for $\lambda=\lambda_0$ (and $\tau=1$). Finally,

$$\inf_{\lambda\in\overline{\mathcal{S}}[\lambda_0,\delta^*]} P_\lambda\left(\tau\in\left(\hat{\tau}-\frac{r^2}{\delta^{*2}},\ \hat{\tau}+\frac{1}{M}\right]\right)\ge 1-\alpha-\beta\ .$$

Proof of Lemma 4.10

Let α in $(0,1/2)$, $\lambda_0>0$, $R>\lambda_0$ and $L\ge1$. Taking $\phi(N)=a=b=1/2$, we obtain $\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0,R])\le1$. Assume now that $\mathcal{L}_\alpha(\overline{\mathcal{S}}[\lambda_0,R])<1$. We may fix ϕ in $\mathcal{MD}$ such that $\mathcal{L}_\alpha(\phi,\overline{\mathcal{S}}[\lambda_0,R])<1$, and then there exists $a>0$ and $b>0$ such that $a+b<1$ and

$$\inf_{\lambda\in\overline{\mathcal{S}}[\lambda_0,R]} P_\lambda(\tau\in(\phi(N)-a,\phi(N)+b])\ge1-\alpha\ .$$

On the one hand, for $\lambda=\lambda_0$ we get

$$P_{\lambda_0}(1\in(\phi(N)-a,\phi(N)+b])\ge1-\alpha\ ,$$

hence $P_{\lambda_0}(\phi(N)\ge1-b)\ge1-\alpha$. Since $1-b>a$, this yields $P_{\lambda_0}(\phi(N)>a)\ge1-\alpha$ and

$$P_{\lambda_0}(\phi(N)\le a)\le\alpha\ . \tag{4.100}$$

On the other hand, for every δ in $(0,R-\lambda_0]$ and every τ in $(0,1)$,

$$P_\lambda(\tau\in(\phi(N)-a,\phi(N)+b])\ge1-\alpha$$

with $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$. Using the Girsanov lemma (see Lemma 2.1), one computes

$$P_\lambda(\tau\in(\phi(N)-a,\phi(N)+b])=E_{\lambda_0}\left[\exp\left(\int_\tau^1\log\left(\frac{\lambda_0+\delta}{\lambda_0}\right)dN_t-\delta(1-\tau)L\right)\mathbb{1}_{\tau\in(\phi(N)-a,\phi(N)+b]}\right]\ ,$$

so letting δ tends to 0, we obtain by dominated convergence that for every τ in $(0,1)$:

$$P_{\lambda_0}(\tau\in(\phi(N)-a,\phi(N)+b])\ge1-\alpha\ .$$

In particular, when τ tends to 0, this leads to

$$P_{\lambda_0}(\phi(N)\le a)\ge1-\alpha\ . \tag{4.101}$$

Therefore, (4.100) and (4.101) leads to

$$1-\alpha\le P_{\lambda_0}(\phi(N)\le a)\le\alpha\ .$$

This entails $\alpha\ge1/2$ which is a contradiction.

Proof of Lemma 4.11

Let $\lambda_0 > 0$ and ϕ_α a level-α test of (H_0) "$\lambda = \lambda_0$" versus (H_1) "$\lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0]$", with $\mathcal{S}_{\geq\Delta}[\lambda_0]$ defined by (4.18).
Let $r > 0$ and λ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$ such that $d_2(\lambda, \{\lambda_0\}) \geq r$. We have

$$\begin{aligned} P_\lambda(\phi_\alpha(N) = 0) &= 1 - P_\lambda(\phi_\alpha(N) = 1) + P_{\lambda_0}(\phi_\alpha(N) = 1) - P_{\lambda_0}(\phi_\alpha(N) = 1) \\ &\geq 1 - \alpha - |P_{\lambda_0}(\phi_\alpha(N) = 1) - P_\lambda(\phi_\alpha(N) = 1)| \\ &\geq 1 - \alpha - V\left(P_\lambda, P_{\lambda_0}\right) , \end{aligned}$$

where $V\left(P_\lambda, P_{\lambda_0}\right)$ is the total variation distance between the probability measures P_λ and P_{λ_0}. Then, using the Pinsker inequality (see for example Lemma 2.5 in [Tsy08]),

$$P_\lambda(\phi_\alpha(N) = 0) \geq 1 - \alpha - \sqrt{\frac{K\left(P_\lambda, P_{\lambda_0}\right)}{2}} ,$$

where $K\left(P_\lambda, P_{\lambda_0}\right)$ is the Kullback divergence between the probability measures P_λ and P_{λ_0}. As in the proof of Lemma 2.11, we thus deduce that if there exists λ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$ such that $d_2(\lambda, \{\lambda_0\}) \geq r$ satisfying $1 - \alpha - \sqrt{K\left(P_\lambda, P_{\lambda_0}\right)/2} \geq \beta$, then $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) \geq r$.
For $r \geq \Delta$, let us introduce for all τ in $(0,1)$, $\lambda_r = \lambda_0 + r(1-\tau)^{-1/2}\mathbb{1}_{(\tau,1]}$ in $\mathcal{S}_{\geq\Delta}[\lambda_0]$, and such that $d_2(\lambda_r, \{\lambda_0\}) = r$. Then, the Girsanov lemma (see Lemma 2.1) entails

