Combinatorics and a priori estimates for rough S(P)DEs

Lorenzo Zambotti
Sorbonne Université
(joint work with Lorenzo Agabiti and Alberto Bonicelli)

Nancy, 18th June 2026

Controlled ODE

Fix $T > 0$, a path $X : [0,T] \to \mathbb{R}^d$ and a smooth function $\sigma : \mathbb{R}^k \to \mathbb{R}^k \otimes (\mathbb{R}^d)^*$. Consider

$$\dot{Z}_t = \sigma(Z_t)\,\dot{X}_t, \qquad Z_0 = z_0. \tag{1}$$

- If $X \in \mathscr{C}^1, \sigma$ Lipschitz: well-posedness and $Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s$
- If $X \in \mathscr{C}^\alpha$ Hölder-continuous for $\alpha \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ for any $n \geq 1$. Need to reformulate (1) in order for it to be meaningful.

Taylor Expansion and Rough Paths

Consider $X \in C^1$. Taylor expand $\sigma(Z)$ around s and use (1):

$$Z_t - Z_s = \sigma(Z_s)\,(X_t - X_s) + \nabla\sigma(Z_s)\sigma(Z_s)\int_s^t (X_u - X_s) \otimes \dot{X}_u \,\mathrm{d}u + R_{st}$$

with the remainder satisfying $|R_{st}| = O(|t-s|^3)$.

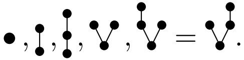
For $X \in \mathscr{C}^\alpha$ and $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$ and $n \geq 2$, the integral is ill-defined: Rough Paths Theory is needed.

If one really wants to avoid rough paths, it is possible to consider $X \in \mathscr{C}^1$ but look for estimates depending only on the $\mathscr{C}^\alpha$-norm.

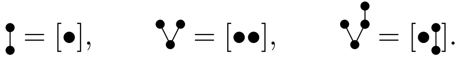
Rooted Trees

Idea from Numerical Analysis: use trees to encode the expansion.

Non-planar rooted trees $\mathcal{T}$:



- $|\tau|$ the number of vertices
- $\mathcal{F}$ the space of forests $\tau_1 \cdot \ldots \cdot \tau_k$, where $\cdot$ is the disjoint union of graphs, $\mathbf{1}$ is the empty forest and $\mathcal{T}^{\leq n}, \mathcal{F}^{\leq n}$ the trees and forests with n vertices or less
- $[\cdot] : \mathcal{F} \to \mathcal{T}$ the operation of attaching the trees of a forest to a common new root $\bullet$ by adding new edges:



Davie's Solution

Trees allow to give a general definition of a solution *à la* Davie of the rough equation:

$$Z_t - Z_s - \sum_{\tau \in \mathscr{T}^{\leq n}} \Upsilon_\tau(Z_s)\, \mathbb{X}_{st}(\tau) = o\left(|t-s|\right), \qquad 0 \leq s \leq t \leq T,$$

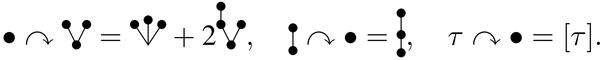
where $\mathbb{X}_{st} : \mathscr{F} \to \mathbb{R}$ (the rough path) is defined by

- $\mathbb{X}_{st}(\mathbf{1}) = 1$
- $\mathbb{X}_{st}([w]) = \int_s^t \mathbb{X}_{su}(w)\, \dot{X}_u \, \mathrm{d}u$
- $\mathbb{X}_{st}(\tau_1 \cdots \tau_k) = \mathbb{X}_{st}(\tau_1) \cdots \mathbb{X}_{st}(\tau_k)$,

and $\Upsilon : \mathscr{F} \to C(\mathbb{R}^k; \mathbb{R}^k)$ (the elementary differentials) are coefficients defined in the next slides.

