General inverses for matrices

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Sommaire

Notations

1 Introduction

1.1 Inverse of a non-singular matrix

It is well known that a non-singular matrix A has a unique inverse denoted by A^{-1} with:

$$AA^{-1} = A^{-1}A = I,$$

There are many properties such as

$$(A^{-1})^{-1} = A,$$
$$(A^T)^{-1} = (A^{-1})^T,$$
$$(A^*)^{-1} = (A^{-1})^*,$$
$$(AB)^{-1} = B^{-1}A^{-1}.$$

1.2 Inverse for singular matrix

We know that a non-singular matrix must be square, so for a general matrix A we need to associate a matrix X which must have to respect :

- exists for a class of matrices larger than the class of nonsingular matrices
- has some of the properties of the usual inverse
- reduces to the usual inverse when A is nonsingular.

A example of generalized inverse for a matrix A is a matrix X which verifies $AXA = A$. If A is non-singular then we see that $X = A^{-1}$

1.3 Illustration with linear system

One of the most common applications of matrices is solving systems of linear equations. Consider the system

$$Ax = b, \tag{1.1}$$

where b is a known vector and x is the unknown. If A is nonsingular, the system has a unique solution:

$$x = A^{-1}b.$$

In the more general case, where A may be singular or non-square, the system may have no solution or infinitely many.
A solution x exists if and only if b lies in the column space of A—that is, if b is a linear combination of the columns of A. If $A \in \mathbb{R}^{m\times n}$ and has rank less than m, this may not be true. When it is, there exists some vector h such that

$$b = Ah.$$

Now, suppose X is a matrix satisfying

$$AXA = A. \tag{1.2}$$

If we define

$$x = Xb,$$

then

$$Ax = AXb = AXAh = Ah = b,$$

so this x solves the system.
More generally, when the system may admit multiple solutions, we often seek a complete description of the solution set. It has been shown that if X satisfies $AXA = A$, then $Ax = b$ has a solution if and only if

$$AXb = b,$$

in which case the general solution is:

$$x = Xb + (I - XA)y,$$

where y is an arbitrary vector.

2 Preliminaries

First we will define general algebric definitions and properties concerning matrices.

2.1 Definitions, properties

Definition 1.
The matrix $A \in \mathbb{F}^{m\times n}$ is:

- ***diagonal*** *if $A[i,j] = 0$ for all $i \neq j$,*
- ***upper triangular*** *if $A[i,j] = 0$ for all $i > j$,*
- ***lower triangular*** *if $A[i,j] = 0$ for all $i < j$.*

An $m \times n$ diagonal matrix $A = [a_{ij}]$ is denoted as:

$$A = \operatorname{diag}(a_{11}, a_{22}, \dots, a_{pp}),$$

where $p = \min\{m, n\}$.

Definition 2. ***Special Classes of Square Matrices.***
A square matrix $A \in \mathbb{C}^{n\times n}$ is said to be:

- ***Hermitian*** *if $A = A^*$,*
- ***Symmetric*** *if A is real and $A = A^\top$,*
- ***Normal*** *if $AA^* = A^*A$,*
- ***Unitary*** *if $A^* = A^{-1}$,*
- ***Orthogonal*** *if A is real and $A^\top = A^{-1}$.*

Definition 3. ***Range and Null Space.***
For any matrix $A \in \mathbb{C}^{m\times n}$, we define:

$$\mathcal{R}(A) = \{y \in \mathbb{C}^m \ : \ y = Ax \text{ for some } x \in \mathbb{C}^n\}\,, \tag{2.1}$$
$$\mathcal{N}(A) = \{x \in \mathbb{C}^n \ : \ Ax = 0\}\,, \tag{2.2}$$

where:

- *$\mathcal{R}(A)$ is called the **range** (or* column space*) of A,*
- *$\mathcal{N}(A)$ is called the **null space** (or* kernel*) of A.*

Theorem 1. *For any matrix $A \in \mathbb{C}^{m\times n}$, we have*

$$\mathcal{N}(A) = \mathcal{R}(A^*)^\perp, \quad \text{and} \quad \mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \tag{2.3}$$

Definition 4. ***Eigenvalue and Eigenspace***
Let $A \in \mathbb{C}^{n\times n}$, $\mathbf{0} \neq \mathbf{x} \in \mathbb{C}^n$, and $\lambda \in \mathbb{C}$ satisfy

$$A\mathbf{x} = \lambda\mathbf{x}. \tag{2.4}$$

Then λ is called an eigenvalue *of A corresponding to the* eigenvector $\mathbf{x}$.
The set of all eigenvalues of A is called the spectrum *of A, and is denoted by $\lambda(A)$.*
If λ is an eigenvalue of A, the subspace

$$\{\mathbf{x} \in \mathbb{C}^n : A\mathbf{x} = \lambda\mathbf{x}\} \tag{2.5}$$

is called the eigenspace *of A associated with λ. Its dimension is called the* geometric multiplicity *of the eigenvalue λ.*

We admit the next proposition to leave space for the rest.

Proposition 1. *Let $H \in \mathbb{C}^{n\times n}$ be a Hermitian matrix. Then:*

(a) The eigenvalues of H are real.

(b) Eigenvectors corresponding to distinct eigenvalues are orthogonal.

(c) There exists an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of H.

(d) The eigenvalues of H, ordered as

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

and corresponding eigenvectors x_j satisfying

$$Hx_j = \lambda_j x_j, \quad j = 1, \ldots, n,$$

can be computed recursively by the Courant-Fischer *characterization:*

$$\lambda_1 = \max_{\|x\|=1} \langle Hx, x\rangle = \langle Hx_1, x_1\rangle,$$

$$\lambda_j = \max_{\substack{\|x\|=1 \\ x\perp\{x_1,\ldots,x_{j-1}\}}} \langle Hx, x\rangle = \langle Hx_j, x_j\rangle, \quad j = 2, \ldots, n.$$

Definition 5. ***Positive Semidefinite and Positive Definite Hermitian Matrices.*** *A Hermitian matrix $H \in \mathbb{C}^{n\times n}$ is called*

- ***positive semidefinite*** *(PSD) if*

$$\langle Hx, x\rangle \geq 0 \quad \text{for all } x \in \mathbb{C}^n,$$

or equivalently, if all eigenvalues of H are nonnegative. The set of all $n\times n$ PSD matrices is denoted by PSD_n.

- ***positive definite*** *(PD) if*

$$\langle Hx, x\rangle > 0 \quad \text{for all } x \in \mathbb{C}^n,\ x \neq 0,$$

or equivalently, if all eigenvalues of H are positive. The set of all $n \times n$ PD matrices is denoted by PD_n.

The next result/definition is deeply rooted to the construction of the Moore-Penrose inverse that we shall see later.

