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HIGHEST WEIGHT CATEGORIES

- intership supervised by Alessio CIPRIANI[(i)]-

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Notations & definitions

Ab will denote the category of abelian groups.

Algebra : We will only consider unital and associative algebra over a field K. Algebras may be non-commutative.

Arrows: we use special arrows for monomorphisms, epimorphisms and isomorphisms (see definition A.1.4), respectively $\hookrightarrow$, $\twoheadrightarrow$ and $\xrightarrow{\sim}$. These arrows may be used to implicitely mean that some arrows are epic, monic, etc.

Artinian poset: A poset $(P, \preccurlyeq)$ is said to be *artinian* if there is no infinite decreasing sequence of elements of P.

Categories : We will usually denote a category with cursif letters such as $\mathscr{A}$. Usual categories will be denoted with bold letters, like **Set**. We will often use the notation $A \in \mathscr{A}$ to mean "A is an object in the category $\mathscr{A}$". Given two objects $A, B \in \mathscr{A}$, we will write $\mathscr{A}(A, B)$ or $\mathrm{Hom}_{\mathscr{A}}(A, B)$ the class of arrows from A to B and we shall omit the ambiant category when the context is clear. In the special case $A = B$, we may write $\mathrm{End}(A) = \mathrm{Hom}(A, A)$.

Classes are defined as in the set theory presented in [9], chapter 1.8, therefore there is a distinction between *sets* and *classes*.

Domains, codomains: given an arrow $f : A \to B$ in any category, we write $\mathrm{Dom}(f)$ for its domaine (that is, A) and $\mathrm{Codom}(f)$ for its codomain (that is, B).

Field: In this report, K will always denote an algebraically closed field.

Integer : by convention, we assume that $0 \in \mathbb{N}$ and 0 is considered an integer. We will write $\mathbb{N}^*$ for the set of strictly positive integers.

Integer intervals: we will write $[\![k, n]\!]$ for the set $\{l \in \mathbb{Z} \mid k \leqslant l \leqslant n\}$. When $k = 1$, we write shortlier $[\![n]\!]$.

KRONECKER **symbol:** we use $\delta_{i,j}$ for the KRONECKER symbol, which is 1 if $i = j$, 0 otherwise.

Mod(R) will denote the category of (right) modules over R. We will also write **mod**(R) for the full subcategory of finitely generated modules over R.

Noetherian poset: A poset $(P, \preccurlyeq)$ is said to be *noetherian* if there is no infinite decreasing sequence of elements of P.

Poset: A *poset* is a couple $(P, \preccurlyeq)$ with P a set and $\preccurlyeq$ a partial order on P.

Retraction: for an arrow f, a *retraction* of f is an arrow r such that $r \circ f = 1_{\mathrm{Dom}(f)}$.

Ring will denote the category of rings. A ring is always supposed to be unital and associative, though we will often consider non-commutative rings.

Section: for an arrow f, a *section* of f is an arrow s such that $f \circ s = 1_{\mathrm{Codom}(f)}$.

Set will denote the category of sets.

Vect(K) will denote the category of vector spaces over a field K.

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I think the author of this tool(ii) deserves some thanks for the hours of commutative diagram drawing he spared me.

Introduction

The study of algebras and modules over them had been an important subject of research for decades. In 1988, E. Cline, B., Parshall and L. Scott used the language of category theory to define a certain type of categories they called *highest weight*, which has several strong implications in the study of algebras, in their article [14]. It gives for instance strong criterions to detect hereditary algebras, and a sufficient property of quasi-hereditary algebras, as it is shown by E. Green and S. Schroll in [10].

The present report summarises the work I've done on highest weight categories under the supervision of Alessio Cipriani in the Università di Verona. Our study focused on the case of categories of modules over a finite dimensional algebra over an algebraically closed field. It does not follow the approach of [14] but rather the one found in [7].

As the reader may expect it, we will make abundant use of notions of category theory, and for the sake of conciseness, we can't make a complete introduction to category theory in this report. Instead, the reader is invited to refer to the appendix for an almost exhaustive list of what we'll need, and references to more complete introductions. We assume the reader knows at least what a category and a functor are, and is familiar with very basic notions of category theory.

Our work is split in three parts. The first one introduces the context in which we will mostly work: abelian categories, which serve as a generalisation of categories of modules. It will contain a quite long introduction to several fundamental notions, like composition series, exact sequences and projectivity, in a very abstract context that will use the language of categories. We will prove along the way two famous and crucial theorems: Jordan-Hölder theorem that gives sense to the notion of the length of a composition series, and Gabriel-Mitchell theorem that gives a sufficient criterion for an abelian category to be identified with a category of modules. Both will be essential for our purpose.

The second part will focus on categories of modules. Results from the first section will specialise in this more precise context, and we will develop new tools granted by representation theory. The language of categories will make precise the connection between categories of modules and representations of quivers, and this point of view will give much information about the categories we want to study, often by using combinatorial arguments.

The last part will introduce highest weight categories as a certain collection of categories of modules over a finite-dimensional algebra. It will combine everything we did in the previous section to prove a weaker version of a theorem discovered by H. Krause that explains how highest weight categories are, in a sense that will be made clear, built by gluing together copies of the simplest module category: $\mathbf{Vect}(K)$. Our work will be concluded by a criterion discovered by E. Green and S. Schroll. When restricting to categories of representations of a nice quiver, it gives a necessary and sufficient way to detect highest weight categories.

(ii) https://tikzcd.yichuanshen.de/

1 Abelian categories

Our ultimate goal is to study algebras and modules over algebras, therefore categories in their full generality are a bit too abstract for our purpose. This first section will introduce a certain type of categories with more structure, which will allow us to retrieve familliar properties we're accustomed to use in modules categories, like the possibility to study morphisms by searching their kernels, or to use quotients and isomorphism theorems. Abelian categories offer a proper setting to do so, and will be our main tool in this report.

We will throughout this report limit ourselves to locally small categories (definition A.1.1), which will be highly sufficient for our purpose.

1.1 Overview

Definition 1.1.1 (Additive category). *A locally small category*[iii] $\mathscr{A}$ *is said to be* additive *if it satisfied the followings:*

(i) *For all pairs of objects $A, B \in \mathscr{A}$, the set $\mathscr{A}(A, B)$ admits an abelian group structure for which the composition is $\mathbb{Z}$-bilinear.*

(ii) *$\mathscr{A}$ admits a special object, denoted 0, which is both initial and terminal.*

(iii) *$\mathscr{A}$ admits all finite coproducts.*

Remark 1.1.2:

(i) Terminal and initial objects being particular cases of limits and colimits (see remark A.4.6), the zero element is unique up to isomorphism.

(ii) We will usually denote by 0 both the zero-object and the neutral element for the group structure of <u>any</u> Hom-set. The context is usually clear enough to prevent ambiguity, and it allows the following natural equalities to hold for any arrow $f : A \to B$:

$$f \circ 0 = 0 \quad 0 \circ f = 0 \quad 0 \cdot f = 0 \tag{1.1.1}$$

♦

Additive categories are sufficient enough to define the familiar notion of a kernel. As an abstract category may countains arrows which are not applications in the sense of the set theory, we can't rely on equalities like:

$$x \in \mathrm{Ker}(f) \iff f(x) = 0 \tag{1.1.2}$$

(even the evaluation of f makes no sense!) we will instead define it as a universal object:

[iii] Since a group can't have a proper class as its unerlying set, the hypothesis that $\mathscr{A}$ is locally small is in fact countained in the first axiom.

Definition 1.1.3 (Kernel & cokernel). *Given an arrow $f : A \to B$ in an additive category $\mathscr{A}$, we define its kernel as the equalizer of f and $0 : A \to B$ (definition A.4.7). In other words, it is an object $\mathrm{Ker}(f)$ together with a morphism $\kappa : \mathrm{Ker}(f) \to A$ with the property that $f \circ \kappa = 0$. We want this property to be universal in the sense that for any arrow $z : Z \to A$ such that $f \circ z = 0$, z factorise uniquely through $\mathrm{Ker}(f)$ in a way that the following diagram commutes:*

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, "\iota"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B \\ & Z \arrow[u, "z"] \arrow[ul, dashed, "\bar{z}"] & \end{tikzcd} \tag{1.1.3}$$

We naturally define the cokernel as the dual notion.

Remark 1.1.4:

- The arrows that come with the kernel and cokernel are always monic and epic respectively (it is easy to show that it is true for any equalizer and coequalizer).
- Unintuitively, kernels and cokernels might not exist for all arrows.
- We will sometimes use the abuse of notation that the kernel of f is the object $\mathrm{Ker}(f)$, but the definition makes no sense at all without the morphism ι. On the other hand, defining the kernel as the arrow ι would suffice in some way as it countains the information of its domain. Therefore, we will (rarelier) mixed up $\mathrm{Ker}(f)$ and the arrow κ, as it will allows us to write convenient expressions like $\mathrm{Coker}(\mathrm{Ker}(f))$.

♦

When working in usual categories, we often invoke the kernel together with the image:

Definition 1.1.5 (Image & coimage). *Let f be an arrow admitting a kernel ι and a cokernel π. Then, we define its image as the kernel of π and its coimage as the kernel of ι, and write $\mathrm{Im}(F)$ and $\mathrm{Coim}(f)$ for the underlying objects.*

<u>Example 1.1.6:</u> Let's consider $f : G \to H$ a homomorphism of abelian groups. Then the usual kernel with the natural inclusion $\mathrm{Ker}(f) \hookrightarrow G$ is the kernel of f in the sense of definition 1.1.3. Its cokernel is given by the projection :

$$\pi : H \twoheadrightarrow H \big/ \mathrm{Im}(f) \tag{1.1.4}$$

where the image is taken in the usual sense. Now the image of f is defined as $\mathrm{Ker}(\pi)$, which is indeed the image in the usual sense! The coimage is the cokernel of the inclusion of $\mathrm{Ker}(f)$ in G, that is, it is $G \big/ \mathrm{Ker}(f)$, which, according to NOETHER's isomorphism theorem, is isomorphic to $\mathrm{Im}(f)$!

This example is more profound than it seems. First, it reflects that cokernels act somehow as quotients. Second, we saw that $\mathrm{Coim}(f) \cong \mathrm{Im}(f)$. This is not true in any additive category, but will hold as an axiom for abelian categories. ♦

Proposition 1.1.7. *Let $f : A \to B$ be an arrow in an additive category $\mathscr{A}$. We suppose that f admits a kernel, a cokernel, an image and a coimage. Then there is a canonical arrow* $\operatorname{Coim}(f) \to \operatorname{Im}(f)$ *such that there is a commutative diagram:*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, two heads] & B \\
\operatorname{Coim}(f) \arrow[r] & \operatorname{Im}(f) \arrow[u, hook]
\end{tikzcd}
\tag{1.1.5}
$$

Proof.
We consider the following diagram, which is commutative:

$$
\begin{tikzcd}
\operatorname{Coim}(f) & & \operatorname{Coker}(f) \\
A \arrow[u, two heads, "p"'] \arrow[urr, "0"] \arrow[rr, "f"] & & B \arrow[u, two heads, "c"'] \\
\operatorname{Ker}(f) \arrow[u, hook, "\kappa"'] \arrow[urr, "0"] \arrow[uu, bend left, "0"] & & \operatorname{Im}(f) \arrow[u, hook, "i"'] \arrow[uu, bend right, "0"']
\end{tikzcd}
$$

Since $f \circ \kappa = 0$ by definition, using the universal property of the cokernel, we obtains an arrow $\bar{f} : \operatorname{Coim}(f) \to B$ which makes the diagram commutative again:

$$
\begin{tikzcd}
\operatorname{Coim}(f) \arrow[drr, dashed, "\bar{f}"] & & \operatorname{Coker}(f) \\
A \arrow[u, two heads, "p"'] \arrow[rr, "f"] & & B \arrow[u, two heads, "c"'] \\
\operatorname{Ker}(f) \arrow[u, hook, "\kappa"'] \arrow[uu, bend left, "0"] & & \operatorname{Im}(f) \arrow[u, hook, "i"'] \arrow[uu, bend right, "0"']
\end{tikzcd}
$$

Then we must notice that :

$$c \circ \bar{f} \circ p = c \circ f = 0 = 0 \circ p$$

and since p is epic, we may simplify:

$$c \circ \bar{f} = 0$$

Using the universal property of the kernel, we obtain the arrow we want. □

This can now introduce the definition of an abelian category:

Definition 1.1.8 (Abelian category). *A locally small category $\mathscr{A}$ is said to be abelian when:*

(i) *$\mathscr{A}$ is additive.*

(ii) *Every arrow of $\mathscr{A}$ admits a kernel and a cokernel.*

(iii) *For any arrow f,* $\operatorname{Coim}(f) \cong \operatorname{Im}(f)$ via *the canonical arrow given by proposition 1.1.7.*

Example 1.1.9:

- **Ab** is an abelian category (who might have guessed?...).
- Given any ring R, $\mathbf{mod}(R)$ and $\mathbf{Mod}(R)$ are abelian category.
- **Ring** is not abelian; it is not even additive! Indeed, the terminal object of **Ring** is the one-element ring whereas its initial object is $\mathbb{Z}$.
- **Grp** is not abelian too, as Hom-sets almost never bear an abelian group structure.

♦

In every example we gave, the axiom 3 is exactly NOETHER's ismorphism theorem.

Categories of modules, in which we will often work, tend to be a bit more than abelian when their base ring is an algebra over a field K:

Definition 1.1.10 (K-linear category). *An abelian category $\mathscr{A}$ is called K-linear for a field K when, for any pair of objects (A, B), $\mathscr{A}(A, B)$ is a K-vector space for which the composition is K-bilinear.*

As always in category theory, a new type of object comes with its arrows. Simple functors don't preserve the structure of abelian categories, hence we will need to specialise them:

Definition 1.1.11 (Additive / K-linear functor). *Let $\mathscr{A}$ and $\mathscr{B}$ be two abelian categories (resp. K-linear categories). A functor $F : \mathscr{A} \to \mathscr{B}$ is said to be* additive *(resp. K-*linear*) if for any pair of objects $(A, A') \in \mathscr{A}$, F induces a group homomorphism between $\mathscr{A}(A, A')$ and $\mathscr{B}(F(A), F(A'))$ (resp. a K-linear map).*

Proposition 1.1.12 (Properties of additive functors). *Let $F : \mathscr{A} \to \mathscr{A}$ be an additive functor. Then:*

(i) *F preserves finite coproducts, i.e.:*

$$\forall A_1, \ldots, A_n \in \mathscr{A},\ F\left(\bigoplus_{i=1}^{n} A_i\right) \cong \bigoplus_{i=1}^{n} F(A_i) \tag{1.1.6}$$

(ii) *F sends the zero element of $\mathscr{A}$ to the zero element of $\mathscr{B}$*

Proof. These are quite elementary verifications. We don't treat them in detail. □

1.2 A first toolbox to work in abelian categories

We shall now prove some basic results about abelian categories which will be very useful in our study. Indeed, abelian categories are built in a way that is intended to look like abelian groups, and we will retrieve familiar results, though in a way more abstract and general context.

Proposition 1.2.1. *Let $\mathscr{A}$ be an abelian category and $f : A \to B$ be any arrow. Then the following hold:*

(i) *f is monic if and only if* $\mathrm{Ker}(f) = 0$

(ii) *f is epic if and only if* $\mathrm{Coker}(f) = 0$

(iii) *f is an isomorphism if and only if it is both monic and epic.*

Proof. (i) Let's first assume that f is monic and consider κ its kernel. Then, by definition:

$$f \circ \kappa = 0 \tag{1.2.1}$$

But by monicity of f, we must have $\kappa = 0$.[iv] We will show that $\mathrm{Ker}(f) = 0$ by proving that it is terminal and then use the fact that limits are unique up to isomorphism (recall that terminal objects are a particular case of limits). Let $z_1, z_2 : Z \to \mathrm{Ker}(f)$ be arrows for some domain Z. The following diagram arises:

$$\begin{tikzcd} Z \arrow[r, "0"] \arrow[d, shift right, "z_1"'] \arrow[d, shift left, "z_2"] & A \arrow[r, "f"] & B \\ \mathrm{Ker}(f) \arrow[ur, "\kappa=0"'] & & \end{tikzcd} \tag{1.2.2}$$

It commutes. Hence, z_1 and z_2 are two factorisations of $0 : Z \to A$ into $\mathrm{Ker}(f)$. By universal property of the kernel, they must be equal, therefore for any Z there can be only one arrow with domain Z and codomain $\mathrm{Ker}(k)$. Hence $\mathrm{Ker}(f)$ is terminal and is therefore isomorphic to 0.

Conversely, taking $z_1, z_2 : Z \to A$ such that $f \circ z_1 = f \circ z_2$, we have that $f \circ (z_1 - z_2) = 0$ and hence $z_1 - z_2$ factorise uniquely through the zero object. It is therefore 0 and f is monic.

(ii) We may observe that this sentence is exactly the dual of the one we have juste proved. We will prove below that the collection of abelian category is stable under duality, hence we have nothing else to prove.

(iii) We already know that if f is an iso, it is also an epi and a mono. Suppose that f is both monic and epic. Using proposition 1.1.7 and the fact that $\mathscr{A}$ is an abelian category, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, hook, "\kappa"] & A \arrow[r, "f"] \arrow[d, two heads] & B \arrow[r, two heads, "\pi"] & \mathrm{Coker}(f) \\ & \mathrm{Coim}(f) \arrow[r, "\sim"] & \mathrm{Im}(f) \arrow[u, hook] & \end{tikzcd} \tag{1.2.3}$$

with the arrow $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ being an isomorphism. But since f is monic and epic, $\mathrm{Ker}(f) = 0$, $\mathrm{Coker}(f) = 0$ and it si straightforward to prove that 1_A and 1_B are respectively

[iv] Our intuition says that if 0 is monic then its domain must be 0, and it is indeed telling us the truth, but as we can't reason on the elements of $\mathrm{Ker}(f)$, the proof will be more tricky than it would be in $\mathbf{mod}(R)$ for instance.

a cokernel and a kernel of κ and π. Hence we have:

$$\begin{tikzcd} \mathrm{Ker}(f) \arrow[r, hook, "\kappa"] & A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, two heads, "\pi"] \arrow[d, equal] & \mathrm{Coker}(f) \\ & A \arrow[r, "\sim"] & B & \end{tikzcd} \tag{1.2.4}$$

and f is an isomorphism.

□

Proposition 1.2.2. *Let $\mathscr{A}$ be an additive category. For any finite familly of objects $A_1, \ldots A_n$, the inclusions $\iota_j : A_j \hookrightarrow \bigoplus_{i=1}^{n} A_i$*[v] *admit retractions p_j, and $\bigoplus_{i=1}^{n} A_i$ together with the (p_j) is a product of the (A_i).*

Proof. It is sufficient to prove it for $n = 2$. The general case follows be immediate induction. Let's consider the following diagram:

$$\begin{tikzcd} A \arrow[r, hook, "\iota_A"] \arrow[dr, "1_A"'] & A \oplus B \arrow[d, dashed] & B \arrow[l, hook', "\iota_B"'] \arrow[dl, "0"] \\ & A & \end{tikzcd} \tag{1.2.5}$$

where the dashed arrow is the only arrow that makes the diagram commute, given by the universal property of the coproduct. Let's call it p_A. By definition, $p_A \circ \iota_A = 1_A$ and $p_A \circ \iota_B = 0$. The same way, we may define $p_B : A \oplus B \to B$.

We may now show that the coproduct with these two projections form a product. Let's take $z_A : Z \to A$ and $z_B : Z \to B$. We define:

$$\bar{z} := \iota_A \circ z_A + \iota_B \circ z_B \tag{1.2.6}$$

Now we have to compute:

$$\begin{aligned} p_A \circ \bar{z} &= p_A \circ \iota_A \circ z_A + p_A \circ \iota_B \circ z_B && \text{by bilinearity of the composition} && (1.2.7) \\ &= 1_A \circ z_A + 0 \circ z_B && && (1.2.8) \\ &= z_A && && (1.2.9) \end{aligned}$$

Simillarly, $p_B \circ \bar{z} = z_B$. Proving that $\bar{z}$ is unique, we have our result. Given another arrow $\tilde{z}$ with the same property □

With the same kind of argument, we may prove the following fact:

[v] As for the equalizers, it is easy to show that in full generality, the inclusion in the coproduct is always monic.

Proposition 1.2.3. *In an additive category, given families of objects $(A_i)_{1\leqslant i\leqslant k}$ and $(B_j)_{1\leqslant j\leqslant l}$, we have an isomorphism of groups:*

$$\operatorname{Hom}\left(\bigoplus_{i=1}^{k} A_i, \bigoplus_{j=1}^{l} B_j\right) \to \bigoplus_{i=1}^{k}\bigoplus_{j=1}^{l} \operatorname{Hom}(A_i, B_j) \cong \prod_{i=1}^{k}\prod_{j=1}^{l} \operatorname{Hom}(A_i, B_j)$$
$$f \mapsto (\iota_j \circ f \circ p_i)_{\substack{1\leqslant j\leqslant l\\ 1\leqslant i\leqslant k}}$$

with $\iota_j : B_j \hookrightarrow \bigoplus_{j=1}^{l} B_j$ and $p_i : \bigoplus_{i=1}^{k} A_i \twoheadrightarrow A_i$ the natural injection and projection.

Remark 1.2.4: This is somehow equivalent to the matricial notation we often use in the case of finite dimensional vector spaces over a field K:

$$\operatorname{Hom}(K^k, K^l) \cong \mathbb{M}_{l,k}(K) \tag{1.2.10}$$

♦

Proposition 1.2.2 enables us to make this very simple yet powerful remark:

Lemma 1.2.5. *The opposite category of an abelian category is still abelian.*

As a corollary, every statement we prove in the context of abelian category imply that its dual statement is also true in all abelian categories, and we won't need more effort to prove it than writing "by duality". We give an example immediatetly:

Theorem 1.2.6. *Let $\mathscr{A}$ be an abelian category. Then $\mathscr{A}$ admits all finite limits and colimits.*

Proof. We shall use theorem A.4.10 to prove this result. We already know that $\mathscr{A}$ admits a terminal object 0 and every binary products thanks to the previous proposition. We just have to show that $\mathscr{A}$ admits all equalizers.

