Linear Landau equation from a mean field particle system

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Combinatorics from singular SPDEs to kinetic equations
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Joint work with: Thierry Bodineau *(IHES)*

I. Introduction

Mean-field

Particle system: For $i \in \{1, ..., N\}$

$$\frac{d}{dt} X_t^{i,N} = V_t^{i,N}, \quad \frac{d}{dt} V_t^{i,N} = \frac{1}{N} \sum_{j=1}^{N} K(X_t^{i,N} - X_t^{j,N}).$$

Question: Limit $N \to \infty$?

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$$\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{i,N}, V_t^{i,N})}.$$

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$$LLN: \quad \frac{1}{N}\sum_{j=1}^N K(x - X_t^{j,N}) \xrightarrow[N\to\infty]{a.s.} \mathbb{E}\left[K(x - \cdot)\right]$$

where the expectation is taken over a "typical particle" of the system

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$$\frac{d}{dt} \bar{X}_t = \bar{V}_t, \quad \frac{d}{dt} \bar{V}_t = K *_x \bar{\rho}_t(\bar{X}_t), \quad \bar{\rho}_t = \mathrm{Law}(\bar{X}_t, \bar{V}_t).$$

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$$\longleftrightarrow \quad \boxed{\partial_t \rho_t^N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} \rho_t^N = -\frac{1}{N} \sum_{i,j} K(x_i - x_j) \cdot \nabla_{v_i} \rho_t^N.}$$

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$$\longleftrightarrow \quad \boxed{\partial_t \bar{\rho}_t + v \cdot \nabla_x \bar{\rho}_t = -(K * \bar{\rho}_t) \cdot \nabla_v \bar{\rho}_t}$$

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Assume K **is of mean 0** (*ex:* $K = \nabla\Phi$ with Φ radially symmetric), and the **distribution of particles is uniform** (*ex:* uniform on the torus, or Poisson Point Process),

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$\implies$ The better approximation comes not when neglecting correlations *(order 0)*, but when keeping only the 2 particle correlations *(order 1)*.

Why do we speak of order?

BBGKY hierarchy: For $k \leq N$,

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$$\partial_t\rho_t^N + \sum_{i=1}^N v_i\cdot\nabla_{x_i}\rho_t^N = -\frac{1}{N}\sum_{i,j}K(x_i-x_j)\cdot\nabla_{v_i}\rho_t^N,$$

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$\Longrightarrow$ Linked to controls on correlations, or Higher order propagation of chaos.

Time-rescaling

Denoting $f_t = \text{Law}(v_t^1)$.

Goal: Finding the limit of $((f_{\tau N})_{\tau\in[0,1]})_N$.

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Convergence to Lenard-Balescu notoriously difficult $\implies$ Simplification.

A Rayleigh gas

Consider:

- $(x_i)_{i\in\mathbb{N}}$ a homogeneous Poisson Point Process in $\mathbb{R}^d$ of intensity $N \in \mathbb{N}$ *(Not exactly the final framework...)*,
- $(v_i)_{i\in\mathbb{N}}$ independent Gaussian random variables $\mathcal{N}(0, I_d)$ *(Denote* γ*=Gaussian density)*.

$\Longrightarrow$ **Background particles**.

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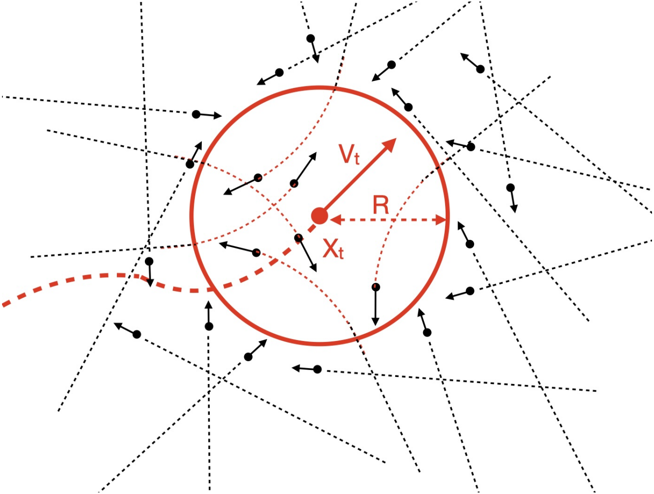
Introduce a tagged particle (X_t, V_t)