Grafting

We introduce the grafting operator $\curvearrowright : \mathcal{T} \otimes \mathcal{T} \to \mathcal{T}$:



Extend $\curvearrowright$ to forests by the Guin-Oudom construction, for $\tau, \tau_1, \ldots, \tau_k \in \mathcal{T}$

$$
\begin{aligned}
&\tau \curvearrowright (\tau_1 \cdot \ldots \cdot \tau_k) = \\
&\qquad = \sum_{i=1}^{k} \tau_1 \cdot \ldots \cdot (\tau \curvearrowright \tau_i) \cdot \ldots \cdot \tau_k \, , \\
&(\tau_1 \cdot \ldots \cdot \tau_k) \curvearrowright \tau = \\
&\qquad = \tau_1 \curvearrowright ((\tau_2 \cdot \ldots \cdot \tau_k) \curvearrowright \tau) - (\tau_1 \curvearrowright (\tau_2 \cdot \ldots \cdot \tau_k)) \curvearrowright \tau \, .
\end{aligned}
$$

Elementary Differentials

We define the *elementary differential* $\Upsilon : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ by

$$\Upsilon_{\bullet} := \sigma, \quad \Upsilon_{[\tau_1 \cdot \ldots \cdot \tau_k]} := \nabla^k \sigma \prod_{j=1}^{k} \Upsilon_{\tau_j} .$$

Recall that $[\tau_1 \cdot \ldots \cdot \tau_k] = (\tau_1 \cdot \ldots \cdot \tau_k) \curvearrowright \bullet$.

Lemma

For $\rho, \tau_1, \ldots, \tau_k \in \mathcal{T}$

$$\Upsilon_{(\tau_1 \cdot \ldots \cdot \tau_k) \curvearrowright \rho} = \nabla^k \Upsilon_\rho \prod_{j=1}^{k} \Upsilon_{\tau_j} .$$

Branched Rough Paths

Let now $X : [0,T] \to \mathbb{R}^d$ be $\mathscr{C}^\alpha$ with $\alpha \in (0,1]$.

A branched α-rough path over X is a map $\mathbb{X} : [0,T]^2_\leq \to \mathscr{F}^*$ such that $\mathbb{X}_{st}(\bullet) = X_t - X_s$ and

- For any $(s,u,t) \in [0,T]^3_\leq$, the Chen relation holds:

$$\mathbb{X}_{st} = \mathbb{X}_{su} \star \mathbb{X}_{ut}$$

  (where $\star$ is an associative product associated with the grafting $\curvearrowright$).

- For any $\tau \in \mathscr{T}$

$$|\mathbb{X}_{st}(\tau)| \leq |t-s|^{|\tau|\alpha} \, .$$

- For any forest $w = \tau_1 \cdot \ldots \cdot \tau_k \in \mathscr{F}$,

$$\mathbb{X}_{st}(w) = \prod_{i=1}^{k} \mathbb{X}_{st}(\tau_i) \, .$$

Davie's Solution

$$Z_t = Z_0 + \int_0^t \sigma(Z_s)\,\dot{X}_s\,\mathrm{d}s$$

$X \in \mathscr{C}^\alpha$ Hölder-continuous for $\alpha \in (\frac{1}{n+1}, \frac{1}{n}]$, $n \geq 1$

$$Z_t - Z_s - \sum_{\tau \in \mathscr{T}^{\leq n}} \Upsilon_\tau(Z_s)\,\mathbb{X}_{st}(\tau) = o\left(|t-s|\right), \qquad 0 \leq s \leq t \leq T,$$

Define the remainders

$$Z^{[k]}_{st} := Z_t - Z_s - \sum_{\tau \in \mathscr{T}^{\leq k-1}} \Upsilon_\tau(Z_s)\,\mathbb{X}_{st}(\tau), \quad k \in \{1, \ldots, n+1\}.$$

Z solution *à la Davie* $\iff Z^{[n+1]}_{st} = o(|t-s|)$

Goal

- A priori estimates: bounds on $\|Z^{[k]}\|_{k\alpha}$, $k = 1, \ldots, n+1$, where

$$\|R\|_\eta := \sup_{0 \le s < t \le T} \frac{|R_{st}|}{|t-s|^\eta}.$$

- Local and Global solution with minimal hypothesis on the elementary differentials
- Since in the equation

$$Z_t - Z_s - \sum_{\tau \in \mathcal{T}^{\le n}} \Upsilon_\tau(Z_s)\, \mathbb{X}_{st}(\tau) = o\left(|t-s|\right), \qquad 0 \le s \le t \le T,$$

only elementary differentials Υ_τ appear, it would be natural to assume only that they are all (for $\tau \in \mathcal{T}^{\le n}$) Lipschitz continuous. For example, not necessarily bounded.
- Previous works: [Lejay 08] for $\alpha > \frac{1}{3}$. [Bonnefoi, Chandra, Moinat, Weber 22] and [Chevyrev, Gubinelli 26] using coercivity.