Definition 6. *Singular values*
Let $A \in \mathbb{C}^{m\times n}$ and let the eigenvalues $\lambda_j(AA^*)$ of the positive semidefinite matrix AA^* be ordered as

$$\lambda_1(AA^*) \geq \lambda_2(AA^*) \geq \cdots \geq \lambda_r(AA^*) > \lambda_{r+1}(AA^*) = \cdots = \lambda_n(AA^*) = 0, \tag{2.6}$$

where $r = \operatorname{rank}(A)$.
The singular values *of A, denoted by $\sigma_j(A)$ or simply σ_j, for $j = 1, \dots, r$, are defined by*

$$\sigma_j(A) = \sqrt{\lambda_j(AA^*)}, \quad j = 1, \dots, r, \tag{2.7a}$$

$$\text{or equivalently,} \quad \sigma_j(A) = \sqrt{\lambda_j(A^*A)}, \quad j = 1, \dots, r, \tag{2.7b}$$

and are ordered as

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0. \tag{2.8}$$

The set of singular values of A is denoted by

$$\sigma(A) = \{\sigma_1, \sigma_2, \dots, \sigma_r\}.$$

Proposition 2. *Let A and $\{\sigma_j\}$ be as above. Let $\{u_i : i = 1, \dots, m\}$ be an orthonormal basis of $\mathbb{C}^m$ consisting of eigenvectors of AA^*, such that*

$$AA^*u_i = \sigma_i^2 u_i, \quad i = 1, \dots, r, \tag{2.9a}$$

$$AA^*u_i = 0, \quad i = r+1, \dots, m. \tag{2.9b}$$

Define

$$v_j = \frac{1}{\sigma_j} A^* u_j, \quad j = 1, \dots, r, \tag{2.10}$$

and let $\{v_j : j = r+1, \dots, n\}$ be an orthonormal set of vectors orthogonal to $\{v_j : j = 1, \dots, r\}$. Then the set $\{v_j : j = 1, \dots, n\}$ is an orthonormal basis of $\mathbb{C}^n$ consisting of eigenvectors of A^*A, satisfying

$$A^*Av_j = \sigma_j v_j, \quad j = 1, \dots, r, \tag{2.11a}$$

$$A^*Av_j = 0, \quad j = r+1, \dots, n. \tag{2.11b}$$

Conversely, starting from an orthonormal basis $\{v_j : j = 1, \dots, n\}$ of $\mathbb{C}^n$ satisfying (13a)–(13b), we can construct an orthonormal basis $\{u_i : i = 1, \dots, m\}$ of $\mathbb{C}^m$ satisfying (11a)–(11b) by

$$u_i = \frac{1}{\sigma_i} A v_i, \quad i = 1, \dots, r, \tag{2.12}$$

and completing it to an orthonormal set.

Proposition 3. ***SVD decomposition***
Let A, $\{u_i : i = 1, \dots, m\}$, and $\{v_j : j = 1, \dots, n\}$ be as above. Then equation (14) can be written in matrix form as

$$A \begin{bmatrix} v_1 & \cdots & v_r & v_{r+1} & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1 & \cdots & u_r & u_{r+1} & \cdots & u_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \mathbf{0} \\ & & \sigma_r & & \\ & \mathbf{0} & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix},$$

Let

$$U = \begin{bmatrix} u_1 & \cdots & u_m \end{bmatrix}, \quad V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$

Since V is unitary, we have the singular value decomposition *(SVD) of A:*

$$A = U\Sigma V^*, \tag{2.13}$$

where Σ is the $m \times n$ diagonal matrix with singular values $\sigma_1, \dots, \sigma_r$ on the diagonal.

The next decompositions are useful to construct others generalized inverses that's why we are introducing them now.

Proposition 4. ***The QR factorization.***
Let the orthonormal set $\{q_1, \ldots, q_r\}$ *be obtained from the set of vectors* $\{a_1, \ldots, a_n\}$ *by the Gram–Schmidt orthogonalization (GSO) process. Let*

$$Q_r = \begin{bmatrix} q_1 & \cdots & q_r \end{bmatrix} \in \mathbb{C}^{m \times r} \quad \text{and} \quad A = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} \in \mathbb{C}^{m \times n}$$

be the corresponding matrices.
Then,

$$A = \tilde{Q}\tilde{R}, \tag{2.14}$$

where the columns of $\tilde{Q}$ *form an orthonormal basis of* $\mathcal{R}(A)$ *(the range of* A*) and* $\tilde{R} \in \mathbb{C}^{r \times n}$ *is an upper triangular matrix.*
If $r < m$*, it is possible to complete* $\tilde{Q}$ *to a unitary matrix*

$$Q = \begin{bmatrix} \tilde{Q} & Z \end{bmatrix},$$

where the columns of Z *form an orthonormal basis of* $\mathcal{N}(A^*)$ *(the null space of* A^**).*
Then equation (2.14) *can be written as*

$$A = QR, \tag{2.15}$$

where

$$R = \begin{bmatrix} \tilde{R} \\ 0 \end{bmatrix}$$

is an upper triangular matrix.
The expression (2.15) *is called the* QR-factorization *of* A*. By analogy, we call* (2.14) *a* $\tilde{Q}\tilde{R}$*-factorization of* A*.*

Proposition 5. ***Schur Triangularization.***
Any matrix $A \in \mathbb{C}^{n \times n}$ *is unitarily similar to an upper triangular matrix; that is, there exists a unitary matrix* $U \in \mathbb{C}^{n \times n}$ *such that*

$$U^* A U = T,$$

where T *is upper triangular.*

Proof. It is well known that every complex matrix is triangularizable, so $\exists P \in \mathcal{GL}(\mathbb{C})$ and a upper triangular matrix T such as $A = PTP^{-1}$. Thanks to proposition 4 and the fact that P is invertible, $\exists Q$ unitary and R upper triangular such as $P = QR$. We now define $U = QRSR^{-1}Q^*$ and $T = RSR^{-1}$.

2.2 Hermite form and related items

The Hermite form is a very powerful way to represent matrix, and different elements can be easily seen thanks to this form such as its kernel or its range.

Definition 7. ***Hermite form*** *A matrix $H \in \mathbb{C}_r^{m\times n}$ is said to be in* ***Hermite normal form*** *(also called reduced row-echelon form) if:*

(a) The first r rows contain at least one nonzero element, while the remaining rows consist entirely of zeros;

(b) There exist r integers
$$1 \leq c_1 < c_2 < \cdots < c_r \leq n,$$
such that the first nonzero element in row $i \in \{1, \ldots, r\}$ appears in column c_i;

(c) All other elements in column c_i are zero, for each $i \in \{1, \ldots, r\}$.

By an appropriate permutation of its columns, a matrix $H \in \mathbb{C}_r^{m\times n}$ in Hermite normal form can be brought into block partition form

$$H = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix},$$

where I_r is the $r \times r$ identity matrix, K is some matrix.

Proposition 6. *Let $A \in \mathbb{C}^{m\times n}$, and let $E_k, E_{k-1}, \ldots, E_2, E_1$ be elementary row operations. Also, let P be a permutation matrix such that*

$$EAP = \begin{bmatrix} I_r & K \\ O & O \end{bmatrix},$$

where
$$E = E_k E_{k-1} \cdots E_2 E_1,$$

and r is the rank of A.
Then we have
$$A = E^{-1} \begin{bmatrix} I_r & K \\ O & O \end{bmatrix} P^{-1}.$$

A basis for the range of A, denoted $\mathcal{R}(A)$, consists of the c_1st, c_2nd, ..., c_rth columns of A, where the indices $\{c_j : j \in \{1, \ldots, r\}\}$ are as defined in the Hermite normal form.
To see this, let P_1 be the submatrix consisting of the first r columns of the permutation matrix P used in the decomposition
$$EAP = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix}.$$

Then, we have:
$$EAP_1 = \begin{bmatrix} I_r \\ 0 \end{bmatrix}.$$

Proposition 7.
Hence, $AP_1 \in \mathbb{C}^{m\times r}$ is a rank-r matrix whose columns form a basis for $\mathcal{R}(A)$. Since AP_1 consists of the c_1st, c_2nd, ..., c_rth columns of A, these columns form a basis for the range of A.
Moreover, the columns of the $n \times (n-r)$ matrix

$$P \begin{bmatrix} -K \\ I_{n-r} \end{bmatrix}$$

form a basis for the null space $\mathcal{N}(A)$.