Let's consider a fork $A \overset{f}{\underset{g}{\rightrightarrows}} B$ and set $E := \operatorname{Ker}(f-g)$ and e the natural inclusion of E into A. Then:

$$g \circ e = g \circ e + 0 = g \circ e + (f-g) \circ e = f \circ e \tag{1.2.11}$$

and given $z : Z \to A$ equalizing the fork, we must have:

$$(f-g) \circ z = 0 \tag{1.2.12}$$

by bilinearity. Hence z factorise uniquely through E and E is an equalizer of f and g.

We just proved that every abelian category admits all finite limits. By duality, it also admits all finite colimits. □

1.3 Composition series, the lattice of subobjects and Jordan-Hölder theorem

In **Set** it is more than common to consider subsets of a given set. In other categories, like $\mathbf{mod}(R)$, we're not truely interested in *subsets*, but rather in *submodules* thanks to which we often deduce facts about the ambiant module. We would like to generalise this idea of a subobject to any category, and see what we can do with it in the case of abelian categories.

In this paragraph, we will introduce the basic tools to study the structure of subobjects, how they interact with the ambiant object, and we shall eventually prove Jordan-Hölder theorem, which is an important and well known result in the context of finitely generated modules over a finite-dimensional algebra, yet we will present those results in a more abstract and general context. Our presentation is highly inspired by [1].

Definition 1.3.1 (Subobject). *Let $\mathscr{A}$ be any category and A be an object in $\mathscr{A}$. Given two monomorphisms i and j with codomain A, we say that i and j are equivalent whenever there exists an isomorphism such that the following diagram commutes:*

$$\begin{tikzcd} I \arrow[r, hook, "i"] \arrow[d, "\wr"'] & A \\ J \arrow[ur, hook, "j"'] & \end{tikzcd} \tag{1.3.1}$$

A subobject *of A is an equivalence class of monomorphisms with codomain A under this relation.*

Given two subobjects i and j, we say that $i \subset j$ whenever there exist a monomorphism $\mathrm{Dom}(i) \hookrightarrow \mathrm{Dom}(j)$ such that the obtained triangle commutes. When the class of subobjects of A is a set, the inclusion embbed it with the structure of a poset. **From now on, we will always assume that the class of subobjects is a set to avoid set-theoretical issues**, and we will denote by $\mathrm{Sub}(A)$ the poset of subobjects of A.

Remark 1.3.2: It is very common to identify a subobject i with its domain, although there can be two non-equal subobjects with the same domain. For instance, $2\mathbb{Z}$ can be seen as a subobject of $\mathbb{Z}$ in the category of abelian groups by the natural inclusion or by $\mathbb{Z} \ni 2n \mapsto 4n \in \mathbb{Z}$, and there can't be an automorphism of $2\mathbb{Z}$ as in (1.3.1). We may use the abuse of notation when there is no ambiguity. ♦

As always, there is a dual notion which is quotients. We define quotients as an equivalence class of epimorphisms. Yet we will more often work directly with subobjects, hence we introduce the following notation:

Definition 1.3.3 (Quotient by a subobject). *Given $i : I \hookrightarrow A$ a subobject of A, we will write:*

$$A/I := \mathrm{Coker}(i) \tag{1.3.2}$$

whenever this notation is clear in the context.

Remark 1.3.4: In **Ab** for instance, if we have a subgroup $H \lhd G$, then G/H is indeed the cokernel of the natural inclusion. ♦

The poset $\mathrm{Sub}(A)$ has, in the case of abelian categories, a richer structure than just a poset:

Proposition 1.3.5. *Given A an object in an abelian category $\mathscr{A}$, $\mathrm{Sub}(A)$ is a lattice with respect to the order $\subset$. In other words, given subobjects i and j of A, there exist a unique subobject $i \wedge j$ which is the largest lower bound of i and j, and a unique subobject $i \vee j$ which is the smallest upper bound of i and j.*

Proof (sketch). We are given a corner of arrows:

$$\begin{tikzcd} & I \arrow[d, hook, "i"] \\ J \arrow[r, hook, "j"] & A \end{tikzcd} \tag{1.3.3}$$

Since $\mathscr{A}$ is abelian, it admits all finite limits (theorem A.4.10). Therefore we may take the pullback of i and j (definition A.4.9) and obtain:

$$\begin{tikzcd} P \arrow[r, hook, "p_i"] \arrow[d, hook, "p_j"] \arrow[dr, hook, "i\wedge j"] & I \arrow[d, hook, "i"] \\ J \arrow[r, hook, "j"] & A \end{tikzcd} \tag{1.3.4}$$

The fact that p_i and p_j are monic is a common property of pullbacks of monic arrows. If $s : S \hookrightarrow A$ is a common lower bound of i and j, we have a following diagram:

$$\begin{tikzcd} S \arrow[drr, hook, bend left] \arrow[ddr, hook, bend right] & & \\ & P \arrow[r, hook, "p_i"] \arrow[d, hook, "p_j"] \arrow[dr, hook, "i\wedge j"] & I \arrow[d, hook, "i"] \\ & J \arrow[r, hook, "j"] & A \end{tikzcd} \tag{1.3.5}$$

And therefore s must factorise uniquely into P due to the universal property of pullbacks.

We obtain $i \vee j$ by taking the pushout of p_1 and p_2. The proof is almost the same. □

Remark 1.3.6: When conflating subobjects with their domains, it will be common to denote

$$I \cap J := i \wedge j \quad I + J := i \vee j \tag{1.3.6}$$

as these notations coincide with the one used for abelian groups, vector spaces or modules (be sure to mind the difference between $I + J$ and $I \oplus J$). ♦

We will start to study objects by studying their subobjects and quotients. We need an additional couple of definitions:

Definition 1.3.7 (Simple and semi-simple object). *An object A is called:*

(i) simple *if it's non-zero and its only subobjects are the zero map and the indentity.*

(ii) semi-simple *if it is the coproduct of a finite family of simple objects.*

There are two important results about simple objects we must state now:

Lemma 1.3.8 (Schur). *Let f be any non-zero arrow in an abelian category $\mathscr{A}$. Then, the following hold:*

(i) *If* $\mathrm{Dom}(f)$ *is simple, f is monic.*

(ii) *If* $\mathrm{Codom}(f)$ *is simple, f is epic.*

(iii) *If both* $\mathrm{Dom}(f)$ *and* $\mathrm{Codom}(f)$ *are simple, f is an isomorphism.*

Proof.

(i) Suppose $\mathrm{Dom}(f)$ is simple. $\mathrm{Ker}(f)$ is a subobject of $\mathrm{Dom}(f)$ via the canonique monomorphism κ. Hence κ is either 0 or can be identified with 1_A in $\mathrm{Sub}(A)$. In the former case, f is monic due to proposition 1.2.1. In the latter, f must be zero which is not the case by assumption.

(ii) Immediate by duality.

(iii) It's an immediate consequence of the two previous point and of proposition 1.2.1.

□

In the category if modules, there is a well known theorem that establishes a bijective correspondance between the submodules of a quotient M/N and the intermediate modules between N and M simply by taking the inverse image by the projection onto the quotient. This result still holds in the context of abstract abelian categories:

Theorem 1.3.9 (Correspondance of subobjects). *Let i be a subobject of some A in an abelian category $\mathscr{A}$. Then, there is an order-preserving bijective corespondance between the intermediate subobjects $i \subset j \subset 1_A$ and the subobject of* $\mathrm{Coker}(i)$ *given by taking the cokernel of j.*

Proof. Let's consdier $i : A_i \to A$ and a subobject $b : B \hookrightarrow \mathrm{Coker}(i)$. We may then take $\pi_i^{-1}(B)$ the pullback of b and π_i the cokernel of i[(vi)]. This corresponds to the following diagram:

$$\begin{tikzcd} & \pi_i^{-1}(B) \arrow[r, two heads] \arrow[d, hook, "\bar{b}"'] & B \arrow[d, hook, "b"] \\ A_i \arrow[r, hook, "i"] & A \arrow[r, two heads, "\pi_i"] & \mathrm{Coker}(i) \end{tikzcd} \tag{1.3.7}$$

Then, we must notice that $\pi_i \circ i = 0$ and therefore there is a commutative square:

$$\begin{tikzcd} A_i \arrow[r, "0"] \arrow[d, hook, "i"'] & B \arrow[d, two heads] \\ A \arrow[r, two heads, "\pi_i"'] & \mathrm{Coker}(i) \end{tikzcd} \tag{1.3.8}$$

(vi) This construction is the way to generalise in abstract categories the notion of inverse image which justify the notation $\pi_i^{-1}(B)$

and applying the universal property of the pullback, we obtain an arrow $j : A_i \to P$ such that $\bar{b} \circ j = i$. It remains to prove that j is monic. It is easy to show that:

$$\mathrm{Ker}(j) = \mathrm{Ker}(\bar{b} \circ j) = \mathrm{Ker}(i) = 0 \tag{1.3.9}$$

by monicity of $\bar{b}$.

Now, we only have to show that b is a cokernel of j and that conversely, if $i \subset k \subset 1_A$ is a subobject, applying our construction to the cokernel of k gives k up to equivalence of subobjects, which are quite straightforward. □

Applying this result to simple quotients gives to very useful lemma:

Lemma 1.3.10. *Given a subobject $A_i \hookrightarrow A$, the quotient ${}^{A}\!/_{A_i}$ is simple if and only if there is no intermediate subobject between A_i and A.*

Simple objects will serve as basic bricks to construct more complicated objects. Alas, every object is not semi-simple, hence we'll need something more flexible. It will motivate the introduction of a *composition series.*

Definition 1.3.11 (Filtration). *A filtration of an object A is a tower of subobjets:*

$$0 = i_0 \subset i_1 \subset \cdots \subset i_n = 1_A \tag{1.3.10}$$

The cokernels $\mathrm{Coker}(\iota_i)$ *are called the* factors *of the filtration.*

Definition 1.3.12 (Composition serie). *A* composition series *of an object A is a filtration in which every factor is simple.*

Remark 1.3.13: Informally (or in concrete categories), a composition series becomes a tower of objects:

$$0 = A_0 \subset \cdots \subset A_n = A \tag{1.3.11}$$

such that every quotient ${}^{A_{j+1}}\!/_{A_j}$ is simple. ♦

We wish to know whether an object admits a composition series or not. There is indeed a pretty simple cirterion:

Theorem 1.3.14. *Let $\mathscr{A}$ be an abelian category and A a non-zero object in $\mathscr{A}$. We suppose:*

(i) $\mathrm{Sub}(A)$ *is a set.*

(ii) *A is noetherian, that is, every ascending chain of subobjects is ultimely stationary.*

(iii) *A is artinian, thet is, every descending chain of subobjects is ultimely stationary.*

Then A admits a composition series.

Proof. We will prove this theorem by exhibiting a terminating algorithm which returns a composition series of A.

```
Algo: CompositionSerie:

Entry: a filtration of A:  0 ⊂ i_1 ⊂ ··· ⊂ 1_A
Return: a composition series of A
if the entry is a composition series:
   return the entry
else:
   chose the term of highest index i_j whose cokernel is not simple
   add an intermediate term i_j ⊊ ĩ ⊊ i_{j+1} in the filtration such that the factor
of ĩ ⊂ i_{j+1} is simple
   return CompositionSerie applied on the extended filtration.
```

Note that this algorithm is possible due to the application of ZORN's lemma on the poset of intermediate subobjects to extract a simple factor. Indeed, $\mathrm{Sub}(A)$ being a noetherian poset, ZORN's lemma allows us to extract a maximal intermediate subobject, and the obtained factor must be simple thanks to lemma 1.3.10.

If this algorithm terminates, it is crystal clear that it returns a composition series, and we may give it the trivial filtration $0 \subset 1_A$ as an entry.

Suppose, for the sake of contradiction, that it does not terminate on some entry. There is then an inductive procedure which alows us to build iteratively a descending sequence of maximal subobjects. Since we supposed that $\mathrm{Sub}(A)$ is a set, we may apply the axiom of dependant choice to obtain an infinite sequence of such subobjects, which contradicts the artinian condition. □

Remark 1.3.15: The converse is true, yet we won't need it for our work. ♦

We will often work in categories in which every object admit a composition series, and those will allow us to prove various results using the simples. The main result about composition series is certainly JORDAN-HÖLDER's theorem. We will state and prove it in a quite general context although it remains true even with weaker hypothesis, but for the purpose of our work, the formulation will be highly sufficient. Our proof is motivated by [1], and will almost follow the proof that is given in [8] with adapted arguments in the context of abelian categories.

Theorem 1.3.16 (JORDAN-HÖLDER). *Let $\mathscr{A}$ be an abelian category in which every object admits a composition series*(vii)*. Then, given an object A and two composition series :*

$$\begin{gathered} 0 = A_0 \xhookrightarrow{i_0} A_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_s} A_{s+1} = A \\ 0 = B_0 \xhookrightarrow{j_0} B_1 \xhookrightarrow{j_1} \dots \xhookrightarrow{j_t} B_{t+1} = A \end{gathered} \tag{1.3.12}$$

we have that $s = t$ and there exists $\sigma \in \mathfrak{S}_s$ such that:

$$\forall i \in [\![0, s-1]\!],\ A_{i+1}/A_i \cong B_{\sigma(i)+1}/B_{\sigma(i)} \tag{1.3.13}$$

(vii) The theorem is still true if we only suppose that A admits a composition serie, but it is much harder to prove. See [1].

Proof. We will start our proof by a lemma:

Lemma 1.3.17 (Parallelogram rule). *Given two subobjects A_i and A_j of A, we have an isomorphism:*

$$(A_i + A_j)\Big/A_j \cong A_j\Big/(A_i \cap A_j) \tag{1.3.14}$$

The proof, though not very complicated, is quite technical. The reader may find it in [1], proposition 4.27.

We will reason by induction on s. If $s = 1$, then A is a simple object and t must be 1 too.

Let's suppose the result true for any integer inferior to a certain $s - 1$, and consider the composition series (1.3.12). If $i_s = j_t$, then we have two composition series of A_s, one of length $s - 1$, and the result follows by induction. We suppose then that $i_s \neq j_t$. It implies that $i_s \vee j_t \neq i_s$. But since $\mathrm{Coker}(i_s)$ is simple, there is no intermediate object between i_s and 1_A (lemma 1.3.10), hence $i_s \vee j_t = 1_A$. We may now apply the parallelogram rule (lemma 1.3.17):

$$A/A_s \cong B_t/A_s \cap B_t \quad A/B_t \cong A_s/A_s \cap B_t \tag{1.3.15}$$

But by assumption, $A_s \cap B_t$ admits a compositon serie too:

$$0 = C_0 \xhookrightarrow{k_0} C_1 \xhookrightarrow{k_1} \dots \xhookrightarrow{k_r} C_{r+1} = A_s \cap B_t \tag{1.3.16}$$

Hence we obtain another composition series of A_s:

$$0 = C_0 \xhookrightarrow{k_0} C_1 \xhookrightarrow{k_1} \dots \xhookrightarrow{k_r} A_s \cap B_t \xhookrightarrow{} A_s \tag{1.3.17}$$

And by the induction hypothesis, we have $r = s-1$ and every factor are pairwise isomorphic up to a certain permutation. We may now compare the two composition series:

$$\begin{gathered} 0 = C_0 \xhookrightarrow{k_0} C_1 \xhookrightarrow{k_1} \dots \xhookrightarrow{k_r} A_s \cap B_t \xhookrightarrow{} A_s \xhookrightarrow{i_s} A \\ 0 = C_0 \xhookrightarrow{k_0} C_1 \xhookrightarrow{k_1} \dots \xhookrightarrow{k_r} A_s \cap B_t \xhookrightarrow{} B_T \xhookrightarrow{j_t} A \end{gathered} \tag{1.3.18}$$

We proved previously that the last factors are isomorphic. Therefore, every factors of these two composition series are pairwise isomorphic. It remains only to do with B_t the same trick we did with A_s to prove that $s = t$ and to complete the proof. ☐

Jordan-Hölder theorem has several powerful implications. In particular, we can now define properly the length of an object in an abelian category which meet the hypothesis as the integer s. A category in which every object admits a composition series will be called a *length category.*

1.4 Exact sequences & Yoneda's extension groups

Exact sequences hold a major role in modern algebra. Abelian categories provide us a proper context to define them and work with them. We will also introduce the extension groups. As many of the properties of these groups rely on proofs heavy on notations while not very intersting for our subject, we shall often redirect the reader to some references.

In this paragraph, every considered category will be supposed locally small and abelian.

Definition 1.4.1 (Exact sequence). *Let* $\mathscr{A}$ *be an (abelian) category. An* exact sequence *is a sequence of morphisms:*

$$\dots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \xrightarrow{f_{n+2}} \dots \tag{1.4.1}$$

such that:

$$\forall n \in \mathbb{Z},\ \operatorname{Ker}(f_{n+1}) = \operatorname{Im}(f_n) \tag{1.4.2}$$

When the exact sequence is of the form(viii)*:*

$$0 \longrightarrow K \overset{\kappa}{\hookrightarrow} A \overset{\pi}{\twoheadrightarrow} C \longrightarrow 0 \tag{1.4.3}$$

we call it a short exact sequence.

Remark 1.4.2:

(i) In a short exact sequence, the left morphism is always monic since its kernel is 0 and the right one is epic since its cokernel is 0. Furhermore, κ is the kernel of π and π is the cokernel of κ. Hence, a short exact sequence represent the diagram of a quotient by a subobject:

$$0 \longrightarrow A_i \overset{i}{\hookrightarrow} A \overset{p}{\twoheadrightarrow} {}^{A}\!/_{A_i} \longrightarrow 0 \tag{1.4.4}$$

Conversely, every quotient or subobject of A gives birth to such an exact sequence.

(ii) The condition $\operatorname{Ker}(f_{n+1}) = \operatorname{Im}(f_n)$ is stronger than having an isomorphism between the underlying objects of the kernel and the image. The iso must be compatible with the morphisms, otherwise a sequence like

$$0 \longrightarrow 4\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{Z} \overset{p}{\twoheadrightarrow} {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \longrightarrow 0 \tag{1.4.5}$$

would be exact in **Ab**.

♦

(viii) To dissipate all ambiguity, K denotes in this sequence an object of the category $\mathscr{A}$, not a field. The letter K has been chosen to remind that κ is the kernel of π.

Definition 1.4.3 ((Left/Right-)exact functor). *Let $F : \mathscr{A} \to \mathscr{B}$ be an additive functor. We say that F is* left-exact *(resp.* right-exact*) when, given any short exact sequence in $\mathscr{A}$:*

$$0 \longrightarrow A' \overset{f}{\hookrightarrow} A \overset{g}{\twoheadrightarrow} A'' \longrightarrow 0 \tag{1.4.6}$$

F sends it to an exact sequence:

$$0 \longrightarrow F(A') \overset{F(f)}{\hookrightarrow} F(A) \xrightarrow{F(g)} F(A'') \tag{1.4.7}$$

reps.

$$F(A') \xrightarrow{F(f)} F(A) \overset{F(g)}{\twoheadrightarrow} F(A'') \longrightarrow 0 \tag{1.4.8}$$

We say that F is exact *when it is both left and right exact, thet is, it sends short exact sequences to short exact sequences.*

Remark 1.4.4: It follows immediately from the definition that an exact functor preserves epimorphisms, monomorphisms, subobjects and quotients. ♦

Proposition 1.4.5. *Left (resp. right) adjoints are right-exact (resp. left-exact).*

Proof. We consider a pair of adjoints:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "G"'] & \mathscr{B} \end{tikzcd} \tag{1.4.9}$$

and a short exact sequence in $\mathscr{A}$:

$$0 \longrightarrow A' \overset{f}{\hookrightarrow} A \overset{g}{\twoheadrightarrow} A'' \longrightarrow 0 \tag{1.4.10}$$

We already know by theorem A.4.12 and lemma A.4.13 that F preserves epimorphisms as a left adjoint. Therefore we only have to prove that $\mathrm{Im}(F(f)) = \mathrm{Ker}(F(g))$. By assimilating the kernels and cokernels to the underlying morphisms:

$$\mathrm{Im}(F(f)) := \mathrm{Ker}(\mathrm{Coker}(F(f))) = \mathrm{Ker}(F(\mathrm{Coker}(f))) \tag{1.4.11}$$

because F preserves colimits as a left adjoint (theorem A.4.12). But because the sequence (1.4.10) is exact,

$$\mathrm{Coker}(f) = g \tag{1.4.12}$$

which gives exactly:

$$\mathrm{Im}(F(f)) = \mathrm{Ker}(F(g)) \tag{1.4.13}$$

and F is right-exact. The other statement follows by duality. □

We will sometimes use this well known lemma of homological alebra:

Lemma 1.4.6 (Short five lemma). *Consider a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, hook, "i"] \arrow[d, "f'"] & A \arrow[r, two heads, "p"] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r, hook, "j"] & B \arrow[r, two heads, "q"] & B'' \arrow[r] & 0 \end{tikzcd} \tag{1.4.14}$$

Then, f is monic (resp. epic, an isomorphism) if and only if f' and f'' are monic (resp. epic, isomorphisms).

Proof.

- We suppose that f' and f'' are monic. We will prove that f is monic too using proposition 1.2.1 and showing that $\mathrm{Ker}(f)$ is a terminal object. Let's denote by κ the inclusion of the kernel of f and consider $z : Z \to \mathrm{Ker}(f)$ an arrow. Then we have:

  $$q \circ f \circ \kappa \circ z = 0 \tag{1.4.15}$$

  and hence:

  $$f'' \circ p \circ \kappa \circ z = 0 \tag{1.4.16}$$

  But by monicity of f'', $p \circ \kappa \circ z = 0$ and $\kappa \circ z$ factorise uniquely through the kernel of p, which is the image of i. i being monic, its image is isomorphic to A' and hence we have a relation:

  $$i \circ \bar{z} = \kappa \circ z \tag{1.4.17}$$

  Therefore, composing by f gives:

  $$j \circ f' \circ \bar{z} = 0 \tag{1.4.18}$$

  and by monicity of j and f', we have $\bar{z} = 0$ and therefore:

  $$\kappa \circ z = 0 \tag{1.4.19}$$

  κ being monic too, we finally obtain $z = 0$. As it holds for any z with codomain $\mathrm{Ker}(f)$, we have that $\mathrm{Ker}(f)$ is zero and that f is monic.

