Novikov algebras and multi-indices in regularity structures

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Multi-indices

Let $(z_k)_{k\in\mathbb{N}}$, where the variable z_k encodes nodes of the tree that have k children. Multi-indices β over $\mathbb{N}$ are given

$$z^\beta := \prod_{k\in\mathbb{N}} z_k^{\beta(k)}.$$

Pre-Lie product:

$$z^\beta \triangleright z^{\beta'} = z^\beta D(z^{\beta'}),$$

where D is the derivation given by

$$D = \sum_{k\in\mathbb{N}} (k+1) z_{k+1} \partial_{z_k}.$$

Populated multi-indices

$$[\beta] = \sum_{k\in\mathbb{N}} (1-k)\beta(k) = 1.$$

ODEs in one dimension

We consider

$$y' = f(y), \quad y(0) = y_0 \in \mathbb{R},$$

where $f \in \mathcal{C}^\infty$ is a smooth function. One can formally expand the solution as

$$y(t) = \sum_{[\beta]=1} \alpha(z^\beta) F_f[z^\beta](y_0)$$

where

$$F_f[z^\beta](y) = \prod_{k\in\mathbb{N}} \left(f^{(k)}(y)\right)^{\beta(k)}.$$

Novikov algebras

A Novikov algebra is a vector space equipped with a bilinear product $x, y \mapsto x \triangleright y$, satisfying the identities

$$(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z),$$
$$(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright y.$$

Theorem

The Novikov algebra of populated multi-indices is isomorphic to the free algebra on one generator.

Conjectured by Dominique Manchon in 2022.
Goes back to A. Dzhumadil'daev and C. Löfwall (2002).

Singular SPDEs

We are looking at the class of subcritical semi-linear SPDEs of the form

$$(\partial_t - \mathcal{L})\, u = \sum_{\mathfrak{l} \in \mathfrak{L}^-} a^{\mathfrak{l}}(\mathbf{u}) \xi_{\mathfrak{l}}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

$a^{\mathfrak{l}}(\mathbf{u})$ is a function of u and its iterated partial derivatives. $\mathcal{L}$ is a differential operator, $\mathfrak{L}^-$ is a finite set and the $\xi_{\mathfrak{l}}$ are space-time noises. For $\mathbf{n} \in \mathbb{N}^{d+1}$, one considers:

$$u^{(\mathbf{n})} := \frac{\partial_{x_0}^{n_0} \cdots \partial_{x_d}^{n_d}}{n_0! \cdots n_d!}(u),$$

In the epxansion of the solution, one will have to deal with

$$\prod_{(\mathfrak{l},\mathbf{n}) \in \mathfrak{L}^- \times \mathbb{N}^{d+1}} \partial_{u^{(\mathbf{n})}}^{\beta(\mathfrak{l},\mathbf{n})} a^{\mathfrak{l}}(\mathbf{u}).$$

General multi-indices

New formal variables $z_{(\mathfrak{l},w)}$, $(\mathfrak{l},w) \in \mathfrak{L}^- \times M(\mathbb{N}^{d+1})$, and define the *general multi-indices* β as

$$z^\beta := \prod_{(\mathfrak{l},w)\in\mathfrak{L}^-\times M(\mathbb{N}^{d+1})} z_{(\mathfrak{l},w)}^{\beta(\mathfrak{l},w)}.$$

Introduced in B.-Linares (2023). For each $\mathbf{n} \in \mathbb{N}^{d+1}$, the derivation $D^{(\mathbf{n})}$ is given by

$$D^{(\mathbf{n})} = \sum_{(\mathfrak{l},w)\in\mathfrak{L}^-\times M(\mathbb{N}^{d+1})} (w(\mathbf{n})+1)\, z_{(\mathfrak{l},\mathbf{n}w)} \partial_{z_{(\mathfrak{l},w)}}.$$

We can define products $\triangleright_{\mathbf{n}}$ by setting

$$z^\beta \triangleright_{\mathbf{n}} z^{\beta'} = z^\beta D^{(\mathbf{n})}(z^{\beta'}).$$

Populated general multi-indices:

$$\sum_{(\mathfrak{l},w)} (1-|w|)\beta(\mathfrak{l},w) = 1.$$

Multi-Novikov

A multi-Novikov algebra is a vector space equipped with bilinear products $x, y \mapsto x \triangleright_a y$ indexed by a set A satisfying

$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (y \triangleright_a x) \triangleright_b z - y \triangleright_a (x \triangleright_b z),$$
$$(x \triangleright_a y) \triangleright_b z - x \triangleright_a (y \triangleright_b z) = (x \triangleright_b y) \triangleright_a z - x \triangleright_b (y \triangleright_a z),$$
$$(x \triangleright_a y) \triangleright_b z = (x \triangleright_b z) \triangleright_a y,$$

for all $a, b \in A$.

