Complex numbers : applications to geometry

I) Euclidian geometry

Let's denote $\mathcal{P}$ an euclidian plane, $\mathcal{R}=(0,\vec{i},\vec{j})$ the orthonormed system and P the vectorial euclidian plane.

Considering $M\in\mathcal{P}$ is equivalent to considering $\vec{OM}$ in $\mathbb{R}^2$, so $\mathbb{R}^2$ will be identified to $\mathbb{C}$.

1) Writing objects in $\mathcal{P}$ as objects in $\mathbb{C}$.

definition 1.1 Let $M\in\mathcal{P}$ represented by its coordinates (x,y), its affix is $z_M := x+iy \in \mathbb{C}$.

For all $z\in\mathbb{C}$, we have $M\in\mathcal{P}$ with coordinates $(\mathrm{Re}\, z, \mathrm{Re}\, z)$, we denote this point $M(z)$.

proposition 1.2 Let $M, M'\in\mathbb{R}^2$.

$\boxed{\mathrm{Re}(\bar{z}_M z_{M'}) = xx' + yy' = \vec{OM}\cdot\vec{OM'}}$, $\boxed{\mathrm{Im}(\bar{z}_M z_{M'}) = xy' - x'y = \det(\vec{OM},\vec{OM'})}$

proposition 1.3 If $z = x+iy\in\mathbb{C}$, then $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

application 1.4 : Line equation

$\bar{A}z + A\bar{z} + B = 0$, $A\in\mathbb{C}^*$, $B\in\mathbb{R}$.

application 1.5 • Equation of the circle $\mathcal{C}$, with center $M(\omega)$ and radius r:

$\bar{z}z - \bar{\omega}z - \omega\bar{z} + \bar{\omega}\omega = r^2$

• Equation of the circle (general form):

$\bar{z}z + \bar{\lambda}z + \lambda\bar{z} = \mu$, $(\lambda,\mu)\in\mathbb{C}\times\mathbb{R}$.

• Equation of the conic

$A\bar{z}z + \bar{B}z^2 + B\bar{z}^2 + \bar{C}z + C\bar{z} + D = 0.$

We denote $\Delta = A^2 - 4(\mathrm{Re}(B)^2 + \mathrm{Im}\, B^2)$

Δ sign	+	0	-
conic type	ellipse	parabol	hyperbol

2) The case of the triangle

Let $M(a), M(b), M(c)\in\mathbb{C}$ three non aligned points. Let's denote $\mathcal{T}$ the $M(a)M(b)M(c)$ triangle.

proposition 1.6 The triangle area, denoted S, of $M(a)M(b)M(c)$ is :

$S = \frac{1}{4i}\left((a-b)\bar{c} + (b-c)\bar{a} + (c-a)\bar{b}\right)$

definition 1.7

- The 3 perpendicular bissectors of the triangle $\mathcal{T}$ meet at the circumcenter. This point is the center of the circumscribed circle, which is the only circle on which A, B and C lines.

proposition 1.8 The affix of the following points of $\mathbb{C}$ are :

- Center of gravity G : $g = \frac{a+b+c}{3}$
- Altitude H : $h = a+b+c$.

proposition 1.9

$\mathcal{T}$ is a right triangle in A if and only if $b+c=0$.

$\mathcal{T}$ is isocoles in A if and only if $a^2+bc=0$

3) Angles.

proposition 1.10. Exponential function can be defined on $\mathbb{C}$ by the power series $\sum_{n\geq 0}\frac{z^n}{n!}$

This power series comes with absolute convergence for all $z\in\mathbb{C}$ and uniform convergence over all compact spaces of $\mathbb{C}$.

$\exp : \mathbb{C}\to\mathbb{C}^*$ is a group morphism with kernel $\ker\exp = 2\pi i$

$\mathbb{U} = \{z\in\mathbb{C}, |z|=1\}$, $\varphi : \mathbb{R}\to\mathbb{U}$, $\theta\mapsto e^{i\theta}$ is a group morphism.

proposition 1.11 Let $\vec{u},\vec{v}$ be two unit vectors of S^1.

