Cactus groups, twin groups, and right-angled Artin groups

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CACTUS GROUPS, TWIN GROUPS, AND RIGHT-ANGLED ARTIN GROUPS

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Abstract. Cactus groups J_n are currently attracting considerable interest from diverse mathematical communities. This work explores their relations to right-angled Coxeter groups, and in particular twin groups Tw_n and Mostovoy's Gauss diagram groups D_n, which are better understood. Concretely, we construct an injective group 1-cocycle from J_n to D_n, and show that Tw_n (and its k-leaf generalisations) inject into J_n. As a corollary, we solve the word problem for cactus groups, determine their torsion (which is only even) and center (which is trivial), and answer the same questions for pure cactus groups, PJ_n. In addition, we yield a 1-relator presentation of the first non-abelian pure cactus group PJ_4. Our tools come mainly from combinatorial group theory.

1. Introduction

Cactus groups appeared under the name of *quasibraid groups* in the study of the *mosaic operad*; this latter governs the moduli space of configurations of smooth points on punctured stable real algebraic curves of genus zero [Dev99, EHKR10, KW19]. They were immediately generalised to other Coxeter types, and renamed *mock reflection groups* [DJS03].

It was later realised that the same groups control *coboundary categories*, just as braid groups control braided categories [HK06a]. That paper launched the term *cactus groups*, inspired by the Opuntia-cactus-like form of the moduli spaces above. Coboundary categories were designed to study the crystals of finite-dimensional reductive Lie algebras and, more generally, the representations of coboundary Hopf algebras.

Cactus groups also appear in the context of hives and octahedron recurrence [KTW04, HK06b]. Together with their generalisations to other Coxeter types, they have become a recurrent tool in representation theory [Bon16, Los19, CGP20].

Concretely, the *cactus group* J_n is defined by its generators[1] $s_{p,q}$, where $1 \leq p < q \leq n$, and relations

$$s_{p,q}^2 = 1, \tag{j1}$$

$$s_{p,q}s_{m,r} = s_{m,r}s_{p,q} \quad \text{if} \quad [p,q] \cap [m,r] = \emptyset, \tag{j2}$$

$$s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q} \quad \text{if} \quad [m,r] \subset [p,q]. \tag{j3}$$

The generator $s_{p,q}$ can be diagrammatically represented as the braid on n strands where the strands p, $p+1$, …, q intersect at one common point, and reverse their order after that point. The relations are depicted in Fig. 1. Here and below the diagrams are drawn from left to right, in order to match the order of generators in a word representing a cactus. These diagrams make one think of cacti once again—saguaros these time. For this reason we will often call *cacti* the elements of J_n, and use the term *leaf number* for the parameter $q-p+1$ of the generator $s_{p,q}$.

One should handle such diagrams with care: the braid relation $s_{1,2}s_{2,3}s_{1,2} = s_{2,3}s_{1,2}s_{2,3}$ from Fig. 2, natural in braid and knot theories (where it corresponds to the Reidemeister III move), does not hold in cactus groups.

The closure of such braids yields *cactus doodles*, i.e. curves with self-intersections [MR22].

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[1]We should have written $s_{p,q;n}$ here. However, we systematically drop the subscript n since it is always clear from the context. The same is done for the maps s and d, and for the Gauss diagrams τ_I below.

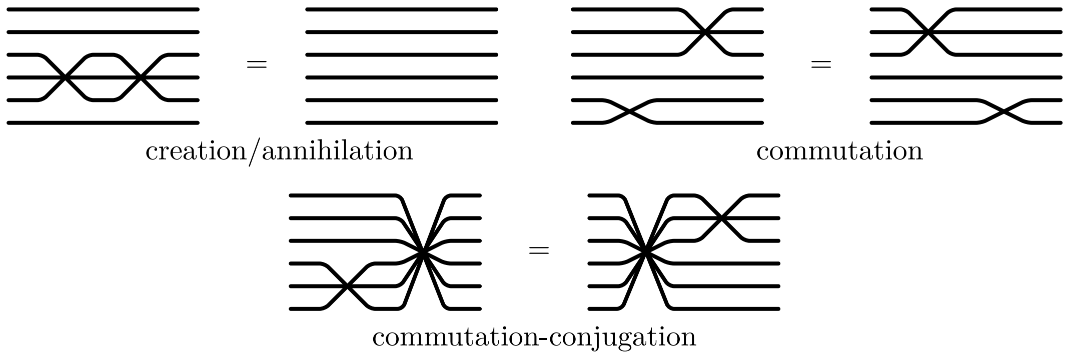

FIGURE 1. Three types of relations in cactus groups

FIGURE 2. The braid relation, false in cactus groups

Looking at how such braids permute their strands, one obtains a group morphism

$$\begin{aligned} &s\colon J_n \to S_n, \\ &s_{p,q} \mapsto (1, 2, \ldots, p-1,\ \boldsymbol{q, q-1, \ldots, p+1, p},\ q+1, q+2, \ldots, n). \end{aligned}$$

The kernel of this map is the *pure cactus group* PJ_n, sometimes denoted as Γ_{n+1}. It is the fundamental group of the real locus of the Deligne–Mumford compactification $\overline{\mathcal{M}}_{0,n+1}$ of the moduli space of rational curves with $n+1$ marked points [Dev99]. This explains why these groups are particularly interesting.

Our braid-like diagrams can be read in another way: label the strands from 1 (top) to n (bottom), and then at each multiple point write down the set of the labels of the intersecting strands. This yields a set-theoretic map

$$d\colon J_n \to D_n.$$

Here D_n is the *Gauss diagram group* from [Mos19][2]. Concretely, it has one generator τ_I for each subset I of $\{1, 2, \ldots, n\}$ of size ≥ 2, and the relations are

$$\tau_I^2 = 1, \tag{d1}$$

$$\tau_I \tau_J = \tau_J \tau_I \quad \text{if} \quad I \cap J = \emptyset \quad \text{or} \quad I \subset J. \tag{d2}$$

It is a *right-angled Coxeter group* (*RACG*), that is, it is generated by idempotents with only commutation relations between them. Its elements will be called *Gauss diagrams*, since they are related to the Gauss diagrams from virtual knot theory.

The map d is not a group morphism. In Section 2, we explain that it is a *group* 1-*cocycle*, injective by a theorem from [Mos19] (see also [Yu22] for a proof for other Coxeter types). However, its restriction to the pure part PJ_n becomes a group morphism.

The "reading" maps s and d can be assembled into a single injective group morphism

$$\rho = d \times s\colon J_n \to D_n \rtimes S_n.$$

The semi-direct product on the right can be seen as the *virtual cactus group*, where any, not necessarily neighboring, collection of strands can come together (using the S_n part) and form a multi-strand intersection (using the D_n part). Note that, contrary to the usual approach to virtuality in similar settings [BSV19, KNS21], in $D_n \rtimes S_n$ a diagram $\tau_I \in D_n$ and a permutation $\sigma \in S_n$ do commute when σ permutes elements from I only.

In Section 2, we use the map d to reduce the *word problem* in J_n to its much easier analogue in the RACG D_n, and describe an efficient solution.

[2]In [Mos19], the D_n were simply called *diagram groups*. Following a suggestion of Mostovoy, we use a more precise term, in order to avoid confusion with Guba and Sapir's diagram groups.

Similarly, in Section 3 we work on the D_n side to study certain subgroups of J_n. Concretely, given some $2 \le i \le j \le n$, consider the group $J_n^{i,j}$ defined by the generators $s_{p,q}$, where $1 \le p < q \le n$ and $i \le q - p + 1 \le j$, and the cactus relations (j1)-(j3). In other words, we keep only those generators whose leaf number is between i and j. The groups $Tw_n = J_n^{2,2}$ appeared as *Grothendieck cartographical groups* in [Voe90]; further as *twin groups* in [Kho97], as a diagrammatic description of the motion of n points on the plane without triple collisions, but also as a tool to study *doodles* (closed plane curves without triple intersections, see [FT79]); later under the name of *flat braids* [Mer99] and *planar braids* [MRM20]; and finally under the name of *traids* in physics literature [HK20]. By definition, the twin groups are RACGs, just like any group $J_n^{i,i}$. As other RACG families, they appear in several contexts such as topological robotics [GLMRM21]. Our first results is

Theorem A. *For all $2 \le i \le j \le n$, the natural maps*

$$J_n^{i,j} \to J_n,$$
$$s_{p,q} \mapsto s_{p,q}$$

are injective.

Thus cactus groups contain twin groups and their higher-leaf analogues. This result is actually established for a wider class of partial presentations of J_n. It can also be seen as the braid-like counterpart of the recent proof that the space of doodles embeds into that of cactus doodles [MR22].

We further exploit the injectiveness of d in Section 4 to study the *torsion* and the *center* of J_n and PJ_n. We prove

Theorem B. *The cactus groups J_n have no odd torsion. Moreover, for any k they contain torsion of order 2^k provided that n is big enough.*

Theorem C. *The pure cactus groups PJ_n is torsionless.*

This last result can also be obtained topologically, by interpreting PJ_n as the fundamental group of an aspherical manifold [EHKR10]. Our proof, of combinatorial nature, has the advantage of being elementary and self-contained.

Theorem D. *The groups J_n and PJ_n are centerless whenever $n > 2$ and $n > 3$ respectively.*

We have seen that the cactus group J_n contains several important RACGs (Tw_n and more generally all the $J_n^{i,i}$). In the opposite direction, it injects (non-homomorphically) into the RACG D_n. In some sense, it can be thought of as a deformation of a RACG, where some commutation relations are deformed to commutation-conjugation relations. In fact, it can be seen as the Coxeter-like finite quotient, in the sense of [LV19], of the structure group of a partial solution to the Yang–Baxter equation, in the sense of [Cho21]. It inherits some properties of the RACG D_n, but loses others: for instance, it has less center (the center of D_n is $\langle \tau_{1,2,\dots,n} \rangle \simeq \mathbb{Z}_2$) and more torsion ($D_n$ has torsion of order 2 but not of order 4).

