May 28^{th} - June 22^{nd}, 2018
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OSCILLATORY INTEGRALS

$$\int_a^b u'(t)v(t)dt = [u(t)v(t)]_{t=a}^{t=b} - \int_a^b f(t)v'(t)dt$$

$$\hat{f}(\xi) = \int_{-\infty}^{+\infty} f(t)e^{-i\xi t}dt$$

Acknowledgements

I thank Vincent Duchene for its support along my internship.

Abstract

The goal of this internship is the study of oscillatory integrals of one variable, in particular give an asymptotic expansion (with one term), and apply this result to partial differential equations linked with fluids mechanics. We will use two main tools in this study: The Fourier transform and the Integration by parts.

Key-words: Stationnary phase method, integration by parts, partial differential equations, Fourier transform, long time behaviour.

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Part I

Oscillatory integrals of one variable

Chapter 1

First properties and observations

1.1 Generalities

Definition. *An* ***oscillatory integral of one variable*** *is a parametric integral where the form is given by:*

$$F(\lambda) = \int_a^b e^{i\lambda\Phi(t)} g(t)\mathrm{d}t$$

Where g and Φ are two functions respectively complex and real-valued, and Φ is assumed of $\mathcal{C}^2$-class.

We have too $-\infty \leq a < b \leq +\infty$

The goal of this part is the study of the behaviour of $F(\lambda)$ when $\lambda \to +\infty$. By changing Φ by $-\Phi$, we can study the case $\lambda \to -\infty$.

Definition. *Let f and Φ two $\mathcal{C}^1$-class functions. We define the differential operator D by:*

$$Df(t) = \frac{1}{i\lambda\Phi'(t)} \frac{df}{dt}$$

and its transpose:

$$D^T f(t) = \frac{-d}{dt} \left(\frac{f}{i\lambda\Phi'(t)} \right)$$

Remark. The name of transpose comes from similarities with matricies and endomorphisms in linear algebra. Indeed, if we consider this dot product:

$$\langle f|g\rangle = \int_a^b f(t)g(t)\mathrm{d}t$$

With an integration by parts, and good hypothesis (**cf. appendix**), we can show this:

$$\langle Df|g\rangle = \left\langle f|D^T g\right\rangle$$

Property. Let Φ and g two smooth functions. We assume a and b have finite values, g has a compact support in $]a;b[$ and for all $t \in [a;b]$, $\Phi'(t) \neq 0$. We have:

$$\forall N \in \mathbb{N}, \qquad F(\lambda) = \underset{\lambda \to +\infty}{\mathcal{O}} \left(\frac{1}{\lambda^N} \right)$$

Proof. Let $N \in \mathbb{N}$. We have $D^N(e^{i\lambda\Phi(t)}) = e^{i\lambda\Phi(t)}$ (this function is an eigenvector associated to the eigenvalue 1 of the operator D).
We have:

$$\int_a^b e^{i\lambda\Phi(t)} g(t)\mathrm{d}t = \int_a^b D^N \left(e^{i\lambda\Phi(t)} \right) g(t)\mathrm{d}t = \int_a^b e^{i\lambda\Phi(t)} \left(D^T \right)^N (g(t))\, \mathrm{d}t$$

We obtain this result by using N integrations by parts (**cf. appendix**). □

If this results gives a valuation for some sorts of integrals, it does not allows study all sorts of integrals that we defined at the beginning of this part.

1.2 Observations through an exemple

By studying oscillatory integrals, we can observe we have $F(\lambda) \underset{\lambda \to +\infty}{\longrightarrow} 0$.

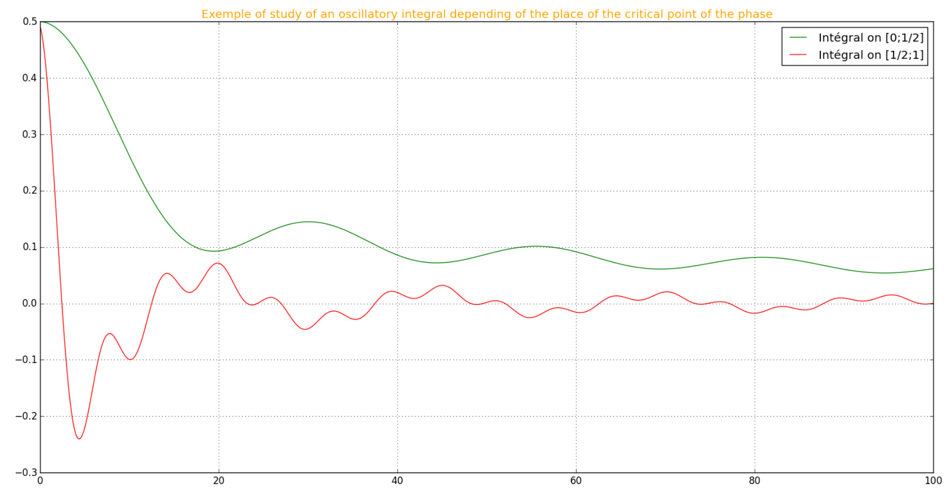
We consider this exemple:

$$F(\lambda) = \int_0^1 \cos \left(\frac{1}{3} t^3 + \lambda t^2 \right) \mathrm{d}t = \mathfrak{Re} \int_0^1 e^{\frac{1}{3} i t^3 + \lambda t^2} \mathrm{d}t$$

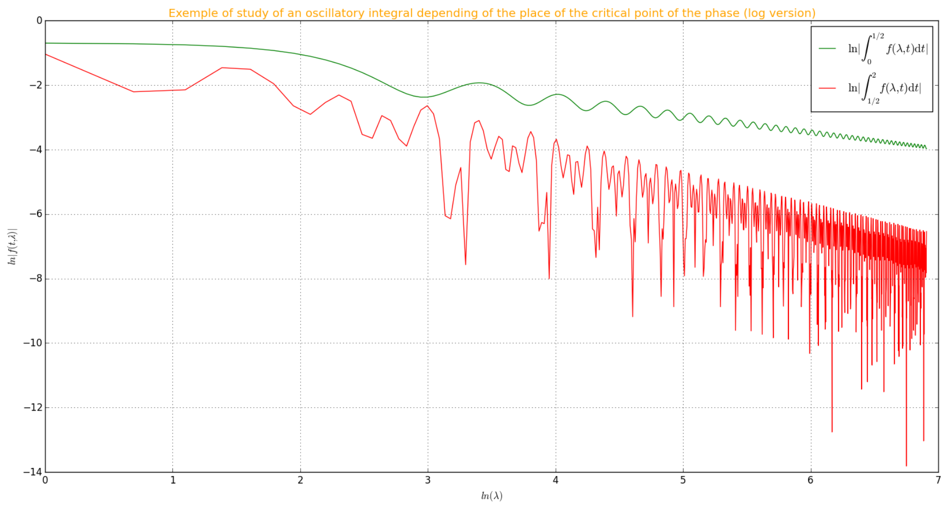
We can cut this integral in two parts:

$$F(\lambda) = \int_0^{\frac{1}{2}} \cos \left(\frac{1}{3} t^3 + \lambda t^2 \right) \mathrm{d}t + \int_{\frac{1}{2}}^1 \cos \left(\frac{1}{3} t^3 + \lambda t^2 \right) \mathrm{d}t$$

