Top of the spectrum of the Anderson Hamiltonian with correlated Gaussian noise

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Work in progress with Giuseppe Cannizzaro and Willem van Zuijlen

Nancy - December 6th 2024

The operator

$$\mathcal{H}_L f = \Delta f + f \cdot \xi \,, \qquad \text{on } Q_L := \mathbb{Z}^d \cap [-L, L]^d \,,$$

where $\xi : \mathbb{Z}^d \to \mathbb{R}$ is a random field.

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Main questions

As $L \to \infty$

1. Statistics of the largest eigenvalues?
2. What do the associated eigenfunctions look like?
3. Relationships between largest eigefunctions and maximas of the field?

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1. eigenvalues and eigenfunctions are explicit
2. eigenfunctions put mass everywhere (delocalised).

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If we turn off the Laplacian (i.e. $\mathcal{H}_L f = \xi \cdot f$) then for any $k \geq 1$:

1. $\lambda_{k,L} = \xi(y_{k,L})$
2. $\varphi_{k,L} = \delta_{y_{k,L}}$ (localised).

A motivation

Continuous Anderson Hamiltonian with white noise potential:

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Very precise information in $d = 1$.

Theorem (Dumaz-L.'18)

In $d = 1$, take $a_L = (\frac{3}{8} \ln L)^{2/3}$. Then:

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In $d = 1$, take $a_L = (\frac{3}{8} \ln L)^{2/3}$. Then:

1. $\Big(4\sqrt{a_L}(\lambda_{k,L} - a_L)\Big)_{k\geq 1}$ *CV in law as $L \to \infty$ to a P.P.P. on $\mathbb{R}$ of intensity $e^{-u} du$.*

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2. *For any $k \geq 1$:*

$$\Big(\frac{\sqrt{2}}{a_L^{1/4}} \varphi_{k,L}\big(x_{k,L} + \frac{x}{\sqrt{a_L}}\big), x \in \mathbb{R}\Big) \xrightarrow[\text{loc. unif.}]{(\mathbb{P})} \Big(\frac{1}{\cosh(x)}, x \in \mathbb{R}\Big) \ .$$

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3. *The centers of localisation are uniform:*

$$\left(\frac{x_{k,L}}{L}\right)_{k \geq 1} \Rightarrow \text{ i.i.d. Uniform}[-1, 1] \ .$$

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a.s. $\lambda_{1,L} \underset{L\to\infty}{\sim} a_L$

where $a_L := C_d(\log L)^{\frac{1}{2-\frac{d}{2}}}$ $\begin{cases} d = 2 \text{ Chouk-van Zuijlen'20} \\ d = 3 \text{ Hsu-L.'21} \end{cases}$

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where Q is the optimizer of Gagliardo-Nirenberg inequality
$\|f\|_{L^4(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/4} \|f\|_{L^2(\mathbb{R}^d)}^{1-d/4}$.

Literature on the i.i.d. case

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Two settings are well-understood:

1. *Weibull tail*: for some parameter $q > 0$

$$\mathbb{P}(\xi(0) > x) = \exp(-x^q) \ , \quad x \geq 0 \ .$$

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2. *Doubly-exponential tail*: for some parameter $\rho > 0$

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Theorem (Grenkova-Molchanov-Sudarev '90, Astrauskas '07, '08, Sidorova-Twarowski '14)

1. *There exist a_L, b_L s.t. $\left(\frac{\lambda_{k,L} - a_L}{b_L}\right)_{k \geq 1}$ CV in law as $L \to \infty$ to a P.P.P. on $\mathbb{R}$ of intensity $e^{-u}du$.*

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3. $(x_{k,L}/L)_{k \geq 1} \Rightarrow$ *i.i.d. Uniform*$[-1, 1]^d$.

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Question: relationship between $\varphi_{k,L}$ and the maximas of ξ on Q_L?

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Theorem (Astrauskas'12)

In the Weibull case: $\mathbb{P}(\xi(0) > x) = \exp(-x^q)\ , \quad x \geq 0$. *For any given* $k \geq 1$*:*

1. *if* $q < 3$, *then* $\ell_L(k) = k$ *w.l.p.*
2. *if* $q = 3$, *then* $\ell_L(k)$ *of order* 1 *w.l.p.*
3. *if* $q > 3$, *then* $\ell_L(k) \to +\infty$ *in probability.*

Not much explanation in Astrauskas'12 paper...

