Horn's inequalities from a geometric point of view

A refinement using Belkale's method

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Summary

Eigenvalues of a sum

Definition

- $\mathcal{H}(r) = \{A | A^* = A\} \subset \mathrm{M}_r(\mathbb{C})$.
- $\mathbb{R}^r_{\geqslant}$ the decreasing real r-tuples.
- $\lambda(A) \in \mathbb{R}^r_{\geqslant}$ the spectrum with multiplicity of $A \in \mathcal{H}(r)$.

▶ What is the link between $\lambda(A), \lambda(B)$ and $\lambda(A+B)$?

Eigenvalues of a sum

Trace of $A+B=C$: with $\alpha=\lambda(A)$, $\beta=\lambda(B)$ and $\gamma=\lambda(C)$,

$$\sum_{i=1}^{r}\alpha(i)+\sum_{i=1}^{r}\beta(i)=\sum_{i=1}^{r}\gamma(i).$$

A sufficient condition for $r=1$: if $\alpha,\beta,\gamma\in\mathbb{R}^r$ satisfie this last equation,

$$\exists A,B,C\in\mathcal{H}(r),\ A+B=C$$
$$\lambda(A)=\alpha,\lambda(B)=\beta,\lambda(C)=\gamma.$$

- Admissible spectra in the Hermitian case ?

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- ▶ Admissible spectra in the Hermitian case ?

Horn's conjecture

Weyl (1912) : if $1 \leqslant i+j-1 \leqslant r$,

$$\gamma(i+j-1) \leqslant \alpha(i)+\beta(j).$$

Case $r \in \{2,3\}$: the Weyl inequalities with the trace are necessary and sufficient conditions.
Other inequalities : Ky Fan (1949), Lidskii (1950), etc.
Conjecture (1962) : inductive description on r of the cone of admissible spectra with inequalities of the form

$$\sum_{k\in K}\gamma(k) \leqslant \sum_{i\in I}\alpha(i)+\sum_{j\in J}\beta(j).$$

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$$\sum_{k \in K} \gamma(k) \leqslant \sum_{i \in I} \alpha(i) + \sum_{j \in J} \beta(j).$$

Formulating the problem

Definition

Kirwan's cone :

$$\mathrm{LR}(r,s) := \left\{ (\lambda(X_k))_k ; X_k \in \mathcal{H}(r), \sum_{k=1}^{s} X_k = 0 \right\} \subset (\mathbb{R}^r)^s.$$

$$\mathrm{LR}(r,1) = \{0\}$$

$$\mathrm{LR}(r,2) = \{(\lambda, (-\lambda(n), \dots, -\lambda(1))); \lambda \in \mathbb{R}^r_{\geqslant}\}$$

$$\mathrm{LR}(1,s) = \left\{ (\Lambda_1, \dots, \Lambda_s) \in \mathbb{R}^s | \sum_{k=1}^{s} \Lambda_k = 0 \right\}.$$

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The conjecture is true

Notation : for $\lambda \in \mathbb{R}^r$,

$$|\lambda| = \sum_{j=1}^{r} \lambda(j)$$

$$\forall J \subset [r], |\lambda|_J = \sum_{j \in J}^{r} \lambda(j)$$

1998-99 Klyachko and Knutson-Tao prove that the conjecture is true for inequalities of the form $\Lambda_k(i) \geqslant \Lambda_k(i+1)$ and $\sum_{k=1}^{s} |\Lambda_k|_{\mathcal{J}_k} \leqslant 0$..

2000 Belkale reduces the number of inequalities.

2004 Knutson-Tao-Woodward prove that these inequalities are irredundant if $s = 3$.

The conjecture is true

Notation : $[r] := \{1, \ldots, r\} \subset \mathbb{N}^*$ and, for all $J = \{J(1) < \cdots < J(d)\} \subset [r]$,

$$\mu(J) := \left(J(d) - d - (r-d)\frac{s-1}{s}, \ldots, J(1) - 1 - (r-d)\frac{s-1}{s} \right) \in \mathbb{R}^r_{\geqslant}.$$

Theorem (Horn's inequalities, Klyachko-Knutson-Tao)

The cone $\mathrm{LR}(r,s)$ is the set of all $\Lambda \in (\mathbb{R}^r_{\geqslant})^s$ such that, for all $d \in [r-1]$ and all s-tuple $(\mathcal{J}_k)_{k \in [s]}$ of subsets of $[r]$ of cardinality d such that $(\mu(\mathcal{J}_k))_k \in \mathrm{LR}(d,s)$,

$$\sum_{k=1}^{s} |\Lambda_k| = 0 \text{ and } \sum_{k=1}^{s} |\Lambda_k|_{\mathcal{J}_k} \leqslant 0.$$

Saturation

Theorem (s=3)

The Littlewood-Richardson coefficients are saturated.

