Mixing of geodesic flow on infinite volume

With some 🍌 and 🍓

Baptiste Dugué

IMJ-PRG

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Flows

Definition

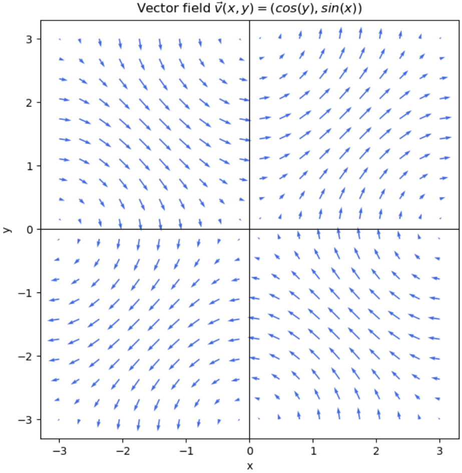
A **flow** on a set X is a map

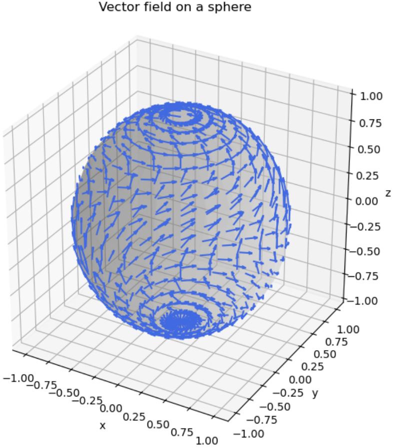
$$\varphi : X \times \mathbb{R} \to X$$

such that for all $x \in X$ and $s, t \in \mathbb{R}$,

$$\varphi_0(x) = x, \quad \text{and } \varphi_{s+t}(x) = \varphi_s(\varphi_t(x)).$$

Flows


Vector field $\vec{v}(x,y) = (cos(y), sin(x))$
y
x
−3 −2 −1 0 1 2 3
3 2 1 0 −1 −2 −3


Vector field on a sphere
x
y
z
−1.00 −0.75 −0.50 −0.25 0.00 0.25 0.50 0.75 1.00
1.00 0.75 0.50 0.25 0.00 −0.25 −0.50 −0.75 −1.00

Geodesics

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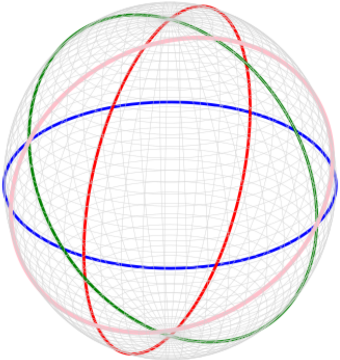
- geodesics on the euclidean plane are straight lines,
- geodesics on the euclidean sphere are **great circles**.



Geodesics

Geodesics on a torus can look like this :

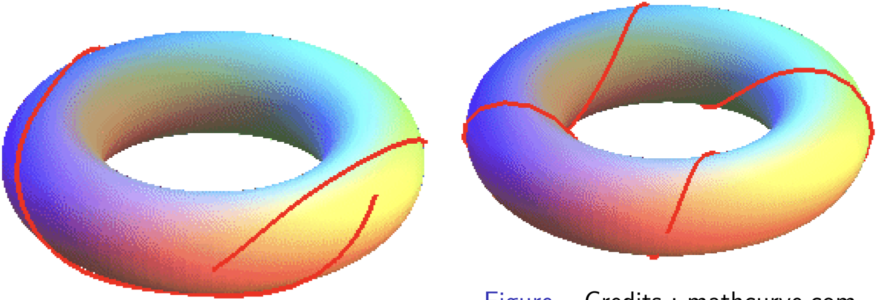
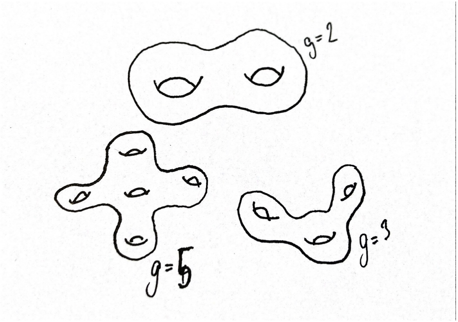


Figure – Credits : mathcurve.com

Such geodesics can be constructed as projections of geodesics of $\mathbb{R}^2$ (i.e. straight lines) by remembering that $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Geodesics

What about surfaces of genus $g \geqslant 2$?