$$\frac{d}{dt}X_t = V_t, \qquad \frac{d}{dt}V_t = -\frac{1}{N}\sum_{i\in\mathbb{N}} \nabla\Phi(X_t - x_t^i),$$

for $\Phi \in \mathcal{C}_c^\infty$, and

$$x_0^i = x_i, \quad v_0^i = v_i, \quad \frac{d}{dt}x_t^i = v_t^i, \quad \text{and} \quad \frac{d}{dt}v_t^i = \frac{1}{N}\nabla\Phi(X_t - x_t^i).$$

A picture = a thousand words



Goal

The goal is to understand how $\left((V_{\tau N})_{\tau\in[0,1]}\right)_N$ behaves when the density N of background particles increases to infinity.

False

What happens?

Dynamics from the fluctuations. Define the fluctuations

$$\mu_t^N(h) := \frac{1}{N}\sum_{i=1}^{N} h(x_t^i, v_t^i), \quad \xi_t^N(h) := \sqrt{N}\left(\mu_t^N(h) - \int h(x,v)dx\gamma(dv)\right).$$

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Therefore

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By the Central limit theorem

$$\xi_t^N(\nabla\Phi(X_t-\cdot)) \xrightarrow[N\to\infty]{law} \mathcal{N}(0,\Sigma) \implies \frac{1}{N}\sum_{i\in\mathbb{N}}\nabla\Phi(X_t - x_t^i) \simeq \mathcal{N}\left(0, \frac{1}{N}\Sigma\right)$$

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$$\frac{1}{N}\sum_{i\in\mathbb{N}} \nabla\Phi(X_t - x_t^i) = \frac{1}{\sqrt{N}}\xi_t^N(\nabla\Phi(X_t - \cdot)).$$

By the Central limit theorem

$$\xi_t^N(\nabla\Phi(X_t-\cdot)) \xrightarrow[N\to\infty]{law} \mathcal{N}(0,\Sigma) \implies \frac{1}{N}\sum_{i\in\mathbb{N}}\nabla\Phi(X_t - x_t^i) \simeq \mathcal{N}\left(0, \frac{1}{N}\Sigma\right)$$

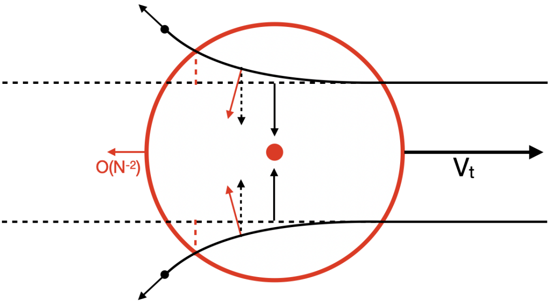
Sum of (independent) Gaussian increments:

$$V_{\tau N} - V_{\sigma N} = -\int_{\sigma N}^{\tau N}\frac{1}{N}\sum_{i\in\mathbb{N}}\nabla\Phi(X_s - x_s^i)ds \simeq -\sum_{s=\sigma N}^{\tau N-1}\frac{1}{N}\sum_{i\in\mathbb{N}}\nabla\Phi(X_s - x_s^i)$$
$$\simeq \mathcal{N}(0, (\tau-\sigma)\Sigma)$$

False

What happens?

Appearance of a friction.