$$K\left(P_{\lambda_r}, P_{\lambda_0}\right) = \int \log\left(\frac{dP_{\lambda_r}}{dP_{\lambda_0}}\right) dP_{\lambda_r} = \log\left(1 + \frac{r}{\lambda_0\sqrt{1-\tau}}\right)\left(\lambda_0 + \frac{r}{\sqrt{1-\tau}}\right)(1-\tau)L - Lr\sqrt{1-\tau} .$$

Hence choosing τ close enough to 1 – which is allowed as long as λ_r is not constrained to be upper bounded by some given constant, $K\left(P_{\lambda_r}, P_{\lambda_0}\right) \leq 2(1-\alpha-\beta)^2$. This entails

$$\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) \geq r, \quad \text{for all } r \geq \Delta ,$$

which allows to conclude that $\mathrm{mSR}_{\alpha,\beta}^{\{\lambda_0\}}(\mathcal{S}_{\geq\Delta}[\lambda_0]) = +\infty$.

Proof of Theorem 4.14

By now, for $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$, the simple hypothesis $H_k[\lambda_0, \Delta, R]$ is simply written H_k for short. The following lemma gives an upper bound for the quantiles $s_{\Delta,k}(1-\alpha/2)$ (respectively $s_{-\Delta,k}(1-\alpha/2)$) of $S_{\Delta,k}$ (respectively of $S_{-\Delta,k}$) under H_k, and its proof follows the same lines as the one of Lemma 4.17, just replacing δ^* by Δ when $\delta^* > 0$ and by $-\Delta$ when $\delta^* < 0$. It highlights in particular that we can bound these quantiles by some constants which do not depend on k, L and M.

Lemma 4.24 (Control of the quantiles).
Let α in $(0,1)$, $\lambda_0 > 0$, $R > 0$ and Δ in $(0, \lambda_0 \wedge (R - \lambda_0))$. For all M in $\mathbb{N}^*$, for all $L \geq 1$ and for all $k \in \{1, \ldots, M\}$, one has

$$\begin{cases} s_{\Delta,k}\left(1 - \frac{\alpha}{2}\right) \leq q_{\lambda_0}\left(1 - \frac{\alpha}{2}, \frac{\Delta}{2}\right) , \\ s_{-\Delta,k}\left(1 - \frac{\alpha}{2}\right) \leq \dfrac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0 - \Delta/2}\right)} , \end{cases} \tag{4.102}$$

where $q_{\lambda_0}(1-\alpha/2, \Delta/2)$ is defined in Lemma 4.2.

Let us prove the Theorem 4.14 and first recall that for λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$, $\mathcal{T}(\lambda) = \{H_k \in \mathcal{H}_{M,\Delta,R}\ ,\ \lambda \in H_k\}$ is the set of true hypotheses.

Proof of Theorem 4.14.
For all k in $\{1,\ldots,M\}$, recall that H_k stands for $H_k[\lambda_0,\Delta,R]$. We start with the control of FWER($\mathcal{R}_3$) over $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$, and for λ in $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0, R]$ we compute to this end

$$P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = P_\lambda\left(\exists k \in \{1,\ldots,\lfloor\tau M\rfloor\},\ k \geq \hat{k}_3 + 1,\ \lambda \in H_k\right)$$

because λ belongs to $H_{\lfloor\tau M\rfloor}$ and not to $H_{\lfloor\tau M\rfloor+1}$. If $\tau < 1/M$ then $P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) = 0$, and if $\tau \geq 1/M$ one has

$$\begin{aligned}
P_\lambda\left(\mathcal{R}_3 \cap \mathcal{T}(\lambda) \neq \varnothing\right) &= P_\lambda\left(\hat{k}_3 + 1 \leq \lfloor\tau M\rfloor\right) \\
&= P_\lambda\left(\phi_{3,\lfloor\tau M\rfloor} = 1\right) \\
&\leq P_\lambda\left(S_{\Delta,\lfloor\tau M\rfloor}(N) > s_{\Delta,\lfloor\tau M\rfloor}(1-\alpha/2)\right) + P_\lambda\left(S_{-\Delta,\lfloor\tau M\rfloor}(N) > s_{-\Delta,\lfloor\tau M\rfloor}(1-\alpha/2)\right) \\
&\leq \alpha\ ,
\end{aligned}$$

that is FWER($\mathcal{R}_3$) is bounded by α.