Lemma 1. *Full-rank factorization*
Let $A \in \mathbb{C}^{m\times n}$ be a matrix of rank $r > 0$. Then there exist matrices $F \in \mathbb{C}_r^{m\times r}$ and $G \in \mathbb{C}_r^{r\times n}$ such that

$$A = FG.$$

Proof. We use the $\tilde{Q}\tilde{R}$ factorization Let F be any matrix whose columns form a basis for $\mathcal{R}(A)$. Then $F \in \mathbb{C}^{m\times r}$. The matrix $G \in \mathbb{C}^{r\times n}$ is then uniquely determined by $A = FG$, since every column of A can be uniquely represented as a linear combination of the columns of F.
Finally, $\operatorname{rank}(G) = r$, since

$$\operatorname{rank}(G) \;\geq\; \operatorname{rank}(FG) \;=\; r.$$

The columns of F can, in particular, be chosen as any maximal linearly independent set of columns of A.

2.3 Jordan Form

Let the matrices $A \in \mathbb{C}^{n\times n}$, $X \in \mathbb{C}^{n\times k}$, and the scalar $\lambda \in \mathbb{C}$ satisfy

$$AX = XJ_k(\lambda),$$

where

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & \cdots & \cdots & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{k\times k}.$$

Writing X by its columns, $X = [x_1\ x_2\ \cdots\ x_k]$, the equation becomes:

$$Ax_1 = \lambda x_1, \tag{2.16}$$
$$Ax_j = \lambda x_j + x_{j-1}, \quad j = 2, \ldots, k. \tag{2.17}$$

It follows, for $j = 1, \ldots, k$, that:

$$(A - \lambda I)^j x_j = 0, \quad (A - \lambda I)^{j-1} x_j = x_1 \neq 0,$$

where we interpret $(A - \lambda I)^0$ as the identity matrix I.
The vector x_1 is therefore an eigenvector of A corresponding to the eigenvalue λ. We call x_j a *λ-vector of grade j* or, following Wilkinson, a *principal vector* of A of grade j associated with the

eigenvalue λ. Evidently, principal vectors generalize eigenvectors: an eigenvector is a principal vector of grade 1.
The matrix $J_k(\lambda)$ is called a $k \times k$ *Jordan block* corresponding to the eigenvalue λ.

Theorem 2. ***Jordan decomposition***
Any matrix $A \in \mathbb{C}^{n\times n}$ is similar to a block diagonal matrix J with Jordan blocks on its diagonal. That is, there exists a nonsingular matrix P such that

$$P^{-1}AP = J = \begin{bmatrix} J_{k_1}(\lambda_1) & 0 & \cdots & 0 \\ 0 & J_{k_2}(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{k_p}(\lambda_p) \end{bmatrix},$$

where each $J_{k_j}(\lambda_j)$ is a Jordan block of size $k_j \times k_j$ corresponding to the eigenvalue λ_j, for $j = 1, \ldots, p$.
The matrix J is unique up to a permutation of its Jordan blocks.

Proposition 8. *Define A as above, then If an eigenvalue λ_i appears in q Jordan blocks*

$$J_{j_1}(\lambda_i),\ J_{j_2}(\lambda_i),\ \ldots,\ J_{j_q}(\lambda_i),$$

with sizes ordered as $j_1 \geq j_2 \geq \cdots \geq j_q$, then the characteristic polynomial of A is the product

$$\chi(x) = c_1(x)\, c_2(x) \cdots c_p(x),$$

where

$$c_i(x) = (x - \lambda_i)^{a_i}, \quad a_i = j_1 + j_2 + \cdots + j_q.$$

*The exponent a_i is called the **algebraic multiplicity** of the eigenvalue λ_i. It equals the sum of the sizes of all Jordan blocks associated with λ_i. The size of the largest Jordan block j_1 is called the **index** of the eigenvalue λ_i, and is denoted by $\nu(\lambda_i)$.*
We can notice the the polynomial

$$m(x) = m_1(x)\, m_2(x) \cdots m_p(x)$$

*satisfies $m(A) = 0$, and is of the smallest possible degree among all such polynomials. It is called the **minimal polynomial** of A.*

2.4 The Smith normal form

Theorem 3. ***The Smith normal form*** *Let $A \in \mathbb{Z}^{m\times n}$ be an integer matrix of rank r. Then A is equivalent over $\mathbb{Z}$ to a matrix $S = [s_{ij}] \in \mathbb{Z}^{m\times n}$ such that:*

(a) $s_{ii} \neq 0$ *for all* $i \in \{1, \ldots, r\}$*;*

(b) $s_{ij} = 0$ *for all* $i \neq j$*;*

(c) $s_{ii} \mid s_{i+1,i+1}$ *for all* $i \in \{1, \ldots, r-1\}$.

Remark. *The matrix S is called the* ***Smith normal form*** *of A, and the nonzero diagonal entries s_{ii}, for $i = 1, \ldots, r$, are called the* ***invariant factors*** *of A.*

2.5 Nonnegative Matrices

Definition 8. *A matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is defined as:*

(a) ***Nonnegative*** *if all entries satisfy $a_{ij} \geq 0$;*

(b) ***Reducible*** *if there exists a permutation matrix Q such that:*

$$Q^\top AQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix},$$

where A_{11} and A_{22} are square submatrices. Otherwise, A is called ***irreducible****.*

Theorem 4. ***(The Perron–Frobenius Theorem)***
Let $A \in \mathbb{R}^{n\times n}$ be a nonnegative and irreducible matrix. Then:

(a) A has a positive real eigenvalue ρ, which is equal to the spectral radius of A, i.e., $\rho = \rho(A)$.

(b) The eigenvalue ρ has algebraic multiplicity 1.

(c) There exists a corresponding eigenvector with strictly positive entries, i.e., a positive eigenvector associated with ρ.

3 Existence and construction of generalized inverses

3.1 The Penrose equations

In 1955, Penrose showed that for every finite matrix A (square or rectangular) with real or complex entries, there exists a unique matrix X satisfying the four equations, known as the *Penrose equations*:

$$AXA = A, \tag{1}$$
$$XAX = X, \tag{2}$$
$$(AX)^* = AX, \tag{3}$$
$$(XA)^* = XA, \tag{4}$$

Because this unique generalized inverse had previously been studied (though defined differently) by E. H. Moore, it is commonly known as the *Moore–Penrose inverse* and is often denoted by $A^\dagger$.

Definition 9. *For any $A \in \mathbb{C}^{m\times n}$, let $A_{\{i,j,\dots,k\}}$ denote the set of matrices $X \in \mathbb{C}^{n\times m}$ that satisfy equations (i), (j), …, (k) from among equations (1)–(4). A matrix $X \in A_{\{i,j,\dots,k\}}$ is called an $\{i,j,\dots,k\}$-inverse of A, and is also denoted by $A^{(i,j,\dots,k)}$.*

3.2 Construction of {1}-inverse

Theorem 5. *Let $A \in \mathbb{C}_r^{m\times n}$ and let $E \in \mathbb{C}^{m\times m}$ and $P \in \mathbb{C}^{n\times n}$ be such that*

$$EAP = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix}. \tag{0.72}$$

Then, for any $L \in \mathbb{C}^{(n-r)\times(m-r)}$, the $n \times m$ matrix

$$X = P \begin{bmatrix} I_r & 0 \\ 0 & L \end{bmatrix} E \tag{5}$$

is a $\{1\}$-inverse of A. The partitioned matrices and (5) must be interpreted suitably in the cases $r = m$ or $r = n$.

