A toy model for the Lenard–Balescu problem

with Mitia Duerinckx

Corentin Le Bihan

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Particle system

- Particles follows the Newton's laws

$$\frac{\mathrm{d}}{\mathrm{d}t}x_i = v_i, \ \frac{\mathrm{d}}{\mathrm{d}t}v_i = -\frac{1}{N}\sum_{j\neq i}^{N}\nabla\mathcal{V}(x_i - x_j).$$

- Denote $F_N(t, x_{[1,N]}, v_{[1,N]})$ the distribution of particles. Then F_N follows the Liouville equation:

$$\partial_t F_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} F_N - \frac{1}{N}\sum_{i\neq j}\nabla\mathcal{V}(x_i - x_j)\nabla_{v_j} F_N = 0.$$

Radial interaction potential: $\mathcal{V}(x) = \mathcal{V}(|x|)$, $\mathcal{V} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Description on short time scales $t = O(1)$

We make the chaos assumption at $t = 0$: for $f_0 \in (L^1 \cap \mathcal{P})\big((\mathbb{R}/L\mathbb{Z})^d \times \mathbb{R}^d\big)$,

$$F_N(t = 0, x_{[1,N]}, v_{[1,N]}) = \prod_{i=1}^{N} f_0(x_i, v_i)$$

First marginal:

$$F_N^{(1)}(t, x_1, v_1) = \int F_N(t, x_{[1,N]}, v_{[1,N]}) \mathrm{d}x_{[2,N]} \mathrm{d}v_{[2,N]}.$$

Theorem (Debroshin 1979)

For any $t \in \mathbb{R}$, $F_N^{(1)}(t, x_1, v_1) \underset{N\to\infty}{\overset{*}{\rightharpoonup}} F(t, x_1, v_1)$ *in sens of measure, where* $F(t, x, v)$ *is solution Vlasov equation:*

$$\partial_t F + v_1 \cdot \nabla_{x_1} F - \nabla_x \big(\mathcal{V} *_x \int F \mathrm{d}v\big) \nabla_v F = 0, \tag{1}$$

The error terms are of order $O(\frac{1}{N} e^{Ct})$ for some $C > 0$.

Conservative equation, weak dissipation (Landau Damping).

Long time description of the system

Consider distribution of particles is invariant under translation:

$$F_N(t=0, x_{[1,N]}, v_{[1,N]}) = \prod_{i=1}^{N} f_0(v_i)$$

Conjecture

For all $\tau > 0$, $F_N^{(1)}(N\tau, v_1) \underset{N\to\infty}{\to} F^{(1)}(\tau, v_1)$ where $F^{(1)}(\tau)$ is solution of the Lenard–Balescu equation:

$$\partial_\tau F^{(1)}(\tau, v_1) = \nabla_{v_1} \cdot \int \frac{k \otimes k \delta_{k\cdot(v_1-v_2)}}{|\varepsilon(F, v_1, k)|^2} (\nabla_{v_1} - \nabla_{v_2}) \left(F^{(1)}(v_1) F^{(1)}(v_2) \right) \mathrm{d}v_2$$

$$\varepsilon(F, v_1, k) = 1 + \hat{\mathcal{V}}(k) \int \frac{k \cdot \nabla F(v_*)}{k \cdot (v_* - v) + i0} \mathrm{d}v_*.$$

This equation look like the Fokker-Planck-Landau equation:

- there is a H-theorem: $\frac{d}{d\tau} \int F(t, v_1) \log F(t, v_1) \mathrm{d}v_1 \leq 0$,
- the Gaussian are stationary solutions: $M(v) := \frac{1}{(2\pi)^{d/2}} \exp(-\frac{|v|^2}{2})$.

Heuristic of the derivation

Cumulant decomposition: (writing $z = (x, v)$)

$$F_N^{(2)}(z_1, z_2) = F_N^{(1)}(z_1) F_N^{(1)}(z_2) + \tfrac{1}{N} G_N^{(2)}(z_1, z_2)$$

$$\forall n \in [1, N], \quad F_N^{(n)}(z_{[1,n]}) = \sum_{r=1}^{n} \sum_{\substack{(\rho_1, \cdots, \rho_r) \\ \in \mathcal{P}^r([1,n])}} \prod_{i=1}^{r} \frac{1}{N^{|\rho_i|-1}} G_N^{(|\rho_i|)}(z_{\rho_i})$$

Formally, the G_N^r are of order C_r (independent from N).