Theorem (B.-Dotsenko, 2023)

The multi-Novikov algebra of populated general multi-indices is isomorphic to free algebra generated by the set $\mathfrak{L}^-$.

Extension of the proof of A. Dzhumadil'daev and C. Löfwall (2002).

Derivatives

New derivatives ∂_{x_i} have to be considered computed via the chain rule

$$\partial_{x_i} = \sum_{\mathbf{n}\in\mathbb{N}^{d+1}} (n_i + 1) u^{(\mathbf{n}+e_i)} \partial_{u^{(\mathbf{n})}}.$$

One has the following relations:

$$\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}, \quad \partial_{u^{(\mathbf{n})}}\partial_{u^{(\mathbf{m})}} = \partial_{u^{(\mathbf{m})}}\partial_{u^{(\mathbf{n})}}, \quad \partial_{x_i}\partial_{u^{(\mathbf{n})}} = n_i \partial_{u^{(\mathbf{n}-e_i)}} + \partial_{u^{\mathbf{n}}}\partial_{x_i},$$

where e_i is the standard basis vector of $\mathbb{N}^{d+1}$.

We introduce an abstract associative algebra $\mathcal{A}$ generated by the letters $\mathbf{n} \in \mathbb{N}^{d+1}$ and d_i, and impose the relations

$$d_i d_j = d_j d_i, \quad \mathbf{n}\mathbf{m} = \mathbf{m}\mathbf{n}, \quad d_i \mathbf{n} = n_i(\mathbf{n} - e_i) + \mathbf{n} d_i.$$

SPDE multi-indices

We consider the set of formal variables $(z_{(\mathfrak{l},\alpha)})_{(\mathfrak{l},\alpha)\in\mathfrak{L}^-\times\mathcal{A}}$. Each $z_{(\mathfrak{l},\alpha)}$ corresponds to $\mathrm{D}^\alpha a^{\mathfrak{l}}(\mathbf{u})$, where D^α is obtained by

$$d_i \to \partial_{x_i}, \quad \mathbf{n} \to \partial_{u^{(\mathbf{n})}}.$$

Multi-indices β are given by

$$z^\beta := \prod_{(\mathfrak{l},\alpha)\in\mathfrak{L}^-\times\mathcal{A}} z_{(\mathfrak{l},\alpha)}^{\beta(\mathfrak{l},\alpha)}.$$

Populated SPDE multi-indices:

$$\sum_{(\mathfrak{l},\alpha)}(1-|\alpha|)\beta(\mathfrak{l},\alpha) = 1.$$

where $|\alpha|$ is the number of letters $\mathbf{n} \in \mathbb{N}^{d+1}$ in α.

Coding

Usually, one encodes the ∂_{x_i} by another set of variables $z_{\mathbf{n}}$, $\mathbf{n} \in \mathbb{N}^{d+1}$. Our coding is more compact. For example, $z_{(\mathfrak{l}, d_i)}$ corresponds to

$$\partial_i a^{\mathfrak{l}}(\mathbf{u}) = \sum_{\mathbf{n}} u^{\mathbf{n}+e_i} \partial_{u^{(\mathbf{n}+e_i)}} a^{\mathfrak{l}}(\mathbf{u})$$

which would otherwise corresponds to $\sum_{\mathbf{n}} z_{(\mathbf{n}+e_i)} z_{(\mathfrak{l},\mathbf{n})}$.

Derivations

Family of derivations: $D^{(\mathbf{n})}$, $\mathbf{n} \in \mathbb{N}^{d+1}$, and ∂_i, $0 \leq i \leq d$.

$$D^{(\mathbf{n})} z_{(\mathfrak{l},\alpha)} = z_{(\mathfrak{l},\mathbf{n}\alpha)}, \quad \partial_i z_{(\mathfrak{l},\alpha)} = z_{(\mathfrak{l},d_i\alpha)}$$

One has

$$\partial_i D^{(\mathbf{n})} = D^{(\mathbf{n})} \partial_i + n_i D^{(\mathbf{n}-e_i)}.$$

We define a family of products $\triangleright_{\mathbf{n}}$ by setting

$$z^{\gamma} \triangleright_{\mathbf{n}} z^{\gamma'} = z^{\gamma} D^{(\mathbf{n})}(z^{\gamma'}).$$

They define a multi-Novikov algebra structure.