We note that since P and E are both nonsingular, the rank of X is the rank of the partitioned matrix on the right-hand side of (5). In view of the form of that matrix,

$$\operatorname{rank} X = r + \operatorname{rank} L. \tag{6}$$

Since L is arbitrary, it follows that a $\{1\}$-inverse of A exists that has any rank between r and $\min\{m,n\}$, inclusive.
For example, define A as :

3.3 Properties of {1}-inverses

For any scalar λ, we define $\lambda^\dagger$ by

$$\lambda^\dagger = \begin{cases} \lambda^{-1}, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$$

Lemma 2. *Let $A \in \mathbb{C}_r^{m\times n}$ and $\lambda \in \mathbb{C}$. Then:*

(a) $(A^{(1)})^* \in A^*\{1\}$.

(b) If A is nonsingular, then $A^{(1)} = A^{-1}$ uniquely

(c) $\lambda^\dagger A^{(1)} \in (\lambda A)_{\{1\}}$.

(d) $\text{rank}(A^{(1)}) \geq \text{rank}(A)$.

(e) If S and T are nonsingular, then $T^{-1}A^{(1)}S^{-1} \in (SAT)\{1\}$.

(f) $AA^{(1)}$ and $A^{(1)}A$ are idempotent and have the same rank as A.

Proof. Verify the equations

Lemma 3. *Let $A \in \mathbb{C}_r^{m\times n}$. Then:*

(a) $A^{(1)}A = I_n$ if and only if $r = n$.

(b) $AA^{(1)} = I_m$ if and only if $r = m$.

Proof. We will prove a) since b) is similarly proved.

($\Longrightarrow$) If $A^{(1)}A = I_n$ then $rank(A^{(1)}A) = rank(I_n) = n$ and since $rank(A^{(1)}A) = rank(A)$ with lemma 2, we have $n = r$.
($\Longleftarrow$) If $r = n$ then with lemma 2 $A^{(1)}A$ is a idempotent matrix and is inversible since it is a n-square matrix of rank n.

Proposition 9. *Let $A = FHG$ where F is full column rank and G is full column rank. Then $rankA = rankH$*

Proof. Thanks to Lemma 3, if $A = FGH$ multiplying by $F^{(1)}$ we have $F^{(1)}F = I_n$ so we have $F^{(1)}A = HG$. Similarly multipliying by $G^{(1)}$ and using Lemma 3 gives $F^{(1)}AG^{(1)} = H$. Now the rank of a product never exceeds the rank of each matrices we have $rank(A) = rank(H)$.

3.4 Construction of {1,2}-inverses

Lemma 4. *Let $Y, Z \in A\{1\}$, and let*

$$X = YAZ.$$

Then $X \in A\{1, 2\}$.

Proof. Verify its a (2)-inverse.
So we can notice that we have an easy way to construct $\{1, 2\}$-inverses using only $\{1\}$-inverses. Since the matrices A and X occur symmetrically in (1) and (2), $X \in A\{1, 2\}$ and $A \in X\{1, 2\}$ are equivalent statements, and in either case we can say that A and X are $\{1, 2\}$ inverses of each other.

Theorem 6. ***(Bjerhammar)***. *Given A and $X \in A^{\{1\}}$,*

$$X \in A\{1,2\} \quad \textit{if and only if} \quad \mathrm{rank}(X) = \mathrm{rank}(A).$$

Proof.
If: Clearly, $\mathcal{R}(XA) \subseteq \mathcal{R}(X)$. But $\mathrm{rank}(XA) = \mathrm{rank}(A)$ by Lemma 2(f). So, if $\mathrm{rank}(X) = \mathrm{rank}(A)$, then $\mathcal{R}(XA) = \mathcal{R}(X)$. Thus,

$$XAY = X$$

for some matrix Y. Premultiplying both sides by A,

$$AX = AXAY = AY,$$

and therefore,

$$XAX = X.$$

Only if: This follows immediately from equations (1) and (2).

Corollary 1. *Any two of the following three statements imply the third:*

1. $X \in A\{1\}$,
2. $X \in A\{2\}$,
3. $\mathrm{rank}(X) = \mathrm{rank}(A)$.

Remark. In view of Theorem 6, equation (6) shows that the $\{1\}$-inverse obtained from the Hermite normal form becomes a $\{1,2\}$-inverse if we choose $L = 0$. That is, the matrix

$$X = P \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} E \tag{11}$$

is a $\{1,2\}$-inverse of A, where P and E are nonsingular matrices satisfying

$$EAP = \begin{bmatrix} I_r & K \\ 0 & 0 \end{bmatrix} \tag{0.72}$$

3.5 Construction of {1,2,3},{1,2,4} and {1,2,3,4}-inverses

It this subsection we will see ways of creating special generalized inverses using previously created one.

Lemma 5. *For any finite matrix A, rank(AA^*) = rank(A) = rank(A^*A)*

Corollary 2. *For any finite matrix A, $R(AA^*) = R(A)$ and $N(AA^*) = N(A)$*

Theorem 7. ***Urquhart*** *For every finite matrix A with complex elements,*

$$Y = (A^*A)^{(1)}A^* \in A\{1,2,3\}$$

$$Z = A^*(AA^*)^{(1)} \in A\{1,2,4\}$$

.

Proof. Applying Corollary 2 to A^* gives:

$$\mathcal{R}(A^*A) = \mathcal{R}(A^*),$$

and hence, there exists a matrix U such that

$$A^* = A^*AU.$$

Taking the conjugate transpose of both sides yields:

$$A = U^*A^*A.$$

Consequently, we compute:

$$AYA = U^*A^*A(A^*A)^{(1)}A^*A = U^*A^*A = A.$$

Thus, $Y \in A\{1\}$.
Now, by Lemma 2(d), we know that $\text{rank}(Y) \geq \text{rank}(A)$, and by the definition of Y, we also have $\text{rank}(Y) \leq \text{rank}(A^*) = \text{rank}(A)$. Therefore:

$$\text{rank}(Y) = \text{rank}(A),$$

and by Theorem 6, this implies $Y \in A\{1,2\}$.
Finally, from the last two equations , we deduce:

$$AY = U^*A^*A(A^*A)^{(1)}A^*AU = U^*A^*AU,$$

which is clearly Hermitian, since it equals its own conjugate transpose. The case for Z is similarly proved.

Theorem 8. ***Urquhart*** *For any finite matrix A with complex elements,*

$$A^{(1,4)}AA^{(1,3)} = A^\dagger,$$

Proof. Let X denote the left-hand side of equation above. It follows at once from Lemma 4 that $X \in A^{\{1,2\}}$. Moreover, the equation gives

$$AX = AA^{(1,3)}, \quad XA = A^{(1,4)}A.$$

But both $AA^{(1,3)}$ and $A^{(1,4)}A$ are Hermitian, by the definitions of $A^{(1,3)}$ and $A^{(1,4)}$. Thus,

$$X \in A^{\{1,2,3,4\}}.$$

However, $A^{\{1,2,3,4\}}$ contains at most one element. Therefore, it contains exactly one element, namely the Moore–Penrose inverse $A^\dagger$, and hence $X = A^\dagger$.

3.6 Exact formula for $A^\dagger$

Theorem 9. ***McDuffee*** *If $A \in \mathbb{C}_r^{m\times n}$, with $r > 0$, has a full-rank factorization*

$$A = FG, \tag{16}$$

then the Moore–Penrose inverse of A is given by

$$A^\dagger = G^* \left(F^*AG^*\right)^{-1} F^*.$$

Proposition 10. *Given A a matrix and a, b column vectors,*

(a) $(A^\dagger)^\dagger = A$;

(b) $(A^*)^\dagger = (A^\dagger)^*$;

(c) $(A^T)^\dagger = (A^\dagger)^T$;

(d) $A^\dagger = (A^*A)^\dagger A^* = A^*(AA^*)^\dagger$.