- The converse require the same kind of "diagram chase". We will skip it.

The other statements follow by duality and proposition 1.2.1. □

Proposition 1.4.7 (Split sequence). *Let's consider an exact sequence:*

$$0 \longrightarrow K \xhookrightarrow{\kappa} A \xrightarrow{\pi} C \longrightarrow 0 \tag{1.4.20}$$

the following statements are equivalent:

(i) κ *admits a retraction* r *(i.e.* $r \circ \kappa = 1_K$*).*

(ii) π *admits a section* s *(i.e.* $\pi \circ s = 1_C$*)/*

(iii) $A \cong K \oplus C$ *and the following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\kappa"] \arrow[d, equal] & A \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"', "\wr"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, hook, "\iota_k"] & K \oplus C \arrow[r, two heads, "p_C"] & C \arrow[r] & 0
\end{tikzcd} \tag{1.4.21}$$

When these hold, we say that the sequence is split.

Proof. • We suppose that κ admits a retraction r and we will prove that $A \cong K \oplus C$. Let's take $\alpha : A \to K \oplus C$ using the universal property of the product on r and π, thanks to proposition 1.2.2. We then have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, shift left, "\kappa"] \arrow[d, equal] & A \arrow[l, two heads, shift left, "r"] \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K \arrow[r, hook, shift left, "\iota_k"] & K \oplus C \arrow[l, two heads, shift left, "p_k"] \arrow[r, two heads, "p_C"] & C \arrow[r] & 0
\end{tikzcd} \tag{1.4.22}$$

We just have to show that $\alpha \circ \kappa = \iota_k$ and that $p_C \circ \alpha = \pi$ and then apply the five lemma.

$$(p_K \circ \alpha) \circ \kappa = r \circ \kappa = 1_K, \quad (p_C \circ \alpha) \circ \kappa = \pi \circ \kappa = 0 \tag{1.4.23}$$

and then $\alpha \circ \kappa$ has the same components as ι_K therefore these arrows must be equal (and we used the fact that $p_C \circ \alpha = \pi$ which is by definition of α. The diagram commutes and α is an isomorphism thanks to lemma 1.4.6.

- Proving that (2) $\implies$ (3) is simillar.
- We suppose (3). Then taking $s = p_C \circ \alpha$ and $r = p_K \circ \alpha$ gives a section and a retraction for κ and π.

□

Definition 1.4.8 (Extension group). *Let A and B be two objects in a locally small abelian category $\mathscr{A}$. We say that two short exact sequences $\mathfrak{E}_1$ and $\mathfrak{E}_2$ with border objects B and A are equivalent whenever there exists a morphism f such that the following diagram commutes:*

$$
\begin{tikzcd}
(\mathfrak{E}_1 :) & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, "f"] & A \arrow[r] \arrow[d, equal] & 0 \\
(\mathfrak{E}_2 :) & 0 \arrow[r] & B \arrow[r, hook] & E' \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
\tag{1.4.24}
$$

We write $\operatorname{Ext}^1_{\mathscr{A}}(A, B)$ for the set of all equivalence class of such short exact sequences under this relation and call it YONEDA's first extension group of A and B.

Remark 1.4.9:

- it follows from the five lemma that such an arrow f is an isomorphism, and therefore this relation is indeed an equivalence relation.
- in the short exact sequence $\mathfrak{E}_1$, the object E is called an *extension* of A and B. In the category $\mathbf{Mod}(R)$, B identifies as submodule of E and A identifies as the quotient ${}^{E}/_{B}$, therefore it makes sense to percieve E as a bigger object built in some sense by gluing A and B together.

♦

It is unclear how these sets can be seen as "groups" for now. It is nonetheless possible to define a group law on the extension groups, but it will require some additional work.

Definition 1.4.10. *Let $\alpha : A' \to A$ and $\beta : B \to B'$ be two arrows in $\mathscr{A}$ (a locally small abelian category). Given $\mathfrak{E} \in \operatorname{Ext}^1_{\mathscr{A}}(A, B)$, we define $\beta \cdot \mathfrak{E}$ and $\mathfrak{E} \cdot \alpha$ by taking P the pushout of b and β and P' the pullback of a and α in the following diagram.*

$$
\begin{tikzcd}
(\mathfrak{E} :)\alpha & 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & P' \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\alpha"] & 0 \\
(\mathfrak{E} :) & 0 \arrow[r] & B \arrow[r, hook, "b"] \arrow[d, "\beta"] & E \arrow[r, two heads, "a"] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\
\beta(\mathfrak{E} :) & 0 \arrow[r] & B' \arrow[r, hook] & P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
\tag{1.4.25}
$$

It makes $\operatorname{Ext}^1_{\mathscr{A}}$ a bifunctor.

In order to define a group law on $\operatorname{Ext}^1_{\mathscr{A}}(A, B)$, we need two important arrows for any object $Z \in \mathscr{A}$:

$$\Sigma_Z : Z \oplus Z \to Z \tag{1.4.26}$$

defined by using the product's universal property on $(1_Z, 1_Z)$, and

$$\Delta_Z : Z \oplus Z \to Z \tag{1.4.27}$$

obtained by using the universal property of the coproduct on $(1_Z, 1_Z)$. In the special case where $\mathscr{A}$ is a module category, we have explicit and simple expressions :

$$\Sigma_Z(z, z') = z + z' \quad \Delta_Z(z) = (z, z) \tag{1.4.28}$$

Furthermore, given two exact sequences:

$$\begin{aligned} &(\mathfrak{E}_1 :) & 0 \longrightarrow B \hookrightarrow E \twoheadrightarrow A \longrightarrow 0 \\ &(\mathfrak{E}_2 :) & 0 \longrightarrow B' \hookrightarrow E' \twoheadrightarrow A \longrightarrow 0 \end{aligned} \tag{1.4.29}$$

We define $\mathfrak{E}_1 \oplus \mathfrak{E}_2$ as the element of $\mathrm{Ext}^1_{\mathscr{A}}(A \oplus A', B \oplus A')$:

$$(\mathfrak{E}_1 \oplus \mathfrak{E}_2 :) \qquad 0 \longrightarrow A \oplus A' \hookrightarrow E \oplus E' \twoheadrightarrow B \oplus E' \longrightarrow 0 \tag{1.4.30}$$

with all the arrows given by applying the isomorphism of proposition 1.2.3.

Definition 1.4.11 (Baer's sum). *Let A and B be two objects of $\mathscr{A}$. We equip $\mathrm{Ext}^1_{\mathscr{A}}$ with the internal law:*

$$\mathfrak{E}_1 + \mathfrak{E}_2 := \Sigma_A \cdot (\mathfrak{E}_1 \oplus \mathfrak{E}_2) \cdot \Delta_A \tag{1.4.31}$$

We call it the Baer*'s sum of $\mathfrak{E}_1$ and $\mathfrak{E}_2$.*

Theorem 1.4.12. *For any pair of object $A, B \in \mathscr{A}$, $\mathrm{Ext}^1_{\mathscr{A}}(A, B)$, the set $\mathrm{Ext}^1_{\mathscr{A}}(A, B)$ equipped with the* Baer*'s sum is an abelian group with identity:*

$$0 \longrightarrow A \xhookrightarrow{\iota_A} A \oplus B \xrightarrow{p_B} B \longrightarrow 0 \tag{1.4.32}$$

Furthermore, if $\mathscr{A}$ is a K-linear category, the $\mathrm{Ext}^1_{\mathscr{A}}$ becomes a K-vector space with respect to the scalar multiplication:

$$(\lambda \cdot \mathfrak{E} :) \qquad 0 \longrightarrow A \xhookrightarrow{\frac{1}{\lambda} a} E \xrightarrow{b} B \longrightarrow 0 \tag{1.4.33}$$

As one may have guessed, the proof is very heavy while not being particularly intersting. A full proof is given in the case of modules in [4], section 5.2, and can be easily adapted in the context of any abelian category.

We will finish this paragraph by giving a short survey on properties comming from homological algebra. We will avoid the details as they rely on notions of cohomology we did not study for the purpose of this intership, but it will sometimes be useful:

Proposition 1.4.13. *Given $B \in \mathscr{A}$ and an exact sequence:*

$$0 \longrightarrow A' \overset{f}{\hookrightarrow} A \overset{g}{\twoheadrightarrow} A'' \longrightarrow 0 \tag{1.4.34}$$

there exists a long exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_{\mathscr{A}}(B, A') \arrow[r, "\mathrm{Hom}(B\,f)"] & \mathrm{Hom}_{\mathscr{A}}(B, A) \arrow[r, "\mathrm{Hom}(B\,g)"] & \mathrm{Hom}_{\mathscr{A}}(B, A'') \arrow[dll] \\ & \mathrm{Ext}^1_{\mathscr{A}}(B, A') \arrow[r, "\mathrm{Ext}^1_{\mathscr{A}}(B\,f)"] & \mathrm{Ext}^1_{\mathscr{A}}(B, A) \arrow[r, "\mathrm{Ext}^1_{\mathscr{A}}(B\,g)"] & \mathrm{Ext}^1_{\mathscr{A}}(B, A'') \end{tikzcd} \tag{1.4.35}$$

Proof. See [6], proposition 2.5.2. □

One may wish to continue this sequence on the right, and indeed it's possible using the extension groups of higher order $\mathrm{Ext}^p_{\mathscr{A}}$, but a proper introduction would requires some tools from cohomology, hence we won't do more than mentionning their existence.

1.5 Projectivity

When comming to highest weight categories, we will often need special kind of objects named *projectives*. This paragraph aims at introducing the required definitions and basic properties we will need for our study in a very general context.

Throughout this paragraph, we set an abelian category $\mathscr{A}$.

Definition 1.5.1 (Projective object). *An object $P \in \mathscr{A}$ is said to be* projective *when, for any given epimorphism $\pi : A \twoheadrightarrow B$, the map* $\mathrm{Hom}_{\mathscr{A}}(P, \pi)$ *is surjective (equivalently, the functor* $\mathrm{Hom}_{\mathscr{A}}(P, -)$ *preserves epimorphisms).*

In other words, it means that given a corner of arrows:

$$\begin{tikzcd} & P \arrow[d, "f"] \\ A \arrow[r, two heads, "\pi"] & B \end{tikzcd} \tag{1.5.1}$$

There exists an arrow such that the following triangle commutes:

$$\begin{tikzcd} & P \arrow[d, "f"] \arrow[dl, dashed] \\ A \arrow[r, two heads, "\pi"] & B \end{tikzcd} \tag{1.5.2}$$

As always, we define the dual notion of *injective* objects.

Example 1.5.2:

- Given a field K, every vector space is projective in $\mathbf{Vect}(K)$. More generally, if R is a ring, every free R-module is projective in $\mathbf{Mod}(R)$.

- The statement that every set is projective in **Set** is equivalent to the axiom of choice.

⧫ There are several criterions used to detect projective objects:

Proposition 1.5.3 (Projectivity criterions). *Let $P \in \mathscr{A}$ be an object. Then, the following are equivalent:*

(i) *P is projective.*

(ii) *Every epimorphism $\pi : Z \to P$ admits a section.*

(iii) $\mathrm{Hom}_{\mathscr{A}}(P, -)$ *is an exact functor.*

(iv) $\mathrm{Ext}^1_{\mathscr{A}}(P, -)$ *is the 0 functor.*

Proof.

(i) $\implies$ (iii) : Given an exact sequence in $\mathscr{A}$:

$$0 \longrightarrow A' \xhookrightarrow{\iota} A \xrightarrow{\pi} A'' \longrightarrow 0 \tag{1.5.3}$$

we may apply $\mathrm{Hom}_{\mathscr{A}}(P,-)$ on it to obtain the following sequence in **Ab**:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(P, A') \xrightarrow{\iota\circ-} \mathrm{Hom}_{\mathscr{A}}(P, A) \xrightarrow{\pi\circ-} \mathrm{Hom}_{\mathscr{A}}(P, A'') \longrightarrow 0 \tag{1.5.4}$$

We now shall prove it is exact. $\iota \circ -$ is monic by monicity of ι:

$$f \in \mathrm{Ker}(\iota \circ -) \iff \iota \circ f = 0 \iff f = 0 \tag{1.5.5}$$

The projectivity of π gives the surjectivity of $\pi \circ - = \mathrm{Hom}_A(P, \pi)$. As $\iota \circ \pi = 0$, it is immediate that $\mathrm{Im}(\iota) \subset \mathrm{Ker}(\pi)$. For f such that $\pi \circ f = 0$, we have that f factorise uniquely through the kernel of π, which is $\mathrm{Im}(\iota)$, and we find a factorisation $f = \iota \circ \bar{f}$. So our sequence is exact. Note that (iii) $\implies$ (i) is clear since an exact functor always preserves epimorphisms.

(iii) $\implies$ (iv) : Assume that $\mathrm{Hom}_A(P,-)$ is exact. Then, given an epi $\pi : Z \to P$, the exactness of $\mathrm{Hom}_A(P,-)$ implies that it preserves epimorphism, so $\mathrm{Hom}_{\mathscr{A}}(P, \pi) : \mathrm{Hom}_A(P, Z) \to \mathrm{Hom}_A(P, P)$ is surjective. In particular, there exis $s \in \mathrm{Hom}_A(P, Z)$ such that $\pi \circ s = 1_P$.

(ii) $\implies$ (ii) : By functoriality, it is sufficient to prove that for any object A, $\mathrm{Ext}^1_{\mathscr{A}}(P, A)$ is the one-element group, that is, every short sequence:

$$0 \longrightarrow A \lhook\joinrel\longrightarrow E \longrightarrow\!\!\!\!\!\rightarrow P \longrightarrow 0 \tag{1.5.6}$$

splits. Using lemma 1.4.7 and hypothesis (ii), it is trivial.

(iv) $\implies$ (iii) : We will use proposition 1.4.13 and the fact that $\mathrm{Ext}_{\mathscr{A}}(P,-)$ is zero. Then, applying $\mathrm{Hom}_A(P,-)$ to an exact sequence gives another exact sequence, which is exactly (iii).

An elementary proof of (iv) $\implies$ (i) can be given when $\mathscr{A}$ is a category of modules by showing first that a module is projective if and only if it is the direct summand of a free module. $\square$

Lemma 1.5.4.

(i) *A coproduct of projectives is still projective.*

(ii) *A summand of a projective is projective.*

Proof.

(i) Let $(P_i)_{i\in I}$ be a family of projectives in $\mathscr{A}$ that admits a coproduct and denote by $\iota_j : P_j \to \bigoplus_{i\in I} P_i$ the natural injection. Let's consider an epimorphism $\pi : A \twoheadrightarrow B$ and an arrow $f : \bigoplus_{i\in I} P_i \to B$. Then, taking $f_j := f \circ \iota_j : P_j \to B$, there exist $\bar{f}_j : P_j \to A$ such that $\bar{f}_j \circ \pi = f_j$ by projectivity of P_j. We may then take $\bar{f} : \bigoplus_{i\in I} P_i \to A$ given by universal property of the coproduct applyed on the $(\bar{f}_i)$. By unicity, $\pi \circ \bar{f} = f$ and $\bigoplus_{i\in I} P_i$ is projective.

(ii) Let $A, B \in \mathscr{A}$ be two objects and assume that P admits a decomposition $P = Q \oplus R$. Given an epimorphism $\pi : A \to B$ and an arrow $f : Q \to B$, we may extend f to P by using the universal property of the coproduct on f and $0 : R \to B$. Using the projectivity of P, we construct an arrow $\bar{f} : P \to A$ such that $\pi \circ \bar{f} \circ \iota_Q = f$ and so Q is projective.

$\square$

Something we will often try to do is to see arbitrary objects as quotient of a projective.

Definition 1.5.5 (Category with enough projectives / injectives). *An abelian category $\mathscr{A}$ is said to* have enough projectives *when every object is the quotient of a projective, that is, for any $A \in \mathscr{A}$, there exists an epimorphism $P \twoheadrightarrow A$ with P projective.*
Dually, $\mathscr{A}$ is said to have enough injectives *when every object is a subobject of an injective, that is, given any object $A \in \mathscr{A}$, there exist a monomorphism $A \hookrightarrow I$ with I injective.*

We will have two main tools to study such categories:

Definition 1.5.6 (Projective generator). *An object $P \in \mathscr{A}$ is said to be a (finite)* projective generator *when for any object A, there exist a (finite) set I such that A is a quotient of $P^{\oplus I}$.*

Remark 1.5.7: We will only work with finite projective generators in this report, so we will now use the term projective generator for a finite projective generator. ♦

Definition 1.5.8 (Projective cover). *Let $A \in \mathscr{A}$ be an object. We call a pair (P, p) a projective cover of A when:*

(i) *P is a projective object.*

(ii) *$p : P \twoheadrightarrow A$ is an epimorphism.*

(iii) $\forall \alpha \in \mathrm{End}(P),\ p \circ \alpha = p \implies \alpha \in \mathrm{Aut}(P)$

Projective covers are, in some sense, minimal projective objects such that A can be seen as a quotient of it. As often in category theory, when an object satisfies some sort of minimality property, it is unique up to isomorphism:

Proposition 1.5.9 (Unicity of the projective cover). *Let $A \in A$ be any object and $(P, p), (P', p')$ be two projective covers of A. Then there exists an isomorphism $\alpha : P \to P'$ such that the following diagram commutes:*

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "p"'] & & P' \arrow[dl, two heads, "p'"] \\ & A & \end{tikzcd} \tag{1.5.7}$$

Proof. Using the projectivity of P and P', we find two arrows $\alpha : P \to P'$ and $\beta : P \to P'$ such that these diagrams commute:

$$\begin{tikzcd} P \arrow[rr, "\alpha"] \arrow[dr, two heads, "p"'] & & P' \arrow[dl, two heads, "p'"] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} P \arrow[dr, two heads, "p"'] & & P' \arrow[ll, "\beta"'] \arrow[dl, two heads, "p'"] \\ & A & \end{tikzcd} \tag{1.5.8}$$

since p and p' are both epic. But now:

$$p \circ (\beta \circ \alpha) = p' \circ \alpha = p \tag{1.5.9}$$

which implies that $\beta \circ \alpha$ is an isomorphism and therefore α must be monic. Doing the same with $\alpha \circ \beta$, we show that α is epic. By proposition 1.2.1, α is an isomorphism. □

Remark 1.5.10: The isomorphism need not be unique. ♦

We may now talk about the projective cover. Whenever this exists, we will note $\mathbb{P}_{\mathscr{A}}(A)$ a projective cover of A in the category $\mathscr{A}$, or just $\mathbb{P}(A)$ when the context is clear enough.

Projectives and simple objects act together in a convenient way, and the projective covers of simple objects will be fundamental for the study of highest weight categories. We will give here some intersting properties:

Proposition 1.5.11. *Let $\mathscr{A}$ be an abelian length category with finitely many isoclasses of simple objects $S_1, \dots S_n$ and such that each of the S_i is a quotient of some projective P_i. Then, the object:*

$$\mathbb{P} := \bigoplus_{i=1}^{n} P_i \tag{1.5.10}$$

is a projective generator of $\mathscr{A}$.

Proof. Let $A \in \mathscr{A}$ be an object of length l. We have a composition series:

$$0 = A_0 \xhookrightarrow{i_0} A_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_l} A_{l+1} = A \tag{1.5.11}$$

with ${A_{i+1}}/{A_i} \cong S_{k(i)}$ for some $k(i) \in [\![n]\!]$. We will reason by induction on l.

For $l = 1$, A is simple and is therefore a quotient of $\mathbb{P}(S_{k(1)})$. It is clear that it is a quotient of $\mathbb{P}$.

Assume it is true for any object of length $l - 1$. Then by induction, A_l is a quotient of $\mathbb{P}^{l-1}$. We have therefore the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{P}^{l-1} \arrow[r, hook] \arrow[d, two heads] & \mathbb{P}^{l-1} \oplus \mathbb{P} \arrow[r, two heads] & \mathbb{P} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & A_l \arrow[r, hook] & A \arrow[r, two heads] & S_{k(l)} \arrow[r] & 0
\end{tikzcd} \tag{1.5.12}$$

But then by projectivity of $\mathbb{P}$ (lemma 1.5.4), there exists an arrow $\mathbb{P} \to A$ such that the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{P}^{l-1} \arrow[r, hook] \arrow[d, two heads] \arrow[dr, dashed] & \mathbb{P}^{l-1} \oplus \mathbb{P} \arrow[r, two heads] & \mathbb{P} \arrow[r] \arrow[d, two heads] \arrow[dl, dashed] & 0 \\
0 \arrow[r] & A_l \arrow[r, hook] & A \arrow[r, two heads] & S_{k(l)} \arrow[r] & 0
\end{tikzcd} \tag{1.5.13}$$

(the left dashed-arrow is simply obtained by composition) and using the universal property of the coproduct on the two dashed arrows, we obtain an arrow $\mathbb{P}^{l-1} \oplus \mathbb{P} \to A$ such that the diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{P}^{l-1} \arrow[r, hook] \arrow[d, two heads] & \mathbb{P}^{l-1} \oplus \mathbb{P} \arrow[r, two heads] \arrow[d, dashed] & \mathbb{P} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & A_l \arrow[r, hook] & A \arrow[r, two heads] & S_{k(l)} \arrow[r] & 0
\end{tikzcd} \tag{1.5.14}$$

The short five lemma (1.4.6) asserts that this arrow is epic. Hence A is a quotient of $\mathbb{P}^l$ and the induction is complete.