Finally, in Appendix A we yield a (complicated) one-relator presentation for the first interesting pure cactus group PJ_4. In particular, it confirms the absence of torsion in this group.

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2. Word problem for cactus groups

Recall that, given a cactus diagram t representing a cactus $c \in J_n$, the Gauss diagram $d(c) \in D_n$ is constructed as follows: label the left endpoints of the strands of t from 1 (top) to n (bottom); at each crossing reverse the order of the strands, and hence of the labels; at the ith crossing write down the ()unordered) set I_i of the labels of the intersecting strands; finally, multiply the generators of D_n corresponding to these label sets from left to right, setting $d(c) = \tau_{I_1}\tau_{I_2}\cdots$. See Fig. 3 for an example[3].

[3]Here and below we write $\tau_{a,b,\dots}$ instead of $\tau_{\{a,b,\dots\}}$ for simplicity.

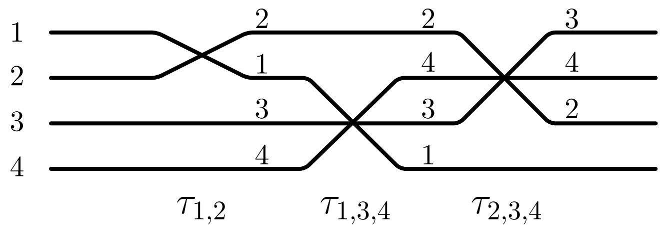

FIGURE 3. A cactus c with $s(c) = (4312)$ and $d(c) = \tau_{1,2}\tau_{1,3,4}\tau_{2,3,4}$

Theorem 2.1 ([Mos19, Yu22]). *The above procedure yields a well-defined injective map*

$$d\colon J_n \hookrightarrow D_n.$$

This is a consequence of the following two lemmas, which we will also need below.

Notation 2.2. Let FJ_n (resp., FD_n) be the free group on the generators $s_{p,q}$ (resp., τ_I).

The above procedure defines a map

$$\overline{d}\colon FJ_n \hookrightarrow FD_n,$$

which is clearly injective, but not surjective for $n > 2$ (for instance, $\tau_{1,3}$ is not in its image). A careful comparison of the relations defining J_n and D_n yields

Lemma 2.3. *(a) If a word $w' \in FJ_n$ is obtained from $w \in FJ_n$ by applying a relation of type* (j1) *(resp.,* (j2) *or* (j3)*), then $\overline{d}(w') \in FD_n$ is obtained from $\overline{d}(w) \in FD_n$ by applying a relation of type* (d1) *(resp.,* (d2)*).*
(b) Conversely, if a word $v' \in FD_n$ is obtained from some $\overline{d}(w) \in FD_n$ by applying an annihilation relation $\tau_I^2 \rightsquigarrow 1$ (resp., a commutation relation (d2)*), then $v' = d(w')$ for the word $w' \in FJ_n$ obtained from w by applying a corresponding relation of type* (j1) *(resp.,* (j2) *or* (j3)*).*

In other words, both commutation and commutation-conjugation relations for cacti are translated by commutation relations for Gauss diagrams.

Note that the statement (b) is false for creation relations $1 \rightsquigarrow \tau_I^2$, since these latter can lead outside of the image of $\overline{d}$.

The following result is standard in the theory of RACGs:

Lemma 2.4. *Let G be a RACG, and w a word in the standard generators (called* letters*) representing an element $g \in G$. Consider the following procedure: as long as w contains two copies of the same letter l separated by letters commuting with l, move one copy towards the other by commutation, then annihilate them by applying $l^2 \rightsquigarrow 1$; repeat. The result of this procedure is independent, up to commutation in G, of the choice of the annihilated couples and of the choice of the word w representing g.*

In particular, one can transform any two words representing the same element of a RACG into the same word without ever applying the creation relation.

These two lemmas immediately imply that $\overline{d}$ induces an injective map $d\colon J_n \to D_n$.

In fact, the second lemma yields more. Choose any order on the set of generators of a RACG G, and extend it lexicographically to the words in these generators. Since by Lemma 2.4 all minimal-length representatives of an element g of a RACG G are equivalent up to commutation, the minimal word among such representatives yields an easily computable *normal form* on G. According to Lemma 2.3, this normal form can be pulled back from D_n to J_n. This gives a solution to the *word problem* in J_n, which we summarise as follows:

Proposition 2.5. *Let $w \in FJ_n$ be a word representing a cactus $c \in J_n$. Consider the following procedure: if w contains two letters l and l' such that l' can be (conjugation-)commuted all the way to l (according to the rules* (j2)-(j3)*) and in the process becomes l, then do this (conjugation-)commutation and annihilate ll (according to the rule* (j1)*); repeat as long as possible. The result is the empty word if and only if the cactus c is trivial.*

This procedure has a nice diagrammatic interpretation if one works with i-leaf cacti only (that is, with elements from $J_n^{i,i}$). It then consists in *bigon killing*, as illustrated in Fig. 4.

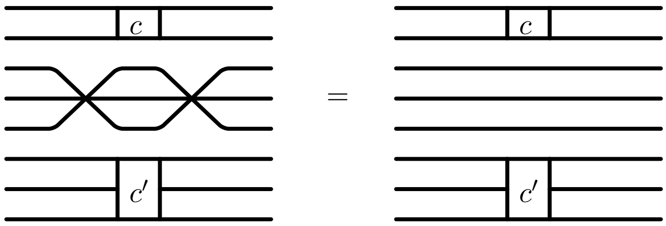

FIGURE 4. Bigon killing; here c is any cactus from $J_a^{3,3}$, and c' is any cactus from $J_b^{3,3}$

Definition 2.6. A word $w \in FJ_n$ is called *irreducible* if it contains no two letters that can be (conjugation-)commuted together and annihilated.

Pulling back form D_n to J_n the results of Lemma 2.4, one sees that all irreducible representatives of a cactus $c \in J_n$ are related by (conjugation-)commutation. In particular, they have the same length, which is minimal for representatives of c.

Remark 2.7. The conjugacy problem in J_n is much more delicate. In particular, conjugation may shorten even very simple irreducible words. The word $s_{3,4}s_{1,2}s_{1,4}s_{3,6} \in J_6$ illustrates this phenomenon:

$$\begin{aligned} s_{5,6}s_{3,4} \cdot (s_{3,4}s_{1,2}s_{1,4}s_{3,6}) \cdot s_{3,4}s_{5,6} &= s_{5,6}s_{1,2}s_{1,4}s_{3,6}s_{3,4}s_{5,6} = s_{5,6}s_{1,4}s_{3,4}s_{3,6}s_{5,6}s_{3,4} \\ &= s_{5,6}s_{1,4}s_{3,6}s_{5,6}s_{5,6}s_{3,4} = s_{1,4}s_{5,6}s_{3,6}s_{3,4} \\ &= s_{1,4}s_{3,6}s_{3,4}s_{3,4} = s_{1,4}s_{3,6}. \end{aligned}$$

We finish this section with a remark on the nature of the map d. It is not a group morphism: for example, applied to the cactus from Fig. 3, it yields $\tau_{1,2}\tau_{1,3,4}\tau_{2,3,4}$, whereas a group morphism would have given $\tau_{1,2}\tau_{2,3,4}\tau_{1,2,3}$. However, it is not so far from being one. It is in fact a *group* 1-*cocycle*, that is, it satisfies the twisted compatibility relation

$$d(c_1c_2) = d(c_1)\ {}^{c_1}d(c_2), \qquad c_1, c_2 \in J_n,$$

where the left group action of J_n on D_n is induced from the label-permuting S_n-action on D_n: ${}^{c}t = {}^{s(c)}t$, with $c \in J_n$, $t \in D_n$. In the example from Fig. 3, we obtain

$$d(s_{1,2}s_{2,4}s_{1,3}) = \tau_{1,2}\ {}^{(2134)}\tau_{2,3,4}\ {}^{(4132)}\tau_{1,2,3} = \tau_{1,2}\tau_{1,3,4}\tau_{2,3,4}.$$

Note that, restricted to the pure part PJ_n, the map d becomes a group morphism, since $s(c) = \mathrm{Id}$ for a pure cactus c.

3. Twin groups are subgroups of cactus groups

Consider a group $G = \langle S \mid R \rangle$ defined by a set of generators S and a set of relations R. For any subset of generators $I \subseteq S$, one can extract from R all the relations R_I involving the generators from I only. This defines a new group $G_I := \langle I \mid R_I \rangle$, with the obvious map

$$\begin{aligned} \iota_I \colon G_I &\to G, \\ g &\mapsto g \qquad \text{for all } g \in I. \end{aligned}$$

Such maps will be called *ι-type maps* in what follows. They need not be injective.

Definition 3.1. A subset of generators I is called *complete* if the above map ι_I is injective.

Example 3.2. In a RACG or a RAAG with its standard presentation, any generator subset is complete. It follows from Lemma 2.4 and its analogue for RAAGs.

Example 3.3. All generator subsets are complete in braid and symmetric groups with their standard presentations as well.

This is actually true for more general Artin–Tits and Coxeter groups.