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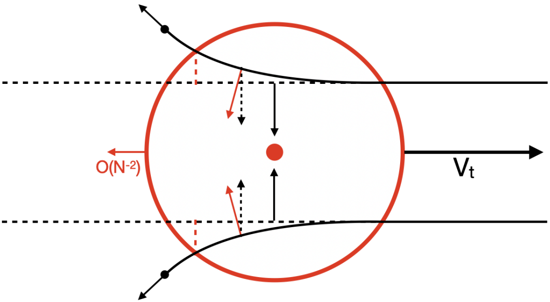

Friction $O(N^{-2})$ per background particle.
During time T, NT background particle met
$\implies$ Force $O(1)$ for $T \simeq N$.

What happens?

The rescaled process $(V_{\tau N})_{\tau \in [0,1]}$ should behave, in the limit $N \to \infty$, like a diffusion process $(\mathcal{V}_\tau)_{\tau \in [0,1]}$

$$d\mathcal{V}_\tau = 2\Lambda(\mathcal{V}_\tau)d\tau + \sqrt{2}\Sigma(\mathcal{V}_\tau)dB_\tau,$$

where $(B_\tau)_{\tau \in [0,1]}$ is a d dimensional Brownian motion, and Λ, Σ are functions to be identified.

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(γ=Gaussian) and construct

$$\mathbb{E}\left[F(Z, z_{1:\mathcal{N}})\right] = \frac{1}{\mathcal{Z}} \sum_{M=0}^{\infty} \frac{N^M}{M!} \int_{\mathbb{T}\times\mathbb{R}^d} dZ \int_{\mathbb{T}^M\times\mathbb{R}^{Md}} dz_{1:M}\, \rho_M\left(Z, z_{1:M}\right) F\left(Z, z_{1:M}\right).$$

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$\implies$ the system is stationary (equilibrium).

Main result

Theorem (Bodineau-P.L.B. ('26))

Provided $d \geq 4$, under the grand canonical ensemble, the family of processes $((V_{\tau N})_{\tau\in[0,1]})_N$ converges in law as $N \to \infty$ to a diffusion process $(\mathcal{V}_\tau)_{\tau\in[0,1]}$ defined by $\mathcal{V}_0 \sim \mathcal{N}(0, I_d)$ and*

$$d\mathcal{V}_\tau = 2\Lambda(\mathcal{V}_\tau)d\tau + \sqrt{2}\Sigma(\mathcal{V}_\tau)dB_\tau,$$

where B is a Brownian motion in $\mathbb{R}^d$. Here Λ is defined by

$$\Lambda(V) = -\int_{\mathcal{B}(0,R)} dx \int_{\mathbb{R}^d} dv\gamma(v+V)Hess\Phi(x)\int_{-\infty}^{0}\int_{-\infty}^{s}\nabla\Phi(x+uv)duds,$$

and we define Σ as the square root of D, an $\mathbb{R}^{d\times d}$ positive symmetric matrix given by

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* i.e for all continuous bounded functions $F : \mathcal{C}([0,1],\mathbb{R}^d) \mapsto \mathbb{R}$, we have

$$\mathbb{E}\left[F(V_{\cdot N})\right] \xrightarrow[N\to\infty]{} \mathbb{E}\left[F(\mathcal{V})\right].$$

Landau equation (linear)

Lemma

The Fokker-Planck equation associated with the SDE for the limit $\mathcal{V}$ is the linear Landau equation, i.e. $f_t = Law(\mathcal{V}_t)$ is a weak solution of

$$\partial_t f_t = \nabla \cdot (D(V)(\nabla f_t + V f_t)),$$

where

$$D(V) = \frac{\pi}{(2\pi)^d} \int_{\mathbb{R}^d} (k \otimes k) |\hat{\Phi}(k)|^2 \int_{\mathbb{R}^d} \gamma(v + V) \delta_{k \cdot v} dv dk.$$

A bit of literature

- On controls of correlation/higher order PoC

  Bogolyubov ('46), Duerinckx ('21), Rowan - Hess-Childs ('23), Duerinckx - Bernou ('24), Ren - Wang ('25)

- On Lenard-Balescu and Landau

  Guernsey, Lenard, Balesu ('60s), Duerinckx ('21), Duerinckx - Saint-Raymond ('21), Nota - Simonella - Velázquez ('18), Nota - Velázquez - Winter ('21, '22), Duerinckx - Le Bihan ('25).