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2. *For any given $k \geq 1$, $\varphi_{k,L}$ puts a macroscopic mass at distance $O(1)$ from $x_{k,L}$.*
3. $(x_{k,L}/L)_{k \geq 1} \Rightarrow$ *i.i.d. Uniform*$[-1, 1]^d$.

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Our goal:

1. cover the counterpart of the "Weibull tail" i.i.d. case.
2. obtain a precise understanding of the relationship of the top of the spectrum with the maxima of the fields.

Our framework

The (sequence of) potential(s)

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   only depends on Euclidean norm $|x|$ of x.

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- correlation length:

$$c_L := \inf\{r \geq 1 : \forall r' \geq r, v_L(r') = 0\} \,.$$

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- decay parameter:

$$d_L \in [1, \infty) \text{ s.t. } v_L(1) = 1 - \frac{1}{d_L}\ .$$

Three examples:

1. i.i.d. $\mathcal{N}(0,1)$ r.v. Then $c_L = d_L = 1$.

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   Let η be a white noise on $\mathbb{R}^d$. Let u be a radial function supported in $B(0,1/2)$. Set
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   $$\zeta := \eta * u \, .$$
   For some sequence $c_L \to \infty$, set
   $$\xi_L(x) := \zeta(x/c_L) \, , \quad x \in \mathbb{Z}^d \, .$$
   Then $d_L \to +\infty$.

Analysis of ξ_L

To understand the top of the spectrum of

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one needs to understand the largest peaks of ξ_L on Q_L.

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where

1. $S_L(y) = a_L(1 - v_L(y))$ is a deterministic *shape*
2. ζ_{L,x_0} is a Gaussian field independent of $\xi_L(x_0)$.

Local eigenproblem

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Competition between two terms:

1. $\xi_L(x_0)$ which is of order a_L and fluctuates at scale $1/a_L$,
2. $\sum_{x \textbf{ close to } x_0} \bar{\varphi}_L(x - x_0)^2 \zeta_{L,x_0}(x)$ which fluctuates at scale τ_L where

$$\tau_L^2 := \mathrm{var}\left[\sum_{x \textbf{ close to } x_0} \bar{\varphi}_L(x - x_0)^2 \zeta_{L,x_0}(x)\right] .$$

Main results - Cannizzaro-L.-van Zuijlen (in progress)

Assume $d_L \ll a_L$.

Theorem (Eigenvalue order statistics)

The point process

$$\left(\frac{x_{k,L}}{L}, a_L(\lambda_{k,L} - a_L\sqrt{1+\tau_L^2} - \bar{\lambda}_L)\right)_{1 \le k \le \#Q_L},$$

CV in law as $L \to \infty$ *towards a P.P.P. on* $[-1,1]^d \times \mathbb{R}$ *of intensity* $dx \otimes e^{-u}du$.

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Theorem (Localisation)

For any $k \geq 1$*, the r.v.*

$$\frac{a_L}{d_L}\left\|\varphi_{k,L}(\cdot) - \bar{\varphi}_L(\cdot - x_{k,L})\right\|_{\ell^2(Q_L)},$$

converges to 0 *in probability.*

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Theorem (Relationship with the maxima of ξ_L)

1. *if* $\tau_L \ll \frac{1}{a_L}$ *then for any given* $k \geq 1$, $\mathbb{P}(\ell_L(k) = k) \to 1$ *as* $L \to \infty$,

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Let $u_1 > u_2 > \dots$ be distributed according to a P.P.P. of intensity $e^{-u}du$. Draw an independent sequence $(v_i)_{i\geq 1}$ of i.i.d. $\mathcal{N}(0,1)$ r.v. Let $(w_i)_{i\geq 1}$ be the order statistics of $(u_i + bv_i)_{i\geq 1}$. Then for any $k \geq 1$, $\ell_{\infty,b}(k)$ is defined through $w_k = u_{\ell_{\infty,b}(k)}$.

Thank you for your attention!