Example 3.4. In the virtual braid group $G = VB_3$ with its classical generators σ_1, σ_2 and virtual generators τ_1, τ_2 and the usual relations, the set $I = \{\sigma_1, \tau_1, \tau_2\}$ is not complete. Indeed, since there are no relations relating σ_1 only to the τ's, G_I is the direct product $G_I = \mathbb{Z} * S_3$, and it includes two distinct elements

$$\sigma_1\tau_1\tau_2\sigma_1\tau_2\tau_1\sigma_1 \quad \text{and} \quad \tau_1\tau_2\sigma_1\tau_2\tau_1\sigma_1\tau_1\tau_2\sigma_1\tau_2\tau_1,$$

sent by ι_I to the same element $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$ of G, since $\sigma_2 = \tau_1\tau_2\sigma_1\tau_2\tau_1$ in G.

Cactus groups provide some more interesting counterexamples:

Example 3.5. In the cactus group $G = J_4$ with its usual presentation, there are no relations involving only $s_{1,2}$ and $s_{1,4}$, except for idempotence relations. Thus, for $I = \{s_{1,2}, s_{1,4}\}$, one gets $G_I = \mathbb{Z}_2 * \mathbb{Z}_2$. However, in G these generators satisfy the relation

$$s_{1,2}s_{1,4}s_{1,2}s_{1,4} = s_{1,4}s_{1,2}s_{1,4}s_{1,2},$$

which expresses the commutation of $s_{1,2}$ and $s_{3,4} = s_{1,4}s_{1,2}s_{1,4}$.

In the example above, one should add the generator $s_{3,4} = s_{1,4}s_{1,2}s_{1,4}$ to make the set $I = \{s_{1,2}, s_{1,4}\}$ complete. We will now prove that this is the only possible completeness defect in cactus groups. More precisely, a generator subset is complete al long as it is stable by certain conjugations.

Definition 3.6. A collection $\mathcal{C}$ of sub-intervals of the integer interval $[1, n]$ is called *symmetric* if together with any two nested sub-intervals $[m, r] \subset [p, q]$ it contains the sub-interval $[p+q-r, p+q-m]$, symmetric to $[m, r]$ with respect to the middle of $[p, q]$.

Theorem 3.7. *For any symmetric collection $\mathcal{C}$ of sub-intervals of $[1, n]$, the family $\{s_I | I \in \mathcal{C}\}$ of generators of the cactus group J_n (with its standard presentation) is complete.*

Before giving a proof, let us describe several important particular cases.

Corollary 3.8. *The group J_n can be viewed as a subgroup of J_{n+k}, via the map $s_{p,q} \mapsto s_{p,q}$.*

Corollary 3.9. *The twin group Tw_n can be viewed as a subgroup of the cactus group J_n, via the map $s_{p,p+1} \mapsto s_{p,p+1}$.*

More generally, given some $2 \le i \le j \le n$, the sub-interval collection

$$\mathcal{C}_{i,j} := \{[p, q] \mid 1 \le p < q \le n,\ i \le q - p + 1 \le j\}$$

is clearly symmetric. Theorem 3.7 thus applies to the group $J_n^{i,j}$ defined by the generators $s_{p,q}$, where $[p, q] \in \mathcal{C}_{i,j}$, and the cactus relations (j1)-(j3). In other words, in $J_n^{i,j}$ we keep only those generators whose leaf number is between i and j. We get

Corollary 3.10. *The group $J_n^{i,j}$ can be viewed as a subgroup of J_n, via the map $s_{p,q} \mapsto s_{p,q}$.*

Proof of Theorem 3.7. Take a symmetric collection $\mathcal{C}$ of sub-intervals of $[1, n]$. Consider a word $w \in FJ_n$ which contains only generators s_I with $I \in \mathcal{C}$, and which represents the trivial element in J_n. We need to show that it also represents the trivial element in $(J_n)_I$. According to Proposition 2.5, the word w can be turned into the trivial word by applying commutation, commutation-conjugation and annihilation relations. But all these relation are also available in the group $(J_n)_I$; in fact, the symmetry condition on I was imposed precisely to preserve all commutation-conjugation relations from J_n in $(J_n)_I$. $\square$

Remark 3.11. If one is interested in the i-leaf group $J_n^{i,i}$ only (for instance, the twin group $Tw_n = J_n^{2,2}$), then in the arguments above the Gauss diagram group D_n can be replaced with a smaller RACG. Concretely, consider the *width i Gauss diagram group D_n^i* generated by the idempotents τ_I for all i-element subsets I of $\{1, 2, \ldots, n\}$, which commute if the corresponding subsets are disjoint. The symmetric group S_n still acts on such subsets I, and hence on D_n^i. Consider the following *eraser map*:

$$\begin{aligned}
&\varepsilon_i\colon J_n \to D_n^i \rtimes S_n, \\
&\quad s_{p,q} \mapsto 1 && \text{if } q-p+1<i, \\
&\quad s_{p,q} \mapsto (\tau_{[p,q]}, s(s_{p,q})) && \text{if } q-p+1=i, \\
&\quad s_{p,q} \mapsto (1, s(s_{p,q})) && \text{if } q-p+1>i.
\end{aligned}$$

Going through the defining relations (j1)-(j3) of J_n, one checks that this map is well defined. Now, in the diagram below, the rectangle and the square clearly commute:

$$\begin{tikzcd}
J_n^i \arrow[r, "\iota"] & J_n \arrow[r, "\rho"] & D_n \rtimes S_n \arrow[r, "\pi_1"] & D_n \\
J_n^i \arrow[u, "\mathrm{Id}"] \arrow[r, "\iota"] & J_n \arrow[r, "\varepsilon_i"] & D_n^i \rtimes S_n \arrow[u, "\iota"] \arrow[r, "\pi_1"] & D_n^i. \arrow[u, "\iota"]
\end{tikzcd}$$

Here we abusively use the same notation ι for all ι-type maps, and the same notation π_1 for all (set-theoretic) projections onto the first component of a semi-direct product. Then the injectivity of the total map of the first line implies the injectivity for the second line. In other words, we obtain an injective group 1-cocycle $J_n^i \to D_n^i$.
In the same vein, the symmetric group S_n above can be replaced with the subgroup S_n^i generated by all the size i flops $s(s_{p,q})$. It would be interesting to understand the structure of these permutation subgroups.

The eraser map from the above remark admits the following variation:

$$\begin{aligned}
&\epsilon_i\colon J_n \to J_n^{i,n}, \\
&\quad s_{p,q} \mapsto 1 && \text{if } q-p+1<i, \\
&\quad s_{p,q} \mapsto s_{p,q} && \text{if } q-p+1\geq i.
\end{aligned}$$

In other words, it erases all generators with leaf number $< i$. A quick direct verification shows that it is well defined and surjective; the map $\iota_i\colon J_n^{i,n} \to J_n$, $s_{p,q} \mapsto s_{p,q}$, is its section (cf. Corollary 3.10). The subgroup $J_n^{2,i-1} \overset{\iota_i'}{\hookrightarrow} J_n$, and hence its normal closure $\langle\langle J_n^{2,i-1}\rangle\rangle$, is by construction in the kernel of ϵ_i. We will now prove that this is the whole kernel. In particular, this yields the following semi-direct decompositions of the cactus groups:

$$J_n \simeq \langle\langle J_n^{2,i-1}\rangle\rangle \rtimes J_n^{i,n}.$$

Proposition 3.12. *The maps above define the following split exact sequence:*

$$0 \longrightarrow \langle\langle J_n^{2,i-1}\rangle\rangle \xrightarrow{\iota_i'} J_n \underset{\iota_i}{\overset{\epsilon_i}{\rightleftarrows}} J_n^{i,n} \longrightarrow 0.$$

Proof. It remains to prove that any cactus c in the kernel of the eraser map ϵ_i lies in fact in the normal closure $\langle\langle J_n^{2,i-1}\rangle\rangle$. Take a word $w \in FJ_n$ representing c. Since $c \in \mathrm{Ker}(\epsilon_i)$, one can erase all the letters from w with leaf number $< i$, and then (permutation-)commute together and annihilate pairs of remaining letters, in well-chosen order, until the word becomes empty, as explained in Proposition 2.5. Now, this (permutation-)commutation and annihilation can still be performed when the "small" letters are not erased: to move a letter l over a small letter m (or its conjugate), simply replace m by its l-conjugate, since $lm = (lml)l$ and $ml = l(lml)$. When the process stops, one is left with a product of conjugates of small letters representing c. $\square$

One can push the above arguments slightly further and show that, for any $2 \leq i' \leq i \leq j \leq j' \leq n$, $J_n^{i,j}$ can be viewed as a subgroup of $J_n^{i',j'}$, via the map $s_{p,q} \mapsto s_{p,q}$. This defines a functor from the poset of integer sub-intervals of $[1,n]$ to the category of subgroups of J_n. Moreover, for any $2 \leq i' \leq i \leq j \leq n$, one has the decomposition

$$J_n^{i',j} \simeq \langle\langle J_n^{i',i-1}\rangle\rangle \rtimes J_n^{i,j}.$$

A possible application of these constructions is the filtration

$$\langle\langle J_n^{2,n-2}\rangle\rangle \lhd J_n^{2,n-1} \lhd J_n$$

with RACG quotients

$$J_n/J_n^{2,n-1} \simeq J_n^{n,n} \simeq \mathbb{Z}_2,$$
$$J_n^{2,n-1}/\langle\langle J_n^{2,n-2}\rangle\rangle \simeq J_n^{n-1,n-1} \simeq \mathbb{Z}_2 * \mathbb{Z}_2 \text{ if } n \geq 3.$$

However, understanding the structure of the next piece, $\langle\langle J_n^{2,n-2}\rangle\rangle$, seems difficult even for $n = 4$.