Let us compute now an upper bound for $\mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0, R])$.
Let $r > 0$ and M in $\mathbb{N}^*$. Recall that for λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$,

$$\mathcal{F}_r(\lambda) = \{H_k[\lambda_0,\Delta,R] \in \mathcal{H}_{M,\Delta,R},\ d_2(\lambda, H_k[\lambda_0,\Delta,R]) \geq r\}$$

with $d_2(\lambda, H_k[\lambda_0,\Delta,R]) = |\delta|\sqrt{k/M - \tau}\mathbb{1}_{\tau\leq k/M}$ for all k in $\{1,\ldots,M\}$. One has the following straightforward assertion

$$\forall \lambda \in \mathcal{S}_{\geq\Delta}[\lambda_0, R],\ \mathcal{F}_r(\lambda) = \varnothing \iff r \geq \lambda_0 \vee (R - \lambda_0)\ ,$$

and in particular, for all multiple testing procedure $\mathcal{R}$,

$$\mathrm{FWSR}_\beta(\mathcal{R}, \mathcal{S}_{\geq\Delta}[\lambda_0, R]) \leq \lambda_0 \vee (R - \lambda_0).$$

Indeed, let $r > 0$ be such that $r \geq \lambda_0 \vee (R-\lambda_0)$. Then for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$, $\mathcal{F}_r(\lambda) = \varnothing$ and then $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) = 1 \geq 1 - \beta$ for all multiple testing procedure $\mathcal{R}$ and for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$.

Now, assume that $L > (C(\alpha,\beta,\lambda_0,\Delta,R)/(\lambda_0 \vee (R-\lambda_0)))^2$ and let $r > 0$ be such that

$$\lambda_0 \vee (R - \lambda_0) > r \geq \frac{C(\alpha,\beta,\lambda_0,\Delta,R)}{\sqrt{L}}\ , \tag{4.103}$$

where $C(\alpha,\beta,\lambda_0,\Delta,R)$ is defined by

$$\begin{aligned}
C(\alpha,\beta,\lambda_0,\Delta,R) = 2\max\Bigg(&\sqrt{R-\lambda_0}\sqrt{q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) + \frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)}}\ , \\
&\sqrt{\lambda_0}\sqrt{q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right) + \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\ ,\ 2\sqrt{\frac{3R}{\beta}}\Bigg)\ ,
\end{aligned} \tag{4.104}$$

where $q_{\lambda_0}(1-\alpha/2, \Delta/2)$ and $q_{\lambda_0-\Delta}(1-\beta/3,\Delta/2)$ are two positive constants defined by Lemma 4.2.
To bound $\mathrm{FWSR}_\beta(\mathcal{R}_3, \mathcal{S}_{\geq\Delta}[\lambda_0, R])$ by r, it is sufficient to obtain $P_\lambda\left(\mathcal{F}_r(\lambda) \subset \mathcal{R}_3\right) \geq 1-\beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0, R]$.

We consider λ in $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$ of the form $\lambda=\lambda_0+\delta\mathbb{1}_{(\tau,1]}$ with τ in $(0,1)$ and δ in $\{(-\lambda_0,-\Delta]\cup[\Delta,R-\lambda_0]\}$. If $\mathcal{F}_r(\lambda)=\emptyset$, we easily get $P_\lambda(\mathcal{F}_r(\lambda)\subset\mathcal{R}_3)=1\geq 1-\beta$. We therefore assume by now that λ is satisfying $\mathcal{F}_r(\lambda)\neq\emptyset$, and we define

$$k_r=\min\{\tau M<k'\leq M,\ \delta^2(k'/M-\tau)\geq r^2\}\ . \tag{4.105}$$

By virtue of $\{\mathcal{F}_r(\lambda)\subset\mathcal{R}_3\}=\{k_r\geq\hat{k}_3+1\}$, we want to prove the following inequality

$$P_\lambda\left(\hat{k}_3\geq k_r\right)\leq\beta \tag{4.106}$$

to obtain the expected result.

First, if $k_r=M$ then

$$\begin{aligned}P_\lambda\left(\hat{k}_3\geq k_r\right)&=P_\lambda\left(\hat{k}_3=M\right)\\&=P_\lambda\left(\phi_{3,M}=0\right)\\&=P_\lambda\left(S_{\Delta,M}(N)\leq s_{\Delta,M}(1-\alpha/2),\ S_{-\Delta,M}\leq s_{-\Delta,M}(1-\alpha/2)\right)\ .\end{aligned}\tag{4.107}$$

Assume that $\Delta\leq\delta\leq R-\lambda_0$ and notice that (4.107) ensures

$$\begin{aligned}P_\lambda\left(\hat{k}_3\geq k_r\right)&\leq P_\lambda\left(S_{\Delta,M}(N)\leq s_{\Delta,M}(1-\alpha/2)\right)\\&\leq P_\lambda\left(S_{\Delta,M}(N)\leq q_{\lambda_0}(1-\alpha/2,\Delta/2)\right)\ \text{ using (4.102)}\\&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\Delta/2)(1-\tau)L\leq q_{\lambda_0}(1-\alpha/2,\Delta/2)\right)\ .\end{aligned}\tag{4.108}$$

