Internship report
A symplectic structure on coadjoint orbits

Antoine Médoc, supervised by professor Anton Alekseev

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Abstract. This report is the result of a three month internship at the end of the first year of a master's degree in mathematics. First, we introduce the definition of a Lie group, its associated Lie algebra and the coadjoint action linking them. Then we prove that the coadjoint orbits of a Lie group admit a canonical symplectic structure. Most of the examples given are about matrix Lie groups. See Subsection 1.1 for an abstract with more details.

Contents

1 Introduction

1.1 Abstract and motivations

About this report. This internship has taken place in the section of mathematics of the University of Geneva for three month at the end of a first year of a master's degree in fundamental mathematics at the École normale supérieure de Rennes. It is about a symplectic structure on the coadjoint orbits of a Lie group. We start from the level of a bachelor's degree in mathematics in addition to a basic course in differential geometry. In particular, we use some results about smooth functions on manifolds, k-forms on manifolds and immersed and embedded submanifolds. We add in the bibliography a course about smooth manifolds ([10]).

About its subject. Why studying coadjoint orbits in this report ? From a mathematical point of view, *Lie groups* are very interesting and useful objects. They combine the structure of a group and of a smooth manifold, hence join algebra, analysis and geometry. Lie groups are very important as they have a useful application in a lot of fields, such that functional analysis, number theory, quantum mechanics or string theory. This internship started with the reading of [2], which present several features of interest about the use of smooth manifolds and *symplectic manifolds* in physics. The notions of Lie groups and *Lie algebra* are also introduced. These two important structures are linked by the *coadjoint action* of a Lie group (on its associated Lie algebra). The corresponding orbits, called the *coadjoint orbits* of the Lie group, have been studied by Alexandre Kirillov in the nineteen-sixties and can also be used in physics. The main goal of this report is to state Theorem 4.26, to prove it and to illustrate it. This theorem assures that any coadjoint orbit admit a canonical invariant *symplectic form* (and in particular is a *symplectic manifold*). We briefly introduce two main examples of coadjoint orbits : in the special unitary group of degree 2 SU_2 (which is often used in quantum mechanics to describe spins or angular momentums) and the real special linear group of degree 2 $\mathrm{SL}_2(\mathbb{R})$.

About its content. First, we give the useful notations that we use in the rest of the report in 1.2. Secondly, we see a very short introduction to symplectic geometry. We define symplectic vector spaces in 2.1 and symplectic manifolds in 2.2 ([14]). Thirdly, we explain what is a Lie group and its associated Lie algebra. We define Lie groups and Lie algebras in 3.1 ([2, 8]). In 3.2 we define the Lie algebra $\mathfrak{X}(M)$ of vector fields on a manifold M using the correspondance between derivations on the smooth functions algebra $\mathcal{C}^\infty(M, \mathbb{R})$ and vector fieds $\mathfrak{X}(M)$ ([4]). In 3.3 we define the Lie algebra of a Lie group, seen as the set of its left-invariant vector fields or as the tangent space at the neutral element ([11]). In 3.4 we define the exponential map from the Lie algebra $\mathfrak{g}$ of a Lie group G in this Lie group ([12]), and with this new tool we give a new interpretation of the Lie bracket on $\mathfrak{g}$ with the commutativity of the flow ([2]) and we give the Cartan magic formula ([14]). Fourthly, we define the coadjoint orbits of a Lie group and state and talk about their geometry. In 4.1 we define a Lie group action and see the classical diffeomorphism between an orbit and the quotient of the Lie group by a stabilizer of an element of the orbit ([3, 8, 10, 16]). In 4.2 we define the adjoint and coadjoint actions associated to a Lie group ([11, 12, 14, 16]) and see that they carry a canonical symplectic structure (this is Theorem 4.26, the main result of this report) ([1, 6, 7, 9]). In 4.3 we give two classical examples of Lie groups with their coadjoint orbits represented in $\mathbb{R}^3$ to illustrate the previous subsection ([1, 5, 13, 15, 17]). Fifthly, we mention some news and applications about this subject and give the bibliography. Sixthly and finally, we give in the appendix some figures to illustrate this report.

About proofs. We do not give any proof of a result that is not about symplectic structures, Lie groups or Lie algebras. We admit Cartan's theorem 3.3 about Lie subgroups, Theorem 4.3 about quotient manifolds, and Theorem 4.4 about passing a smooth function to the quotient. We only prove Proposition 3.29 in the particular case of a Lie subgroup of the general linear group of degree $n \in \mathbb{N}^*$ over $\mathbb{K}$ and we admit it in the general case.

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part of a completely new field for me. I also would like to thank research associate Élise Raphael, events officer Patricia Parraga, administrator Isabelle Bretton, administrator Joselle Besson and all the members of the section of mathematics of the University of Geneva who helped me and made my stay in Geneva very easy.

1.2 Settings and notation

We give in this section some useful notations and reminders to read this report. In addition to them, we will use some classical results of differential geometry without proof : most of them can be found in [10].

◇ Notations. For all this report, let $n \in \mathbb{N}^*$ and $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$.

◇ Notations. About linear algebra.

- The *conjugate transpose* of a matrix $M \in \mathrm{M}_n(\mathbb{C})$ is M^*, the transpose of a matrix $M \in \mathrm{M}_n$ (resp. of a vector $v \in \mathbb{K}^n$) is M^T (resp. v^T). We will often use the identification between $\mathrm{M}_n(\mathbb{K})$ and $\mathbb{K}^{n^2}$.
- The set of *Hermitian* matrices is $\mathcal{H}_n$ and the set of *symmetric* matrices is $\mathcal{S}_n$. The *unitary group* is $\mathrm{U}_n := \left\{U \in \mathrm{M}_n(\mathbb{C}) \;\middle|\; U^*U = UU^* = \mathrm{I}_n\right\}$ and the *special unitary* group is $\mathrm{SU}_n := \left\{U \in \mathrm{U}_n \;\middle|\; \det U = 1\right\}$. The special linear group is $\mathrm{SL}_n(\mathbb{K}) = \det^{-1}(\{1\})$.
- Given E a $\mathbb{K}$-vector space, we denote by $E^* := \mathcal{L}_{\mathbb{K}}(E, \mathbb{K})$ its dual space.

◇ Remarks. An open set of $\mathbb{R}^n$ is a submanifold of $\mathbb{R}^n$. We can notice that $M_n(\mathbb{K})$ is isomorphic to $\mathbb{R}^{n^2}$ and $\mathrm{M}_n(\mathbb{C})$ to $\mathbb{R}^{(2n)^2}$, hence is a manifold. As an open set of M_n, GL_n is then a manifold.

With $F : U \in \mathrm{M}_n(\mathbb{C}) \mapsto U^*U - \mathrm{I}_n \in \mathcal{H}_n$, the set $\mathrm{U}(n)$ is given by the equation $F = 0$ and for all $U \in \mathrm{U}_n$ d_UF is surjective, hence U_n is a submanifold of $\mathrm{M}_n(\mathbb{C})$. The set SU_n is given by the equation $\det|_{\mathrm{U}(n)} = 1$ and for all $U \in \mathrm{SU}_n$ $d\det_U$ is surjective, hence SU_n is a submanifold of U_n. Likewise, $\mathrm{SL}_n(\mathbb{R})$ is a submanifold of $\mathrm{M}_n(\mathbb{R})$.

◇ Notations. About manifolds.

- A manifold is a differential real manifold of unique dimension. A n-manifold is a manifold such that $\dim M = n$. A smooth function between two $\mathbb{K}$-manifold M and N is an element of $\mathcal{C}^\infty(M, N)$.
- Let M be a manifold. For every $x \in M$, the tangent space to M at x is denoted by T_xM. The tangent bundle of M is denoted by TM and the tangent cobundle of M is denoted T^*M.
- Let $(U, x_1, \ldots, x_n)$ a chart of a manifold M and $i \in [\![1, n]\!]$. For all $x \in U$, we denote the *unitary vector* at x associated to the coordinate x_i by $\frac{\partial}{\partial x_i}\big|_x \in T_xM$. We also denote the function $\frac{\partial}{\partial x_i} : x \in U \mapsto \frac{\partial}{\partial x_i}\big|_x \in TM$. Given a manifold N, this is the same notation as the operator partial derivation with respect to the coordinate x_i on the vector space $\mathcal{C}^\infty(M, N)$.

◇ Reminder. For $(U, x_1, \ldots, x_n)$ a chart of a manifold M and $x \in U$, $\left(\frac{\partial}{\partial x_1}\big|_x, \ldots, \frac{\partial}{\partial x_n}\big|_x\right)$ is a basis of T_xM.

◇ Notations. About smooth function on manifolds. Let M and N be manifolds on $\mathbb{K}$.

- For $f : M \to N$ a smooth function, we denote its *derivative map* by df and for all $x \in M$ $df_x := df(x) : T_xM \to T_{f(x)}N$. For J an open subset of $\mathbb{R}$ and $\gamma : J \to N$ a smooth function, for all $t \in J$ we denote $\gamma'(t) := \frac{d}{ds}(\gamma(s))\big|_{s=t} := d\gamma_t(1)$.
- The set of *diffeomorphisms* of M (i.e. the set of every bijective smooth functions $M \to M$ such that its inverse is smooth) is denoted by $\mathrm{Diff}(M)$.

- A *vector field* on M is a smooth function $X : M \to TM$ such that, for all $x \in M$, $X(x) \in T_xM$. The vector space of all vector fields on M is denoted by $\mathfrak{X}(M)$. Given $X \in \mathfrak{X}(M)$ and a chart $(U, x_1, \ldots, x_n)$ on M, saying that $(u_1, \ldots, u_n)$ is the component of X in the coordinates $(x_1, \ldots, x_n)$ means that $(u_1, \ldots, u_n) \in \mathcal{C}^\infty(U, \mathbb{K})^n$ and $X|_U = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$.

- Suppose there is an embedding $i : M \to N$. We consider i as an inclusion : for all $x \in N$, $x = i(x)$ and $T_xN = di_x(T_xN) \subset T_xM$.

We remind here a useful proposition about submanifolds :

◇ Notations. About k-forms. Let $k \in \mathbb{N}$ and M be a manifold.

- The set of k-forms on M is $\Omega^k(M)$. The set of forms on M is $\Omega(M) := \bigoplus_{l \in \mathbb{N}} \Omega^l(M)$. For $\omega \in \Omega(M)$ and $x \in M$, we denote $\omega(x)$ by ω_x.

- For $\omega \in \Omega(M)$, its exterior derivative is denoted by $d\omega$. The exterior product on $\Omega(M)$ is denoted by $\wedge$. Let $(U, x_1, \ldots, x_n)$ be a chart of M and $I := (i_1, \ldots, i_k) \in [\![1, n]\!]^k$ such that $i_1 < \ldots < i_n$. We denote by dx^I the k-form $dx_{i_1} \wedge \ldots \wedge dx_{i_k}$.

- For $t \in \mathbb{R}$, $J \subset \mathbb{R}$ a t-neighbourhood in $\mathbb{R}$ and $(\omega^t)_{t \in J} \in \Omega^k(M)^J$ a differential form family such that for all $x \in M$ and $v_1, \ldots, v_k \in T_xM$ the map $s \in \mathbb{R} \mapsto \omega^s_x(v_1, \ldots, v_k) \in \mathbb{R}$ is smooth, we denote by $\frac{d}{ds}(\omega^s)\big|_{s=t}$ the k-form on M such that, for all $x \in M$ and $v_1, \ldots, v_k \in T_xM$, $\frac{d}{ds}(\omega^s)\big|_{s=t}{}_x(v_1, \ldots, v_k) = \frac{d}{ds}(\omega^s_x(v_1, \ldots, v_k))\big|_{s=t}$.

- For $f \in \mathcal{C}^\infty(M, N)$ and $\omega \in \Omega(M)$, we denote by $f^*\omega \in \Omega(M)$ the pullback of ω by f.

- Let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^k(M)$. We denote by $i_X(\omega)$ the interior product of X and ω.

◇ Remark. Let $k \in \mathbb{N}$, M be a manifold, $\omega \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$. If $k \geqslant 1$, $i_X\omega$ is the $(k-1)$-form on M defined by : for all $x \in M$ and $v_1, \ldots, v_{k-1} \in T_xM$, $i_X(\omega)_x(v_1, \ldots, v_{k-1}) = \omega_x(X(x), v_1, \ldots, v_{k-1})$. If $k = 0$, $\omega \in \mathcal{C}^\infty(M, \mathbb{R})$ and $i_X\omega = 0$.

◇ Reminder. Let M be a manifold and $X \in \mathfrak{X}(M)$ be a complete vector field. The interior product by X is linear on $\Omega(M)$. In addition to that, for all $k \in \mathbb{N}$, $\omega_1 \in \Omega^k(M)$ and $\omega_2 \in \Omega(M)$, we have $i_X(\omega_1 \wedge \omega_2) = (i_X\omega_1) \wedge \omega_2 + (-1)^k \omega_1 \wedge (i_X\omega_2)$.

2 Symplectic geometry

We give here some basic definitions about symplectic geometry, based on [14].

2.1 Symplectic vector spaces

DEFINITIONS 2.1.
- Let V be a finite-dimensional $\mathbb{K}$-vector space. A *sympectic form* (or *non-degenerate form*) on V is a skew-symmetric bilinear form $\Omega : V \times V \to \mathbb{K}$ such that

$$\tilde{\Omega} : \left| \begin{array}{ccc} V & \longrightarrow & V^* \\ v & \longmapsto & u \in V \mapsto \Omega(v,u) \in \mathbb{K} \end{array} \right.$$

  is bijective.

- A *sympectic vector space* is a finite-dimensional $\mathbb{K}$-vector space V associated to a symplectic form Ω on V, denoted (V, Ω).

▷ EXAMPLE. For all $p \in \mathbb{N}$, the map

$$\begin{array}{ccc} \mathbb{R}^{2n+p} \times \mathbb{R}^{2n+p} & \longrightarrow & \mathbb{R} \\ (u,v) & \longmapsto & u^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{I}_n \\ 0 & -\mathrm{I}_n & 0 \end{pmatrix} v \end{array}$$

is symplectic if, and only if, $p = 0$.

PROPOSITION 2.2. A symplectic vector space has an even dimension.

Proof We prove this statement by induction : for all $n \in \mathbb{N}$, a symplectic vector space V such that $\dim V \leqslant n$ has an even dimension. The basic case, $n = 0$, is clear.

Let (V, Ω) be a sympectic vector space such that $\dim V \geqslant 1$. Let $e_1 \in V \setminus \{0\}$. Since Ω is symplectic, there is $f_1 \in V$ such that $\Omega(e_1, f_1) \neq 0$. Let $V_1 = \{e_1, f_1\}$ and $V_1^\Omega := \left\{v \in V \,\middle|\, \forall v_1 \in V_1, \Omega(v, v_1) = 0\right\}$. We want to show that $V = V_1 \oplus V_1^\Omega$ and that V_1^Ω is symplectic.

For all $(a, b) \in \mathbb{K}^2$ such that $ae_1 + bf_1 \in V_1^\Omega$, $0 = \Omega(ae_1 + bf_1, e_1) = -b$ and $0 = \Omega(ae_1 + bf_1, f_1) = a$, hence $a = b = 0$. We deduce than $V_1 \cap V_1^\Omega = \{0\}$. In addition to that, for all $v \in V$, with $a := \Omega(v, e_1)$ and $b := \Omega(v, f_1)$, we have $v = (-af_1 + be_1) + (v + af_1 - be_1)$, $-af_1 + be_1 \in V_1$ and $v + af_1 - be_1 \in V_1^\Omega$. Hence $V = V_1 + V_1^\Omega$ and $V = V_1 \oplus V_1^\Omega$. Secondly, we want to show that V_1^Ω is symplectic. Let $\tilde{v}_1 \in V_1^\Omega$ such that for all $\tilde{w}_1 \in V_1^\Omega$ $\Omega(\tilde{v}_1, \tilde{w}_1) = 0$. By definition of V_1^Ω, for all $w \in V$, $\Omega(\tilde{v}_1, w) = 0$, hence $\tilde{v}_1 = 0$. Thus $\left(V_1^\Omega, \Omega|_{V_1^\Omega \times V_1^\Omega}\right)$ is symplectic.

To sum up, $\dim V = \dim V_1 + \dim V_1^\Omega$. We have $\dim V_1 = 2$ and, by induction hypothesis, $\dim V_1^\Omega$ is even. This concludes. □

2.2 Symplectic manifolds

DEFINITIONS 2.3. Let M be a manifold. A *symplectic form* on M is a closed 2-form ω on M such that, for all $x \in M$, ω_x is symplectic. A symplectic manifold is a manifold M associated to a symplectic form ω on M, denoted (M, ω).

◇ REMARK. Using Proposition 2.2, a symplectic manifold has an even dimension.

▷ EXAMPLE. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the canonical coordinates on $\mathbb{R}^{2n}$. We can check that $dx \wedge dy = \sum_{i=1}^n dx_i \wedge dy_i$ is a symplectic form on this manifold.

◇ REMARK. If you want to know more about the example above, see the Darboux theorem to see that every symplectic form looks like this one : see Subsection 8.1 of [14].

3 Lie Groups and Lie algebras

In this section we present two main new structures : Lie groups and Lie algebra. We see in Subsection 3.3 how these two can be linked with the Lie algebra of a Lie group.

3.1 Definition and examples

This subsection is mainly based on [2, 8].

3.1.1 Basic definitions about Lie groups

A Lie group combines two important structures (as a group and as a manifold) linked by the fact that the group operations are smooth. This subsection is mainly based on [14, 2].

Definition 3.1. We call *Lie group* a manifold G with a group law such that

$$\begin{array}{ccc} G^2 & \longrightarrow & G \\ (g_1, g_2) & \longmapsto & g_1 g_2 \end{array} \quad \text{and} \quad \begin{array}{ccc} G & \longrightarrow & G \\ g & \longmapsto & g^{-1} \end{array}$$

are smooth.

▷ Examples. (classical Lie groups)

- The Euclidean space $\mathbb{R}^n$ with the law $+$ and the usual manifold structure is a Lie group : $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto x + y \in \mathbb{R}^n$ and $x \in \mathbb{R}^n \mapsto -x \in \mathbb{R}^n$ are polynomial, hence smooth.
- The circle $\mathbb{S}^1 \subset \mathbb{C}$ with $\times$: $(\theta_1, \theta_2) \in \mathbb{R}^2 \mapsto \theta_1 + \theta_2 \in \mathbb{R}$ and $\theta \in \mathbb{R} \mapsto -\theta \in \mathbb{R}$ are smooth. We notice that $\mathbb{S}^1 \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$.
- The sets of matrices $\mathrm{GL}_n(\mathbb{K})$, SU_n and $\mathrm{SL}_n(\mathbb{R})$ with $\times$: the group operations are polynomial, hence smooth. Using the continuity of the determinant, we notice that $\mathrm{GL}_n(\mathbb{R})$ is a non-connected Lie group.
- Let G be a Lie group. The cartesian product manifold $G \times G$ together with the cartesian product group $G \times G$ laws is a Lie group.

We can adapt the definition of a group morphism and a group action to this new notion :

Definitions 3.2.

- A morphism of Lie groups from G_1 to G_2 is a smooth group morphism from G_1 to G_2.
- Given G a Lie group, a *Lie subgroup* of G is a subgroup $H < G$ with a Lie group structure such that $h \in H \mapsto g \in G$ is an injective immersion.

◇ Remark. We have define here a Lie subgroup as immersed manifold : in fact, in this report, we only consider closed Lie subgroup i.e., by Cartan's theorem below, embedded Lie subgroups.

Theorem 3.3. (Cartan's theorem)
Let $H < G$ a closed subgroup of G for G-topology. There is a unique manifold structure on H such that H is an embedded Lie subgroup of G.

Proof See Theorem 15.29 page 392 in [10]. □

◇ Notation. Given G a Lie group and H a closed subgroup of G, we consider that H is equiped with the embedded Lie group structure of Theorem 3.3.

In Subsection 4.1 we will see the definition of a Lie group action (and some of its properties). It is another important notion coming from classical groups we use later in this report.

3.1.2 Basic definitions about Lie algebras

DEFINITIONS 3.4.
- Given a vector space L and a skew-symmetric map $[\cdot,\cdot] : L \times L \to \mathrm{L}$, $[\cdot,\cdot]$ satisfies the *Jacobi identity* if for all $(A, B, C) \in L^3$
$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0.$$
- A *Lie algebra* is a vector space L associated to a bilinear skew-symmetric map which satisfies the Jacobi identity. Such a map is called a *Lie bracket.*
- A *Lie subalgebra* of a Lie algebra $(L, [\cdot,\cdot]_L$ is a Lie algebra $(M, [\cdot,\cdot]_M)$ such that $M \subset L$ is a linear subspace of L and for all $(A, B) \in M^2$, $[A, B]_M = [A, B]_L$.

▷ EXAMPLES. (of Lie algebras)

- Let L a vector space. The map $(u, v) \in L^2 \mapsto 0 \in L$ is a Lie bracket. Any linear subspace of L with the null Lie bracket is a Lie subalgebra of L.
- Let $\wedge$ be the vector multiplication in $\mathbb{R}^3$. It is bilinear and skew-symmetric. In addition to that, we know that for all $u, v, w \in \mathbb{R}^3$ we have $(u \wedge v) \wedge w = (u \cdot w)v - (v \cdot w)u$. Hence, $\wedge$ satisfies the Jacobi identity and $(\mathbb{R}^3, \wedge)$ is a Lie algebra.
- Let $(A, +, \times, \cdot)$ an associative $\mathbb{K}$-algebra. For all $(a, b) \in A^2$ we denote $[a, b] = ab - ba$. This bilinear operation gives A a Lie algebra structure. If $A = \mathrm{M}_n(\mathbb{K})$, $[\cdot,\cdot]$ is the commutator (used in quantum mechanics, for example). If A is commutative, this Lie bracket is null.
- With the commutator on $\mathrm{M}_n(\mathbb{K})$ and on $\mathfrak{su}_n := \left\{A \in \mathrm{M}_n(\mathbb{K}) \mid A + A^* = 0, \operatorname{tr} A = 0\right\}$, $\mathfrak{su}_n$ is a Lie subalgebra of $\mathrm{M}_n(\mathbb{K})$.

DEFINITION 3.5. Let $(L_1, [\cdot,\cdot]_1)$ and $(L_2, [\cdot,\cdot]_2)$ be Lie algebra on $\mathbb{K}$. A Lie algebra morphism from L_1 to L_2 is a linear map $\varphi \in \mathcal{L}(L_1, L_2)$ such that for all $A_1, B_1 \in L_1$
$$\varphi([A_1, B_1]_1) = [\varphi(A_1), \varphi(B_1)]_2 .$$

▷ EXAMPLE. Let $(L, [\cdot,\cdot])$ be a Lie algebra. Thanks to the Jacobi identity, the map $\varphi : A \in L \mapsto [A, \cdot] \in \mathcal{L}(L)$ is a Lie algebra morphism. We see this example in Proposition 4.15 with the map ad.

3.2 The Lie algebra of vector fields on a manifold

Let M be a manifold of dimension n. In this subsubsection we present a Lie bracket on the vector space made of all the vector fields on M, using a correspondance between vector fields and derivations of smooth functions. This will allow us to construct the Lie algebra associated to a Lie group. This subsection is mainly based on [4].

DEFINITIONS 3.6.
- Let A be a $\mathbb{K}$-algebra. A *derivation* on A is $D \in \mathcal{L}_{\mathbb{K}}(A)$ such that, for all $(a, b) \in A^2$, $D(ab) = D(a)b + aD(b)$. The set of all derivations on A is denoted by $\mathrm{Der}(A)$.
- Given $X \in \mathfrak{X}(M)$, the *directional derivative* in the direction X is
$$\begin{array}{ccc} \mathcal{C}^\infty(M, \mathbb{K}) & \longrightarrow & \mathcal{C}^\infty(M, \mathbb{K}) \\ f & \longmapsto & x \in M \mapsto df_x(X(x)) \in \mathbb{K} \end{array}.$$
The directional derivative of f in the direction X is denoted $X(f)$ or $df \circ X$. The directionnal derivative in the direction X is also denoted X.

◇ NOTATION. Let $X \in \mathfrak{X}(M)$. Given another manifold N and $f \in \mathcal{C}^\infty(M, N)$, we also denote $x \in M \mapsto df_x(X(x)) \in TN$ by $df \circ X$.