Remark 3.13. In this section, we have seen that J_n contains many RACG subgroups. It would be interesting to find out whether **all** RACGs can be realised inside cactus groups. For instance, a tedious direct verification shows that one can include any RACG with ≤ 5 generators into a (sufficiently big) cactus group by sending each generator to a generator (as usual for the standard presentation), except for the "pentagon" group

$$\langle g_1, \ldots, g_5 \mid \forall i, g_i^2 = 1 \text{ and } g_i g_{i+1} = g_{i+1} g_i \rangle.$$

Here g_6 is identified with g_1.

4. Torsion and center of cactus groups

Many basic group-theoretic questions are easy to answer for a RACG G. For instance,

(1) Its center $Z(G)$ is generated by all its *friendly* generators (that is, the generators of G commuting with all other generators). Thus $Z(G) \simeq \mathbb{Z}_2^f$, f being the number of the friendly generators of G.
(2) The only torsion G has is of order 2. More precisely, 2-torsion elements are the conjugates of products of pairwise commuting generators.

In particular, for the Gauss diagram group D_n, we have the center

$$Z(D_n) = \langle \tau_{1,2,\ldots,n} \rangle \simeq \mathbb{Z}_2,$$

and a big 2-torsion part, without any other torsion.

The aim of this section is to determine the center and the torsion of the cactus group J_n and its pure part PJ_n. Our main tool is the connection between J_n and the RACG D_n. Curiously, the answers are close to but different from those for D_n.

Theorem 4.1. *The cactus group J_n is centerless whenever $n > 2$.*

In the case $n = 2$, we have $J_2 \simeq \mathbb{Z}_2$, and PJ_2 is trivial. We will no longer mention this case in what follows.

Proof. Let $w \in FJ_n$ be a word representing a non-trivial central element $c \in J_n$. It can be assumed to be of minimal length among representatives of non-trivial central elements. We will show that the Gauss diagram $d(c)$ is then central in D_n. As recalled above, this would imply $d(c) = \tau_{1,2,\ldots,n}$ or 1. Since d is injective and c non-trivial, this means $c = s_{1,n}$. But the generator $s_{1,n}$ does not commute with $s_{1,2}$ when $n > 2$, since

$$d(s_{1,n}s_{1,2}) = \tau_{1,2,\ldots,n}\tau_{n,n-1} \neq \tau_{1,2,\ldots,n}\tau_{1,2} = \tau_{1,2}\tau_{1,2,\ldots,n} = d(s_{1,2}s_{1,n}).$$

Thus there are no non-trivial central elements in J_n.

Take a generator $s_{p,q}$ of J_n. Since $cs_{p,q} = s_{p,q}c$, Lemmas 2.3 and 2.4 leave us with two options.

Option 1: The words $ws_{p,q}$ and $s_{p,q}w$ are irreducible. Then $s_{p,q}w$ can be transformed to $ws_{p,q}$ by commutation and commutation-conjugation relations only. In particular, a letter l in $s_{p,q}w$ can be (conjugation-)commuted to the end of the word and yield the letter $s_{p,q}$.

- **Case 1:** The letter l is the initial letter $s_{p,q}$ of $s_{p,q}w$. At the level of Gauss diagrams, this means that all letters in the word $\overline{d}(w) \in FD_n$ commute with ${}^{s(c)}s_{p,q}$ in D_n.
- **Case 2:** The letter l is from the word w. This means that (conjugation-)commutation can transform w into $w's_{p,q}$. But then $ws_{p,q} = w's_{p,q}s_{p,q} = w'$ in J_n, and the word $ws_{p,q}$ is no longer minimal.

Option 2: The words $ws_{p,q}$ and $s_{p,q}w$ are reducible (simultaneously, since all minimal representatives of a cactus have the same length). Recalling that the word w is minimal, and looking what this means for $ws_{p,q}$ on the D_n side, one concludes that we are in the situation of the Case 2 above: a letter l' can be (conjugation-)commuted to the end of w, so that w becomes $w's_{p,q}$. But the same argument applied to $s_{p,q}w$ shows that a letter l'' can be (conjugation-)commuted to the beginning of w, so that w becomes $s_{p,q}w''$. Again, two cases are possible.

Case 1: The letters l' and l'' occupy the same position in w. At the level of Gauss diagrams, this means that all letters in the word $\overline{d}(w) \in FD_n$ commute with ${}^{s(c)}s_{p,q}$ in D_n.

Case 2: The letters l' and l'' occupy different positions in w. Then (conjugation-)commutation can transform w into $s_{p,q}us_{p,q}$. The relation $ws_{p,q} = s_{p,q}w$ implies $us_{p,q} = s_{p,q}u$ in J_n, hence $w = s_{p,q}us_{p,q} = us_{p,q}s_{p,q} = u$ in J_n. We get a shorter word u representing the same cactus as w, which contradicts the minimality of w.

Since these arguments work for any letter $s_{p,q}$, one concludes that the diagram $d(c) \in D_n$ represented by the word $\overline{d}(w) \in FD_n$ is central, as claimed. □

Theorem 4.2. *The pure cactus subgroup PJ_n has trivial centralizer in J_n whenever $n > 3$. In particular, its center is trivial.*

The exceptional case $n = 3$ can be treated by hand. We have

$$J_3 \simeq FC_2 \rtimes \mathbb{Z}_2,$$

where the free Coxeter subgroup $\langle s_{1,2}, s_{2,3}\rangle \simeq FC_2$ is generated by the 2-leaf cacti, and the generator $s_{1,3}$ of the $\mathbb{Z}_2$ part acts on the FC_2 by permuting $s_{1,2}$ and $s_{2,3}$. Further,

$$PJ_3 = \langle a := s_{1,2}s_{2,3}s_{1,2}s_{1,3}\rangle \simeq \mathbb{Z},$$

and its centralizer in J_3 is

$$\mathrm{C}_{J_3}(PJ_3) = \langle b := s_{1,2}s_{1,3}\rangle \simeq \mathbb{Z}.$$

Note that $a = b^3$. See Appendix A for more detail.

Proof. Let $w \in FJ_n$ be a word representing a non-trivial element $c \in J_n$ commuting with every pure cactus. It can be assumed of minimal length among such words.

Any generator $s_{p,q}$ with leaf number > 2 can be transformed into a pure cactus by attaching some 2-leaf generators:

$$\tilde{s}_{p,q} := s_{p,q}s_{p_1,p_1+1}s_{p_2,p_2+1}\cdots,$$

since neighbouring transpositions generate the symmetric group S_n. This can be done in multiple ways; any choice will work for us. The commuting relation $c\tilde{s}_{p,q} = \tilde{s}_{p,q}c$ can be analysed along the lines of the proof of Theorem 4.1. One concludes that all letters in the word $\overline{d}(w) \in FD_n$ commute with ${}^{s(c)}s_{p,q}$ in D_n. Thus all letters in $\overline{d}(w)$ are *almost friendly*, that is, commute with all the τ_I of size $|I| > 2$.

Let us now prove that $\tau_{1,2,\ldots,n}$ is the only almost friendly generator of D_n when $n > 3$. Indeed, given a proper subset $I \subsetneq \{1, 2, \ldots, n\}$ of size > 2, one can replace one of its elements with another element from $\{1, 2, \ldots, n\}$, and get a subset I' of size > 2 such that τ_I and $\tau_{I'}$ do not commute in D_n. For a subset of size 2 the argument is similar, except that one replaces an element with two new ones; there is enough place for it in $\{1, 2, \ldots, n\}$ since $n > 3$.

Thus the centralizer of PJ_n can contain only the d-preimage $s_{1,n}$ of $\tau_{1,2,\ldots,n}$. But this element does not commute with $s_{1,3}s_{1,2}s_{2,3}s_{1,2} \in PJ_n$, since

$$d(s_{1,n}s_{1,3}s_{1,2}s_{2,3}s_{1,2}) = \tau_{1,2,\ldots,n}\tau_{n-2,n-1,n}\tau_{n-2,n-1}\tau_{n-2,n}\tau_{n-1,n} \quad \text{and}$$
$$d(s_{1,3}s_{1,2}s_{2,3}s_{1,2}s_{1,n}) = \tau_{1,2,3}\tau_{2,3}\tau_{1,3}\tau_{1,2}\tau_{1,2,\ldots,n} = \tau_{1,2,\ldots,n}\tau_{1,2,3}\tau_{2,3}\tau_{1,3}\tau_{1,2}$$

are distinct in D_n when $n > 3$. □

Theorem 4.3. *The cactus group J_n has no odd torsion.*

Proof. Fix an odd prime p. Among non-trivial p-torsion elements in J_n (if they exist), choose an element c with the shortest possible representative w. According to Proposition 2.5, the triviality of c^p implies that in the word w^p a letter l can be (conjugation-)commuted to the right towards a letter l', so that the two get annihilated.

Case 1: The letters l and l' occupy different positions in the qth and q'th copies of w respectively. We have $q < q'$ since w is irreducible. One can assume l to be in the last position, and l' in the first position (otherwise the letters of w should be (conjugation-)commuted accordingly). Remove the first letter of w and put it to the end; let w' be the resulting word. It represents a non-trivial p-torsion element $c' \in J_n$ (which is a conjugate of c). The word w'^p is obtained from w^p by moving the first letter to the end. In w'^p, the letters l and l' can still be (conjugation-)commuted together and annihilated. The letter l remains in the qth copy of w', whereas the letter l' is now in the $(q'-1)$st copy. They are still in different positions in their respective copies. Repeating this argument, one gets a non-trivial p-torsion element represented by a word $\tilde{w}$ with an annihilation possibility inside $\tilde{w}$ (case $q = q'$), hence with a representative shorter than w. This contradicts the minimality of w.