By definition of k_r, the condition (4.103) entails in particular that

$$\delta\sqrt{1-\tau}\geq\frac{2}{\sqrt{L}}\max\left(\sqrt{(R-\lambda_0)q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)},\ 2\sqrt{\frac{R}{\beta}}\right)\ ,$$

and then

$$\delta\sqrt{1-\tau}\geq\frac{2}{\sqrt{L}}\max\left(\sqrt{\delta q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)},\ 2\sqrt{\frac{\lambda_0+\delta}{\beta}}\right)\ . \tag{4.109}$$

Therefore, we get on the one hand $\delta\sqrt{1-\tau}\geq 2\sqrt{\delta q_{\lambda_0}(1-\alpha/2,\Delta/2)}/\sqrt{L}$ and then $\delta(1-\tau)\geq 4q_{\lambda_0}(1-\alpha/2,\Delta/2)/L$, and on the other hand $\delta\sqrt{1-\tau}\geq 4\sqrt{(\lambda_0+\delta)/(\beta L)}$, hence $\delta(1-\tau)\geq 4\sqrt{(\lambda_0+\delta)(1-\tau)/(\beta L)}$. Thus, using $2\max(a+b)\geq a+b$ for all $a,b\geq 0$, (4.109) leads to

$$\delta(1-\tau)\geq 2\left(\frac{q_{\lambda_0}(1-\alpha/2,\Delta/2)}{L}+\sqrt{\frac{(\lambda_0+\delta)(1-\tau)}{\beta L}}\right)\ ,$$

and since $\delta/2\leq\delta-\Delta/2$, we get

$$\left(\delta-\frac{\Delta}{2}\right)(1-\tau)L\geq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)+\sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}\ . \tag{4.110}$$

Moreover, one has $E_\lambda[N(\tau,1]-(\lambda_0+\Delta/2)(1-\tau)L]=(\delta-\Delta/2)(1-\tau)L$, $\mathrm{Var}_\lambda[N(\tau,1]-(\lambda_0+\Delta/2)(1-\tau)L]=(\lambda_0+\delta)(1-\tau)L$, and then (4.108) gives

$$\begin{aligned}P_\lambda\left(\hat{k}_3\geq k_r\right)&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta)(1-\tau)L\leq q_{\lambda_0}(1-\alpha/2,\Delta/2)-(\delta-\Delta/2)(1-\tau)L\right)\\&\leq P_\lambda\left(N(\tau,1]-(\lambda_0+\delta)(1-\tau)L\leq-\sqrt{(\lambda_0+\delta)(1-\tau)L/\beta}\right)\ \text{ with (4.110)}\\&\leq\beta\ \text{ with the Bienayme-Chebyshev inequality}\ .\end{aligned}$$

Secondly, assume that $-\lambda_0 < \delta \le -\Delta$ and notice that (4.107) yields

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \ge k_r\right) &\le P_\lambda\left(S_{-\Delta,M}(N) \le s_{-\Delta,M}(1-\alpha/2)\right) \\
&\le P_\lambda\left(S_{-\Delta,M}(N) \le \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right) \quad \text{using (4.102)} \\
&\le P_\lambda\left(-N(\tau,1] + \left(\lambda_0 - \frac{\Delta}{2}\right)(1-\tau)L \le \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right) \quad . \qquad (4.111)
\end{aligned}$$

By definition of k_r, the condition (4.103) also entails in particular that

$$|\delta|\sqrt{1-\tau} \ge \frac{2}{\sqrt{L}} \max\left(\sqrt{\frac{\lambda_0 \log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\,,\, 2\sqrt{\frac{R}{\beta}}\right) \ ,$$

hence

$$|\delta|\sqrt{1-\tau} \ge \frac{2}{\sqrt{L}} \max\left(\sqrt{\frac{|\delta| \log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\,,\, 2\sqrt{\frac{\lambda_0+\delta}{\beta}}\right) \ ,$$

and then, using the inequality $2\max(a+b) \ge a+b$ for all $a, b \ge 0$ again,

$$\frac{|\delta|}{2}(1-\tau)L \ge \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}} \ .$$

Since $|\delta|/2 \le |\delta| - \Delta/2$, we finally get

$$\left(|\delta| - \frac{\Delta}{2}\right)(1-\tau)L \ge \frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} + \sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}} \ , \qquad (4.112)$$

and since $E_\lambda[-N(\tau,1] + (\lambda_0 - \Delta/2)(1-\tau)L] = (|\delta| - \Delta/2)(1-\tau)L$ and $\mathrm{Var}_\lambda[-N(\tau,1] + (\lambda_0 - \Delta/2)(1-\tau)L] = (\lambda_0+\delta)(1-\tau)L$, we conclude with the following inequalities

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \ge k_r\right) &\le P_\lambda\left(-N(\tau,1] + (\lambda_0 - |\delta|)(1-\tau)L \le \frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)} - \left(|\delta| - \frac{\Delta}{2}\right)(1-\tau)L\right) \\
&\le P_\lambda\left(-N(\tau,1] + (\lambda_0 - |\delta|)(1-\tau)L \le -\sqrt{\frac{(\lambda_0+\delta)(1-\tau)L}{\beta}}\right) \quad \text{with (4.112)} \\
&\le \beta \quad \text{with the Bienayme-Chebyshev inequality} \ ,
\end{aligned}$$

which concludes the first part of the proof when $k_r = M$.