- On tagged particles

  Holley ('71), Kesten - Papanicolaou ('79, '80), Dürr - Goldstein - Lebowitz ('81, '83, '87), Spohn ('91) Kusuoka - Liang ('10), Dobson - Legoll - Lelièvre - Stoltz ('13), Bodineau - Gallagher - Saint-Raymond ('16).

II. Sketch of proof

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NB We distinguish between the microscopic time $t \in [0, N]$ and the macroscopic time $\tau \in [0, 1]$: $t = \tau N$.

Martingale problem

Itô's formula applied to the SDE $d\mathcal{V}_\tau = 2\Lambda(\mathcal{V}_\tau)d\tau + \sqrt{2}\Sigma(\mathcal{V}_\tau)dB_\tau$, states in particular that for any test function f the quantity

$$\mathcal{M}^f_\tau := f(\mathcal{V}_\tau) - \int_0^\tau \left(2\Lambda(\mathcal{V}_\sigma)\cdot\nabla f(\mathcal{V}_\sigma) + D(\mathcal{V}_\sigma) : \mathrm{Hess} f(\mathcal{V}_\sigma)\right) d\sigma,$$

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$$\mathbb{E}\left[\prod_{i=1}^n g_i(\mathcal{V}_{\tau_i})\left(f(\mathcal{V}_{\tau_{n+1}}) - f(\mathcal{V}_{\tau_n}) - \int_{\tau_n}^{\tau_{n+1}} \mathcal{L}f(\mathcal{V}_\sigma)d\sigma\right)\right] = 0.$$

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We have

$$\begin{aligned}\mathbb{E}\left[g(V_0)(f(V_{\tau N}) - f(V_0))\right] &= -\,\mathbb{E}\left[g(V_0)\int_0^{\tau N} \nabla f(V_t)\frac{1}{N}\sum_i \nabla\Phi(X_t - x_t^i)dt\right] \\ &= -\,\hat{\mathbb{E}}\left[g(V_0)\int_0^{\tau N} \nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right],\end{aligned}$$

where $\hat{\mathbb{E}}[\cdot] = \mathbb{E}[\frac{\mathcal{N}}{N}\cdot]$.

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$$\frac{d}{dt}v_t^1 = \frac{1}{N}\nabla\Phi(X_t - x_t^1) = O\left(\frac{1}{N}\right),$$

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$$\begin{aligned}\nabla f(V_t)=&\nabla f(\overline{V}_t)+\mathrm{Hess}f(\overline{V}_t)(V_t-\overline{V}_t)+O\left(\frac{1}{N^2}\right),\\ =&\nabla f(\overline{V}_t)-\mathrm{Hess}f(\overline{V}_t)\int_0^t\frac{1}{N}\nabla\Phi(\overline{X}_s-x_0^1-sv_0^1)ds+O\left(\frac{1}{N^2}\right).\end{aligned}$$

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$$\begin{aligned}\nabla f(V_t) =& \nabla f(\overline{V}_t) + \text{Hess} f(\overline{V}_t)(V_t - \overline{V}_t) + O\left(\frac{1}{N^2}\right), \\ =& \nabla f(\overline{V}_t) - \text{Hess} f(\overline{V}_t)\int_0^t \frac{1}{N}\nabla\Phi(\overline{X}_s - x_0^1 - sv_0^1)ds + O\left(\frac{1}{N^2}\right).\end{aligned}$$