Theorem (saturation, Knutson-Tao)

$$\Lambda \in \mathrm{LR}(r,s) \cap (\mathbb{Z}^r)^s \Leftrightarrow \left(\bigotimes_k V(\Lambda_k)\right)^{\mathrm{U}(r)} \neq \{0\}$$

Summary

Reducing the number of inequalities

Three sets of s-tupes of subsets of $[n]$ of cardinality r :

$$\mathrm{Horn}^{00}(r,n,s) \subset \mathrm{Horn}^{0}(r,n,s) \subset \mathrm{Horn}(r,n,s).$$

Example

$\mathrm{Horn}(1,2,3) = \{(\{1\},\{2\},\{2\}),(\{2\},\{1\},\{2\}),(\{2\},\{2\},\{1\}),(\{2\},\{2\},\{2\})\}.$

Theorem (Horn's inequalities, Belkale-Klyachko-Knutson-Tao)

The cone $\mathrm{LR}(r,s)$ is the set of all $\Lambda \in (\mathbb{R}^r_{\geqslant})^s$ such that

$$\sum_{k=1}^{s} |\lambda_k| = 0 \text{ and } \forall d \in [r-1], \forall J \in \mathrm{Horn}^*(d,r,s), \sum_{k=1}^{s} \sum_{i \in \mathcal{J}_k} \Lambda_k(i) \leqslant 0.$$

Horn^{00} is harder to compute. Error in Klyachko's article.

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Reducing the number of inequalities

Theorem (Knutson-Tao-Woodward)

For $s = 3$, the inequalities parametrized by Horn^{00} are irredundant.

Ressayre : computation of $\mathrm{Horn}^{00}(r, n, 3)$.

r	1	2	3	4	5	6	7	8	9	10
$I^0(r,3)$	2	8	20	52	156	539	2,082	8,775	39,742	191,382
$I_{\min}(r,3)$	2	5	20	52	156	538	2,062	8,522	37,180	168,602

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Inequalities with repetitions

We choose a partition of the number of matrices : $s = s_1 + \cdots + s_a$. The associated repetitions are imposed : the s_1 five spectra are identical, the next s_2 are identical, and so on.

Example

$$\mathrm{LR}(r,3)^{(3)} = \{\Lambda \in \mathrm{LR}(r,3) | \Lambda_1 = \Lambda_2 = \Lambda_3\}$$
$$\mathrm{LR}(r,3)^{(2,1)} = \{\Lambda \in \mathrm{LR}(r,3) | \Lambda_1 = \Lambda_2\}$$

- Are certain equations becoming redundant ?

Inequalities with repetition

Theorem (Horn's inequalities with repetition)

- The $\mathrm{LR}(r, s)$ cone with repetitions admits the same inductive description as in the solution of the Horn conjecture.
- The inequalities describing the elements of $\mathrm{LR}(r, s)$ with repetitions can be reduced to those with the same repetitions.
- Horn's tuples with repetitions are parameterised by the smallest tuples verifying the same repetitions.

Examples

Number of equations required to describe LR$(r,3)$ and LR$(r,3)$ with $\Lambda_1 = \Lambda_2 = \Lambda_3$.

r	1	2	3	4	5	6	7	8	9	10
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$l_3^0(r,3)$	2	3	4	7	10	10	18	25	24	51
$l_3^{00}(r,3)$	2	3	4	7	10	9	16	21	18	35

Is $l_3^{00}(r,3)$ minimal ?