▷ EXAMPLES. (of derivations and directional derivative)

- The directional derivative in the direction of a vector field $X \in \mathfrak{X}(M)$ is a derivation on the $\mathbb{K}$-algebra $\mathcal{C}^\infty(M, \mathbb{K})$: the Proposition 3.7 allows us to prove that for all $f, g \in \mathcal{C}^\infty(M, \mathbb{K}), X(fg) = X(f)g + fX(g)$.
- Let $(E, ||\cdot||_E)$ be a normed $\mathbb{R}$-vector space and $(F, ||\cdot||_F)$ be a normed $\mathbb{R}$-algebra. Let $h \in E$ and U an open set of E. We have, for all $f_1, f_2 \in \mathcal{C}^\infty(U, F)$ and $x \in \Omega$, $d(f_1 f_2)(x)(h) = df_1(x)(h)f_2(x) + f_1(x)df_2(x)(h)$. Hence the differential in the direction h is a derivation on the $\mathbb{R}$-algebra $\mathcal{C}^\infty(U, F)$: $f \in \mathcal{C}^\infty(U, F) \mapsto df(\cdot)(h) \in \mathcal{C}^\infty(U, F)$. We notice that if $E = \mathbb{R}^n$ (and then is a connected manifold) and $F = \mathbb{R}$ this is the derivative in the direction of the constant vector field equal to h.

We often want to compute directional derivatives using coordinates $(x_1, \dots, x_n)$ on M. For every $X \in \mathfrak{X}(M)$ there are $u_1, \dots, u_n \in \mathcal{C}^\infty(M, \mathbb{R})$ such that $X = \sum_{i=1}^n u_i \frac{\partial}{\partial x_i}$. Using this result with the following proposition allow us to compute directional derivatives.

Proposition 3.7. Let $X \in \mathfrak{X}(M)$, $(U, x_1, \dots, x_n)$ be a chart of M, $u_1, \dots, u_n \in \mathcal{C}^\infty(U, \mathbb{K})$, $f \in \mathcal{C}^\infty(M, \mathbb{K})$ a smooth function. Let $(u_i)_i$ be the component of X in $(x_i)_i$. We have, on U,

$$X(f) = \sum_{i=1}^n u_i \frac{\partial f}{\partial x_i}.$$

Proof For all $x \in U$, $X(f)(x) = df_x \left(\sum_{i=1}^n u_i(x) \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^n u_i(x) df_x \left(\frac{\partial}{\partial x_i} \right) = \sum_{i=1}^n u_i(x) \frac{\partial f}{\partial x_i}(x)$. □

In the following Theorem 3.9, we see a very interesting and useful correspondance between the vector fields on M and the derivation operators on smooth functions. In order to prove it, we need the following lemma :

Lemma 3.8.

1. Let A be a $\mathbb{K}$-algebra. The set $\mathrm{Der}(A)$ is a $\mathbb{K}$-linear subspace of $\mathcal{L}(A)$.
2. For all $D \in \mathrm{Der}(\mathcal{C}^\infty(M))$ and $\lambda : M \to \mathbb{R}$ a constant smooth function, $D(\lambda) = 0$.
3. Let $D \in \mathrm{Der}(\mathcal{C}^\infty(M))$ and $U \subset M$ an open subset of M. Let $g_1, g_2 \in \mathcal{C}^\infty(M)$ such that $g_1|_U = g_2|_U$. We have $D(g_1)|_U = D(g_2)|_U$.
4. Let $U \subset M$ and open set of M, $f \in \mathcal{C}^\infty(U)$ and $x \in M$. There is an open neighboorhood $V \subset U$ of x and $\tilde{f} \in \mathcal{C}^\infty(M)$ such that $f|_V = \tilde{f}|_V$.

Proof

1. We have $0 \in \mathrm{Der}(A)$, so $\mathrm{Der}(A) \neq \emptyset$.
2. Let $\mathbf{1} : x \in M \mapsto 0 \in \mathbb{K}$. We have $D(\mathbf{1}) = D(\mathbf{1} \times \mathbf{1}) = D(\mathbf{1}) \times \mathbf{1} + \mathbf{1} \times D(\mathbf{1}) = 2 \cdot D(\mathbf{1})$ so $D(\mathbf{1}) = 0$. By linearity of D, it is null on the set of constant functions $M \to \mathbb{K}$.
3. Let $x \in U$. There is $\chi \in \mathcal{C}^\infty(M)$ and an open neighboorhood $V \subset U$ of x such that $\operatorname{supp} \chi \subset U$ and $\chi(V) = \{1\}$. We have $g_1 - g_2 = (g_1 - g_2) \times (1 - \chi)$, hence $Dg_1 - Dg_2 = D(g_1 - g_2)(1 - \psi) + (g_1 - g_2)(0 - D\psi)$, hense $(Dg_1 - Dg_2)(x) = D(g_1 - g_2) \times 0 - 0 \times (D\psi)(x)$, i.e. $(Dg_1)(x) = (Dg_2)(x)$.
4. Since U is open, there is $V \subset U$ an open neighboorhood of x and $\chi \in \mathcal{C}^\infty(M)$ such that $\operatorname{supp} \chi \subset U$ and $\chi(V) = \{1\}$. Let $\tilde{f} : x \in M \mapsto \begin{cases} \chi(x) f(x) & \text{if } x \in U \\ 0 & \text{if } x \in M \backslash U \end{cases} \in \mathbb{R}$.

□

Theorem 3.9. The following map is an isomorphism of $\mathbb{R}$-vector spaces

$$\begin{array}{ccc} \mathfrak{X}(M) & \longrightarrow & \mathrm{Der}(\mathcal{C}^\infty(M, \mathbb{R})) \\ X & \longmapsto & f \in \mathcal{C}^\infty(M) \mapsto df \circ X \in \mathcal{C}^\infty(M) \end{array}.$$

Proof We use the method seen in [4]. We denote by ψ this map. By Proposition 3.7, ψ is well defined. The derivation is linear so ψ is linear.

Let $X \neq 0$: there is $x \in M$ such that $X(x) \neq 0$. Let $(U, x_1, \ldots, x_n)$ a chart of M such that $x \in U$, and $u_1, \ldots, u_n \in \mathbb{R}$ the coordinates of $X(x)$ in $\left(\frac{\partial}{\partial x_i}|_x\right)_i$. There is $i \in [\![1, n]\!]$ such that $u_i \neq 0$. There is $V_i \subset U_i$ a compact set of M and $\chi_i : M \to \mathbb{K}$ such that $\chi_i|_{V_i} = x_i|_{V_i}$ and $\chi|_{M \setminus U_i} = 0$. We have $\chi_i \in \mathcal{C}^\infty(M)$ and $X(\chi_i)(x) = dx_i(X(x)) = u_i \neq 0$, thus $X(\chi) \neq 0$. We deduce that $\operatorname{Ker}\psi = \{0\}$ and ψ is injective.

For surjectivity, we start to prove it on an convex open set $M \subset \mathbb{R}^n$. Let $(x_1, \ldots, x_n)$ the canonical global coordinates on M. Let D a derivation on $\operatorname{Der}(\mathbb{C}^\infty(M))$. For all $i \in [\![1, n]\!]$, let $d_i := D(x_i)$. Let $X := \sum_{i=1}^n d_i \frac{\partial}{\partial x_i}$. We have $X \in \mathfrak{X}(M)$ and we want to show that $\psi(X) = D$. Let $f \in \mathcal{C}^\infty$. Let $x \in M$. By Taylor's theorem with the remainder under integral form at order 1, for all $y \in M$,

$$
\begin{aligned}
f(y) &= f(x) + \int_0^1 df_{x+t(y-x)}(y-x)dt \\
&= f(x) + \sum_{i=1}^n x_i(y-x) df_{x+t(y-x)}\left(\frac{\partial}{\partial x_i}|_{y-x}\right) \\
&= f(x) + \sum_{i=1}^n (x_i(y) - x_i(x)) \frac{\partial f}{\partial x_i}(x + t(y-x)).
\end{aligned}
$$

For all $i \in [\![1, n]\!]$ we denote $f_i : y \in M \mapsto \frac{\partial f}{\partial x_i}(x + t(y-x)) \in \mathbb{K}$. Hence for all $y \in M$, $(Df)(y) = 0 + \sum_{i=1}^n (D(x_i)(y) - 0) f_i(y) + (x_i(y) - x_i(x)) D(f_i)(y)$. All the functions in this equality are continuous so the limit as $y \to x$ gives us $(Df)(x) = \sum_{i=1}^n D(x_i)(x) f_i(x) + 0 = \sum_{i=1}^n u_i(x) \frac{\partial f}{\partial x_i}(x) = \psi(X)(f)(x)$. We deduce that $\psi(X) = D$.

Finally, we verify the surjectivity in the general case. Let $(U_i, \varphi_i)_{i \in I}$ an atlas on M such that for all $i \in I$, U_i is convex. By the theorem of the partition of unity, there is $(\chi_i)_i \in \mathcal{C}^\infty(M, \mathbb{R}^+)$ such that

$$
\begin{cases}
\forall i \in I, \operatorname{supp}\chi_i \subset U_i \\
\forall x \in M, \exists V_x \subset M \text{ open neighboorhood of } x, \operatorname{Card}\left\{i \in I \mid U_i \cap V_x \neq \emptyset\right\} < +\infty \\
\sum_{i \in I} \chi_i = 1
\end{cases}.
$$

Let $i \in I$. Let $x \in U_i$. By point 4 of the lemma, there is $V_x \subset U_i$ an open neighboorhood of x and $g_x \in \mathcal{C}^\infty(M)$ such that $f \circ \varphi_i|_V = g_x|_V$. Let $D_i(f)(x) := \left((\chi_i \times D(g_x)) \circ \varphi_i^{-1}\right)(x)$. By point 3, $D_i(f)(x)$ does not depend on the choice of g_x, only on (f, i, x). This allows us to define $D_i(f) : V_i \to \mathbb{R}$. By composition, $D_i(f)$ is smooth on an open neighboorhood of every $x \in V_i$, so $D_i(f)$ is smooth. Let $D_i : f \in \mathcal{C}^\infty(V_i) \mapsto D_i(f) \in \mathcal{C}^\infty(V_i)$. We want to show that D_i is a derivation. Let $(f_1, f_2) \in \mathcal{C}^\infty(V_i)^2$ and $x \in M$. For $j \in \{1, 2\}$, there is $V_x^{(j)} \subset U_i$ and open neighboorhood of x and $g_x^{(j)} \in \mathcal{C}^\infty(M)$ such that $g_x^{(j)}|_{V_x^{(j)}} = f_j|_{V_x^{(j)}}$. Let $V_x := V_x^{(1)} \cap V_x^{(2)}$. We have $g_x^{(1)} g_x^{(2)}|_V = f_1 f_2|_V$ so

$$
\begin{aligned}
D_i(f_1 f_2)(x) &= \left(\left(\chi_i \times D\left(g_x^{(1)} g_x^{(2)}\right)\right) \circ \varphi_i^{-1}\right)(x) \\
&= \left(\left(\chi_i \times D\left(g_x^{(1)}\right) g_x^{(2)}\right) \circ \varphi_i^{-1}\right)(x) + \left(\left(\chi_i \times g_x^{(1)} D\left(g_x^{(2)}\right)\right) \circ \varphi_i^{-1}\right)(x) \\
&= (D_i f_1)(x) f_2(x) + f_2(x)(D_i f_2)(x)
\end{aligned}
$$

We deduce that $D_i \in \operatorname{Der}(\mathcal{C}^\infty(V_i))$. We already proved the surjectivity in the case of a convex open subset of $\mathbb{R}^n$, so there is $X_i \in \mathfrak{X}(V_i)$ such that for all $f \in \mathcal{C}^\infty(V_i)$, $D_i(f) = X_i(f)$. Let $\tilde{X}_i = \left(d\varphi_i^{-1} \circ X_i\right) \circ \varphi_i \in \mathfrak{X}(U_i)$. We want to extend $\tilde{X}_i$ on M in order to add it to the other similar vector fields on the maps of the atlas. Let $W_i := U_i \setminus \operatorname{supp}\chi_i$. For all $f \in \mathcal{C}^\infty(W_i)$, $x \in W_i$ and $\tilde{f} \in \mathcal{C}^\infty(M)$ coinciding with f on a neighboorhood of x, $X_i|_{W_i}(f)(x) = X_i\left(\tilde{f}|_{U_i}\right)(x) = D_i\left(\tilde{f}|_{U_i}\right)(x) = 0 \times D\left(\tilde{f}\right)\left(\varphi^{-1}(x)\right) = 0$. Using the beginning of the proof about injectivity, we then deduce that for all $x \in W_i$, $X_i(x) = 0$. For all $x \in V_i \setminus \varphi_i(\operatorname{supp}\chi_i)$, $\tilde{X}_i(x) = \left(d\varphi_i^{-1}\right)_{\varphi_i(x)} X_i(\varphi_i(x)) = \left(d\varphi_i^{-1}\right)_{\varphi_i(x)}(0) = 0$. Hence we can extend $\tilde{X}_i$ by zero on $M \setminus U_i$, and we still denote it by $\tilde{X}_i$, which is now an element of $\mathfrak{X}(M)$. Since for all $x \in M$

there is a neighboor of x intersecting a finite number of maps of the atlas, we can define $X := \sum_{i\in I} \tilde{X}_i$. We now want to show that $\psi(X) = D$ to conclude. For all $f \in \mathcal{C}^\infty(M)$ and $x \in M$, with $I_x := \{i \in I \mid x \in U_i\}$, $X(f)(x) = \sum_{i\in I} \tilde{X}_i(f)(x) = \sum_{i\in I_x} df_x \left(d\left(\varphi_i^{-1}\right)_{\varphi_i(x)}(X_i(\varphi_i(x)))\right) = \sum_{i\in I_x} d\left(f\circ\varphi_i^{-1}\right)_{\varphi_i(x)}(X_i(\varphi_i(x))) = \sum_{i\in I_x} X_i\left(f\circ\varphi_i^{-1}\right)(\varphi_i(x)) = \sum_{i\in I_x} D_i(f\circ\varphi_i^{-1})(\varphi_i(x)) = \sum_{i\in I_x}(\chi_i D(f))(x) = D(f)(x)$. Hence $D = \psi(X)$. This concludes.

□

◇ Remark. Let $X, Y \in \mathfrak{X}(M)$. Using this theorem, to show that $X = Y$ it is enough to show that for all $f \in \mathcal{C}^\infty(M,\mathbb{K})$ $X(f) = Y(f)$.

We now see a lemma that will allow us to define the Lie bracket on $\mathfrak{X}(M)$, which look like the classical Lie bracket on $\mathrm{M}_n(\mathbb{K})$.

Lemma 3.10. For all $X, Y \in \mathfrak{X}(M)$, $f \in \mathcal{C}^\infty(M) \mapsto Y(X(f)) - X(Y(f)) \in \mathcal{C}^\infty(M)$ is a derivation.

Proof Let $(X,Y) \in \mathfrak{X}(M)^2$ and $(U, x_1, \ldots, x_n)$ a chart on M. Let $(u_i)_i, (v_i)_i$ be the component of X, Y in this coordinate system. Let $f \in \mathcal{C}^\infty(M,\mathbb{K})$. At first sight $Y(X(f)) - X(Y(f))$ includes second order partial derivative of f : let's use Schwarz's theorem to see how these disappear.

On U we have $X(Y(f)) = X\left(\sum_{j=1}^n v_j \frac{\partial f}{\partial x_j}\right) = \sum_{i,j=1}^n u_i \frac{\partial v_j}{\partial x_i}\frac{\partial f}{\partial x_j} + u_i v_j \frac{\partial^2 f}{\partial x_i \partial x_j}$ and likewise $Y(X(f)) = \sum_{i,j=1}^n v_i \frac{\partial u_j}{\partial x_i}\frac{\partial f}{\partial x_j} + v_i u_j \frac{\partial f}{\partial x_i \partial x_j}$, hence $Y(X(f)) - X(Y(f)) = \sum_{i,j=1}^n \left(v_i \frac{du_j}{\partial x_i} - u_i \frac{\partial v_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j} + \sum_{i,j=1}^n u_j v_i \frac{\partial^2 f}{\partial x_i \partial x_j} - u_i v_j \frac{\partial^2 f}{\partial x_j \partial x_i} = \sum_{i,j=1}^n \left(v_i \frac{du_j}{\partial x_i} - u_i \frac{\partial v_j}{\partial x_i}\right)\frac{\partial f}{\partial x_j}$. □

Definition 3.11. For all $X, Y \in \mathfrak{X}(M)$, we define the Lie bracket of X and Y as the unique vector field $[X,Y]$ such that for all $f \in \mathcal{C}^\infty(M,\mathbb{K})$

$$[X,Y](f) = X(Y(f)) - Y(X(f)).$$

Before we give an example, the proof of Lemma 3.10 and Theorem 3.9 gives us the

Corollary 3.12. Let $X, Y \in \mathfrak{X}(M)$, $(U, x_1, \ldots, x_n)$ a chart on M and $(u_i)_i, (v_i)_i$ the components of X, Y in these coordinates. We have on U

$$[X,Y] = \sum_{j=1}^n \left(\sum_{i=1}^n v_i \frac{\partial u_j}{\partial x_i} - u_i \frac{\partial v_j}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

▷ Examples. (of Lie brackets of vector fields)

- For all $X, Y \in \mathfrak{X}(M)$ such that their components in the chart $(U, x_1, \ldots, x_n)$ are constants, $[X,Y]$ is null on U.
- On $\mathbb{R}^2$ with the global coordinates (x,y), let $X := x^2 \frac{\partial}{\partial x}$ and $Y := y^2 \frac{\partial}{\partial y}$. We have $[X,Y] = \left(0\times 2x - x^2 \times 0 + y^2 \times 0 - 0\times 0\right)\frac{\partial}{\partial x} + \left(0\times 0 - x^2\times 0 + y^2\times 0 - 0\times 2y\right)\frac{\partial}{\partial y} = 0$. With $Z := y^2 \frac{\partial}{\partial x}$ we have $[X,Z] = \left(y^2\times 2x - x^2\times 0 + 0\times 0 - 0\times 2y\right)\frac{\partial}{\partial x} + \left(y^2\times 0 - x^2\times 0 + 0\times 0 - 0\times 0\right)\frac{\partial}{\partial y} = 2xy^2\frac{\partial}{\partial x} = 2xZ$.

Corollary 3.13. The vector space $\mathfrak{X}(M)$ associated to $[\cdot,\cdot]$ of definition 3.11 is a Lie algebra.

Proof We want to show that $[\cdot,\cdot] : \mathfrak{X}(M)^2 \to \mathfrak{X}(M)$ satisfies definition 3.4, using the last corollary. It is easy to show that $[\cdot,\cdot]$ is bilinear and skew-symmetric. A longer direct calculus allows us to prove the Jacobi identity on each chart of M : the development of each sum gives us 3×6 terms, which can be paired by opposite sign. □

This last proposition gives us an important example of Lie algebra, and will be used to define the Lie algebra associated to a Lie group.

3.3 The Lie algebra of a Lie group

Let G be a Lie group over $\mathbb{K}$ with an identity element e. The aim of this subsubsection is to define the Lie algebra of G, which is a useful tool to describe G. There are two main points of view on this Lie algreba : we can see it as a set of particular vector fields or as the tangent plane at the neutral element. This subsection is mainly based on [11].

3.3.1 Definition of the Lie algebra of a Lie group

Definition 3.14. For all $g \in G$ we define $L_g : h \in G \mapsto gh \in G$ the left multiplication by g and $R_g : h \in G \mapsto hg \in G$ the right multiplication by g. A vector field $X \in \mathfrak{X}(G)$ is *left invariant* (resp. *right invariant*) if for all $h \in G$, $(dL_g)_h(X(h)) = X(L_g(h))$ (resp. $(dR_g)_h(X(h)) = X(R_g(h))$). The vector space of all left invariant vector fields on G is denoted by $\mathfrak{g}$.

▷ Example. We take the example of the Lie group $G = \mathrm{GL}_n(\mathbb{K})$. Let $B \in G$. We can show that $T_B G = B \cdot T_{\mathrm{I}_n} G$ and we see in Proposition 3.21 that $T_{\mathrm{I}_n} \mathrm{GL}_n(\mathbb{K}) = \mathrm{M}_n(\mathbb{K})$. Hence $T_B G = \mathrm{M}_n(\mathbb{K})$. For all $A \in \mathrm{M}_n(\mathbb{K})$, $B \in G \mapsto BA \in TG$ is an invariant vector field on G.

◇ Remarks.

- In this report we mostly look at the left invariant vector fields.
- Let $X \in \mathfrak{X}(G)$. Given $g \in G \backslash \{e\}$, the map $h \in G \mapsto X(gh) \in TM$ is not necessarily a vector field : it is smooth, valued in TM but we have for all $g \in G$ $X(gh) \in T_{gh}M$ while we demand $X(gh) \in T_h M$. However, we notice that X is left invariant if an only if for all $h \in G$, $(dL_g)_{g^{-1}h}(X(g^{-1}h)) = X(h)$. This last condition is an equality between two vector fields : we can use Theorem 3.9. We use this in the proof of Proposition 3.16.

Proposition 3.15. Let $X \in \mathfrak{g}$, $g \in G$ and $f \in \mathcal{C}^\infty(G, \mathbb{K})$. We have $X(f \circ L_g) = X(f) \circ L_g$.

Proof For all $h \in G$, $X(f \circ L_g)(h) = (d(f \circ L_g) \circ X)(h) = df_{gh}(X(L_g)(h)) = df_{gh}(X(gh)) = X(f)(gh)$. □

Proposition 3.16. The set $\mathfrak{g}$ is a $\mathbb{K}$-vector space and for all $X, Y \in \mathfrak{g}$ we have $[X, Y] \in \mathfrak{g}$. Hence $\left(\mathfrak{g}, [\cdot,\cdot]\,|^{\mathfrak{g}}_{\mathfrak{g}\times\mathfrak{g}}\right)$ is a $\mathbb{K}$-Lie algebra.

Proof For all $(g, h) \in G^2$, $(dL_g)_h$ is linear, hence $\mathfrak{g}$ is a $\mathbb{K}$-vector space. Let $X, Y \in \mathfrak{g}^2$. We want to prove that $[X, Y]$ is left invariant : by the remark above, if we introduce $Z : h \in G \mapsto (dL_g)_{g^{-1}h}(X(g^{-1}h)) \in TG$ then it is equivalent to show that for all $f \in \mathcal{C}^\infty(M, \mathbb{K})$ and $h \in G$, $Z(f)(h) = [X, Y](f)(h)$.

Let $f \in \mathcal{C}^\infty(M, \mathbb{K})$, $h \in G$ and $\tilde{h} := g^{-1}h$. We have $Z(f)(h) = df_h\left(d(L_g)_{\tilde{h}}\left([X, Y]\left(\tilde{h}\right)\right)\right) = d(f \circ L_g)_{\tilde{h}}([X, Y](\tilde{h})) = [X, Y](f \circ L_g)(\tilde{h}) = Y(X(f \circ L_g))(\tilde{h}) - X(Y(f \circ L_g))(\tilde{h})$. Hence by Proposition 3.15, $Z(f)(h) = Y(X(f) \circ L_g)\left(\tilde{h}\right) - X(Y(f) \circ L_g)\left(\tilde{h}\right) = (Y(X(f)) \circ L_g)\left(\tilde{h}\right) - (X(Y(f)) \circ L_g)\left(\tilde{h}\right) = Y(X(f))(h) - X(Y(f))(h) = [X, Y]\,(f)(h)$. □

Definition 3.17. We also denote $[\cdot,\cdot] : (X, Y) \in \mathfrak{g}^2 \mapsto [X, Y] \in \mathfrak{g}$. The *Lie algebra* of the Lie group G is $(\mathfrak{g}, [\cdot,\cdot])$.

3.3.2 The Lie algebra of a Lie group as a tangent space

We now see another point of view of the Lie algebra $\mathfrak{g}$, as the tangent space at the identity $T_e G$.

Proposition 3.18. The evaluation at the identity

$$\mathrm{ev}_e : \left|\begin{array}{ccc} \mathfrak{g} & \longrightarrow & T_e G \\ X & \longmapsto & X(e) \\ g \in G \mapsto (dL_g)_e X_e \in TG & \longleftarrow & X_e \end{array}\right.$$

is a vector space isomorphism.

Proof The evaluation ev_e is clearly well defined and linear. Let $X, Y \in \mathfrak{g}$ such that $X(e) = Y(e)$. For all $g \in G$, $X(g) = X(ge) = (dL_g)_e(X(e)) = (dL_g)_e(Y(e)) = Y(g)$, hence $X = Y$. So ev_e is injective.

Let $X_e \in T_eG$. Let $X : g \in G \mapsto (dL_g)_e(X_e) \in TG$. Since G is a Lie group, X is smooth and for all $g \in G$, $X(g) \in T_gG$. So $X \in \mathfrak{X}(G)$. Since $L_e = \mathrm{Id}_G$, $X(e) = X_e$. Finally, we want to show that X is left invariant. For all $g, h \in G$, $(dL_g)_h X(h) = (dL_g)_h(dL_h)_e X_e = (dL_g)_h(dL_h)_e X_e = d(L_g \circ L_h)_e X_e = d(L_{gh})X_e = X(gh)$. So $X \in \mathfrak{g}$ and $\mathrm{ev}_e(X) = X_e$. Hence ev_e is surjective, and this concludes. □

▷ Example. We see in the proof of Proposition 3.22 that for all $X_e \in T_{\mathrm{I}_n} \mathrm{GL}_n$, $\mathrm{ev}_e^{-1} X_e : A \in \mathrm{GL}_n \mapsto AX_e \in T_{\mathrm{I}_n} \mathrm{GL}_n$.