Evolution equation: (we set $\tau = t/N$)

$$\tfrac{1}{N}\partial_\tau F_N^{(1)} = \tfrac{1}{N} C_{1,2} G_N^{(2)}, \quad \tfrac{1}{N}\partial_\tau G_N^{(2)} + \underbrace{iT}_{\simeq \sum_i v_i \cdot \nabla_{x_i}} G_N^{(2)} = C_{2,1} F_N^{(1)} \otimes F_N^{(1)} + O(\tfrac{1}{N^2}),$$

We deduce that

$$G_N^{(2)}(\tau) \simeq N \int_0^\tau e^{iN(\tau'-\tau)T} C_{2,1} (F_N^{(1)}(\tau'))^{\otimes 2} \mathrm{d}\tau' \simeq \tfrac{1}{iT+0} C_{2,1} (F_N^{(1)}(\tau))^{\otimes 2}$$

Finally:

$$\partial_\tau F_N^{(1)} \simeq C_{1,2} \tfrac{1}{iT+0} C_{2,1} (F_N^{(1)}(\tau))^{\otimes 2}.$$

Bibliography

Validity times reached

- Non-linear case: $N\tau = O((\log N)^{1-\delta})$, $\delta > 0$ (Duerinckx 19),
- Linear case (interacting background) : $N\tau = O(N^{\delta})$, $\delta < \frac{1}{18}$ (Duerinckx–Saint-Raymond 19),
- Linear case (non-interacting background) : $N\tau = O(N)$ (Bodineau–Le Bris 26)

Cauchy problem for Lenard–Balescu equation

- Linearized equation with Coulomb potential: (Merchant–Liboff 73)
- Non linear equation: (Duerinckx–Winter 23)

Quantum setting

- Non-physical (Boltzmann) statistic: (LB 23)

The linear version of the problem

We tagged by one particle 0 in an equilibrium bath: the initial data is

$$F_N(t=0, x_{[0,N]}, v_{[0,N]}) = f_0(v_0) \prod_{i=0}^{N} M(v_i).$$

Linear cumulants: We construct $(G_N^{(n)})_{n=0,\cdots,N}$ such that $\forall n \leq N$

$$F_N^{(n)}(x_{[0,N]}, v_{[0,N]}) = M^{n+1}(v_{[0,N]}) \sum_{k=0}^{N} \sum_{\substack{\sigma \subset [0,n] \\ |\sigma|=k+1 \\ 0 \in \sigma}} G_N^{(k)}(x_\sigma, v_\sigma).$$

Orthogonality properties:

as $\int F_N^{(n)}(x_{[0,n]}, v_{[0,n]}) \mathrm{d}x_n \mathrm{d}v_n = F_N^{(n-1)}(x_{[0,n-1]}, v_{[0,n-1]})$,

$$\forall\ 0 < n \leq N, \int G_N^{(n)}(x_{[0,n]}, v_{[0,n]}) M(v_n) \mathrm{d}x_n \mathrm{d}v_n = 0$$

Bound on the cumulant

L^2 structure:

$$\int \frac{|F_N|^2}{M^{N+1}} \mathrm{d}x_{[0,N]} \mathrm{d}v_{[0,N]} = \sum_{n=0}^{N} \binom{N}{n} \int \left| G_N^{(n)} \right|^2 M^{n+1} \mathrm{d}x_{[0,n]} \mathrm{d}v_{[0,n]}.$$

Conservation of norm by Liouville equation:

$$\int \frac{|F_N(t)|^2}{M^{N+1}} \mathrm{d}x_{[0,N]} \mathrm{d}v_{[0,N]} \lesssim \left(\int \frac{|F_N|^{2q}}{M_{N+1}} \mathrm{d}x_{[0,N]} \mathrm{d}v_{[0,N]} \right)^{\frac{1}{q}} = \text{constant}.$$

where $M_{N+1} = \frac{1}{\mathcal{Z}_N} \exp\left(-\frac{|v_{[0,N]}|^2}{2} - \frac{1}{N} \sum_{i \neq j} \mathcal{V}(x_i - x_j) \right)$, the Gibbs measure.