There exists a unique rotation sending $\vec{u}$ on $\vec{v}$

Then, we define the equivalence relation $\mathcal{R}$:

$(\vec{u},\vec{v})\mathcal{R}(\vec{u'},\vec{v'}) \Leftrightarrow$ there is a rotation f such that $\begin{cases} f(\vec{u}) = f(\vec{u'}) \\ f(\vec{v}) = f(\vec{v'}) \end{cases}$

definition 1.12.

The equivalence class of (u,v) is called oriented angle.

proposition 1.13 $\mathbb{R}\to\mathbb{U} \hookrightarrow O^+(\mathbb{R}) \hookrightarrow (S^1)^2/\mathcal{R}$.

$\theta\mapsto e^{i\theta} \mapsto f:(z\mapsto ze^{i\theta}) \mapsto ((0,1),(\cos\theta,\sin\theta))$.

examples 1.14

polar coordinates $\forall z\in\mathbb{C}^*$, $\exists!(r,\theta)\in\mathbb{R}^{+*}\times[0,2\pi[$, $z = re^{i\theta}$

vertex affix : $\mathcal{T}$ is a direct equilateral triangle $\Leftrightarrow a+bj^2+cj=0$ with $j=e^{\frac{2i\pi}{3}}$

$\mathcal{T}$ is an indirect equilateral triangle $\Leftrightarrow a+bj+cj^2=0$.

4) Transformations

definition 1.15 A similitude of $\mathcal{P}$ is a bijection $f:\mathcal{P}\to\mathcal{P}$ such that $\forall x,y\in\mathbb{R}^2$, $d(f(x),f(y)) = r\,d(x,y)$ where $r\in\mathbb{R}$ and independant from x and y.

In the complex plane, these are the applications of the form $z\mapsto az+b$ (direct) or $z\mapsto a\bar{z}+b$ (indirect) where $(a,b)\in\mathbb{C}^*\times\mathbb{C}$.

proposition 1.16 There exists a unique direct similitude s, of center A such that $s: \begin{matrix} B\mapsto B' \\ C\mapsto C' \end{matrix}$ if and only if there exists one, of center A, s' such that $s' : \begin{cases} B\mapsto C \\ B'\mapsto C' \end{cases}$

examples 1.17 : homothetics, direct isometries

proposition 1.18 : Direct similitudes preserve oriented angles.

proposition 1.19 If $f : P\to P$, a linear application, preserves angles then f is a direct similitude

definition 1.20 An application $f:\mathbb{C}\to\mathbb{C}$ such that $\forall a,b\in\mathbb{C}$, $|f(a)-f(b)| = |a-b|$ is called an isometry.

examples 1.21 : orthogonal symetries, central symetries are isometries

proposition 1.23 An isometry can be written as the composition of one, two or three reflexions

See appendix for the type of isometries we can find

4) inversions

definition 1.24 Applications $h : \begin{array}{l} \mathbb{C} \to \mathbb{C} \\ z \mapsto z' \\ a \mapsto a \end{array}$ such that

- $(z'-a)(z-a) = k$ if $z \neq a$ are called analytic inversions
- $\overline{(z'-a)}(z-a) = k$ if $z \neq a$ are called geometric inversions

we denote its center a and ratio $k \in \mathbb{R}^*$

proposition 1.25 $\overline{(z'-a)}(z-a) = k \iff M(z), M(z')$ and $M(a)$ are aligned

proposition 1.26 A geometric inversion can by written as the composition of an analytic inversion and a conjugate

proposition 1.27 Let's take I an inversion of center $M(a)$ and ratio $k \neq 0$

1) $\{M \in \mathcal{P}, I(M) = M\} = \{M \in \mathcal{P}, \Omega M^2 = k\}. = \begin{cases} \emptyset & \text{if } k < 0 \\ \mathcal{C} & \text{if } k \geq 0 \end{cases}$ where $\mathcal{C}$ is the circle of center $M(a)$ and radius $\sqrt{k}$.

this set is called the inversion circle of I.