◇ Remarks.
- We can see that $(X_e, Y_e) \in (T_eG)^2 \mapsto \mathrm{ev}_e\left(\left[\mathrm{ev}_e^{-1}(X_e), \mathrm{ev}_e^{-1}(Y_e)\right]\right) \in T_eG$ defines a Lie bracket on the tangent space T_eG. With this Lie bracket, ev_e is a Lie algebra isomorphism.
- The same proposition holds for right invariant vector fields :

$$\begin{array}{ccc} \left\{\tilde{X} \in \mathfrak{X}(G) \;\middle|\; X \text{ right invariant}\right\} & \longrightarrow & T_eG \\ X & \longmapsto & X(e) \\ g \in G \mapsto (dR_g)_e X_e \in TG & \longleftarrow & X_e \end{array}$$

  is a bijection. To prove it, we use the same technique than in the proof above, with right translation instead of left translation. This side result is used in the proof of Lemma 4.25.

Corollary 3.19. We have $\dim G = \dim \mathfrak{g}$.

Proof It is a direct application of Proposition 3.18. □

◇ Notations.
- We also denote the tangent space T_eG by $\mathfrak{g}$ and its Lie bracket (given by ev_e, cf last remark) by $[\cdot, \cdot]$. Depending on the context, the Lie algebra $\mathfrak{g}$ of G can be the set of set of left invariant vector fields on G or the set of tangent vector at the identity.
- We denote the Lie algebra of a Lie group by the notation of the Lie group in Fraktur lowercases. Given a finite dimensional vector space V, the Lie algebra of $\mathrm{GL}(V)$ is $\mathfrak{gl}(V)$.

Proposition 3.20. The Lie algebra of a Lie subgroup of G is a Lie subalgebra of $\mathfrak{g}$.

Proof Let $H < G$ a Lie subgroupd of G and $\mathfrak{h}$ its Lie algebra. Let $X, Y \in \mathfrak{h}$. There is $(U, x_1, \ldots, x_n)$ a chart of M and $m \in [\![1, n]\!]$ such that $e \in U$ and $N \cap U = \{x_{m+1} = \ldots = x_n = 0\}$. Let $(u_i)_i$ (resp. $(v_i)_i$) the component of X (resp. Y) in the coordinates $(U \cap N, x_1, \ldots, x_m)$. Let $\tilde{X} \in \mathfrak{g}$ (resp. $\tilde{Y} \in \mathfrak{g}$) the left invariant vector field associated to $X(e)$ (resp. $Y(e)$) by Proposition 3.18. Let $(u_i)_i$ (resp. $(\tilde{u}_i)_i$) the component of $\tilde{X}$ (resp. $\tilde{Y}$) in $(U, x_1, \ldots, x_n)$. By unicity, we have $u_1 = \tilde{u}_1|_{U\cap N}, \ldots, u_m = \tilde{u}_m|_{U\cap N}$ and $\tilde{u}_{m+1}|_{U\cap N} = \ldots = \tilde{u}_n|_{U\cap N} = 0$. Hence, by Corollary 3.12, $[X, Y](e) = \left[\tilde{X}, \tilde{Y}\right](e)$, i.e. $[X(e), Y(e)] = \left[\tilde{X}(e), \tilde{Y}(e)\right]$. □

Proposition 3.21. We have

$$\begin{cases} \mathfrak{gl}_n(\mathbb{K}) & = \mathrm{M}_n(\mathbb{K}) \\ \mathfrak{sl}_n(\mathbb{K}) & = \left\{X \in \mathrm{M}_n(\mathbb{K}) \;\middle|\; \operatorname{tr} X = 0\right\} \\ \mathfrak{su}_n & = \left\{X \in \mathrm{M}_n(\mathbb{C}) \;\middle|\; X + X^* = 0, \operatorname{tr} X = 0\right\} \end{cases}.$$

Proof
- We described the Lie group $\mathrm{GL}_n(\mathbb{K})$ as a submanifold of $\mathrm{M}_n(\mathbb{K})$. Hence $T_{\mathrm{I}_n} \mathrm{GL}_n(\mathbb{K}) \subset \mathrm{M}_n(\mathbb{K})$.
  Let $X \in \mathrm{GL}_n(\mathbb{K})$. The path $\gamma : t \in]-1, 1[\mapsto \exp(tX) \in \mathrm{GL}_n(\mathbb{K})$ is smooth and statisfies $\gamma(0) = \mathrm{I}_n$ and $\gamma'(0) = X$. Hence, $X \in T_{\mathrm{I}_n} \mathrm{GL}_n(\mathbb{K})$.
- Let $X \in \mathfrak{sl}_n$. There is $\gamma :]-1, 1[\to \mathrm{SL}_n$ such that $\gamma(0) = \mathrm{I}_n$ and $\gamma'(0) = X$. For all $t \in]-1, 1[$, $\det \gamma(t) = 1$, hence $d(\det)_{\gamma(0)}\gamma'(0) = 0$, i.e. $\operatorname{tr}\left(\mathrm{Com}(\mathrm{I}_n)^T X\right) = 0$, i.e. $\operatorname{tr} X = 0$.

Thus $\mathfrak{sl}_n \subset \operatorname{Ker}\operatorname{tr}$.
Let $X \in \operatorname{Ker}\operatorname{tr}$. For all $t \in]-1,1[$, tX is triangularizable on $\mathbb{C}$ so $\det\exp(tX) = \exp(\operatorname{tr} tX) = 1$. Thus we can define the path $\gamma : t \in]-1,1[\mapsto \exp(tX) \in \mathrm{SL}_n(\mathbb{K})$ and conclude that $X \in \mathfrak{sl}_n$.

- Let $X \in \mathfrak{su}_n$. There is $\gamma :]-1,1[\to \mathrm{SU}_n$ such that $\gamma(0) = \mathrm{I}_n$ and $\gamma'(0) = X$. By the previous point, $\operatorname{tr} X = 0$. For all $t \in]-1,1[$, $\gamma(t)\gamma(t)^* = \mathrm{I}_n$ hence $\gamma'(0)\gamma(0)^* + \gamma(0)\gamma'(0)^* = 0$, i.e. $X + X^* = 0$.
  Let $X \in \mathrm{M}_n(\mathbb{K})$ such that $X + X^* = 0$ and $\operatorname{tr} X = 0$. For all $t \in]-1,1[$, $\exp(tX)\exp(tX)^* = \exp(tX)\exp(tX^*) = \exp(t(X + X^*)) = \mathrm{I}_n$. Thus we can define the path $\gamma : t \in]-1,1[\mapsto \exp(tX) \in \mathrm{SU}_n(\mathbb{K})$ and conclude that $X \in \mathfrak{su}_n$.

□

Proposition 3.22. Let $G < \mathrm{GL}_n(\mathbb{K})$ be a Lie subgroup of $\mathrm{GL}_n(\mathbb{K})$. We have $\mathfrak{g} \subset \mathrm{M}_n(\mathbb{K})$ and all $X, Y \in \mathfrak{g}$, $[X, Y] = XY - YX$ (the Lie bracket of its Lie algebra is the commutator).

Proof Let $X_e, Y_e \in T_{\mathrm{I}_n}G$. For all $A \in G$, $B \in \mathrm{M}_n(\mathbb{K}) \mapsto AB \in \mathrm{M}_n(\mathbb{K})$ is linear so $d(L_A)_e : B \in G \mapsto AB \in TG$. Thus, if we denote $X := \mathrm{ev}_e^{-1}(X_e)$ and $Y := \mathrm{ev}_e^{-1}(Y_e)$, $X : A \in G \mapsto AX_e \in TG$ and $Y : A \in G \mapsto AY_e \in TG$. Let $(i,j) \in [\![1,n]\!]^2$ and $x_{i,j} : (a_{k,l})_{k,l \in [\![1,n]\!]^2} \in \mathfrak{g} \mapsto a_{i,j} \in \mathbb{K}$. It is clear that $x_{i,j} \in \mathcal{C}^\infty(\mathfrak{g})$. In addition to that, the associated map on the whole space M_n is linear, hence for all $Z \in \mathfrak{X}(G)$ and $g \in G$, $Z(x_{i,j})(g) = x_{i,j}(Z(g))$. We deduce that $x_{i,j}([X_e, Y_e]) = x_{i,j}([X,Y](\mathrm{I}_n)) = [X,Y](x_{i,j})(\mathrm{I}_n) = X(Y(x_{i,j}))(\mathrm{I}_n) - Y(X(x_{i,j}))(\mathrm{I}_n) = x_{i,j}(X(\mathrm{I}_n)Y_e) - x_{i,j}(Y(\mathrm{I}_n)X_e) = x_{i,j}(X_eY_e - Y_eX_e)$. This shows that $[X_e, Y_e] = X_eY_e - Y_eX_e$. □

◇ Remarks.

- Using Proposition 3.20, it would have been enough to show that the Lie bracket on $\mathrm{GL}_n(\mathbb{K})$ is the commutator.
- The Lie algebra $\mathfrak{su}_n$ is a Lie subalgebra of $\mathfrak{sl}_n(\mathbb{C})$, which is a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{C})$.
- We could have shown an intersting thing about the correspondance between Lie group morphisms and Lie algebra motphisms. Let H be a Lie subgroup and $\mathfrak{h}$ its Lie algebra. For all $f : G \to H$ Lie group morphism, $df_e : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra morphism.

3.4 The exponential map on the Lie algebra of a Lie group

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra and $[\cdot,\cdot]$ its Lie bracket. This subsection is mainly based on [2, 12, 14].

3.4.1 Definition of the exponential map

We can define the exponential on a Banach algebra B (such that $\mathrm{M}_n(\mathbb{K})$) as a conveging series $\exp : x \in B \mapsto \sum_{i=1}^{+\infty} \frac{x^n}{n!} \in B$. In this subsubsection we see a generalization to Lie groups : it is an interesting link between a Lie group and its Lie algebra. To understand the rest of the report, it is possible to skip this subsubsection about the exponential by reading Proposition 3.27. This proposition is useful to define the Lie derivative in Subsubsection 3.4.3 and in some proofs in Subsection 4.2.

Definitions 3.23.

- Given M a manifold, $x \in M$ and $X \in \mathfrak{X}(M)$, an *integral curve* of X throught x is a smooth function γ from an open neighboorhood of $0 \in \mathbb{R}$ to M such that

$$\begin{cases} \gamma' &= X \circ \gamma \\ \gamma(0) &= x \end{cases}.$$

  A vector field X on a manifold M is *complete* if for all $x \in M$ there is an integral curve $\mathbb{R} \to M$ of X throught x.

- A *one-parameter group* of G is a smooth homomorphism $(\mathbb{R}, +) \to (G, \cdot)$. The set of all one-parameter groups of G is denoted by $\mathcal{I}$.

▷ EXAMPLES.

- If we consider the manifold $\mathbb{R}^n$, looking for an integral curve of a vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ is looking for a solution to an ordinary differential equation. Let M a manifold and $X \in \mathfrak{X}(M)$: in local coordinates, the condition of γ beeing an integral curve of X can be written as an ordinary differential equation. We see in the remark below an example of how important this observation is.
- By the Cauchy-Lipschitz theorem, every smooth Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ is a complete vector field. In the proof of Proposition 3.21, we have seen some examples of integral curves of complete vector fields on matrices (we also notice that we have used the exponential on a Banach algebra). On the other hand, the vector field $x \in \mathbb{R} \mapsto x^2 \in \mathbb{R}$ has integral curves and yet is not complete.
- The real exponential $\exp : \mathbb{R} \to \mathbb{R}^{+*}$ is a one-parameter subgroup of $(\mathbb{R}^{+*}, \times)$.

◇ REMARKS.

- Since every vector field X on a manifold M is smooth by definition and because of the first point of the example we have just seen, we can use the Cauchy-Lipschitz theorem : for all $x \in M$, there is a unique maximal integral curve of X throught x. See Theorem 12.9 page 314 of [10]. For example, we use this result in the proof of Lemma 3.24 below.
- Let $\gamma \in \mathcal{I}$. Since $\mathbb{R}$ is commutative, $\mathrm{Im}\gamma$ is commutative : for all $s, t \in \mathbb{R}$, $\gamma(t)\gamma(s) = \gamma(s)\gamma(t)$.

LEMMA 3.24. With $\mathcal{I}$ the set of all one-parameter groups of G, the map

$$D : \left| \begin{array}{ccc} \mathcal{I} & \longrightarrow & T_eG \\ \gamma & \longmapsto & \gamma'(0) \end{array} \right.$$

is a bijection.

Proof Let $\gamma_1, \gamma_2 \in \mathcal{I}$ such that $\gamma_1'(0) = \gamma_2'(0) =: X_e$. For all $(s,t) \in \mathbb{R}^2$, $\gamma_1(s+t) = \gamma_1(s)\gamma_1(t) = L_{\gamma_1(s)}(\gamma_1(t))$, so $\gamma_1'(s+t) = d\left(L_{\gamma_1(s)}\right)_{\gamma_1(t)}(\gamma_1'(t))$. Hence, for all $t \in \mathbb{R}$, $\gamma_1'(t) = d\left(L_{\gamma_1(s)}\right)_e(X_e)$. Let $X := \mathrm{ev}_e^{-1}(X_e)$: we have, for all $t \in \mathbb{R}$, $\gamma_1'(t) = X(\gamma_1(t))$ and in the same way $\gamma_2'(t) = X(\gamma_2(t))$. In addition to that, $\gamma_1(0) = \gamma_2(0) = e$ so by the Cauchy-Lipschitz theorem $\gamma_1 = \gamma_2$. We have shown that D is injective.

Let $X_e \in T_eG$ and $X := \mathrm{ev}_e^{-1}(X_e) \in \mathfrak{X}(G)$. There is $\varepsilon > 0$ and an integral curve $\gamma :]-\varepsilon, \varepsilon[\to G$ of X through e. Let $s \in]-\varepsilon, \varepsilon[$, $J^s =]-s-\varepsilon, -s+\varepsilon[\cap]-\varepsilon, \varepsilon[$, $\gamma_1^s : t \in J^s \mapsto \gamma(s+t) \in G$ and $\gamma_2^s : t \in J^s \mapsto \gamma(s)\gamma(t) \in G$. We want to show that these two functions are equal. For all $t \in J^s$, $(\gamma_1^s)'(t) = \gamma'(t+s) = X(\gamma(t+s)) = X(\gamma_1^s)$ and as we saw earlier $(\gamma_2^s)'(t) = d\left(L_{\gamma(s)}\right)_{\gamma(t)}(\gamma'(t)) = d\left(L_{\gamma(s)}\right)_{\gamma(t)}(X(\gamma(t))) = X(\gamma(s)\gamma(t)) = X(\gamma_2^s(t))$. In addition to that, $\gamma_1^s(0) = \gamma_2^s(0) = \gamma(t)$, hence γ_1^s and γ_2^s are two integal curves of X through $\gamma(t)$. By the Cauchy-Lipschitz theorem, $\gamma_1^s = \gamma_2^s$. We deduce that for all $s, t \in]-\varepsilon, \varepsilon[$ such that $s+t \in]-\varepsilon, \varepsilon[$, $\gamma(s+t) = \gamma(s)\gamma(t)$. We now want to extend γ to $\mathbb{R}$. For all $m \in \mathbb{N}$ we define $J_m :=]-2^m\varepsilon, 2^m\varepsilon[$. Let $m \in \mathbb{N}$ and suppose that there is an integral curve $\gamma_m : J_m \to G$ of X through e such that

$$\left\{ \begin{array}{l} \gamma_m|_{]-\varepsilon,\varepsilon[} = \gamma \\ \forall s, t \in J_m, s+t \in J_m \Rightarrow \gamma_n(s+t) = \gamma_n(s)\gamma_n(t) \end{array} \right. .$$

Let $\gamma_{m+1} : t \in J_{m+1} \mapsto \gamma_m(t/2)^2 \in G$. For all $s, t \in J_{m+1}$ such that $s+t \in J_{m+1}$, $\gamma_{m+1}(s+t) = \gamma_m\left(\frac{s+t}{2}\right)^2 = \gamma_m\left(\frac{s}{2}\right)^2 \gamma_m\left(\frac{t}{2}\right)^2 = \gamma_{m+1}(s)\gamma_{m+1}(t)$. For all $t \in]-\varepsilon, \varepsilon[$, $\gamma_{m+1}(t) = \gamma_m\left(\frac{t}{2}\right)^2 = \gamma\left(\frac{t}{2}\right)^2 = \gamma(t)$. Let $t_0 \in J_{m+1}$. There is $\eta \in]0, \varepsilon[$ such that $]t_0 - \eta, t_0 + \eta[\subset J_{m+1}$. For all $t \in]-\eta, +\eta[$, $\gamma_{m+1}(t_0 + t) = \gamma_{m+1}(t_0)\gamma_{m+1}(t) = L_{\gamma_{m+1}}(\gamma(t))$, hence $\gamma_{m+1}'(t_0 + t) = d(L_{\gamma_{m+1}(t_0)\gamma(h)}(\gamma'(h))$. In particular, $\gamma_{m+1}'(t_0) = d(L_{\gamma_{m+1}(t_0)})_e(X(e)) = X(\gamma_{m+1}(t_0))$. Since $\gamma_{m+1}(0) = e$, we deduce that γ_{m+1} is an integral curve of X through e. By induction, with $\gamma_0 := \gamma$, for all $m \in \mathbb{N}$ there is such a curve γ_m. By the Cauchy-Lipschitz theorem, for all $m_1, m_2 \in \mathbb{N}$ such that $m_1 \leqslant m_2$, $\gamma_{m_2}|_{J_{m_1}} = \gamma_{m_1}$. Hence there is a unique $\tilde{\gamma} : \mathbb{R} \to G$ such that for all $m \in \mathbb{N}$, $\tilde{\gamma}|_{J_m} = \gamma_m$. Using the properties we have just shown in the induction, $\tilde{\gamma} \in \mathcal{I}$ and $D(\tilde{\gamma}) = \gamma'(0) = X_e$. We conclude that D is surjective. □

Corollary 3.25. Let $X \in \mathfrak{g}$. The vector field X is complete and for all $g \in G$ there is a unique integral curve of X through g.

Proof Let $g \in G$ and $\gamma_e := D^{-1}(X(e))$. The map $g\gamma_e$ defined on the whole line $\mathbb{R}$ is an integral curve of X through g, and by the Cauchy-Lipschitz theorem it is the only one. $\square$

The lemma 3.24 allows us to introduce the

Definition 3.26. The exponential on $\mathfrak{g}$ is

$$\exp : \left| \begin{array}{ccc} \mathfrak{g} & \longrightarrow & G \\ X_e & \longmapsto & D^{-1}(X_e)(1) \end{array} \right.$$

3.4.2 Basic properties about the exponential map on a Lie algebra

The exponential on a Lie group has some of the classic properties of the exponential on the matrices, as we see for example in the propositions below.

Propositions 3.27. Let $X_e \in \mathfrak{g}$.

1. We have $D^{-1}(X_e) : t \in \mathbb{R} \mapsto \exp(tX_e) \in G$. In particular,

$$\begin{cases} \exp(0) & = e \\ \forall s, t \in \mathbb{R}, \exp((s+t)X_e) & = \exp(tX_e)\exp(sX_e) \\ \frac{d}{dt}\exp(tX_e)\big|_{t=0} & = X_e \end{cases}$$

2. We have $\exp(X_e)^{-1} = \exp(-X_e)$.

Proof Let $\gamma_1 := D^{-1}(X_e)$. Let $t \in \mathbb{R}$ and $\gamma_t : s \in \mathbb{R} \mapsto \gamma_1(st) \in G$. It is clearly smooth and for all $s \in \mathbb{R}$, $\gamma_t'(s) = d(\gamma_1)_{st}(t) = t\gamma'(st) = tX(\gamma_t(s))$, hence $\gamma_t = D^{-1}(tX_e)$ and $\exp(tX_e) = \gamma_1(t)$. We deduce the first statement. From this statement, we have $\exp(X_e)\exp(-X_e) = \exp((1-1)X_e) = e$, hence we have the second statement. $\square$

▷ Example. We also have, for all $X_e, Y_e \in \mathfrak{g}$ such that $[X_e, Y_e] = 0$, $\exp(X_e+Y_e) = \exp(X_e)\exp(Y_e)$. We do not prove it (but we will not use it).

▷ Example. Let $f : t \in \mathrm{M}_n(\mathbb{K}) \mapsto \sum_{k=0}^{+\infty} \frac{(tA)^k}{k!} \in \mathrm{GL}_n(\mathbb{K})$. We have $f(0) = \mathrm{I}_n$ and $f'(0) = A$, hence $\exp(A) = f(1)$, i.e.

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{A^k}{k!}.$$

In the same way, for all $t \in \mathbb{R}$, $\exp(t) = \sum_{k=0}^{+\infty} \frac{t^k}{k!}$. We notice that we used the exponential of a matrix in the context of Lie groups in the proof of Proposition 3.21.

Proposition 3.28. The map exp is smooth.

Proof Let $X_e \in \mathfrak{g}$ and $X \in \mathfrak{X}(G)$ the left-invariant vector field such that $X(e) = X_e$. For all $Y_e \in \mathfrak{g}$ we denote by $\gamma_{Y_e} : \mathbb{R} \to G$ the integral curve of X through e. By the theorem of smooth dependance on initial condition of the solutions of an ordinary differential equation, there is $U \subset \mathfrak{g}$ a neighboorhood of X_e and $\varepsilon > 0$ such that $Y_e \in U \mapsto \gamma|_{[-\varepsilon,\varepsilon]} \in \mathcal{C}^0([-\varepsilon,\varepsilon], G)$ is smooth. In particular, $Y_e \in U \mapsto \exp(Y_e) \in G$ is smooth. Hence exp is smooth. $\square$

Proposition 3.29. For all $X_e, Y_e \in \mathfrak{g}$,

$$[X_e, Y_e] = \frac{d}{dt}\left(\frac{d}{ds}\left(\exp(tX_e)\exp(sY_e)\exp(-tX_e)\right)\big|_{s=0}\right)\big|_{t=0}.$$

Proof We only prove it for Lie group of matrices. The general result is admitted. We suppose that $G < \mathrm{GL}_n(\mathbb{K})$ is a Lie subgroup of $\mathrm{GL}_n(\mathbb{K})$. Let $X_e, Y_e \in \mathfrak{g}$. We have $\frac{d}{dt}\left(\frac{d}{ds}\left(\exp(tX_e)\exp(sY_e)\exp(-tX_e)\right)\Big|_{s=0}\right)\Big|_{t=0} = \frac{d}{dt}\left(\exp(tX_e)Y_e\exp(-tX_e)\right)\Big|_{t=0} = X_eY_e\,\mathrm{I}_n + \mathrm{I}_n\,Y_e(-X_e) = [X_e, Y_e]$. □

◇ REMARKS.

- What does the formula given above means ? Let $X_e, Y_e \in \mathfrak{g}$. For all $s, t \in \mathbb{R}$ we have $\exp(tX_e)\exp(tY_e)\exp(-tX_e) \in G$, hence for all $t \in \mathbb{R}$ we have $\left(\exp(tX_e)\exp(tY_e)\exp(-tX_e)\right)\big|_{s=0} \in \mathfrak{g}$. By the identification $T_0\mathfrak{g} = \mathfrak{g}$ we make, the second derivation of the formula correspond to the classical derivation of a function from $\mathbb{R}$ to a normed vector space. Thus $\frac{d}{dt}\left(\frac{d}{ds}\left(\exp(tX_e)\exp(tY_e)\exp(-tX_e)\right)\Big|_{s=0}\right)\Big|_{t=0}$ is indeed an element of $\mathfrak{g}$ and the formula is coherent.

- Let $X, Y \in \mathfrak{g}$ left-invariant vector fields. Since the flow of X (resp. Y) trough e is $t \in \mathbb{R} \mapsto \exp(tX(e)) \in G$, Proposition 3.29 allows us to see the Lie bracket of two elements of $\mathfrak{g}$ as a "measure of the degree of noncommutativity of the flows of these two elements" : see page 210 in [2].

- How to describe the Lie algebra $\mathfrak{g}$ of the Lie group G ? We have seen two ways of looking at $\mathfrak{g}$: as the set of the left invariant vector fields on G or as the tangent space at the neutral element. We have also seen three ways of looking at its Lie bracket : in algebra as an operation on the corresponding derivations of $\mathcal{C}^\infty(G)$, in calculus with a local explicit formula, and now in geometry as the "degree of noncommutativity" of two flows. This last one has also interesting consequences : for example see Propositions 4.15.