(e) $a^\dagger = (a^*a)^\dagger a^*$;

(f) $(ab^*)^\dagger = (a^*a)^\dagger (b^*b)^\dagger ba^*$.

(g) if H is Hermitian and idempotent, then $H^\dagger = H$.

(h) $H^\dagger = H$ *if and only if* H^2 *is Hermitian and idempotent and* $\operatorname{rank}(H^2) = \operatorname{rank}(H)$.

(i) If $D = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, $D^\dagger = \operatorname{diag}(d_1^\dagger, d_2^\dagger, \ldots, d_n^\dagger)$

Proof. For $a), b), c)$ we just need to check if the candidate satisfies the four equations. For $d)$ we use the SVD-decomposition of A. Indeed $A = U\Sigma V^*$ where U, V are unitarian matrices and $\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & \mathbf{0} & \\ & & \sigma_r & & \\ & \mathbf{0} & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$ thanks to Proposition 13 we have $A^\dagger = V\Sigma^\dagger U^{-1}$ or since Σ is diagonale we have $\Sigma^\dagger = \begin{bmatrix} (\sigma_1)^{-1} & & & & \\ & \ddots & & \mathbf{0} & \\ & & (\sigma_r)^{-1} & & \\ & \mathbf{0} & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix}$ Then $A^*A = V\Sigma^2 V^*$ and

$$(A^*A)^\dagger A^* = V \begin{bmatrix} (\sigma_1)^{-2} & & & & \\ & \ddots & & \mathbf{0} & \\ & & (\sigma_r)^{-2} & & \\ & \mathbf{0} & & 0 & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} \Sigma U^* = A^\dagger.$$

For $g)$ it is easy to see that H satifies the Moore-Penrose equations since it is hermitian and idempotent. Same thing for i). Let's proove h). ($\Longrightarrow$) We suppose that $H^\dagger = H$ so equations (1),(2),(3) and (4) give $(H^2)^* = H^2$ and $H^3 = H$ so H^2 is hermitian and multiply the last equality by H give $H^4 = H^2$ in other words, H^2 is idempotent. ($\Longleftarrow$) We suppose that H^2 is idempotent and hermitian and $rank(H^2) = rank(H)$ we need to check if $H^\dagger = H$. We need to verify if H satisfies the Moore-Penrose equations, since H^2 is hermitian (3) and (4) equations are satisfied, to proove (1) and (2) equations we need to proove $H^3 = H$. We have $rank(H^2) = rank(H)$ so $\forall x, \exists y, Hx = H^2y$, we multiply the last equation by H^2 and we have $H^3x = H^4y = H^2y = Hx$ since it is true for all x we have $H^3 = H$ and we proove $H^\dagger = H$.

Proposition 11. *If U and V are unitary matrices, then $(UAV)^\dagger = V^*A^\dagger U^*$for any matrix A for which the product UAV is defined. In particular, if $A, B \in \mathbb{C}^{n\times n}$ are unitarily similar, i.e., $B = U^{-1}AU$ for some unitary matrix U, then $B^\dagger = U^{-1}A^\dagger U$*

3.7 Construction of {2}-inverses of Prescribed rank

Following the proof of Theorem 5, we described the construction of a {1}-inverse of a given matrix $A \in \mathbb{C}_r^{m\times n}$ having any prescribed rank between r and $\min(m, n)$, inclusive. From equation (2), it is easily deduced that

$$\operatorname{rank} A^{(2)} \leq r.$$

We also note that the $n \times m$ zero matrix is a {2}-inverse of rank 0, and any $A^{(1,2)}$ is a {2}-inverse of rank r, by Theorem 2.
For $r > 1$, is there a construction analogous to Fisher's for a {2}-inverse of rank s, for arbitrary s between 0 and r?
Using a full-rank factorization, we can readily answer this question affirmatively.
Let $X_0 \in A^{\{1,2\}}$ have a rank factorization

$$X_0 = YZ,$$

where

$$Y \in \mathbb{C}_r^{m\times r} \quad \text{and} \quad Z \in \mathbb{C}_r^{r\times n},$$

then

$$YZAYZ = YZ.$$

In view of Lemma 3, multiplication on the left by $Y^{(1)}$ and on the right by $Z^{(1)}$ gives

$$ZAY = I_r.$$

Let Y_s denote the submatrix of Y consisting of the first s columns, and let Z_s denote the submatrix of Z consisting of the first s rows. Then both Y_s and Z_s are of full rank s, and it follows that

$$Z_sAY_s = I_s.$$

Proposition 12. *Given $A = FG \in \mathbb{C}_r^{m\times n}$ a full-rank factorization. Then :*

a) $G^{(i)}F^{(1)} \in A\{i\}, i = 1, 2, 4$

b) $G^{(1)}F^{(j)} \in A\{j\}, j = 1, 2, 3$

c) $A^\dagger = G^\dagger F^{(1,3)} = G^{(1,4)}F^\dagger$

Proof. Verify the equations

4 Linear Systems and Characterization of Generalized Inverses

Generalized inverses are a strong tool to resolve linear systems, thanks to them we can easily describe the solutions of them.

4.1 Solutions of Linear Systems

Theorem 10. *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, and $D \in \mathbb{C}^{m\times q}$. Then the matrix equation*

$$AXB = D \tag{4.1}$$

is consistent if and only if, for some $A^{(1)}$, $B^{(1)}$,

$$AA^{(1)}DB^{(1)}B = D, \tag{4.2}$$

in which case the general solution is

$$X = A^{(1)}DB^{(1)} + Y - A^{(1)}AYBB^{(1)} \tag{4.3}$$

for arbitrary $Y \in \mathbb{C}^{n\times p}$.

Proof. If (2) holds, then

$$X = A^{(1)}DB^{(1)}$$

is a solution of (1).
Conversely, if X is any solution of (1), then

$$D = AXB = AA^{(1)}AXBB^{(1)}B = AA^{(1)}DB^{(1)}B.$$

Moreover, it follows from (2) and the definition of $A^{(1)}$ and $B^{(1)}$ that every matrix X of the form (3) satisfies (1).
On the other hand, let X be any solution of (1). Then, clearly

$$X = A^{(1)}DB^{(1)} + X - A^{(1)}AXBB^{(1)}.$$

Next results are a direct consequence of Theorem 10 :

Corollary 3. *Let $A \in \mathbb{C}^{m\times n}$, $A^{(1)} \in A\{1\}$. Then*

$$A\{1\} = \left\{A^{(1)} + Z - A^{(1)}AZAA^{(1)} : Z \in \mathbb{C}^{n\times m}\right\}. \tag{4.4}$$

Corollary 4. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. Then the equation*

$$Ax = b \tag{4.5}$$

is consistent if and only if, for some $A^{(1)}$,

$$AA^{(1)}b = b, \tag{4.6}$$

in which case the general solution of (5) is

$$x = A^{(1)}b + (I - A^{(1)}A)y \tag{4.7}$$

for arbitrary $y \in \mathbb{C}^n$.

4.2 Characterization of $A\{1,3\}$ and $A\{1,4\}$

We can now describe the set of $\{1,3\}$ and {1,4} inverses.

Theorem 11. *The set $A\{1,3\}$ consists of all solutions X of*

$$AX = AA^{(1,3)}, \tag{4.8}$$

where $A^{(1,3)}$ is an arbitrary element of $A\{1,3\}$.