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Examples

Let $\lambda \in \mathbb{Z}^6$. Representation $V(\lambda) \otimes V(\lambda) \otimes V(\lambda)$ has a U(6) invariant non-zero vector if and only if $\lambda(1) \geqslant \cdots \geqslant \lambda(6)$ and

$$\lambda(1) + \lambda(2) + \lambda(3) + \lambda(4) + \lambda(5) + \lambda(6) = 0$$
$$\lambda(1) + \lambda(5) + \lambda(6) \leqslant 0$$
$$\lambda(2) + \lambda(4) + \lambda(6) \leqslant 0 \ (*)$$
$$\lambda(3) + \lambda(4) + \lambda(5) \leqslant 0$$

Summary

Flags and positions

Definition

- A *flag* is a sequence of vector subspaces of $\mathbb{C}^n$ such that

$$E(j) \subset E(j+1)$$
$$\dim E(j) = j$$

- The position of $V \subset \mathbb{C}^n$ with respect to the flag E is the subset $\mathrm{Pos}(V, E)$ of $[n]$ with r elements composed of the jumps of

$$0 = \dim E(0) \cap V \leqslant \cdots \leqslant \dim E(n) \cap V = r.$$

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Cells

Definition

$$\Omega_I^0(E) := \{V \in \mathrm{Gr}(r, \mathbb{C}^n) |\, \mathrm{Pos}(V, E) = I\}$$
$$\mathrm{Flag}_I^0(V, \mathbb{C}^n) := \{E \in \mathrm{Flag}(\mathbb{C}^n) |\, \mathrm{Pos}(V, E) = I\}.$$

Remark

Decomposition into cells :

$$\mathrm{Gr}(r, \mathbb{C}^n) = \bigsqcup_{I \subset [n], \# I = r} \Omega_I^0(E)$$
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Definition

$$\Omega_I(E) := \overline{\Omega_I^0(E)} \subset \mathrm{Gr}(r, \mathbb{C}^n).$$

Proposition

Algebraic variety satisfying

$$\Omega_I(E) = \bigcup_{J \leqslant I} \Omega_J^0(E) \text{ and } \dim \Omega_I(E) = \sum_{i=1}^{r} I(i) - i := \dim I.$$

Example

$$\Omega_{[n-r+1,n]}(E) = \mathrm{Gr}(r, \mathbb{C}^n) \text{ and } \Omega_{[r]}(E) = \{E(r)\}.$$

Schubert varieties

Definition

$$\Omega_I(E) := \overline{\Omega_I^0(E)} \subset \mathrm{Gr}(r, \mathbb{C}^n).$$

Proposition

Algebraic variety satisfying

$$\Omega_I(E) = \bigcup_{J \leqslant I} \Omega_J^0(E) \text{ and } \dim \Omega_I(E) = \sum_{i=1}^{r} I(i) - i := \dim I.$$

Example

$$\Omega_{[n-r+1,n]}(E) = \mathrm{Gr}(r, \mathbb{C}^n) \text{ and } \Omega_{[r]}(E) = \{E(r)\}.$$

Definition

Cohomology : $\Omega_I \subset \mathrm{Gr}(r, \mathbb{C}^n)$, $\omega_I \in H^{2m}(\mathrm{Gr}(r, \mathbb{C}^n))$ with $m = \mathrm{codim}_{\mathbb{C}} \Omega_I(E)$.

$$H^*(\mathrm{Gr}(r, \mathbb{C}^n)) = \bigoplus_{m=1}^{r(n-r)} H^{2m}(\mathrm{Gr}(r, \mathbb{C}^n)) = \bigoplus_{\#I=r} \mathbb{R}\omega_I.$$

Definition

- $(\mathcal{I}_k)_k \in$ Horn if $\prod_k \omega_{\mathcal{I}_k} \neq 0$.
- $(\mathcal{I}_k)_k \in \mathrm{Horn}^0$ if $\prod_k \omega_{\mathcal{I}_k} = x[\mathrm{pt}]$, $x \neq 0$.
- $(\mathcal{I}_k)_k \in \mathrm{Horn}^{00}$ if $\prod_k \omega_{\mathcal{I}_k} = [\mathrm{pt}]$.