Assume by now that $k_r \le M-1$, and we compute

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \ge k_r\right) &= P_\lambda\left(\exists k \ge k_r, \phi_{3,k} = 0\right) \\
&= P_\lambda\left(\exists k \ge k_r,\ S_{\Delta,k}(N) \le s_{\Delta,k}(1-\alpha/2),\ S_{-\Delta,k} \le s_{-\Delta,k}(1-\alpha/2)\right) \ . \qquad (4.113)
\end{aligned}$$

Assume first that $\Delta \le \delta \le R - \lambda_0$.
The equality (4.113) leads to

$$P_\lambda\left(\hat{k}_3 \ge k_r\right) \le P_\lambda\left(\exists k \ge k_r,\ S_{\Delta,k} \le s_{\Delta,k}(1-\alpha/2)\right) \ ,$$

and we use (4.102) in Lemma 4.24 to get

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) &\leq P_\lambda\left(\exists k \geq k_r,\ \sup_{t\in(0,k/M)}\left(N\left(t,\frac{k}{M}\right] - \left(\lambda_0+\frac{\Delta}{2}\right)L\left(\frac{k}{M}-t\right)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right)\\
&\leq P_\lambda\left(\exists k \geq k_r,\ N\left(\tau,\frac{k}{M}\right] - \left(\lambda_0+\frac{\Delta}{2}\right)L\left(\frac{k}{M}-\tau\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right)\\
&= P_\lambda\left(\inf_{k\in[k_r,M]}\left(N\left(\tau,\frac{k}{M}\right] - \left(\lambda_0+\frac{\Delta}{2}\right)L\left(\frac{k}{M}-\tau\right)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right)\\
&\leq P_\lambda\left(\inf_{s\in[k_r/M,1]}\left(N(\tau,s] - \left(\lambda_0+\frac{\Delta}{2}\right)L(s-\tau)\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right)\\
&= P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] - \left(\lambda_0+\frac{\Delta}{2}\right)L\left(\frac{k_r}{M}-\tau\right) + \inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)\right)\\
&= P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right)\ ,
\end{aligned}$$

where $Z_t = N(k_r/M,t] - (\lambda_0+\Delta/2)(t-k_r/M)L$ for t in $(k_r/M,1]$. Let us write J for the interval

$$J = \left[(\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}};\ (\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right]\ . \tag{4.114}$$

Using the total probability formula, we get

$$\begin{aligned}
P_\lambda\left(\hat{k}_3 \geq k_r\right) \leq{}& P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\
&+ P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] < (\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L - \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\
&+ P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right] > (\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\
\leq{}& P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right) + \frac{2\beta}{3}
\end{aligned} \tag{4.115}$$

with the Bienayme-Chebyshev inequality. To compute this last probability, we consider a simple Poisson process $(N_t^{\lambda_0+\delta})_{t\geq 0}$ of intensity $(\lambda_0+\delta)L$ with respect to the Lebesgue measure on $\mathbb{R}^+$, which is the distribution of N_t for t greater than k_r/M. We then obtain

$$\begin{aligned}
&P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - N\left(\tau,\frac{k_r}{M}\right] + \left(\lambda_0+\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\
&\quad\leq P_\lambda\left(\inf_{s\in(k_r/M,1]} Z_s \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - \left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L + \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\
&\quad= \mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta}(0,t] - \left(\lambda_0+\frac{\Delta}{2}\right)Lt\right) \leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right) - \left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right.\\
&\qquad\qquad\left. + \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\ .
\end{aligned}$$

By definition of k_r in (4.105), the condition (4.103) gives with (4.104),

$$\delta\sqrt{\frac{k_r}{M}-\tau} \geq \frac{2}{\sqrt{L}}\max\left(\sqrt{\delta}\sqrt{q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)+\frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)}}\ ,\ 2\sqrt{3}\sqrt{\frac{\lambda_0+\delta}{\beta}}\right)\ . \tag{4.116}$$