Proof. If X satisfies (8), then clearly

$$AXA = AA^{(1,3)}A = A,$$

and, moreover, AX is Hermitian since $AA^{(1,3)}$ is Hermitian by definition. Thus,

$$X \in A\{1,3\}.$$

On the other hand, if $X \in A\{1,3\}$, then

$$AA^{(1,3)} = AXAA^{(1,3)} = (AX)^*AA^{(1,3)} = X^*A^*\left(A^{(1,3)}\right)^*A^*.$$

Since $\left(A^{(1,3)}\right)^* = A^{(1,3)}$ by definition, it follows that

$$AA^{(1,3)} = X^*A^* = AX.$$

Corollary 5. *Let $A \in \mathbb{C}^{m\times n}$, $A^{(1,3)} \in A\{1,3\}$. Then*

$$A\{1,3\} = \left\{A^{(1,3)} + (I - A^{(1,3)}A)Z : Z \in \mathbb{C}^{n\times m}\right\}. \tag{4.9}$$

Since (3) and (4) inverses are relatively close we have :

Theorem 12. *The set $A\{1,4\}$ consists of all solutions X of*

$$XA = A^{(1,4)}A, \tag{4.10}$$

where $A^{(1,4)}$ is an arbitrary element of $A\{1,4\}$.

Corollary 6. *Let* $A \in \mathbb{C}^{m\times n}$, $A^{(1,4)} \in A\{1,4\}$. *Then*

$$A\{1,4\} = \left\{A^{(1,4)} + Y(I - AA^{(1,4)}) : Y \in \mathbb{C}^{n\times m}\right\}. \tag{4.11}$$

Corollary 7. *Let* $A \in \mathbb{C}^{m\times n}$, *and suppose* $A^{(1,3,4)} \in A\{1,3,4\}$. *Then*

$$A\{1,3,4\} = \left\{A^{(1,3,4)} + (I - A^{(1,3,4)}A)Y(I - AA^{(1,3,4)}) : Y \in \mathbb{C}^{n\times m}\right\}. \tag{4.12}$$

If $A^{(1,2,3)} \in A\{1,2,3\}$, *then*

$$A\{1,2,3\} = \left\{A^{(1,2,3)} + (I - A^{(1,2,3)}A)ZA^{(1,2,3)} : Z \in \mathbb{C}^{n\times m}\right\}. \tag{4.13}$$

If $A^{(1,2,4)} \in A\{1,2,4\}$, *then*

$$A\{1,2,4\} = \left\{A^{(1,2,4)} + A^{(1,2,4)}Z(I - AA^{(1,2,4)}) : Z \in \mathbb{C}^{m\times m}\right\}. \tag{4.14}$$

4.3 Characterization of $A\{2\}, A\{1,2\}$, and Other Subsets of $A\{2\}$

Theorem 13. *Let* $A \in \mathbb{C}_r^{m\times n}$, *and* $0 < s \leq r$. *Then*

$$A\{2\}_s = \{YZ : Y \in \mathbb{C}^{n\times s},\ Z \in \mathbb{C}^{s\times m},\ ZAY = I_s\}. \tag{4.15}$$

Proof. Let

$$X = YZ,$$

where the conditions on Y and Z in (18) are satisfied. Then Y and Z are of rank s, and X is of rank s by Proposition 9. Moreover,

$$XAX = YZAYZ = YZ = X.$$

On the other hand, let $X \in A\{2\}_s$ and let $X = YZ$ be a full-rank factorization. Then

$$Y \in \mathbb{C}_s^{n\times s}, \quad Z \in \mathbb{C}_s^{s\times m},$$

and

$$YZAYZ = YZ. \tag{20}$$

Moreover, if $Y^{(1)}$ and $Z^{(1)}$ are any $\{1\}$-inverses, then by Lemma 1.2

$$Y^{(1)}Y = ZZ^{(1)} = I_s.$$

Thus, multiplying (20) on the left by $Y^{(1)}$ and on the right by $Z^{(1)}$ gives

$$ZAY = I_s.$$

Corollary 8. *Let* $A \in \mathbb{C}_r^{m\times n}$. *Then*

$$A\{1,2\} = \left\{YZ : Y \in \mathbb{C}^{n\times r},\ Z \in \mathbb{C}^{r\times m},\ \textit{such that } ZAY = I_r\right\}. \tag{4.16}$$

Proof. We use Theorem 13 to gives the solution and Bjerhammar's theorem which tells a $(1,2)$ inverse is a (2)-inverse which has the same rank as A.

Theorem 14. *Let $A \in \mathbb{C}_r^{m\times n}$, and let $0 < s \leq r$. Then*

$$A\{2,3\}_s = \left\{Y(AY)^\dagger : Y \in \mathbb{C}^{n\times s},\ AY \in \mathbb{C}_s^{m\times s}\right\}. \tag{4.17}$$

Theorem 15. *Let $A \in \mathbb{C}_r^{m\times n}$, and let $0 < s \leq r$. Then*

$$A\{2,4\}_s = \{(YA)^\dagger Y : Y \in \mathbb{C}^{s\times m},\ YA \in \mathbb{C}_s^{s\times n}\}. \tag{4.18}$$

4.4 Idempotent Matrices and Projectors

In the next subsection we will see generalized inverse are deeply rooted to projector on specific subspaces. Since idempotent matrices are linked to projectors we will admit few results concerning the later.

Proposition 13. *Let $E \in \mathbb{C}^{n\times n}$ be an idempotent matrix, i.e., $E^2 = E$. Then:*

(a) E^* *and* $I - E$ *are idempotent.*

(b) *The eigenvalues of* E *are* 0 *and* 1. *The multiplicity of the eigenvalue* 1 *is* $\operatorname{rank}(E)$.

(c) $\operatorname{rank}(E) = \operatorname{trace}(E)$.

(d) $E(I - E) = (I - E)E = 0$

(e) $Ex = x$ *if and only if* $x \in \mathcal{R}(E)$.

(f) $E \in E\{1, 2\}$, *i.e.,* E *satisfies properties (1) and (2) of generalized inverses.*

(g) $\mathcal{N}(E) = \mathcal{R}(I - E)$.

Lemma 6. *Let a square matrix E have the full-rank factorization*

$$E = FG,$$

where $F \in \mathbb{C}^{n\times r}$, $G \in \mathbb{C}^{r\times n}$, and $\operatorname{rank}(E) = r$. Then E is idempotent if and only if

$$GF = I_r.$$

Proof. ($\Longleftarrow$) obvious
($\Longrightarrow$) Suppose E is idempotent then $E^2 = E$ so $FGFG = FG$ thanks to Lemma 3 and the fact that F, G are full-rank row and column matrices multiplying by $F^{(1)}$ and $G^{(1)}$ gives the result.