Extended Algebras

Let some type of algebras $\mathcal{P}_A$ with operations indexed by a set A, $f_a, a \in A$ ($D^{(\mathbf{n})}, A = \mathbb{N}^{d+1}$). Let $\mathcal{P}_A^{\mathrm{lin}}$ its linearised version. We suppose that $V = \mathrm{Vect}(A)$ carries a representation of a Lie algebra $\mathfrak{g}$.

The class of $\mathfrak{g}$-*extended* $\mathcal{P}_A^{\mathrm{lin}}$-*algebras* has α_g, $g \in \mathfrak{g}$ satisfying $\alpha_g\alpha_h - \alpha_h\alpha_g = \alpha_{[g,h]}$ and the identities

$$\begin{aligned}\alpha_g f_v(x_1, \dots, x_n) = \sum_{i=1}^{n} f_v(x_1, \dots, x_{i-1}, \alpha_g(x_i), x_{i+1}, \dots, x_n) \\ + f_{g(v)}(x_1, \dots, x_n).\end{aligned}$$

We apply this to $\alpha_g = \partial_i$ and $g(\mathbf{n}) = n_i(\mathbf{n} - e_i)$ where $g = i$ and $\mathfrak{g}$ is the $d+1$-dimensional abelian Lie algebra.

Free Multi-Novikov

Proposition (B.-Dotsenko, 2023)

As a $\mathcal{P}_A^{\mathrm{lin}}$-algebra, the free $\mathfrak{g}$-extended $\mathcal{P}_A^{\mathrm{lin}}$-algebra generated by a vector space W is isomorphic to the free algebra generated by $U(\mathfrak{g}) \otimes W$, the free $\mathfrak{g}$-module on W.

As a consequence ($W = \mathrm{Vect}(\mathfrak{L}^-)$), one has

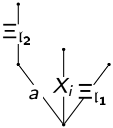
Theorem (B.-Dotsenko, 2023)

The multi-Novikov algebra of populated SPDE multi-indices is isomorphic to the free algebra generated by the set $\mathbb{N}^{d+1} \times \mathfrak{L}^-$.

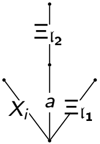
Connection with decorated trees

We consider planar decorated trees such that

$$\mathcal{I}_a(\Xi_{\mathfrak{l}_2})X_i\Xi_{\mathfrak{l}_1} =$$



$$\neq$$



$$= X_i\mathcal{I}_a(\Xi_{\mathfrak{l}_2})\Xi_{\mathfrak{l}_1}.$$

We quotient these decorated trees by the following relations:

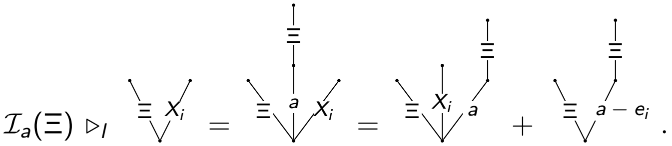
$$X_iX_j = X_jX_i, \quad \mathcal{I}_a(\tau)\mathcal{I}_b(\sigma) = \mathcal{I}_b(\sigma)\mathcal{I}_a(\tau)$$

$$\mathcal{I}_a(\tau)X_i = X_i\mathcal{I}_a(\tau) + \mathcal{I}_{a-e_i}(\tau).$$

We denote by $\mathcal{T}$ the linear span of these decorated trees.

Multi-pre-Lie structure

Left grafting products:



Theorem

The multi-pre-Lie algebra $(\mathcal{T}, \triangleright_l)$ is isomorphic to the free pre-Lie algebra generated by all elements $X^k \Xi_{\mathfrak{l}}$.

Perspectives

- Multi-indices are free-Novikov: more knowledge on Novikov algebra (free Lie algebra?).
- Unique definition of renormalisation maps.
- Other free structures than multi-indices for the expansion of solutions for singular SPDEs.
- Connection with post-Lie algebra and deformation theory.
- Geometric interpretation.