Case 2: The letters l and l' occupy the same position i in different copies of w. Consider the word

$$\overline{d}(w^p) = \overline{d}(w)\ {}^{t}\overline{d}(w)\ {}^{t^2}\overline{d}(w)\ldots\ {}^{t^{p-1}}\overline{d}(w),$$

where $t = s(c)$. Then the letters of $\overline{d}(w^p) \in FD_n$ corresponding to the p copies of the ith letter l from w are $\tau,\ {}^{t}\tau,\ {}^{t^2}\tau, \ldots,\ {}^{t^{p-1}}\tau$. Since p is prime, the permutation t is of order p or 1 (as $t^p = s(c)^p = s(c^p) = s(1) = \mathrm{Id}$). In its orbit containing τ, two elements, corresponding to l and l' in w^p, can be commuted together and annihilated, thus coincide. The p letters in $\overline{d}(w^p)$ corresponding to l are thus all identical, and can be commuted all through the word $\overline{d}(w^p)$. Since the word $\overline{d}(w^p)$ represents the trivial Gauss diagram, it contains an even number of copies of the letter τ, and thus at least one copy different from the p copies mentioned above; here we used that p is odd for the first time in this proof. Thus τ appears at least twice in one of the words $\overline{d}(w),\ {}^{t}\overline{d}(w),\ {}^{t^2}\overline{d}(w), \ldots,\ {}^{t^{p-1}}\overline{d}(w)$, where its two occurrences can be moved together and annihilated. This contradicts the minimality of w. ☐

Theorem 4.4. *The cactus group J_{2^k} has torsion of order 2^k.*

Proof. Consider the cacti defined inductively by

$$\begin{aligned} t_1 &= s_{1,2}, \\ t_2 &= s_{1,2}s_{1,4}, \qquad \cdots \\ t_{k+1} &= t_k s_{1,2^{k+1}}. \end{aligned}$$

The cactus t_k is defined in the group J_n whenever $n \geq 2^k$. Let us prove by induction that t_k is of order 2^k. For $k = 1$, this is just the idempotence of $s_{1,2}$. To move from k to $k+1$, observe that

$$t_{k+1}^2 = (t_k s_{1,2^{k+1}})^2 = t_k(s_{1,2^{k+1}} t_k s_{1,2^{k+1}}) = t_k t'_k.$$

In the word $t'_k := s_{1,2^{k+1}} t_k s_{1,2^{k+1}}$, the first letter $s_{1,2^{k+1}}$ can be conjugate-commuted all the way to the right and annihilated with the last letter. In the resulting word, the indices of all letters are $> 2^k$. Thus the cactus t_k and its conjugate t'_k commute. Since both are of order 2^k by assumption, so is their product t_{k+1}^2. Hence the order of t_{k+1} is 2^{k+1}. ☐

Theorem 4.5. *The pure cactus group PJ_n is torsionless.*

Proof. By Theorem 4.3, it is sufficient to show that PJ_n has no 2-torsion. Assume that there is some. Among non-trivial 2-torsion elements in PJ_n, choose an element c with the shortest possible representative w. Following the proof of Theorem 4.3, one concludes that $\overline{d}(w)$ is a product of pairwise commuting generators. In particular, one can reorganise the word w by (conjugation-)commutation into a word w' so that the leaf number of its letters never increases from left to right. Let $s_{p,q}$ be the first letter of w. Viewing $s(c)$ as a permutation on the set $\{1, 2, \ldots, n\}$, let us trace what it does to the element p. First, $s(s_{p,q})$ sends p to position q. The next letter whose associated permutation moves this element has to be of the form $s_{p',q}$ with $p' > p$, due to the pairwise commutativity of the letters of $\overline{d}(w')$. The next letter moving this element is $s_{p',q'}$ with $q' < q$, and so on. We observe a retracting ping-pong-like trajectory. Overall, the permutation $s(c)$ moves our element strictly to the right, and thus cannot be trivial. Hence the cactus c cannot be pure. $\square$

Appendix A. A one-relator presentation for PJ_4

The goal of this appendix is to provide explicit group presentations for PJ_3 and PJ_4. First let us state simpler presentations for J_3 and J_4 which can be easily obtained using Tietze transformations:

Proposition A.1. *The cactus groups J_3 and J_4 admit the following group presentations:*

$$J_3 \simeq \langle s_{1,2}, s_{1,3} \mid s_{1,2}^2 = s_{1,3}^2 = 1 \rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$$

$$J_4 \simeq \left\langle \; s_{1,2}, s_{1,3}, s_{1,4} \;\middle|\; \begin{array}{c} s_{1,2}^2 = s_{1,3}^2 = s_{1,4}^2 = 1, \\ s_{1,2}s_{1,4}s_{1,2}s_{1,4} = s_{1,4}s_{1,2}s_{1,4}s_{1,2}, \\ s_{1,4}s_{1,3}s_{1,2}s_{1,3} = s_{1,3}s_{1,2}s_{1,3}s_{1,4} \end{array} \right\rangle.$$

Similar presentations can be produced for general n.

Corollary A.2. *The pure cactus group PJ_3 admits the following group presentation:*

$$PJ_3 = \langle (s_{1,2}s_{1,3})^3 \rangle \simeq \mathbb{Z}.$$

Proof. The cactus group J_3 is the RACG $\mathbb{Z}_2 * \mathbb{Z}_2$ generated by $s_{1,2}$ and $s_{1,3}$, and the symmetric group S_3 admits the presentation $S_3 \simeq \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, (s_1s_2)^3 = 1 \rangle$, which, when s_2 is replaced with $s_2' = s_1s_2s_1$, becomes $S_3 \simeq \langle s_1, s_2' \mid s_1^2 = (s_2')^2 = 1, (s_1s_2')^3 = 1 \rangle$. Since $s(s_{1,2}) = s_1$ and $s(s_{1,3}) = s_1s_2s_1 = s_2'$, the kernel of s is freely generated by $(s_{1,2}s_{1,3})^3$. $\square$

Note that the generator of PJ_3 above can be rewritten in a shorter form:

$$a := (s_{1,2}s_{1,3})^3 = s_{1,2}(s_{1,3}s_{1,2}s_{1,3})s_{1,2}s_{1,3} = s_{1,2}s_{2,3}s_{1,2}s_{1,3}.$$

Theorem A.3. *The pure cactus group PJ_4 admits the following group presentation:*

$$PJ_4 = \langle \alpha,\ \beta,\ \gamma,\ \delta,\ \epsilon \mid \alpha\gamma\epsilon\beta\epsilon\alpha^{-1}\delta^{-1}\beta\gamma\delta^{-1} = 1 \rangle,$$

where
$$\begin{aligned} \alpha &= (s_{1,3}s_{1,2})^3, \\ \beta &= s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4} = s_{1,3}s_{1,4}s_{1,3}(s_{1,2}s_{1,4})^2, \\ \gamma &= s_{1,2}s_{1,4}s_{1,2}(s_{1,3}s_{1,4})^2, \\ \delta &= s_{1,3}(s_{1,2}s_{1,4})^2 s_{1,3}s_{1,4}, \\ \epsilon &= (s_{1,4}s_{1,2}s_{1,3}s_{1,2})^2. \end{aligned}$$

We will derive this presentation by hand, using the Reidemeister–Schreier method. Our computations were verified in GAP by Neha Nanda and John Guaschi.

The group PJ_4 is thus a one-relator group, where the relation is not a power. Applying Theorem 4.12 of [MKS04], we then obtain another proof of the absence of torsion in PJ_4. One can also derive several other nice properties of PJ_4, using the classical theory of one-relator groups (see for instance [MKS04, LS01, Put] and references therein): PJ_4 is locally indicable and of cohomological dimension ≤ 2; it has algorithmically decidable word problem; it satisfies the Tits alternative: every its subgroup is either solvable or contains a free group of rank 2; its presentation 2-complex is aspherical.

Note that in our presentation of PJ_4, we used the generating set $\{s_{1,2}, s_{1,3}, s_{1,4}\}$ of J_4. One obtains shorter and more manageable expressions by including the generators $s_{i,j}$ with $i > 1$:

$$\begin{aligned} \alpha &= (s_{1,3}s_{1,2})^3 = s_{1,3}s_{1,2}s_{2,3}s_{1,2}, & \beta &= s_{1,4}s_{2,4}s_{1,3}s_{1,2}s_{3,4}, \\ \gamma &= s_{1,2}s_{3,4}s_{2,4}s_{1,3}s_{1,4}, & \delta &= s_{1,3}s_{1,2}s_{3,4}s_{1,3}s_{1,4}, \\ \epsilon &= s_{3,4}s_{2,3}s_{2,4}s_{1,2}s_{2,3}s_{1,3}. \end{aligned}$$

In particular, α is the inverse of the (image of the) generator a of PJ_3 from Corollary A.2. Also, some generators can be replaced with shorter and/or more meaningful ones:

(a) $\epsilon \rightsquigarrow \zeta = \epsilon\alpha^{-1} = s_{2,4}s_{2,3}s_{3,4}s_{2,3}$, which is the generator α "shifted" to the right (in other words, the inclusion of J_3 into J_4 given by $s_{p,q} \mapsto s_{p+1,q+1}$ sends a^{-1} to ζ);
(b) $\gamma \rightsquigarrow \eta = \beta\gamma = (s_{1,3}s_{2,4})^2$, which is the commutator of $s_{1,3}$ and $s_{2,4}$;
(c) $\delta \rightsquigarrow \theta = \alpha^{-1}\delta = s_{1,2}s_{2,3}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{3,4}s_{1,3}s_{1,4} = s_{1,2}s_{2,3}s_{3,4}s_{1,3}s_{1,4}$;
(d)
$$\begin{aligned}\beta \rightsquigarrow \kappa = \theta\eta^{-1}\beta &= s_{1,2}s_{2,3}s_{3,4}s_{1,3}s_{1,4} \cdot s_{2,4}s_{1,3}s_{2,4}s_{1,3} \cdot s_{1,4}s_{2,4}s_{1,3}s_{1,2}s_{3,4}\\ &= s_{1,2}s_{2,3}s_{3,4}s_{1,4}s_{2,4} \cdot s_{2,4}s_{1,3}s_{2,4}s_{1,3} \cdot s_{1,3}s_{2,4}s_{1,4}s_{1,2}s_{3,4}\\ &= s_{1,2}s_{2,3}s_{3,4}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{3,4}\\ &= s_{1,2}s_{2,3}s_{3,4}s_{2,4}s_{3,4}s_{1,2}\\ &= s_{1,2} \cdot s_{2,3}s_{3,4}s_{2,3}s_{2,4} \cdot s_{1,2},\end{aligned}$$

which is ζ^{-1} conjugated by $s_{1,2}$.