We can observe the first part has a slower decay than the second, and this graphic illustrates this phenomenon:



It is possible to have a better observation of the phenomenon if we use a ”log version” of the graphic:

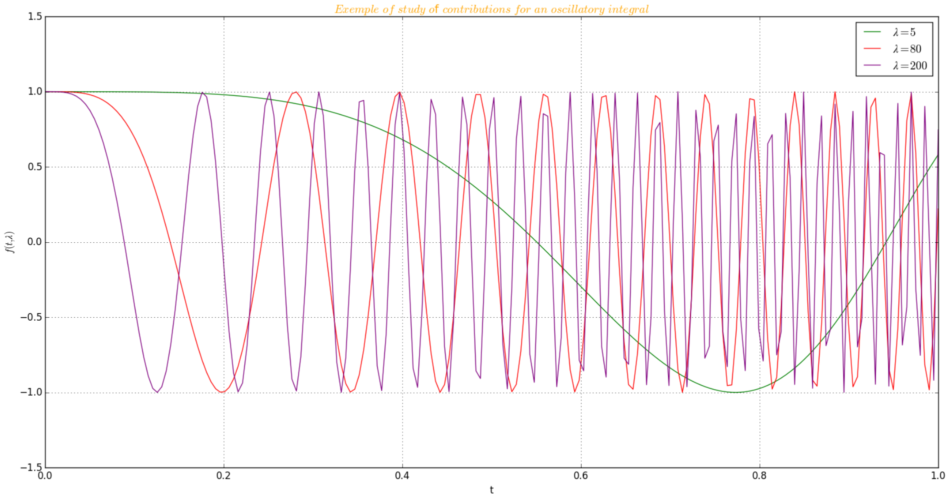


Where

$$f(t, \lambda) = \cos\left(\frac{1}{3}t^3 + \lambda t^2\right) \mathrm{d}t$$

.

On the next graphic, we can observe the contributions to the value of the integral, when $\lambda \to +\infty$, are more important near 0, and, 0 is the only critical point of the phase which is the function $t \longmapsto t^2$.



The goal of the next part will be to determine the order of the decay of an oscillatory integral, and justify why this decay is less important near the critical points of the phase.

Chapter 2

Stationary phase method theorem

We conserve the notations of the previous chapter

2.1 Statement of the Theorem

Theorem. *VALUATION OF OSCILLATORY INTEGRALS*
Let $t_0 \in \mathbb{R}$. *We make these hypothesis:*

- Φ *is* $\mathcal{C}^k$*-class (whith* $k \geq 2$*) and* $\forall j \in [\![1; k-1]\!]$, $\Phi^{(j)}(t_0) = 0$ *and* $\Phi^{(k)}(t_0) \neq 0$.
- $g(t_0) \neq 0$.
- $\exists N \in \mathbb{N}^* :$ g *is* $\mathcal{C}^N$*-class,* $\left(D^T\right)^N (g) \in L^1(\mathbb{R};\mathbb{C})$ *and* $\forall j \in [\![0; N-1]\!]$; $\frac{\left(D^T\right)^j (g)(t)}{\Phi'(t)} \underset{|t|\to+\infty}{\longrightarrow} 0$.

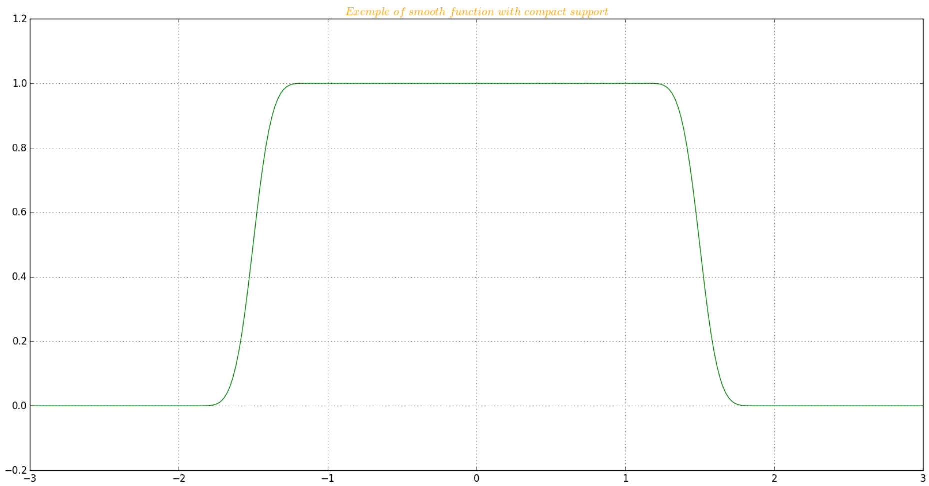
We have:

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mathrm{d}t \underset{\lambda\to+\infty}{\sim} 2e^{i\lambda\Phi(t_0)} g(t_0) \left(\frac{k!}{\Phi^{(k)}(t_0)}\right)^{\frac{1}{k}} \Gamma\left(\frac{k+1}{k}\right) e^{i\frac{\pi}{2k}} \lambda^{-\frac{1}{k}}$$

2.2 Proof of the theorem

Definition. *We define this* ***smooth function with compact support*** *defined by:*

$$\mu(t) = \begin{cases} 1 & \textit{if } |t| \leq 2 \\ \frac{1}{2}\left(1 + \tanh\left(\frac{1}{|t|-1} + \frac{1}{|t|-2}\right)\right) & \textit{if } 1 \leq |t| \leq 2 \\ 0 & \textit{if } |t| \geq 2 \end{cases}$$



Besides, for$\varepsilon > 0$ *and* $t_0 \in \mathbb{R}$*, we define an adapted function named* μ_{ε,t_0} *and defined by:* $\mu_{\varepsilon,t_0} = \mu\left(\frac{t-t_0}{\varepsilon}\right)$

By using our smooth function with compact support, we choose $\varepsilon > 0$ like $\Phi :]t_0 - \varepsilon; t_0 + \varepsilon[\to \Phi(]t_0 - \varepsilon; t_0 + \varepsilon[)$ is a local diffeomorphism, and we have:

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t) \mu_{\varepsilon,t_0}(t) \mathrm{d}t + \int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t) \left(1 - \mu_{\varepsilon,t_0}(t)\right) \mathrm{d}t \qquad (1)$$

We can easily estimate the second integral in (1), by cutting it in three parts:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)}g(t)\left(1-\mu_{\varepsilon,t_0}(t)\right)\mathrm{d}t = \int_{[t_0-2\varepsilon;t_0-\varepsilon]\cup[t_0+\varepsilon;t_0+2\varepsilon]} e^{i\lambda\Phi(t)}g(t)\left(1-\mu_{\varepsilon,t_0}(t)\right)\mathrm{d}t$$

$$+\int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}g(t)\mathrm{d}t + \int_{-\infty}^{-t_0-2\varepsilon} e^{i\lambda\Phi(t)}g(t)\mathrm{d}t$$

If $\Phi' \neq 0$ on the domain $[t_0-2\varepsilon;t_0-\varepsilon]\cup[t_0+\varepsilon;t_0+2\varepsilon]$, the fisrt integral in decreasing on $\underset{\lambda\to+\infty}{\mathcal{O}}(\frac{1}{\lambda^q})$ for all $q\in\mathbb{N}$.

