Perturbative renormalization of Φ^4_d as generalized Wick renormalization

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1. Φ^4_d model

$\Lambda = \mathbb{R}^d/\mathbb{Z}^d$, $\phi: \Lambda \to \mathbb{R}$, $H_\alpha(\phi) = \int_\Lambda \left(\frac{1}{2}\|\nabla\phi\|^2 + \frac{1}{2}\phi^2 + \alpha\phi^4\right) dx$

Aim: $\mu_\alpha(\phi) \sim e^{-H_\alpha(\phi)}$ Compute $\mathbb{E}^{\mu_\alpha}[\phi(x)\phi(y)] \ldots$

a) Case $\alpha = 0$ $H_0(\phi) = \frac{1}{2}\langle \phi, [-\Delta+1]\phi \rangle$

Fourier: $\phi(x) = \sum_{k\in\mathbb{Z}^d} \phi_k e_k(x)$ $\Delta e_k = -\lambda_k e_k$, $\lambda_k \sim \|k\|^2$

$$\Rightarrow H_0(\phi) = \frac{1}{2}\sum_k (1+\lambda_k)|\phi_k|^2$$

$$\mathbb{E}^{\mu_0}[\phi(x)\phi(y)] = \sum_k \frac{e_k(x)e_{-k}(y)}{1+\lambda_k} = G(x-y) \quad \text{Green Fct.}$$

Rem: $\phi(x) = \sum_k \frac{Z_k}{\sqrt{1+\lambda_k}} e_k(x)$ $Z_k \sim \mathcal{N}(0,1)$

b) Case $\alpha > 0$: $H_\alpha(\phi) = H_0(\phi) + \alpha X_0$ $X_0 = \int_\Lambda \phi^4 dx$

$$\mathbb{E}^{\mu_\alpha}[F(\phi)] = \int F(\phi) d\mu_\alpha \qquad d\mu_\alpha = \frac{1}{Z_\alpha} e^{-H_\alpha} d\phi = \frac{Z_0}{Z_\alpha} e^{-\alpha X_0} d\mu_0$$

$$= \frac{Z_0}{Z_\alpha}\mathbb{E}^{\mu_0}[F(\phi) e^{-\alpha X_0}]$$

In part, $F(\phi) = 1$ $\Rightarrow$ $\frac{Z_\alpha}{Z_0} = \mathbb{E}^{\mu_0}[e^{-\alpha X_0}] = \sum_{n\geq 0} \frac{(-\alpha)^n}{n!}\mathbb{E}^{\mu_0}[X_0^n]$

d=1: $$\mathbb{E}^{\mu_0}[X_0] = \sum_{k_1\ldots k_4} \frac{\mathbb{E}[Z_{k_1}\ldots Z_{k_4}]}{\sqrt{1+\lambda_{k_1}}\ldots\sqrt{1+\lambda_{k_4}}} \int e_{k_1}\ldots e_{k_4} dx$$

$$\lesssim \sum_{k_1,k_2} \frac{1}{(1+\lambda_{k_1})(1+\lambda_{k_2})} = \left(\sum_k \frac{1}{1+\lambda_k}\right)^2 \sim \left(\sum_k \frac{1}{1+k^2}\right)^2 < \infty$$

d=2: $$\sum_{k\in\mathbb{Z}^2} \frac{1}{1+\lambda_k} \sim \sum_k \frac{1}{1+\|k\|^2} \sim \int_1^\infty \frac{r\,dr}{r^2} = +\infty$$

$\Rightarrow$ renormalization

2. Wick calculus

X r.v. $\quad \mu_X(x^n) = \mathbb{E}[X^n]$

$\mathbb{E}[e^{tX}] = \sum_{n\geq 0} \frac{t^n}{n!} \mu_X(x^n) =: \Lambda(\mu_X)(t)$ moment generating fct

Wick map: $(t,x) \mapsto W(t,x) = \frac{e^{tx}}{\mathbb{E}[e^{tX}]} = \sum_{n\geq 0} \frac{t^n}{n!} W(x^n)$

Property: $\mathbb{E}[W(X^n)] = 0 \quad \forall n \geq 1$

Case $X \sim \mathcal{N}(0,\sigma^2)$: $\mathbb{E}[e^{tX}] = e^{\sigma^2 t^2/2} \qquad W(x^n) = H_n(x,\sigma^2)$ Hermite polg.