Sewing Bound

In this business there is a fundamental analytic tool:

Theorem (Sewing Bound (Gubinelli, Feyel-de la Pradelle))

For any $R \in C([0,T]^2_\leq, \mathbb{R}^k)$ such that $R_{st} = o(t-s)$. Then, for any $\eta \in (1,\infty)$,

$$\|R\|_\eta \leq K_\eta \|\delta R\|_\eta \qquad \textit{where} \qquad K_\eta := (1 - 2^{1-\eta})^{-1}.$$

Here:

$$\delta R_{sut} := R_{st} - R_{su} - R_{ut}, \qquad \|\delta R\|_\eta := \sup_{0 \leq s < u < t \leq T} \frac{|\delta R_{sut}|}{|t-s|^\eta}$$

Important fact: in general δR is simpler than R.

Remainder Formula

Lemma

We have, for a combinatorial coefficient $\pi(w)$,

$$\delta Z_{sut}^{[n+1]} = \sum_{\tau \in \mathcal{T}^{\leq n}} \underbrace{\left(\Upsilon_\tau(Z_u) - \sum_{w \in \mathcal{F}^{\leq n-|\tau|}} \frac{\Upsilon_{w \curvearrowright \tau}(Z_s)}{\pi(w)} \mathbb{X}_{su}(w) \right)}_{B_{su}^{[n+1-|\tau|]}(\tau)} \mathbb{X}_{ut}(\tau).$$

We want

$$|\delta Z_{sut}^{[n+1]}| \lesssim |t-s|^{(n+1)\alpha},$$

we know that

$$|\mathbb{X}_{ut}(\tau)| \lesssim |t-s|^{|\tau|\alpha},$$

we need to prove

$$|B_{su}^{[n+1-|\tau|]}| \lesssim |t-s|^{(n+1-|\tau|)\alpha}.$$

A first formula

Theorem

For any $\tau \in \mathcal{T}$ *and* $n \geq 0$

$$B^{[n+1]}_{st}(\tau) = \Upsilon_\tau(Z_t) - \sum_{w \in \mathcal{F}^{\leq n}} \frac{\Upsilon_{w \curvearrowright \tau}(Z_s)}{\pi(w)} \mathbb{X}_{st}(w)$$

$$= \sum_{k=0}^{n} \sum_{\substack{\boldsymbol{\tau} \in (\mathcal{T})^k, l \in \mathbb{N}_* \\ |\boldsymbol{\tau}|+l=n+1}} \int_{[0,1]} \nabla^{k+1} \Upsilon_\tau((1-r)Z_s + rZ_t) \cdot \left[Z^{[l]}_{st} \, [\Upsilon\mathbb{X}]^{\boldsymbol{\tau}}_{st} \right] \frac{(1-r)^k}{k!} \, \mathrm{d}r\,.$$

Here

$$[\Upsilon\mathbb{X}]^{\boldsymbol{\tau}}_{st} = \prod_{i=1}^{k} \Upsilon_{\tau_i}(Z_s)\, \mathbb{X}_{st}(\tau_i).$$

This result gives a priori estimates if $\|\nabla^{k+1}\Upsilon_\tau\|_\infty < +\infty$ and $\|\Upsilon_{\tau_i}\|_\infty < +\infty$.

Towards another representation

The simplest case if for $n = 0$:

$$B^{[1]}_{st}(\tau) = \Upsilon_\tau(Z_t) - \Upsilon_\tau(Z_s) = \int_0^1 \nabla\Upsilon_\tau((1-r)Z_s + rZ_t)\, Z^{[1]}_{st}\, \mathrm{d}r, \qquad Z^{[1]}_{st} := Z_t - Z_s.$$

This is already optimal: $|B^{[1]}_{st}(\tau)| \lesssim |t-s|^\alpha$, but it is sufficient only for $\alpha > \frac{1}{2}$.