□

Proposition 1.5.12. *Let $S \in \mathscr{A}$ be a simple objet which admits a projective cover. Then $\mathbb{P}(S)$ is indecomposable. Furthermore, the converse is true under some assumptions (see section 2).*

Proof. First, let's write:

$$\mathbb{P}(S) \cong A \oplus B \tag{1.5.15}$$

Since we have an epimorphism $p : \mathbb{P}(S) \twoheadrightarrow S$ with S simple, we must have either $p \circ \iota_A \neq 0$ or $p \circ \iota_B \neq 0$ otherwise p would be 0 and S would be 0 by epicity of p. Let's assume that $p \circ \iota_A \neq 0$. By SCHUR's lemma (1.3.8), $p \circ \iota_A$ is epic. Furthermore, A is projective by lemma 1.5.4. Taking $\alpha : A \to A$ such that $p \circ \iota_B \circ \alpha = p \circ \iota_B$, we may apply proposition 1.2.3 to construct the endomorphism:

$$\Phi := \begin{pmatrix} \alpha & 0 \\ 0 & 1_B \end{pmatrix} : A \oplus B \to A \oplus B \tag{1.5.16}$$

But then, we have:

$$p \circ \Phi \circ \iota_A = p \circ \iota_A \circ \alpha = p \circ \iota_A \tag{1.5.17}$$

Similarly:

$$p \circ \Phi \circ \iota_B = p \circ \iota_B \tag{1.5.18}$$

By unicity of a morphism given its components:

$$p \circ \Phi = p \tag{1.5.19}$$

and since $(\mathbb{P}(S), p)$ is a projective cover, Φ is an isomorphism. It is then straightforward to show that α is an isomorphism. But it proves that $A, p \circ \iota_A$ is a projective cover of $\mathbb{P}(S)$. By lemma 1.5.9, $A \cong \mathbb{P}(S)$ and $\mathbb{P}(S)$ is indecomposable. □

Proposition 1.5.13. *Let (P, p) be the projective cover of some simple S in an abelian category $\mathscr{A}$. Then,* $\mathrm{Ker}(p)$ *is a maximal subobject in* $\mathrm{Sub}(P)$ *that countains every other proper subobject.*

Proof. Let $u : U \hookrightarrow P$ be a subobject of P. Then, the composition $\varphi := p \circ u$ is either zero or epic due to SHUR's lemma 1.3.8. If it is zero, then the universal property of the kernel of p shows that u is countained in $\mathrm{Ker}(p)$. If it is epic, by projectivity of P, there is an arrow $v : P \to U$ such that $\varphi \circ v = p$. But this means:

$$\varphi \circ v = p \circ (u \circ v) = p \tag{1.5.20}$$

and since (P, p) is a projective cover, $u \circ v$ is an isomorphism. It implies in particular that u is epic. But by definition, u is also monic. Therefore it is an isomorphism and u is not proper. □

We will conclude this first section by a powerful theorem, which is a result due to P. GABRIEL and B. MITCHELL, given in [5], chapter 2, theorem 1.3. We shall prove this result in a quite elementary way.

Theorem 1.5.14 (Gabriel-Mitchell). *Let $\mathscr{A}$ be an abelian (resp. K-linear) category which admits a projective generator $\mathbb{P}$. Setting $\Lambda := \mathrm{End}(\mathbb{P})$, we have that the functor:*

$$\mathscr{A}(\mathbb{P}, -) : \mathscr{A} \to \mathbf{mod}(\Lambda) \tag{1.5.21}$$

is an equivalence of abelian (resp. K-linear) categories.

Proof. Let's first remark that given $A \in \mathscr{A}$, $\mathscr{A}(\mathbb{P}, A)$ is indeed a right-Λ-module We define the external multiplication by elements of Λ by right-composition, and since the composition is $\mathbb{Z}$-bilinear, it gives $\mathrm{Hom}(\mathbb{P}, A)$ a well difined structure of Λ-module. The fact that this module is finitely generated will come later in the proof.

By theorem A.1.10, we have to prove that $\mathscr{A}(\mathbb{P}, -)$ is full, faithfull and essentially surjective on objects.

- Let's first prove that it is full, that is, for any objects A and B and $f : \mathscr{A}(\mathbb{P}, A) \to \mathscr{A}(\mathbb{P}, B)$ Λ-linear, there exist $\Phi : A \to B$ such that $f = \Phi \circ -$.

  Since $\mathbb{P}$ is a projective generator, there exist subobjects $\iota_I : I \hookrightarrow \mathbb{P}^r$ and $\iota_J : J \hookrightarrow \mathbb{P}^s$ such that $A \cong \mathbb{P}^r/I = \mathrm{Coker}(\iota_I)$ and $B \cong \mathbb{P}^s/J = \mathrm{Coker}(\iota_J)$. Let's consider $\varphi \in \mathscr{A}(\mathbb{P}, A)$ and the following diagram:

$$\begin{tikzcd} & \mathbb{P} \arrow[ddl, "\varphi"] \arrow[ddr, "f(\varphi)"'] & \\ \mathbb{P}^r \arrow[d, two heads, "\pi_I"] & & \mathbb{P}^s \arrow[d, two heads, "\pi_J"] \\ \mathbb{P}^r/I & & \mathbb{P}^s/J \end{tikzcd} \tag{1.5.22}$$

  By projectivity of $\mathbb{P}$, the arrows φ and $f(\varphi)$ admit a lift such that the following diagram commutes:

$$\begin{tikzcd} & \mathbb{P} \arrow[dl, dashed, "\bar{\varphi}"'] \arrow[dr, dashed, "\overline{f(\varphi)}"] \arrow[ddl, "\varphi"] \arrow[ddr, "f(\varphi)"'] & \\ \mathbb{P}^r \arrow[d, two heads, "\pi_I"] & & \mathbb{P}^s \arrow[d, two heads, "\pi_J"] \\ \mathbb{P}^r/I & & \mathbb{P}^s/J \end{tikzcd} \tag{1.5.23}$$

  We may now notice that:

$$\mathscr{A}(\mathbb{P}, \mathbb{P}^r) \cong \Lambda^r \tag{1.5.24}$$

  and $\mathscr{A}(\mathbb{P}, \mathbb{P}^r)$ is hence a free Λ-module. Let's take $(\bar{\varphi}_i)_{1 \leqslant i \leqslant r}$ a basis of $\mathscr{A}(\mathbb{P}, \mathbb{P}^r)$ and denote by $\varphi_i = \pi_I \circ \bar{\varphi}_i$. $(\varphi_i)_{1 \leqslant i \leqslant r}$ is then a generative family of $\mathscr{A}(\mathbb{P}, A)$ and it is therefore sufficient to find Φ such that $f(\varphi_i) = \Phi \circ \varphi_i$ for all i (note that it proves that $\mathscr{A}(\mathbb{P}, A)$ is finitely generated as a Λ module).

  By universal property of free modules, there exists a unique $\mathcal{F} : \mathscr{A}(\mathbb{P}, \mathbb{P}^r) \to \mathscr{A}(\mathbb{P}, \mathbb{P}^s)$ Λ-linear such that:

$$\forall i \in [\![1,\ r]\!],\ \mathcal{F}(\bar{\varphi}_i) = \overline{f(\varphi_i)} \tag{1.5.25}$$

  We may notice now that there exist $F : \mathbb{P}^r \to \mathbb{P}^s$ such that $\mathcal{F} = F \circ -$. Indeed, taking $F_i := \mathcal{F}(\bar{\varphi}_i)\ :\ \mathbb{P} \to \mathbb{P}^s$ and applying the universal property of the coproduct on this family gives us F such that $F \circ \bar{\varphi}_i = \mathcal{F}(\bar{\varphi}_i)$, and we conclude by Λ-linearity.

We then have the following commutative diagram:

$$
\begin{tikzcd}
 & \mathbb{P} \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\overline{f(\varphi)}"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "f(\varphi)"] & \\
\mathbb{P}^r \arrow[rr, "F"] \arrow[d, two heads, "\pi_I"] & & \mathbb{P}^s \arrow[d, two heads, "\pi_J"] \\
\mathbb{P}^r/I & & \mathbb{P}^s/J
\end{tikzcd}
\tag{1.5.26}
$$

Our proof will be complete if we find Φ which makes the following diagram commute:

$$
\begin{tikzcd}
 & \mathbb{P} \arrow[dl, "\bar{\varphi}"'] \arrow[dr, "\overline{f(\varphi)}"] \arrow[ddl, bend right, "\varphi"'] \arrow[ddr, bend left, "f(\varphi)"] & \\
\mathbb{P}^r \arrow[rr, "F"] \arrow[d, two heads, "\pi_I"] & & \mathbb{P}^s \arrow[d, two heads, "\pi_J"] \\
\mathbb{P}^r/I \arrow[rr, dashed, "\Phi"] & & \mathbb{P}^s/J
\end{tikzcd}
\tag{1.5.27}
$$

Let's recall that $\mathbb{P}^r/I$ is the cokernel of $\iota_I : I \to \mathbb{P}^r$. We may therefore prove that $\pi_J \circ F \circ \iota_I$ is zero and then apply the universal property of the cokernel on $\pi_J \circ F$ to obtain Φ.

Let's consider any arrow $a : \mathbb{P} \to I$. We may write:

$$\iota_I \circ a = \sum_{i=1}^{r} \bar{\varphi}_i \lambda_i \tag{1.5.28}$$

with the $\lambda_i \in \Lambda$. Applying π_I gives:

$$\pi_I \circ \iota_I \circ a = \sum_{i=1}^{r} \varphi_i \lambda_i = 0 \tag{1.5.29}$$

But we also have:

$$
\begin{aligned}
\pi_J \circ F \circ \iota_I \circ a &= \pi_J \left(\sum_{i=1}^{r} \overline{f(\varphi_i)} \lambda_i \right) \\
&= \sum_{i=1}^{r} f(\varphi_i) \lambda_i \\
&= f\left(\sum_{i=1}^{r} \varphi_i \lambda_i \right) \\
&= 0
\end{aligned}
$$

As it holds for any $a \in \mathscr{A}(\mathbb{P}, I)$, it is sufficient to prove that $\pi_J \circ F \circ \iota_I = 0$ since $\mathbb{P}$ is a projective generator: indeed we may choose $a : \mathbb{P}^k \twoheadrightarrow I$ an epimorphism an consider the composition $\pi_J \circ F \circ \iota_I \circ a$ component by component to prove that it is zero, and the fact that a is epic suffices to get the result.

- We now prove that it is faithful. Given two arrows $f, g : A \to B$ in $\mathscr{A}$, we have :

$$\begin{aligned}\mathscr{A}(\mathbb{P}, f) = \mathscr{A}(\mathbb{P}, g) &\iff \forall a : \mathbb{P} \to A,\ f \circ a = g \circ a \\ &\iff \forall a : \mathbb{P} \to A,\ (f - g) \circ a = 0 \\ &\iff f - g = 0 \\ &\iff f = g\end{aligned}$$

  as $\mathbb{P}$ is a projective generator. Hence the functor is faithfull.

- To prove the essential surjectivity on objects, we will use the theory of presentations of modules. Let's take $M \in \mathbf{mod}(\Lambda)$. As M is finitely generated, there exists $r, s \in \mathbb{N}$ and a set of *relations* $\rho_1, \dots \rho_r \in \Lambda^s$ such that:

$$M \cong \Lambda^s \Big/ \langle \rho_1, \dots \rho_r \rangle \tag{1.5.30}$$

  Hence taking $\varphi : \Lambda^r \to \Lambda^s$ as the Λ-linear map mapping a basis of Λ^r to the the set of relations, we have that:

$$M \cong \mathrm{Coker}(\varphi) \tag{1.5.31}$$

  But we also have:

$$\varphi \in \mathrm{Hom}_\Lambda(\Lambda^r, \Lambda^s) \cong \mathrm{Hom}_\Lambda(\mathscr{A}(\mathbb{P}, \mathbb{P}^r), \mathscr{A}(\mathbb{P}, \mathbb{P}^s)) \tag{1.5.32}$$

  and by fullness of $\mathscr{A}(\mathbb{P}, -)$, there exists $f : \mathbb{P}^r \to \mathbb{P}^s$ is $\mathscr{A}$ such that $\varphi = \mathscr{A}(\mathbb{P}, f)$. But $\mathbb{P}$ being projective, $\mathscr{A}(\mathbb{P}, -)$ is an exact functor (proposition 1.5.3), and therefore it preserves cokernels. By unicity of the cokernel:

$$M \cong \mathrm{Coker}(\varphi) = \mathrm{Coker}(\mathscr{A}(\mathbb{P}, f)) \cong \mathscr{A}(\mathbb{P}, \mathrm{Coker}(f)) \tag{1.5.33}$$

  and the proof is complete.

$\square$

This result states that whenever the conditions are met, $\mathscr{A}$ may be identified to a category of modules, which will allows us to use properties specific to modules to work on more abstract structures. In particular, it allows us to treat objects as sets of elements, and arrows as concrete applications whith nice properties. We will have to be carefull though, as aditive functors does not in general preserve arbitrary products and coproducts, and thus we cannot extrapolate to many properties of modules onto $\mathscr{A}$.

2 Quivers and representation of algebras

Highest weight categories find a concrete application in the study of algebras and categories of modules over algebras. In this section, we will introduce some basic tools from the representation theory, which will be used in the final part.

The entire goal of this section is to allow us to represent modules over an algebra as a set of vector spaces linked together by K-linear arrows. This will be the role of quivers and their representations. After giving some general considerations on algebras and modules, we will introduce the quivers and see how they can be a powerful tool to study an algebra and its modules. The final part will show how general is this point of view by using the language of categories.

In this section, we fix an alegbraically closed field K. We will always consider finite dimensional algebras over K and modules over such an alegbra. In this context, our modules will always consist of an underlying K-vector space and an action of the algebra compatible with the scalar multiplication. We insist on the fact that our algebras need not be commutative. By default, modules are always considered to be right-modules.

When $\mathcal{A}$ is a K-algebra, we write $\mathbf{Mod}(\mathcal{A})$ for the category of (right) modules over $\mathcal{A}$, and $\mathbf{mod}(\mathcal{A})$ for the full subcategory of finitely generated modules. A module in $\mathbf{mod}(\mathcal{A})$ has therefore a finite dimensional underlying vector space when $\mathcal{A}$ is finite dimensional.

We will write $(\mathcal{A})_{\mathcal{A}}$ when we want to consider $\mathcal{A}$ as a right $\mathcal{A}$-module. We shall often use the language introduced in the first section (simple objects, projectives, etc.) as categories of modules are examples of abelian categories.

2.1 The category of modules

This first paragraph will introduce some tools for the study of categories of modules. As it is mainly intended to create context, we will skip some of the proofs and redirect to a reference. Let $\mathcal{A}$ be a finite dimensional K-algebra. We will begin by applying some of the notions we introduced before in the case of module catgeories:

Theorem 2.1.1 (Properties of module categories).

(i) $\mathbf{mod}(\mathcal{A})$ *is a K-linear category.*

(ii) $\mathbf{mod}(\mathcal{A})$ *is a length category, that is, every object admits a compostion serie.*

Proof. The first point consists only in elementary verifications. For the second, it is a corollary of theorem 1.3.14 and the fact the we're considering only **finitely generated** modules over a finite dimensional algebra. As their underlying vector space is finite dimensional, they're all both noetherian and artinian and the result follows. □

We have the additional and very useful:

Definition 2.1.2 (Standard duality). *Let $D : \mathbf{mod}(\mathcal{A}) \to \mathbf{mod}(\mathcal{A}^{op})$ be the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, K)$ sending right $\mathcal{A}$-modules to left-$\mathcal{A}$-modules. Given a right module M, $D(M) = M^*$ is endowed with a left action of $\mathcal{A}$ by the formula:*

$$\forall \varphi \in M^*,\ \forall a \in \mathcal{A},\ \forall m \in M,\ (a \cdot \varphi)(m) := \varphi(m \cdot a) \tag{2.1.1}$$

We call this functor the standard duality.

Proposition 2.1.3. *The standard duality functor is an equivalence of categories.*

Remark 2.1.4: This equivalence establish two results: the category of right $\mathcal{A}$ modules is equivalent to the one of left $\mathcal{A}$-modules, which implies that even if we arbitrarily choose to consider right modules, the theory would be exactly the same with left modules. Second, $\mathbf{mod}(A)^{op}$ can be indetified with the category of left-$\mathcal{A}$-modules, which, as we said, behave the same as

mod$(\mathcal{A})$. Therefore we will be able to prove many results by duality when dealing with categories of modules, since every statement about **mod**$(\mathcal{A})$ is equivalent to its dual statement. ♦

We introduced in the previous section the notions of simple, semi-simple and indecompsable objects, which are central in many proof and statements. In the particular case of modules, we have additional tools to study them.

We will first mark a distinction between two ambiguous terms:

Definition 2.1.5 (Semi-simple alegbra). *An algebra $\mathcal{A}$ is said to be* semi-simple *when $(\mathcal{A})_{\mathcal{A}}$ is semi-simple in* **mod**$(\mathcal{A})$.

Definition 2.1.6 (Jacobson radical). *Let M be a $\mathcal{A}$-module. We define:*

(i) $\mathrm{Rad}(\mathcal{A})$ *as the intersection of every maximal right-ideal of $\mathcal{A}$.*

(ii) $\mathrm{Rad}(M)$ *as the intersection of every maximal right sub-module of M.*

[2] gives an in-depth study of the radical of an algebra and of a module. We give some of the most useful properties without a proof, as they can be entirely found in this book.

Lemma 2.1.7 (Properties of the radical).

(i) $\mathrm{Rad}(\mathcal{A})$ *is the largest bilateral nilpotent ideal of $\mathcal{A}$.*

(ii) $\mathrm{Rad}(\mathcal{A}) = \mathrm{Rad}((\mathcal{A})_{\mathcal{A}})$

(iii)
$$\forall x \in M,\ x \in \mathrm{Rad}(M) \iff \forall f : M \to S,\ f(x) = 0$$
where $\forall f$ is to be understanded as "for all S a simple module, for all $f : M \to S$ $\mathcal{A}$-linear.

(iv) $\forall M, N \in \mathbf{mod}(\mathcal{A}),\ \mathrm{Rad}(M \oplus N) = \mathrm{Rad}(M) \oplus \mathrm{Rad}(N)$

(v) *For any morphism of $\mathcal{A}$-modules $f : M \to N$, $f(\mathrm{Rad}(M)) \subset \mathrm{Rad}(N)$.*

5 justifies the definition of a functor top which sends a module M to $M/\mathrm{Rad}(M)$. One can show that this functor is K-linear.

The radical serves as a tool to measure "how far" an arbitrary module is from being semi-simple. Indeed we have:

Theorem 2.1.8. *Let M be any $\mathcal{A}$-module. Then, the following are equivalent:*

(i) *M is semi-simple.*

(ii) $\mathrm{Rad}(M) = 0$

Proof.

(i) If M is semi-simple, let's write:

$$M \cong \bigoplus_{i=1}^{n} S_i \tag{2.1.2}$$

with the (S_i) being simple. Then using lemma 2.1.7, we obtain:

$$\mathrm{Rad}(M) = \bigoplus_{i=1}^{n} \mathrm{Rad}(S_i) = 0 \tag{2.1.3}$$

since the radical of a simple, being a proper submodule, is always 0.

(ii) Conversely if $\mathrm{Rad}(M) = 0$, we may reason by induction of the length of M. If $l(M) = 1$, M is simple and therefore is semi-simple. Now, let M be of any finite dimension. Because M admits a composition series, it admits a simple sub-module S. Because S is non-zero by definition, there exists a maximal sub-module N of M such that $S \not\subset N$ and by maximality of N, $N + S = M$. By simplicity of S, $S \cap N = 0$ hence there is a direct sum decomposition:

$$M = S \oplus N \tag{2.1.4}$$

and $l(N) < l(M)$. But we have:

$$0 = \mathrm{Rad}(M) = \mathrm{Rad}(S) \oplus \mathrm{Rad}(N) \tag{2.1.5}$$

hence $\mathrm{Rad}(N) = 0$ and by induction, N is semi-simple, and so is M.

□

As a consequence, the top of every module is semi-simple:

Proposition 2.1.9. *Let $M \in \mathbf{mod}(\mathcal{A})$ be any $\mathcal{A}$-module. Then:*

$$\mathrm{Rad}\left(M \big/ \mathrm{Rad}(M) \right) = 0 \tag{2.1.6}$$

Proof. It is an immediate consequence from the fact that submodules of $M \big/ \mathrm{Rad}(M)$ are in bijective and order-preserving correspondance with submodules of M countaining $\mathrm{Rad}(M)$ by taking their inverse image through the natural projection π. In fact, $\pi^{-1}\left(\mathrm{Rad}\left(M \big/ \mathrm{Rad}(M)\right)\right) = \mathrm{Rad}(M)$ and we have our proof. □

We will conclude by introducing two important results about the structure of algebras and modules. The first one is linked to the notion of idempotents of $\mathcal{A}$. As these results are not in the core of our study, we will skip most of the proofs, however they will give a more perspective on what we will do next. A full treatment is done in the fourth section of the first chapter of [2].

Definition 2.1.10 (Idempotents). *An element $e \in \mathcal{A}$ is said to be* idempotent *when $e^2 = e$. Furthermore, we say that:*

(i) *e is* non-trivial *if $e \neq 1,\ e \neq 0$.*

(ii) *e is* primitive *if e can't be written as a non-trivial sum of idempotents.*

(iii) *A family $e_1, \dots e_n$ of idempotents is* orthogonal *if $e_i e_j = 0$ whenever $i \neq j$.*

Remark 2.1.11:

(i) 0 and 1 are always idempotents.

(ii) If e is a non-trivial idempotent, e and $1 - e$ are non-trivial orthogonal idempotent.

♦

Lemma 2.1.12. *Let $M_1, M_2 \in \mathbf{mod}(\mathcal{A})$. The following are equivalent:*

(i) $(\mathcal{A})_{\mathcal{A}} \cong M_1 \oplus M_2$

(ii) *There exists an idempotent $e \in \mathcal{A}$ such that:*

$$M_1 \cong e \cdot \mathcal{A} \ \wedge \ M_2 \cong (1 - e) \cdot \mathcal{A} \tag{2.1.7}$$

Furthermore, M_1 is indecomposable if and only if e is primitive.