We deduce the bound

$$\forall t > 0,\ n \leq N,\ \left(\int \left| G_N^{(n)} \right|^2 M^{n+1} \mathrm{d}x_{[0,n]} \mathrm{d}v_{[0,n]} \right)^{\frac{1}{2}} \lesssim \frac{\sqrt{n!}}{N^{\frac{n}{2}}}. \qquad (2)$$

Evolution of cumulant:

The cumulant are solution of

$$\frac{1}{N}\partial_\tau G_N^n + i \underbrace{L_n}_{=\sum_j v_j \nabla_{x_j} + \text{compact}} G_N^n = S^{n,+} G_N^{n+1} + \frac{1}{N}\left(S^{n,0} G_N^{n+1} + S^{n,-} G_N^{n-1} + S^{n,=} G_N^{n-2}\right) \quad (3)$$

It is a variation of the BBGKY on the marginal $F_N^{(n)}$.

Each $S^{n\pm}$ corresponds to a derivation in v.

Simplification of the linear formulation

1. we add some diffusion in v at order $\frac{\kappa}{N}$ for some $\kappa > 0$,
2. we perform the large box limit: $(\mathbb{R}/L\mathbb{Z})^d \underset{L\to\infty}{\to} \mathbb{R}^d$,
3. we simplify the hierarchy (remove the non tridiagonal terms),
4. we take $S^{n,+}$ and $S^{n+1,-}$ anti-selfadjoint,
5. we truncate brutally: $\forall n > m_0,\ G_N^n = 0$.

$$\frac{1}{N}\partial_\tau G_N^n + \Big(i \underbrace{L_n}_{=\sum_j v_j \nabla_{x_j} + \text{compact}} - \frac{\kappa}{N} D_n\Big) G_N^n = S^{n,+} G_N^{n+1}$$
$$+ \frac{1}{N}\left(S^{n,0} G_N^{n+1} + S^{n,-} G_N^{n-1} + S^{n,=} G_N^{n-2}\right)$$

Normalization of the cumulants :

$$g_N^n(k_{[0,N]}, v_{[0,N]}) = \sqrt{\frac{N^n}{M^{n+1}}}\,\hat{G}_N^n(k_{[0,N]}, v_{[0,N]}).$$

The studied hierarchy

We want to describe $(g_N^n(t))_{n\in[0,m_0]}$, with

$$g_N^n(t) : \{(v_{[0,n]}, k_{[0,n]} \in \mathbb{R}^{2(n+1)d}, \sum_i k_i = 0\} \to \mathbb{C},$$

evolving with respect to

$$\frac{1}{N}\partial_\tau g_N^n + \left(i\sum_{j=0}^{n} k_j\cdot v_j - \frac{\kappa}{N}\Delta_n\right) g_N^n = \frac{1}{\sqrt{N}}\left(S^{n+} g_N^{n+1} + S^{n-} g_N^{n-1}\right),$$

$$S^{n+} = \sum_{j=0}^{n} S^{n+}_{j,n+1}, \quad S^{n-} = \sum_{j=0}^{n}\sum_{\substack{\ell=1\\ \ell\neq j}}^{n} S^{n-}_{j,\ell},$$

$$S^{n+}_{j,n+1} g_N^{n+1} = -i\int \mathrm{d}k_{n+1}\mathrm{d}v_{n+1}\sqrt{M(v_{n+1})}\hat{\mathcal{V}}(k_{n+1})$$
$$\times\, k_{n+1}\cdot\nabla_{v_j} g_N^{n+1}(\cdots, \underbrace{k_j - k_{n+1}, v_j}_{j}, \cdots, \underbrace{k_{n+1}, v_{n+1}}_{n+1}),$$