Characterisation

Theorem

$(\mathcal{I}_k)_k \in \mathrm{Horn}(r, n, s)$ if and only if

$$\forall (\mathcal{E}_k)_k \in \mathrm{Flag}(\mathbb{C}^n)^s, \bigcap_{k=1}^{s} \Omega_{\mathcal{I}_k}(\mathcal{E}_k) \neq \emptyset.$$

Characterisation

- Inductive description of Horn(r, n, s) ?

$$\omega_{\mathcal{I}}^0 : \left| \begin{array}{ccc} \mathrm{GL}(n) \times \prod_{k=1}^s \mathrm{Flag}_{\mathcal{I}_k}^0(V, \mathbb{C}^n) & \longrightarrow & \mathrm{Flag}(\mathbb{C}^n)^s \\ (\gamma, \mathcal{E}) & \longmapsto & (\gamma \mathcal{E}_k)_k \end{array} \right.$$

Characterisation of an Horn's tuple by the image of $\omega_{\mathcal{I}}^0$. If $\mathcal{I}$ is a Horn's tuple, there is an inequality on the dimensions :

$$\mathrm{edim}\, \mathcal{I} := r(n-r) - \sum_{k=1}^{s} (r(n-r) - \dim \mathcal{I}_k) \geqslant 0.$$

Slopes

Lemma (Harder-Narasimhan)

There is a unique linear subspace with minimum slope and maximum dimension.

Notation : $\mathcal{I}_k : [r] \nearrow [n]$, $\mathcal{I}\mathcal{J} = (\mathcal{I}_k \circ \mathcal{J}_k)_k$.

Proposition (Algorithmic point of view)

If $(\mathcal{I}_k)_k$ is a Horn's tuple then $\operatorname{edim} \mathcal{I} \geqslant 0$ and

$$\forall d \in [r-1], \forall \mathcal{J} \in \mathrm{Horn}^*(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant 0.$$

Somes tools for the reciprocal

$$
\begin{array}{ccccccccc}
\mathsf{P}_{\mathsf{kpt}} & \overset{\text{ouvert}}{\subset} & \mathsf{P}_{\mathsf{kt}} & \overset{\text{ouvert}}{\subset} & \mathsf{P}_{\mathsf{t}} & \overset{\text{ouvert}}{\subset} & \mathsf{P} & \overset{\text{fermé}}{\subset} & \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s \times \mathcal{L}(V, Q) \\
\downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
\mathsf{B}_{\mathsf{kpt}} & \overset{\text{ouvert}}{\subset} & \mathsf{B}_{\mathsf{kt}} & \overset{\text{ouvert}}{\subset} & \mathsf{B}_{\mathsf{t}} & \overset{\text{ouvert}}{\subset} & \mathsf{B} & = & \mathrm{Flag}(V)^s \times \mathrm{Flag}(Q)^s
\end{array}
$$

Proof of the reciprocal

Induction on r by verifying Horn's inequalities on a smaller tuple.

Theorem (Belkale)

$(\mathcal{I}_k)_k$ is a Horn's tuple if and only if $\operatorname{edim} \mathcal{I} \geqslant 0$ and

$$\forall d \in [r-1], \forall \mathcal{J} \in \mathrm{Horn}^*(d, r, s), \operatorname{edim} \mathcal{I}\mathcal{J} \geqslant 0.$$

- Computing Horn's tuples is "easy" using a computer.

Remark

$$\mathrm{Horn}^0 = \{\mathcal{I} \in \mathrm{Horn} \mid \operatorname{edim} \mathcal{I} = 0\}$$

Summary

The Hersch-Zahlen-Klyachko lemma

Lemma (Hersch-Zahlen-Klyachko)

If $\Lambda \in \mathrm{LR}(r, s)$, it verifies Horn's inequalities for Horn's tuples.

Proof : minimisation of a continuous function.

Working with integers

Spectra with integers : seen as weights.

Definition

$$c(\Lambda) := \dim \left(\bigotimes_{k=1}^{s} V(\Lambda_k) \right)^{\mathrm{U}(r)}.$$

Lemma (Kempf-Ness)

For all Λ made of integers,

$$c(\Lambda) > 0 \Rightarrow \Lambda \in \mathrm{LR}(r, s).$$

- Find an invariant for the reciprocal.

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Searching for invariants

If $\operatorname{edim} \mathcal{I} = 0$, $d\omega_{\mathcal{I}}$ is between spaces of the same dimensions.

Definition

$$\delta_{\mathcal{I}} : \left| \begin{array}{ccc} \mathrm{GL}(r)^s \times \mathrm{GL}(n-r)^s & \longrightarrow & \mathbb{C} \\ (g,h) & \longmapsto & \det \Delta_{\mathcal{I},g,h} \end{array} \right. .$$

Proposition

δ is an invariant for $\Lambda(\mathcal{I})$ and any integer Λ satisfying Horn's inequalities comes from a Horn's tuple of zero expected dimension.

- Reciprocal.

Summary

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