Theorem 2.1.13. *For any finite-dimensional K-algebra $\mathcal{A}$, there exists a set of primitive orthogonal idempotents $e_1, \dots e_n$ such that:*

$$(\mathcal{A})_{\mathcal{A}} \cong \bigoplus_{i=1}^{n} e_i \mathcal{A} \tag{2.1.8}$$

Such a set is called a complete *familiy of primitive idempotents.*

Proof. It's an immediate consequence of the previous lemma and the fact that $\mathcal{A}$ is artinian. □

Primitive idempotents are linked with the isoclasses of simples in $\mathbf{mod}(\mathcal{A})$ in a very strong way:

Proposition 2.1.14. *Let S be a simple $\mathcal{A}$-module and $(e_1, \dots e_n)$ be a complete set of primitive idempotents of $\mathcal{A}$. Then there exists $i \in [\![1, n]\!]$ such that:*

$$S \cong \operatorname{top}(e_i \mathcal{A}) \tag{2.1.9}$$

Proof (sketch). The hard part of this proof is to show that $\mathrm{top}(e_i\mathcal{A})$ is simple. It can be found in [2], chapter 2, proposition 4.5. Once this is done, we may consider a non-zero arrow $f : \mathcal{A} \to S$, since S is a non-zero $\mathcal{A}$-module. We may then apply the functor top. Since S is simple, its radical is zero by 2.1.8. Therefore, we have a non-zero arrow:

$$\mathrm{top}(f) : \mathrm{top}(\mathcal{A}) \to S \tag{2.1.10}$$

But by K-linearity of top:

$$\begin{align} \mathrm{top}(\mathcal{A}) &= \mathrm{top}\left(\bigoplus_{i=1}^{n} e_i\mathcal{A}\right) \tag{2.1.11} \\ &\cong \bigoplus_{i=1}^{n} \mathrm{top}(e_i\mathcal{A}) \tag{2.1.12} \\ & \tag{2.1.13} \end{align}$$

But $\mathrm{top}(f)$ being non-zero, there exist $i \in [\![1, n]\!]$ such that $\mathrm{top}(f)|_{\mathrm{top}(e_i\mathcal{A})}$ is non-zero. By SCUR's lemma 1.3.8, this restriction is an isomorphism because it's $\mathcal{A}$-linear between two simples. ☐

We finish with this last theorem, which is much harder to prove:

Theorem 2.1.15. *Let $M \in \mathbf{mod}(\mathcal{A})$ be a module. Then, M admits a decomposition into indecomposable modules, and such a decomposition is unique up to permutation and isomorphisms.*

Proof. See [2], chapter 1, theorem 4.10. ☐

2.2 Quivers

We are mainly interested in the study of the category of modules over an algebra $\mathcal{A}$. The representation theory gives us a quite powerful tool: quivers and their representations. In this paragraph, we will introduce the definitions of concepts we will used in our study of representations of a quiver.

Definition 2.2.1 (Quiver). *A* quiver *is a quadruple $Q = (Q_0, Q_1, s, t)$ where:*

(i) *Q_0 is a set, called the set of* vertices.

(ii) *Q_1 is a set, called the set of* arrows.

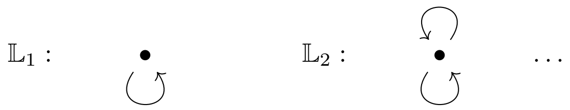
(iii) *s and t are two functions $Q_1 \to Q_0$ called the* source *and the* target.

If both Q_0 and Q_1 are finite, we say that Q is a finite quiver.

A quiver is really close to be a graph, though a quiver is always assumed to be oriented. Furthermore, there can be multiple arrows between two vertices, and loops are allowed.

Example 2.2.2:

(i) The *loop quiver* with n loops:



(ii) The 2-Kronecker :

$$K_2 \qquad \bullet \rightrightarrows \bullet$$

(iii) The quiver family of type A_n composed of all the quivers which reduce to the following graph when forgetting the direction of the arrows:

$$A_n \qquad \begin{tikzcd} 1 \arrow[r, no head] & 2 \arrow[r, dashed, no head] & {} \arrow[r, dashed, no head] & n-1 \arrow[r, no head] & n \end{tikzcd}$$

(iv) The quiver family $\tilde{A}_n$ of quivers which reduce to:

$$A_n \qquad \begin{tikzcd} 1 \arrow[r, no head] \arrow[drr, no head] & 2 \arrow[r, dashed, no head] & {} \arrow[r, dashed, no head] & n-1 \arrow[r, no head] & n \arrow[dll, no head] \\ & & n+1 & & \end{tikzcd}$$

and without a cyclic orientation.

♦

In a very instinctive way, we will define the notion of a path in a quiver:

Definition 2.2.3 (Path). *Let Q be a quiver. A* path *in Q is a finite sequence of arrows $(\alpha_1, \dots \alpha_n)$ such that $t(\alpha_i) = s(\alpha_{i+1})$ for all $i \in [\![1, n-1]\!]$.*
In addition, for each vertex $a \in Q_0$, we define ε_a as the empty path from a to a composed of zero arrow.

We will set back some common terminology of the graph theory:

Definition 2.2.4. *A quiver is said to be:*

(i) connected *when for all $(a, b) \in Q_0^2$, there exists a path from a to b* **or** *from b to a.*

(ii) acyclic *if for all $A \in Q_0$, the only path from a to a is ε_a.*

To every quiver can be associated an algebra, and very simple combinatorics consideration about the quiver will give much information about the said algebra.

Definition 2.2.5 (Path algebra). *Let Q be a quiver. We define the* path algebra *of Q in the following way:*

(i) *We take the free K-vector space generated by the paths of Q as the underlying vector space.*

(ii) *We define a product of paths in the following way: given two paths $p_1 = (\alpha_1, \ldots, \alpha_n)$ and $p_2 = (\beta_1, \ldots, \beta_m)$, we set:*

$$p_1 \cdot p_2 = \begin{cases} (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots \beta_m) \text{ if } t(\alpha_n) = s(\beta_1) \\ 0 \text{ otherwise} \end{cases} \tag{2.2.1}$$

(iii) *We extend the product by bilinearity.*

We denote by KQ the path algebra generated by Q.

Example 2.2.6:

(i) $K\mathbb{L}_1 \cong K[X]$ via the isomorphism given by the evaluation of polynmials of $K[X]$ on the loop of $\mathbb{L}_1$. In a simillar way, $K\mathbb{L}_n$ is isomorphic to $K\langle X_1, \ldots X_n\rangle$ the algebra of non-commutative polynomials in n indeterminates.

(ii) The path algebra of K_2 is the quite strange matrix algebra:

$$\left\{ \begin{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} & 0 \\ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} & \begin{pmatrix} y \\ y \end{pmatrix} \end{pmatrix}, (x, y, z_1, z_2) \in K^4 \right\} \tag{2.2.2}$$

where the multiplication of vectors is performed component-wise. x represent the ε_1 coordinate, y the ε_2 coordinate, and z_1 and z_2 represent each one of the component in the two parallel arrows.

(iii) In [2], it is shown that nice enough quivers give birth to a path algebra isomorphic to such an algebra of matrices with vector as entry which reflect the structure of the quiver, taking in account the multiplicity of possible paths from one vertex to another (lemma 1.12 of chapter 2).

♦

We shall limit ourselves to finite quivers, as we're interested in the study of finite dimensional algebras, and it is clear that if Q is infinite, KQ cannot be finite dimensional. In this case it admits $\sum_{a \in Q_0} \varepsilon_a$ as a unit.

Our dream would be that any finite dimensional algebra can be represented in the form of a finite quiver. It is, unfortunately, not the case. That's what we will allow the path algebra to be quotiented by some special ideals.

Definition 2.2.7 (Admissible ideal). *A bilateral ideal $I \subset KQ$ is said to be* admissible *when:*

$$\exists m \geqslant 2 \ : \ RQ_m \subset I \subset RQ_2 \tag{2.2.3}$$

where RQ_k is the bilateral ideal of KQ generated by all path of length at least k.

The goal of admissible ideals is to add relations on the paths, for instance in the quiver:

$$\begin{tikzcd} 1 \arrow[r, bend right, "b"'] & 2 \arrow[l, bend right, "a"'] \arrow[r, bend right, "d"'] & 3 \arrow[l, bend right, "c"'] \end{tikzcd} \tag{2.2.4}$$

one may wish to impose that taking the two loops starting on 2 is exactly the same. Therefore, we should add the relation $ab - dc$[ix] in the ideal I. In this context, we call a *relation* a linear combination of paths **all sharing their starting and ending point**. Non trivially, every admissible ideal is always generated by finitely many relations:

Lemma 2.2.8. *Let I be an admissible ideal of a path algebra KQ. Then their exists a finite set of relations generating I.*

Proof. Let's take $m \in \mathbb{N}^*$ such that $RQ_m \subset I$. Then the KQ-module RQ_m is generated by paths of length m and since Q is finite, it is finitely generated. Furthermore, ${}^{I}\!/_{RQ_m}$ is a K-vector space of finite dimension since it is countained in the free vector space generated by paths of length at most $m-1$. Therefore, KQ need to be finitely generated. Taking generators $\sigma_1, \dots \sigma_s$, we may impose that they are combination of paths with the same starting point and the same ending point by considering the family:

$$\{\varepsilon_a \cdot \sigma_i \cdot \varepsilon_b, \ (a, b, i) \in Q_0^2 \times [\![1, s]\!]\} \tag{2.2.5}$$

which gives us a finite set of relations. □

Definition 2.2.9 (Monomial ideal). *An admissible ideal I is said to be* monomial *if it admits a finite set of generators composed only of paths rather than sums of paths.*

Lemma 2.2.10. *If I is an admissible ideal, then ${}^{KQ}\!/_{I}$ is a finite dimensional algebra.*

Proof. Let's take $m \in \mathbb{N}^*$ such that $RQ_m \subset I$. Q being finite, the number of paths of length at most m is bounded by $\displaystyle\sum_{l=0}^{m} |Q_0||Q_1|^l$, and the class of these paths in the quotient by I form a generative family. Therefore, ${}^{KQ}\!/_{I}$ is finite dimensional. □

[ix] Note that when writting concatenation of paths, the order of arrows is the opposite of the order of composition. ab is the path that goes first on a and then on b.

Remark 2.2.11: If the quiver Q is acyclic, then the admissible condition $RQ_m \subset I$ becomes trivial, taking m as the length of the longest path in Q plus 1. Hence, the zero ideal is admissible and KQ is finite-dimensional (note that in this context, the converse is true: KQ is finite dimensional if and only if Q is acyclic). ♦

There are some properties of the algebra ${}^{KQ}/_{I}$ that can simply be read in the quiver using combinatorial arguments:

Proposition 2.2.12 (Properties of KQ/I). *Let's I be an admissible ideal of a finite quiver Q. Then:*

(i) *The family $([\varepsilon_a])_{a\in Q_0}$ obtained by projecting the empty paths in the quotient is a complete family of primitive orthogonal idempotents.*

(ii) *The radical of ${}^{KQ}/_{I}$ is ${}^{RQ}/_{I}$, where RQ is the ideal generated by all non-empty paths.*

Proof.

(i) the $([\varepsilon_A])$ are clearly idempotent and orthogonal. Proving that they are primitive is a bit technical. The reader may find a complete proof in [2], lemma 2.4 of chapter 2.

(ii) Since $\mathcal{A} := {}^{KQ}/_{I}$ is finite dimensional, we must show that ${}^{RQ}/_{I}$ is the largest nilpotent ideal of $\mathcal{A}$, according to lemma 2.1.7. Since there exist a positive integer m such that $RQ_m \subset I$, we have that $RQ^m \subset I$ and therefore $\left({}^{RQ}/_{I}\right)^m = 0$. Conversely, given an ideal J which is not countained in ${}^{RQ}/_{I}$, we have that $\pi^{-1}(J) \not\subset RQ$ by the ideals correspondance theorem, with π the natural projection. Let's take $x \in \pi^{-1}(J)\setminus RQ$. There must exist $a \in Q_0$ such that x has a non-zero component in ε_a. Since a is idempotent, no positive power of x is countained in RQ and therefore J is not nilpotent. By contrapositive, ${}^{RQ}/_{I} = \operatorname{Rad}\mathcal{A}$.

□

2.3 The category of representations

In this paragraph, we set a finite quiver Q and an admissible ideal I of KQ. We note $\mathcal{A} := {}^{KQ}/_{I}$. The interest of linking $\mathcal{A}$ with its quiver is that it will allow us to represent modules over $\mathcal{A}$ as a set of vector spaces (which are much nicer) linked together by K-linear morphisms.

Definition 2.3.1 (Representation of a quiver). *A* representation *of (Q, I) is a couple $V = ((V_a)_{a\in Q_0}, (f_\alpha)_{\alpha\in Q_A})$, where:*

(i) *$(V_a)_{a\in Q_0}$ is a family of K-vector spaces.*

(ii) *$(f_\alpha)_{\alpha\in Q_A}$ is a family of K-linear maps such that f_α's domain and codomain are $s(\alpha)$ and $t(\alpha)$. We also impose that they satisfy the relations of I, which means that given a path $(\alpha_1, \ldots, \alpha_n) \in I$, the composition $f_{\alpha_n} \circ \cdots \circ f_{\alpha_1}$ is zero, naturally extended by K-linearity to combinations of paths.*

If all the (V_a) are finite dimensional, we say that the representation is finite.

Example 2.3.2:

(i)

$$K \underset{0}{\overset{1}{\rightrightarrows}} K \qquad\qquad K^2 \underset{(2\ \ -7)}{\overset{(1\ 0)}{\rightrightarrows}} K \tag{2.3.1}$$

are two representations of K^2 with no relations.

(ii) Let's consider the quiver:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[loop right, "c"] \end{tikzcd} \tag{2.3.2}$$

with the admissible ideal $I := \langle ba, c^2, cb \rangle$. A possible representation would be:

$$\begin{tikzcd} K^3 \arrow[r, bend left, "f"] & K^2 \arrow[l, bend left, "g"] \arrow[loop right, "h"] \end{tikzcd} \tag{2.3.3}$$

with $f = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $h = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

♦

Definition 2.3.3 (Morphism of representation). *Given two representations $V = ((V_a), (f_\alpha))$ and $W = ((W_a), (g_\alpha))$, a morphism of representations $H : V \to W$ is a family $(H_a : V_a \to W_a)_{a \in Q_0}$ of K-linear maps such that, for any arrow $\alpha : a \to b$ in Q_1, the following square commutes:*

$$\begin{tikzcd} V_a \arrow[r, "f_\alpha"] \arrow[d, "H_a"] & V_b \arrow[d, "H_b"] \\ W_a \arrow[r, "g_\alpha"] & W_b \end{tikzcd} \tag{2.3.4}$$

We note $\mathbf{Rep}(Q, I)$ the category of all the representations of ${KQ}/{I}$ with their morphisms and $\mathbf{rep}(Q, I)$ its full subcategory whose objects are the finite representations.

Proposition 2.3.4. $\mathbf{Rep}(Q, I)$ *and* $\mathbf{rep}(Q, I)$ *are K-linear categories.*

Proof. It's only a matter of straightforward verifications. □

Remark 2.3.5: As part of the proof, one may observe that if $f : V \to W$ is a morphism of representations, then $\mathrm{Ker}(f) = ((\mathrm{Ker}(f_a))_{a \in Q_0}, (\bar{f}_\alpha)_{\alpha \in Q_A})$, with $\bar{f}_\alpha$ being f_α restricted and corestricted to the kernels. The cokernel is built in a simillar way. Indeed, most of common constructions work in the category of representations in an instinctive manner: coproducts, quotients, etc... In particular, a morphism H is monic (resp. epic, an isomorphism) if and only if each of its component is monic (resp. epic, an isomorphism), as the zero representation is the representation whith all vector spaces being 0. ♦

We will now try to identify the different kind of objects we introduced above (simples, projectives...). Since Q is finite, we will write $n \in \mathbb{N}$ the number of vertices in Q and we shall assimilate Q_0 with the integer interval $[\![n]\!]$.

Proposition 2.3.6 (Simples of $\mathbf{rep}(Q,I)$). *A representation $V = ((V_i),(f_\alpha)) \in \mathbf{rep}(Q,I)$ is simple if and only if:*

(i) $\exists i \in [\![n]\!],\ :\ \forall j \in [\![n]\!],\ V_j \cong \delta_{i,j} \cdot K \cong \begin{cases} K \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$

(ii) $\forall \alpha \in Q_1,\ f_\alpha = 0$

We then write S_i for the simple which only non-zero vector space is of index i.

Proof. If W is a subrepresentation of S_i, there exist a monomorphism $W \hookrightarrow S_i$, which is a family of injective K-linear maps. Therefore:

$$\forall j \in Q_0,\ \dim_K(W_j) \leqslant \dim_K((S_i)_j) \tag{2.3.5}$$

and it is clear that W is isomorphic either to 0 or to S_i. S_i is simple in $\mathbf{rep}(Q,I)$.

Conversely, proposition 2.1.14 shows that there is at most $|Q_0|$ isoclasses of simples since the empty paths give a complete family a primitive idempotents of $\mathcal{A}$, which finishes the proof. □

Proposition 2.3.7 (Projective covers of $\mathbf{rep}(Q,I)$ (admited)). *Let $i \in Q_0$ be a vertex and S_i be the associated simple. Then S_i admits a projective cover in $\mathbf{rep}(Q,I)$ given by:*

(i) *For $j \in Q_0$, $(\mathbb{P}(S_i))_j$ is the sub-vector space of KQ/I generated by all the classes in the quotient of the paths starting at i and ending at j.*

(ii) *For an arrow $\alpha : j \to k$ in Q_1, f_α is the linear map induced from $\mathbb{P}(S_i)_j$ to $\mathbb{P}(S_i)_k$ by the multiplication by α in KQ/I*

(iii) *The epimorphism $p : \mathbb{P}(S_i) \twoheadrightarrow S_i$ has for the i-component the only K-linear map $\mathbb{P}(S_i)_i \to K$ that map ε_i to 1 and every other path to 0. The other components must be 0*

We will write P_i for the projective cover of S_i rather that $\mathbb{P}(S_i)$.

Remark 2.3.8: We saw that $\mathbf{rep}(Q,I)$ has finitely many isoclasses of simples and that each of them admits a projective cover. Since a representation is composed of finitely many finite dimensional vector spaces, $\mathbf{rep}(Q,I)$ is a category in which every object is both noetherian and artinian, and therefore it is a length category. Using proposition 1.5.11, we have that $\mathbf{rep}(Q,I)$ admits a projective generator and is therefore equivalent to a category of modules thanks to Gabriel-Mitchell theorem 1.5.14. We will see much better in the following paragraph, but this is to keep in mind. ♦

Let's dig a bit more in the projective covers of the simples in $\mathbf{rep}(Q, I)$, as they will be very important for the study of highest weight categories. First, our new tools allow us to prove a partial converse of proposition 1.5.12.

Proposition 2.3.9. *Every indecomposable projective P is the projective cover of some simple.*

Proof. We saw that $\mathbf{rep}(Q, I)$ is length-finite, has finitely many simples $(S_1, \dots S_n)$ and that each of them admits a projective cover P_i. Therefore we have a projective generator given by:

$$\mathbb{P} := \bigoplus_{i=1}^{n} P_i \tag{2.3.6}$$

using proposition 1.5.11. By definition, there exists an integer r such that P is a quotient of $\mathbb{P}^r$. In other words, we have a short exact sequence:

$$0 \longrightarrow Z \hookrightarrow \mathbb{P}^r \twoheadrightarrow P \longrightarrow 0 \tag{2.3.7}$$

But by projectivity of P, this sequence splits (proposition 1.5.3), and therefore, using proposition 1.4.7, we have:

$$\mathbb{P}^r \cong \bigoplus_{i=1}^{n} P_i^{\oplus r} \cong P \oplus Z \tag{2.3.8}$$

But by Gabriel-Mitchell theorem 1.5.14, $\mathbf{rep}(Q, I)$ is equivalent to a category of modules, which in this case is over the finite dimensional K-algebra $\mathrm{End}_{\mathbf{rep}(Q,I)}(\mathbb{P})$. Using proposition 1.5.12, we know that the $(\mathbb{P}(S_i))$ are indecomposable. Since the decomposition into indecomposable modules is unique up to isomorphism (theorem 2.1.15), P must be isomorphic to one of the $(\mathbb{P}(S_i))$. $\square$

We deduce an alternative characterisation of the projective cover of a simple in $\mathbf{rep}(Q, I)$:

Proposition 2.3.10 (Alternative characterisation of the projective cover). *Let P be a finite representation in $\mathbf{rep}(Q, I)$. We denote the simples by $S_1, \dots, S_n$. Let's consider some $i \in [\![n]\!]$. Then, the following are equivalent:*

(i) *There exists $p : P \to S_i$ such that (P, p) is the projective cover of S_i.*

(ii) *P is projective and:*

$$\forall j \in [\![n]\!], \ \mathrm{Hom}(P, S_i) \cong \delta_{i,j} \cdot K \tag{2.3.9}$$

Proof.

(i) P admits a unique maximal subobject by 1.5.13. But using lemma 1.3.10 together with the correspondance theorem, it is easy to show that there is a one to one bijection between the simple quotients of P and its maximal subobjects, therefore P admits a single simple as a quotient, which must be S_i. To show that $\mathrm{Hom}(P, S_i)$ is one-dimensional, let's take a non zero morphism $p' : P \to S_i$. Its kernel is a maximal subobject, and since it's proper, it is a subobject of $\mathrm{Ker}(p)$. But by maximality, we have that $\mathrm{Ker}(p) \cong \mathrm{Ker}(p')$. Hence we have a factorisation:

$$\begin{tikzcd} P \arrow[r, two heads, shift left, "p"] \arrow[r, two heads, shift right, "p'"'] \arrow[d, two heads, "\pi"'] & S \\ P\big/\mathrm{Ker}(p) \arrow[ur, shift left, "\bar{p}"] \arrow[ur, shift right, "\bar{p'}"'] & \end{tikzcd} \tag{2.3.10}$$

But just looking at the shape of the simples, it is clear that $\mathrm{End}(S_i)$ is one dimensional. Therefore, $(\bar{p}, \bar{p'})$ is a bounded family and we have a non-zero scalar λ such that $\bar{p} + \lambda \cdot \bar{p'} = 0$. It implies:

$$0 = \pi \circ (\bar{p} + \lambda \bar{p'}) = p + \lambda p' \tag{2.3.11}$$

Combining these two facts, we obtain that $\mathrm{Hom}(P, S_j) \cong \delta_{i,j} \cdot K$.