The next case is for $n = 1$: setting $u_0 := (1-r_0)Z_s + r_0 Z_t$,

$$B^{[2]}_{st}(\tau) = \Upsilon_\tau(Z_t) - \Upsilon_\tau(Z_s) - \Upsilon_{\bullet\curvearrowright\tau}(Z_s)\, \mathbb{X}_{st}(\bullet) =$$
$$= \int_0^1 \nabla\Upsilon_\tau(u_0)\, Z^{[1]}_{st}\, \mathrm{d}r_0 - \nabla\Upsilon_\tau(Z_s)\, \sigma(Z_s)\, \mathbb{X}_{st}(\bullet).$$

Recall that we want $|B^{[2]}_{st}(\tau)| \lesssim |t-s|^{2\alpha}$.

Towards another representation

$$
\begin{aligned}
B^{[2]}_{st}(\tau) &= \int_0^1 \nabla\Upsilon_\tau(u_0)\, Z^{[1]}_{st}\, \mathrm{d}r_0 - \nabla\Upsilon_\tau(Z_s)\,\sigma(Z_s)\,\mathbb{X}_{st}(\bullet) \\
&= \int_0^1 \nabla\Upsilon_\tau(u_0)\, Z^{[2]}_{st}\, \mathrm{d}r_0 + \int_0^1 \nabla\Upsilon_\tau(u_0)\,\sigma(Z_s)\,\mathbb{X}_{st}(\bullet)\,\mathrm{d}r_0 - \nabla\Upsilon_\tau(Z_s)\,\sigma(Z_s)\,\mathbb{X}_{st}(\bullet)
\end{aligned}
$$

since

$$
Z^{[1]}_{st} = Z^{[2]}_{st} + \sigma(Z_s)\,\mathbb{X}_{st}(\bullet)\,.
$$

By assumption, $|Z^{[2]}_{st}| \lesssim |t-s|^{2\alpha}$, therefore the first term is fine. Moreover $|\mathbb{X}_{st}(\bullet)| \lesssim |t-s|^{\alpha}$. Therefore it remains to prove that

$$
\left|\int_0^1 \left(\nabla\Upsilon_\tau(u_0) - \nabla\Upsilon_\tau(Z_s)\right)\sigma(Z_s)\,\mathrm{d}r_0\right| \lesssim |t-s|^{\alpha}.
$$

Towards another representation

We want to show:

$$\left| \int_0^1 \left(\nabla\Upsilon_\tau(u_0) - \nabla\Upsilon_\tau(Z_s)\right) \sigma(Z_s)\, \mathrm{d}r_0 \right| \lesssim |t-s|^\alpha.$$

This is easy, if we assume $\nabla^2\Upsilon_\tau$ and $\sigma = \Upsilon_\bullet$ bounded, but the game is to assume only that Υ_τ is Lipschitz.

Here is another idea:

$$\begin{aligned}\int_0^1 \nabla\Upsilon_\tau(u_0)\, \sigma(Z_s)\, \mathrm{d}r_0 &= \int_0^1 \nabla\Upsilon_\tau(u_0)\, \sigma(u_0)\, \mathrm{d}r_0 + \int_0^1 \nabla\Upsilon_\tau(u_0)\, (\sigma(Z_s) - \sigma(u_0))\, \mathrm{d}r_0 \\ &= \int_0^1 \Upsilon_{\bullet\curvearrowright\tau}(u_0)\, \mathrm{d}r_0 - \int_0^1 \nabla\Upsilon_\tau(u_0) \int_0^{r_0} \nabla\Upsilon_\bullet(u_1)\, Z_{st}^{[1]}\, \mathrm{d}r_1\, \mathrm{d}r_0\end{aligned}$$

with $u_i := (1-r_i)Z_s + r_iZ_t$, $i = 0, 1$.