The second integral can easily be estimated. Indeed, a mathematical induction (**cf. appendix**) shows that:

$$\int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}g(t)\mathrm{d}t = \frac{e^{i\lambda\Phi(t_0+2\varepsilon)}}{i\lambda\Phi'(t_0+2\varepsilon)}\sum_{j=0}^{N-1}\left(D^T\right)^j(g)(t_0+2\varepsilon) + \int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}\left(D^T\right)^N(g)(t)\mathrm{d}t$$

And if $\left(D^T\right)^N(g)\in L^1(\mathbb{R};\mathbb{C})$, whe have:

$$\int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}\left(D^T\right)(g)(t)\mathrm{d}t = \underset{\lambda\to+\infty}{\mathcal{O}}\left(\frac{1}{\lambda^N}\right)$$

A similar calculation shows the same result for the third integral, and finally, we obtain:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)}g(t)\left(1-\mu_{\varepsilon,t_0}(t)\right)\mathrm{d}t = \underset{\lambda\to+\infty}{\mathcal{O}}\left(\frac{1}{\lambda}\right)$$

We can now study the first integral in (1)

We can cut our integral in two parts:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t = \int_{0}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t + \int_{-\infty}^{0} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t$$

By making the variable change $t \longmapsto -t$ for the second integral, we obtain:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t = \int_{0}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t + \int_{0}^{+\infty} e^{i\lambda\Phi(-t)} g(-t)\mu_{\varepsilon,t_0}(-t)\mathrm{d}t$$

We can assume $t_0 > 0$ (but the symetric case is similar). The second integral has a quicker decay than the first. If $t_0 = 0$, we can obtain the case $t_0 \neq 0$ by making a variable change $u = t + c$, where $c \neq 0$.

2.2.1 Study of a particular case of phase

Fisrt, we will focus on the case where $\Phi(t) = t^k$, and will use three lemmas:

Lemma 1. *Let $k, l \in \mathbb{N}$, with $k \geq 2$. There exists $\left(c_j^{(l)}\right) \in \mathbb{C}^{\mathbb{N}}$ like:*

$$\int_0^{\infty} e^{i\lambda t^k} t^l e^{-t^k}\mathrm{d}t \underset{\lambda\to+\infty}{\sim} \lambda^{-\frac{1}{k}(l+1)} \sum_{j=0}^{+\infty} c_j^l \lambda^{-j}$$

Remark. In reality, an equivalent gives only on term, and, in our case, we have an equality when λ is taller than a given value.

Proof. In the integral, we make the variable change $t = (1-i\lambda)^{-\frac{1}{k}} z$, and we obtain $e^{i\lambda t^k} t^l e^{-t^k}\mathrm{d}t = (1-i\lambda)^{-\frac{1}{k}(l+1)} z^l e^{-z^k}\mathrm{d}z$, and this:

$$\int_0^{\infty} e^{i\lambda t^k} t^l e^{-t^k}\mathrm{d}t = (1-i\lambda)^{-\frac{1}{k}(l+1)} \int_{(1-i\lambda)^{\frac{1}{k}}\mathbb{R}} z^l e^{-z^k}\mathrm{d}z$$

.

We note:

$$I_{l,k} = \int_{(1-i\lambda)^{\frac{1}{k}}\mathbb{R}} z^l e^{-z^k}\mathrm{d}z$$

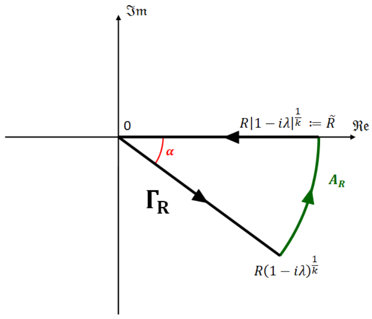
which is a line integral of a holomorphic function.

By convention, we have $\alpha := \arg\left((1-i\lambda)^{\frac{1}{k}}\right) = \frac{1}{k}\arg(1-i\lambda) \in \left]-\frac{\pi}{2k}; 0\right[$.

By using complex analysis, we will show that:

$$I_{l,k} = \int_0^{+\infty} z^l e^{-z^k}\mathrm{d}z$$

We consider this complex contour, named Γ_R, with $R > 0$:



The function $z \mapsto z^l e^{-z^k}$ is holomorphic and without singularity on $\overline{Int\,(\Gamma_R)}$, and, according to the Cauchy's integral theorem, we have:

$$\oint_{\Gamma_R} z^l e^{-z^k} \mathrm{d}z = 0$$

We cut our line integral in three parts, corresponding to the three parts or our contour:

$$\int_{(1-i\lambda)^{\frac{1}{k}}} z^l e^{-z^k} \mathrm{d}z - \int_0^{\widetilde{R}} z^l e^{-z^k} \mathrm{d}z + \int_{A_R} z^l e^{-z^k} \mathrm{d}z = 0$$

If we have $A_R = e^{i\theta},\ \ \theta \in [-\alpha; 0]$, the third integral writes this way:

$$\int_{A_R} z^l e^{-z^k} \mathrm{d}z = -\int_0^{\alpha} i\widetilde{R}e^{i\theta}\widetilde{R}^l e^{li\theta} e^{-\widetilde{R}^k e^{ki\theta}} \mathrm{d}\theta = -\widetilde{R}^{l+1} \int_0^{\alpha} e^{(l+1)i\theta} e^{-\widetilde{R}^k e^{ki\theta}} \mathrm{d}\theta$$

And, by using a standard majoration, we obtain:

$$\left| \int_{A_R} z^l e^{-z^k} \mathrm{d}z \right| \leq |\alpha|\, \widetilde{R}^{l+1} \sup_{\theta \in [\alpha;0]} \left| e^{-\widetilde{R}^k \cos(k\theta)} \right|$$