2) Let's denote $M(m) = I(M(m))$ and $M(m') = I(M(m'))$

$$m' - m' = \frac{(m-m)k}{(m-a)(m'-a)}.$$

remark 1.28. I can by written $\begin{cases} z : x+iy \mapsto k\left(\frac{x-iy}{x^2+y^2}\right) & \text{if analytic} \\ z : x+iy \mapsto k\left(\frac{x+iy}{x^2+y^2}\right) & \text{if geometric} \end{cases}$

- geometric inversion preserve only oriented angles

application 1.29 Ptolemy's theorem

Let Π be an euclidian plane, and $ABCD$ a (convex) quadrilateral $ABCD$ is inscribable in a circle if and only if $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

II/ Complex projective line and homographies

1) Definition of $\mathbb{P}_1(\mathbb{C})$

Definition 2.1. The equivalence relation on $\mathbb{C}^2 \setminus \{(0,0)\}$ defined by "$x \sim y \iff x$ and y are colinear" allows the definition of the complex projective line : $\mathbb{P}_1(\mathbb{C}) = \dfrac{\mathbb{C}^2 \setminus \{(0,0)\}}{\sim}$

Remark 2.2. $\mathbb{P}_1(\mathbb{C})$ is homeomorphic to $\mathbb{C} \cup \{\infty\}$ and to the Riemann sphere thanks to the stereographic projection.

For more details, see appendix A

2) Homographies

Remark 2.3. $GL_2(\mathbb{C})$ preserves the lines of $\mathbb{C}^2$. Thus, there is an action of $GL_2(\mathbb{C})$ on $\mathbb{P}_1(\mathbb{C})$, namely $\begin{cases} GL_2(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) \longrightarrow \mathbb{P}_1(\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, [z:z'] \longmapsto [az+bz' : cz+dz'] \end{cases}$

Definition 2.4. The complex projectif linear group, denoted by $PGL_2(\mathbb{C})$, is defined by: $PGL_2(\mathbb{C}) = \dfrac{GL_2(\mathbb{C})}{\mathbb{C} I_2} \simeq \left\{ \begin{array}{l} \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \\ z \mapsto \dfrac{az+b}{cz+d} \end{array} \right.$, $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0\}$. Its elements are called homographies.

Remark 2.5. We will take the following conventions :

$a \times \infty = \infty$; $b + \infty = \infty$; $0 \times \infty = 0$; $\frac{a}{0} = \infty$; $\dfrac{a\infty + b}{c\infty + d} = \dfrac{a}{c}$

where $a, c \neq 0$ and $b \in \mathbb{C}$

Proposition 2.6. • A homography is a bijection

• Since $PGL_2(\mathbb{C})$ is a group, the inverse of a homography is a homography

Theorem 2.7 Let a_1, a_2, a_3 be three distinct points of $\mathbb{P}_1(\mathbb{C})$, let b_1, b_2, b_3 be three distinct points of $\mathbb{P}_1(\mathbb{C})$. Then there is one and only one homography h such that $h(a_1) = b_1$, $h(a_2) = b_2$ and $h(a_3) = b_3$

3) Cross - ratios.

Definition 2.8. Let a, b, c be three distinct points of $\mathbb{P}_1(\mathbb{C})$ and $d \in \mathbb{P}_1(\mathbb{C})$. The theorem 2.7 ensures that there is one and only one homography h such that $h(a) = \infty$, $h(b) = 0$ and $h(c) = 1$. The cross-ratio of a, b, c and d is defined by $[a,b,c,d] = h(d)$.

Remark 2.9 For instance, $[a,b,c,a] = \infty$; $[a,b,c,b] = 0$ and $[a,b,c,c] = 1$

Proposition 2.10. With the conventions of remark 2.6,

$$[a,b,c,d] = \frac{(c-a)/(c-b)}{(d-a)/(d-b)}$$

Proposition 2.11. Let a, b, c be three distinct points of $\mathbb{P}_1(\mathbb{C})$ and $d \in \mathbb{P}_1(\mathbb{C})$. Then $[a,b,c,d] \in \overline{\mathbb{R}}$ if and only if a, b, c, d are aligned or cocyclic.