Towards another representation

So let us restart:

$$B^{[2]}_{st}(\tau) = \Upsilon_\tau(Z_t) - \Upsilon_\tau(Z_s) - \Upsilon_{\bullet\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\bullet) =$$

$$= \int_0^1 \nabla\Upsilon_\tau(u_0)\,\mathrm{d}r_0\,Z^{[1]}_{st} - \Upsilon_{\bullet\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\bullet)$$

$$= \int_0^1 \nabla\Upsilon_\tau(u_0)\,\mathrm{d}r_0\,Z^{[2]}_{st} + \left[\int_0^1 \nabla\Upsilon_\tau(u_0)\,\Upsilon_\bullet(Z_s)\,\mathrm{d}r_0 - \Upsilon_{\bullet\curvearrowright\tau}(Z_s)\right]\mathbb{X}_{st}(\bullet)$$

$$= \int_0^1 \nabla\Upsilon_\tau(u_0)\,\mathrm{d}r_0\,Z^{[2]}_{st} +$$

$$+ \left[\int_0^1 \left[(\Upsilon_{\bullet\curvearrowright\tau}(u_0) - \Upsilon_{\bullet\curvearrowright\tau}(Z_s)) - \nabla\Upsilon_\tau(u_0)\,(\Upsilon_\bullet(u_0) - \Upsilon_\bullet(Z_s))\right]\mathrm{d}r_0\right]\mathbb{X}_{st}(\bullet)$$

Towards another representation

We conclude introducing $[0,1]^n_{\leq} := \{(r_0,\dots,r_{n-1}) \in \mathbb{R}^n : 0 \leq r_0 \leq \cdots \leq r_{n-1} \leq 1\}$:

$$B^{[2]}_{st}(\tau) = \int_0^1 \nabla\Upsilon_\tau(u_0)\,\mathrm{d}r_0\, Z^{[2]}_{st} +$$
$$+ \left[\int_{[0,1]^2_{\leq}} [\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_\tau(u_0)\nabla\Upsilon_\bullet(u_1)]\,\mathrm{d}r_0\,\mathrm{d}r_1\right] Z^{[1]}_{st}\,\mathbb{X}_{st}(\bullet).$$

Bingo!

Indeed we have by assumption:

$$\left|Z^{[2]}_{st}\right| + \left|Z^{[1]}_{st}\,\mathbb{X}_{st}(\bullet)\right| \lesssim |t-s|^{2\alpha}$$

and all $\nabla\Upsilon_\tau, \nabla\Upsilon_{\bullet\curvearrowright\tau}, \nabla\Upsilon_\tau, \nabla\Upsilon_\bullet$ bounded.

The next step

$$
\begin{aligned}
B^{[3]}_{st}(\tau) &= B^{[2]}_{st}(\tau) - \Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\overset{\bullet}{\underset{\bullet}{|}}) - \frac{1}{2}\Upsilon_{(\bullet\bullet)\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\bullet\bullet) \\
&= \int_0^1 \nabla\Upsilon_\tau(u_0)\,\mathrm{d}r_0\, Z^{[2]}_{st} + \\
&+ \left[\int_{[0,1]^2_{\le}} [\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_\tau(u_0)\nabla\Upsilon_\bullet(u_1)]\,\mathrm{d}r_0\,\mathrm{d}r_1\right] Z^{[1]}_{st}\,\mathbb{X}_{st}(\bullet) \\
&- \Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\overset{\bullet}{\underset{\bullet}{|}}) - \frac{1}{2}\Upsilon_{(\bullet\bullet)\curvearrowright\tau}(Z_s)\,\mathbb{X}_{st}(\bullet\bullet).
\end{aligned}
$$

Now, $Z^{[i]}_{st}$ must be upgraded to $Z^{[i+1]}_{st}$ and we get additional terms.