On the one hand, (4.116) leads to $\delta\sqrt{k_r/M-\tau} \geq 2\sqrt{\delta/L}\sqrt{q_{\lambda_0}(1-\alpha/2,\Delta/2)+\log(3/\beta)/\log((\lambda_0+\Delta)/(\lambda_0+\Delta/2))}$, and then $\delta(k_r/M-\tau)L \geq 4\big(q_{\lambda_0}(1-\alpha/2,\Delta/2)+\log(3/\beta)/\log((\lambda_0+\Delta)/(\lambda_0+\Delta/2))\big)$. On the other hand, (4.116) yields $\delta\sqrt{k_r/M-\tau} \geq 4\sqrt{3(\lambda_0+\delta)/(\beta L)}$ and then $\delta(k_r/M-\tau)L \geq 4\sqrt{3(\lambda_0+\delta)(k_r/M-\tau)L/\beta}$. This leads to

$$\delta\left(\frac{k_r}{M}-\tau\right)L \geq 4\max\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)+\frac{\log(3/\beta)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)}\ ,\ \sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\ ,$$

and using the fact that $a+b \leq 2\max(a,b)$ for all $a,b \geq 0$, one obtains

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\frac{\delta}{2}\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}} \leq \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\Delta}{\lambda_0+\Delta/2}\right)}\ ,$$

hence using the fact that $\delta \geq \Delta$,

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}} \leq \frac{\log(\beta/3)}{\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)}\ . \tag{4.117}$$

The inequality (4.117) is equivalent to

$$\exp\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)\right) \leq \frac{\beta}{3}\ . \tag{4.118}$$

Notice that since $\beta < 1$, (4.117) ensures

$$q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}} \leq 0\ . \tag{4.119}$$

We therefore may apply Theorem 3 and equation (15) in [Pyk59] to obtain

$$\begin{aligned}\mathbb{P}\Bigg(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right) &\leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\Bigg)\\ &\leq \exp\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\omega\right)\ ,\end{aligned}$$

where ω is the largest real root of the equation $(\lambda_0+\delta)(1-e^{-\omega}) = \omega(\lambda_0+\Delta/2)$. The root ω satisfies $\omega > \log((\lambda_0+\delta)/(\lambda_0+\Delta/2))$, and then

$$\begin{aligned}\mathbb{P}\Bigg(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right) &\leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\Bigg)\\ &\leq \exp\left(\left(q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\log\left(\frac{\lambda_0+\delta}{\lambda_0+\Delta/2}\right)\right)\ ,\end{aligned}$$

which entails with (4.118)

$$\mathbb{P}\left(\inf_{t\in(0,1-k_r/M]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0+\frac{\Delta}{2}\right)Lt\right)\leq q_{\lambda_0}\left(1-\frac{\alpha}{2},\frac{\Delta}{2}\right)-\left(\delta-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\leq\frac{\beta}{3}\ .$$

Gathering this inequality with (4.115) leads finally to $P_\lambda\left(\hat{k}_1\geq k_r\right)\leq\beta$.

Now, assume that $-\lambda_0<\delta<-\Delta$.

We proceed as in the case $\delta>0$. Let us define to this end $X_t=N\left(k_r/M,t\right]-\left(\lambda_0-\Delta/2\right)\left(t-k_r/M\right)L$ for all t in $(k_r/M,1]$.

The equality (4.113) leads to

$$P_\lambda\left(\hat{k}_3\geq k_r\right)\leq P_\lambda\left(\exists k\geq k_r,\ S_{-\Delta,k}(N)\leq s_{-\Delta,k}(1-\alpha/2)\right)\ ,$$

and applying the inequality (4.102),

$$\begin{aligned}P_\lambda\left(\hat{k}_3\geq k_r\right)&\leq P_\lambda\left(\exists k\geq k_r,\ \left(\lambda_0-\frac{\Delta}{2}\right)L\left(\frac{k}{M}-\tau\right)-N\left(\tau,\frac{k}{M}\right]\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}\right)\\&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\right)\ .\end{aligned}$$

Using the interval J defined by (4.114), we obtain using the total probability formula and the Bienayme-Chebyshev inequality

$$\begin{aligned}P_\lambda\left(\hat{k}_3\geq k_r\right)\leq{}&P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\&+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]<(\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\&+P_\lambda\left(N\left(\tau,\frac{k_r}{M}\right]>(\lambda_0+\delta)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\\leq{}&P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)+\frac{2\beta}{3}\ .\end{aligned}\tag{4.120}$$

We conclude the proof giving an upper bound for this last probability. We compute

$$\begin{aligned}&P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+N\left(\tau,\frac{k_r}{M}\right]-\left(\lambda_0-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L\ ,\ N\left(\tau,\frac{k_r}{M}\right]\in J\right)\\&\leq P_\lambda\left(\inf_{s\in(k_r/M,1]}(-X_s)\leq\frac{-\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}-\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\&=P_\lambda\left(\sup_{s\in(k_r/M,1]}X_s\geq\frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\&\leq\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right)\geq\frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\ ,\end{aligned}\tag{4.121}$$