$(1, x, x^2-\sigma^2, x^3-3\sigma^2 x, x^4-6\sigma^2x^2+3\sigma^4 \ldots)$

Remark: $\Lambda: \mathcal{L}(\mathbb{R}[x], \mathbb{R}) \longrightarrow \mathbb{R}[[t]]$

$\varphi \longmapsto \sum_n \varphi(x^n) \frac{t^n}{n!}$

isomorphism between convolution algebra & algebra of power series

$\Lambda(\varphi)\Lambda(\psi) = \Lambda(\varphi * \psi) \qquad$ with $\quad \varphi * \psi = m(\varphi \otimes \psi)\Delta$

$\Rightarrow W = (\mu_X^{-1} \otimes \mathrm{id})\Delta \quad [= (\exp_*(-\varkappa_X) \otimes \mathrm{id})\Delta \quad$ where $\mu_X = \exp_*(\varkappa_X)]$

Lemma: X, Y jointly Gaussian, centred, var σ_1^2, σ_2^2

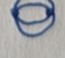
$\Rightarrow \mathbb{E}[H_n(X,\sigma_1^2) H_m(Y,\sigma_2^2)] = n!\, \delta_{nm}\, \mathbb{E}[XY]^n$

Consequence: $\Phi_N(x) = \sum_{|k|\leq N} \frac{Z_k}{\sqrt{1+\lambda_k}} e_k(x) \qquad C_N = \mathbb{E}[\Phi_N(x)^2] = \sum_{|k|\leq N} \frac{1}{1+\lambda_k} \sim \log N$

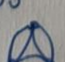
In H, $\quad \Phi^4(x) \rightarrow \,:\Phi^4:(x) = H_4(\Phi_4(x), C_N) \qquad X = \int_\Lambda :\Phi_N^4:(x)\,dx$

$\mathbb{E}^{\mu_0}[X] = 0$

$\mathbb{E}^{\mu_0}[X^2] = 4! \iint \mathbb{E}^{\mu_0}[\Phi_N(x)\Phi_N(y)]^4\, dx\, dy$

$= 4! \int_\Lambda G_N(x)^4\, dx = 4!$ 

$\lesssim \int_0^1 r \log(r)^4\, dr < \infty \qquad (\log(r+\frac{1}{N}))$

$\mathbb{E}^{\mu_0}[X^3] \sim$  $< \infty$

In general: sum over pairings

3. The case $d=3$

$$G_N(x) \sim \frac{1}{\|x\| + \frac{1}{N}} \Rightarrow \int_\Lambda G_N(x)^4 dx \sim \int_0^1 \frac{r^2 dr}{(r+\frac{1}{N})^4} \sim N$$

Theorem: Measure is well-defined for

$$H_\alpha(\phi) = H_0(\phi) - \alpha^2 C_N^{(2)} \int_\Lambda \phi^2 dx + \alpha \int_\Lambda :\phi^4: dx + \alpha^2 C_N^{(3)} - \alpha^3 C_N^{(4)}$$

$C_N^{(2)}, C_N^{(4)} \sim \log N$, $C_N^{(3)} \sim N$

Write $H_\alpha = H_0 + \alpha X + \beta Y + \gamma$

$$\Rightarrow \frac{Z_\alpha}{Z_0} = \underbrace{\mathbb{E}^{\mu_0}[e^{-\alpha X - \beta Y}]}\cdot e^{-\gamma}$$

$$= \sum_{n,m \geq 0} \frac{(-\alpha)^n}{n!} \frac{(-\beta)^m}{m!} \underbrace{\mathbb{E}[X^n Y^m]}_{\text{sum of Feynman graphs}}$$

e.g. $\mathbb{E}[X^2 Y]$ $= c\, \triangle$ $(c = 2\cdot(4!)^2)$

Fact 1: $\log \mathbb{E}^{\mu_0}[e^{-\alpha X - \beta Y}]$ = projection on connected graphs (linked-cluster thm)