(ii) Conversely, suppose that P is projective and $\mathrm{Hom}(P, S_j) \cong \delta_{i,j} \cdot K$. Then, if we write $P \cong A \oplus B$, we have:

$$\mathrm{Hom}(P, S_i) \cong \mathrm{Hom}(A, S_i) \oplus \mathrm{Hom}(B, S_i) \cong K \tag{2.3.12}$$

by proposition 1.2.3. Therefore, either $\mathrm{Hom}(A, S_i)$ or $\mathrm{Hom}(A, S_i)$ is 0. Assuming it holds for A, we have that:

$$\forall j \in [\![n]\!],\ \mathrm{Hom}(A, S_j) = 0 \tag{2.3.13}$$

Then A is zero, because otherwise it would have a composition series and admit one of the simples as a quotient. Thus P is indecomposable and projective, and is therefore the projective cover of some simple due to the previous proposition. Since S_i is the only simple into which P has maps, P is the projective cover of S_i

□

2.4 Equivalence of categories

This final paragraph aims at showing in which generality we can study quivers and their representations instead of studying an algebra and its modules. We will often give only references to the proofs of the theorem as they are often long, technical and less important than the results.

Definition 2.4.1 (Morita equivalence). *Let $\mathcal{A}$ and $\mathcal{B}$ be two finite dimensional K-algebras. We say that $\mathcal{A}$ and $\mathcal{B}$ are* Morita-equivalent *when:*

$$\mathbf{mod}(\mathcal{A}) \sim \mathbf{mod}(\mathcal{B}) \tag{2.4.1}$$

in the sense of K-linear categories.

There are two main results that show how large is the scope of representing algebras by quivers with relations.

Theorem 2.4.2 (Gabriel). *Let $\mathcal{A}$ be any finite dimensional K-algebra. Then, there exist a finite quiver Q and an admissible ideal I of KQ such that $\mathcal{A}$ is Morita-equivalent to KQ/I.*

For the proof, see [2], chapter 2, theorem 3.7, or [4], theorem 3.28. In [2], the authors restrict to a basic algebra, yet it is shown in [4] that every finite dimensional algebra is MORITA-equivalent to some basic algebra.

Remark 2.4.3: The quiver Q and the ideal I can be explicitely constructed knowing $\mathcal{A}$. ♦

Theorem 2.4.4. *Let Q be a finite connected quiver and I an admissible ideal of KQ. Then, there is an equivalence of K-linear categories:*

$$\mathbf{Rep}(Q, I) \sim \mathbf{Mod}\left({}^{KQ}/_{I}\right) \tag{2.4.2}$$

which restricts to an equivalence of K-linear categories:

$$\mathbf{rep}(Q, I) \sim \mathbf{mod}\left({}^{KQ}/_{I}\right) \tag{2.4.3}$$

Proof (sketch). We will give here the construction of the equivalence, although we let the details of the proofs to [2], chapter 3, theorem 1.6. Let's note $\mathcal{A} := {}^{KQ}/_{I}$.

- Given a representation $V = ((V_i), (f_\alpha))$, we associate it with a module with underlying vector space:
$$M_V := \bigoplus_{i=1}^{n} V_i \tag{2.4.4}$$
equiped with a $\mathcal{A}$-module structure. Given a path $p = (\alpha_1, \dots \alpha_k)$ in $\mathcal{A}$, we set:
$$\forall x \in M_V,\ x \cdot p := \begin{cases} f_{\alpha_k} \circ \cdots \circ f_{\alpha_1}(x) \text{ if } x \in s(\alpha_1) \\ 0 \text{ otherwise} \end{cases} \tag{2.4.5}$$
Note that if $V \in \mathbf{rep}(Q, I)$, then M_V is finitely generated since it is a finite-dimensional vector space.
Given a morphism $H = (H_i) : V \to W$, our functor will send H to the obvious map $\bigoplus_{i=1}^{n} H_i : \bigoplus_{i=1}^{n} V_i \to \bigoplus_{i=1}^{n} W_i$ (prop 1.2.3). A straightforward calculation prove that it is indeed $\mathcal{A}$-linear.
- Conversely, given a module $M \in \mathbf{Mod}(\mathcal{A})$, we define a representation V_M with vector spaces:
$$\forall i \in [\![n]\!],\ (V_M)_i := M \cdot \varepsilon_i \tag{2.4.6}$$
and arrows:
$$\forall \alpha : i \to j,\ f_\alpha : M_i \ni x \mapsto x \cdot \alpha \in M_j \tag{2.4.7}$$
which is well defined since for $x \in M_i$, x can be written $m \cdot \varepsilon_i$ with $m \in M$ and therefore $x \cdot \alpha = m \cdot (\varepsilon_i \alpha) = x \cdot (\alpha \varepsilon_j)$.
Note that if M is finitely generated, its underlying vector space is finite-dimensional and therefore $V_M \in \mathbf{rep}(Q, I)$.
Given a morphism of modules $\varphi : M \to M'$, we define the morphism of representations $H : V_m \to V_{M'}$ by:
$$\forall i \in [\![n]\!],\ H_i = \varphi|_{M \cdot \varepsilon_i}^{M' \cdot \varepsilon_i} \tag{2.4.8}$$

□

These two theorems mean that as far as we're only interested in the categories of modules over $\mathcal{A}$ rather than $\mathcal{A}$ itself, we can equivalently work on $\mathbf{rep}(Q,I)$, as most of the properties of $\mathbf{mod}(\mathcal{A})$ will be transported to $\mathbf{rep}(Q,I)$ by the K-linear equivalence. In particular, an immediate consequence of our work on $\mathbf{rep}(Q,I)$ is the following corollary:

Corollary 2.4.5. *The category* $\mathbf{mod}(\mathcal{A})$ *admits a projective generator. In particular, it has enough projectives.*

Furthermore, it gives an explicit way of constructing the projective cover of any module. Indeed, it is not that hard to prove that given a module M, if $\operatorname{top}(M)$ admits a projective cover, then $\mathbb{P}(M) = \mathbb{P}(\operatorname{top}(M))$. But using the equivalence with $\mathbf{rep}(Q,I)$, we know how to construct the projective cover of simple modules and by noticing that $\mathbb{P}\left(\bigoplus_{i=1}^{k} M_i\right) = \bigoplus_{i=1}^{k} \mathbb{P}(M_i)$ for any family of modules admitting a projective cover (M_i) (easy check), we can construct the projective cover of $\operatorname{top}(M)$, which is semi-simple due to proposition 2.1.9.

3 Highest weight categories

This final section will present the core of our work on highest weight categories and, at last, introduce the definition of a highest weight categories, using everything we did previously. Our approach will mainly follow the one found in [7], and conceals the reason why these categories are called *highest weight*. This term makes sense in the original approach given in the founding article [14], in which the definition of a highest weight categories countains a poset of *weights*.

We will give two crucial results: Kreuse's theorem that describes the construction of a highest weight categories in a language that connects to the theory of representations of quivers, and Green-Schroll's criterion to detect, in some cases, highest weight categories using combinatorial arguments.

3.1 Recollements of categories

Given a category of representations of some finite quiver with relations, it would be very convenient to see this category as something built iteratively by adding the vertices one by one. Although it is not true in full generality, our intuition will become reality when $\mathbf{rep}(Q,I)$ is highest weight.

We'll see it later, highest weight categories tell us something about the way we can build our category by "gluing" simpler categories together, in a process that can be compared to a composition series in abelian categories. To give a precise sense to to this intuition, we need to define a *recollement* of categories.

Definition 3.1.1 (Recollement). *A* recollement *of catageories is a diagram of abelian categories and additive functors:*

$$\begin{tikzcd} \mathscr{A}' \arrow[r, tail, "i_*"] & \mathscr{A} \arrow[l, bend right, two heads, "i^*"'] \arrow[l, bend left, two heads, "i^!"] \arrow[r, two heads, "j^*"] & \mathscr{A}'' \arrow[l, bend right, tail, "j_!"'] \arrow[l, bend left, tail, "j_*"] \end{tikzcd} \tag{3.1.1}$$

such that:

(i) i_* *is exact, full and faithful.*

(ii) j^* *is exact.*

(iii) $j_!$ *and* j_* *are full and faithfull.*

(iv) $i^* \dashv i_* \dashv i^!$ *and* $j_! \dashv j^* \dashv j_*$ *are adjoint triplets.*

(v) *An object* $A \in \mathscr{A}$ *is annihilated by* j^* *(i.e.* $j^*(A) \cong 0$*) if and only if it is in the essential image of* i_* *(i.e. it is isomorphic to some* $i_*(A')$*).*

This definition may seem heavy and obscure at first. The idea is really that the bigger category $\mathscr{A}$ can be built by gluing together objects comming from the two smaller categories $\mathscr{A}'$ and $\mathscr{A}''$, which we inject in $\mathscr{A}$ using the functors i_*, $j_!$ and j_*. One sometimes write:

$$\mathscr{A}' \overset{i_*}{\rightarrowtail} \mathscr{A} \overset{j^*}{\twoheadrightarrow} \mathscr{A}/\mathscr{A}' \tag{3.1.2}$$

in a way that recall short exact sequences to follow the intuition that $\mathscr{A}$ is an extension of $\mathscr{A}'$ and $\mathscr{A}''$.

Before moving to the core of highest weight category theory, we will develop some definition and lemma on recollements in the particular case of module categories.

Definition 3.1.2 (SERRE subcategory). *A subcategory* $\mathscr{C} \subset \mathscr{A}$ *of an abelian category is said to be* SERRE *if it is non empty and stable under subobjects, quotients and extensions, that is, given a short exact sequence:*

$$0 \longrightarrow A' \hookrightarrow A \twoheadrightarrow A'' \longrightarrow 0 \tag{3.1.3}$$

we have that:

$$A \in \mathscr{C} \iff A' \in \mathscr{C} \wedge A'' \in \mathscr{C} \tag{3.1.4}$$

Example 3.1.3: Let $F : \mathscr{A} \to \mathscr{B}$ be an exact functor between abelian categories. Then the annihilator of F $\mathrm{Ann}(F)$, defined as the full subcategory of $\mathscr{A}$ whose objects are sent by F to 0 up to isomorphism is a SERRE subcategory of $\mathscr{A}$. Indeed, given a short exact sequence in $\mathscr{A}$, we may apply F on it to obtain another short exact sequence in $\mathscr{B}$ by exactness of F:

$$0 \longrightarrow F(A') \hookrightarrow F(A) \twoheadrightarrow F(A'') \longrightarrow 0 \tag{3.1.5}$$

and then it is clear that $F(A) \cong 0 \iff [F(A') \cong 0 \wedge F(A'') \cong 0]$ ♦

Lemma 3.1.4. *Let $\mathscr{A}$ be the category of finitely generated modules over a finite dimensional K-algebra with simples $S_1, \ldots, S_n$ and projective covers $(P_i = \mathbb{P}_{\mathscr{A}}(S_i))$. Then there exists a recollement:*

$$\begin{tikzcd} \mathrm{Ann}(\mathrm{Hom}(P_n, -)) \arrow[r, tail, "i_*"] & \mathbf{mod}(\mathcal{A}) \arrow[l, bend right, two heads] \arrow[l, bend left, two heads] \arrow[r, two heads, "j^*"] & \mathbf{mod}(\mathrm{End}(P_n)) \arrow[l, bend right, tail] \arrow[l, bend left, tail] \end{tikzcd} \tag{3.1.6}$$

with i_* being the naturlal inclusion with a left and a right adjoint given by taking respectively the maximum quotient and maximum subobject belonging to $\mathrm{Ann}(\mathrm{Hom}(P_n, -))$

Proof (sketch). Everything is explicit here so it's just a matter of verifications. The functor j^* is simply $\mathrm{Hom}(P_n, -)$ which is exact because P_n is projective (proposition 1.5.3). It is clear that the inclusion i_* is fully faithfull and that an object is annihilated by $\mathrm{Hom}(P_n, -)$ if and only if it is in the essential image of i_*. The left adjoint of j^* is given by $- \otimes_{\mathrm{End}(P_n)} P_n$. The right one is obtained by duality (see [11], lemma 2.5). $\square$

3.2 Highest weight categories

Our approach of highest weight categories will follow the one given in [7].

Definition 3.2.1 (Deligne**-finite category).** *A K-linear category $\mathscr{A}$ is said to be* Deligne*-finite when the following holds:*

(i) *$\mathscr{A}$ is* Hom *and* Ext^1 *finite, that is, for any pair of objects $(A, B) \in \mathscr{A}$,* $\mathrm{Hom}(A, B)$ *and* $\mathrm{Ext}^1(A, B)$ *are finite-dimensional vector spaces.*

(ii) *$\mathscr{A}$ has finitely many isoclasses of simples.*

(iii) *$\mathscr{A}$ has enough projectives.*

(iv) *$\mathscr{A}$ is a length category (every object admits a composition series).*

In fact, Deligne-finite categories are just a way to apply categorical language on something that is well known: module categories.

Proposition 3.2.2. *Let $\mathscr{A}$ be a K-linear category. Then, the following are equivalent:*

(i) *$\mathscr{A}$ is* Deligne*-finite.*

(ii) *There exists a finite dimensional K-algebra $\mathcal{A}$ such that $\mathscr{A} \sim \mathbf{mod}(\mathcal{A})$.*

Proof.

(i) Let's assume that $\mathscr{A}$ is Deligne-finite. Since $\mathscr{A}$ is a length category with finitely many simples and enough projectives, we may apply proposition 1.5.11 to get a projective generator $\mathbb{P}$ of $\mathscr{A}$. Then, applying GABRIEL-MITCHELL theorem 1.5.14, we obtain that $\mathscr{A}$ is equivalent to $\mathbf{mod}(\mathrm{End}_{\mathscr{A}}(\mathbb{P}))$. But since $\mathscr{A}$ is K-linear and Hom-finite, $\mathrm{End}_{\mathscr{A}}(\mathbb{P})$ is a finite dimensional K-algebra.

(ii) Conversely, if $\mathscr{A}$ is a module category, our work on section 2 already prove every axiom except the fact that $\mathscr{A}$ is Ext^1-finite. We will only give a sketch of the proof. The tedious part is to prove that it is true for simple modules. Thanks to GABRIEL-theorem 2.4.2, we may assimilate $\mathscr{A}$ with some $\mathbf{rep}(Q, I)$. Given two simples S_i and S_j, it is possible to prove that $\dim_K(\mathrm{Ext}^1(S_i, S_j))$ is the number of paths from i to j in Q which are not in I by adapting the argument of [4], example 5.11. Once it is done, we prove that if A is any representation and S is a simple, then $\mathrm{Ext}^1(S, A)$ is finite-dimensional by induction on the length of A. By taking a composition series on A, we obtain a short exact sequence:

$$0 \longrightarrow A' \hookrightarrow A \twoheadrightarrow S' \longrightarrow 0 \tag{3.2.1}$$

with S' simple and A' of length strictly less than A. Therefore, applying $\mathrm{Hom}(S, -)$ to this sequence gives an exact sequence in $\mathbf{Vect}(K)$, thanks to proposition 1.4.13:

$$\mathrm{Ext}^1(S, A') \xrightarrow{f} \mathrm{Ext}^1(S, A) \xrightarrow{g} \mathrm{Ext}^1(S, S') \tag{3.2.2}$$

and using the rank-nullity theorem, we have:

$$\dim_k(\mathrm{Ext}^1(S, A)) = \dim_K(\mathrm{Im}(g)) + \dim_K(\mathrm{Ker}(g)) = \dim_K(\mathrm{Im}(g)) + \dim_K(\mathrm{Im}(f)) \tag{3.2.3}$$

and both terms are finite by the induction hypothesis. A dual argument may now be used to generalise in the first argument of Ext^1.

□

Remark 3.2.3: As an immediate consequence, every object of a DELIGNE-finite category admits a projective cover. ♦

Highest weight categories are a sub-family of DELIGNE-finite categories (a sub-family of modules categories so) which have certain additional properties which, when met, will allow us to deduce things on the category of modules we're trying to study. These properties concern a certain family of objects called the standard objects that gives another nice filtration than the filtration by the simples.

Definition 3.2.4 (Filtration closure). *Let $\mathscr{A}$ be an abelian category and $X_1, \dots X_n$ be a finite family of objects of $\mathscr{A}$. The* filtration closure *generated by the (X_i), written $\mathcal{F}ilt(X_1, \dots, X_n)$ is the full subcategory of $\mathscr{A}$ such that an object $A \in \mathscr{A}$ is in $\mathcal{F}ilt(X_1, \dots X_n)$ if and only if there is a tower of subobjects:*

$$0 = A_0 \subset A_1 \subset \cdots \subset A_{l+1} = A \tag{3.2.4}$$

such that every factor A_{j+1} / A_j is one of the (X_i).

Definition 3.2.5 (Standard objects). *Let $\mathscr{A}$ be a* Deligne*-finite category with simples $S_1, \dots S_n$ on which we have chosen an order of the simples, and indecomposable projectives $(P_i = \mathbb{P}(S_i))$. For $i \in [\![n]\!]$, the i-th standard object Δ_i is the maximum quotient of P_i which belongs to $\mathcal{F}ilt(S_1, \dots S_i)$. "Maximal" must be understood in the sense that, given another quotient $P_i \twoheadrightarrow Q$, there is a morphism $\Delta_i \to Q$ which makes the triangle commutes:*

$$\begin{tikzcd} & P_i \arrow[dl, two heads] \arrow[dr, two heads] & \\ \Delta_i \arrow[rr] & & Q \end{tikzcd} \tag{3.2.5}$$

Remark 3.2.6:

- Each simple S_i gives a unique standard object in a Deligne-finite category. Indeed, subsection 1.3 tells us that the set of quotients of P_i (it is a set because P_i is somehow equivalent to a module) is a lattice, and due to $\mathscr{A}$ being equivalent to a module category over a finite dimensional K-algebra, this lattice is noetherian. Because the noetherian poset made of such these quotients belonging to $\mathcal{F}ilt(S_i, \dots, S_n)$ is non-empty (it countains 0), Zorn's lemma can be applied to extract a standard object. Given two of them, let's say Δ_i and Δ_i', we have two epimorphisms $\Delta_i \rightleftarrows \Delta_i'$. Considering them as K-linear maps between finite dimensional vector spaces, it proves that these vector spaces have the same dimension, and therefore these maps are in fact isomorphisms.
- Δ_n is always given by P_n because our categories are length-finite, and therefore P_n is already in $\mathcal{F}ilt(S_1, \dots, S_n)$. In particular, Δ_n is projective.

♦

Definition 3.2.7 (Highest weight category). *A* Deligne*-finite category with simples $S_1, \dots, S_n$ put in a certain order is said to be* highest weight **for the chose order** *if:*

(i) $\forall i \in [\![n]\!],\ \mathrm{End}(\Delta_i) \cong K$

(ii) $\forall i \in [\![n]\!],\ P_i \in \mathcal{F}ilt(\Delta_i, \dots \Delta_n)$

Remark 3.2.8: Note that the definition of standard objects depend on the chosen order on the simples, and a category can be highest for a certain order while not being highest weight for another. ♦

Example 3.2.9:

(i) Let us consider A_3 with no relation and the natural order on the vertices $1 < 2 < 3$:

$$1 \longrightarrow 2 \longrightarrow 3 \tag{3.2.6}$$

Then the simples and the projectives can be computed using propositions 2.3.6 and 2.3.7:

$$\begin{array}{llll} (S_1 :) & K \longrightarrow 0 \longrightarrow 0 & (P_1 :) & K \xrightarrow{1} K \xrightarrow{1} K \\ \\ (S_2 :) & 0 \longrightarrow K \longrightarrow 0 & (P_2 :) & 0 \longrightarrow K \xrightarrow{1} K \\ \\ (S_3 :) & 0 \longrightarrow 0 \longrightarrow K & (P_3 :) & 0 \longrightarrow 0 \longrightarrow K \end{array} \tag{3.2.7}$$

Let us compute Δ_1. We know that Δ_1 is an object of $\mathcal{F}ilt(S_1)$. But it is easy to see that every object of $\mathcal{F}ilt(S_1)$ are supported in 1 in the sence that every vector space associated to another vertex must be 0. It is quite clear that the only potential quotients of P_1 that are supported in 1 are the zero representation and S_1. Let us show that S_1 is indeed a quotient of P_1. There is only one possible choice of epimorphism $P_1 \twoheadrightarrow S_1$ up to isomorphism:

$$\begin{tikzcd} (P_1 :) & K \arrow[r, "1"] \arrow[d, two heads, "1"] & K \arrow[r, "1"] \arrow[d, two heads] & K \arrow[d, two heads] \\ (S_1 :) & K \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{3.2.8}$$

an the obtained diagram commutes. Hence, S_1 is a quotient of P_1 and we obtain that $\Delta_1 = S_1$. Using the same kind of arguments, we obtain the standards:

$$\begin{array}{ll} (\Delta_1 :) & K \longrightarrow 0 \longrightarrow 0 \\ \\ (\Delta_2 :) & 0 \longrightarrow K \longrightarrow 0 \\ \\ (\Delta_3 :) & 0 \longrightarrow 0 \longrightarrow K \end{array} \tag{3.2.9}$$

They are just the simples! Therefore, it is clear that their endomorphism ring is one-dimensional. Furthermore, given $i \in [\![3]\!]$, we have the tower:

$$0 \subset P_3 \subset \cdots \subset P_i \tag{3.2.10}$$

with the quotient ${P_{j+1}}\big/{P_j}$ being S_{j+1}. Hence $P_i \in \mathcal{F}ilt(\Delta_i, \ldots \Delta_3)$ and $\mathbf{rep}(A_3)$ is highest weight for the chosen order. In fact, one can show that it is highest weight for every possible order.