The generators $\alpha, \zeta, \kappa, \theta$ and η are depicted in Fig. 5.

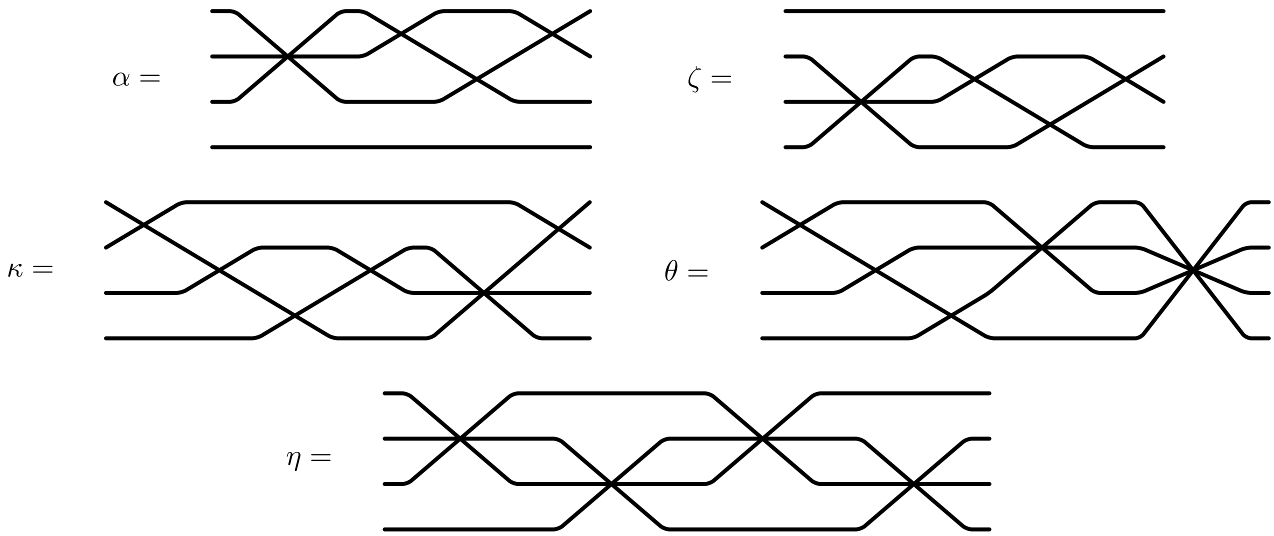

FIGURE 5. Generators of the group PJ_4

Observe that the squares α^2, ζ^2 and κ^2 can be rewritten using 2-leaf generators only, and yield 3 out of the 7 free generators of the *pure twin group* (also called the *planar pure braid group*) PTw_4 from [Mos20].

Remark A.4. In [Dev99], the group PJ_4 was given a topological interpretation. It is the fundamental group of the connected sum of five real projective planes. This yields its one-relator presentation of the following form:
$$PJ_4 = \langle\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5 \mid \alpha_1^2\alpha_2^2\alpha_3^2\alpha_4^2\alpha_5^2 = 1\ \rangle.$$
However, it seems difficult to find explicit expressions of the generators α_k in terms of the generators $s_{i,j}$ of the whole cactus group J_4.

Proof of Theorem A.3. Let us first recall the Reidemeister–Schreier method, in order to fix notations.

Let G be a group with presentation $G = \langle X \mid R \rangle$, where $X = \{x_1, \ldots, x_p\}$ is the set of generators and $R = \{r_1, \ldots, r_q\}$ is the set of relations. Let $F(X)$ be the free group on X. A *Schreier transversal* of a subgroup H in G is a set T of reduced words in the generators $\{x_1, \ldots, x_p\}$ containing exactly one representative of every right coset of H, and together with each word containing all its prefixes. Any subgroup of G admits a Schreier transversal [MKS04].

Fix a Schreier transversal T of H. Denote by $^-$ the map $F(X) \to T$ sending w to its representative $\overline{w} \in T$. For any $k \in T$ and $x_i \in X$, put

$$a_{k,x_i} = (kx_i)(\overline{kx_i})^{-1}.$$

According to Theorem 2.9 of [MKS04], H admits a group presentation having as generators all the non-trivial a_{k,x_i}. A system of relations is constructed as follows. Let $w = x_{i_1}^{\varepsilon_1} x_{i_1}^{\varepsilon_2} \dots x_{i_m}^{\varepsilon_m}$, where $\varepsilon_l = \pm 1$ and $x_{i_l} \in X$ for $l = 1, \dots, m$. The *rewriting function* is defined as

$$\tau(w) = a^{\varepsilon_1}_{k_{i_1}, x_{i_1}} \, a^{\varepsilon_2}_{k_{i_2}, x_{i_2}} \cdots a^{\varepsilon_m}_{k_{i_m}, x_{i_m}},$$

$$\text{where } k_{i_j} = \begin{cases} \overline{x_{i_1}^{\varepsilon_1} \cdots x_{i_{j-1}}^{\varepsilon_{j-1}}} & \text{if } \varepsilon_j = 1, \\ \overline{x_{i_1}^{\varepsilon_1} \cdots x_{i_j}^{\varepsilon_j}} & \text{if } \varepsilon_j = -1. \end{cases}$$

A complete set of relations for H is given by $\{\tau(kr_jk^{-1}) \mid 1 \leq j \leq q, \ k \in T\}$.

We now turn to our concrete subgroup $H = PJ_4$ of $G = J_4$. Fix the following Schreier transversal:

$$K = \left\{ \begin{array}{l} k_1 = 1, \ k_2 = s_{1,2}, \ k_3 = s_{1,3}, \ k_4 = s_{1,4}, \ k_5 = s_{1,2}s_{1,3}, \ k_6 = s_{1,2}s_{1,4}, \ k_7 = s_{1,3}s_{1,2}, \\ k_8 = s_{1,3}s_{1,4}, \ k_9 = s_{1,4}s_{1,2}, \ k_{10} = s_{1,4}s_{1,3}, \ k_{11} = s_{1,2}s_{1,3}s_{1,2}, \ k_{12} = s_{1,2}s_{1,3}s_{1,4}, \\ k_{13} = s_{1,2}s_{1,4}s_{1,2}, \ k_{14} = s_{1,2}s_{1,4}s_{1,3}, \ k_{15} = s_{1,3}s_{1,2}s_{1,4}, \ k_{16} = s_{1,3}s_{1,4}s_{1,2}, \\ k_{17} = s_{1,3}s_{1,4}s_{1,3}, \ k_{18} = s_{1,4}s_{1,2}s_{1,3}, \ k_{19} = s_{1,4}s_{1,2}s_{1,4}, \ k_{20} = s_{1,4}s_{1,3}s_{1,2}, \\ k_{21} = s_{1,4}s_{1,3}s_{1,4}, \ k_{22} = s_{1,2}s_{1,3}s_{1,2}s_{1,4}, \ k_{23} = s_{1,2}s_{1,3}s_{1,4}s_{1,2}, \ k_{24} = s_{1,2}s_{1,4}s_{1,3}s_{1,2} \end{array} \right\}.$$

The non-trivial generators of PJ_4 are:

$$\begin{aligned}
a_{k_7,s_{1,3}} &= s_{1,3}s_{1,2}s_{1,3}s_{1,2}s_{1,3}s_{1,2} \\
a_{k_{11},s_{1,3}} &= s_{1,2}s_{1,3}s_{1,2}s_{1,3}s_{1,2}s_{1,3} \\
a_{k_{12},s_{1,3}} &= s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4} \\
a_{k_{13},s_{1,3}} &= s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4} \\
a_{k_{13},s_{1,4}} &= s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,3} \\
a_{k_{14},s_{1,4}} &= s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4} \\
a_{k_{15},s_{1,2}} &= s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4} \\
a_{k_{15},s_{1,3}} &= s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2} \\
a_{k_{16},s_{1,3}} &= s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,2} \\
a_{k_{16},s_{1,4}} &= s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4} \\
a_{k_{17},s_{1,2}} &= s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4} \\
a_{k_{17},s_{1,4}} &= s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2} \\
a_{k_{18},s_{1,2}} &= s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,2} \\
a_{k_{18},s_{1,4}} &= s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2} \\
a_{k_{19},s_{1,2}} &= s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3} \\
a_{k_{19},s_{1,3}} &= s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,2} \\
a_{k_{20},s_{1,3}} &= s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,2} \\
a_{k_{20},s_{1,4}} &= s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3} \\
a_{k_{21},s_{1,2}} &= s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3} \\
a_{k_{21},s_{1,3}} &= s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2} \\
a_{k_{22},s_{1,2}} &= s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,4} \\
a_{k_{22},s_{1,3}} &= s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4} \\
a_{k_{23},s_{1,3}} &= s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,3} \\
a_{k_{23},s_{1,4}} &= s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,2} \\
a_{k_{24},s_{1,3}} &= s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,3} \\
a_{k_{24},s_{1,4}} &= s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,2}
\end{aligned}$$