Fact 2: $\deg \Gamma = d(|\mathcal{V}|-1) - |\mathcal{E}|$ eg $\deg(\triangle) = 2d - 5 = 1$, $\deg(\ominus) = d - 3 = 0$

$$A(\Gamma) = -\Gamma - \sum_{\substack{1 \neq \bar\Gamma \subsetneq \Gamma \\ \deg \bar\Gamma \leq 0}} A(\bar\Gamma)\cdot(\Gamma/\bar\Gamma)$$

$A(\ominus) = -\ominus$ $A(\triangle) = -\triangle + \ominus\,\bigcirc$

$$\Pi_N(\Gamma) := \int_{\Lambda^{\mathcal{V}}} \prod_{e \in \mathcal{E}} G_N(x_{e_+} - x_{e_-}) dx$$

$$\Pi_N^{BPHZ}(\Gamma) = (\Pi_N \tilde{A} \otimes \Pi_N)\Delta_{ck}(\Gamma)$$

$$\begin{cases} \tilde{A}(\Gamma) = A(\Gamma)\, 1_{\deg \Gamma \leq 0} \\ \Delta_{ck}(\Gamma) = \sum_{\bar\Gamma \subset \Gamma} \bar\Gamma \otimes (\Gamma/\bar\Gamma) \end{cases}$$

Then $\Pi_N^{BPHZ}(\Gamma)$ bdd unif N if $\deg \Gamma \geq 0$

Our approach: (inspired by Ebrahimi-Fard, Patras, Tapia & Zambotti 2020)

$$\begin{array}{ccccc} H = \mathrm{span}(X^n) & \xrightarrow{\mathcal{P}} & \mathcal{G} = \mathrm{span}(\Gamma) & \xrightarrow{\Pi_N^{BPHZ}} & \mathbb{R} \\ & & \downarrow (\tilde{A}\otimes id)\Delta_{ck} & & \\ & & \mathcal{G}\otimes\mathcal{G} & & \\ & & \downarrow \Pi_N \otimes id & & \\ & & \mathcal{G} & \xrightarrow{\Pi_N} & \mathbb{R} \end{array}$$

$$\begin{array}{ccc} e^{-\alpha X} & \stackrel{\mathcal{P}}{\longmapsto} & \sum_n \frac{(-\alpha)^n}{n!} \mathcal{P}(X^n) \\ ?\ \downarrow & & \downarrow (\Pi_N \tilde{A} \otimes id)\Delta \\ e^{-\alpha X - \beta Y} & \stackrel{\mathcal{P}}{\longmapsto} & \sum_{n,m} \frac{(-\alpha)^n}{n!} \frac{(-\beta)^m}{m!} \mathcal{P}(X \end{array}$$

$$\Pi_N \circ \mathcal{P}(e^{-\alpha X - \beta Y}) = \Pi_N^{BPHZ} \circ \mathcal{P}(e^{-\alpha X})$$

Thm: [B, Klose, Tapia]

$$\begin{array}{ccccc} e^{-\alpha X} & \xmapsto{\mathcal{P}} & (\ldots) & \xmapsto{\pi_N^{BPHZ}} & \\ W \big\downarrow & & \big\downarrow (\pi_N \tilde{A} \otimes id)\Delta_{ex} & & (\ldots) \\ e^{-\alpha X - \beta Y} & \xmapsto{\mathcal{P}} & (\ldots) & \xmapsto{\pi_N} & \end{array}$$

commutes, where W is a Wick map given by

$$W(X^n) = (\mu^{-1} \otimes id)\Delta$$

$$\mu^{-1} = \exp_*(-\varkappa) \qquad \varkappa(X^n) = \begin{cases} \sigma^2 Y & n=2 \\ 0 & \text{else} \end{cases} \qquad \sigma^2 = \frac{2\beta}{\alpha^2}$$

Examples:
$$\begin{cases} W(X) = X \\ W(X^2) = X^2 - \sigma^2 Y \\ W(X^3) = X^3 - 3\sigma^2 XY \\ W(X^4) = X^4 - 6\sigma^2 X^2 Y + 3\sigma^4 Y^2 \end{cases}$$

Seems to remain true for $3 < d < 4$ with different $\varkappa$