# Algorithms for modular correspondences between abelian varieties with level structure

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April 1st 2025, Journées C2

# Abelian varieties

## Definition

An abelian variety is a complete connected group variety over a base field $k$.

- Abelian variety = points on a projective space (locus of homogeneous polynomials) + algebraic group law.
- An abelian variety is projective, smooth, irreducible and its group law is abelian.

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## Example

- Elliptic curves = Abelian varieties of dimension 1,
- Jacobians of genus $g$ (smooth) curves are abelian varieties of dimension $g$,
- The inclusion is strict for $g \geq 4$.

# Isogenies

**Definition**

An isogeny is a finite surjective morphisme between abelian verieties.

- Isogenies $=$ Rational map $+$ group morphism $+$ finite kernel.
- Isogenies $\iff$ Finite subgroups :

$$(f : A \to B) \mapsto \operatorname{Ker} f$$
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**Example**

Multiplication by $\ell$ ($\mapsto\ A[\ell]$, the $\ell$-torsion of $A$).

# Complex abelian varieties

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A projective embedding of $A = \mathbb{C}^g/\Lambda$ can be given by quasi-periodic functions with respect to $\Lambda$.

**Definition**

The space $\mathcal{L}_m$ of $\Lambda$-quasi-periodic function of level $m$ is the space of analytic function satisfying, for $z \in \mathbb{C}^g$ and $\lambda \in \mathbb{Z}^g$:

$$f(z+\lambda) = f(z) \quad f(z+\Omega\lambda) = \exp(-m \cdot \pi\imath {}^t\lambda\Omega\lambda - m \cdot 2\pi\imath {}^t z\lambda) f(z).$$

# Theta functions

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A theta function with rational characteristics $a, b \in \mathbb{Q}^g$ is given by :

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z, \Omega) = \sum_{n \in \mathbb{Z}^g} \exp\left(\imath\pi\,{}^t(n+a)\Omega(n+a) + 2\imath\pi\,{}^t(n+a)(z+b)\right).$$

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For $m \geq 2$, let $Z(m) = \mathbb{Z}^g / m\mathbb{Z}^g$. A basis of $\mathcal{L}_m$ is given by :

$$\left\{\theta_i := \theta\left[\begin{smallmatrix} 0 \\ i/m \end{smallmatrix}\right](\cdot, \Omega/m)\right\}_{i \in Z(m)}.$$

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If $m \geq 3$, it gives us an embedding :

$$\varphi_{m,\Omega} : \left( \begin{array}{ccc} A & \longrightarrow & \mathbb{P}^{Z(m)} \\ z & \longmapsto & (\theta_i(z))_{i \in Z(m)} \end{array} \right).$$

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The point $\varphi_{m,\Omega}(0_A)$ is called the theta null point of $\varphi_{m,\Omega}$.

# Relations

**Theorem (Mumford)**

The level $m$ theta null point $(a_i)_{i\in Z(m)}$ satisfy the Riemann equations of evel $m$:

$$L(x,y)L(u,v) = L(x+z,y-z)L(u-z,v-z),$$

with $L(x,y)$ of the form $\sum_{t\in Z(2)} \chi(t) a_{x+t} a_{y+t}$.

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## Definition

There is an action by translation of $Z(m) \times Z(m)$ on the theta basis :

$$(i,j) \cdot \theta_k = \theta_k(\cdot - i/m - \Omega j/m) = e_{\mathcal{L}_m}(i+k, j)\theta_{i+k},$$

where $e_{\mathcal{L}_m}$ is the commutator paring.

# The isogeny theorem

## Theorem

- Let $\psi : Z(m) \to Z(dm)$ be the canonical embedding. Let $K = (\{0\} \times Z(m)) \cdot 0_A \subset A[m] \subset A[dm]$,

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## Proof.

$$\theta\left[\begin{smallmatrix} 0 \\ i/m \end{smallmatrix}\right](\cdot, (\Omega/m)/d) = \theta\left[\begin{smallmatrix} 0 \\ di/dm \end{smallmatrix}\right](\cdot, \Omega/dm).$$

□

# Change of level algorithms and isogeny computation

**Definition : Changing level**

A change of level algorithm takes the theta null point of level $m$ of $A$, and $K = A[dm]$, and computes the theta null point of level $dm$ of $A$ (going up) or the other way around (going down).

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**Definition : Computing isogeny**

An isogeny computation algorithm takes the theta null point of a marked abelian variety $A$ of level $m$, and $K \subset A[dm]$ a subgroup isomorphic to $Z(d)$, and computes the theta null point of an abelian variety $B$ of level $m$, where $B = A/K$, and the isogeny $f : A \to B$.

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## Can we do better ?

Previous results :

- Duplication formula : going up form level $m$ to level $2m$;
- Koizumi formula : going down from level $dm$ to level $m$;
- Lubicz and Robert, 2022 : change of level algorithms and isogeny computation for $d$ prime to $m$.