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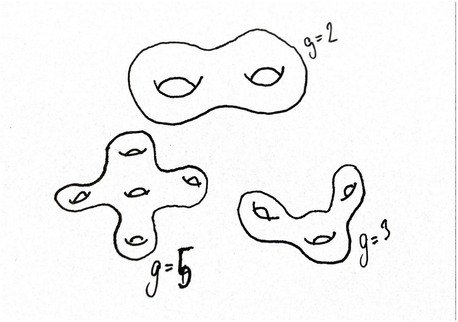
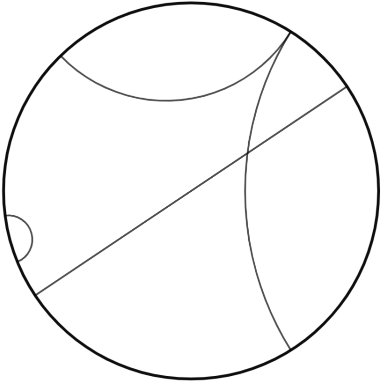


Theorem (Henri Poincaré)

Every such (orientable) surface can always be built as a quotient of the hyperpolic plane $\mathbb{H}^2$ by a lattice Γ, that is, a discrete subgroup of $Isom^+(\mathbb{H}^2) \simeq PSL_2(\mathbb{R})$ with no global fixed point.

Geodesics

In the Poincaré disk model, the geodesics of $\mathbb{H}^2$ are diameters and arcs of circles orthogonal to the boundary.



Geodesics

For example, a genus 2 surface can be built by glueing sides of a geodesic octogon :

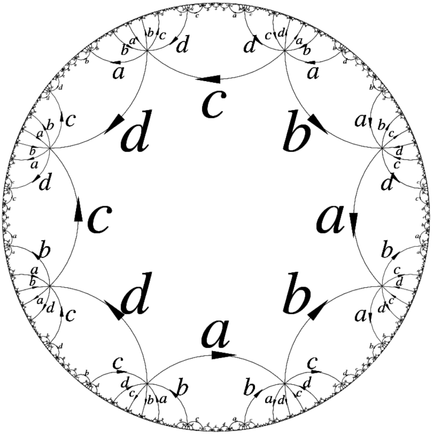


Figure – Credit : Fisher-Hasselblatt, Hyperbolic flows

Geodesic flow

Informal definition

The **geodesic flow** on a manifold M is the flow which moves points along geodesics.

Geodesic flow

Definition

The **tangent bundle** of a manifold M is

$$TM := \bigsqcup_{x \in M} T_x M = \{(x, v) \mid x \in M,\ v \in T_x M\}.$$

The **unit tangent bundle** of M is the unit sphere bundle for TM, namely

$$T^1 M = \{(x, v) \in TM \mid \|v\| = 1\}.$$

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Formal definition

For $(x, v) \in T^1 M$, let $g_{(x,v)} : \mathbb{R} \to M$ be the (unique) geodesic (parametrized by arc length) such that $g(0) = x$ and $g'(0) = v$.
The **geodesic flow** on M, denoted φ_t, is the flow **defined on** $T^1 M$ by :

$$\forall t \in \mathbb{R},\ \varphi_t : \left| \begin{array}{ccc} T^1 M & \longrightarrow & T^1 M \\ (x, v) & \longmapsto & (g_{(x,v)}(t), g'_{(x,v)}(t)) \end{array} \right. .$$

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Definition

Dynamical systems

Definition(informal)

A dynamical system is a bunch of stuff moving around.

Dynamical systems

Let X be a topological space. (Feel free to think only about metric spaces.)

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Discrete dynamical system

Let $\Phi : X \to X$ be a diffeomorphism. The couple (X, Φ) is a **discrete dynamical system**.
In this case, the trajectory of a point $x \in X$ in the future is the set

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Continuous dynamical system

Let $\varphi : (X, \mathbb{R}) \to X$ be a flow. The couple (X, φ) is a **continuous dynamical system**.
In this case, the trajectory of a point $x \in X$ in the future is the set

$$\{\varphi_t(x) \mid t \in \mathbb{R}_+\}.$$

Dynamical systems

- The cat map : $(\mathbb{T}^2, \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix})$
- The geodesic flow on a (hyperbolic) manifold : (T^1M, φ_t)

The cat map and the geodesic flow on a hyperbolic manifold behave a lot like blenders (*mixeurs* in french).

Measure-preserving dynamical system

Measure preserving transformation

Let $(X, \mathcal{A}, \mu)$ be a measured space and $\Phi : X \to X$ be a measurable transformation. One says that Φ **preserves the measure** μ if

$$\forall A \in \mathcal{A},\ \mu(\Phi^{-1}(A)) = \mu(A).$$

Measure-preserving dynamical system

A measure-preserving dynamical system is a dynamical system equipped with a measure and a measure-preserving transformation $(X, \mathcal{A}, \mu, \Phi)$.

How to make a smoothie ?

Strawberry-banana smoothie recipe : 200g of 🍓, 100g of 🍌. Put it all in a good blender (🇫🇷 mixeur 🇫🇷) and press the Φ button for long enough.