This new tool can also help us to describe the Lie algebra of a Lie subgroup of G, for example by

PROPOSITION 3.30. Let $H < G$ be a Lie subgroup of G and $\mathfrak{h}$ its Lie algebra. We have

$$\mathfrak{h} = \left\{X_e \in \mathfrak{g} \;\middle|\; \forall t \in \mathbb{R}, \exp(tX_e) \in H\right\}.$$

Proof Let $X_e \in \mathfrak{g}$ such that for all $t \in \mathbb{R}$, $\exp(tX_e) \in H$. We have $\frac{d}{dt}\left(\exp(tX_e)\right)\big|_{t=0} \in T_{\exp(0)}H = \mathfrak{h}$. Reciprocally, let $X_e \in \mathfrak{h}$. Let $i : h \in H \mapsto h \in G$ the inclusion map. Since H is a Lie group, there is $\gamma : \mathbb{R} \to H$ a one-parameter group of H such that $\gamma'(0) = X_e$. Hence $i \circ \gamma$ is a one-parameter group of G such that $(i \circ \gamma)'(0) = di_e(X_e) = X_e$. By unicity, for all $t \in \mathbb{R}$ we have $i(\gamma(t)) = \exp(tX_e)$, and in particular $\exp(tX_e) \in H$. □

3.4.3 The Lie derivative and the Cartan magic formula

In differential geometry, two important objects are vector fields and differential forms. They are naturally linked by the interior product of a vector field and a diffential form (see 1.2). In this subsubsection, we use the exponential map to define another tool : the Lie derivative of a differential form by a vector field. In fact, the interior product and the Lie derivative are linked by the Cartan magic formula : see Proposition 3.36. To understand the rest of the report, it is possible to skip this subsubsection about the exponential by reading this last proposition : it is very useful in Subsubsection 4.2.2 to compute the exterior derivative of an interior product.

DEFINITION 3.31. Let $X \in \mathfrak{X}(G)$ a complete vector field. For all $g \in G$ there is a unique integral curve $\gamma_g \in \mathcal{C}^\infty(\mathbb{R}, G)$ of X through g. For all $t \in \mathbb{R}$ we denote by $\exp(tX) : g \in G \mapsto \gamma_g(t) \in G$. The *exponential map* (or *flow*) of X is $(t, g) \in \mathbb{R} \times G \mapsto \exp(tX)(g) \in G$.

◇ REMARK. This definition has a natural generalisation on a manifold (and not only on a Lie group). See this interesting definition in Subsection 6.1 of [14].

▷ EXAMPLE. Let $m \in \mathbb{N}^*$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^m)$ a smooth Lipschitz map. It is a complete vector field and its exponential map (or flow) is its flow as seen in the ordinary differential equations theory.

Proposition 3.32. Let $X \in \mathfrak{X}(G)$ a complete vector field.

1. The exponential map of X is in $\mathcal{C}^\infty(\mathbb{R} \times G)$ (i.e. is smooth).
2. For all $t, s \in \mathbb{R}$, $\exp(tX) \circ \exp(sX) = \exp((t+s)X)$.
3. For all $t \in \mathbb{R}$ and $g \in G$, $\frac{d}{ds}\left(\exp(tX)(g)\right)\big|_{s=t} = X(\exp(tX)(g))$.

Proof The exponential map of X is the flow of a $\mathcal{C}^\infty$ map. The first and second statement are a reminder of some important properties about such flows. The third one comes directly from the definition of the exponential of X and of an integral curve. $\square$

Using the exponential defined on $\mathfrak{g}$, we can give a formula for the exponential map of some classical complete vector fields.

Proposition 3.33. Let $X^L \in \mathfrak{X}(G)$ (resp. $X^R \in \mathfrak{X}(G)$) a left-invariant (resp. right-invariant) vector field. Let $X_e^L := X^L(e)$ and $X_e^R := X^R(e)$. For all $t \in \mathbb{R}$,

$$\exp\left(tX^L\right) = R_{\exp(tX_e^L)} \text{ and } \exp\left(tX^R\right) = L_{\exp(tX_e^R)}.$$

Proof Let $g \in G$ and $t \in \mathbb{R}$. We have $\frac{d}{ds}\left(R_{\exp(tX_e^L)}g\right)\big|_{s=t} = \frac{d}{dt}\left(L_g \exp(tX_e^L)\right)\big|_{s=t} = d(L_g)_{\exp(tX_e^L)}(X^L(\exp(tX_e^L))) = X^L(g\exp(tX_e^L)) = X^L(R_{\exp(tX_e^L)}g)$ and, likewise, $\frac{d}{ds}\left(L_{\exp(tX_e^R)}g\right)\big|_{s=t} = \frac{d}{dt}\left(R_g \exp(tX_e^R)\right)\big|_{s=t} = d(R_g)_{\exp(tX_e^R)}(X^R(\exp(tX_e^R))) = X^R(\exp(tX_e^R)g) = X^R(L_{\exp(tX_e^R)}g)$. Using the unicity of such integral curves, this concludes. $\square$

◇ Remark. This last proposition explains why the notation for the exponential of a complete vector field is convenient : for $X \in \mathfrak{g}$ a left invariant vector field, $\exp(tX)g$ can be the group element $\exp(tX) \in G$ times $g \in G$ or the image of $g \in G$ by the map $\exp(tX) \in \mathcal{C}^\infty(G, G)$, and these two are equals.

Definition 3.34. Let $X \in \mathfrak{X}(G)$ be a complete vector field. The Lie derivative of X is

$$\mathcal{L}_X : \left|\begin{array}{ccc} \Omega(G) & \longrightarrow & \Omega(G) \\ \omega & \longmapsto & \frac{d}{dt}\left((\exp(tX))^*\omega\right)\big|_{t=0} \end{array}\right. .$$

▷ Example. We consider the case $G = \mathrm{GL}_n(\mathbb{R})$. Let $M \in \mathrm{M}_n(\mathbb{R})$ and $X : A \in G \mapsto MA \in TG$. This map is a left invariant vector field and for all $t \in \mathbb{R}$, $\exp(tX) : B \in G \mapsto \exp(tA)B \in G$. The map det is smooth on G and for all $B \in G$ $\mathcal{L}_X(\det)(B) = \frac{d}{dt}((\exp(tX))^* \det)\big|_{t=0}(B) = \frac{d}{dt}(\det(\exp(tA)B))\big|_{t=0} = \frac{d}{dt}(\exp(\operatorname{tr} tA))\big|_{t=0} \det B = (\operatorname{tr} A)(\det B)$.

◇ Remark. Let $X \in \mathfrak{X}(G)$ be a complete vector field. We can quickly notice two things : the Lie derivative $\mathcal{L}_X$ of X is linear and for all $k \in \mathbb{N}$ we have $\mathcal{L}_X(\Omega^k(G)) \subset \Omega^k(G)$.

Given $X \in \mathfrak{X}(G)$ a complete vector field, the lie derivative $\mathcal{L}_X$ on $\Omega^0(G)$, i.e. on $\mathcal{C}^\infty(G)$, is the directional derivative in the direction X we saw in definition 3.6. In addition to that, it is a derivation on the $\mathbb{R}$-algebra of the differential form.

Propositions 3.35. Let $X \in \mathfrak{X}(G)$ be a complete vector field.

1. For all $f \in \mathcal{C}^\infty(G)$,
$$\mathcal{L}_X f = X(f).$$
2. We have $\mathcal{L}_X \in \mathrm{Der}((\Omega(M), \wedge))$, i.e. the Lie derivative of X is a derivation on the algebra $\Omega(M)$ equiped with the exterior product $\wedge$.

Proof 1. For all $f \in \mathcal{C}^\infty(G)$ and $g \in G$, $(\mathcal{L}_X f)(g) = \frac{d}{dt}\left(((\exp(tX))^* f)(g)\right)\big|_{t=0} = \frac{d}{dt}\left(f(\exp(tX)g)\right)\big|_{t=0} = df_g(X(g))$.

2. For all $\omega_1, \omega_2 \in \Omega(G)$, $\mathcal{L}_X(\omega_1 \wedge \omega_2) = \frac{d}{dt}\left(\exp(tX)^*(\omega_1 \wedge \omega_2)\right)\big|_{t=0} = \frac{d}{dt}\left((\exp(tX)^*\omega_1) \wedge (\exp(tX)^*\omega_2)\right)\big|_{t=0} = \frac{d}{dt}\left((\exp(tX)^*\omega_1)\right)\big|_{t=0} \wedge (\mathrm{Id}_G^*\,\omega_2) + (\mathrm{Id}_G^*\,\omega_1) \wedge \frac{d}{dt}\left((\exp(tX)^*\omega_2)\right)\big|_{t=0} = \mathcal{L}_X(\omega_1) \wedge \omega_2 + \omega_1 \wedge \mathcal{L}_X\omega_1$.

□

Proposition 3.36. (Cartan magic formula)
Let $X \in \mathfrak{X}(G)$ be a complete vector field and $\omega \in \Omega(G)$. We have

$$\mathcal{L}_X\omega = i_X d\omega + d i_X \omega.$$

Proof In this proof we use the calculus rules about the interior product we have reminded in the subsection 1.2, and the method seen in the exercise page 34 of [14]. Let $D_1 : \omega \in \Omega(G) \mapsto \mathcal{L}_X\omega \in \Omega(G)$ and $D_2 : \omega \in \Omega(G) \mapsto i_X d\omega + d i_X\omega \in \Omega(G)$. For all $k \in \mathbb{N}$ we denote by $\mathcal{I}_k$ the set $\left\{(i_1, \ldots, i_k) \in [\![1, n]\!] \;\middle|\; i_1 < \ldots < i_k\right\}$.

We start proving Cartan magic formula on $\Omega^0(M)$. Let $f \in \Omega^0(G) = \mathcal{C}^\infty(G)$. For all $g \in G$, $D_1(f)(g) = \frac{d}{dt}\left(f(\exp(tX)g)\right)\big|_{t=0} = df_g\left(X(\exp(tX)g)\big|_{t=0}\right) = df_g(X(g)) = i_X df = i_X df + d i_X f = D_2(f)(g)$. Hence the Cartan magic formula for f.

Secondly, we want to prove that d commutes with D_1 and D_2. Let $k \in \mathbb{N}$ and $\omega \in \Omega^k(G)$. We have $D_2(d\omega) = i_X dd\omega + d i_X d\omega = d i_X d\omega = d i_X d\omega + d d i_X \omega = dD_2(\omega)$. It is harder to prove it for D_1. Let $(U, x_1, \ldots, x_n)$ a chart of G. There is $(\omega_I)_I \in \mathcal{C}^\infty(U \times \mathbb{R})^{\mathcal{I}_k}$ such that for all $t \in \mathbb{R}$ we have $((\exp(tX)^*)\omega)|_U = \sum_{I \in \mathcal{I}_k} \omega_I(\cdot, \ldots, \cdot, t)dx^I$. We have $dD_1(\omega)|_U = d\frac{d}{dt}\left(\sum_{I \in \mathcal{I}_k} \omega_I(\cdot, \ldots, \cdot, t)dx^I\right)\big|_{t=0} = d\sum_{I \in \mathcal{I}_k} \frac{d}{dt}\left(\omega_I(\cdot, \ldots, \cdot, t)\right)\big|_{t=0} dx^I$, hence

$$\begin{aligned}
dD_1(\omega)|_U &= \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{\partial}{\partial x_i}\frac{d}{dt}\left(\omega_I(\cdot, \ldots, \cdot, t)\right)\big|_{t=0} dx_i \wedge dx^I \\
&= \sum_{I \in \mathcal{I}_k} \sum_{i=1}^n \frac{d}{dt}\left(\frac{\partial}{\partial x_i}\omega_I(\cdot, \ldots, \cdot, t)\right)\big|_{t=0} dx_i \wedge dx^I \\
&= \frac{d}{dt}\left(\sum_{I \in \mathcal{I}_k} d(\omega_I(\cdot, \ldots, \cdot, t) \wedge dx^I)\right)\big|_{t=0},
\end{aligned}$$

hence $dD_1(\omega)|_U = \frac{d}{dt}\left(d(\exp(tX)^*\omega)\right)\big|_{t=0} = \frac{d}{dt}\left(\exp(tX)^*(d\omega)\right)\big|_{t=0} = D_1(d\omega)|_U$. This concludes our second step.

Thirdly, we want to prove that $D_1, D_2 \in \mathrm{Der}((\Omega(G), \wedge))$. We saw this result for D_1 in Propositions 3.35. Let $k \in \mathbb{N}$, $\omega_1 \in \Omega^k(G)$ and $\omega_2 \in \Omega(G)$. We have $D_2(\omega_1 \wedge \omega_2) = i_X(d(\omega_1 \wedge \omega_2)) + d i_X(\omega_1 \wedge \omega_2) = i_X(d\omega_1 \wedge \omega_2 + (-1)^k\omega_1 \wedge d\omega_2) + d((i_X\omega_1) \wedge \omega_2 + (-1)^k\omega_1 \wedge i_X\omega_2) = i_X(d\omega_1 \wedge \omega_2) + (-1)^k i_X(\omega_1 \wedge d\omega_2) + d((i_X\omega_1) \wedge \omega_2) + (-1)^k d(\omega_1 \wedge i_X\omega_2) = (i_X d\omega_1) \wedge \omega_2 + (-1)^{k+1}(d\omega_1) \wedge i_X\omega_2 + (-1)^k(i_X\omega_1) \wedge d\omega_2 + (-1)^{2k}\omega_1 \wedge i_X d\omega_2 + (d i_X\omega_1) \wedge \omega_2 + (-1)^{k-1}(i_X\omega_1) \wedge d\omega_2 + (-1)^k(d\omega_1) \wedge i_X\omega_2 + (-1)^{2k}\omega_1 \wedge d i_X\omega_2 = (i_X d\omega_1) \wedge \omega_2 + \omega_1 \wedge i_X d\omega_2 + (d i_X\omega_1) \wedge \omega_2 + \omega_1 \wedge d i_X\omega_2 = (i_X d\omega_1 + d i_X\omega_1) \wedge \omega_2 + \omega_1 \wedge (i_X d\omega_2 + d i_X\omega_2) = D_2(\omega_1) \wedge \omega_2 + \omega_1 \wedge D_2(\omega_2)$. Hence D_2 is also a derivation of $(\Omega(G), \wedge)$.

Finally, we prove the Cartan magic formula in the general case. Let $k \in \mathbb{N}^*$ and $\omega \in \Omega^k(G)$. Let $(U, x_1, \ldots, x_n)$ a chart of G. There is $(\omega_I)_I \in \mathcal{C}^\infty(G)^{\mathcal{I}_k}$ such that $\omega|_U = \sum_{I \in \mathcal{I}_k} \omega_I dx^I$. The maps D_1 and D_2 are naturally defined on $\Omega(U)$ too, and we still denote them D_1 and D_2 : for all $j \in \{1, 2\}$, $D_j(\omega|_U) = D_j(\omega)|_U$ and the steps 1, 2 and 3 of our proof are still true for D_j on $\Omega(U)$. We have $D_1(\omega)|_U = \sum_{I \in \mathcal{I}_k} D_1(\omega_I)dx^I + \omega_I D_1(dx^I)$. By the first step, for all $I \in \mathcal{I}_k$, $D_1(\omega_I) = D_2(\omega_I)$. In addition to that, for all $i \in [\![1, n]\!]$ we have $D_1(x_i) = D_2(x_i)$, hence $dD_1(x_i) = dD_2(x_i)$ and by the second step $D_1(dx_i) = D_2(dx_i)$. Hence, because D_1 and D_2 are derivations, for all $I \in \mathcal{I}_k$ we have $D_1(dx^I) = D_2(dx^I)$. Hence $D_1(\omega)|_U = \sum_{I \in \mathcal{I}_k} D_2(\omega_I)dx^I + \omega_I D_2(dx^I) = D_2(\omega)|_U$. We deduce that $D_1 = D_2$, i.e. the Cartan magic formula. □

◇ Remark. This proof is an example of a classical method to prove the equality of two maps defined on differential forms. If we notice that these maps are derivations commuting with the exterior derivative, it is enough to show that they are equals on 0-forms (i.e. on smooth functions) to show they are equals.

4 Coadjoint orbits

Let G be a Lie group with an identity e and $\mathfrak{g}$ its Lie algebra with $[\cdot,\cdot]$ the associated Lie bracket.

4.1 Lie group actions

In this subsection we define a group action in the particular case of a Lie group and see some of the geometric properties that come from it. This subsection is mainly based on [3, 8, 10, 16].

4.1.1 Definition of a Lie group action on a manifold

We start to give many definitions, but a lot of them are from the group action theory seen without any Lie groups.

Definition 4.1.

- A *left action* of a Lie group G on a manifold M is a group homomorphism $\psi : G \to \mathrm{Diff}(M)$. We denote it by $G \curvearrowright_\psi M$ and for all $(g,x) \in G \times M$ we denote $\psi(g)(x)$ by $\psi_g(x)$ or $g \cdot x$.
- A *right action* of a Lie group G on a manifold M is a map $\psi : G \to \mathrm{Diff}(M)$ such that for all $(g_1, g_2) \in G^2$, $\psi_{g_1 g_2} = \psi_{g_2} \circ \psi_{g_1}$. We denote it by $G \curvearrowleft_\psi M$ and for all $(g,x) \in G \times M$ we denote $\psi(g)(x)$ by $\psi_g(x)$ or $x \cdot g$.
- Let ψ be an action from a Lie group G on a manifold M. The action is *transitive* (or M is *homogeneous*) if for all $x, y \in M$ there is $g \in G$ such that $g \cdot x = y$. The action is *smooth* if

$$\mathrm{ev}_\psi : \left| \begin{array}{ccc} G \times M & \longrightarrow & M \\ (g,x) & \longmapsto & g \cdot x \end{array} \right.$$

  is smooth. The action is *free* if for all $(g,x) \in G \times M$, $g \cdot x = x \Rightarrow g = e$. The action is *proper* if the inverse images of compacts by the map $(g,x) \in G \times M \mapsto (g \cdot x, g) \in G \times G$ are compacts.
- A *representation* of a $\mathbb{K}$-Lie group G is a $\mathbb{K}$-vector space V associated to a group morphism $\psi : G \to \mathrm{GL}(V)$.

▷ Examples. (of actions and morphisms)

- The map $\theta \in \mathbb{R} \mapsto 2\theta \in \mathbb{R}$ is smooth, hence $z \in \mathbb{S}^1 \mapsto z^2 \in \mathbb{S}^1$ is a Lie group morphism. The determinant $\det : \mathrm{GL}_n(\mathbb{K}) \to \mathbb{K}^*$ is a Lie group morphism.
- With $\mathbb{S}^1 := \left\{ z \in \mathbb{C} \;\middle|\; |z| = 1 \right\}$ the unitary circle, the scalar multiplication $\mathrm{ev} : (\lambda, z) \in \mathbb{S}^1 \times \mathbb{C}^n \mapsto \lambda z \in \mathbb{C}^n$ defines an action $\mathbb{S}^1 \curvearrowright \mathbb{C}^n$.
- Given a Lie group G, it acts on itself on the left by conjugation $\mathrm{ev} : (g,h) \in G \times G \mapsto ghg^{-1} \in G$. If for all $g \in G$ we denote $L_g : h \in G \mapsto gh \in G$ and $R_g : h \in G \mapsto hg \in G$, we have $G \curvearrowright_L G$ and $G \curvearrowleft_R G$.
- See Lemma 4.7 for another interesting example of a Lie group action.

◇ Notations. Given an action of a Lie group G on a manifold M, we talk about the orbits and stabilizers of the action of the group G on the set M. For all $x \in M$ we denote $\mathcal{O}_x := \{g \cdot x \;;\; g \in G\}$ its orbit, and $G_x := \left\{ g \in G \;\middle|\; g \cdot x = x \right\}$ its stabilizer.

◇ Reminder. Let an action of a group G on a set X. Let $x \in X$ and the action $G \curvearrowleft G_x$ given by for all $(h,g) \in G_x \times G$, $h \cdot g = R_h(g) = gh$. The map $g \in G \mapsto g.x \in \mathcal{O}_x$ induces a bijection $\varphi : G/G_x \to \mathcal{O}_x$ such that for all $g \in G$, $\varphi(gG_x) = g \cdot x$.

Lemma 4.2. Let M and N two manifolds, a transitive smooth action ψ^M from G on M and a smooth action ψ^N from G on N. Let $f : M \to N$ a smooth map such that for all $g \in G$, $f \circ \psi^M_g = \psi^N_g \circ f$. The map f has a constant rank, i.e. for all $x_1, x_2 \in M$ we have $df_{x_1} = df_{x_2}$.

Proof Let $x_1, x_2 \in M$. There is $g \in G$ such that $g \cdot x_1 = x_2$. We have $d(\psi^N_g)_{f(x_1)} \circ df_{x_1} = df_{x_2} \circ d\left(\psi^M_g\right)_{x_1}$. But $d(\psi^N_g)_{f(x_1)}$ and $d\left(\psi^M_g\right)_{x_1}$ are bijective, so $df_{x_1} = \mathrm{rg}\, df_{x_2}$. □

◇ Remark. In Subsubsection 4.1.2, we use this lemma together with a classical result of differential geometry that we do not prove in this report : Proposition 5.17 at page 111 of [10].

4.1.2 Basic geometry about Lie group actions

In this subsubsection we see some basic properties about the geometry of smooth Lie group actions. First, Theorem 4.3 gives us a manifold structure on the quotient space. In addition to that, Propositions 4.8 give us a manifold structure on each orbit and a link between an orbit and the quotient space associated to a stabilizer.

Let M be a manifold of unique dimension and a smooth action of G on M.

Theorem 4.3. Let M be a manifold of unique dimension and a smooth, free and proper action from the Lie group G on M. The quotient M/G has a unique manifold structure such that the canonical surjection $\pi : x \in M \mapsto G \cdot x \in M/G$ is a smooth submersion. The quotient space M/G is then a manifold of dimension $\dim M - \dim G$.

Proof See Theorem 7.10 page 153 in [10]. □

◇ Remark. From now on, if we have a smooth, free and proper action from a Lie group on a manifold of unique dimension, we equip the quotient space with the manifold structure given in Theorem 4.3.

As always with quotient structures, we now see a theorem that allows us to "pass a map to the quotient".

Theorem 4.4. Let M_1, M_2, N smooth manifolds, $\pi : M_1 \to M_2$ a surjective submersion, and $f : M_1 \to N$ such that for all $x_1, x_2 \in M$, $\pi(x_1) = \pi(x_2) \Rightarrow f(x_1) = f(x_2)$. There is a unique smooth map $\tilde{f} : M_2 \to N$ such that $\tilde{f} \circ \pi = f$.

Proof It is Proposition 5.20 page 112 of [10]. □

Hence theorems 4.3 and 4.4 gives us

Corollary 4.5. Let M be a manifold of unique dimension and a smooth, free and proper action from the Lie group G on M. Let $\pi : x \in M \mapsto G \cdot x \in M/G$ be the canonical surjection. Let N a manifold and $f : M \to N$ a smooth map such that for all $x_1, x_2 \in M$, $\pi(x_1) = \pi(x_2) \Rightarrow f(x_1) = f(x_2)$. There is a unique smooth map $\tilde{f} : M/G \to N$ such that $\tilde{f} \circ \pi = f$.

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\pi"'] & N \\ M/G \arrow[ur, "\exists! \tilde{f}"'] & \end{tikzcd}$$

Proposition 4.6. Let H be a closed subgroup of G and $G \curvearrowleft H$ the action of H on G by right translation : for all $(h, g) \in H \times G$, $h \cdot g = R_h(g) = gh$. This action is smooth, free and proper.

Proof By definition of a Lie group, $(h, g) \in G \times G \mapsto gh \in G$ is smooth. Since H is an emmeded submanifold of G, $(h, g) \in H \times G \mapsto gh \in G$ is smooth, i.e. $G \curvearrowleft H$ is smooth. For all $(h, g) \in H \times G$, $gh = g \Rightarrow h = e$, hence $G \curvearrowleft H$ is free. Let $m : (h, g) \in H \times G \mapsto (gh, g) \in G \times G$ and $K \subset G \times G$ a compact subset. Since a manifold is in particular a separated topological space, we can use the sequential characterisation of compactness. Let $(h_i, g_i)_i \in (H \times G)^{\mathbb{N}}$ such that for all $i \in \mathbb{N}$,

$(g_ih_i, g_i) \in K$. There is $\varphi : \mathbb{N} \to \mathbb{N}$ a strictly increasing map such that $(g_{\varphi(i)}h_{\varphi(i)})_i$ and $(g_{\varphi(i)})_i$ are convergent in G. By continuity, $(h_{\varphi(i)})_i = \left(g_{\varphi(i)}h_{\varphi(i)}h^{-1}_{\varphi(i)}\right)_i$ is convergent in G, and since H is closed, is convergent in H. Hence $(h_{\varphi(i)}, g_{\varphi(i)})_i$ converges in $H \times G$. □

◇ Remark. Given H a closed subgroup of G, we now have a Lie group structure on H (as an embedded submanifold) and a manifold structure on G/H such that $\dim G/H = \dim G - \dim H$.