We have:

$$\begin{aligned}
-\frac{\pi}{2k} < \alpha \leq \theta \leq 0 \\
-\frac{\pi}{2} < k\alpha \leq k\theta \leq 1 \\
0 < \cos(k\alpha) \leq \cos(k\theta) \leq 1 \\
1 > e^{-\widetilde{R}^k \cos(k\alpha)} \geq e^{-\widetilde{R}^k \cos(k\theta)} \geq e^{-\widetilde{R}^k}
\end{aligned}$$

And we obtain:

$$\left| \int_{A_R} z^l e^{-z^k} \mathrm{d}z \right| \leq |\alpha| \underbrace{\widetilde{R}^{l+1} e^{-\widetilde{R}^k \cos(k\alpha)}}_{\substack{\longrightarrow 0 \\ \widetilde{R} \to +\infty \Leftrightarrow R \to +\infty}}, \quad \cos(k\alpha) > 0$$

And if $R \to +\infty$, we have:

$$I_{l,k} = \int_0^{+\infty} z^l e^{-z^k} \mathrm{d}z$$

Now, if we make the variable change $z = s^{\frac{1}{k}}$, we have:

$$I_{l,k} = \frac{1}{k} \int_0^{+\infty} s^{\frac{l+1-k}{k}} e^{-s} \mathrm{d}s = \frac{1}{k} \Gamma\left(\frac{l+1}{k}\right)$$

Where Γ is the Euler Gamma function:

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} \mathrm{d}t$$

And we obtain:

$$\begin{aligned}
\int_0^{+\infty} e^{i\lambda t^k} t^l e^{-t^k} \mathrm{d}t = (1 - i\lambda)^{-\frac{1}{k}(l+1)} \frac{1}{k} \Gamma\left(\frac{l+1}{k}\right) \\
\int_0^{+\infty} e^{i\lambda t^k} t^l e^{-t^k} \mathrm{d}t = \lambda^{-\frac{1}{k}(l+1)} \underbrace{(\lambda^{-1} - i)^{-\frac{1}{k}(l+1)}}_{(*)} \frac{1}{k} \Gamma\left(\frac{l+1}{k}\right)
\end{aligned}$$

When $\lambda > 1$, it is possible to write the function($*$) as a power serie (1 is its radius of convergence). For this reason, there exists $\left(\tilde{c_j}^{(l)}\right) \in \mathbb{C}^{\mathbb{N}}$ like:

$$\forall \lambda > 1; \quad \left(\lambda^{-1} - i\right)^{-\frac{1}{k}(l+1)} = \sum_{j=0}^{+\infty} \tilde{c_j}^{(l)} \lambda^{-j}$$

That's why we obtain:

$$\int_0^{+\infty} e^{i\lambda t^k} t^l e^{-t^k} \mathrm{d}t \underset{\lambda \to +\infty}{\sim} \lambda^{-\frac{1}{k}(l+1)} \sum_{j=0}^{+\infty} c_j{}^{(l)} \lambda^{-j}$$

With $c_j{}^{(l)} = \frac{1}{k}\Gamma\left(\frac{l+1}{k}\right) \tilde{c_j}^{(l)}$.

This equivalent is in reality an equality when $\lambda > 1$. □

Lemma 2. *Let η a smooth function with compact support, and $l \in \mathbb{N}^*$. There exists $A \geq 0$ like:*

$$\left| \int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \mathrm{d}t \right| \leq A \lambda^{-\frac{1}{k}(l+1)}$$

Proof. Let $\varepsilon > 0$. We consider our function $\mu_{\varepsilon,0}$ defined at the begining of this section, and write our integral like this:

$$\int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \mathrm{d}t = \int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \mu_{\varepsilon,0}(t) \mathrm{d}t + \int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right) \mathrm{d}t$$

The first integral can easily be bounded from above:

$$\left| \int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \mu_{\varepsilon,0}(t) \mathrm{d}t \right| = \left| \int_0^{2\varepsilon} e^{i\lambda t^k} t^l \eta(t) \mu_{\varepsilon,0}(t) \mathrm{d}t \right| \leq \underbrace{2 \left\| \eta \right\|_{L^\infty}}_{:=C} \varepsilon^{l+1} \qquad [1]$$

The second integral is more difficult to be bounded from above. We consider the operator D defined at the begining of the chapter 1, and a positive integer N checking $N > \frac{1}{k}(l+1)$. We have:

$$\int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right) \mathrm{d}t = \int_0^{+\infty} e^{i\lambda t^k} \left(D^T\right)^N \left(t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right) \mathrm{d}t$$

which is possible because the function $t \longmapsto t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)$ vanishes near 0 and when $t \to +\infty$.

$$\int_0^{+\infty} e^{i\lambda t^k} t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right) \mathrm{d}t = \int_\varepsilon^{+\infty} e^{i\lambda t^k} \left(D^T\right)^N \left(t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right) \mathrm{d}t$$

Besides, a mathematical induction (**cf. appendix**) shows that:

$$\left(D^T\right)^N \left(t^l \eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right) = \frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l}\, t^j \left(\frac{d}{dt}\right)^j \left[\eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right]$$

with the complex numbers $C_{N,j,l}$ depending on N, j and l.
By the Leibniz rule, we have too:

$$\left(\frac{d}{dt}\right)^j \left[\eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right] = \sum_{p=0}^{j} \binom{j}{p} \left(1 - \mu_{\varepsilon,0}\right)^{(p)} (t) \eta^{(j-p)}(t)$$

$$\left(\frac{d}{dt}\right)^j \left[\eta(t) \left(1 - \mu_{\varepsilon,0}(t)\right)\right] = \sum_{p=0}^{j} \binom{j}{p} \frac{1}{\varepsilon^p} \left(1 - \mu\right)^{(p)} (t) \eta^{(j-p)}(t)$$

And we obtain:

$$\left(D^T\right)^N \left(t^l \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right) =$$

$$\frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l}\; t^j \sum_{p=0}^{j} \binom{j}{p} \frac{1}{\varepsilon^p} (1-\mu)^{(p)} \left(\frac{t}{\varepsilon}\right) \eta^{(j-p)}(t)$$

$$= \frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l}\; \left(\frac{t}{\varepsilon}\right)^j \sum_{p=0}^{j} \binom{j}{p} \varepsilon^{j-p} (1-\mu)^{(p)} \left(\frac{t}{\varepsilon}\right) \eta^{(j-p)}(t)$$

$$= \frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l} \sum_{p=0}^{j} \binom{j}{p} \varepsilon^{j-p} \left(\frac{t}{\varepsilon}\right)^j (1-\mu)^{(p)} \left(\frac{t}{\varepsilon}\right) \eta^{(j-p)}(t)$$

$$= \frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l} \sum_{p=0}^{j} \binom{j}{p} \left[\tau \mapsto \tau^j (1-\mu)^{(p)}(\tau)\right] \left(\frac{t}{\varepsilon}\right) \eta^{(j-p)}(t) \varepsilon^{j-p}$$