Theorem 16. *For every idempotent matrix $E \in \mathbb{C}^{n\times n}$, the subspaces $\mathcal{R}(E)$ and $\mathcal{N}(E)$ are complementary, and*

$$E = P_{\mathcal{R}(E),\mathcal{N}(E)}. \tag{23}$$

Conversely, if L and M are complementary subspaces of $\mathbb{C}^n$, then there exists a unique idempotent matrix $P_{L,M}$ such that

$$\mathcal{R}(P_{L,M}) = L, \qquad \mathcal{N}(P_{L,M}) = M.$$

Proposition 14. *Let $A \in \mathbb{C}^{m\times n}$ and let $A^{(1)} \in A\{1\}$ be a $\{1\}$-inverse of A. Then:*

1. $\mathcal{R}(AA^{(1)}) = \mathcal{R}(A)$
2. $\mathcal{N}(A^{(1)}A) = \mathcal{N}(A)$
3. $\mathcal{R}((A^{(1)}A)^*) = \mathcal{R}(A^*)$

Proof. Lemma 2 and inclusion

Corollary 9. *If A and X are $\{1,2\}$-inverses of each other, AX is the projector on $\mathcal{R}(A)$ along $\mathcal{N}(X)$, and XA is the projector on $\mathcal{R}(X)$ along $\mathcal{N}(A)$.*

Proof. Lemma 2 and theorem 16

Theorem 17. *Let $A \in \mathbb{C}^{n\times n}$ have s distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s$. Then A is diagonalizable if and only if there exist projectors $E_1, E_2, \dots, E_s \in \mathbb{C}^{n\times n}$ such that:*

$$E_iE_j = \delta_{ij}E_i \quad \text{for all } i,j = 1,\dots,s, \tag{4.19}$$

$$I_n = \sum_{i=1}^{s} E_i, \tag{4.20}$$

$$A = \sum_{i=1}^{s} \lambda_i E_i. \tag{4.21}$$

4.5 Generalized Inverses with Prescribed Range and Null Space

We can know construct generalized inverses with prescribed range and null space if those satisfy certain properties.

Proposition 15. *$P_{L,M}A = A$ if and only if $\mathcal{R}(A) \subset L$ and $AP_{L,M} = A$ if and only if $M \subset \mathcal{N}(A)$.*

Proof. Obvious

Proposition 16. *The matrix equations $AX = B, XD = E$ (S) have a common solution if and only if each equation separately has a solution and $AE = BD$.*

Proof. Only if, for any $A^{(1)}$, $D^{(1)}$,

$$X = A^{(1)}B + ED^{(1)} - A^{(1)}AED^{(1)},$$

then X is a common solution of both equations (S) provided

$$AE = BD, \quad AA^{(1)}B = B, \quad ED^{(1)}D = E.$$

By Theorem 10, the latter two equations are equivalent to the consistency of equations (S) considered separately.
If : obvious.

Proposition 17. *Let the equations (S) have a common solution $X_0 \in \mathbb{C}^{m\times n}$. Then, the general solution is given by*

$$X = X_0 + (I - A^{(1)}A)Y(I - DD^{(1)}),$$

for arbitrary $A^{(1)} \in A\{1\}$, $D^{(1)} \in D\{1\}$, and $Y \in \mathbb{C}^{m\times n}$.

We admit the next proposition :

Proposition 18. *There is at most one matrix X satisfying the three equations $AX = B, XA = D, XAX = X$*

Theorem 18. *Let $A \in \mathbb{C}^{m\times n}$ have rank r, with*

$$\mathcal{R}(A) = L, \quad \mathcal{N}(A) = M, \quad L \oplus S = \mathbb{C}^m, \quad \textit{and} \quad M \oplus T = \mathbb{C}^n.$$

Then:

(a) X is a $\{1\}$-inverse of A such that

$$\mathcal{N}(AX) = S \quad \textit{and} \quad \mathcal{R}(XA) = T$$

if and only if

$$AX = P_{L,S}, \quad XA = P_{T,M}. \tag{4.22}$$

(b) The general solution is

$$X = P_{T,M}A^{(1)}P_{L,S} + (I_n - A^{(1)}A)Y(I_m - AA^{(1)}), \tag{4.23}$$

where $A^{(1)}$ is a fixed (but arbitrary) element of $A\{1\}$, and $Y \in \mathbb{C}^{n\times m}$ is arbitrary.

(c)

$$A^{(1,2)}_{T,S} = P_{T,M}A^{(1)}P_{L,S}$$

is the unique $\{1,2\}$-inverse of A having range T and null space S.

Proof. (a) The "if" part of the statement follows at once from Theorem 16 and Proposition 13(e), while the "only if" part follows from Proposition 12(f), (28), and Theorem 16.
(b) Thanks to proposition 14 and 15, we can easily verify that (23) is satisfied by

$$X = P_{T,M}A^{(1)}P_{L,S}.$$

The result then follows from Proposition 17.
(c) Since $P_{T,M}A^{(1)}P_{L,S}$ is a $\{1\}$-inverse of A, its rank is at least r by Lemma 2(d), while its rank does not exceed r, since rank $P_{L,S} = r$ by (22) and Lemma 1.1(f). Thus it has the same rank as A, and is therefore a $\{1, 2\}$-inverse by Theorem 6.
It follows from parts (a) and (b) that it has the required range and null space. On the other hand, a $\{1, 2\}$-inverse of A having range T and null space S satisfies (58) and also

$$XAX = X.$$

By proposition 18, these three equations have at most one common solution.

Corollary 10. *Under the hypotheses of Theorem 18, let $A^{(1)}_{T,S}$ be some $\{1\}$-inverse of A such that*

$$\mathcal{R}(A^{(1)}_{T,S}A) = T, \quad \mathcal{N}(AA^{(1)}_{T,S}) = S,$$

and let $A\{1\}_{T,S}$ denote the class of such $\{1\}$-inverses of A. Then

$$A\{1\}_{T,S} = \left\{ A^{(1)}_{T,S} + \left(I_n - A^{(1)}_{T,S}A\right) Y \left(I_m - AA^{(1)}_{T,S}\right) : Y \in \mathbb{C}^{n\times m} \right\}.$$

Theorem 19. *Let $A \in \mathbb{C}^{m\times n}$ with rank r, and let $U \in \mathbb{C}^{n\times p}$, $V \in \mathbb{C}^{q\times m}$. Define*

$$X = U(VAU)^{(1)}V,$$

where $(VAU)^{(1)}$ is a fixed, but arbitrary element of $(VAU)\{1\}$. Then:

(a) $X \in A\{1\}$ if and only if

$$\operatorname{rank}(VAU) = r.$$

(b) $X \in A\{2\}$ and $\mathcal{R}(X) = \mathcal{R}(U)$ if and only if

$$\operatorname{rank}(VAU) = \operatorname{rank}(U).$$

(c) $X \in A\{2\}$ and $\mathcal{N}(X) = \mathcal{N}(V)$ if and only if

$$\operatorname{rank}(VAU) = \operatorname{rank}(V).$$

(d) $X = A^{(1,2)}_{\mathcal{R}(U),\mathcal{N}(V)}$ if and only if

$$\operatorname{rank}(U) = \operatorname{rank}(V) = \operatorname{rank}(VAU) = r.$$

Proof. If: We have

$$\operatorname{rank}(AU) = r,$$

since

$$r = \operatorname{rank}(VAU) \le \operatorname{rank}(AU) \le \operatorname{rank}(A) = r.$$

Therefore, $\mathcal{R}(AU) = \mathcal{R}(A)$, and so

$$A = AUY$$

for some Y. Thus by Lemma 3,

$$AXA = AU(VAU)^{(1)}VAUY = AUY = A.$$

Only if: Since $X \in A\{1\}$,

$$A = AXAXA = AU(VAU)^{(1)}VAU(VAU)^{(1)}VA,$$

and therefore

$$\operatorname{rank}(VAU) = \operatorname{rank}(A) = r.$$

(b) If:

$$XAU = U(VAU)^{(1)}VAU = U,$$

from which it follows that

$$XAX = X \quad \text{and} \quad \operatorname{rank}(X) = \operatorname{rank}(U).$$

Then,

$$\mathcal{R}(X) = \mathcal{R}(U).$$

Only if: Since $X \in A\{2\}$,

$$X = XAX = U(VAU)^{(1)}VAU(VAU)^{(1)}V.$$

Therefore,

$$\operatorname{rank}(X) \leq \operatorname{rank}(VAU) \leq \operatorname{rank}(U) = \operatorname{rank}(X).$$

(c) Similar to (b).
(d) Follows from (a), (b), and (c).