Lemma 4.7. *Let H a closed subgroup on G and the left action from G on G/H given by for all $g_1, g_2 \in G$, $g_1 \cdot (g_2H) = (g_1g_2)H$. This is a smooth transitive action from the Lie group G on the manifold G/H.*

Proof We already know that it is transitive. We want to show that it is a smooth Lie group action. Let $\pi : g \in G \mapsto gH \in H$, $\tilde{\pi} : (g_1, g_2) \in G^2 \mapsto (g_1, \pi(g_2) \in G \times G/H$ and $f : (g_1, g_2) \in G^2 \mapsto (g_1, \pi(g_1g_2)) \in G \times G \times G/H$. Both Id_G and π are surjective submersions (by Theorem 4.3), hence $\tilde{\pi}$ is a surjective submersion. The multiplication and π are smooth, hence f is smooth. In addition to that, for all $(g_1, g_2), (g_1', g_2') \in G^2$ such that $\tilde{\pi}(g_1, g_2) = \tilde{\pi}(g_1', g_2')$, $g_1 = g_1'$ and $g_2H = g_2'H$, hence $g_1 = g_1'$ and $g_1g_2H = g_1'g_2'H$, i.e. $f(g_1, g_2) = f(g_1', g_2')$. By Theorem 4.4, there is a unique $\tilde{f} : G \times G/H \to G \times G/H$ such that $\tilde{f} \circ \tilde{\pi} = f$, i.e. for all $g_1, g_2 \in G$, $\tilde{f}(g_1, g_2H) = (g_1, g_1g_2H)$. In particular, the unique application $L : G \times G/H \to G/H$ such that for all $g_1, g_2 \in G \times G/H$ $L(g_1, g_2H) = g_1g_2$ is smooth. This means that the action $G \curvearrowright G/H$ given by L is smooth. This proof is inspired by Proposition 5.21 page 113 in [10].

In addition to that, for all $g \in G$, $L(g, \cdot)$ is smooth. But for all $g \in G$, $L(g, \cdot) \circ L(g^{-1}, \cdot) = L(g^{-1}, \cdot) \circ L(g, \cdot) = \mathrm{Id}_{G/H}$, hence $L(g, \cdot) \in \mathrm{Diff}(G)$. We decuce that L defines a Lie group action from G on G/H. □

Propositions 4.8. *Let $x \in M$.*

1. *The stabilizer G_x is closed in G.*
2. *The orbit $\mathcal{O}_x$ has the structure of an immersed submanifold of M and the unique map $\varphi_x : G/G_x \to \mathcal{O}_x$ such that for all $g \in G$, $\varphi_x(gG_x) = g \cdot x$ is a diffeomorphism.*

Proof We denote by ψ the smooth action from G on M. Let $\mathrm{ev}_x : g \in G \mapsto g \cdot x \in M$.

1. Since the action is smooth, ev_x is smooth. We have $G_x = \mathrm{ev}_x^{-1}(\{x\})$, hence G_x is closed in G. We conclude by Theorem 3.3.
2. By Theorem 4.3, the quotient space G/G_x is a smooth manifold. There is $\overline{L}$ an action $G \curvearrowright G/G_x$ given by for all $g_1, g_2 \in G$, $g_1 \cdot g_2G_x = (g_1g_2)G_x$. By Lemma 4.7, it is transitive and smooth. For all $g_1, g_2 \in G$, $\pi(g_1) = \pi(g_2) \Rightarrow g_1g_2^{-1} \in G_x \Rightarrow (g_1g_2^{-1}) \cdot x = x \Rightarrow \mathrm{ev}_x(g_1) = \mathrm{ev}_x(g_2)$. Passing smoothly to the quotient, there is a unique smooth map $\varphi_x : G/G_x \to M$ such that $\mathrm{ev}_x = \varphi_x \circ \pi$. We know that φ_x is injective and that its image is the orbit $\mathcal{O}_x$. For all $g_1, g_2 \in G$, $g_1 \cdot \varphi_x(g_2G_x) = g_1 \cdot (g_2 \cdot x) = (g_1 \cdot g_2) \cdot x = \varphi_x((g_1g_2)G_x) = \varphi_x(g_1 \cdot g_2G_x)$. Hence for all $g \in G$ we have $\psi_g \circ \varphi_x = \varphi_x \circ \overline{L}_g$. By Lemma 4.2, φ_x has a constant rank.

   To resume, φ_x is an injection with a constant rank. By Proposition 5.17 at page 111 of [10], φ_x is an injective immersion and $\mathcal{O}_x$ is an immersed submanifold of M. We now have a smooth bijective map $\varphi_x : G/G_x \to \mathcal{O}_x$ of constant rank between two manifolds. By Proposition 6.5.(b) page 132 of [10], it is a diffeomorphism.

□

◇ Remark. Let $x \in M$. The diffeomorphism $G/G_x \simeq \mathcal{O}_x$ allows us to "replace" $\mathcal{O}_x$ by G/G_x in some results or proofs : see Lemma 4.25 for an example.

◇ Notation. For all $x \in M$, we denote the Lie algebra of the stabilizer G_x by $\mathfrak{g}_x$.

◇ Remark. Let $x \in M$. Since we do the notation abuse $\mathfrak{g}_x \subset \mathfrak{g}$, we can consider the quotient space $\mathfrak{g}/\mathfrak{g}_x = \{u + \mathfrak{g}_x \; ; \; u \in \mathfrak{g}\}$ equiped with its canonical linear space structure. We remind that $\dim \mathfrak{g}/\mathfrak{g}_x = \dim \mathfrak{g} - \dim \mathfrak{g}_x$.

Corollary 4.9. Let $x \in M$ and $\pi : g \in G \mapsto G/G_x \in gG_x$.

1. The manifold $\mathcal{O}_x$ has a unique dimension and $\dim \mathcal{O}_x = \dim G - \dim G_x$.
2. For all $y \in \mathcal{O}_x$, $\dim \mathfrak{g}_x = \dim \mathfrak{g}_y$.
3. We have $\operatorname{Ker} d\pi_e = \mathfrak{g}_x$.

Proof 1. Let φ_x the diffeomorphism saw in Proposition 4.8. The map $d(\varphi_x)_{\varphi_x^{-1}(y)} : T_{\varphi_x^{-1}(y)}G/G_x \to T_y\mathcal{O}_x$ is an isomorphism, hence $\dim T_y\mathcal{O}_x = \dim G/G_x$.

2. Let $y \in \mathcal{O}_x$. We have $\mathcal{O}_x = \mathcal{O}_y$ hence, by the first item of this corollary, $\dim \mathfrak{g}_x = \dim \mathfrak{g} - \dim \mathcal{O}_x = \dim g_y$.

3. Because π is a submersion, $d\pi_e$ is surjective and by the rank formula $\dim \operatorname{Ker} d\pi_e + \dim T_{G/G_x} = \dim \mathfrak{g}$. Hence $\dim \operatorname{Ker} d\pi_e = \dim G_x = \dim \mathfrak{g}_x$. Let $X_e \in \mathfrak{g}_x$. By Proposition 3.30, for all $t \in \mathbb{R}$, $\exp(tX_e) \in G_x$. Hence $d\pi_e(X_e) = \frac{d}{dt}\left(\pi(\exp(tX_e))\right|_{t=0} = \frac{d}{dt}\left(G_x\right)\Big|_{t=0} = 0$, hence $X_e \in \operatorname{Ker} d\pi_e$. By dimension equality, $\operatorname{Ker} \pi_e = \mathfrak{g}_x$.

□

◇ Remarks.
- Let $x \in M$ and $\pi : G \to G/G_x$ the canonical surjection. By Corollary 4.9, $d\pi_e$ induces a isomorphism $\mathfrak{g}/\mathfrak{g}_x \simeq T_{G_x}G/G_x$.
- In this subsubsection we have seen some interesting properties about the geometry of orbits. See Subsubsection 4.2.2 for an example of special orbits with even more geometric properties.

Another interesting thing we can notice about the geometry of Lie group actions is the possibility to associate to each vector of the Lie algebra $\mathfrak{g}$ a vector field on the manifold on which the Lie group is smoothly acting.

Definition 4.10. Let M be a manifold, a smooth action from G on M and $\mathrm{ev} : (g, x) \in G \times M \mapsto g \cdot x \in M$. For all $X_e \in \mathfrak{g}$, the *associated vector field* to X_e on M is

$$X : \left| \begin{array}{ccc} M & \longrightarrow & TM \\ x & \longmapsto & d(\mathrm{ev}(\cdot, x))_e(X_e) \end{array} \right. .$$

◇ Remarks.
- In this definition the action is smooth : hence the associated vector field to an element of $\mathfrak{g}$ is a vector field on the manifold.
- This point of view is useful in the proof of Proposition 4.12.

▷ Example. Let $X_e \in \mathfrak{g}$. For the action of G on itself by left translation (resp. right translation), the associated vector field to X_e on G is the left invariant (resp. right invariant) vector field associated to X_e through Proposition 3.18.

Lemma 4.11. Suppose that the action of G on M is smooth and left. Let $X_e \in \mathfrak{g}$ and $X \in \mathfrak{X}(M)$ the associated vector field to X_e on M. The vector field X is complete and, for all $x \in M$, $\gamma : t \in \mathbb{R} \mapsto \exp(tX_e) \cdot x \in M$ is the only integral curve of X through x on $\mathbb{R}$. In particular, for all $x \in M$,

$$X(x) = \frac{d}{dt}\left(\exp(tX_e) \cdot x\right)\Big|_{t=0}.$$

Proof For all $y \in$ M we denote $\mathrm{ev}_y : g \in G \mapsto g \cdot y \in M$. Let $x \in M$. By definition of a left group action, for all $g_1, g_2 \in G$ and $y \in M$, $g_1 \cdot (g_2 \cdot y) = (g_1 g_2) \cdot y$, i.e. $\mathrm{ev}_{g_2 \cdot y}(g_1) = \mathrm{ev}_y(g_1 g_2)$, i.e. $\mathrm{ev}_{\mathrm{ev}_y(g_2)}(g_1) = \mathrm{ev}_y(g_1 g_2)$.

Let $\gamma : t \in \mathbb{R} \mapsto \exp(tX_e) \cdot x \in M$. We have $\gamma(0) = x$ and for all $t \in \mathbb{R}$, $\gamma'(t) = \frac{d}{ds}\left(\exp(sX_e) \cdot x\right)\Big|_{s=t} = d(\mathrm{ev}_x)_{\exp(tX_e)}(d(L_{\exp(tX_e)})_e(X_e)) = d(\mathrm{ev}_x \circ L_{\exp(tX_e)})_e(X_e) = d(\mathrm{ev}_{\mathrm{ev}_x(\exp(tX_e))})_e(X_e) = X(\mathrm{ev}_x \exp(tX_e)) = X(\gamma(t))$. Hence γ is an integral curve of X through x and by the Cauchy-Lipschitz theorem it is the only one (we already saw this kind of argument in Subsubsection 3.4).

In particular, $X(x) = X(\gamma(0)) = \gamma'(0) = \frac{d}{dt}\left(\exp(tX_e) \cdot x\right)\Big|_{t=0}$. □

Proposition 4.12. Suppose that the action of G on M is smooth and left. Let $x \in M$ and $\mathrm{ev} : (g, y) \in G \times M \mapsto g \cdot y \in M$. The Lie algebra of the stabilizer G_x is

$$\mathfrak{g}_x = \left\{ X_e \in \mathfrak{g} \;\middle|\; d(\mathrm{ev}_x)_e(X_e) = 0 \right\}.$$

Proof Let $X_e \in \mathfrak{g}_x$, i.e. $X_e \in \mathfrak{g}$ and for all $t \in \mathbb{R}$ we have $\exp(tX_e) \in G_x$. Hence $d(\mathrm{ev}_x)_e(X_e) = \frac{d}{dt}\left(\mathrm{ev}_x(\exp(tX_e))\right)\big|_{t=0} = \frac{d}{dt}(x)\big|_{t=0} = 0$.

Reciprocally, let $X_e \in \mathfrak{g}$ such that $d(\mathrm{ev}_x)_e(X_e) = 0$. Let $X \in \mathfrak{X}(M)$ be the associated vector field to X_e on M. We have $X(x) = 0$. Let $\gamma : t \in \mathbb{R} \mapsto x \in M$. It is a smooth map and for all $t \in \mathbb{R}$, $\gamma'(t) = 0 = X(x) = X(\gamma(t))$. Hence γ is an integral curve of X through x. By Lemma 4.11, for all $t \in \mathbb{R}$, $\gamma(t) = \exp(tX_e) \cdot x$, i.e. $\exp(tX_e) \cdot x = x$, i.e. $\exp(tX_e) \in G_x$. Hence $X_e \in \mathfrak{g}_x$. $\square$

4.2 Adjoint and coadjoint orbits

Here we see two important examples of Lie group actions (the adjoint and coadjoint actions) and we study their orbits. In particular, we prove Theorem 4.26, which is the main result of this report. This subsection is mainly based on [11, 12, 14, 16] for the first subsubsection and on [1, 6, 7, 9] for the second subsubsection.

4.2.1 Definition of the adjoint and coadjoint orbits

◇ Notations. For all $\mathfrak{g} \in G$, we define $\psi_g : h \in G \mapsto ghg^{-1} \in G$ the conjugation and $\mathrm{Ad}_g := d(\psi_g)_e : T_eG \to T_eG$ its derivative at the identity.

The natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$ is $\langle \cdot, \cdot \rangle$: for all $(\xi, X) \in \mathfrak{g}^* \times \mathfrak{g}$, $\langle \xi, X \rangle = \xi(X)$. For all $g \in G$ we define $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ by : for all $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$,

$$\left\langle \mathrm{Ad}^*_g \xi, X \right\rangle = \left\langle \xi, \mathrm{Ad}_{g^{-1}} X \right\rangle.$$

◇ Remark. We notice a similarity between the definition of Ad^* and the definition of the Hermitian adjoint of a linear operator. However, we do not define Ad by $\left\langle \mathrm{Ad}^*_g \xi, X \right\rangle = \langle \xi, \mathrm{Ad}_g X \rangle$ in order to have the property below (Ad is a group morphism).

Proposition 4.13. The maps

$$\mathrm{Ad} : \left| \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(\mathfrak{g}) \\ g & \longmapsto & \mathrm{Ad}_g \end{array} \right. \quad \text{and} \quad \mathrm{Ad}^* : \left| \begin{array}{ccc} G & \longrightarrow & \mathrm{GL}(\mathfrak{g}^*) \\ g & \longmapsto & \mathrm{Ad}^*_g \end{array} \right.$$

are representations of G (on repectively $\mathfrak{g}$ and $\mathfrak{g}^*$).

Proof We have $\mathrm{Ad}_e = d(\mathrm{Id}_G)_e = \mathrm{Id}_{\mathfrak{g}}$. Plus, for all $g_1, g_2 \in G$, $\mathrm{Ad}_{g_1g_2} = d(\psi_{g_1g_2})_e = d(\psi_{g_1} \circ \psi_{g_2})_e = d(\psi_{g_1})_e \circ d(\psi_{g_2})_e = \mathrm{Ad}_{g_1} \mathrm{Ad}_{g_2}$. Hence Ad is well defined and is a group morphism.

We have $\mathrm{Ad}_e = \mathrm{Id}_{\mathfrak{g}}$ hence $\mathrm{Ad}^*_e = \mathrm{Id}_{\mathfrak{g}^*}$. Let $g_1, g_2 \in G$, $\xi \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$. We have $\left\langle \mathrm{Ad}^*_{g_1g_2} \xi, X \right\rangle = \left\langle \xi, \mathrm{Ad}_{g_2^{-1}g_1^{-1}} X \right\rangle = \left\langle \xi, \mathrm{Ad}_{g_2^{-1}} \mathrm{Ad}_{g_1^{-1}} X \right\rangle = \left\langle \mathrm{Ad}^*_{g_1} \mathrm{Ad}^*_{g_2} \xi, X \right\rangle$. Hence Ad^* is well defined and is a group morphism. $\square$

Definitions 4.14.
- The *adjoint representation* of G is $\mathrm{Ad} : g \in G \mapsto \mathrm{Ad}_g \in \mathrm{GL}(\mathfrak{g})$. The *coadjoint representation* of G is $\mathrm{Ad}^* : g \in G \mapsto \mathrm{Ad}^*_g \in \mathrm{GL}(\mathfrak{g}^*)$.
- An *adjoint orbit* (resp. *coadjoint orbit*) of G is an orbit of the adjoint representation (resp. coadjoint representation) of G.

▷ Example. (of adjoint and coadjoint representations of a Lie group)

Here we suppose $G = \mathrm{GL}_n(\mathbb{K})$. We have $\mathfrak{g} = \mathrm{M}_n(\mathbb{K})$ and for all $A \in G$, $d(\psi_A)_{\mathrm{I}_n} : B \in \mathrm{M}_n(\mathbb{K}) \mapsto ABA^{-1} \in \mathrm{M}_n(\mathbb{K})$. Hence for all $(A, B) \in \mathrm{GL}_n(\mathbb{K}) \times \mathrm{M}_n(\mathbb{K})$, $\mathrm{Ad}_A(B) = ABA^{-1}$. See Subsection 4.3 for more examples.

◇ Remark. By propositions 3.27, for all $g \in G$ and $X_e \in \mathfrak{g}$, $\mathrm{Ad}_g(X_e) = \frac{d}{dt}\left(g \exp(tX_e) g^{-1}\right)\big|_{t=0}$.

We now define similar representations for the Lie algebra $\mathfrak{g}$. Since $\dim \mathfrak{g} < +\infty$, by Proposition 3.21 the Lie algebra of $\mathrm{GL}(\mathfrak{g})$ is $\mathcal{L}(\mathfrak{g})$.

◇ Notation. We denote the derivative of Ad at the identity e by $\mathrm{ad} := d(\mathrm{Ad})_e : \mathfrak{g} \to \mathcal{L}(g)$ and for all $X \in \mathfrak{g}$, $\mathrm{ad}_X := \mathrm{ad}(X)$. We denote the derivative of Ad^* at the identity e by $\mathrm{ad}^* := d(\mathrm{Ad}^*)_e : \mathfrak{g} \to \mathcal{L}(g^*)$ and for all $X \in \mathfrak{g}$, $\mathrm{ad}^*_X := \mathrm{ad}^*(X)$.

Propositions 4.15. 1. We have

$$\mathrm{ad} : \left| \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathcal{L}(\mathfrak{g}) \\ X & \longmapsto & [X, \cdot] \end{array} \right.$$

and ad is a Lie algebra morphism.

2. The map $\mathrm{ad}^* : \mathfrak{g} \to \mathcal{L}(\mathfrak{g}^*)$ is a Lie algebra morphism and for all $\xi \in \mathfrak{g}^*$ and $X, Y \in \mathfrak{g}$,

$$\langle \mathrm{ad}^*_X \xi, Y \rangle = \langle \xi, -\mathrm{ad}_X Y \rangle .$$

Proof 1. Let $X_e, Y_e \in \mathfrak{g}$. We have $ad_{X_e}(Y_e) = d(\mathrm{Ad})_e(X_e)(Y_e) = \frac{d}{dt}\left(Ad_{\exp(tX_e)}\right)\big|_{t=0}(Y_e) = \frac{d}{dt}\left(Ad_{\exp(tX_e)}(Y_e)\right)\big|_{t=0} = \frac{d}{dt}\left(\frac{d}{ds}\left(\exp(tX_e)\exp(sY_e)\exp(-tX_e)\right)\big|_{s=0}\right)\big|_{t=0}$ i.e., by Proposition 3.29, $ad_{X_e}(Y_e) = [X_e, Y_e]$.

Let $X, Y \in \mathfrak{g}$. We have $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X = [X, [Y, \cdot]] - [Y, [X, \cdot]] = -\left[[\cdot, X], Y\right] - \left[[Y, \cdot], X\right]$ hence, by the Jacobi identity, $[\mathrm{ad}_X, \mathrm{ad}_Y] = \left[[X, Y], \cdot\right] = \mathrm{ad}_{[X,Y]}$.

2. Let $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. We have $\langle \mathrm{ad}^*_X \xi, Y \rangle = \left\langle \frac{d}{dt}\left(\mathrm{Ad}_{\exp(tX)}\right)_{t=0} \xi, Y \right\rangle = \frac{d}{dt}\left(\left\langle \mathrm{Ad}_{\exp(tX)} \xi, Y \right\rangle\right)_{t=0} = \frac{d}{dt}\left(\left\langle \xi, \mathrm{Ad}_{\exp(-tX)} Y \right\rangle\right)_{t=0} = \left\langle \xi, \frac{d}{dt}\left(\mathrm{Ad}_{\exp(-tX)}\right)_{t=0} Y \right\rangle = \langle \xi, \mathrm{ad}_X Y \rangle = \langle \xi, -\mathrm{ad}_X Y \rangle$.

□

◇ Remark. The second statement of Propositions 4.15 shows us that there is a link between ad and ad^* like the link between Ad and Ad^* : for all $X \in \mathfrak{g}$, $\mathrm{ad}^*_X = (\mathrm{ad}_{-X})^*$.

Definition 4.16. The map ad (resp. ad^*) is called *adjoint representation* (resp. *coadjoint representation*) of the Lie algebra $\mathfrak{g}$.

4.2.2 Geometry of the coadjoint orbits

This subsubsection is almost exclusively dedicated to the explanation of Theorem 4.26, which is a very intesting property about the geometry of coadjoint orbits : they carry a symplectic structure. This last notion has been introduced in Subsection 2.2. Let $\xi \in \mathfrak{g}^*$.

First, we define a 1-form on G using the Lie group structure of G and the natural pairing $\langle \cdot, \cdot \rangle$ between $\mathfrak{g}^*$ and $\mathfrak{g}$. By taking its derivative, we now have a 2-form on G. By applying pullbacks on this last 2-form, we obtain a 2-form on the coadjoint orbits associated to ξ. Finally, we prove that this 2-form is symplectic. Proving that it is non-denegerated is the hardest point : we use a classic result of this field, the KKS formula. This is the method used in [1], which is itself based on [9].

See appendix 7.1 for a graph summarising this proof.

◇ Notation. For all $g \in G$ we denote $L_g : g' \in G \mapsto gg' \in G$ and $R_g : g' \in G \mapsto g'g \in G$. We denote by G_ξ (resp. $\mathcal{O}_\xi$) the stabilizer (resp. orbit) of ξ for the coadjoint action of G and by π the canonical projection $G \to G/G_\xi$. We denote by $\mathfrak{g}_\xi$ the Lie algebra of the Lie group G_ξ. We denote by φ_ξ the unique map $\mathcal{O}_\xi \to G/G_\xi$ such that for all $g \in G$, $\varphi_\xi(g \cdot \xi) = gG_\xi$.

In this subsection we consider three Lie group actions :

$$\left\{ \begin{array}{lll} G \curvearrowright G : & \forall (g_1, g_2) \in G^2, & g_1 \cdot g_2 = L_{g_1}(g_2) = g_1 g_2 \\ G \curvearrowleft G_\xi : & \forall (g, h) \in G \times G_\xi, & g \cdot h = R_h(g) = gh \\ G \curvearrowright \mathcal{O}_\xi : & \forall (g, \nu) \in G \times \mathcal{O}_\xi, & g \cdot \nu = \mathrm{Ad}^*_g \nu \end{array} \right.$$

LEMMA 4.17. For all $\nu \in \mathfrak{g}^*$, $\mathfrak{g}_\nu = \left\{X_e \in \mathfrak{g} \;\middle|\; \mathrm{ad}^*_{X_e} \nu = 0\right\}$.