We detail only non-trivial relations of type $\tau(kRk^{-1})$:

$\tau(k_1(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_1^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} = a_{k_{12},s_{1,3}}a_{k_{19},s_{1,2}}$
$\tau(k_1(s_{1,2}s_{1,4})^4k_1^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} = a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}$
$\tau(k_2(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_2^{-1}) = \tau(s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,2}}a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{17},s_{1,2}}a_{k_{19},s_{1,3}}$
$\tau(k_2(s_{1,2}s_{1,4})^4k_2^{-1}) = \tau(s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,2}}a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17}s_{1,4}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{19},s_{1,2}}a_{k_{17}s_{1,4}}$
$\tau(k_3(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_3^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,3}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}}a_{k_1,s_{1,3}}a_{k_3,s_{1,3}} = a_{k_7,s_{1,3}}a_{k_{22},s_{1,3}}$
$\tau(k_3(s_{1,2}s_{1,4})^4k_3^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} = a_{k_{15},s_{1,2}}a_{k_{20},s_{1,4}}$
$\tau(k_4(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_4^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{18},s_{1,4}}a_{k_{13},s_{1,3}}$
$\tau(k_4(s_{1,2}s_{1,4})^4k_4^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}}a_{k_1,s_{1,4}}a_{k_4,s_{1,4}} = a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}$
$\tau(k_5(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_5^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,3}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,3}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{11},s_{1,3}}a_{k_{15},s_{1,3}}$
$\tau(k_5(s_{1,2}s_{1,4})^4k_5^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{24},s_{1,4}}$
$\tau(k_6(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_6^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} =$
$a_{k_{13},s_{1,3}}a_{k_{18},s_{1,4}}$
$\tau(k_6(s_{1,2}s_{1,4})^4k_6^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}}a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} =$
$a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}$
$\tau(k_7(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_7^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}}a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,3}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,3}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} =$
$a_{k_{20},s_{1,3}}a_{k_{11},s_{1,3}}$
$\tau(k_7(s_{1,2}s_{1,4})^4k_7^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,2}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} =$
$a_{k_{16},s_{1,4}}a_{k_{21},s_{1,2}}$
$\tau(k_8(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_8^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{16},s_{1,3}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,4}}$
$\tau(k_8(s_{1,2}s_{1,4})^4k_8^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,4}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{16},s_{1,4}}a_{k_{21},s_{1,2}}$
$\tau(k_9(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_9^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,2}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{21},s_{1,3}}a_{k_{14},s_{1,4}}$
$\tau(k_9(s_{1,2}s_{1,4})^4k_9^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}}a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{13},s_{1,4}}a_{k_{17},s_{1,2}}$

$\tau(k_{10}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{10}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,4}) =$
$a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,3}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,3}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}}a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{20},s_{1,3}}a_{k_{11},s_{1,3}}$
$\tau(k_{10}(s_{1,2}s_{1,4})^4k_{10}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} =$
$a_{k_{20},s_{1,4}}a_{k_{15},s_{1,2}}$
$\tau(k_{11}s_{1,3}^2k_{11}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,3}}a_{k_7,s_{1,3}}a_{k_{11},s_{1,2}}$
$a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{11},s_{1,3}}a_{k_7,s_{1,3}}$
$\tau(k_{11}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{11}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,3}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{24},s_{1,3}}a_{k_7,s_{1,3}}$
$\tau(k_{11}(s_{1,2}s_{1,4})^4k_{11}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{,2}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{23},s_{1,4}}a_{k_{18},s_{1,2}}$
$\tau(k_{12}s_{1,3}^2k_{12}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,4}}$
$a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{12},s_{1,3}}a_{k_{19},s_{1,3}}$
$\tau(k_{12}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{12}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,3}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{23},s_{1,3}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,3}}a_{k_{22},s_{1,2}}a_{k_{19},s_{1,3}}$
$\tau(k_{12}(s_{1,2}s_{1,4})^4k_{12}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} =$
$a_{k_{23},s_{1,4}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,2}}$
$\tau(k_{13}s_{1,3}^2k_{13}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,4}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,2}}$
$a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{13},s_{1,3}}a_{k_{21},s_{1,3}}$
$\tau(k_{13}s_{1,4}^2k_{13}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,4}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,2}}$
$a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{13},s_{1,4}}a_{k_{17},s_{1,4}}$
$\tau(k_{13}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{13}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,2}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} =$
$a_{k_{14},s_{1,4}}a_{k_{21},s_{1,3}}$
$\tau(k_{13}(s_{1,2}s_{1,4})^4k_{13}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}}a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,}}a_{k_2,s_{1,2}} =$
$a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}$
$\tau(k_{14}s_{1,4}^2k_{14}^{-1})\tau(s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,4}s_{1,3}s_{1,4}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}$
$a_{k_2,s_{1,2}} = a_{k_{14},s_{1,4}}a_{k_{18},s_{1,4}}$
$\tau(k_{14}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{14}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,3}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} =$
$a_{k_{24},s_{1,3}}a_{k_7,s_{1,3}}$
$\tau(k_{14}(s_{1,2}s_{1,4})^4k_{14}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} =$
$a_{k_{24},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}$
$\tau(k_{15}s_{1,2}^2k_{15}^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,4}}$
$a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} = a_{k_{15},s_{1,2}}a_{k_{21},s_{1,2}}$
$\tau(k_{15}s_{1,3}^2k_{15}^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,2}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,4}}$
$a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} = a_{k_{15},s_{1,3}}a_{k_{24},s_{1,3}}$
$\tau(k_{15}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{15}^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,3}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} =$
$a_{k_{15},s_{1,2}}a_{k_{21},s_{1,3}}a_{k_{13},s_{1,4}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,3}}$

$\tau(k_{15}(s_{1,2}s_{1,4})^4k_{15}^{-1}) = \tau(s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{1,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,3}} =$
$a_{k_{15},s_{1,2}}a_{k_{20},s_{1,4}}$
$\tau(k_{16}s_{1,3}^2k_{16}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,4}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}$
$a_{k_3,s_{1,3}} = a_{k_{16},s_{1,3}}a_{k_{23},s_{1,3}}$
$\tau(k_{16}s_{1,4}^2k_{16}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,4}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,2}}$
$a_{k_3,s_{1,3}} = a_{k_{16},s_{1,4}}a_{k_{20},s_{1,4}}$
$\tau(k_{16}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{16}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{17},s_{1,4}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,3}}$
$\tau(k_{16}(s_{1,2}s_{1,4})^4k_{16}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{,2}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{15},s_{1,2}}a_{k_{20},s_{1,4}}$
$\tau(k_{17}s_{1,2}^2k_{17}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}$
$a_{k_3,s_{1,3}} = a_{k_{17},s_{1,2}}a_{k_{19},s_{1,2}}$
$\tau(k_{17}s_{1,4}^2k_{17}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,3}s_{1,3}s_{1,4}s_{1,3}) = a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}$
$a_{k_3,s_{1,3}} = a_{k_{17},s_{1,4}}a_{k_{13},s_{1,4}}$
$\tau(k_{17}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{17}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}}a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{17},s_{1,2}}a_{k_{19},s_{1,3}}$
$\tau(k_{17}(s_{1,2}s_{1,4})^4k_{17}^{-1}) = \tau(s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,4}s_{1,3})$
$= a_{k_1,s_{1,3}}a_{k_3,s_{1,4}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}}a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,2}}a_{k_{13},s_{1,4}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}} =$
$a_{k_{17},s_{1,2}}a_{k_{13},s_{1,4}}$
$\tau(k_{18}s_{1,2}^2k_{18}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,2}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,3}}$
$a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} = a_{k_{18},s_{1,2}}a_{k_{22},s_{1,2}}$
$\tau(k_{18}s_{1,4}^2k_{18}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,4}s_{1,3}s_{1,2}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,3}}$
$a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} = a_{k_{18},s_{1,4}}a_{k_{14},s_{1,4}}$
$\tau(k_{18}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{18}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{18},s_{1,2}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,3}}a_{k_{12},s_{1,3}}$
$\tau(k_{18}(s_{1,2}s_{1,4})^4k_{18}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{18},s_{1,2}}a_{k_{23},s_{1,4}}a_{k_{14},s_{1,4}}$
$\tau(k_{19}s_{1,2}^2k_{19}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,2}}a_{k_{19},s_{1,4}}$
$a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} = a_{k_{19},s_{1,2}}a_{k_{17},s_{1,2}}$
$\tau(k_{19}s_{1,3}^2k_{19}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,2}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,4}}$
$a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} = a_{k_{19},s_{1,3}}a_{k_{12},s_{1,3}}$
$\tau(k_{19}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{19}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,3}}a_{k_8,s_{1,4}}a_{k_3,s_{1,3}}a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{19},s_{1,2}}a_{k_{12},s_{1,3}}$
$\tau(k_{19}(s_{1,2}s_{1,4})^4k_{19}^{-1}) = \tau(s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,2}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}}a_{k_1,s_{1,4}}a_{k_4,s_{1,2}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,2}}a_{k_4,s_{1,4}} =$
$a_{k_{19},s_{1,2}}a_{k_{17},s_{1,4}}$
$\tau(k_{20}s_{1,3}^2k_{20}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,3}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,3}}$
$a_{k_{22},s_{1,3}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{20},s_{1,3}}a_{k_{22},s_{1,3}}$
$\tau(k_{20}s_{1,4}^2k_{20}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,3}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,4}}$
$a_{k_{16},s_{1,4}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{20},s_{1,4}}a_{k_{16},s_{1,4}}$