Corollary 11. *Let $A \in \mathbb{C}^{m\times n}$ have rank r. Let T be a subspace of $\mathbb{C}^n$ of dimension $s \leq r$, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - s$.*
Then, A has a $\{2\}$-inverse X such that

$$\mathcal{R}(X) = T \quad \text{and} \quad \mathcal{N}(X) = S$$

if and only if

$$AT \oplus S = \mathbb{C}^m, \tag{4.24}$$

in which case X is unique.

Corollary 12. *Let $A \in \mathbb{C}^{m\times n}$ have rank r. Let T be a subspace of $\mathbb{C}^n$ of dimension r, and let S be a subspace of $\mathbb{C}^m$ of dimension $m - r$.*
Then, the following three statements are equivalent:

(a) $AT \oplus S = \mathbb{C}^m$.

(b) $\mathcal{R}(A) \oplus S = \mathbb{C}^m$ *and* $\mathcal{N}(A) \oplus T = \mathbb{C}^n$.

(c) There exists $X \in A\{1, 2\}$ such that

$$\mathcal{R}(X) = T \quad \text{and} \quad \mathcal{N}(X) = S.$$

4.6 Orthogonal Projections and Orthogonal Projectors

Lemma 7. *Let $\mathbb{C}^n = L \oplus M$. Then $M = L^{\perp}$ if and only if the projector $P_{L,M}$ is Hermitian.*

Theorem 20. *Let $A \in \mathbb{C}^{n\times n}$ have k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then A is normal if and only if there exist orthogonal projectors $E_1, E_2, \ldots, E_k$ such that*

$$E_iE_j = 0, \quad \text{if } i \neq j, \tag{4.25}$$

$$I_n = \sum_{i=1}^{k} E_i, \tag{4.26}$$

$$A = \sum_{i=1}^{k} \lambda_i E_i. \tag{4.27}$$

4.7 Restricted Generalized Inverses

In a linear equation of the form

$$Ax = b,$$

where $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$ are given, the solution x is sometimes constrained to lie in a subspace $S \subseteq \mathbb{C}^n$. This leads to a **constrained** linear system:

$$Ax = b \quad \text{with} \quad x \in S. \tag{1}$$

In principle, this situation poses no fundamental difficulty, since the constrained system (1) is equivalent to the following **unconstrained but enlarged** linear system:

$$\begin{pmatrix} A \\ P_{S^\perp} \end{pmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix}, \tag{2}$$

where $P_{S^\perp} = I - P_S$ is the orthogonal projector onto the orthogonal complement of S.

Definition 10. *Let $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$, and let $S \subseteq \mathbb{C}^n$ be a subspace. The restriction of A to S, denoted by $A_{[S]}$, is the linear transformation from S to $\mathbb{C}^m$ defined by*

$$A_{[S]}x = Ax, \quad x \in S. \tag{1}$$

Conversely, let $B \in \mathcal{L}(S, \mathbb{C}^m)$. The extension of B to $\mathbb{C}^n$, denoted by $\operatorname{ext}(B)$, is the linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$ defined by

$$\operatorname{ext}(B)x = \begin{cases} Bx, & \text{if } x \in S, \\ 0, & \text{if } x \in S^\perp. \end{cases} \tag{2}$$

Restricting a transformation $A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$ to S and then extending back to $\mathbb{C}^n$ yields the operator $\operatorname{ext}(A[S]) \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$, defined by

$$\operatorname{ext}(A_{[S]})x = \begin{cases} Ax, & \text{if } x \in S, \\ 0, & \text{if } x \in S^\perp. \end{cases} \tag{3}$$

Lemma 8. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$, and let $S \subseteq \mathbb{C}^n$ be a subspace. The system*

$$Ax = b, \quad x \in S \tag{1}$$

is consistent if and only if the system

$$AP_S z = b \tag{2}$$

is consistent, where P_S denotes the orthogonal projector onto S. In that case, x is a solution of (1) if and only if

$$x = P_S z,$$

where z is a solution of (2).

5 Minimal Properties of Generalized Inverses

5.1 Least-squares solutions

Theorem 21. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathbb{C}^m$. Then the norm $\|Ax - b\|$ is minimized when*

$$x = A^{(1,3)}b,$$

where $A^{(1,3)} \in A\{1,3\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ has the property that, for all b, the quantity $\|Ax - b\|$ is minimized when $x = Xb$, then

$$X \in A\{1,3\}.$$

Proof. Given b we have :

$$b = \big(P_{\mathcal{R}(A)} + P_{\mathcal{R}(A)^\perp}\big)b,$$

so that

$$b - Ax = \big(P_{\mathcal{R}(A)}b - Ax\big) + P_{\mathcal{N}(A^*)}b.$$

Hence, by the Pythagorean theorem

$$\|Ax - b\|^2 = \|Ax - P_{\mathcal{R}(A)}b\|^2 + \|P_{\mathcal{N}(A^*)}b\|^2. \tag{eq}$$

Evidently, (eq) assumes its minimum value if and only if

$$Ax = P_{\mathcal{R}(A)}b,$$

which holds if

$$x = A^{(1,3)}b$$

for any $A^{(1,3)} \in A\{1,3\}$, since by theorem 16), proposition 14 and lemma 7,

$$AA^{(1,3)} = P_{\mathcal{R}(A)}.$$

Conversely, if X is such that for all b, $\|Ax - b\|$ is minimized when $x = Xb$, then we have

$$AXb = P_{\mathcal{R}(A)}b \quad \text{for all } b,$$

and therefore

$$AX = P_{\mathcal{R}(A)}.$$

Thus, by Theorem 2.3,

$$X \in A\{1,3\}$$

Corollary 13. *A vector x is a least-squares solution of the equation $Ax = b$ if and only if*

$$Ax = P_{\mathcal{R}(A)}b = AA^{(1,3)}b,$$

where $A^{(1,3)} \in A\{1,3\}$. Thus, the general least-squares solution is given by

$$x = A^{(1,3)}b + (I_n - A^{(1,3)}A)y, \tag{8}$$

where $y \in \mathbb{C}^n$ is arbitrary.

5.2 Solutions of minimal norm

When system (1) has a multiplicity of solutions for x, there is a unique solution of the minimum norm.

Lemma 9. *Let $A \in \mathbb{C}^{m\times n}$. Then A defines a one-to-one mapping from $\mathcal{R}(A^*)$ onto $\mathcal{R}(A)$.*

Corollary 14. *Let $A \in \mathbb{C}^{m\times n}$ and $b \in \mathcal{R}(A)$. Then the equation*

$$Ax = b$$

has a unique solution given by the unique solution of previous system that lies in $\mathcal{R}(A^*)$.

Theorem 22. *Let $A \in \mathbb{C}^{m\times n}$, $b \in \mathbb{C}^m$. If the equation $Ax = b$ has a solution, then the unique solution with the smallest norm is given by*

$$x = A^{(1,4)}b,$$

where $A^{(1,4)} \in A\{1,4\}$. Conversely, if $X \in \mathbb{C}^{n\times m}$ is such that, whenever $Ax = b$ has a solution, the solution $x = Xb$ is the minimum-norm solution, then

$$X \in A\{1,4\}.$$

6 Applications of generalized inverse

6.1 Application of the Group Inverse in Finite Markov Chains