Likewise

$$\begin{aligned}\nabla\Phi(X_t - x_t^1) =& \nabla\Phi\left(\overline{X}_t - x_0^1 - tv_0^1\right) \\ & + \text{Hess}\Phi\left(\overline{X}_t - x_0^1 - tv_0^1\right)\left((X_t - \overline{X}_t) - (x_t^1 - x_0^1 - tv_0^1)\right) + O\left(\frac{1}{N^2}\right) \\ =& \nabla\Phi\left(\overline{X}_t - x_0^1 - tv_0^1\right) \\ & - 2\text{Hess}\Phi\left(\overline{X}_t - x_0^1 - tv_0^1\right)\int_0^t\int_0^s \frac{1}{N}\nabla\Phi(\overline{X}_u - x_0^1 - uv_0^1)duds + O\left(\frac{1}{N^2}\right).\end{aligned}$$

In the end

We obtain

$$
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\mathbb{E}\left[g(V_0)(f(V_{\tau N}) - f(V_0))\right] &= -\hat{\mathbb{E}}\left[g(V_0)\int_0^{\tau N} \nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right] \\
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First term disappears and with a bit of re-writing

$$\begin{aligned}
&\mathbb{E}\left[g(V_0)\left(f(V_{\tau N})-f(V_0)-\int_0^{\tau}\left(2\nabla f(V_{\sigma N})\cdot\Lambda(V_{\sigma N})+\mathrm{Hess}f(V_{\sigma N}):D(V_{\sigma N})\right)d\sigma\right)\right] \\
&\quad=O\left(\frac{\tau}{N}\right),
\end{aligned}$$

with (γ=Gaussian)

$$\Lambda(V)=-\int_{\mathcal{B}(0,R)}dx\int_{\mathbb{R}^d}dv\gamma(v+V)\mathrm{Hess}\Phi(x)\int_{-\infty}^0\int_{-\infty}^s\nabla\Phi(x+uv)duds,$$

$$D(V)=\int_{\mathcal{B}(0,R)}dx\int_{\mathbb{R}^d}dv\gamma(v+V)\nabla\Phi(x)\otimes\int_{-\infty}^0\nabla\Phi(x+sv)ds.$$

Just a bit false

Tightness*

Using this estimate, for $f = g = id$ (which you can't do...)

$$\begin{aligned}\mathbb{E}[|V_{\tau N} - V_0|^2] &= -2\mathbb{E}[V_0 \cdot (V_{\tau N} - V_0)] \\ &= -\frac{2}{N}\mathbb{E}\left[V_0 \cdot \int_0^{\tau N} 2\Lambda(V_s)ds\right] + O\left(\frac{\tau}{N}\right) = O(\tau),\end{aligned}$$

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Compute fourth moment, using $f(v) := |v - V_0|^4$ (which you still can't do...)

$$\begin{aligned}\mathbb{E}[|V_{\tau N} - V_0|^4] =& \frac{1}{N}\mathbb{E}\left[\int_0^{\tau N} \nabla f(V_s - V_0) \cdot \Lambda(V_s) + \mathrm{Hess} f(V_s - V_0) : D(V_s)ds\right] + O\left(\frac{\tau}{N}\right) \\ \lesssim& \mathbb{E}\left[\int_0^{\tau} (|V_{sN} - V_0|^4 + |V_{sN} - V_0|^2)ds\right] + O\left(\frac{\tau}{N}\right),\end{aligned}$$

and, using $\mathbb{E}[|V_{\tau N} - V_0|^2] = O(\tau)$, we may conclude that $\mathbb{E}[|V_{\tau N} - V_0|^4] = O(\tau^2)$.

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Tightness, identification and uniqueness of the limit allow us to conclude on the convergence, and thus on the main result.

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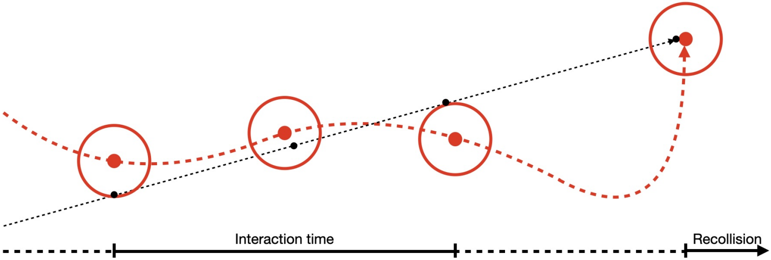
Recall that we wrote $X_t = \overline{X}_t + O\left(\frac{t^2}{N}\right)$

We therefore need to control the two main sources of correlations in time: particles interacting for a long time, and particles recolliding after having left the interaction radius.