(ii) Now let's consider the quiver Q:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd} \tag{3.2.11}$$

with the admissible ideal $I := \langle \alpha\beta \rangle$. Then, we compute again:

$$\begin{array}{llll} (S_1 :) & K \rightleftarrows 0 & (P_1 :) & K \underset{0}{\overset{1}{\rightleftarrows}} K \\ (S_2 :) & 0 \rightleftarrows K & (P_2 :) & K \underset{\begin{pmatrix} 1 & 0 \end{pmatrix}}{\overset{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{\rightleftarrows}} K^2 \end{array} \tag{3.2.12}$$

Now for the natural order $1 < 2$, we obtain straightforwardly that:

$$\Delta = S_1, \quad \Delta_2 = P_2 \tag{3.2.13}$$

But then, just looking at the dimensions of the vector spaces, it is clear that P_2 can't be a quotient of P_1, and since P_1 is not supported in 1, it is not in $\mathcal{F}ilt(\Delta_1, \Delta_2)$. Hence $\mathbf{rep}(Q, I)$ is not highest weight. However, considering the reverse order $2 < 1$, we have that:

$$\Delta_2 = S_2, \quad \Delta_1 = P_1 \tag{3.2.14}$$

The endomorphism ring of S_2 is therefore clearly one-dimensional. An endomorphism of Δ_1 is entirely defined by two scalars x and y:

$$\begin{tikzcd} K \arrow[r, shift left, "1"] \arrow[d, "x"'] & K \arrow[l, shift left, "0"] \arrow[d, "y"] \\ K \arrow[r, shift left, "1"] & K \arrow[l, shift left, "0"] \end{tikzcd} \tag{3.2.15}$$

and since the diagram commutes, we have that $x = y$. Hence $\mathrm{End}(\Delta_2) \cong K$. Furthermore, we have a filtration of P_2:

$$0 \subset \Delta_1 \subset P_2 \tag{3.2.16}$$

with the quotient ${P_2}/{\Delta_1}$ being Δ_2, hence $P_2 \in \mathcal{F}ilt(\Delta_2, \Delta_1)$. As P_1 is Δ_1, it is trivial that $P_1 \in \mathcal{F}ilt(\Delta_1)$ and $\mathbf{rep}(Q, I)$ is highest weight for the reverse order.

♦

Highest weight categories have several nice properties, one of the most beautiful might be one dicovered by Henning Krause, which states that a highest weight category must be built from very simple blocks, in the sense of recollements. We will here state and prove a weaker version of Krause's theorem, although the full statement can be found in [11], and require knowledge about extension groups of higher order.

Theorem 3.2.10 (Krause). *Let $\mathscr{A}$ be a highest weight category with ordered simples $S_1, \dots S_n$. Then, denoting by $\mathscr{A}_i$ the subcategory $\mathcal{F}ilt(S_1, \dots S_i)$, we have a tower of recollements:*

$$0 = \mathscr{A}_0 \rightarrowtail \mathscr{A}_1 \dots \mathscr{A}_{n-1} \rightarrowtail \mathscr{A}_n = \mathscr{A} \tag{3.2.17}$$

such that:

$$\forall i \in [\![n-1]\!], \ {\mathscr{A}_{i+1}}/{\mathscr{A}_i} \sim \mathbf{Vect}(K) \tag{3.2.18}$$

The full version of the theorem states that we can in addition ask the recollements to be homological, that is the inclusion $\iota : \mathscr{A}_i \rightarrowtail \mathscr{A}_{i+1}$ induces an isomorphism $\mathrm{Ext}^p_{\mathscr{A}_i}(A,B) \cong \mathrm{Ext}^p_{\mathscr{A}_{i+1}}(\iota(A),\iota(B))$ for all $p \in \mathbb{N}$, $A, B \in \mathscr{A}_i$, and that the coverse is true: any category that is built this way is highest weight. The converse is way harder to prove and is very rarely used to detect highest weight categories, therefore we will not prove it.

Proof. Let us consider the functor $\mathrm{Hom}_{\mathscr{A}}(\Delta_n, -)$. Since $\Delta_n = P_n$ (remark 3.2.6), and because of proposition 3.2.2, $\mathscr{A}$ can be assimilated to a category of modules. Therefore we can apply lemma 3.1.4 and we obtain the following recollement with $\mathscr{A}' := \mathrm{Ann}(\mathrm{Hom}_{\mathscr{A}}(\Delta_n, -))$ the annihilator of $\mathrm{Hom}_{\mathscr{A}}(\Delta_n, -)$:

$$\begin{tikzcd} \mathscr{A}' \arrow[r, tail, "i_*"] & \mathscr{A} \arrow[l, two heads, bend right, "L"'] \arrow[l, two heads, bend left, "R"] \arrow[r, two heads, "j^*"] & \mathbf{Vect}(K) \arrow[l, tail, bend right, "j_!"'] \arrow[l, tail, bend left, "j_*"] \end{tikzcd} \tag{3.2.19}$$

We will now prove that $\mathscr{A}'$ is a highest weight category with simples $S_1, \dots S_{n-1}$ and standards $\Delta_1, \dots \Delta_{n-1}$.

$\mathscr{A}'$ is Hom, $\mathbf{Ext^1}$ and legth finite: since Δ_n is projective, $\mathrm{Hom}_{\mathscr{A}}(\Delta_n,)$ is an exact functor (proposition 1.5.3) and therefore $\mathscr{A}'$ isSerre. It implies that is stable under quotients and subobjects. Since $\mathscr{A}$ is length-finite, every objects of $\mathscr{A}'$ admit a composition series in $\mathscr{A}$, and the composition series is still valid in $\mathscr{A}'$. Furthermore, i_* is an exact, fully faithfull K-linear functor: it implies that it induces K-linear isomorphisms:

$$\forall (A,B) \in \mathscr{A}', \begin{cases} \mathrm{Hom}_{\mathscr{A}'}(A,B) \cong \mathrm{Hom}_{\mathscr{A}}(i_*(A), i_*(B)) \\ \mathrm{Ext}^1_{\mathscr{A}'}(A,B) \cong \mathrm{Ext}^1_{\mathscr{A}}(i_*(A), i_*(B)) \end{cases} \tag{3.2.20}$$

$S_1, \dots S_{n-1}$ are the simples of $\mathscr{A}'$: we shall first remark that a simple S is $\mathscr{A}'$ is still simple in $\mathscr{A}$ since $\mathscr{A}'$ is Serre. Indeed, a subobject of S in $\mathscr{A}$ is still a subobject of S in $\mathscr{A}'$, therefore the simples of $\mathscr{A}'$ are simples in $\mathscr{A}$. But since $\Delta_n = P_n$ is the projective cover of S_n in $\mathscr{A}$, we have by proposition 2.3.10 that:

$$\forall j \in [\![n]\!],\ \mathrm{Hom}_{\mathscr{A}}(\Delta_n, S_j) \cong \delta_{n,j} \cdot K \tag{3.2.21}$$

and therefore the simples of $\mathscr{A}'$ are exactly the $(S_j)_{1 \leqslant j \leqslant n-1}$.

These points lead to the following observation:

Lemma 3.2.11.

$$\mathscr{A}' = \mathcal{F}ilt(S_1, \dots S_{n-1}) \tag{3.2.22}$$

We prove it by double inclusion. Since $\mathscr{A}'$ is Serre and countains the $(S_j)_{1 \leqslant j \leqslant n-1}$, it must countains $\mathcal{F}ilt(S_1, \dots, S_{n-1})$. Conversely, $\mathscr{A}'$ is length finite, which is sufficient to prove the other inclusion.

We must now prove the key lemma for the end of the proof:

Lemma 3.2.12. *The projective cover of the (S_i) in $\mathscr{A}'$ are given explicitely:*

$$\mathbb{P}_{\mathscr{A}'}(S_i) = L(P_i) \tag{3.2.23}$$

with L the left adjoint of the inclusion i_*.

We will use proposition 2.3.10.

$L(P_i)$ is projective. Indeed, we can observe that by adjunction, there is an isomorphism of K-linear functors:

$$\mathrm{Hom}_{\mathscr{A}'}(L(P_i), -) \cong \mathrm{Hom}_{\mathscr{A}}(P_i, i_*(-)) \tag{3.2.24}$$

and therefore $\mathrm{Hom}_{\mathscr{A}'}(L(P_i), -)$ is exact since i_* and $\mathrm{Hom}_{\mathscr{A}}(P_i, -)$ are by proposition 1.5.3. Now we deduce, still by proposition 1.5.3, that $L(P_i)$ is projective.

It is the projective cover of S_i. For $j \in [\![n-1]\!]$, we have by adjunction:

$$\mathrm{Hom}_{\mathscr{A}'}(L(P_i), S_j) \cong \mathrm{Hom}_{\mathscr{A}}(P_i, i_*(S_j) = S_j) \tag{3.2.25}$$

in the category **Set**. However, the bijection is explicitely given by theorem A.3.5:

$$f \mapsto i_*(f) \circ \eta_{\mathrm{Dom}(f)} \tag{3.2.26}$$

with η the unit of the adjunction. Since the composition is K-bilinear and i_* is a K-linear functor, this bijection is in fact an isomorphism of K-vector spaces and therefore, since P_i is the projective cover of S_i in $\mathscr{A}$, proposition 2.3.10 gives:

$$\mathrm{Hom}_{\mathscr{A}'}(L(P_i), S_j) \cong \delta_{i,j} \cdot K \tag{3.2.27}$$

which is enough to prove that $L(P_i) = \mathbb{P}_{\mathscr{A}'}(S_i)$.

We have proved that $\mathscr{A}'$ has finitely many simples, is length finite and every simple has a projective cover. Therefore, by proposition 1.5.11, $\mathscr{A}'$ has enough projectives and is Deligne-finite.

Lemma 3.2.13. *For all $i \in [\![n-1]\!]$, the i-th standard of $\mathscr{A}'$ is Δ_i.*

To prove this, let's denote by Δ_i' the i-th standard in $\mathscr{A}'$ and show that $\Delta_i \cong \Delta_i'$. By definition, Δ_i is a quotient of P_i, therefore there is an epi $P_i \twoheadrightarrow \Delta_i$. Applying the functor L on this gives another epi $L(P_i) \twoheadrightarrow L(\Delta_i) = \Delta_i$ since a left adjoint preserves epimorphisms (lemma A.4.13) and because by definition, $L(\Delta_i)$ is the maximum quotient of Δ_i that is in $\mathcal{F}ilt(S_1, \ldots, S_{n-1})$, that is, it's Δ_i itself. But then, Δ_i is a quotient of $L(P_i)$ that is in $\mathcal{F}ilt(S_1, \ldots, S_i)$. By maximimality of Δ_i', there must be an epi $\Delta_i' \twoheadrightarrow \Delta_i$. Combining this information gives the following (not necessary commutative) diagram:

$$\begin{tikzcd} P_i \arrow[r, two heads] \arrow[d, two heads] & L(P_i) \arrow[d, two heads] \arrow[dl, two heads] \\ \Delta_i & \Delta_i' \arrow[l, two heads] \end{tikzcd} \tag{3.2.28}$$

In a symmetric way, we have that Δ_i' is a quotient of P_i belonging to $\mathcal{F}ilt(S_1, \ldots, S_i)$. By maximality of Δ_i, there is an epimorphism $\Delta_i \twoheadrightarrow \Delta_i'$.

Now we will use Gabriel-Mitchell theorem 1.5.14: $\mathscr{A}$ is equivalent to a category of finitely generated modules over a finite dimensional K-algebra $\mathcal{A}$ (see proposition 3.2.2). Therefore, the two epimorphisms $\Delta_i \rightleftarrows \Delta_i'$ can be seen as surjective $\mathcal{A}$-linear morphisms. Considering the underlying K-linear maps and vector spaces, it shows that $\dim_K(\Delta_i) = \dim_K(\Delta_i')$ and therefore this surjections are in fact bijections, and we deduce that these modules are isomorphic. Hence the standards of $\mathscr{A}'$ are the (Δ_i).

Lemma 3.2.14. *$\mathscr{A}'$ is a highest weight category.*

This is the final and hardest part. To prove this, we will show that the functor L sends objects of $\mathcal{F}ilt(\Delta_i, \dots, \Delta_n)$ into $\mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$. If this holds, then $L(P_i)$ must be in $\mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$ because $\mathscr{A}$ is highest weight and therefore $\mathscr{A}'$ would be highest weight. We need one last lemma:

Lemma 3.2.15 ([11], 3.2). *For $X \in \mathcal{F}ilt(\Delta_i, \dots, \Delta_n)$, there exists an integer r and a short exact sequence:*

$$0 \longrightarrow \Delta_n^r \hookrightarrow X \twoheadrightarrow X' \longrightarrow 0 \tag{3.2.29}$$

such that $X' \in \mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$.

We will reason by induction on the minimal length l of a filtration of X with factors in $\{\Delta_i, \dots, \Delta_n\}$. If $l = 1$, then X is one of the (Δ_j) and the result is clear. Otherwise, there is an exact sequence:

$$0 \longrightarrow \Delta_j \hookrightarrow X \twoheadrightarrow \bar{X} \longrightarrow 0 \tag{3.2.30}$$

for some $j \in [\![i, n]\!]$ and $\bar{X} \in \mathcal{F}ilt(\Delta_i, \dots, \Delta_n)$ with minimal filtration length less or equal than $l-1$. By induction, there is an exact sequence:

$$0 \longrightarrow \Delta_n^r \hookrightarrow \bar{X} \twoheadrightarrow X' \longrightarrow 0 \tag{3.2.31}$$

with $X' \in \mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$. In terms of quotients, it means:

$$X' \cong \bar{X} \big/ \Delta_n^r \cong \left(X \big/ \Delta_j \right) \big/ \Delta_n^r \tag{3.2.32}$$

Denoting by π the projection of X onto $X \big/ \Delta_j$, and seeing $\mathscr{A}$ as a category of modules the third isomorphism theorem states that:

$$X' \cong X \big/ \pi^{-1}(\Delta_n) \tag{3.2.33}$$

and by the correspondance theorem of submodules, we obtain another short exact sequence:

$$0 \longrightarrow \Delta_j \hookrightarrow \pi^{-1}(\Delta_n^r) \twoheadrightarrow \Delta_n^r \longrightarrow 0 \tag{3.2.34}$$

But by projectivity of Δ_n^r as a coproduct of projectives (lemma 1.5.4) this sequence is split (proposition 1.5.3) and therefore:

$$X' \cong X \big/ \Delta_j \oplus \Delta_n^r \tag{3.2.35}$$

If $j = n$, it gives us exactly the exact sequence we want. If $j \neq n$, then $X \big/ \Delta_n^r$ is already in $\mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$ which gives the results.

Using the lemma, we only have to apply the functor L to the obtained exact sequence. Since left adjoints are right-exact (proposition 1.4.5), we obtain another exact sequence:

$$L(\Delta_n^r) \longrightarrow L(X) \twoheadrightarrow L(X') \longrightarrow 0 \tag{3.2.36}$$

but then, $L(X') = X'$ by definition of L (because $\mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1}) \subset \mathcal{F}ilt(S_1, \dots, S_{n-1})$ by construction of the standards) and $L(\Delta_n^r) = L(\Delta_n)^r = 0$, which proves that $L(X) \cong X' \in \mathcal{F}ilt(\Delta_i, \dots, \Delta_{n-1})$. Therefore, $\mathscr{A}'$ is highest weight with $n-1$ isoclasses of simples. We apply our induction hypothesis, and we proved the weak form of KRAUSE's theorem! $\square$

Let us give an heuristic way of understanding this statement in the case of quivers representations. Given a quiver Q with an admissible ideal I an an order on the verticies, if $\mathbf{rep}(Q, I)$ is highest weight, KRAUSE's theorem gives a precise formulation of the intuition that the representations of Q with relations I can be built iteratively by taking the vertices of Q one by one. In other words, if we denote by Q_i the full subquiver of Q whose vertices are $[\![i]\!]$ and I_i the induced ideal, the category $\mathbf{rep}(Q_{i+1}, I_{i+1})$ is built from $\mathbf{rep}(Q_i, I_i)$ by gluing it with $\mathbf{Vect}(K)$ in the sense of recollements.

3.3 GREEN-SCHROLL's criterion

We saw in the previous examples that checking whether a given category is highest weight or not is quite heavy on computations, even in the case of very small quivers. Our last goal will be to develop an efficient method to detect highest weight categories to be able to profit of their nice properties.

In 2019, Edward GREEN and Sibylle SCHROLL published a combinatorial criterion to detect highest weight categories in the case of categories of representations with some monomial relations in [10]. Although it is not a necessary and sufficient criterion in full generality, it applies in many usual situations.

Let us first introduce some notations, which are those of [7]. We fix from now on an acyclic quiver Q with vertices $[\![n]\!]$ and a monomial admissible ideal I.

Lemma 3.3.1. *There is a canonical set of generators G of I given by:*

$$G := \{p \text{ a path in } I \mid \forall p' \text{ a proper subpath of } p, p' \notin I\} \tag{3.3.1}$$

Definition 3.3.2 (Properly external vertex). *Let $p = (\alpha_1, \dots, \alpha_n)$ be a path in Q. We say that a vertex $a \in Q_0$ is* properly external *in p if:*

(i) $a \in \{s(\alpha_1), t(\alpha_n)\}$

(ii) $\forall i \in [\![n-1]\!],\ t(\alpha_i) \neq a$

Definition 3.3.3 (Maximal vertex). *Given a path $p = (\alpha_1, \dots, \alpha_l)$ in Q, the maximal vertex of p is* $\max\{t(\alpha_i), i \in [\![l]\!]\} \cup \{s(\alpha_1)\}$.

We finely need some combinatorial numbers:

Definition 3.3.4. *For all $i \in [\![n]\!]$, we denote by $Q_{\leqslant i}$ the full subquiver of Q whose set of vertices is $[\![i]\!]$. We then set for all $i, j, k \in [\![n]\!]$ $\pi^k_{i,j}$ to be the number of paths starting in i and ending in j in $Q_{\leqslant k}$ that are not in I.*

Theorem 3.3.5 (Green-Schroll **criterion).** *Let Q be a finite quiver without loop and I a monomial admissible ideal with canonical generator G. Then, the following are equivalent:*

(i) $\mathbf{rep}(Q, I)$ *is a highest weight category.*

(ii) $\forall (i,k) \in [\![n]\!],\ \pi^i_{i,i} = 1 \wedge \sum_{j=1}^{n} \pi^j_{i,j}\pi^j_{j,k} = \pi^n_{i,k}$

(iii) *For all path p in G, the maximal vertex of p is properly external in p.*

Proof. We let the proof to [10] or [7] as it requires tools we did not study. The former follows an approach using non-commutative Gröbner-basis whereas the latter uses more categorical tools. □

When working in the right context, it is manifest that this criterion is easier to check than trying to compute the definition. In fact, it might not be very hard to produce an algorithm that check whether such categories are highest weight or not, even if its complexity would be asymptotically high.

We will conclude our study by reconsidering the previous examples using Green-Schroll-criterions.

Example 3.3.6:

(i) in A_n:

$$1 \longrightarrow 2 \longrightarrow \dots \longrightarrow n \tag{3.3.2}$$

the admissible ideal is 0 since we put no relation. Therefore the condition 3 is met by vacuity and the category of representations of A_n is highest weight, whatever order we choose on the vertices. More generally, any acyclic quiver Q with no relation generates a category of representations that is highest weight for any order on the vertices. This has very strong implications on the alegbra KQ: it is in fact hereditary (see [14], examples 3.3).

(ii) In the quiver Q:

$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"] \end{tikzcd} \tag{3.3.3}$$

with relations $I := \langle ba \rangle$, we see that for the natural order, the maximal vertex of ba is internal whereas it is external for the reverse order, hence it proves what we proved above directly using the definition of a highest weight category. Shorter isn't it?

♦

Appendix

A Elements of category theory

This section aims at introducing some elements of the language of category theory and basic tools that will be used in this report. For the sake of simplicity, it won't be a complete introduction to category theory as one can find in [12] or in [13]. We're only introducing here the tools we will effectively use in our study.

A.1 Categories, functors and natural transformations

We assume in this paragraph the reader at least knows what a category and a functor are. As it is not the purpose of this internship, we won't enter in set-theoretical considerations and often write "a collection of things" when those things might be too numberous to form a set although we will sometimes mark the distinction between a *set* and a *class* as in [9].

Definition A.1.1 (Small and locally small category). *A category $\mathscr{A}$ is said to be:*

(i) small *when the collection of all arrows of $\mathscr{A}$ form a set.*

(ii) locally small *when for any pair of objects $\mathscr{A}(A, B)$, the collection $\mathscr{A}(A, B)$ is a set.*

Remark A.1.2:

- In a small category, the collection of objects is a set too (one has to consider the subset of arrows consisting of all the identities).
- We shall work almost only with locally small categories.

♦

Example A.1.3:

- **Set**, **Grp**, **Ring**, **Vect**(K), **mod**(A) are locally small categories, but not small categories.
- A small category with one object is exactly a monoid. A monoid seen as a category in which every arrow is an isomorphism is exactly a group.

♦

In an abstract category, we will often want to have a generalised notion of injectivity and surjectivity:

Definition A.1.4 (Epimophism, monomorphism, isomorphism). *Let* $\mathscr{A}$ *be any category and* $f : A \to B$ *an arrow of* $\mathscr{A}$*. We say that* f *is:*

(i) monic*, or a* monomorphism*, when:*

$$\forall x_1, x_2 : X \to A,\ f \circ x_1 = f \circ x_2 \iff x_1 = x_2 \tag{A.1.1}$$

We will then write:

$$f : A \hookrightarrow B \tag{A.1.2}$$

(ii) epic*, or an* epimorphism*, when:*

$$\forall x_1, x_2 : B \to X,\ x_1 \circ f = x_2 \circ f \iff x_1 = x_2 \tag{A.1.3}$$

We will then write:

$$f : A \twoheadrightarrow B \tag{A.1.4}$$

(iii) *an* isomorphism *when:*

$$\exists g : B \to A\ :\ f \circ g = 1_B\ \wedge\ g \circ f = 1_A \tag{A.1.5}$$

We will then write:

$$f : A \xrightarrow{\sim} B \tag{A.1.6}$$

Remark A.1.5:

- In **Set**, an application is injective (resp. surjective) if and only if it is monic (resp. epic). It might not be true in other categories, even when the arrows are applications.
- One sometimes refers to arrows of $\mathscr{A}/_A$ i.e. of arrows of $\mathscr{A}$ with target A as *generalised elements* of A. The definition of monomorphism is then exactly the definition of injectivity, remplacing elements by generalised elements.
- Although it is true that an isomorphism is always monic and epic, the converse may not be true in full generality.