Example 2.12. $[1, i, -1, -i] = 2$, so $1, i, -1$ and $-i$ are aligned or cocyclic (here, cocyclic)

Definition 2.13. Let a, b, c be three distinct points of $\mathbb{P}_1(\mathbb{C})$ and d in $\mathbb{P}_1(\mathbb{C})$. The points a, b, c, d form a harmonic range if $[a,b,c,d] = -1$

Proposition 2.14. Let a, b, c be three distinct points on the same line. The points a, b, c and ∞ form a harmonic range if and only if c is the midpoint of $[ab]$

Proposition 2.15. Any cross-ratio is stable under any homography

Application 2.16 Let h be a homography and $\mathcal{C}$ the set of lines and circles in $\mathbb{C}$. Then $h(\mathcal{C}) \subset \mathcal{C}$

4) Circular group

Definition 2.17. The circular group, denoted by G, is the group generated by $z \mapsto \overline{z}$ and the homographies

Theorem 2.18. G is exactly the set of bijections of $\mathbb{P}_1(\mathbb{C})$ that preserve $\mathcal{C}$

III Hyperbolic geometry.

a. Poincaré half-plan.

Definition 3-1: $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$ We set:

$$d_H(z_1 ; z_2) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|\gamma'(t)|}{Im(\gamma(t))} dt \quad \text{for all } z_1, z_2 \text{ in } H$$

where Γ is the set of the piecewise $\mathcal{C}^1$ paths joining z_1 and z_2 in H.

Proposition 3-2: (H, d_H) is a metric space.

Definition 3-3: Let (X, d) be a metric space. A geodesic joining x_1 and x_2 (two points of X) is the shortest piecewise $\mathcal{C}^1$ path joining x_1 and x_2. A metric space is called unique geodesic if between x_1 and x_2 there exists a unique geodesic.

Proposition 3-4. (H, d_H) is a unique geodesic metric space. Its geodesics are contained in circles and lines perpendicular to the border. (see figure B)

Theorem 3-5: The group of the isometries of H is $\mathbb{P}GL(2;\mathbb{R})$
The group of the orientation preserving isometries of H is $\mathbb{P}SL(2;\mathbb{R})$.

Theorem 3-6: A fundamental domain for the action of $\mathbb{P}SL(2;\mathbb{Z})$ on H is:

$$P = \{z \in H \mid |z| > 1 \text{ and } |Re(z)| < \tfrac{1}{2}\}. \text{ (see figure C)}$$

b. Poincaré Disk

Definition 3-7: Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$. For all z_1, z_2 in D, we set:

$$d(z_1 ; z_2) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|\gamma'(t)|}{1 - |\gamma(t)|^2} dt$$

where Γ is the set of the piecewise $\mathcal{C}^1$ paths joining z_1 and z_2 in D.

Proposition 3-8: (D, d) is a metric space isometric to (H, d)

Proposition 3-9: The isometries of (D, d) are:

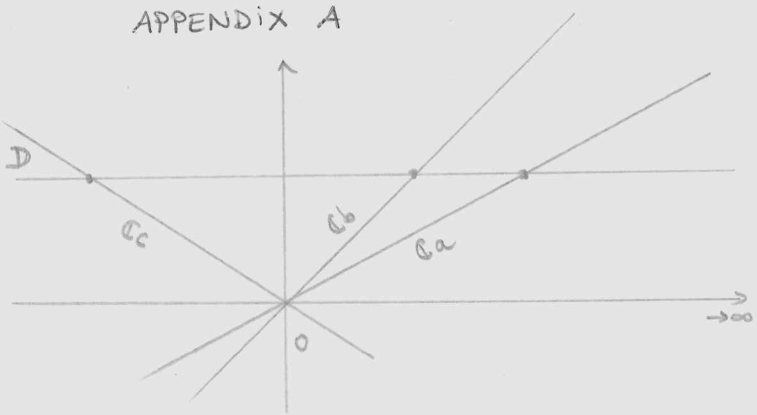
$$f_{\alpha,\beta} : \begin{array}{l} D \longrightarrow D \\ z \longmapsto \dfrac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}} \end{array} \quad \text{where } |\alpha|^2 - |\beta|^2 \neq 0 \quad \text{for some } \alpha, \beta \text{ in } \mathbb{C}.$$