Sources of correlations-2

When dealing with these sources of correlations, we want them to have a probability of occurring $o(1/N)$, so that we can for instance write

$$\begin{aligned}
&\hat{\mathbb{E}}\left[g(V_0)\int_0^{\tau N}\nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right] \\
&= \hat{\mathbb{E}}\left(g(V_0)\mathbb{1}_{x^1\text{doesn't recollide}}\int_0^{\tau N}\nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right) \\
&\quad + \hat{\mathbb{E}}\left(g(V_0)\mathbb{1}_{x^1\text{recollides}}\int_0^{\tau N}\nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right) \\
&= \hat{\mathbb{E}}\left[g(V_0)\mathbb{1}_{x^1\text{doesn't recollide}}\int_0^{\tau N}\nabla f(V_t)\nabla\Phi(X_t - x_t^1)dt\right] + o(1).
\end{aligned}$$

Still a bit false, but less

Stability

Concentration estimates:

$$\frac{1}{N}\sum_i \mathbb{1}_{x_t^i \in \mathcal{B}(X_t,R)} = O(1), \quad \left|\frac{1}{N}\sum_i \nabla\Phi(X_t - x_t^i)\right| = O\left(\frac{1}{\sqrt{N}}\right),$$

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We can thus a priori approximate X_t by $\overline{X}_t$ only on timescales smaller than $N^{1/4}$.

Controlling the interaction time

We have

$$\left|\frac{d}{dt}V_t\right| = \left|\frac{1}{N}\sum_i \nabla\Phi(X_t - x_t^i)\right| \leq O\left(\frac{1}{\sqrt{N}}\right).$$

We can therefore approximate the various motions by straight lines

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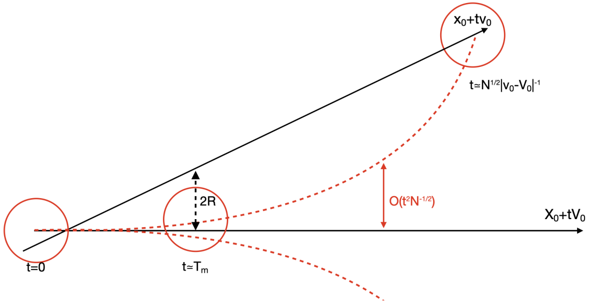

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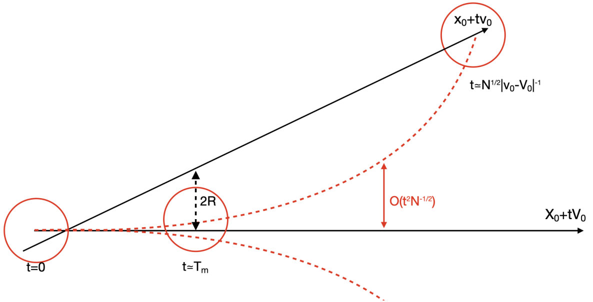

$$\boxed{\textit{If } |x_0^1 - X_0| \leq R, \textit{ then for } t \in \left[|v_0^1 - V_0|^{-1}, N^{1/2}|v_0^1 - V_0|\right], \textit{ we have } |x_t^1 - X_t| > R.}$$

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This approximation is only valid on the first interaction time frame! We thus need to deal with recollisions.

Not too false

Recollisions

A background particle, once it has exited the interaction radius, may only come back at times greater than $\hat{t}_{rec} = N^{1/2}|v_0^1 - V_0|$, with $|v_0^1 - V_0| \geq N^{-\frac{1}{4}-\delta}$.