Proof Let $\nu \in \mathfrak{g}^*$. The adjoint representation of G is in particular a left action of G on a manifold. By Proposition 4.12, $g_\nu = \left\{X_e \in \mathfrak{g} \;\middle|\; d(\mathrm{Ad}^* \nu)_e(X_e)\right\}$. But for all $X_e \in \mathfrak{g}$, $d(\mathrm{Ad}^* \nu)_e(X_e) = d(\mathrm{Ad}^*)_e(X_e)\nu = \mathrm{ad}^*_{X_e} \nu$. □

PROPOSITION 4.18. For all $\nu \in \mathcal{O}_\xi$, the tangent space of the coadjoint orbit $\mathcal{O}_\xi$ at ν is

$$T_\nu \mathcal{O}_\xi = \left\{\mathrm{ad}^*_{X_e} \nu \; ; \; X_e \in \mathfrak{g}\right\}.$$

Proof Let $\nu \in \mathcal{O}_x$. We have $\mathcal{O}_\xi = \mathcal{O}_\nu$, hence $T_\nu \mathcal{O}_\xi = T_\nu \mathcal{O}_\nu$. Let $X_e \in \mathfrak{g}$. The map $\gamma : t \in \mathbb{R} \mapsto \mathrm{Ad}^*_{\exp(tX_e)} \nu \in \mathcal{O}_\nu$ is smooth and verifies $\gamma(0) = \nu$. Hence $\gamma'(0) \in T_\nu \mathcal{O}_\nu$. But $\gamma'(0) = d(\mathrm{Ad}^*)_e(X_e)\nu = \mathrm{ad}^*_{X_e} \nu$. Hence we have a map $u : X_e \in \mathfrak{g} \mapsto \mathrm{ad}^*_{X_e} \nu \in T_\nu \mathcal{O}_\nu$ and it is linear. Lemma 4.17 tells us that $\mathrm{Ker}\, u = \mathfrak{g}_\nu$ hence, by the rank formula, $\dim \mathrm{Im} u = \dim \mathfrak{g} - \dim \mathfrak{g}_\nu = \dim T_\nu \mathcal{O}_\nu$. Hence $\mathrm{Im} u = T\nu \mathcal{O}_\nu$, i.e. $T_\nu \mathcal{O}_\xi = \left\{\mathrm{ad}^*_{X_e} \nu \; ; \; X_e \in \mathfrak{g}\right\}$. □

DEFINITION 4.19. Let $\omega \in \Omega(G)$, M a manifold and an action of G on M.

- The differential form ω is *G-invariant* for this action if for all $g \in G$, $g^*\omega = \omega$.
- The differential form ω is *horizontal* for this action if for all $X \in \mathfrak{g}$, $i_X(\omega) = 0$.
- The differential form ω is *basic* for this action if it is both G-invariant and horizontal.

▷ EXAMPLE. See Lemma 4.25 for an important example of a basic 2-form.

THEOREM 4.20. Let M be a manifold, H a Lie group and an action of H on M which is free and proper. Let $p : M \to M/H$ be the quotient map of this action. The map

$$p^* : \left| \begin{array}{ccc} \Omega^k(M/H) & \longrightarrow & \left\{\omega \in \Omega^k(M) \;\middle|\; \omega \text{ basic}\right\} \\ \omega & \longmapsto & p^*\omega \end{array} \right.$$

is a bijection.

Proof See Theorem 1.1 page 1 in [7]. □

◇ REMARK. We will use Theorem 4.20 with the projection π, i.e. in the case "$M = G$" and "$H = G_\xi$" (we remind that, thanks to Proposition 4.8, G_ξ is a Lie subgroup of G).

LEMMA 4.21. Let $g \in G$ and $\overline{L_g}$ be the unique map $G/G_\xi \to G_\xi$ such that for all $h \in G$, $\overline{L_g}(hG_\xi) = (gh)G_\xi$. We have $\varphi_\xi \circ \psi_g = \overline{L_g} \circ \varphi_\xi$ and $\overline{L_g} \circ \pi = \pi \circ L_g$.

Proof Let $\nu \in \mathcal{O}_\xi$. There is $h \in G$ such that $\nu = h \cdot \xi$. We have $(\varphi_\xi \circ \psi_g)(\nu) = \varphi_\xi(g \cdot (h \cdot \xi)) = ghG_\xi = \overline{L_g}(hG_\xi) = \left(\overline{L_g} \circ \varphi_\xi\right)(h \cdot \xi) = \left(\overline{L_g} \circ \varphi_\xi\right)(\nu)$. In addition to that, for all $h \in G$ we have $\left(\overline{L_g} \circ \pi\right)(h) = \overline{L_g}(hG_\xi) = ghG_\xi = \pi(L_g(h))$. □

We now define the basic material needed to build our symplectic form on a coadjoint orbit.

DEFINITION 4.22. Let M be a manifold and $k \in \mathbb{N}^*$. A *$\mathfrak{g}$-valued k-differential form* on M is an element of the real vector space $\Omega^k(M) \otimes \mathfrak{g}$ denoted by $\Omega^k(M, \mathfrak{g})$.

◇ REMARK. Let $\theta \in \Omega^k(M, \mathfrak{g})$. Let $(u_1, \ldots, u_n)$ be a basis of $\mathfrak{g}$ and U a chart of M. There is $\omega_1, \ldots, \omega_n \in \Omega^k(M)$ such that $\theta = \sum_{i=1}^n \omega_i \otimes u_i$.

▷ EXAMPLE. See in Subsubsection 4.3.1 the $\mathfrak{su}_2$-valued 2-form denoted by θ.

Definition 4.23. The *Maurer-Cartan form* $\theta \in \Omega^1(G, \mathfrak{g})$ on G is

$$\theta : \left| \begin{array}{lcc} G & \longrightarrow & T^*G \otimes \mathfrak{g} \\ g & \longmapsto & v \in T_gG \mapsto d(L_{g^{-1}})_g(v) \in T_eG \end{array} \right.$$

and for all $g \in G$ we often denote $\theta_g := \theta(g)$.

◇ Remark. For all left invariant vector field $X \in \mathfrak{g}$ and $g \in G$, $\theta_g(X(g)) = d(L_{g^{-1}})(X(g)) = X(e)$.

▷ Example. We assume that $G = \mathrm{GL}_n(\mathbb{K})$. For all $A \in G$ and $M \in \mathfrak{g}$, $\theta_g(M) = A^{-1}M$

Lemma 4.24. 1. The Maurer-Cartan form θ on G is invariant for $G \curvearrowright G$.

2. Let $\alpha := -\langle \xi, \theta \rangle$, i.e. for all $g \in G$ and $v \in T_gG$, $\alpha_g(v) = \xi(\theta_g(v)) \in \mathbb{R}$. The map α is a 1-form on G invariant for $G \curvearrowright G$ and $G \curvearrowleft G_\xi$.

Proof 1. For all $g, h \in G$ and $v \in T_gG$, $(h^*\theta)_g(v) = d(L_{(hg)^{-1}})_{hg}(d(L_h)_g(v)) = d(L_{g^{-1}h^{-1}} \circ L_h)_g(v) = d(L_{g^{-1}})_g(v) = \theta_g(v)$.

2. Since θ is G-invariant, it is clear that it holds for α. Let $g \in G$, $h \in G_\xi$ and $v \in T_gG$. We have $(R_h^*\theta)_g(v) = d(L_{(gh)^{-1}})_{gh}(d(R_h)_g(v)) = d\left(L_{(gh)^{-1}} \circ R_h\right)_g(v) = d(\psi_h \circ L_{g^{-1}})_g(v) = d(\psi_h)_e\left(d(L_{g^{-1}})_g(v)\right) = \mathrm{Ad}_h(\theta_g(v))$, hence $(R_h^*\alpha)_g(v) = -\langle \xi, \mathrm{Ad}_h(\theta_g(v))\rangle = -\langle \mathrm{Ad}_h^* \xi, \theta_g(v)\rangle = -\langle \xi, \theta_g(v)\rangle = \alpha_g(v)$.

□

Lemma 4.25. Let $\alpha := -\langle \xi, \theta \rangle$, and $\tilde{\omega} := d\alpha \in \Omega(G)$.

1. The 2-form $\tilde{\omega}$ is basic for $G \curvearrowleft G_\xi$.

2. The 2-form $\omega := \varphi^*(\pi^*)^{-1}\tilde{\omega}$ on O_ξ verifies the Kirillov-Kostant-Souriau (KKS) formula : for all $\nu \in \mathcal{O}_\xi$ and $X_e, Y_e \in \mathfrak{g}$,

$$\omega_\nu(\mathrm{ad}_{X_e}^* \nu, \mathrm{ad}_{Y_e}^* \nu) = \langle \nu, [X_e, Y_e]\rangle$$

Proof 1. For all $f \in \mathcal{C}^\infty(G, G)$, $f^*(d\alpha) = d(f^*\alpha)$. In addition to that, α is G_ξ-invariant. Hence $d\alpha$ is G_ξ-invariant. We want to show that $\tilde{\omega}$ is horizontal. For all $X \in \mathfrak{g}_\xi$, by the Cartan magic formula, $i_X\tilde{\omega} = i_X(d\alpha) = (\mathcal{L}_X\alpha - d(i_X\alpha))$. Because α is invariant for $G \curvearrowleft G_\xi$, $\mathcal{L}_X(\alpha) = \frac{d}{dt}\left((R_{\exp(tX(e))})^*\alpha\right)\Big|_{t=0} = \frac{d}{dt}(\alpha)_{t=0} = 0$, hence $i_X(\tilde{\omega}) = -d(i_X\alpha)$. But $i_X\alpha \in \mathcal{C}^\infty(G)$ and for all $g \in G$, $i_X(\alpha)_g = \alpha_x(X(x)) = -\langle \xi, \theta_g(X(g))\rangle = -\langle \xi, X(e)\rangle$. Hence $i_X\alpha$ is constant and $d(i_X(\alpha)) = 0$, i.e. $i_X(\tilde{\omega}) = 0$. We deduce that $\tilde{\omega}$ is horizontal. To conclude, $\tilde{\omega}$ is G_ξ-basic.

2. Since $\tilde{\omega}$ is G_ξ-basic, we can define $(\pi^*)^{-1}\tilde{\omega} \in \Omega^2(G/G_\xi)$ by Theorem 4.20. Hence ω is well defined and $\omega \in \Omega^2(\mathcal{O}_\xi)$. In fact, this is the differential form we will use to prove Theorem 4.26.

First, we find a similar formula for the two form $\tilde{\omega}$ on G. Let $g \in G$ and $X_g, Y_g \in T_gG$. We will use right invariant vector fields and the remark following Proposition 3.18. Let $X_e := d(R_{g^{-1}})_g(X_g)$ (resp. $Y_e := d(R_{g^{-1}})_g(Y_g)$) and $X \in \mathfrak{X}(G)$ (resp. $Y \in \mathfrak{X}(G)$) the unique right invariant vector field such that $X(e) = X_e$ (resp. $Y(e) = Y_e$). We have $X : h \in G \mapsto d(R_h)_e(X_e) \in TG$ and $X(g) = d(R_g)_e(d(R_{g^{-1}})_g(X_g)) = d(\mathrm{Id}_G)_g(X_g) = X_g$. In the same way, $Y(g) = Y_g$. Hence $(\tilde{\omega})_g(X_g, Y_g) = (i_Y i_X\tilde{\omega})_g$. In addition to that, using Cartan magic formula, $i_Y i_X\tilde{\omega} = i_Y i_X d\alpha = i_Y(-\mathcal{L}_X\alpha + d(i_X\alpha))$. But, because α is invariant for $G \curvearrowright G$, $-\mathcal{L}_X\alpha = \frac{d}{dt}((L_{\exp(tX(e))})^*\alpha)\big|_{t=0} = \frac{d}{dt}(\alpha) = 0$, hence $i_Y i_X\tilde{\omega} = i_Y d(i_X\alpha)$. But for all $h \in G$, $(i_X\alpha)_h = -\langle \xi, \theta_h(X(h))\rangle = -\langle \xi, d(L_{h^{-1}})_g(d(R_h)_e(X_e))\rangle = -\langle \xi, \mathrm{Ad}_{h^{-1}}(X_e)\rangle = -\langle \mathrm{Ad}_h^* \xi, X_e\rangle$, hence $i_Y i_X\tilde{\omega} = -i_Y d(\langle \mathrm{Ad}^* \xi, X_e\rangle) = (\mathcal{L}_Y \langle \mathrm{Ad}^* \xi, X_e\rangle - d(i_Y \langle \mathrm{Ad}^* \xi, X_e\rangle))$.

Since $\langle \mathrm{Ad}^* \xi, X_e\rangle \in \Omega^0(G)$, $i_Y \langle \mathrm{Ad}^* \xi, X_e\rangle = 0$ and

$$\begin{aligned} i_Y i_X \tilde{\omega} &= \mathcal{L}_Y \langle \mathrm{Ad}^* \xi, X_e\rangle \\ &= \frac{d}{dt}\left((L_{\exp(tY_e)})^* \langle \mathrm{Ad}^* \xi, X_e\rangle\right)\big|_{t=0} \end{aligned}$$

with, for all $t \in \mathbb{R}$, $\left\langle \mathrm{Ad}^*_{L_{\exp(tY_e)}} \xi, X_e\right\rangle = \left\langle \mathrm{Ad}^*_{\exp(tY_e)} \mathrm{Ad}^* \xi, X_e\right\rangle = \left\langle \mathrm{Ad}^*_{\exp(tY_e)} \mathrm{Ad}^* \xi, X_e\right\rangle = \left\langle \mathrm{Ad}^* \xi, \mathrm{Ad}_{\exp(-tY_e)} X_e\right\rangle$. Hence $i_Y i_X \tilde{\omega} = \left\langle \mathrm{Ad}^* \xi, \frac{d}{dt}\left(\mathrm{Ad}_{\exp(-tY_e)} X_e\right)\big|_{t=0}\right\rangle = \langle \mathrm{Ad}^* \xi, -\mathrm{ad}_{Y_e}(X_e)\rangle = \langle \mathrm{Ad}^* \xi, -[Y_e, X_e]\rangle = \langle \mathrm{Ad}^* \xi, [X_e, Y_e]\rangle$. In particular, this formula is true in g, i.e. $d\tilde{\alpha}_g(X(g), Y(g)) = \left\langle \mathrm{Ad}^*_g \xi, [X_e, Y_e]\right\rangle$, i.e. $\tilde{\omega}_g(X_g, Y_g) = \left\langle \mathrm{Ad}^*_g \xi, [X_e, Y_e]\right\rangle$. We use this first result to prove the KKS formula.

Let $\nu \in \mathcal{O}_\xi$ and $X_e, Y_e \in \mathfrak{g}$. Let $X \in \mathfrak{X}(G)$ (resp. $Y \in \mathfrak{X}(G)$) the unique right-invariant vector field associated to X_e (resp. Y_e). There is $g \in G$ such that $\nu = \mathrm{Ad}^*_g \xi$. We have $\omega(\mathrm{ad}^*_{X_e} \nu, \mathrm{ad}^*_{Y_e}) = ((\pi^*)^{-1}\tilde{\omega})_{\varphi(\nu)}(d\varphi_\nu(\mathrm{ad}^*_{X_e} \nu), d\varphi_\nu(\mathrm{ad}^*_{Y_e} \nu))$, with $\varphi_\xi(\nu) = gG_\xi = \pi(g)$. We also have $d(\varphi_\xi)_\nu(\mathrm{ad}^*_{X_e} \nu) = d(\varphi_\xi)_\nu(d(\mathrm{Ad}^* \nu)_e(X_e)) = d(\varphi_\xi \circ (\mathrm{Ad}^* \nu))_e(X_e)$ with, for all $h \in G$, $(\varphi_\xi \circ (\mathrm{Ad}^*_\nu))(h) = \varphi_\xi((hg)\cdot \xi) = hgG_\xi = \pi(R_g(h))$. Hence $d(\varphi_\xi)_\nu(\mathrm{ad}^*_{X_e} \nu) = d(\pi \circ R_g)_e(X_e) = d\pi_g(d(R_g)_e(X_e)) = d\pi_g(X(g))$. Likewise, $d(\varphi_\xi)_\nu(\mathrm{ad}^*_{Y_e} \nu) = d\pi_g(Y(g))$, so $\omega(\mathrm{ad}^*_{X_e} \nu, \mathrm{ad}^*_{Y_e}) = ((\pi^*)^{-1}\tilde{\omega})_{\pi(g)}(d\pi_g(X(g)), d\pi_g(Y(g))) = (\pi^*(\pi^*)^{-1}\tilde{\omega})_g(X(g), Y(g)) = \tilde{\omega}_g(X(g), Y(g))$. By the formula we saw above in the second paragraph, we deduce $\omega(\mathrm{ad}^*_{X_e} \nu, \mathrm{ad}^*_{Y_e}) = \left\langle \mathrm{Ad}^*_g \xi, [X_e, Y_e]\right\rangle = \langle \nu, [X_e, Y_e]\rangle$. This is what we wanted to prove.

□

◇ Remarks.
- In this proof we did not use left-invariant vector fields, but right-invariant vector fields : they are the vector fields associated to the elements of the Lie algebra $\mathfrak{g}$ with the right translation action $G \curvearrowleft G$.
- By Proposition 4.18, the KKS formula gives a result on all the fiber bundle $T\mathcal{O}_\xi$.

Now we have the necessary tools to prove the important result of this subsubsection.

Theorem 4.26. For all coadjoint orbit $\mathcal{O}$ of G, there is a G-invariant symplectic form on $\mathcal{O}$.

Proof Let $\xi \in \mathfrak{g}^*$. We want to define a G-invariant symplectic form on $\mathcal{O}_\xi$. We consider the Maurer-Cartan form θ on G, $\alpha := -\langle \xi, \theta\rangle$ and $\tilde{\omega} := d\alpha \in \Omega^2(G)$. With φ_ξ the diffeomorphism $G/G_\xi \simeq \mathcal{O}_\xi$, we consider $\omega := \varphi_\xi^*(\pi^*)^{-1}\tilde{\omega} \in \Omega(\mathcal{O}_\xi)$ the 2-form we have seen in Lemma 4.25. We want to prove that ω is G-invariant, closed, and non-degenerate (this last point is the hardest one).

We start to prove that ω is G-invariant. We denote by ψ the coadjoint action of G. Let $g \in G$. Just like in Lemma 4.21, let $\overline{L_g}$ be the unique map $G/G_\xi \to G_\xi$ such that for all $h \in G$, $\overline{L_g}(hG_\xi) = (gh)G_\xi$. We want to show that $\psi_g^*\omega = \omega$. We have $\psi_g^*\omega = \psi_g^*\varphi_\xi^*(\pi^*)^{-1}d\alpha = (\varphi_\xi \circ \psi_g)^*(\pi^*)^{-1}d\alpha = \left(\overline{L_g} \circ \varphi_\xi\right)^* (\pi^*)^{-1} d\alpha = \varphi_\xi^*\overline{L_g}^*(\pi^*)^{-1}d\alpha$. But $\pi^*\overline{L_g}^*(\pi^*)^{-1} = \left(\overline{L_g} \circ \pi\right)^* (\pi^*)^{-1} = (\pi \circ L_g)^*(\pi^*)^{-1} = L_g^*\pi^*(\pi^*)^{-1} = L_g^*$ so $\overline{L_g}^*(\pi^*)^{-1} = (\pi^*)^{-1}L_g^*$. Hence $\psi_g^*\omega = \varphi_\xi^*(\pi^*)^{-1}L_g^*d\alpha = \varphi_\xi^*(\pi^*)^{-1}d(L_g^*\alpha)$ and, by Lemma 4.24, $\psi_g^*\omega = \varphi_\xi^*(\pi^*)^{-1}d\alpha = \omega$. We deduce that ω is G-invariant.

Now, we want to prove that ω is closed. We have $d\omega = d\left(\varphi_\xi^*(\pi^*)^{-1}d\alpha\right) = \varphi_\xi^* d\left((\pi^*)^{-1}d\alpha\right)$. But $\pi^*\left(d\left((\pi^*)^{-1}d\alpha\right)\right) = d\left(\pi^*(\pi^*)^{-1}d\alpha\right) = d(d\alpha) = 0$ so $d\left((\pi^*)^{-1}d\alpha\right) = 0$. Hence $d\omega = 0$.

Finally, we want to prove that ω is non degenerate using the KKS formula seen in Lemma 4.25. Let $\nu \in \mathcal{O}_\xi$ and $\tilde{X}_e \in T_\nu\mathcal{O}_\xi$ such that $\omega_\nu(\tilde{X}_e, \cdot) = 0$. By propisition 4.18, there is $X_e \in \mathfrak{g}$ such that $\tilde{X}_e = \mathrm{ad}^*_{X_e} \nu$. By the KKS formula, we have for all $Y_e \in \mathfrak{g}$, $\omega_\nu\left(\mathrm{ad}^*_{X_e} \nu, \mathrm{ad}^*_{Y_e} \nu\right) = \langle \nu, [X_e, Y_e]\rangle = \langle \nu, \mathrm{ad}_{X_e} Y_e\rangle = \left\langle \mathrm{ad}^*_{X_e} \nu, Y_e\right\rangle$, hence $\left\langle \mathrm{ad}^*_{X_e} \nu, Y_e\right\rangle = 0$, i.e. $\mathrm{ad}^*_{X_e} \nu = 0$, i.e. $\tilde{X}_e = 0$. We deduce that ω is non degenerate on $\mathcal{O}_\xi$. This concludes. □

◇ Remarks. Let α be the 1-form and ω be the 2-form defined in the precedent proof.

- The 2-form ω is often called *the Kirillov form*.
- We have seen that ω is closed. Yet, while it "comes from" the exact 2-form $d\alpha$, it is not necessarily exact.

- We can say that the 2-form ω is canonical : we only used the Lie group structure of G to define it.
- Why have we not just defined ω by the KKS formula seen in lemma 4.25 ? In the method we used, it is easy to prove that ω is well defined and closed, but hard to prove that it is non-degenerate (we used the KKS formula). With the method where one uses the KKS formula to define ω, it becomes easy to show that it is non-degenerate, but hard to prove that it is well defined and closed. Hence we can say that the main lemma in this subsubsection is lemma 4.25.

4.3 Two examples of coadjoint orbits

We have proved Theorem 4.26 and the fact that the coadjoint orbits of $\mathfrak{g}$ are symplectic manifolds. We give here two important examples of coadjoint orbits, following the second part of the course [1]. This subsection is also based on [5, 13, 15, 17].

4.3.1 The coadjoint orbits of $\mathbf{SU_2}$

First, we consider the case $G := \mathrm{SU}_2$, i.e.

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \; ; \; (\alpha, \beta) \in \mathbb{C}^2, |\alpha|^2 + |\beta|^2 = 1 \right\},$$

which is diffeomorphic to $\mathbb{S}^3$ the 3-sphere, i.e. the unitary ball of $\mathbb{C}^2$ with the 2-norm (this already shows us that $\dim \mathrm{SU}_2 = 3$). What do its coadjoint orbits look like ? What do its coadjoint stabilizers look like ? What is the symplectic form on its coadjoint orbits we defined in the proof of Theorem 4.26 ?

Definition 4.27. We denote

$$\mathbf{1} := \mathrm{I}_2, \mathbf{i} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \mathbf{j} := \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \text{ and } \mathbf{k} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We denote by $\mathbb{H} := \mathrm{Span}_{\mathbb{R}} \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} \subset \mathrm{M}_2(\mathbb{C})$ the set of the *quaternions*, equiped with the matrix addition $+$ and the matrix multiplication $\times$. For all $q \in \mathbb{H}$, we denote by $|q|^2 := \det q$ its *squared absolute value* and by $\bar{q}$ its transpose conjugate.

◇ Notation. Using the injection $\lambda \in \mathbb{R} \mapsto \lambda \mathbf{1} \in \mathbb{H}$, we denote $\mathbb{R} \subset \mathbb{H}$: for all $\lambda \in \mathbb{R}$, $i(\lambda) = \lambda$.

Lemma 4.28.
1. We have $i^2 = j^2 = k^2 = ijk = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$.
2. For all $q \in \mathbb{H}$, $q\bar{q} = |q|^2$.

Proof It is a direct calculus. □

In this subsubsection, we use some basic properties of the quaternions : see Section 1.3 of [15]. As a first remark, we can see that the group SU_2 it the unitary ball of the quaternions :

Lemma 4.29. We have

$$\mathrm{SU}_2 = \left\{ \cos\theta + (\sin\theta)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \; ; \; \theta \in [0, 2\pi[, (x, y, z) \in \mathbb{R}^3, ||(x, y, z)||_2^2 = 1 \right\}$$

and for all $q \in \mathrm{SU}_2$, $q^{-1} = \bar{q}$.

Proof For the first statement, see subsection 1.5 of [15]. For the second one, we use the fact that $q\bar{q} = |q|^2$. □

We denote by $\mathfrak{su}_2$ the Lie algebra of SU_2, and we start by looking for a convenient description of this Lie algebra.

Propositions 4.30. The Lie algebra of SU_2 is

$$\mathfrak{su}_2 = \left\{ \begin{pmatrix} ib & z \\ -\bar{z} & -ib \end{pmatrix} \; ; \; b \in \mathbb{R}, z \in \mathbb{C} \right\}$$

and $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is a real basis of $\mathfrak{su}_2$. In particular, $\dim \mathrm{SU}_2 = 3$.

Proof By Proposition 3.21, $\mathfrak{su}_2 = \left\{ X \in \mathrm{M}_2(\mathbb{C}) \mid X + X^* = 0, \operatorname{tr} X = 0 \right\}$. A quick calculus gives us the first result. The family $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is free and it clear that these three matrices are elements of $\mathfrak{su}_2$, hence it is a real basis of this Lie algebra. □

◇ Remarks.