$\tau(k_{20}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{20}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}}a_{k_1,s_{1,3}}a_{k_3,s_{1,2}}a_{k_7,s_{1,3}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} =$
$a_{k_7,s_{1,3}}a_{k_{22},s_{1,3}}$
$\tau(k_{20}(s_{1,2}s_{1,4})^4k_{20}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,2}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} =$
$a_{k_{21},s_{1,2}}a_{k_{16},s_{1,4}}$
$\tau(k_{21}s_{1,2}^2k_{21}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,3}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,2}}a_{k_{21},s_{1,4}}$
$a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{21},s_{1,2}}a_{k_{15},s_{1,2}}$
$\tau(k_{21}s_{1,3}^2k_{21}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,3}s_{1,4}) = a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,3}}$
$a_{k_{13},s_{1,3}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} = a_{k_{21},s_{1,3}}a_{k_{13},s_{1,3}}$
$\tau(k_{21}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{21}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,2}}a_{k_8,s_{1,3}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,3}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} =$
$a_{k_{21},s_{1,2}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,3}}a_{k_{17},s_{1,4}}a_{k_{13},s_{1,3}}$
$\tau(k_{21}(s_{1,2}s_{1,4})^4k_{21}^{-1}) = \tau(s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,3}s_{1,4})$
$= a_{k_1,s_{1,4}}a_{k_4,s_{1,3}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,2}}a_{k_{15},s_{1,4}}a_{k_7,s_{1,2}}a_{k_3,s_{1,4}}a_{k_8,s_{1,2}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,2}}a_{k_{10},s_{1,4}}a_{k_{21},s_{1,4}}a_{k_{10},s_{1,3}}a_{k_4,s_{1,4}} =$
$a_{k_{21},s_{1,2}}a_{k_{16},s_{1,4}}$
$\tau(k_{22}s_{1,2}^2k_{22}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}$
$a_{k_{18},s_{1,2}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{22},s_{1,2}}a_{k_{18},s_{1,2}}$
$\tau(k_{22}s_{1,3}^2k_{22}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,3}}$
$a_{k_{20},s_{1,3}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{22},s_{1,3}}a_{k_{20},s_{1,3}}$
$\tau(k_{22}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{22}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,3}}a_{k_9,s_{1,4}}a_{k_{19},s_{1,3}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,3}}a_{k_{22},s_{1,4}}$
$a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{22},s_{1,2}}a_{k_{19},s_{1,3}}a_{k_{23},s_{1,3}}a_{k_{16},s_{1,4}}a_{k_{20},s_{1,3}}$
$\tau(k_{22}(s_{1,2}s_{1,4})^4k_{22}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,4}}$
$a_{k_{11},s_{1,2}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{24},s_{1,4}}$
$\tau(k_{23}s_{1,3}^2k_{23}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,3}}$
$a_{k_{16},s_{1,3}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{23},s_{1,3}}a_{k_{16},s_{1,3}}$
$\tau(k_{23}s_{1,4}^2k_{23}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,4}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,4}}$
$a_{k_{24},s_{1,4}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{23},s_{1,4}}a_{k_{24},s_{1,4}}$
$\tau(k_{23}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{23}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,3}}a_{k_{19},s_{1,4}}a_{k_9,s_{1,3}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,3}}a_{k_{23},s_{1,2}}$
$a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{12},s_{1,3}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,3}}a_{k_{20},s_{1,4}}a_{k_{16},s_{1,3}}$
$\tau(k_{23}(s_{1,2}s_{1,4})^4k_{23}^{-1}) = \tau(s_{1,2}s_{1,3}s_{1,4}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,3}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,3}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,2}}a_{k_{12},s_{1,4}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,4}}a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,4}}a_{k_{23},s_{1,2}}$
$a_{k_{12},s_{1,4}}a_{k_5,s_{1,3}}a_{k_2,s_{1,2}} = a_{k_{22},s_{1,2}}a_{k_{18},s_{1,4}}a_{k_{24},s_{1,4}}$
$\tau(k_{24}s_{1,3}^2k_{24}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2})a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,3}}$
$a_{k_{15},s_{1,3}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{24},s_{1,3}}a_{k_{15},s_{1,3}}$
$\tau(k_{24}s_{1,4}^2k_{24}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2}) = a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,4}}$
$a_{k_{23},s_{1,4}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{24},s_{1,4}}a_{k_{23},s_{1,4}}$
$\tau(k_{24}(s_{1,2}s_{1,3}s_{1,4}s_{1,3})^2k_{24}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,3}s_{1,2}s_{1,3}s_{1,4}s_{1,2})$
$= a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,3}}a_{k_5,s_{1,2}}a_{k_{11},s_{1,3}}a_{k_7,s_{1,4}}a_{k_{15},s_{1,3}}a_{k_{24},s_{1,2}}$
$a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{11},s_{1,3}}a_{k_{15},s_{1,3}}$
$\tau(k_{24}(s_{1,2}s_{1,4})^4k_{24}^{-1}) = \tau(s_{1,2}s_{1,4}s_{1,3}s_{1,2}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,4}s_{1,2}s_{1,3}s_{1,4}s_{1,2})$
$a_{k_1,s_{1,2}}a_{k_2,s_{1,4}}a_{k_6,s_{1,3}}a_{k_{14},s_{1,2}}a_{k_{24},s_{1,2}}a_{k_{14},s_{1,4}}a_{k_{18},s_{1,2}}a_{k_{22},s_{1,4}}a_{k_{11},s_{1,2}}a_{k_5,s_{1,4}}a_{k_{12},s_{1,2}}a_{k_{23},s_{1,4}}a_{k_{24},s_{1,2}}$
$a_{k_{14},s_{1,3}}a_{k_6,s_{1,4}}a_{k_2,s_{1,2}} = a_{k_{14},s_{1,4}}a_{k_{18},s_{1,2}}a_{k_{23},s_{1,4}}$

This implies:
$a_{k_7,s_{1,3}} = a^{-1}_{k_{11},s_{1,3}} = a_{k_{15},s_{1,3}} = a_{k_{20},s_{1,3}} = a^{-1}_{k_{22},s_{1,3}} = a^{-1}_{k_{24},s_{1,3}}$; $a_{k_{12},s_{1,3}} = a^{-1}_{k_{13},s_{1,4}} = a_{k_{17},s_{1,2}} = a_{k_{17},s_{1,4}} = a^{-1}_{k_{19},s_{1,2}} = a^{-1}_{k_{19},s_{1,3}}$; $a_{k_{13},s_{1,3}} = a_{k_{14},s_{1,4}} = a^{-1}_{k_{18},s_{1,4}} = a^{-1}_{k_{21},s_{1,3}}$; $a_{k_{15},s_{1,2}} = a_{k_{16},s_{1,4}} = a^{-1}_{k_{20},s_{1,4}} = a^{-1}_{k_{21},s_{1,2}}$; $a_{k_{16},s_{1,3}} = a^{-1}_{k_{23},s_{1,3}}$; $a_{k_{18},s_{1,2}} = a^{-1}_{k_{22},s_{1,2}}$; $a_{k_{23},s_{1,4}} = a^{-1}_{k_{24},s_{1,4}}$.

It follows that PJ_4 is generated by $a_{k_7,s_{1,3}}, a_{k_{12},s_{1,3}}, a_{k_{13},s_{1,3}}, a_{k_{15},s_{1,2}}, a_{k_{16},s_{1,3}}, a_{k_{18},s_{1,2}}, a_{k_{23},s_{1,4}}$, and a complete set of relations is:
$a_{k_7,s_{1,3}} a^{-1}_{k_{23},s_{1,4}} a^{-1}_{k_{16},s_{1,3}} a_{k_{12},s_{1,3}} a_{k_{13},s_{1,3}} a^{-1}_{k_{15},s_{1,2}} = 1$; $a_{k_{13},s_{1,3}} a_{k_{18},s_{1,2}} a_{k_{23},s_{1,4}} = 1$; and
$a_{k_7,s_{1,3}} a^{-1}_{k_{18},s_{1,2}} a^{-1}_{k_{12},s_{1,3}} a^{-1}_{k_{16},s_{1,3}} a_{k_{15},s_{1,2}} = 1$.

Therefore, using $a^{-1}_{k_{23},s_{1,4}} = a_{k_{13},s_{1,3}} a_{k_{18},s_{1,2}}$ and $a^{-1}_{k_{16},s_{1,3}} = a_{k_{12},s_{1,3}} a_{k_{18},s_{1,2}} a^{-1}_{k_7,s_{1,3}} a^{-1}_{k_{15},s_{1,2}}$, we get

$$\begin{aligned} PJ_4 = \langle\ & a_{k_7,s_{1,3}},\ a_{k_{12},s_{1,3}},\ a_{k_{13},s_{1,3}},\ a_{k_{15},s_{1,2}}, a_{k_{18},s_{1,2}}\ \mid \\ & a_{k_7,s_{1,3}} a_{k_{13},s_{1,3}} a_{k_{18},s_{1,2}} a_{k_{12},s_{1,3}} a_{k_{18},s_{1,2}} a^{-1}_{k_7,s_{1,3}} a^{-1}_{k_{15},s_{1,2}} a_{k_{12},s_{1,3}} a_{k_{13},s_{1,3}} a^{-1}_{k_{15},s_{1,2}} = 1 \rangle. \end{aligned}$$ □

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