$$S^{n-}_{j,\ell} g_N^{n-1} = i\sqrt{M(v_\ell)}\hat{\mathcal{V}}(k_\ell) k_\ell\cdot\nabla_{v_j} g_N^{n+1}(\cdots, \underbrace{k_j + k_\ell, v_j}_{j}, \cdots, \underset{\ell}{\emptyset}, \cdots).$$

for some initial data $g_N^n(t=0) = \mathbb{1}_{n=0}\mathfrak{g}$,

L^2 control of the solution

We choose the norms

$$\|g_N^n\|_n^2 = \frac{1}{n!}\int |g_N^n(k_{[n]}, v_{[n]})|^2 \delta_{\sum_j k_j} \mathrm{d}k_{[n]} \mathrm{d}v_{[n]},$$

$$|\!|\!|(g_N^n)_n|\!|\!|^2 = \sum_{n=0}^{m_0} \|g_N^n(t)\|_n^2.$$

We have the equality

$$\left\langle S^{n,+} g^{n+1}, h^n \right\rangle_n = -\left\langle g^{n+1}, S^{n+1,-} h^n \right\rangle_{n+1}.$$

We deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} |\!|\!|(g_N^n(t))_n|\!|\!|^2 = -2\sum_{n=0}^{m_0} \left(-\tfrac{\kappa}{N}\|\nabla g_N^n(t)\|_n^2\right) \leq 0$$

Theorem (Duerinckx–LB. 25)

There exists $C_0 > 0$ such that for any $m_0 > 0$, $\kappa > 0$, assuming that $|\hat{\mathcal{V}}(k)| \leq C_0 \kappa$, $d > C_0 m_0$, $\mathfrak{g} \in H^{C_0 m_0}$, one has

$$g_N^0(\tau, v_0) \underset{N\to\infty}{\to} g^0(\tau, v_0) \text{ in } L^2(e^{-t}\mathrm{d}t\mathrm{d}v_0),$$

where g^0 is solution of

$$\partial_t g^0(\tau, v_0) = (\kappa\Delta + \nabla_{v_0} \cdot A(v_0)\nabla_{v_0})\, g^0(\tau, v_0),$$
$$g^0(t = 0)$$
$$A(v_0) = \pi \int_{(\mathbb{R}^d)^2} (k \otimes k)\hat{\mathcal{V}}(k)^2 M(v_1)\, \delta(k \cdot (v_1 - v_0))\mathrm{d}k\mathrm{d}v_1.$$

Strategy of the proof

We perform a Laplace transform in t:

$$\mathcal{L}g(\alpha) = \int_0^\infty e^{-(1+i\alpha)t} g(t)\mathrm{d}t.$$

$$\frac{1 + i\alpha - \kappa\Delta_n}{N}\mathcal{L}g_N^n + i\sum_{j=0}^{n} k_j \cdot v_j \mathcal{L}g_N^n = \tfrac{1}{\sqrt{N}}\left(S^{n+}\mathcal{L}g_N^{n+1} + S^{n-}\mathcal{L}g_N^{n-1}\right),$$

$$S^{n+} = \sum_{j=0}^{n} S_{j,n+1}^{n+}, \quad S^{n-} = \sum_{j=0}^{n}\sum_{\substack{\ell=1\\ \ell\neq j}}^{n} S_{j,\ell}^{n-},$$

$$S_{j,n+1}^{n+} g_N^{n+1} = -i\int \mathrm{d}k_{n+1}\mathrm{d}v_{n+1}\sqrt{M(v_{n+1})}\hat{\mathcal{V}}(k_{n+1}) \times k_{n+1}\cdot\nabla_{v_j} g_N^{n+1}(\cdots, \underbrace{k_j - k_{n+1}, v_j}_{j}, \cdots, \underbrace{k_{n+1}, v_{n+1}}_{n+1}),$$

$$S_{j,\ell}^{n-} g_N^{n-1} = i\sqrt{M(v_\ell)}\hat{\mathcal{V}}(k_\ell)k_\ell\cdot\nabla_{v_j} g_N^{n+1}(\cdots, \underbrace{k_j + k_\ell, v_j}_{j}, \cdots, \underset{\ell}{\emptyset}, \cdots).$$