If we assume $\varepsilon \geq 1$, we obtain:

$$\left|\left(D^T\right)^N \left(t^l \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right)\right| \leq$$

$$\frac{1}{\lambda^N} t^{l-kN} \underbrace{\sum_{j=0}^{N} C_{N,j,l} \sum_{p=0}^{j} \binom{j}{p} \left\|\tau \mapsto \tau^j (1-\mu)^{(p)}(\tau)\right\|_{L^\infty} \left\|\eta^{(j-p)}\right\|_{L^\infty}}_{:=\widetilde{C_N}}$$

If we integrate this inequality between ε and $+\infty$, we obtain:

$$\int_{\varepsilon}^{+\infty}\left|\left(D^T\right)^N\left(t^l\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right)\right|\mathrm{d}t \leq \frac{\widetilde{C_N}}{\lambda^N}\int_{\varepsilon}^{+\infty}t^{l-kN}\mathrm{d}t = \underbrace{\frac{\widetilde{C_N}}{kN-l+1}}_{:=C_N} \quad [2]$$

By adding the two inequalities [1] and [2], we obtain:

$$\int_0^{+\infty}e^{i\lambda t^k}tl\eta(t)\mathrm{d}t \leq C\varepsilon^{l+1}+C_N\varepsilon^{l-kN+1}\lambda^{-N}$$

and if we choose $\varepsilon=\lambda^{-\frac{1}{k}}$ (the hypothesis $\varepsilon\leq 1$ is checked if λ is very tall), and we have our inequality if we note $A=C+C_N$.

□

Definition. *We define the* ***Schwartz space*** *on this way:*

$$\mathcal{S}(\mathbb{R})=\left\{f\in\mathcal{C}^\infty : \forall\alpha,\beta\in\mathbb{N};\quad \sup_{x\in\mathbb{R}}\left|x^\alpha f^{(\beta)}(x)\right|\right\}$$

Lemma 3. *Let $h\in\mathcal{S}(\mathbb{R})$ like h vanishes near 0 (not only in 0). For all $q\in\mathbb{N}$, we have:*

$$\int_0^{+\infty}e^{i\lambda t^k}h(t)\mathrm{d}t = \underset{\lambda\to+\infty}{\mathcal{O}}\left(\frac{1}{\lambda^q}\right)$$

Proof. If we consider again the differential operator D defined at the beginning of the chapter 1.

$$\int_0^{+\infty}e^{i\lambda t^k}h(t)\mathrm{d}t = \int_0^{+\infty}D^q\left(e^{i\lambda t^k}\right)h(t)\mathrm{d}t$$

$$=\int_0^{+\infty}e^{i\lambda t^k}\left(D^T\right)^q\left(h(t)\right)\mathrm{d}t = \underset{\lambda\to+\infty}{\mathcal{O}}\left(\frac{1}{\lambda^q}\right)$$

Indeed, if we apply the operator D, we gain a $\frac{1}{\lambda}$-factor, and, in our case, we have $\Phi(t) = t^k$. The derivative of the phase reaches 0 when $t = 0$, but the operator is well-defined everywhere because h vanishes near 0, and there is any problem of integrability. When $t \to +\infty$, there is not problem of integrability too, because $h \in \mathcal{S}(\mathbb{R})$.

□

Property. If g is supported in a neighborhood of 0 and $g(0) \neq 0$, we have:

$$\int_0^{+\infty} e^{i\lambda t^k} g(t)\mathrm{d}t \underset{\lambda \to +\infty}{\sim} g(0)\Gamma\left(\frac{k+1}{k}\right) e^{\frac{i\pi}{2k}} \lambda^{-\frac{1}{k}}$$

Proof. We introduce a smooth function $\widetilde{g}$ with compact support like for all $x \in Supp(g)$; $g(x) = 1$, and we have:

$$\int_0^{+\infty} e^{i\lambda t^k} g(t)\mathrm{d}t = \int_0^{+\infty} e^{i\lambda t^k} e^{-t^k} \left[e^{t^k} g(t)\right] \widetilde{g}(t)\mathrm{d}t$$

Let $N \in \mathbb{N}$. If we write a Taylor's approximation to the N^{th} order of the function $t \longmapsto e^{t^k} g(t)$ (which is possible because this function is smooth), we obtain:

$$e^{t^k} g(t) = \underbrace{\sum_{p=0}^{N} b_p t^p}_{:=P(t)} + \underbrace{t^{N+1} R_N(t)}_{\underset{t\to 0}{=} \mathcal{O}(t^{N+1})}$$

By replacing the function $t \longmapsto e^{t^k} g(t)$ by the previous expansion, we obtain:

$$\int_0^{+\infty} e^{i\lambda t^k} g(t)\mathrm{d}t = \int_0^{+\infty} e^{i\lambda t^k} e^{-t^k} \left[P(t) + t^{N+1} R_N(t)\right] \widetilde{g}(t)\mathrm{d}t$$

$$= \int_0^{+\infty} e^{i\lambda t^k} e^{-t^k} P(t) \underset{\sim}{g}(t)\mathrm{d}t + \int_0^{+\infty} e^{i\lambda t^k} t^{N+1} R_N(t) e^{-t^k} \widetilde{g}(t)\mathrm{d}t$$

$$+ \int_0^{+\infty} e^{i\lambda t^k} P(t) e^{-t^k} \left[\widetilde{g}(t) - 1\right] \mathrm{d}t$$

$$= \underbrace{\sum_{p=0}^{N} b_p \int_0^{+\infty} e^{(i\lambda - 1)t^k} t^p \mathrm{d}t}_{:=I_1} + \underbrace{\int_0^{+\infty} e^{i\lambda t^k} t^{N+1} R_N(t) e^{-t^k} \widetilde{g}(t)\mathrm{d}t}_{:=I_2}$$

$$+ \underbrace{\int_0^{+\infty} e^{i\lambda t^k} P(t) e^{-t^k} \left[\widetilde{g}(t) - 1\right] \mathrm{d}t}_{:=I_3}$$

We will first study the integrals I_2 and I_3, and then the integral I_1:

- I_2-Bounding from above:

If we set down $\eta(t) = R_N(t)e^{-t^k}\widetilde{g}(t)$, we have our smooth function with compact support, and, according with the lemma 2, we obtain:

$$|I_2| \leq A\lambda^{-\frac{1}{k}(N+2)}$$

where A is a positive number.