The next step

$$
\begin{aligned}
B_{st}^{[3]}(\tau) = & \int \nabla\Upsilon_\tau(u_0)\, Z_{st}^{[3]}\, \mathrm{d}r_0 + \int \left[\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_\tau(u_0)\nabla\Upsilon_\bullet(u_1)\right] Z_{st}^{[2]}\, \mathbb{X}(\bullet)\, \mathrm{d}r_0\, \mathrm{d}r_1 \\
& + \int \nabla\Upsilon_\tau(u_0)\, \Upsilon_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}}(Z_s)\, \mathbb{X}\big(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\big)\, \mathrm{d}r_0 \\
& + \int \left[\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_\tau(u_0)\nabla\Upsilon_\bullet(u_1)\right]\Upsilon_\bullet(Z_s)\, \mathbb{X}(\bullet)\, \mathbb{X}(\bullet)\, \mathrm{d}r_0\, \mathrm{d}r_1 \\
& - \int \Upsilon_{\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\curvearrowright\tau}(Z_s)\, \mathbb{X}\big(\begin{smallmatrix}\bullet\\|\\\bullet\end{smallmatrix}\big)\, \mathrm{d}r_0 - \int \Upsilon_{(\bullet\bullet)\curvearrowright\tau}(Z_s)\, \mathbb{X}(\bullet\bullet)\, \mathrm{d}r_0\, \mathrm{d}r_1 .
\end{aligned}
$$

The next step

After more rearranging, this can be massaged into

$$
\begin{aligned}
B^{[3]}_{st}(\tau) = {} & \int \nabla\Upsilon_{\tau}(u_0)\, Z^{[3]}_{st}\, \mathrm{d}r_0 + \int \left[\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla \bullet\, (u_1)\right] Z^{[2]}_{st}\, \mathbb{X}(\bullet)\, \mathrm{d}r_0\, \mathrm{d}r_1 \\
& + \int \left[\nabla\Upsilon_{\substack{\bullet\\|\\\bullet}\curvearrowright\tau}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\substack{\bullet\\|\\\bullet}}(u_1)\right] Z^{[1]}_{st}\, \mathbb{X}\big(\substack{\bullet\\|\\\bullet}\big)\, \mathrm{d}r_0\, \mathrm{d}r_1 \\
& + \int [\nabla\Upsilon_{(\bullet\curvearrowright(\bullet\curvearrowright\tau))}(u_2) - \nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\tau}(u_2) - \nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\tau}(u_1) \\
& \quad - \nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1)\nabla \bullet\, (u_2) + \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet\curvearrowright\bullet}(u_1) + \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\nabla\Upsilon_{\bullet}(u_2)] \\
& \quad \mathbb{X}(\bullet\bullet)\, Z^{[1]}_{st}\, \mathrm{d}r_0\, \mathrm{d}r_1\, \mathrm{d}r_2 .
\end{aligned}
$$

Note the term $\nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\tau}$ which is evaluated in two points.

