The multiplicative stochastic heat equation with Lévy stable noise

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Combinatorics from Singular SPDES to kinetic equations, IECL-Nancy

The mSHE

There are very results about SPDEs driven by non-Gaussian noises. We study the multiplicative stochastic heat equation (mSHE)

$$\partial_t u(t,x) = \Delta u(t,x) + \kappa u(t,x)\zeta(t,x), \quad t \geq 0, x \in \mathbb{R}^d, \kappa \geq 0\,,$$

where ζ is a γ-stable space-time white noise with $\gamma \in (1,2]$ (when $\gamma = 2$, ζ is the Gaussian white noise).

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Goal

Find for which (γ, d) the equation is subcritical, and solve the equation in this case. Study the "universality class" of the equation: discrete systems which converge in a scaling limit to mSHE.

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- The equation has an "explicit solution" given by the Feyman-Kac representation

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  This basically implies that there is also no analysis involved.
- But it is singular: the product $u(t,x)\zeta(t,x)$ may be ill-defined for functions u which typically solve the equation.

The litterature

The study of mSHE and its universality class has been the object of a series of works:

Berger-Lacoin ('21), **Berger-Lacoin ('22)**, **Berger-Chong-Lacoin ('23)**.

My contribution in (G'26) is to provide a different approach to these results, more flexible and based on moments, with applications to

- the multiplicative fractional stochastic heat equation with $\alpha \in (0, 2]$

$$\partial_t u(t,x) + (-\Delta)^{\alpha/2} u(t,x) = \kappa u(t,x)\zeta(t,x), \quad t \geq 0, x \in \mathbb{R}^d, \kappa \geq 0, .$$

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- the multiplicative stochastic Volterra equation with $\alpha \in (0,1)$

$$u(t) = 1 + \kappa \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} \zeta(s) \mathrm{d}s, \quad t \geq 0, \kappa \geq 0\,,$$

where ζ is a γ-stable $1d$ white noise.

What is a stable noise ?

For simplicity, let us consider a $1d$ (without space) white noise $\zeta(t)$.

$$\mathbb{E}[e^{i\langle\zeta,f\rangle}] = \exp\Big[- c_\gamma \int_{\mathbb{R}} |f(t)|^\gamma (1 - id_\gamma \mathrm{sgn}(f(t)))\mathrm{d}t\Big].$$

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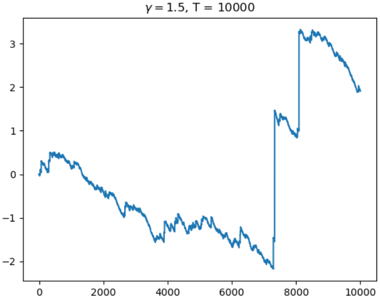
Note that ζ is not symmetric, even if $\mathbb{E}[\langle \zeta, f\rangle] = 0$. Two important properties of stable noises:

1. (Scale invariance) for every $\lambda > 0$, $(\lambda^{(\gamma-1)/\gamma}\zeta(\lambda t))_{t\geq 0} \underset{(d)}{=} (\lambda\zeta(t))_{t\geq 0}$.
2. (Moments) For every smooth function f,

$$\mathbb{E}[|\langle \zeta, f\rangle|^p] < +\infty \iff p < \gamma\,.$$

Lévy processes

Define $L_t = \int_0^t \zeta(\mathrm{d}s)$: this process has stationnary and independent increments, and is not continuous.



How does a stable noise emerge ?

Generalisation of the TCL: take $(\omega_n)_{n\geq 1}$ i.i.d. centered random variables which have a heavy tail:

$$\mathbb{P}[\omega_n \geq x] \sim x^{-\gamma}, \quad \mathbb{P}[\omega_n \leq -x] = o(x^{-\gamma}), \quad \text{as } x \to +\infty.$$

Then,

$$\frac{1}{N^{1/\gamma}} \sum_{n=1}^{N} \omega_n \xrightarrow{N\to+\infty} \int_0^1 \zeta(\mathrm{d}t)\,.$$

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Even in a Gaussian setting: take $(X_i)_{i\geq 1}$ i.i.d. standard Gaussians and define

$$Z_N(\beta) = \sum_{n=1}^{2^N} e^{-\beta\sqrt{N/2}X_i}\,,$$

then for $\beta \in (\frac{\sqrt{\log 2}}{2}, \sqrt{\log 2})$, $Z_N(\beta)/\mathbb{E}[Z_N(\beta)]$ has non-gaussian fluctuations.

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If $\zeta \in B_p^{\tau}$, then (optimistically) $u\zeta \in B_p^{\tau}$ and by Schauder estimates, $u \in B_p^{\tau+2}$. The equation is subcritical if $2 + \tau > 0$, this yields

$$\gamma < \gamma_c := 1 + \frac{2}{d}.$$

The Directed Polymer in a random environment

Consider a simple symmetric random walk $(S_n)_{n\geq 0}$ on $\mathbb{Z}^d$. Denote P_y the law of the random walk started at $S_0 = y$.

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The polymer measure is defined for every realisation of ω by

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where $Z_{N,\beta}(y) = \mathrm{E}_y[\prod_{n=1}^{N}(1+\beta\omega_{n,S_n})]$ is the partition function.
$(Z_{N,\beta}(y))_{N\geq 0, y\in\mathbb{Z}^d}$ solves a discrete version of mSHE, and a very natural result to prove would be that the process

$$u^{(N)}(t,x) := Z_{Nt,\beta}(\sqrt{N}x),$$

converges as $N \to +\infty$ to u.