- I_3-bounding from above:

If we set down $h(t) = P(t)e^{-t^k}\left[\widetilde{g}(t) - 1\right]$, we obtain $h \in \mathcal{S}(\mathbb{R})$, a function of the Shwartz space (we can extend h to the zero function for all $t \leq 0$). According to the lemma 3, we have:

$$I_3 = \underset{\lambda \to +\infty}{\mathcal{O}}\left(\frac{1}{\lambda^q}\right)$$

- Study of I_1:

According to the lemma 1, there exists a sequence of complex-valued sequences $\left(\left(c_j^{(p)}\right)_{j\in\mathbb{N}}\right)_{p\in\mathbb{N}}$ like:

$$I_1 \underset{\lambda\to+\infty}{\sim} \sum_{p=0}^{N} b_p \lambda^{-\frac{1}{k}(p+1)} \sum_{j=0}^{+\infty} c_j^{(p)} \lambda^{-j} \qquad [1]$$

which is in reality an equality when $\lambda > 1$.

If we compare this expansion with the two previous expansions, this term is the biggest, and the integrals I_2 and I_3 have a quick decay (because we choose arbitrarity the integer N). The biggest term of the expansion [1] is given by:

$$b_0 c_0^{(0)} \lambda^{-\frac{1}{k}}$$

whith $c_0^{(0)} = \frac{1}{k}\Gamma\left(\frac{1}{k}\right)\widetilde{c_0^{(0)}}$ and $\widetilde{c_0^{(0)}} = (-i)^{-\frac{1}{k}} = i^{\frac{1}{k}} = e^{\frac{i\pi}{2k}}$.

An integration by parts shows that $\frac{1}{k}\Gamma\left(\frac{1}{k}\right) = \Gamma\left(\frac{k+1}{k}\right)$(**cf. appendix**)

Besides, $b_0 = g(0)$, which gives the good asymprotic expansion (which is correct because $g(0) \neq 0$. □

Remark. If $g(0) = 0$, we will have to compute the coefficients of the next terms of decreasing, which become the biggest terms when λ is big, but this computation is more difficult than the previous.

If we want to have the valuation for the integral

$$\int_{-\infty}^{+\infty} e^{i\lambda t^k} g(t)\mathrm{d}t$$

We will have to consider the part of the integral from $-\infty$ to 0, and a similar reasoning shows:

$$\int_{-\infty}^{0} e^{i\lambda t^k} g(t)\mathrm{d}t \underset{\lambda \to +\infty}{\sim} g(0)\Gamma\left(\frac{k+1}{k}\right) e^{\frac{i\pi}{2k}} \lambda^{-\frac{1}{k}}$$

which gives this expansion (if we consider an asymptotic expansion and not an equivalent in order to add these):

$$\int_{-\infty}^{+\infty} e^{i\lambda t^k} g(t)\mathrm{d}t \underset{\lambda \to +\infty}{\sim} 2g(0)\Gamma\left(\frac{k+1}{k}\right) e^{\frac{i\pi}{2k}} \lambda^{-\frac{1}{k}}$$

2.2.2 Generalization for all phases

The goal of this subsection is the generalization for all phases checking the hypothesis of the theorem of the begining of this chapter.

Property. Let $t_0 \geq 0$. We assume Φ is $\mathcal{C}^k$-class and:
$\forall j \in [\![2; k-1]\!]$; $\Phi^{(j)}(t_0) = 0$ and $\Phi^{(k)}(t_0) \neq 0$. We assume g supported in a neighborhood of t_0 and $g(t_0) \neq 0$ too.

We have:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mathrm{d}t \underset{\lambda\to+\infty}{\sim} 2e^{i\lambda\Phi(t_0)} g(t_0) \left(\frac{k!}{\Phi^{(k)}(t_0)}\right)^{\frac{1}{k}} \Gamma\left(\frac{k+1}{k}\right) e^{\frac{i\pi}{2k}} \lambda^{-\frac{1}{k}}$$

Proof. In the integral

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mathrm{d}t$$

we make the variable change $y = (\Phi(t) - \Phi(t_0))^{\frac{1}{k}}$. If we use the Taylor's formula for Φ, which is:

$$\Phi(t) = \Phi(t_0) + \frac{1}{k!}\Phi^{(k)}(t_0)(t-t_0)^k + (1+\varepsilon(t))(t-t_0)^{k+1}$$

where ε is a smooth function with $\varepsilon(t) \underset{t\to t_0}{\longrightarrow} 0$
we have:

$$y(t) = \left(\frac{1}{k!}\Phi^{(k)}(t_0)(t-t_0)^k + (1+\varepsilon(t))(t-t_0)^{k+1}\right)^{\frac{1}{k}}$$

If we note $U = Supp(g)$, we have y which is a differentiable function on U and its derivative is given by:

$$\begin{array}{rccc} y' : & U & \longrightarrow & \mathbb{C} \\ & t & \longmapsto & \left(\frac{\Phi^{(k)}(t_0)}{k!}\right)^{\frac{1}{k}} \left[(1+\varepsilon(t))^{\frac{1}{k}} + \frac{1}{k}(t-t_0)\left(1+\varepsilon(t)\right)^{\frac{1}{k}-1}\right] \end{array}$$

We have $y'(t_0) = \left(\frac{\Phi^{(k)}(t_0)}{k!}\right)^{\frac{1}{k}} \neq 0$.

By the inverse function theorem, there exist two open sets $\widetilde{U} \subset U$ and $V \subset \mathbb{C}$ with $t_0 \in \widetilde{U}$ and $0 \in V$ like $y : \widetilde{U} \to V$ is a $\mathcal{C}^1$-diffeomorphism.

If we set down $f(y) = g\left(t(y)\right)\frac{dt}{dy}(y)$, which is possible because $t : V \to \widetilde{U}$ is invertible and $\mathcal{C}^1$-class. We will have $f(0) = g(t_0)\left(\frac{k!}{\Phi^{(k)}(t)}\right)^{\frac{1}{k}}$, because $y(t_0) = 0$ and $\frac{dt}{dy}(0) = \left(\frac{dy}{dt}(t_0)\right)^{-1}$. If we extend f to the zero function out of V, we obtain:

$$\int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mathrm{d}t = \int_{-\infty}^{+\infty} e^{i\lambda(\Phi(t)-\Phi(t_0)+\Phi(t_0)} g(t)\mathrm{d}t$$

$$= e^{i\lambda\Phi(t_0)} \int_{-\infty}^{+\infty} e^{i\lambda y^k} g\left(t(y)\right) \frac{dt}{dy}(y)\mathrm{d}y = e^{i\lambda\Phi(t_0)} \int_{-\infty}^{+\infty} e^{i\lambda y^k} f(y)\mathrm{d}y$$

According to the previous property, we have:

$$\int_{-\infty}^{+\infty} e^{i\lambda y^k} f(y)\mathrm{d}y \underset{\lambda\to+\infty}{\sim} 2f(0)\Gamma\left(\frac{k+1}{k}\right) e^{\frac{i\pi}{2k}} \lambda^{-\frac{1}{k}}$$

We know the value of $f(0)$ and are able to deduce the formula of the statement.