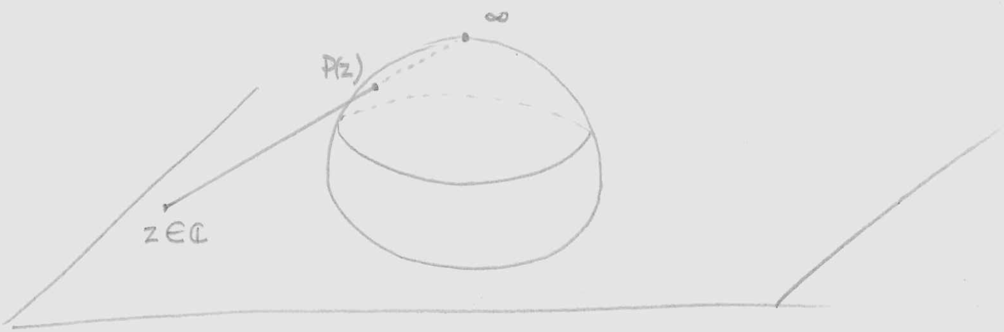
$f_{\alpha,\beta}$ preserves the orientation of D if and only if $|\alpha|^2 - |\beta|^2 > 0$.

Proposition 3-10: Let $f : D \longrightarrow D$ a holomorphic function. Then:

$$\forall (z_1, z_2) \in D \times D : \quad d(f(z_1) ; f(z_2)) \leqslant d(z_1 ; z_2)$$

APPENDIX A





Stereographic projection

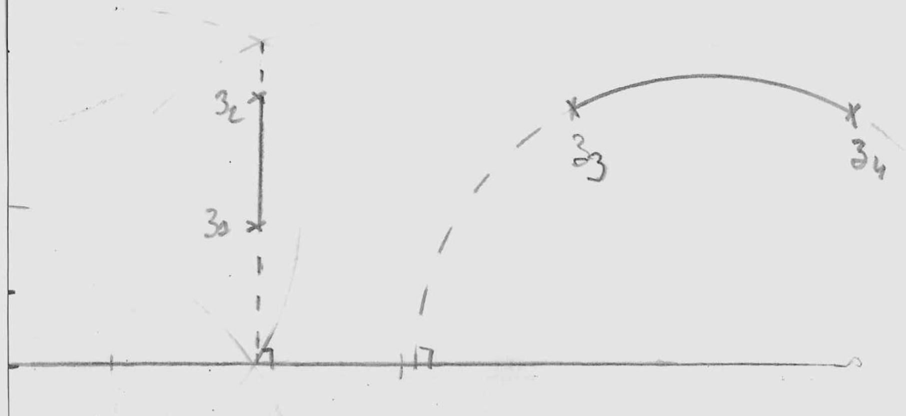


Figure B: Geodesics between z_1 and z_2
z_3 and z_4

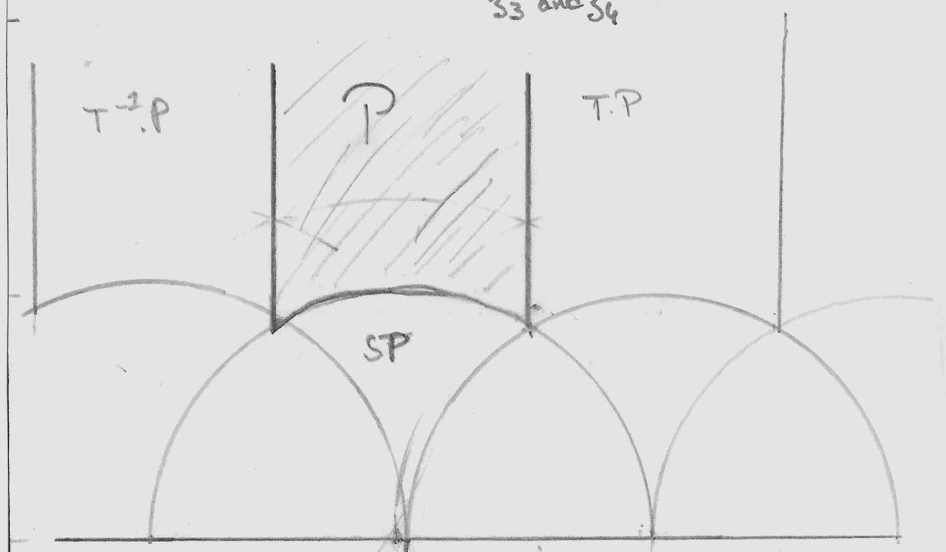


Figure C: P is the fundamental domain:

$T: z \mapsto z+1$

$S: z \mapsto -\dfrac{1}{z}$