# Compatibility and first difference

## Definition

Two theta null points of level $m_1$ and $m_2$, say $\varphi_{m_1,\Omega_1}(0)$ and $\varphi_{m_2,\Omega_2}(0)$, are said to be compatible if there exists $d$ such that $m_1 = dm_2$, and if there exists $\Omega \in \mathcal{H}_g$ such that $\Omega/m_i \simeq \Omega_i \mod \Gamma(m_i, 2m_i)$ for $i = 1, 2$, where $\Gamma(m, 2m)$ is a congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ (Igusa level $m$ subgroups).

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From $A$ an abelian variety of level $m$, and $\varphi_{m,\Omega}(0_A)$ its theta null point :

## Case $d \wedge m = 1$

Any abelian variety of the form $A/K$, where $K \subset A[dm]$ is isomorphic to $Z(d)$, can be equiped with a theta null point compatible with $\varphi_{m,\Omega}(0_A)$.

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## Case $2|d|m$

There is a unique $K_0 \subset A[dm]$, isomorphic to $Z(d)$, such that $A/K_0$ can be equiped with a theta null point compatible with $\varphi_{m,\Omega}(0_A)$ :

$$K_0 = \left(\tfrac{m}{d} Z(d) \times \{0\}\right) \cdot 0_A.$$

# What method for our algorithms?

## Case $d \wedge m = 1$ : Excellent lift

- Compute an affine lift of $K$ (and other groups), consistent with relations on $A$ ;
- Use formulas for theta null point/image by the isogeny.

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Tools :

- Differential addition : $\widetilde{x+y} = \mathtt{DiffAdd}(\tilde{x}, \tilde{y}, \widetilde{x-y})$;
- Action of $Z(m) \times Z(m)$;
- $\mathtt{Inv}$ : $\tilde{x} = (\tilde{x}_i)_{i \in Z(m)} \mapsto \widetilde{-x} = (\tilde{x}_{-i})_{i \in Z(m)}$.

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## Definition

Let $(e_1, \ldots, e_g)$ be a basis of $Z(md)/Z(m)$. We say that $(e_i, e_i + e_j)_{i,j=1,\ldots,g}$ is a chain basis of $Z(d)$.

## Example

For $g = 2$, a chain basis of $Z(d)$ is $((1,0),(0,1),(1,1))$.

# Relations

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In other words, `Inv` acts freely on the set of points we can compute thanks to `DiffAdd` and the action of $Z(m) \times Z(m)$.

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## Case $2|d|m$ : new relations

Let $\phi : Z(dm) \to A[dm]$ be a numbering of $A[dm]$. For $t \in S_{\mathtt{Inv}}$, we have :

$$2\phi(t) = (2dt, 0) \cdot \phi(0),$$

where $dt \in Z(m)$.

# Remedying the obstruction : symmetric compatibility

## Proposition

If there exists $t \in S_{inv}$ such that $\phi(t) \neq (2dt, 0) \cdot \phi(0)$, then $\phi(t) = (2dt, 0) \cdot -\phi(0)$. This property is $Z(m)$-linear in $t$ !

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**Proposition : Changing the theta null point to make it sym. compatible**

For $(e_i)_{i=1,\dots,g}$ a basis of $Z(md)$, if $\phi(e_i) \neq (2de_i, 0) \cdot \phi(0)$, then by replacing $\theta_k$ by $-\theta_k$ for $k \in \langle e_i \rangle$, we get the equality.

**Example**

For $g = 1$, $m = d = 2$ and a theta null point $(a_0 : a_1 : a_2 : a_3)$, either $(a_0 : a_1 : a_2 : a_3)$ or $(a_0 : -a_1 : a_2 : -a_3)$ is symmetric compatible with $K$.

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**Proposition : Changing $K$ to make it symmetric compatible**

For $(e_i)_{i=1,\dots,g}$ a basis of $Z(md)$, if $\phi(e_i) \neq (2de_i, 0) \cdot \phi(0)$, then : $\phi(e_i) + (0, \frac{md}{2} e_i) \cdot \phi(0) = (2de_i, 0) \cdot \phi(0)$.

# Application

**Theorem : Changing level (going up)**

- *Input* : A basis of $K = A[dm]$ and $\varphi_{m,\Omega}(0_A)$ the theta null point of level $m$ of $A$;
- We make $K$ symmetric compatible with $\varphi_{m,\Omega}(0_A)$ (equivalent to a change of numbering or basis);
- We compute an affine lift of $K$ (and other groups), consistent with relations on $A$;
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**Theorem : Computing isogeny**

- *Input* : A basis of $K \subset A[dm]$ a subgroup isomorphic to $Z(d)$ and $\varphi_{m,\Omega}(0_A)$ the theta null point of level $m$ of $A$;
- We make $\varphi_{m,\Omega}(0_A)$ symmetric compatible with it $K$;
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Thank you for your attention !

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