□

In the initial formula where we used our smooth function with compact support,μ_{ε,t_0} named in order to cut our integral in two parts:

$$F(\lambda) = \int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\mu_{\varepsilon,t_0}(t)\mathrm{d}t + \int_{-\infty}^{+\infty} e^{i\lambda\Phi(t)} g(t)\left(1 - \mu_{\varepsilon,t_0}(t)\right)\mathrm{d}t$$

The first integral has a decay in $\frac{1}{\sqrt[k]{\lambda}}$ whereas the second has a decay in $\frac{1}{\lambda}$.

Therefore, the first integral is the biggest term when $\lambda \to +\infty$, which ends the proof of the theorem.

Remarks. - *If the function Φ has several critical points, by using one smooth function with compact support μ_{ε,t_0} for each critical point, and replacing the unique function used for the case of one critical point by the sum of these functions used for each critical point of Φ. For this reason, we have to add the equivalents (if we assume the sum is not equal to 0).*

- *When the oscillatory integral is defined like $a > -\infty$ and $b < +\infty$, the third hypothesis of the theorem is not necessary.*

- *The 2-factor in the theorem's formula comes from 0, which is present as a critical point of the phase $\Phi(t) = t^k$ in the both integrals (between $-\infty$ and 0, and between 0 and $+\infty$) when we study the particular case of phase.*

Chapter 3

Application to special functions

3.1 Bessel functions

Definition. *We define the Bessel functions this way:*

$$J_m(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\lambda \sin\theta - im\theta} \mathrm{d}\theta$$

with $m \in \mathbb{Z}$

Property.

$$J_m(\lambda) \underset{\lambda \to +\infty}{=} \sqrt{\frac{2}{\pi\lambda}} \cos\left(\lambda - \frac{\pi}{4} - m\frac{\pi}{2}\right) + \mathcal{O}\left(\frac{1}{\lambda}\right)$$

Proof. We will use the stationary phase method.
We have:

$$\begin{aligned} \frac{d}{d\theta}(\sin\theta) = 0 \quad &\Leftrightarrow \quad \cos\theta = 0 \\ &\Leftrightarrow \quad \theta \in \left\{\frac{\pi}{2}; \frac{3\pi}{2}\right\} \end{aligned}$$

By using the theorem's formula giving an approximation of the oscillatory integrals when λ is big, we obtain, for $\sin''\left(\frac{\pi}{2}\right) = -1$ and $\sin''\left(\frac{3\pi}{2}\right) = 1$, this approximation:

$$2\pi J_m(\lambda) \underset{\lambda\to+\infty}{\sim} 2e^{i\lambda}e^{-im\frac{\pi}{2}}\left(\frac{2}{-1}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)e^{i\frac{\pi}{4}}\frac{1}{\sqrt{\lambda}}$$

$$+\ 2e^{-i\lambda}e^{-im\frac{3\pi}{2}}\left(\frac{2}{1}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)e^{i\frac{\pi}{4}}\frac{1}{\sqrt{\lambda}}$$

$$2\pi J_m(\lambda) \underset{\lambda\to+\infty}{\sim} e^{i\lambda-im\frac{\pi}{2}-i\frac{\pi}{2}}2\sqrt{2}\Gamma\left(\frac{3}{2}\right)e^{i\frac{\pi}{4}}\frac{1}{\sqrt{\lambda}}$$

$$+\ e^{-i\lambda+im\frac{\pi}{2}}2\sqrt{2}\Gamma\left(\frac{3}{2}\right)e^{i\frac{\pi}{4}}\frac{1}{\sqrt{\lambda}}$$

Besides, we have $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$ (**cf. appendix**) shows that:

For this reason, we have:

$$\pi J_m(\lambda) \underset{\lambda\to+\infty}{\sim} \sqrt{\frac{\pi}{2\lambda}}\left[e^{i\left(\lambda-\frac{\pi}{4}-m\frac{\pi}{2}\right)} + e^{-i\left(\lambda-\frac{\pi}{4}-m\frac{\pi}{2}\right)}\right]$$

$$J_m(\lambda) \underset{\lambda\to+\infty}{\sim} \sqrt{\frac{2}{\pi\lambda}}\cos\left(\lambda-\frac{\pi}{4}-m\frac{\pi}{2}\right)$$

The next term $\mathcal{O}\left(\frac{1}{\lambda}\right)$ comes from the formula [1] given p.25. □

3.2 Airy function

Definition. *The* ***Airy function of the first kind*** *is a solution of this ordinary differential equation:*

$$\frac{d^2y}{dx^2} - xy = 0$$

It is given by this formula:

$$Ai(x) = C\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}t^3+xt\right)}\mathrm{d}t \quad \textit{with } C \in \mathbb{C}$$

Property.

$$\forall N \in \mathbb{N};\ \ Ai(x) \underset{x\to+\infty}{=} \mathcal{O}\left(\frac{1}{x^N}\right)$$

Proof. Let $x > 0$. In the integral

$$Ai(x) = C\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}t^3+xt\right)}\mathrm{d}t$$

we make the variable change $t = \sqrt{x}u$, and we have:

$$dt = \sqrt{x}du$$
$$e^{i\left(\frac{1}{3}t^3+xt\right)}dt = \sqrt{x}e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}du$$

and we obtain:

$$Ai(x) = C\sqrt{x}\int_{-\infty}^{+\infty} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}\mathrm{d}u$$

An integration by parts(**cf. appendix**) shows:

$$Ai(x) = \frac{C}{x}\int_{\mathbb{R}} \frac{2ue^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}}{i\left(u^2+1\right)^2}\mathrm{d}u$$

With a mathematical induction, we will show this property for all $N \in \mathbb{N}$:

$\mathcal{P}_N$: there exist $g_N \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ like $g_N \in L^1(\mathbb{R};\mathbb{C})$ and:

$$Ai(x) = \frac{C}{x^{1+\frac{3N}{2}}}\int_{-\infty}^{+\infty} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}g_N(u)\mathrm{d}u$$

Base case: $\mathcal{P}_0$ is true. Indeed, $\forall u \in \mathbb{R};\ \ , g_0(u) = \frac{2u}{i(u^2+1)}$ and this function is smooth and, clearly, $\|g_0\|_{L^1} < +\infty$.