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One expects to find, anywhere in the blender, 1/3 of banana and 2/3 of strawberry. One wants the **proportions** of fruits to be the same everywhere : for any region R in the blender, one wants

$$\underbrace{\frac{\mu(\Phi^{-n}(\text{🍓}) \cap R)}{\mu(R)}}_{\text{proportion of 🍓 in } R \text{ in time } n} \xrightarrow[n\to\infty]{} \underbrace{\mu(\text{🍓})}_{\text{total proportion of 🍓 (2/3)}} .$$

Mixing

Mixing

A (measured) dynamical system $(X, \mathcal{A}, \mu, \Phi)$ is said to be **mixing** if

$$\forall A, B \in \mathcal{A},\ \mu(\Phi^{-n}(A) \cap B) \xrightarrow[n \to +\infty]{} \mu(A)\mu(B)$$

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Equivalently (this is a theorem), one can also formulate mixing using L^2 functions :

Mixing, alternative definition

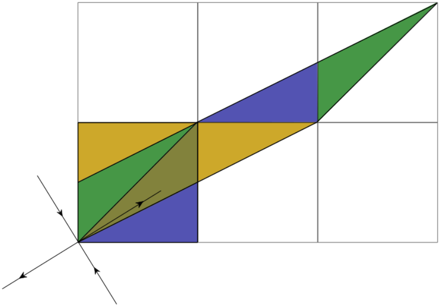
A dynamical system $(X, \mathcal{A}, \mu, \varphi_t)$ is said to be **mixing** if

$$\forall f, g \in L^2(X, \mu),\ \int_X (f \circ \varphi_t) \cdot g d\mu \xrightarrow[t \to \infty]{} \int_X f d\mu \int_X g d\mu.$$

For the probabilists in the room, you could say that the random variables $f \circ \varphi_t$ and g become independant as t grows.

Cat map

The **cat map** is the map induced on $\mathbb{T}^2$ by the action of the matrix $M := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. M has two eigenvalues : $\lambda_1 := \frac{3-\sqrt{5}}{2} \approx 0,4$ and $\lambda_2 := \frac{3+\sqrt{5}}{2} \approx 2,6$. An eigenvector associated to λ_1 is $v_1 := \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix}$.



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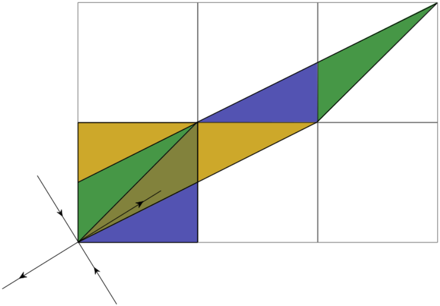

Theorem

The cat map is mixing.

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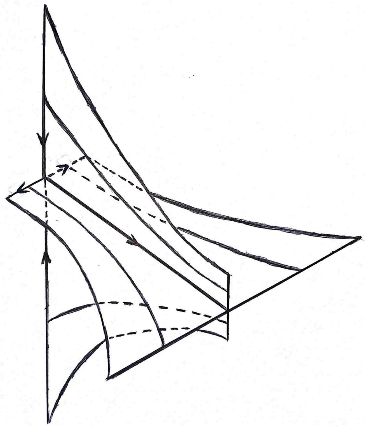
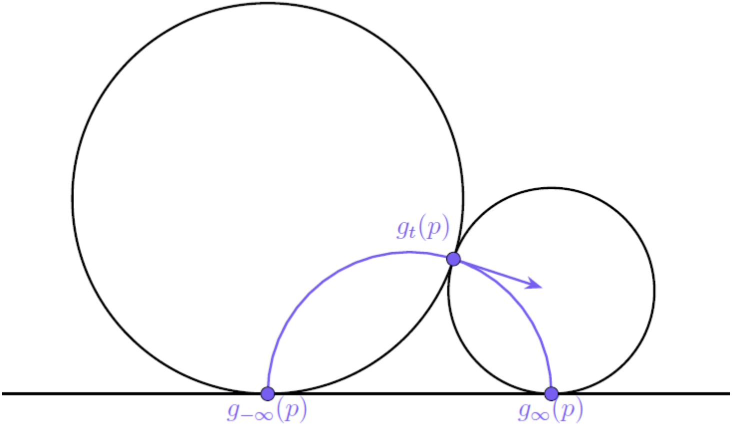
Anosov dynamics

The crucial aspect of the cat map's dynamic is the fact that there are both contracted and dilated directions ; this is an aspect specific to a more general class of dynamic systems, the **Anosov systems**.



Anosov dynamics

It turns out that the geodesic flow on hyperbolic surfaces is also Anosov.
The dilated/contracted directions are constructed from **horocycles** :



Mixing of the geodesic flow

To prove that the geodesic flow on a hyperbolic surface is mixing, one can

- study horocyclic flows, which are closely related to the geodesic flow,
- use an algebraic model : the flows act on $T^1\mathbb{H}^2 \simeq PSL_2(\mathbb{R})$ and can therefore be seen as one-parameter groups of $PSL_2(\mathbb{R})$. We find ourselves manipulating matrices and venturing into Lie theory !

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Theorem

The geodesic flow on a hyperbolic surface is mixing.

...but this is only the beginning of the story !

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Correlation coefficients

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Given a dynamical system (X, μ, φ_t) and two functions $f, g \in L^2(X, \mu)$, the correlation coefficient between f and g in time t is defined as

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Question : How fast does it decay for the geodesic flow ?

What we know so far

Theorem (Marina Ratner)

If M is a **compact** hyperbolic surface, the geodesic flow is **exponentially mixing**.

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And where this presentation ends...

🍓 Thank you very much for your attention ! 🍌

🧋 Any questions ? 🧋