The Gaussian case $d = 1$ and $d \geq 3$

Theorem (Alberts-Khanin-Quastel '14; Caravenna-Sun-Zygouras '16)

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The equation mSHE has a unique solution. Furthermore, in the Directed Polymer Model, choosing $\beta_N = \kappa N^{-1/4}$, the process $u^{(N)}$ converges in law as $N \to +\infty$ to the solution of mSHE with initial condition $u(t = 0, x) = 1$.

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When $d \geq 3$ the equation is supercritical, and what is actually relevant is to study large scales fluctuations: **Cosco, Nakajima, Nakashima ('22)**, **Junk ('25, '25)**, **Junk, Nakajima ('25)** A remark: there are regimes where Lévy stable noises emerge !

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In dimension $d = 2$, the mSHE does not admit a classical solution.

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Théorème (CSZ'23)

But, if in the Directed Polymer model, $\beta_N = \sqrt{\frac{\pi}{\log N}}$, then the process $u^{(N)}$ viewed as measure converges in law as $N \to +\infty$ to a non-trivial object, called the Stochastic Heat Flow.

Polynomial chaos expansion

Duhamel's formulation of mSHE (for simplicity $u(0,x)=1$):

$$u(t,x) = 1 + \kappa \int_0^t \int_{\mathbb{R}^d} \rho(t-s, x-y) u(s,y) \zeta(\mathrm{d}s, \mathrm{d}y)\,,$$

where $\rho(t,x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$.

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where $\rho(t,x) = (2\pi t)^{-d/2} e^{-|x|^2/2t}$. By iteration, at least formally,

$$u(t,x) = 1 + \sum_{k\geq 1} \kappa^k \int_{0<t_1<\cdots<t_k<t} \int_{\mathbb{R}^{dk}} \rho(t_2 - t_1, x_2 - x_1) \cdots$$

$$\cdots \rho(t_k - t_{k-1}, x_k - x_{k-1}) \rho(t - t_k, x - x_k) \prod_{j=1}^k \zeta(\mathrm{d}t_j, \mathrm{d}x_j)$$

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Theorem (Berger,Chong,Lacoin '23; G '26)

When $\gamma < 1 + 2/d$, for every $\kappa \geq 0$, the mSHE with γ-stable noise has a unique solution u, given by the polynomial chaos expansion. Furthermore, for every $p < \gamma$, $\mathbb{E}[|u(t,x)|^p] < +\infty$.

About moments

Consider a function f defined on $[0,T]^k_\leq = \{0 < t_1 < \cdots < t_k < T\}$. How to bound the p-th moment of

$$X_T = \int_{[0,T]^k_\leq} f(t_1, \cdots, t_k) \prod_{j=1}^k \zeta(\mathrm{d}t_j) \quad ?$$

Here X_T is interpreted as

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$$\mathbb{E}[(X_T)^2] = \int_{[0,T]^k_{\leq}} f(t_1,\cdots,t_k)^2 \prod_{j=1}^{k} \mathrm{d}t_j = \|f\|^2_{L^2} \ .$$

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It is a classical result that, when $k = 1$, for every $p < \gamma$

$$\mathbb{E}[|X_T|^p] \leq C_{\gamma,p} \left(\int_0^T |f(t)|^\gamma \mathrm{d}t \right)^{p/\gamma} = C_{\gamma,p} \|f\|^p_{L^\gamma} \, .$$

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Theorem (G'26)

For every $p < \gamma < 2$ and $T > 0$, there exist a constant $C = C_{p,\gamma,T}$ such that for every $q \in (\gamma, 2)$, and every deterministic function $f : [0, T]^k_{\leq} \to \mathbb{R}$,

$$\mathbb{E}\left[\left| \int_{[0,T]^k_{\leq}} f(t_1, \cdots, t_k) \prod_{j=1}^k \zeta(\mathrm{d}t_j) \right|^p \right] \leq \frac{C^k}{(q-\gamma)^{k/q}} \|f\|^p_{L^q} .$$

Applications to mSHE.

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Theorem (G'26)

Furthermore,

- *if* $\lim_{N \to +\infty} \beta_N N^{(\gamma_c - \gamma)d/2\gamma} = 0$*, then* $Z_{Nt,\beta_N}(x\sqrt{N})$ *goes to* 1*.*
- *if* $\lim_{N \to +\infty} \beta_N N^{(\gamma_c - \gamma)d/2\gamma} = +\infty$*, alors* Z_{N,β_N} *then* $Z_{Nt,\beta_N}(x\sqrt{N})$ *goes to* 0*.*

What about the Polymer measure ?

When β_N is adequately scaled, the partition function Z_{N,β_N} converges.
Does it also hold for the Polymer measure

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Theorem (Berger-Lacoin '21,'22, G'26)

When $\gamma < \gamma_c$, choosing $\beta_N = N^{-(\gamma_c-\gamma)d/2\gamma}$, the random measure P_{N,β_N} converges in law as $N \to +\infty$ to a random measure P_κ which is formally given by

$$\frac{\mathrm{dP}_\kappa}{\mathrm{dP}}(B) = \frac{1}{\mathcal{Z}_\kappa} : \exp(\kappa \int_0^1 \zeta(s, B_s)\mathrm{d}s) : ,$$

where ζ is a γ-stable noise, and P is the Wiener measure.

This is only formal since the measure P_κ is actually almost surely singular with respect to P.

Thank you !