By definition of k_r in (4.105) and the condition (4.103), we get with (4.104),

$$|\delta|\sqrt{\frac{k_r}{M}-\tau} \geq \frac{2}{\sqrt{L}} \max\left(\sqrt{\lambda_0}\sqrt{q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\,,\, 2\sqrt{\frac{3R}{\beta}}\right) ,$$

hence

$$|\delta|\sqrt{\frac{k_r}{M}-\tau} > \frac{2}{\sqrt{L}} \max\left(\sqrt{|\delta|}\sqrt{q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}}\,,\, 2\sqrt{\frac{3(\lambda_0+\delta)}{\beta}}\right) .$$

Therefore, as in the case $\delta > 0$, this leads to

$$\frac{|\delta|}{2}\left(\frac{k_r}{M}-\tau\right)L > q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)+\frac{\log(2/\alpha)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}} ,$$

and then, since $|\delta| \geq \Delta$,

$$\frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}} > q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right) . \tag{4.122}$$

We then obtain

$$\begin{aligned}
&\mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right) \geq \frac{\log(\alpha/2)}{\log\left(\frac{\lambda_0}{\lambda_0-\Delta/2}\right)}+\left(|\delta|-\frac{\Delta}{2}\right)\left(\frac{k_r}{M}-\tau\right)L-\sqrt{\frac{3(\lambda_0+\delta)(k_r/M-\tau)L}{\beta}}\right)\\
&\leq \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right) > q_{\lambda_0-\Delta}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right) \quad \text{with (4.122)}\\
&\leq \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-\frac{\Delta}{2}\right)Lt\right) > q_{\lambda_0-|\delta|}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right) \quad \text{with Lemma 4.13}\\
&\leq \mathbb{P}\left(\sup_{t\in(0,1]}\left(N^{\lambda_0+\delta}(0,t]-\left(\lambda_0-|\delta|+\frac{\Delta}{2}\right)Lt\right) > q_{\lambda_0-|\delta|}\left(1-\frac{\beta}{3},\frac{\Delta}{2}\right)\right) \quad \text{since } |\delta| \geq \Delta\\
&\leq \frac{\beta}{3}
\end{aligned}$$

by definition of $q_{\lambda_0-|\delta|}(1-\beta/3,\Delta/2)$ in Lemma 4.2. The proof is then complete using (4.120).

□

Proof of Lemma 4.15

The control of FWER($\mathcal{R}$) over $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]$ by α is a consequence of the second part of Lemma 4.7.
To prove the first part of the lemma, one obtains by the same lines as for Lemma 4.8 that for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$, since $|\delta| \leq \lambda_0 \wedge (R-\lambda_0)$,

$$\begin{aligned}
1-\alpha &\leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\Delta,R},\ \delta^2(k/M-\tau) \geq \delta^2(a+b)\} \subset \mathcal{R}\right)\\
&\leq P_\lambda\left(\{H_k \in \mathcal{H}_{M,\Delta,R},\ \delta^2(k/M-\tau) \geq (\lambda_0^2 \wedge (R-\lambda_0)^2)(a+b)\} \subset \mathcal{R}\right)\\
&= P_\lambda\left(\mathcal{F}_{(\lambda_0\wedge(R-\lambda_0))\sqrt{a+b}}(\lambda) \subset \mathcal{R}\right) .
\end{aligned}$$

Then $\mathrm{FWSR}_\alpha(\mathcal{R},\mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq (\lambda_0 \wedge (R-\lambda_0))\sqrt{a+b}$ and in particular one has $\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq (\lambda_0 \wedge (R-\lambda_0))\sqrt{a+b}$ hence

$$\mathcal{L}_\alpha(\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]) \geq \frac{\mathrm{mFWSR}_{\alpha,\alpha}(\mathcal{S}_{\geq\Delta}[\lambda_0,R])^2}{(\lambda_0 \wedge (R-\lambda_0))^2}$$

which proves the first part of the lemma.

Assume now that $\mathcal{R}$ is a multiple procedure on $\mathcal{H}_{M,\Delta,R}$ satisfying $\mathrm{FWER}(\mathcal{R}) \leq \alpha$ and $\mathrm{FWSR}_\beta(\mathcal{R},\mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq r$. Recall that we define an estimator of τ from $\mathcal{R}$ by $\hat{\tau} = \hat{k}/M$ where $\hat{k} = \sup\{k \in \{1,\ldots,M\},\ H_k \notin \mathcal{R}\} \vee 0$. Following again the same lines as the proof of Lemma 4.8, we get that for all λ in $\overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]$,

$$P_\lambda(\tau > \hat{\tau} + 1/M) \leq \alpha \ \ .$$

Now, since $\mathrm{FWSR}_\beta(\mathcal{R},\mathcal{S}_{\geq\Delta}[\lambda_0,R]) \leq r$, one has $P_\lambda(\mathcal{F}_r(\lambda) \subset \mathcal{R}) \geq 1-\beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$. Since $|\delta| \geq \Delta$, we get then