The next step

$B^{[4]}_{st}(\tau)$ is equal to

$$
\begin{aligned}
&\int \nabla\Upsilon_{\tau}(u_0)\, Z^{[4]}_{st}\, dr_0 + \int \left[\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\right] Z^{[3]}_{st}\, \mathbb{X}(\bullet)\, dr_0\, dr_1 \\
&+ \int \left[\nabla\Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\tau}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla\overset{\bullet}{\underset{\bullet}{|}}(u_1)\right] Z^{[2]}_{st}\, \mathbb{X}(\overset{\bullet}{\underset{\bullet}{|}})\, dr_0\, dr_1 \\
&+ \int \left[\nabla\Upsilon_{\bullet\curvearrowright(\bullet\curvearrowright\tau)}(u_2) - \nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\tau}(u_1) + \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\nabla\Upsilon_{\bullet}(u_2)\right] Z^{[2]}_{st}\, \mathbb{X}(\bullet\bullet)\, dr_0\, dr_1\, dr_2 \\
&+ \sum_{\tau\in\mathcal{T},\ |\tau|=3} \int \left[\nabla\Upsilon_{\tau\curvearrowright\tau}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\right] Z^{[1]}_{st}\, \mathbb{X}(\tau)\, dr_0 \\
&+ \int [\nabla\Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright(\bullet\curvearrowright\tau)}(u_2) + \nabla\Upsilon_{\bullet\curvearrowright(\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\tau)}(u_2) - \nabla\Upsilon_{(\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\bullet)\curvearrowright\tau}(u_1) - \nabla\Upsilon_{(\bullet\curvearrowright\overset{\bullet}{\underset{\bullet}{|}})\curvearrowright\tau}(u_1) \\
&\quad -\nabla\Upsilon_{\bullet\curvearrowright\tau}(u_1)\nabla\overset{\bullet}{\underset{\bullet}{|}}(u_2) - \nabla\Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\tau}(u_1)\nabla\Upsilon_{\bullet}(u_2) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}\curvearrowright\bullet}(u_1) + \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet\curvearrowright\overset{\bullet}{\underset{\bullet}{|}}}(u_1) \\
&\quad +\nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\nabla\Upsilon_{\overset{\bullet}{\underset{\bullet}{|}}}(u_2) + \nabla\Upsilon_{\tau}(u_0)\nabla\overset{\bullet}{\underset{\bullet}{|}}(u_1)\nabla\Upsilon_{\bullet}(u_2)]\, Z^{[1]}_{st}\, \mathbb{X}(\overset{\bullet}{\underset{\bullet}{|}}\,\bullet)\, dr_0\, dr_1\, dr_2 \\
&+ \int [\nabla\Upsilon_{\bullet\curvearrowright(\bullet\curvearrowright(\bullet\curvearrowright\tau))}(u_3) - \nabla\Upsilon_{\bullet\curvearrowright(\bullet\curvearrowright\bullet)\curvearrowright\tau))}(u_2) + \nabla\Upsilon_{((\bullet\curvearrowright\bullet)\curvearrowright\bullet)\curvearrowright\tau}(u_1) \\
&\quad -\nabla\Upsilon_{\bullet\curvearrowright(\bullet\curvearrowright\tau)}(u_2)\nabla\bullet(u_3) + \nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\tau}(u_1)\nabla\Upsilon_{\bullet}(u_2) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{(\bullet\curvearrowright\bullet)\curvearrowright\bullet}(u_1) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet\curvearrowright(\bullet\curvearrowright\bullet)}(u_1) \\
&\quad -\nabla\Upsilon_{\tau}(u_0)\nabla\bullet(u_1)\nabla\Upsilon_{\bullet\curvearrowright\bullet}(u_2) - \nabla\Upsilon_{\tau}(u_0)\nabla\Upsilon_{\bullet}(u_1)\nabla\Upsilon_{\bullet}(u_2)\nabla\Upsilon_{\bullet}(u_3)]\, Z^{[1]}_{st}\, \mathbb{X}(\bullet\ \bullet\ \bullet)\, dr_0\, dr_1\, dr_2\, dr_3
\end{aligned}
$$

Algebraic formula for $B^{[n+1]}(\tau)$

The sought-after Taylor formula for the remainders reads:

$$B^{[n+1]}_{st}(\tau) = \sum_{k=0}^{n} \sum_{w=\tau_k\cdot\ldots\cdot\tau_1\in\mathscr{F}^{\leq n}} H_{st}(\tau_k,\ldots,\tau_1;\tau)\, \mathbb{X}_{st}(w)\, Z^{[n+1-|w|]}_{st}$$

where

$$H_{st}(\tau_k\ldots\tau_1;\tau) := \sum_{p\in\mathscr{P}(I_k)} \sum_{\rho\in\mathscr{T}_b^{p(I_k)}} \Bigg[\sum_{j_1} J_{\rho,1}(j_1) \int_{[0,1]^{k+1}_{\leq}} (-1)^{k-h_\rho}\, \nabla\Upsilon_{G(\rho)}(u_{j_1})\, \mathbf{dt}_k$$

$$+ \sum_{\ell=2}^{m(\rho)+1} \sum_{(j_1,\ldots,j_\ell)} J_{\rho,\ell}(j_1,\ldots,j_\ell) \int_{[0,1]^{k+1}_{\leq}} (-1)^{k+1-h_\rho} \left(\prod_{r=1}^{\ell-1} \nabla\Upsilon_{G(\rho_r)}(u_{j_r})\right) \nabla\Upsilon_{G(P_{\rho,\ell})}(u_{j_\ell})\, \mathbf{dt}_k\Bigg]$$

Bingo!

SPDEs

- We are working on the same problem for SPDEs.
- Here instead of rough paths we have models of regularity structures.
- A priori estimates for such equations form a very active field (F. Otto, H. Weber).
- The algebraic structure is similar but more complex: the grafting operation $\curvearrowright$ of trees is still crucial but it is post-Lie rather than pre-Lie (Y. Bruned, J.-D. Jacques).