- For all $A \in \mathfrak{su}_2 \setminus \{0\}$, $A \in \mathrm{GL}_2(\mathbb{C})$.
- As we saw in Proposition 3.22, if we consider $\mathfrak{su}_2$ as a subset of $\mathrm{M}_2(\mathbb{C})$ (as we did in Proposition 4.30) the Lie bracket is the matrices commutator. Hence, as the unitary ball of the quaternions, the Lie bracket is the communator of this non-commutative $\mathbb{R}$-algebra.

We consider the real linear space isomorphism

$$\varphi : \left| \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathfrak{su}_2 \\ (x, y, z) & \longmapsto & x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \end{array} \right. .$$

We notice that (U, φ^{-1}) is a chart of the manifold $\mathfrak{su}_2$. We denote by $\wedge$ the vector product on $\mathbb{R}^3$. We also denote $e_1 := (1,0,0)^T$, $e_2 := (0,1,0)^T$ and $e_3 := (0,0,1)^T$. With the identification $(\mathbb{R}^3)^* \simeq \mathbb{R}^3$, the natural pairing $\langle \cdot, \cdot \rangle$ between $(\mathbb{R}^3)^*$ and $\mathbb{R}^3$ is the scalar product on $\mathbb{R}^3$. Propositions 4.31 below allows us to identify $\mathfrak{su}_2$ and $\mathfrak{su}_2^*$ with $\mathbb{R}^3$.

Propositions 4.31.

1. For all $u, v \in \mathbb{R}^3$, $[\varphi(u), \varphi(v)] = 2u \wedge v$.
2. For all $u \in \mathbb{R}^3$, $||u||_2^2 = \det \varphi(u)$.
3. For all $u, v \in \mathbb{R}^3$, with $\varphi(u)^*$ the transpose conjugate of the matrix $\varphi(u)$, $\langle u, v \rangle = \frac{1}{2} \operatorname{tr}(\varphi(u)^* \varphi(v))$.
4. For all $u \in \mathbb{R}^3$, $\overline{\varphi(u)} = \varphi(-u)$.
5. For all $u, v \in \mathbb{R}^3$, $\varphi(u)\varphi(v) = -\langle u, v \rangle + u \wedge v$.

Proof It is a direct calculus. □

◇ Remark. Statements 2 and 3 of 4.31 tells us that $(A, B) \in \mathfrak{su}_2 \times \mathfrak{su}_2 \mapsto \frac{1}{2} \operatorname{tr} A^* B \in \mathbb{R}$ is well definded and is a scalar product with an accociated norm equal to the absolute value of the quaternions. With this scalar product, φ preserves the scalar product.

◇ Notation. For all $A, B \in \mathfrak{su}_2$, we denote $\langle A, B \rangle_m = \frac{1}{2} \operatorname{tr} A^* B$.

By the third statement of Propositions 4.31 and Riesz representation theorem, $R : A \in \mathfrak{su}_2 \mapsto \langle A, \cdot \rangle_m \in \mathfrak{su}_2^*$ is a vector space isomorphism.

◇ Notation. With the isomorphism R, we can make the notation abuse $\mathfrak{su}_2^* = \mathfrak{su}_2$: for all $A \in \mathfrak{su}_2$, $R(A) = A$.

At this point, since we know that the coadjoint orbits of SU_2 are even dimensional by Theorem 4.26, we know they are manifolds of dimension 0 (i.e. a point) or 2. What is the coadjoint represensation of SU_2 ?

Proposition 4.32. For all $q \in \mathrm{SU}_2$,

$$\mathrm{Ad}_q^* = \mathrm{Ad}_q : \left| \begin{array}{ccc} \mathfrak{su}_2 & \longrightarrow & \mathfrak{su}_2 \\ A & \longmapsto & qA\bar{q} \end{array} \right. .$$

Proof Let $q \in \mathrm{SU}_2$. We already saw in the example below definition 4.14 that for all $A \in \mathfrak{su}_2$, $\mathrm{Ad}_q A = qAq^{-1} = qA\bar{q}$. For all $A \in \mathrm{SU}_2$ and $B \in \mathfrak{su}_2^*$, $\left\langle \mathrm{Ad}_q^* B, A \right\rangle_m = \langle B, \mathrm{Ad}_{\bar{q}} A \rangle_m = \frac{1}{2} \operatorname{tr} B^* \bar{q} B q = \frac{1}{2} \operatorname{tr} q C^* \bar{q} B = \frac{1}{2} \operatorname{tr}(qC\bar{q})^* B = \langle qC\bar{q}, B \rangle_m$. Hence $\mathrm{Ad}_q^* = \mathrm{Ad}_q$. $\square$

◇ Remark. Proposition 4.32 is very easy to state thanks to the identifications we made earlier.

Lemma 4.33. Let $\theta \in [0, 2\pi[$ and $I \in \mathfrak{su}_2$ such that $|I|^2 = 1$. Let $q := \cos\theta + (\sin\theta)I$. The map $u \in \mathbb{R}^3 \mapsto \varphi^{-1}\big(q\varphi(u)\bar{q}\big) \in \mathbb{R}^3$ is the $\mathbb{R}^3$-rotation of axis $\varphi^{-1}(I)$ and angle -2θ.

Proof See subsection 1.5 of [15]. $\square$

Proposition 4.34. For all $B \in \mathfrak{su}_2$,

$$\mathcal{O}_B = \left\{ C \in \mathfrak{su}_2 \;\middle|\; \det C = \det B \right\}.$$

Proof Let $B \in \mathfrak{su}_2$. For all $q \in \mathrm{SU}_2$, $|qB\bar{q}|^2 = 1 \times (\det B) \times 1 = \det B$. Hence $\mathcal{O}_B \subset \left\{ C \in \mathfrak{su}_2 \;\middle|\; \det C = \det B \right\}$. Let $C \in \mathfrak{su}_2$ such that $\det C = \det B$. The vectors $u := \varphi^{-1}B$ and $v := \varphi^{-1}C$ are such that $||u||_2 = ||v||_2$. Hence there is $a \in \mathbb{R}^3$ and $\theta \in [0, 2\pi[$ such that $||a||_2 = 1$ and v is the rotation of u of axis a and angle θ. By lemmas 4.29 and 4.33, with $q := \cos\left(-\frac{\theta}{2}\right) + \left(\sin\left(-\frac{\theta}{2}\right)\right)\varphi(a)$, $q \in \mathrm{SU}_2$ and $\mathrm{Ad}_q^* B = C$, hence $C \in \mathcal{O}_B$. This concludes. $\square$

Now that we have made some calculus with the Lie algebra isomorphism φ, we do not write it anymore and use the notation abuse below.

◇ Notation. With the isomorphism φ, we can make the notation abuse $\mathfrak{su}_2 = \mathbb{R}^3$: for all $u \in \mathbb{R}^3$, $\varphi(u) = u$.

◇ Remarks.
- In fact, Proposition 4.34 shows us that in $\mathbb{R}^3$ the coadjoint orbits of SU_2 are the origin (of dimension 0) and the spheres of stricly positive radius (of dimension 2). See appendix 7.2 for illustrations.
- Let $r > 0$. We can show that the manifold structure on $\left\{ x \in \mathbb{R}^3 \;\middle|\; ||x||_2 = r \right\}$ given by Theorem 4.8 is the same than the manifold structure on the sphere given by the stereographic projections (i.e. the classic manifold structure on the sphere), in the sense that φ induces a diffeomorphism on each coadjoint orbits of SU_2.

We are now looking for the symplectic strucure of these coadjoint orbits as described in Theorem 4.26 ?

Lemma 4.35. For all $A, B \in \mathfrak{su}_2$, $\mathrm{ad}_A^* B = \mathrm{ad}_A B = 2A \wedge B$.

Proof Let $A, B \in \mathfrak{su}_2$. We have $\mathrm{ad}_A B = [A, B] = 2A \wedge B$. In addition to that, for all $C \in \mathfrak{su}_2$, $\langle \mathrm{ad}_A^* B, C \rangle_m = -\langle B, \mathrm{ad}_A C \rangle_m = -2\langle B, A \wedge C \rangle_m = -2\langle B \wedge A, C \rangle_m = \langle 2A \wedge B, C \rangle_m$. Hence $\mathrm{ad}_A^* B = 2A \wedge B$. $\square$

Proposition 4.36. Let $A \in \mathfrak{su}_2 \setminus \{0\}$ and let $r := \det A$. Let $\omega \in \Omega^2(\mathcal{O}_A)$ the symplectic form defined by the KKS formula : for all $B \in \mathcal{O}_A$ and $u, v \in T_B\mathcal{O}_A$, $\omega_B(\mathrm{ad}_u^* B, \mathrm{ad}_v^* B) = \langle B, [u, v] \rangle$. For all $B \in \mathcal{O}_A$ and $u, v \in T_B\mathcal{O}_A$,

$$\omega_B(u, v) = -\frac{1}{r} u \wedge v.$$

Proof Let $B \in \mathcal{O}_A$. For all $u, v \in \mathfrak{su}_2$, $\omega_B(\mathrm{ad}_u^* B, \mathrm{ad}_v^* B) = \langle B, [u, v] \rangle_m$ i.e., by Lemma 4.35, $\omega_B(u \wedge B, v \wedge B) = \frac{1}{2}\langle B, u \wedge v \rangle_m$, i.e. $\omega_B(u \wedge B, v \wedge B) = \langle B, u \wedge v \rangle$. Let $e_r := \frac{1}{r}B$ and $e_\theta, e_\varphi \in \mathbb{R}^3$ such that $(e_r, e_\theta, e_\varphi)$ is an orthonormal basis of $\mathbb{R}^3$. For all $u \in \mathfrak{su}_2$, $\mathrm{ad}_u^* B = 2u \wedge B = 2ru \wedge e_r$. Hence, by Proposition 4.18, $T_B\mathcal{O}_A = \mathrm{Span}_{\mathbb{R}}\{e_\theta, e_\varphi\}$. Let $u, v \in T_B\mathcal{O}_A$ and $(u_\theta, u_\varphi) \in \mathbb{R}^2$ (resp. $(v_\theta, v_\varphi) \in \mathbb{R}^2$) the coordinate of u (resp. v) in the basis (e_θ, e_φ). Let $\tilde{u} := -\frac{u_\varphi}{r}e_\theta + \frac{u_\theta}{r}e_\varphi$ and $\tilde{v} := -\frac{v_\varphi}{r}e_\theta + \frac{v_\theta}{r}e_\varphi$. We have $\tilde{u} \wedge B = u$ and $\tilde{v} \wedge B = v$, hence $\omega_B(u, v) = \langle B, \tilde{u} \wedge \tilde{v} \rangle = r\frac{-u_\varphi v_\theta + u_\theta v_\varphi}{r^2} = -\frac{1}{r}u \wedge v$. $\square$

◇ Remark. Proposition 4.36 shows us that the symplectic form on a sphere with strictly positive radius is the Euclidean area form on the sphere divided by the diameter of the sphere.

Let $r > 0$ and $S_r \subset \mathbb{R}^3$ the sphere of radius r. What is the area of S_r with this measure ? It is the absolute value of the volume of the compact manifold S_r equiped with the volume form ω defined in the proof of Theorem 4.26. Hence the volume of the sphere is $\left|\int_{S_r} 1\right| = \frac{4\pi r^2}{r} = 4\pi r$ (hence if $r \neq 1$, it is not $4\pi r^2$).

Let $\arg : \mathbb{S}^1 \to [0, 2\pi[$ the unique application such that for all $\theta \in [0, 2\pi[$, $\arg e^{i\theta} = \theta$.

Proposition 4.37. We have $G_{\mathrm{I}_2} = \mathrm{SU}_2$. For all $A \in \mathrm{SU}_2 \setminus \{\mathrm{I}_2\}$,

$$\begin{array}{ccc} \mathbb{S}^1 & \longrightarrow & G_A \\ z & \longmapsto & \cos\arg z + (\sin\arg z)\,\frac{1}{\det A}A \end{array}$$

is a group isomorphism.

Proof Let $A \in \mathrm{SU}_2 \setminus \{\mathrm{I}_2\}$ and $I := \frac{1}{\det A}A$, which is a quaternion of absolute value 1. By Lemma 4.29 we can define $f : z \in \mathbb{S}^1 \mapsto \cos\arg z + (\sin\arg z)\, I \in \mathrm{SU}_2$. Let $z, z' \in \mathbb{S}^1$ and $(\theta, \theta') := (\arg z, argz')$. We have $f(zz') = \cos(\theta + \theta') + \sin(\theta + \theta')I = \pm(\cos\theta)(\cos\theta') - (\sin\theta)(\sin\theta') + ((\cos\theta)(\sin\theta') + (\cos\theta')(\sin\theta))I$. But $I^2 = -I \cdot I + I \wedge I = -\,||I||_2^2 + 0 = -1$, hence by a direct calculus $f(z)f(z') = f(zz')$. Thus f is a group morphism.

For all $z, z' \in \mathbb{S}^1$ such that $f(z) = f(z')$, $\cos\arg z = \cos\arg z'$ and $\sin\arg z = \sin\arg z'$, i.e. $z = z'$, hence f is injective.

Let $q \in G_A$. There is $\theta \in [0, 2\pi[$ and $J \in \mathfrak{su}_2$ such that $|J| = 1$ and $q = \cos\theta + (\sin\theta)J$. We have $qB\bar{q} = B$, i.e. the rotation $r : B \in \mathfrak{su}_2 \mapsto qB\bar{q} \in \mathfrak{su}_2$ of axis J and angle -2π fixes B. Firstly, suppose $J \notin \{I, -I\}$. Then $-2\theta \equiv 0[2\pi]$, i.e. $\theta \equiv 0[\pi]$, i.e. $q \in \{1, -1\} = \{f(1), f(-1)\}$. Secondly, suppose $J \in \{I, -I\}$. We have $q \in \left\{f(e^{i\theta}), f(-e^{i\theta})\right\}$. We deduce that f is surjective. □

4.3.2 The coadjoint orbits of $\mathbf{SL_2}(\mathbb{R})$

Now, we consider the case of the special linear group $G = \mathrm{SL}_2(\mathbb{R})$, which is denoted in this subsubsection by the more convenient notation SL_2 :

$$G = \left\{g \in \mathrm{M}_2(\mathbb{R}) \;\middle|\; \det g = 1\right\}.$$

We denote by $\mathfrak{sl}_2$ its Lie algebra.

Proposition 4.38. The Lie algebra of SL_2 is

$$\mathfrak{sl}_2 = \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \; ; \; (a, b, c) \in \mathbb{R}^3\right\}.$$

In particular, $\dim \mathrm{SL}_2 = 3$.

Proof It is a direct consequence of Proposition 3.21. □

We consider the matrices

$$X := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, Y := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } Z := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The family (X, Y, Z) is clearly a real basis of $\mathfrak{sl}_2$. We see later that this is an interesting choice (which is made in [5]). Like in the precedent example 4.3.1, we consider the isomorphism of linear vector spaces

$$\varphi : \left| \begin{array}{ccc} \mathbb{R}^3 & \longrightarrow & \mathfrak{sl}_2 \\ (x, y, z) & \longmapsto & xX + yY + zZ \end{array} \right. .$$

Lemma 4.39. 1. We have $XY = -YX = Z$, $YZ = -ZY = -X$, $ZX = -XZ = -Y$, $[X,Y] = 2Z$, $[Y,Z] = -2X$, and $[Z,X] = -2Y$.

2. For all $u, v \in \mathbb{R}^3$, $\langle u, v\rangle = \frac{1}{2}\operatorname{tr}\varphi(u)^T\varphi(v)$.

Proof By a direct calculus, and using the fact that the Lie bracket on $\mathfrak{sl}_2$ is the matrix commutator. □

◇ Remark. Lemma 4.39 shows us that $(A,B) \in \mathfrak{sl}_2 \times \mathfrak{sl}_2 \mapsto \frac{1}{2}\operatorname{tr} A^T B \in \mathbb{R}$ is a scalar product and, with this one, φ preserve the scalar product.

◇ Notation. For all $A, B \in \mathfrak{sl}_2$, we denote $\langle A, B\rangle := \frac{1}{2}\operatorname{tr} A^T B$. By the Riesz representation theorem, we make the notation abuse $\mathfrak{sl}_2^* = \mathfrak{sl}_2$: for all $A \in \mathfrak{sl}_2$, $A = \langle A, \cdot\rangle$. Using the isomorphism φ, we make the notation abuse $\mathfrak{sl}_2 = \mathbb{R}^3$: for all $u \in \mathbb{R}^3$, $\varphi(u) = u$.

Proposition 4.40. For all $g \in \mathrm{SL}_2$, $\mathrm{Ad}_g : A \in \mathfrak{sl}_2 \mapsto gAg^{-1} \in \mathfrak{sl}_2$ and

$$\mathrm{Ad}_g^* : \left| \begin{array}{ccc} \mathfrak{sl}_2 & \longrightarrow & \mathfrak{sl}_2 \\ A & \longmapsto & (g^T)^{-1}Ag^T \end{array} \right. = \mathrm{Ad}_{(g^T)^{-1}}.$$

Proof We prove it the same way we proved Proposition 4.32. Let $g \in \mathrm{SL}_2$. By the example below definition 4.14, that for all $A \in \mathfrak{su}_2$, $\mathrm{Ad}_g A = gAg^{-1}$. For all $A \in \mathfrak{su}_2^* = \mathfrak{su}_2$ and $C \in \mathfrak{su}_2$, $\left\langle \mathrm{Ad}_g^* A, C\right\rangle = \left\langle A, g^{-1}Cg\right\rangle = \frac{1}{2}\operatorname{tr} A^T g^{-1}Cg = \left\langle (g^T)^{-1}Ag^T, C\right\rangle$. Hence $\mathrm{Ad}_g^* = \mathrm{Ad}_{(g^T)^{-1}}$. □

Lemma 4.41. (Caculus rules about coadjoint orbits of SL_2.)

1. For all $A \in \mathfrak{su}_2, \mathcal{O}_A = \left\{gAg^{-1} \ ; \ g \in \mathrm{SL}_2\right\}$.
2. For all $(x,y,z) \in \mathbb{R}^3$, $x^2 + y^2 - z^2 = -\det(xX + yY + zZ)$.
3. For all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$,

$$\left\{ \begin{array}{rll} gXg^{-1} = & (ad+cb)X + (cd-ab)Y - (ab+cd)Z & (1) \\ gYg^{-1} = & (-ac+bd)X + \frac{a^2-b^2-c^2+d^2}{2}Y + \frac{a^2+c^2-b^2-d^2}{2} & (2) \\ gZg^{-1} = & -(ac+bd)X + \frac{a^2+b^2-c^2-d^2}{2}Y + \frac{a^2+b^2+c^2+d^2}{2}Z & (3) \end{array} \right.$$

Proof By 4.40 and the fact that $g \in \mathrm{SL}_2 \mapsto (g^T)^{-1} \in \mathrm{SL}_2$ is well defined and is a bijection, we have the first statement. By a direct calculus, we have the second and third statements. □

Proposition 4.42. Let $\mathcal{O} \subset \mathbb{R}^3$. The subset $\mathcal{O}$ is a coadjoint orbit of SL_2 if and only if one and only one of the following statements is true :

1. There is $\lambda > 0$ such that $\mathcal{O} = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = \lambda^2\right\}$.
2. There is $\lambda \geqslant 0$ such that $\mathcal{O} = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = -\lambda^2, z > 0\right\}$.
3. We have $\mathcal{O} = \{0\}$.
4. There is $\lambda \geqslant 0$ such that $\mathcal{O} = \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = -\lambda^2, z < 0\right\}$.

Proof We use Lemma 4.41 and denote by (1) (resp. (2)) (resp. (3)) the formula denoted by (1) (resp. (2)) (resp. (3)) in the third statement of this lemma. First, we want to prove that all the sets defined in the proposition are coadjoint orbits. Let $O_3 = \{0\}$ and, for all $\lambda \geqslant 0$, $O_{1,\lambda} := \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = \lambda^2\right\}$, $O_{2,\lambda} := \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = -\lambda^2, z > 0\right\}$, $O_{4,\lambda} := \left\{(x,y,z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 - z^2 = -\lambda^2, z < 0\right\}$

- Let $\lambda > 0$. We have $\lambda X \in O_{1,\lambda}$. For all $g \in \mathrm{SL}_2$, $-\det(g\lambda Xg^{-1}) = \lambda^2$, i.e. $g\lambda Xg^{-1} \in O_{1,\lambda}$.

Thus $\mathcal{O}_{\lambda X} \subset O_{1,\lambda}$. Reciprocally, let $A := xX + yY + zZ \in O_{1,\lambda}$. Let

$$(a,b,c,d) := \begin{cases} (1, -(y+z)/(2\lambda), (y-z)/(x+\lambda), (x+\lambda)/(2\lambda)) & \text{if } x \neq -\lambda \\ (1, -y/\lambda, \lambda/y, 0) & \text{if } x = -\lambda \text{ and } y = z \neq 0 \\ (0, 1, -1, -y/\lambda) & \text{if } x = -\lambda \text{ and } y = -z \end{cases}.$$

Since $x^2+y^2-z^2 = \lambda^2$, it covers all the possible cases. Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Using $x^2+y^2-z^2 = \lambda^2$ and formula (1), we have $g \in \mathrm{SL}_2$ and $g\lambda X g^{-1} = A$. Hence $A \in \mathcal{O}_{\lambda X}$. Hence $O_{1,\lambda} = \mathcal{O}_{\lambda X}$.

- Let $A_0 := \frac{1}{2}(Y+Z)$. We have $A_0 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \in O_{2,0}$. Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$. We have $-\det gA_0g^{-1} = 0$ and, by formulas (1) and (2), the component of gA_0g^{-1} over Z in the basis (X,Y,Z) is $(a^2+c^2)/2$. But $a = c = 0 \Rightarrow \det g = 0$, hence $(a^2+b^2)/2 > 0$ and $gA_0g^{-1} \in O_{2,0}$. Thus $\mathcal{O}_{A_0} \subset O_{2,0}$. Reciprocally, let $B := xX + yY + zZO_{1,\lambda}$. Since $x^2+y^2 = z^2 > 0$, $(a,b,c,d) := (\sqrt{y+z}, \sqrt{z-y}, -x/\sqrt{y+z}, (\sqrt{z+y} - x\sqrt{z-y})/(z+y))$ is well defined. Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Using $x^2+y^2 = z^2$ and formulas (1) and (2), $g \in \mathrm{SL}_2$ and $gA_0g^{-1} = A$. Thus $A \in \mathcal{O}_{A_0}$. Hence $O_{2,0} = \mathcal{O}_{A_0}$.

  Let $\lambda > 0$. We have $\lambda Z \in O_{2,\lambda}$. Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2$. We have $-\det g\lambda Zg^{-1} = -\lambda^2$ and $(a^2+b^2+c^2+d^2)/2 > 0$ hence, by formula (3), $g\lambda Zg^{-1} \in O_{2,\lambda}$. Hence $\mathcal{O}_{\lambda Z} \subset O_{2,\lambda}$. Reciprocally, let $A := xX + yY + zZO_{2,\lambda}$. Since $x^2+y^2+\lambda^2 = z^2$, we can define $(a,b,c,d) = (-x/\sqrt{\lambda(z-y)}, -1/\sqrt{z-y}, \sqrt{z-y}, d = 0)$. Let $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Using $x^2+y^2+\lambda^2 = z^2$ and formula (3), $g \in \mathrm{SL}_2$ and $g\lambda Zg^{-1} = A$. Thus $A \in O_{2,\lambda}$. Hence $O_{2,\lambda} = \mathcal{O}_{\lambda Z}$.

- It is clear that $O_3 = \mathcal{O}_0$.

- Let $A_0 = \frac{1}{2}(Y+Z)$. We have $-A_0 = \left(\begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix}\right) \in O_{4,0}$. We clearly have $O_{4,0} = \{-B \ ; \ B \in O_{2,0}\}$. But we have shown that $O_{2,0} = \mathcal{O}_{A_0}$, hence $O_{4,0} = \{-A; \ ; A \in \mathcal{O}_{A_0}\} = \mathcal{O}_{-A_0}$.

  Let $\lambda > 0$. We have $-\lambda Z \in O_{4,\lambda}$. Just like in the precedent case, we have $O_{4,\lambda} = \{-A \ ; \ A \in O_{2,\lambda}\}$ and we have shown that $O_{2,\lambda} = \mathcal{O}_{\lambda Z}$, hence $O_{4,\lambda} = \{-A; \ ; A \in \mathcal{O}_{\lambda Z}\} = \mathcal{O}_{-\lambda Z}$.