$$\begin{aligned}\beta \geq P_\lambda(\mathcal{F}_r(\lambda) \cap (\mathcal{H}_{M,\delta^*} \setminus \mathcal{R}) \neq \varnothing) &= P_\lambda(\exists k \in \{1,\ldots,M\},\ d_2(\lambda,H_k) \geq r,\ H_k \notin \mathcal{R}) \\ &= P_\lambda(\exists k \in \{\lceil \tau M \rceil,\ldots,\hat{k}\},\ \delta^2(k/M-\tau) \geq r^2,\ H_k \notin \mathcal{R}) \\ &\geq P_\lambda(\hat{k}/M - \tau \geq r^2/\delta^2) \\ &\geq P_\lambda(\hat{k}/M - \tau \geq r^2/\Delta^2) \ \ ,\end{aligned}$$

and one obtains $P_\lambda\left(\tau \leq \hat{\tau} - r^2/\Delta^2\right) \leq \beta$ for all λ in $\mathcal{S}_{\geq\Delta}[\lambda_0,R]$. This last inequality being true for $\lambda = \lambda_0$ (and $\tau = 1$), we finally get

$$\inf_{\lambda \in \overline{\mathcal{S}}_{\geq\Delta}[\lambda_0,R]} P_\lambda\left(\tau \in \left(\hat{\tau} - \frac{r^2}{\Delta^2},\ \hat{\tau} + \frac{1}{M}\right]\right) \geq 1-\alpha-\beta \ \ .$$

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DOCTORAT BRETAGNE LOIRE / MATHSTIC

UNIVERSITÉ RENNES 2

UNIVERSITÉ DE RENNES 1

Titre : Tests minimax et adaptatifs pour la détection et la localisation d'une rupture dans un processus de Poisson

Mots clés : Processus de Poisson, Détection de ruptures, Tests minimax, Tests adaptatifs, Tests multiples minimax, Tests multiples adaptatifs

Résumé : Motivés par des applications en cybersécurité et en épidémiologie, les résultats présentés dans cette thèse portent sur la détection et la localisation d'une rupture soudaine dans un processus de Poisson. Nous nous intéressons d'abord à la détection d'une rupture caractérisée par un saut (rupture non transitoire) ou un segment (rupture transitoire) dans l'intensité d'un processus de Poisson à partir d'une constante. Nous proposons une étude minimax non asymptotique complète des problèmes de tests associés, lorsque l'intensité de référence est connue ou inconnue. L'adaptation au sens du minimax pour chaque paramètre de la rupture (taille, localisation, longueur) fournit un aperçu exhaustif des différents ordres des vitesses de séparation minimax. En second lieu, on aborde la question de la détection et la localisation simultanées d'un saut dans l'intensité d'un processus de Poisson quand l'intensité de référence est connue, et de la construction d'estimateurs pour l'instant de saut. Ce problème, formulé comme un problème de tests multiples, est étudié d'un point de vue minimax non asymptotique. L'adaptation au sens du minimax en la hauteur du saut et en l'instant de rupture est considérée et deux régimes pour les vitesses de séparation minimax par famille sont établis. Aussi, une correspondance entre des procédures de tests multiples minimax et des intervalles de confiance pour l'instant de rupture est explicitée. Les tests simples et multiples minimax ou minimax adaptatifs sont basés sur des statistiques de comptage inspirées des tests de Neyman-Pearson ou des statistiques quadratiques, pouvant être combinées à des stratégies d'agrégation.

Title : Minimax and adaptive tests for the detection and the localisation of an abrupt change in a Poisson process

Keywords : Poisson process, Change points detection, Minimax tests, Adaptive tests, Minimax multiple tests, Adaptive multiple tests

Abstract : Motivated by applications in cybersecurity and epidemiology, this thesis addresses the problem of detecting and localising an abrupt change in a Poisson process. Firstly, we are interested in the detection of a change in the intensity of a Poisson process characterised by a jump (non transitory change) or a bump (transitory change) from constant. We propose a complete study from the nonasymptotic minimax testing point of view when the constant baseline intensity is known or unknown. The question of minimax adaptation with respect to each parameter (height, location, length) of the change is tackled, leading to a comprehensive overview of the various minimax separation rate regimes. Secondly, we focus on the issue of simultaneously detecting and localising a non transitory change in the intensity of a Poisson process when the baseline intensity is known, and on the construction of change point estimation procedures. This problem, formulated as a multiple testing problem, is studied from the nonasymptotic minimax perspective. The question of minimax adaptation with respect to the change height and location is broached and we identify two regimes in the minimax family-wise separation rate. A correspondence between minimax multiple testing procedures and confidence intervals for the jump localisation is also established. The minimax or minimax adaptive tests and multiple tests are based on simple linear counting statistics derived from Neyman-Pearson tests or quadratic statistics, possibly combined with aggregation strategies.