♦

The notions of surjectivity and injectivity are too restrictive to apply on functors in a useful way. We use the alternative properties:

Definition A.1.6 (Full, faithfull and essentially surjective functor). *Let* $F : \mathscr{A} \to \mathscr{B}$ *be a functor. We say that* F *is:*

(i) full *when given any two objects* $A, B \in \mathscr{A}$*,* F *induces a sujection from* $\mathscr{A}(A, B)$ *to* $\mathscr{B}(F(A), F(B))$*.*

(ii) faithfull *when given any two objects* $A, B \in \mathscr{A}$*,* F *induces an injection from* $\mathscr{A}(A, B)$ *to* $\mathscr{B}(F(A), F(B))$*.*

(iii) essentially surjective on objects *when for all object* $B \in \mathscr{B}$*, there exist* $A \in \mathscr{A}$ *such that* $F(A) \cong B$*.*

We already know that functors are morphisms between categories, and as such, they induce some sort of *meta-morphisms* between regular morphisms. We may want to continue this process and create morphisms between functors. This is the goal of natural transformations:

Definition A.1.7 (Natural transformation). *Let $F, G : \mathscr{A} \to \mathscr{B}$ be functors. A* natural transformation *$\alpha : F \to G$ is a family of arrows of $\mathscr{B}$ $(\alpha_A : F(A) \to G(A))_{A \in \mathscr{A}}$ such that for any arrow $z : A \to B$ in $\mathscr{A}$, the following diagram commutes:*

$$\begin{tikzcd} F(A) \arrow[r, "F(z)"] \arrow[d, "\alpha_A"] & F(B) \arrow[d, "\alpha_B"] \\ G(A) \arrow[r, "G(z)"] & G(B) \end{tikzcd} \tag{A.1.7}$$

We sometimes compile those data in a single diagram:

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd} \tag{A.1.8}$$

Natural transformations may of course be composed together. Given functors $F, G, H : \mathscr{A} \to \mathscr{B}$ and natural transformations $\alpha : F \to G$ and $\beta : G \to H$, we set $\beta \circ \alpha = (\beta_A \circ \alpha_A)_{A \in \mathscr{A}}$. One can then check that functors from $\mathscr{A}$ to $\mathscr{B}$ together with their natural transformations form a category we will denote as $[\mathscr{A}, \mathscr{B}]$ with identities $1_F = (1_{F(A)})_{A \in \mathscr{A}}$. Since it is a category, we may define the idea of an *isomorphism of functors* in the usual way: it is simply an invertible natural transformation. However, we have a more intuitive criterion:

Proposition A.1.8. *A natural transformation $\alpha : F \to G$ is an isomorphism if and only if each of its components is an isomorphism.*

Proof. If for all $A \in \mathscr{A}$, α_A is an isomorphism, then it is clear that $(\alpha_A^{-1})_{A \in \mathscr{A}}$ is an inverse for α. Conversely, we must have:

$$\begin{aligned} 1_F &= \alpha \circ \alpha^{-1} && \text{(A.1.9)} \\ &= (\alpha_A \circ (\alpha^{-1})_A)_{A \in \mathscr{A}} && \text{(A.1.10)} \\ &= ((\alpha^{-1})_A \circ \alpha_A)_{A \in \mathscr{A}} && \text{(A.1.11)} \\ &= (1_{F(A)})_{A \in \mathscr{A}} && \text{(A.1.12)} \end{aligned}$$

and then each component of α is invertible. □

Natural transformations reflect the idea of a transformation on objects of a category that does not require arbitrary choice which depends on the individual structure of objects. For instance, taking $D : \mathbf{Vect}_{fd}(K) \to \mathbf{Vect}_{fd}(K)$ as the functor sending a finite dimensional vector space over the field K to its dual and each linear map to its transposee, this functor is not naturally isomorphic to the identity functor as it requires to take an arbitrary basis of each vector space and then send it to its dual basis. However, the bidual functor is naturally isomorphic to the identity functor which justify the fact that the map:

$$E \to E^{**} \tag{A.1.13}$$
$$x \mapsto (f \mapsto f(x)) \tag{A.1.14}$$

is called the *natural* isomorphism between E and its bidual. [12] gives a proof with full details.

To conclude this paragraph, we will define the notion of *equivalence of categories*. It provides a way to express the sameness of two categories that is less restrictive than a strict isomorphism of categories, while still preserving enough structure. This is simillar to how, in algebraic topology we are often more interested in the homotopy type of spaces than in their homeomorphism class. In fact, the definitions are quite similar:

Definition A.1.9 (Equivalence of categories). *Given two categories $\mathscr{A}$ and $\mathscr{B}$, we say that theese categories are* equivalent *and write $\mathscr{A} \sim \mathscr{B}$ whenever there exist functors $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{A} \to \mathscr{B}$ such that :*

$$F \circ G \cong 1_{\mathscr{B}} \;\wedge\; G \circ F \cong 1_{\mathscr{A}} \tag{A.1.15}$$

we then say that the functor F is an equivalence.

We have the following caracterisation:

Theorem A.1.10. *A functor $F : \mathscr{A} \to \mathscr{B}$ is an equivalence if and only if it is full, faithfull and essentially surjective on objects.*

Proof. See [12], exercice 1.3.32 for a plan to follow. There's nothing hard to prove when doing it in the right order. ☐

Lemma A.1.11. *If $F : \mathscr{A} \to \mathscr{B}$ is an equivalence of categories, then for any arrow f in $\mathscr{A}$, $F(f)$ is an isomorphism if and only if f is an isomorphism.*

Proof. If $F(f)$ is an iso, we may take $G : \mathscr{B} \to \mathscr{A}$ and $\alpha : 1_A \xrightarrow{\sim} GF$. Then by naturality, we have:

$$f = \alpha_B^{-1} \circ GF(f) \circ \alpha_A \tag{A.1.16}$$

But functors preserve isomorphisms, so $G(F(f))$ is an iso and f is an iso as a composed of isos. The converse is trivial. ☐

A.2 Duality

In category theory, a phenomenon called "duality" often appears and tend to englob other instance of duality in mathematics, like duality between vector spaces. Categorical duality is a tool which is often rather simple to use while it has powerful consequences and spares a lot of time in proofs, as it is shown in [3]. We will introduce here the very basic notions of duality we shall use in this report.

Definition A.2.1 (Dual category). *Given any category $\mathscr{A}$, we define its dual, or opposite category, as the category $\mathscr{A}^{op}$ whose objects are the objects of $\mathscr{A}$ and an arrow $B \to A$ in $\mathscr{A}^{op}$ is exactly an arrow $A \to B$ in $\mathscr{A}$.*
Given another category $\mathscr{B}$, a duality *between $\mathscr{A}$ and $\mathscr{B}$ is a choice of an equivalence $\mathscr{A} \sim \mathscr{B}^{op}$.*

Expressed in such a way, duality might not be very impressive, yet it holds some sort of *syntaxic* power. Indeed, we will have duality operate on statements and sentences to produce *dual statements*, which are always equivalent. For instance, monomorphisms in $\mathscr{A}$ are exactly epimorphisms in $\mathscr{A}^{op}$ and conversely. Hence, a statement like "*everyarrow in $\mathscr{A}$ is monic*" is equivalent to "*every arrow in $\mathscr{A}^{op}$ is epic*", which is the dual statement, and hence is equivalent. It means that every time we prove something, we are indeed proving two statements.

True magic appears when we prove some result in a collection of categories which is stable under duality. The dual statement is then true in the same collection of catergories, and will require no more effort to be proved. We will sometimes use this principle to prove statements by duality. The reader may refer to the third chapter of [3] for a more in depth treatment.

A.3 Adjunction

We will now introduce the basic tools to study and use functorial adjunction, which provides a way to express that, in some sense, two functors act as opposite constructions.

Definition A.3.1 (Adjunction). *Let $F : \mathscr{A} \to \mathscr{B}$ and $G : \mathscr{B} \to \mathscr{A}$ be two functors. We say that F is* left adjoint *to G (and that G is* right adjoint *to F whenever for each pair of objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$ there exist an involution :*

$$\begin{aligned} \mathscr{A}(A, G(B)) &\xrightarrow{\sim} \mathscr{B}(F(A), B) \\ f &\mapsto \bar{f} \\ \bar{g} &\leftarrow\!\shortmid g \end{aligned}$$

natural in A and B, which means that the two following conditions hold:

(i) $\forall g : F(A) \to B,\ \forall q : B \to B',\ \overline{q \circ g} = G(q) \circ \bar{q}$

(ii) $\forall f : A \to G(B), \forall p : A' \to a,\ \overline{f \circ p} = \bar{f} \circ F(p)$

We then write $F \dashv G$.

Remark A.3.2: When $\mathscr{A}$ and $\mathscr{B}$ are locally small categories, there is a more abstract way to see adjunction is the following: two functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ form a pair of adjoints $F \dashv G$ if and only the two functors:

$$\mathrm{Hom}_{\mathscr{B}}(F(\cdot), -) : \mathscr{A}^{op} \times \mathscr{B} \to \mathbf{Set} \quad \mathrm{Hom}_{\mathscr{A}}(\cdot, G(-)) : \mathscr{A}^{op} \times \mathscr{B} \to \mathbf{Set} \tag{A.3.1}$$

are naturally isomorphic. ♦

We may notice that right and left adjunction are dual notions.

Given a pair of adjoints $F \dashv G$, an adjunction between them is entierly determined by two special natural transformations, the *unit* and the *couint*:

Definition A.3.3 (Unit & couint). *Let's consider the following pair of adjoints:*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] \arrow[r, bend right, "G"'] & \mathscr{B} \end{tikzcd} \tag{A.3.2}$$

We then define the unit *as* $\eta : 1_{\mathscr{A}} \to GF$ *and its dual the* couint $\varepsilon : FG \to 1_{\mathscr{B}}$ *as:*

(i) $\forall A \in \mathscr{A},\ \eta_A := \overline{1_{F(A)}}$

(ii) $\forall B \in \mathscr{B},\ \varepsilon_B := \overline{1_{G(B)}}$

Let's quickly check that η is a natural transformation: given any arrow $f : A \to A'$ in $\mathscr{A}$, we want the following diagram to commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\eta_A"] & A' \arrow[d, "\eta_{A'}"] \\ GF(A) \arrow[r, "GF(f)"] & GF(A') \end{tikzcd} \tag{A.3.3}$$

We indeed have :

$$GF(f) \circ \eta_A = G(F(f)) \circ \overline{1_{F(A)}} \tag{A.3.4}$$
$$= \overline{F(f) \circ 1_{F(A)}} \quad \text{by adjunction} \tag{A.3.5}$$
$$= \overline{1_{F(A')} \circ F(f)} \tag{A.3.6}$$
$$= \overline{1_{F(A')}} \circ f \quad \text{still by adjunction} \tag{A.3.7}$$
$$= \eta_{A'} \circ f \tag{A.3.8}$$

Which is what we want. ε is also natural by duality.

The unit and counit satisfy the following relations, named *trangle identities*:

Proposition A.3.4 (Triangle identities). *Given an adjunction $F \dashv G$ with unit and counit η and ε, the following diagrams commute for any $A \in \mathscr{A}$ and $B \in \mathscr{B}$:*

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "1_{F(A)}"'] & FGF(A) \arrow[d, "\varepsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "1_{G(B)}"'] & GFG(B) \arrow[d, "G(\varepsilon_B)"] \\ & F(A) \end{tikzcd}$$

We will conclude on adjunction by this beautiful theorem:

Theorem A.3.5. *Let's consider a pair of functors* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ *There is an explicit bijection between adjunctions* $F \dashv G$ *and pairs of natural transformations* $\eta : 1_{\mathscr{A}} \to GF$ *and* $\varepsilon : FG \to 1_{\mathscr{B}}$ *statisfying the triangle identities, given by:*

(i)
$$\forall g : F(A) \to B,\ \bar{g} := G(g) \circ \eta_A \tag{A.3.9}$$

(ii)
$$\forall f : A \to G(B),\ \bar{f} := \varepsilon_B \circ F(f) \tag{A.3.10}$$

Proof. See [12], section 2.2. □

A.4 Limits & colimits

The final part of this appendix will focus of the categorical notion of limit (and its dual the colimit) whose understanding will highly help manipulating abstract constructions.

As we're focusing on conciseness, this introduction will lack examples and will certainly be not very intuitive, yet it shall contain every tool we'll need for our purpose. See [12], chapter 5, for a more understandable approach.

Categorical limit must not be confused with analytic limit. Indeed, we're not taking limit of sequence, but rather of *diagrams.*

Definition A.4.1 (Diagram). *Let* $\boldsymbol{I}$ *be a small category and* $\mathscr{A}$ *be any category. A* diagram *of shape* $\boldsymbol{I}$ *in* $\mathscr{A}$ *is a functor* $D : \boldsymbol{I} \to \mathscr{A}$*.*

In this context, $\boldsymbol{I}$ serves as an anonymous model, abstracting the shape of a particular diagram, while the functor D is a way to find this shape into the category $\mathscr{A}$. For instance, we may consider an anonymous commutative square:

$$\begin{tikzcd}
 & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[dr] & & \bullet \\
 & \bullet \arrow[ur] &
\end{tikzcd} \tag{A.4.1}$$

and define it to be the category $\boldsymbol{I}$ (adding all necessary arrows). Then a diagram of shape $\boldsymbol{I}$ in our favourite category will match a commutative square like this one, but in the category $\mathscr{A}$. In some sense, the category $\boldsymbol{I}$ is an abstraction of the notion of commutative square.

Definition A.4.2 (Cone). *Let $D : \boldsymbol{I} \to \mathscr{A}$ be a diagram and A an object of $\mathscr{A}$. A* cone *of shape D and vertex A is collection of arrows $(\gamma_I : A \to D(I))_{i\in \boldsymbol{I}}$ such that for any arrow $f : I \to J$ in $\boldsymbol{I}$, the following triangle commutes:*

$$\begin{tikzcd}
 & A \arrow[dl, "\gamma_I"'] \arrow[dr, "\gamma_J"] & \\
D(I) \arrow[rr, "D(f)"] & & D(J)
\end{tikzcd}$$

Although it may look a bit heavy at first, this definition is more natural than it seems. Indeed, drawing a cone gives litterally a cone. For instance, let's draw a cone of shape a commutative square like (A.4.1):

$$\begin{tikzcd}
 & & A \arrow[dl] \arrow[drr] \arrow[ddll] \arrow[dd] & \\
 & \bullet \arrow[rr] \arrow[dl] & & \bullet \arrow[dl] \\
\bullet \arrow[rr] & & \bullet &
\end{tikzcd}$$

It's litterally a cone!

The limit of diagram will be some sort of cone of minimal height, which will be transliterated by a universal property:

Definition A.4.3 (Limit). *Given a diagram $D : \boldsymbol{I} \to \mathscr{A}$, a* limit *of D is a cone of shape D $(L, (\lambda_I : L \to D(I))_{i\in \boldsymbol{I}})$ such that for every other cone of shape D $(A, (\gamma : A \to D(I))_{i\in \boldsymbol{I}})$, there is a unique morphism $\varphi : A \to L$ such that every such triangle commutes:*

$$\begin{tikzcd}
 & A \arrow[d, "\varphi"] \arrow[ddl, "\gamma_I"'] \\
 & L \arrow[dl, "\lambda_I"] \\
D(I) &
\end{tikzcd}$$

We then write:

$$L = \varprojlim D \tag{A.4.2}$$

When $\boldsymbol{I}$ is a finite category, we talk about finite limit*.*

Remark A.4.4:

- Note that a limit may not exist.
- When a limit exists, it is unique up to a unique isomorphism, hence we may write "the" limit.

- There is a dual notion of colimit which uses cocones. The definition is the same with every arrow drawn in the opposite direction. We denote a colimit of a diagram D by:

$$\lim_{\rightarrow} D \tag{A.4.3}$$

♦

We will now give some examples of common limits and colimits we will frequently use throughout this report.

Definition A.4.5 (Product & coproduct). *Given a category $\mathscr{A}$ and objects $(A_i)_{i\in I}$ indexed by a set, we may see I as a small category in which the only arrows are the identities and the objects are the elements of I, and the family (A_i) may be seen as a diagram D of shape I.*

(i) *The product $\prod_{i\in I}$ is, when it exists, the limit of D.*

(ii) *The coproduct $\bigoplus_{i\in I} A_i$ is, when it exists, the colimit of D.*

Remark A.4.6: An empty product is, using this definition, an object T and no arrow pointing to every non-object of the empty family, such that for any other object A (which is automatically a cone of shape $\emptyset$), there is a unique arrow $A \to T$ such that nothing needs to commute. We call it a *terminal object.* Simillarly, an empty coproduct is an *initial object.*

Products generalise the cartesian products of **Set**, and coproducts generalise the disjoint union. ♦

Definition A.4.7 (Equalizer & coequalizer). *Given a fork of arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$, we define their equalizer (resp. coequalizer) as the limit of the fork (resp. the colimit).*

Remark A.4.8: Equalizers will serve as a way to generalise kernels, and coequaliser will generalise quotients. ♦

Definition A.4.9 (Pullback and pushout). *Given a corner of arrows:*

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

We define the pullback of f and g as a limit of the corner. Dualizing:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"] & X \\ Y & \end{tikzcd}$$

we define de pushout of f and g as the colimit of this cocorner.

Surprisingly maybe, there are quite simple criterions to know whether or not a given category admits all limits:

Theorem A.4.10. *Let $\mathscr{A}$ be any category. Then:*

(i) *$\mathscr{A}$ admits all limits if and only if $\mathscr{A}$ admits all products and equalizers.*

(ii) *$\mathscr{A}$ admits all finite limits if and only if $\mathscr{A}$ admits binary products, equalizers and a terminal objects.*

Remark A.4.11:

(i) We can prove a dual statement by duality: $\mathscr{A}$ admits all colimits if and only if it admits all coproducts and coequalizers.

(ii) If $\mathscr{A}$ admits all binary products, then it admits all finite non-empty product. Indeed, it is quite easy to show that:

$$(A \times B) \times C \cong A \times (B \times C) \cong A \times B \times C \tag{A.4.4}$$

A terminal object being an empty product as we already remarked it, a category admits binary products and a terminal objects if and only if it admits all finite products.

♦

Proof. See [12], exercice 5.1.38. □

As functors are morphisms of categories, a useful property they could have is the preservation of limits. Functors do not preserve limits in general, but it holds when they are right adjoints.

Theorem A.4.12. *Let*

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, bend left, "G"] \end{tikzcd} \tag{A.4.5}$$

be a pair of adjoints. Then G preserves all limits and F preserves all colimits, that is, if $D : \boldsymbol{I} \to \mathscr{B}$ is a diagram that admits a limit, then the diagram $G \circ D$ admits a limit and we have:

$$G(\lim_{\leftarrow} D) = \lim_{\leftarrow} G \circ D \tag{A.4.6}$$

Proof. We write $(\varphi_I : \lim\limits_{\leftarrow} D \to D(I))_{I \in \boldsymbol{I}}$ the family of morphisms forming the limit cone in $\mathscr{B}$. By functoriality, $(G(\lim\limits_{\leftarrow} D, (G(\varphi_I))_{i \in \boldsymbol{I}}))$ is a cone of shape $G \circ D$ in $\mathscr{A}$. Let's consider $(A, (f_I)_{I \in \boldsymbol{I}})$ another cone of shape $G \circ D$. We have to show two things: first that A factorise through $G(\lim\limits_{\leftarrow} D)$, and second that this factorisation is unique.

Existence of the factorisation : we consider the family $(\bar{f}_I : F(A) \to \lim\limits_{\leftarrow} D)_{I \in \boldsymbol{I}}$ By adjunction, for any $\alpha : I \to J$ in $\boldsymbol{I}$, we have:

$$D(\alpha) \circ \bar{f}_I = \overline{(G \circ D)(\alpha) \circ f_I} = \bar{f}_J \tag{A.4.7}$$

which means that $F(A)$ together with the $(\bar{f}_I)$ is a cone of shape D in $\mathscr{B}$. Hence, there exist a unique $\bar{z} : F(A) \to \lim\limits_{\leftarrow} D$ which factorises the cone into the limit. Again by adjunction, $z : A \to G(\lim\limits_{\leftarrow} D)$ is a factorisation of our cone of shape $G \circ D$ in $\mathscr{A}$.

Unicity of the factorisation : let's consider $z_1, z_2 : A \to G(\lim\limits_{\leftarrow} D)$ two factorisations. Then we have:

$$\begin{aligned} \varphi_I \circ \bar{z}_1 &= \overline{G(\varphi_I) \circ z_1} && \text{by adjunction} && \text{(A.4.8)} \\ &\overline{G(\varphi_I) \circ z_2)} && \text{by assumption} && \text{(A.4.9)} \\ &= \varphi_I \circ \bar{z}_2 && \text{still by adjunction} && \text{(A.4.10)} \end{aligned}$$

Hence, by unicity of the factorisation into $\lim\limits_{\leftarrow} D$, $\bar{z}_1 = \bar{z}_2$ which is sufficient to conclude since $\bar{\cdot}$ is a bijection.

∎

We finish with an unexpected application of limit preservation:

Lemma A.4.13. *A functor that preserves limits (resp. colimits) preserves monomorphisms (resp. epimorphisms).*

Proof. One just have to notice that $f : A \to B$ is monic if and only if the diagram:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \tag{A.4.11}$$

is a pullback, and therefore, if a functor preserves limits, it preserves pullback and must preserve monomorphisms. By duality, it preserves epimorphisms if it preserves colimits. $\square$

$\star \quad \star$

$\star$

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