We have shown that if one and only one of the statements 1., 2., 3., 4. is true, then $\mathcal{O}$ is a coadjoint orbit of SL_2. Reciprocally, we suppose that $\mathcal{O}$ is a coadjoint orbit of SL_2. There is $A \in \mathcal{O}$ and we denote by (x,y,z) its coordinates in the basis (X,Y,Z). We will use the fact that $\mathcal{O} = \mathcal{O}_B$ and the first part of the proof. We make the following case disjonction :

- If $\det A < 0$: $\mathcal{O} = O_{1,\sqrt{-\det A}}$.
- If $\det A \geqslant 0$ and $z > 0$: $\mathcal{O} = O_{2,\sqrt{\det A}}$.
- If $\det A \geqslant 0$ and $z = 0$: $x^2+y^2 = -\det A \leqslant 0$, hence $x = y = z = 0$, i.e. $A = 0$ and $\mathcal{O} = O_3$.
- If $\det A \geqslant 0$ and $z < 0$: $\mathcal{O} = O_{4,\sqrt{\det A}}$.

Finally, this cases are clearly two by two incompatibles. This concludes. ☐

◇ Remarks.
- Proposition 4.42 means that the coadjoint orbits of SL_2, as subsets of $\mathbb{R}^3$ through φ, are the connected components of the surfaces $\left\{(x,y,z) \in \mathbb{R}^3 \ \middle| \ x^2+y^2-z^2 = C\right\}$; $C \in \mathbb{R}$. Hence the coadjoint orbits of SL_2 in $\mathbb{R}^3$ are connected components of hyperboloids of revolution. See appendix 7.3 for illustrations.
- Just like in the case of SU_2, we can show that the manifold structures of coadjoint orbits of SL_2 as embedded submanifolds of $\mathbb{R}^3$ or as immersed submanifolds of $\mathfrak{su}_2$ (see Theorem 4.8) are the same, in the sense that φ induces a diffeomorphism on each coadjoint orbits of SL_2.

Proposition 4.43. Let $A \in \mathfrak{sl}_2$, z its Z-coordinate in the basis (X,Y,Z) and G_B its stabilizer for the coadjoint action of SL_2.

- If $\det A < 0$, $G_A \simeq \left\{\left(\begin{smallmatrix} r & 0 \\ 0 & r^{-1} \end{smallmatrix}\right); \ ; r \in \mathbb{R}^*\right\} \simeq \mathbb{R}^*$.
- If $\det A = 0$ and $z = 0$, $G_A = \mathrm{SL}_2$.

- If $\det A = 0$ and $z \neq 0$, $G_A \simeq \left\{\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix};\ ; a \in \mathbb{R}\right\} \cup \left\{\begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix};\ ; a \in \mathbb{R}\right\} \simeq \{1, -1\} \times \mathbb{R}$.
- If $\det A > 0$, $G_A \simeq \left\{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\ ;\ \theta \in \mathbb{R}\right\} \simeq \mathbb{S}^1$.

Proof Given G a group, X a set and $G \curvearrowright X$ a right group action, we recall that for all $g_0 \in G$ and $x \in X$, $g \in G_x \mapsto g_0 g g_0^{-1} \in G_{g \cdot x}$ is a group isomorphism.

- We suppose that $\det A < 0$. Let $\lambda = \sqrt{-\det A}$. By Proposition 4.42, $\mathcal{O}_B = \mathcal{O}_{\lambda X}$, hence $G_B \simeq G_{\lambda X}$. In addition to that, it is clear that $G_{\lambda X} = G_X$.

  For all $r \in \mathbb{R}^*$, we denote $g_r := \begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \in \mathrm{SL}_2$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_X$. By formula (2) of Lemma 4.41, $ad = 1$ and $b = c = 0$. Let $r := a \in \mathbb{R}$. We have $r \neq 0$ and $g = g_r$. Reciprocally, for all $r \in \mathbb{R}^*$, $g_r X g_r^{-1} = X$. Hence $G_X = \{g_r\ ;\ r \in \mathbb{R}^*\}$. Finally, it is clear that $r \in \mathbb{R} \mapsto g_r \in G_X$ is a group isomorphism.

- We suppose that $\det A = 0$ and $z = 0$. We have $A = 0$ and it is clear that $G_A = \mathrm{SL}_2$.

- We suppose that $\det A = 0$ and $z \neq 0$. Let $A_0 = \frac{1}{2}(X + Z)$. By Proposition 4.42, $\mathcal{O}_A = \mathcal{O}_{A_0}$ or $\mathcal{O}_A = \mathcal{O}_{-A_0}$. But it is clear that $G_{A_0} = G_{-A_0}$, hence $G_A \simeq G_{A_0}$.

  For all $(\delta, a) \in \{1, -1\} \times \mathbb{R}$, we denote $g_{\delta,a} := \begin{pmatrix} \delta & a \\ 0 & \delta \end{pmatrix} \in \mathrm{SL}_2$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{B_0}$. By formula (2) and (3) of Lemma 4.41, $a = d \in \{1, -1\}$ and $c = 0$. Hence $g = g_{a,b}$. Reciprocally, for all $(\delta, a) \in \{1, -1\} \times \mathbb{R}$, $g_{\delta,a} A_0 g_{\delta,a}^{-1} = A_0$. Hence $G_{A_0} = \{g_{\delta,a}\ ;\ (\delta, a) \in \{1, -1\} \times \mathbb{R}\}$. Finally, by a direct calculus, $(\delta, a) \in \{1, -1\} \times \mathbb{R} \mapsto g_{\delta,a} \in G_{A_0}$ is a group isomorphism.

- We suppose that $\det A > 0$. Let $\lambda := \sqrt{\det A}$. By Proposition 4.42, $\mathcal{O}_A = \mathcal{O}_{\lambda Z}$ or $\mathcal{O}_A = \mathcal{O}_{-\lambda Z}$. But it is clear that $G_{\lambda Z} = G_{-\lambda Z} = G_Z$, so $G_A \simeq G_Z$.

  For all $\theta \in \mathbb{R}$, we denote $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in \mathrm{SL}_2$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_Z$. By formula (2) and (3) of Lemma 4.41, $a^2 + b^2 = 1$ and $c^2 + d^2 = 1$. Hence there is $\theta, \theta' \in \mathbb{R}$ such that $(a, b, c, d) = (\cos\theta, \sin\theta, \sin\theta', \cos\theta')$. But $\det A = 1$, i.e. $\cos(\theta + \theta') = 1$, i.e. $\theta \equiv -\theta'[2\pi]$. Hence $A = R_\theta$. Reciprocally, for all $\theta \in \mathbb{R}$, $R_\theta A_0 R\theta^{-1} = A_0$. Hence $G_Z = \{R_\theta\ ;\ \theta \in \mathbb{R}\}$. Finally, for all $z \in \mathbb{S}^1$, we denote by $\arg z$ the only element of $[0, 2\pi[$ such that $z = e^{i\theta}$. The map $z \in \mathbb{S}^1 \mapsto R_{\arg z} \in G_Z$ is a group isomorphism.

$\square$

To conclude this example, we are now looking for the symplectic form defined in the proof of Theorem 4.26.

◇ Remark. With the notation abuse $\mathfrak{sl}_2 = \mathbb{R}^3$, for all $u := (x, y, z) \in \mathbb{R}^3$, $u^T := (x, y, -z)$.

Lemma 4.44. For all $u, v \in \mathbb{R}^3$, $\mathrm{ad}_u v = 2u^T \wedge v^T$ and $\mathrm{ad}_u^* v = \mathrm{ad}_v u^T = 2v^T \wedge u$.

Proof Using Propositions 4.15 about ad and Lemma 4.39 about some useful calculus rules, for all $u := (x, y, z), v := (x', y', z') \in \mathbb{R}^3$, we compute $\mathrm{ad}_u v = [u, v] = (yz' - zy')\,[Y, Z] + (zx' - z'x)\,[Z, X] + (xy' - yx')\,[X, Y] = -2(yz' - zy')X - 2(zx' - z'x)Y + 2(xy' - yx')Z = 2u^T \wedge v^T$.

But for all $A, B, C \in \mathfrak{su}_2$, $\langle \mathrm{ad}_A^* B, C\rangle_m = -\langle B, [A, C]\rangle = -\frac{1}{2}\operatorname{tr} B^T(AC - CA) = -\frac{1}{2}\operatorname{tr} B^T AC - AB^T C = -\frac{1}{2}\operatorname{tr}(A^T B - BA^T)^T C = \left\langle \left[B, A^T\right], C\right\rangle$, hence $\mathrm{ad}_A^* B = \left[B, A^T\right] = 2B^T \wedge A$. $\square$

Proposition 4.45. Let $A \in \mathfrak{sl}_2 \setminus \{0\}$ and $\omega \in \Omega^2(\mathcal{O}_A)$ the unique symplectic form defined by the KKS formula : for all $B \in \mathcal{O}_A$ and $u, v \in T_B\mathcal{O}_A$, $\omega_B(\mathrm{ad}_u^* B, \mathrm{ad}_v^* B) = \langle B, [u, v]\rangle$. Let $B := (x, y, z) \in \mathcal{O}_A$, $r := \sqrt{x^2 + y^2}$ and $(e_x, e_y, e_z) := (X, Y, Z)$. If $r \neq 0$, let $e_r, e_\theta \in \mathbb{R}^3$ such that (e_r, e_θ, e_z) is a direct orthonormal basis of $\mathbb{R}^3$ and $B = re_r + ze_z$. Let $dx = \langle e_x, \cdot\rangle$, $dy := \langle e_y, \cdot\rangle$ and $dz = \langle e_z, \cdot\rangle$. If $r = 0$, let $dr = \langle e_r, \cdot\rangle$ and $d\theta = \frac{1}{r}\langle e_\theta, \cdot\rangle$. We have $z \neq 0$ and

$$\omega_B = \begin{cases} \frac{1}{2} dz \wedge d\theta & \text{if } r \neq 0 \\ \frac{1}{2z} dx \wedge dy & \text{if } r = 0 \end{cases}.$$

Proof To begin with, by proposition 4.42 and the fact that $A \neq 0$, $z \neq 0$.

Let $T_B := T_B \mathcal{O}_A$ the tangent space. If $r = 0$ we denote $(e_r, e_\theta) := (e_x, e_y)$, hence (e_r, e_θ, e_z) is a direct orthonormal basis and we have $B = re_r + ze_z$ (just like in the case $r \neq 0$). For all $u, v \in \mathfrak{sl}_2$, $\omega_B(\mathrm{ad}_u^* B, \mathrm{ad}_v^* B) = \langle B, [u,v] \rangle$ i.e., by Lemma 4.44, $2\omega_B(B^T \wedge u, B^T \wedge v) = \left\langle B, u^T \wedge v^T \right\rangle$. By Lemma 4.44 and Proposition 4.18, $T_B = \left\{ 2B^T \wedge u \; ; \; u \in \mathfrak{sl}_2 \right\} = \left\{ B^T \right\}^\perp := \left\{ w \in \mathbb{R}^3 \;\middle|\; \left\langle B^T, w \right\rangle = 0 \right\}$. But $B^T \neq 0$, hence $w \in T_B \mapsto B^T \wedge w \in T_B$ is a vector space isomorphism.

Let $u, v \in T_B$. There is a unique $(\tilde{u}, \tilde{v}) \in T_B^2$ such that $u = B^T \wedge \tilde{u}$ and $v = B^T \wedge \tilde{v}$. In particular, $2\omega_B(u,v) = \left\langle B, \tilde{u}^T \wedge \tilde{v}^T \right\rangle$. We denote $(\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_z)$ (resp. $(\tilde{v}_r, \tilde{v}_\theta, \tilde{v}_z)$) the coordinate of $\tilde{u}$ (resp. $\tilde{v}$) in the orthonormal basis (e_r, e_θ, e_r). On one hand, by definition of $\tilde{u}$ and $\tilde{v}$, we have $u_r = z\tilde{u}_\theta$, $u_\theta = -(z\tilde{u}_r + r\tilde{u}_z)$, $u_z = r\tilde{u}_\theta$, $v_r = z\tilde{v}_\theta$, $v_\theta = -(z\tilde{v}_r + r\tilde{v}_z)$, $v_z = r\tilde{v}_\theta$. On the other hand, we have $2\omega_B(u,v) = \left\langle B, \tilde{u}^T \wedge \tilde{v}^T \right\rangle = r(-\tilde{u}_\theta\tilde{v}_z + \tilde{u}_z\tilde{v}_\theta) + z(\tilde{u}_r\tilde{v}_\theta - \tilde{u}_\theta\tilde{v}_r)$.

We suppose that $r = 0$, i.e. $B = ze_z$. Hence $\tilde{u}, \tilde{v}$ are in $T_B = \mathrm{Span}_{\mathbb{R}} \{e_r, e_\theta\}$, i.e. $\tilde{u}_z = \tilde{v}_z = 0$. Thus we have $2\omega_B(u,v) = 0 + z(\tilde{u}_r\tilde{v}_\theta - \tilde{u}_\theta\tilde{v}_r)$. But we have $(dx \wedge dy)(u,v) = u_r v_\theta - v_r u_\theta = z\tilde{u}_\theta(-z - r\tilde{v}_z) - z\tilde{v}_\theta(-z\tilde{u}_r - r\tilde{u}_z) = z_0^2(-\tilde{u}_\theta\tilde{v}_r + \tilde{v}_\theta\tilde{u}_r) = 2z_0\omega_B(u,v)$, hence $\omega_B(u,v) = \frac{1}{2z}(dx \wedge dy)(u,v)$.

We suppose that $r \neq 0$. We have $(dz \wedge d\theta)(u,v) = u_z \frac{v_\theta}{r} - \frac{u_\theta}{r} v_z = \frac{1}{r}(r\tilde{u}_\theta - \tilde{u}_r\theta_0)(-\tilde{v}_r z - r\tilde{v}_z) + \frac{1}{r}(-r\tilde{v}_\theta + \tilde{v}_r\theta_0)(-\tilde{u}_r z - r\tilde{u}_z) = \frac{1}{r} r(r(-\tilde{u}_\theta\tilde{v}_z + \tilde{u}_z\tilde{v}_\theta) + z(\tilde{u}_r\tilde{v}_\theta - \tilde{u}_\theta\tilde{v}_r)) = 2\omega_B(u,v)$, hence $\omega_B(u,v) = \frac{1}{2}(dz \wedge d\theta)(u,v)$. □

◇ Remarks.

- With the notation A and ω of the proprosition above, if $A = 0$ then $\mathcal{O}_A = \{0\}$ and $\omega = 0$, which is non degenerate because $\mathrm{Ker}\,\omega_0 = \{0\}$. This is why we only consider the case $A \neq 0$ in this proposition.

- With the notation e_r, e_θ, e_z, r of the proposition above, (e_r, e_θ, e_z) are the circular coordinates we often see in physics and we have $\langle e_\theta, \cdot \rangle = rd\theta$ as expected. Attention, in our proof, x, y, z, r do not "vary", they are fixed by B.

- Let $A \in \mathfrak{sl}_2 \setminus \{0\}$. There is $C \in \mathbb{R}$ such that for all $(x,y,z) \in \mathcal{O}_A$, $x^2 + y^2 - z^2 = C$, hence $d(z^2) = d(x^2 + y^2 - C)$, i.e. $dz = \frac{1}{z}(xdx + ydy)$ (we notice that if $x = y = 0$, it is the null form). For all $(x,y,z) \in \mathcal{O}_A$ such that $x^2 + y^2 \neq 0$, we denote (e_r, e_θ, e_z), $d\theta$ and r just like in the proposition above and we can show that $d\theta = \frac{1}{r^2}(xdy - ydx)$ (we notice that if $r \to +\infty$, it "diverges"). Hence for all $(x,y,z) \in \mathcal{O}_A$ such that $x^2 + y^2 \neq 0$, $\frac{1}{2} dz \wedge d\theta = \frac{1}{2z} dx \wedge dy$, which is non degenerate and is coincident with the case $x^2 + y^2 = 0$. We notice that this gives us another intersesting formulation for the proposition above.

In the next (very short) section of this report, we see some news and application of the subject of this internship report.

5 Conclusion

Overview. We have seen that, with the basic properties of a Lie group, we can define its Lie algebra and some useful Lie group actions. Using algebra and calculus notions, we have shown that the coadjoint orbits of a Lie group carry a canonical symplectic structure. This is a useful geometric property. Finally, with the example of the unitary group of dimension 2 over $\mathbb{C}$ and the special linear group of dimension 2 over $\mathbb{R}$, we have seen two concrete illustrations.

Applications. The Lie algebra of a Lie group is in fact very useful to study this Lie group. Lie algebras are often used to classify Lie groups and to link them. It is very interesting, as Lie groups often appear in several other fields in mathematics or in physics. For example, SU_2 is very useful in quantum mechanics (see 1.1). This internship was initially aiming to study the Virasoro algebra (but the first part presented in this report took too much time and space), which is for example used in string theory. About the Virasoro algebra, see the paper *Coadjoint orbits of the Virasoro algebra and the global Liouville equation* by József Balog, László Fehér and Laszlo Palla published in 1998. In addition to this, we can mention that there is an extension of the theory we have seen in this internship to infinite dimension.

News. There is still research about Lie groups and coadjoint orbits. For example, the non homogeneous Lorentz group, which comes from physics and special relativity, is still studied in mathematics for it has some unconvenient properties.

6 Bibliography

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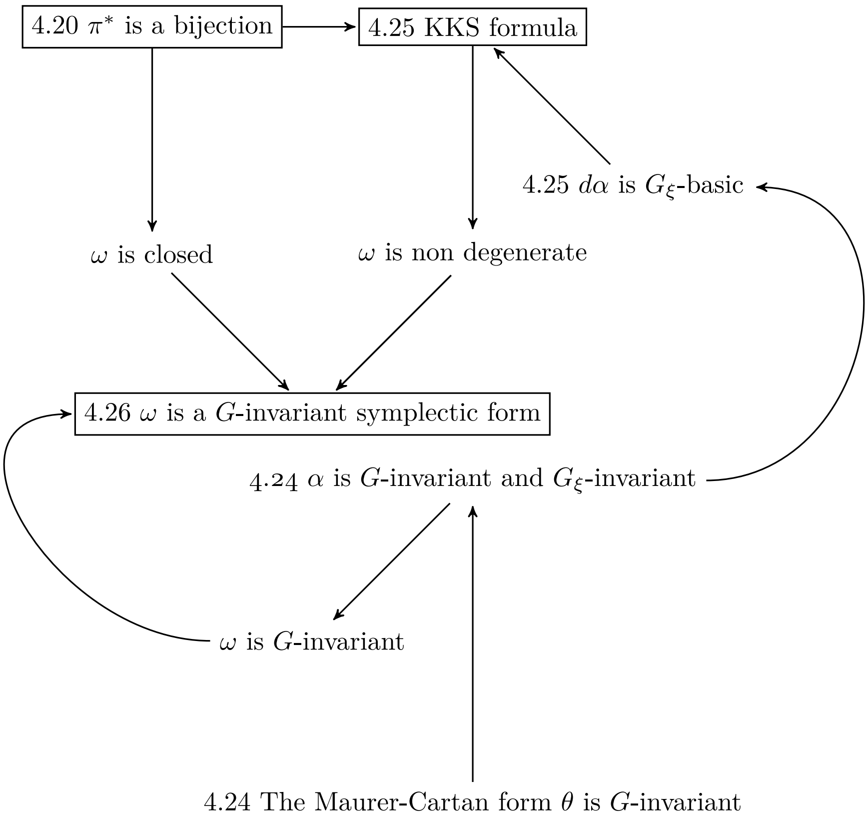
7 Appendix

We give here some figures to illustrate different results of this report.

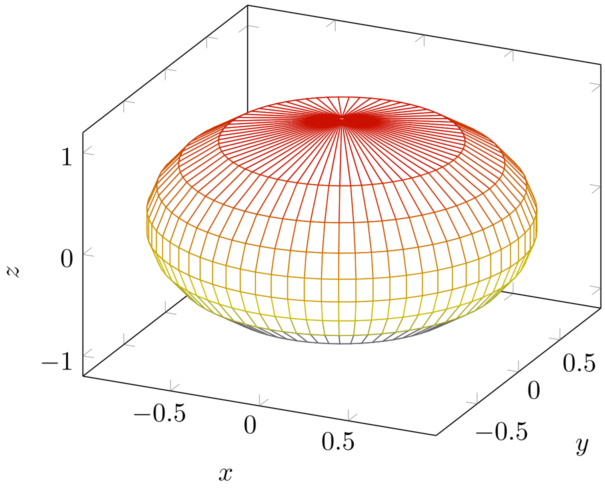
7.1 Proof of Theorem 4.26

Here is a graph summarising the proof of Theorem 4.26 explained in Subsubsection 4.2.2. The framed results are the most important ones. An arrow from a result to another one indicates that the first result is used to prove the second one.

Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. Let $\xi \in \mathfrak{g}^*$ and θ the Maurer-Cartan form on G. Let π the canonical projection $G \to G/G_\xi$, $\alpha := -\langle \xi, \theta \rangle$ and, with φ_ξ the canonical diffeomorphism $G/G_\xi \simeq \mathcal{O}_\xi$, we consider $\omega := \varphi_\xi^*(\pi^*)^{-1} d\alpha \in \Omega(\mathcal{O}_\xi)$. These are the notations used in Subsubsection 4.2.2.



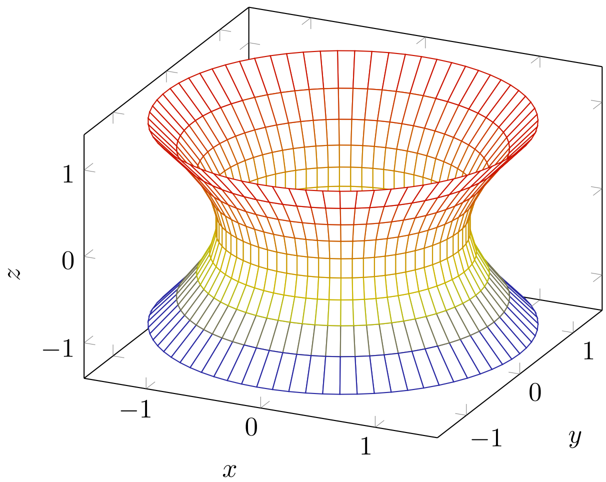
7.2 Coadjoint orbits of the Lie group SU_2, $\mathcal{O} = \{x^2 + y^2 + z^2 = 1\}$



This is one coadjoint orbit.

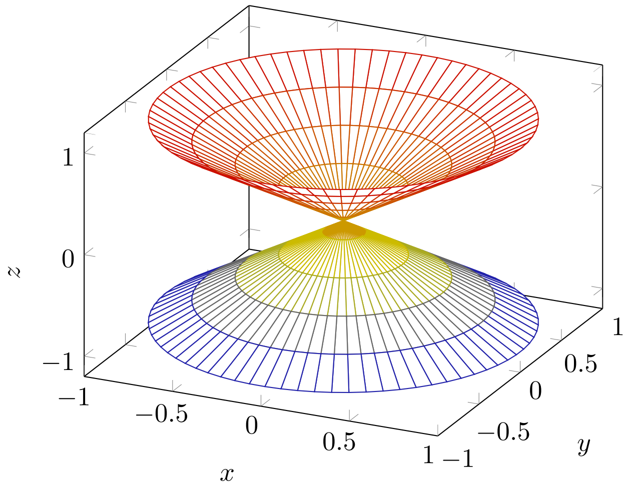
7.3 Coadjoint orbits of the Lie group SL_2

Drawing $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}$



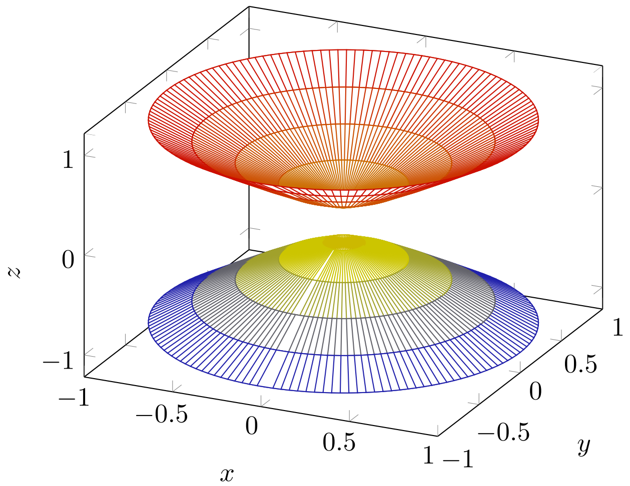
This is one coadjoint orbit.

Drawing $\left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \mathbb{R}^3 \;\middle|\; \boldsymbol{x^2+y^2-z^2=0}\right\}$



This is the union of three coadjoint orbits.

Drawing $\left\{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \in \mathbb{R}^3 \;\middle|\; \boldsymbol{x^2+y^2-z^2=-\frac{1}{30}}\right\}$



This is the union of two coadjoint orbits.