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In other words, since it does not interact and thus goes in a straight line, we need

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This yields a probability of recollision smaller than N^{-1} for $d \geq 5$.

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$\implies$ Improve the controls.

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$\Longrightarrow$ Use also a time averaging.

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$\implies$ Use also a time averaging.

A perfect averaging in space and time would lead to

$$|V_T - V_0| = -\frac{1}{N}\int_0^T \sum_i \nabla\Phi(X_t - x_t^i)dt \simeq \sqrt{\frac{T}{N}}.$$

False and complicated

Hoeffding

Write, for some $\delta_v > 0$, $\delta_t = \delta_v^{-1}$ and $T = N^{1/2}\delta_v$

$$
\begin{aligned}
\frac{1}{N}\int_0^T \sum_i \nabla\Phi(X_t - x_t^i)dt &= \sum_{k=1}^{T/\delta_t} \frac{1}{N}\int_{t_k}^{t_{k+1}} \sum_i \nabla\Phi(X_t - x_t^i)dt \\
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&\quad + \frac{1}{N}\int_0^T \sum_i \nabla\Phi(X_t - x_t^i)\mathbb{1}_{|V_t - v_t^i| < \delta_v} dt.
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Fast particles: By Hoeffding

$$\left| \sum_{k=1, k \text{ odd}}^{T/\delta_t} \frac{1}{N}\int_{t_k}^{t_{k+1}} \sum_i \nabla\Phi(X_t - x_t^i)\mathbb{1}_{|V_t - v_t^i| \geq \delta_v} dt \right| \leq \sqrt{\frac{T}{N\delta_v}}.$$

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Slow particles: By a concentration inequality

$$\left| \frac{1}{\sqrt{N\delta_v^d}} \sum_i \nabla\Phi(X_t - x_t^i)\mathbb{1}_{|V_t - v_t^i|<\delta_v} \right| \leq 1$$

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Sum the controls and optimize for δ_v.

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Sum the controls and optimize for δ_v.

Let $\alpha = \frac{1}{4(d+2)}$ and $\beta = \frac{d+1}{2(d+2)}$, we have for $T \leq N^\beta$

$$|V_T - V_0| = \left|\frac{1}{N}\int_0^T \sum_i \nabla\Phi(X_t - x_t^i)dt\right| \leq \sqrt{\frac{T}{N}}N^\alpha.$$

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$\Longrightarrow$ You can also bootstrap this!

Conclusion

We may now conclude in dimensions $d \geq 4$.

III. Closing remarks

In dimensions 3

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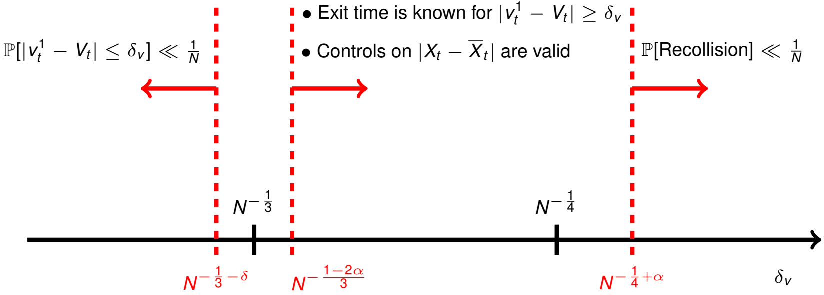

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Recall that, if $|V_t - V_0| \simeq \sqrt{\frac{t}{N}}$, we have

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For $d = 3$ and relatively slow particles, it is the tagged particle which exits, through fluctuations, the radius of influence of a background particle.

We can show that, indeed, $\sqrt{\frac{N}{t^3}}(X_t - X_0 - tV_0)$ behaves like a Gaussian for $t \geq N^{1/3}$, but we cannot yet exploit this fact...

Open questions

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- Interaction potentials that are not in $\mathcal{C}_c^\infty$.

Thank you!