Inductive step: Assume $\mathcal{P}_N$ holds (for some unspecified value of N).We have:

$$Ai(x) = \frac{C}{x^{1+\frac{3N}{2}}} \int_{-\infty}^{+\infty} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)} g_N(u)\mathrm{d}u$$

Let $T > 0$. With an integration by parts, we have:

$$\begin{aligned}
\frac{C}{x^{1+\frac{3N}{2}}} \int_{-T}^{T} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)} g_N(u)\mathrm{d}u &= \frac{C}{x^{1+\frac{3N}{2}}} \int_{-T}^{T} \frac{e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)} ix^{\frac{3}{2}}(u^2+1)}{ix^{\frac{3}{2}}(u^2+1)} g_N(u)\mathrm{d}u \\
&= \frac{C}{x^{1+\frac{3N}{2}}} \left[\frac{e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)} g_N(u)}{ix^{\frac{3}{2}}(u^2+1)} \right]_{u=-T}^{u=T} \\
&+ \frac{C}{x^{1+\frac{3(N+1)}{2}}} \int_{-T}^{T} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)} \left(-\frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{g_N(u)}{i(u^2+1)} \right) \right) \mathrm{d}u
\end{aligned}$$

When $T \to +\infty$, we obtain the good formula, with:

$$\begin{aligned}
\forall u \in \mathbb{R};\ \ g_{N+1}(u) = -\frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{g_N(u)}{i(u^2+1)} \right) &= i \frac{g_N'(u)(u^2+1) - 2g_N(u)u}{(u^2+1)^2} \\
&= i \frac{g_N'(u)}{u^2+1} - 2i \frac{u g_N(u)}{(u^2+1)^2}
\end{aligned}$$

First, there exists $C_1 > 0$ like:

$$\forall u \in \mathbb{R};\ \ \frac{|u|}{(u^2+1)^2} \leq C_1$$

Then, let $T > 0$. We have, by using an integration by parts:

$$\int_{-T}^{T} \frac{g_N'(u)}{u^2+1}\mathrm{d}u = \left[\frac{g_N(u)}{u^2+1} \right]_{u=-T}^{u=+T} - 2 \int_{-T}^{T} \frac{u g_N(u)}{(u^2+1)^2}\mathrm{d}u$$

if $g_N \in L^1(\mathbb{R};\mathbb{C})$ and $g_N \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$, we have $g_N(u) \to 0$ when $|u| \to +\infty$ and, if $T \to +\infty$, we have this expansion:

$$\int_{\mathbb{R}} |g_{N+1}(u)|\,\mathrm{d}u = \int_{\mathbb{R}} \left| \frac{-4u g_N(u)}{(u^2+1)^2} \right| \mathrm{d}u = 4 \int_{\mathbb{R}} \frac{|u g_N(u)|}{(u^2+1)^2}\mathrm{d}u \leq 4C_1 \int_{\mathbb{R}} |g_N(u)|\mathrm{d}u$$

and we have $\|g_{N+1}\|_{L^1} < +\infty$ by the hypothesis of $\mathcal{P}_N$

Besides, we have clearly $g_{N+1} \in \mathcal{C}^\infty(\mathbb{R};\mathbb{C})$
The property $\mathcal{P}_{N+1}$ holds.
We have:

$$\forall N \in \mathbb{N}, \forall x > 0;\ \ |Ai(x)| \leq \frac{C\|g_N\|_{L^1}}{x^{1+\frac{3N}{2}}}$$

which concludes the proof

□

Part II

Application to linear partial differential equations

Chapter 4

Research of solutions

4.1 Generalities

We study the partial differential equations of this kind:

$$P\left(-i\frac{\partial}{\partial x};i\frac{\partial}{\partial t}\right)u(x;t)=0$$

where $(k;\omega)\longmapsto P(k;\omega)$ is a polynomial function of two variables.

Definition. *The relation $P(k;\omega)=0$ is named the **dispertion relation** of the equation.*

If we find first kinds of solutions of this kind:

$$u(x;t)=e^{i(kx-\omega t)}\qquad [E]$$

With the dispersion relation, we can find solutions to the equation $P(k;\omega)=0$, giving a M solutions (where M is a positive integer), named $\omega_1(k),\dots,\omega_M(k)$.

Property. Let d the degree of P of the first variable and f the degree of P of the second variable. Let $\omega\in\{\omega_1;\dots;\omega_M\}$ and A a function like for all $j\in[\![0;d]\!]$; $k\longmapsto k^jA(k)e^{i(kx-\omega t)}\in L^1(\mathbb{R};\mathbb{C})$ and for all $l\in[\![0;f]\!]$; $k\longmapsto\omega(k)^lA(k)e^{i(kx-\omega t)}\in L^1(\mathbb{R};\mathbb{C})$.

The function

$$u:(x;t)\longmapsto\int_{-\infty}^{+\infty}A(k)e^{i(kx-\omega t)}\mathrm{d}k$$

is a solution of the equation $[E]$.

Proof. If we derive j times the integrand from x, we obtain the function $k\longmapsto(ik)^jA(k)e^{i(kx-\omega t)}$, and if we derive l times from t, we obtain the function $k\longmapsto(i\omega(k))^lA(k)e^{i(kx-\omega t)}$. By using the hypothesis, these functions are all in $L^1(\mathbb{R};\mathbb{C})$ (and can be bounded from above by integrable functions if we take the absolute value) and continuous.

According to the paraetric integral differentiability's rule, we obtain:

$$(-i\partial_x)^ju(x;t)=\int_{-\infty}^{+\infty}k^jA(k)e^{i(kx-\omega t)}\mathrm{d}k$$

and

$$(i\partial_t)^l u(x;t) = \int_{-\infty}^{+\infty} \omega(k)^l A(k) e^{i(kx-\omega t)} \mathrm{d}k$$

We obtain finally:

$$P\left(-i\frac{\partial}{\partial x}; i\frac{\partial}{\partial t}\right) u(x;t) = \int_{-\infty}^{+\infty} P(k;\omega(k)) A(k) e^{i(kx-\omega t)} \mathrm{d}k = 0$$

which concludes this proof.

□

With the linearity of the equation, we obtain a solution of this kind:

$$u(x;t) = \sum_{j=1}^{M} \int_{-\infty}^{+\infty} A_j(k) e^{i(kx-\omega_j(k)t)} \mathrm{d}k$$

where A_j are functions which check the hypothesis of the property.

Remark. We can calculate the functions A_j by using initial datas and inverse Fourier transform.

4.2 Application to the Wave equation

We consider the wave equation in one space dimension with initial datas:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ u(x;0) &= f(x) \\ \frac{\partial u}{\partial t}(x;0) &= g(x) \end{cases} \quad \text{with } f, g \in L^1(\mathbb{R};\mathbb{C})$$

where c is the velocity.

The dispersion relation is $c^2 k^2 = \omega^2$, which has two solutions: $\omega_+(k) = ck$ and $\omega_-(k) = -ck$.

If we find a solution of the kind studied in the previous section, we have to find two functions A_+ and A_- cheking the hypothesis of the property of the previous part like:

$$u(x;t) = \int_{-\infty}^{+\infty} A_+(k) e^{i(kx-\omega_+(k)t)} \mathrm{d}k + \int_{-\infty}^{+\infty} A_+(k) e^{i(kx-\omega_-(k)t)} \mathrm{d}k$$

which is identic to:

$$\int_{-\infty}^{+\infty} A_+(k) e^{ik(x-ct)} \mathrm{d}k + \int_{-\infty}^{+\infty} A_-(k) e^{ik(x+ct)} \mathrm{d}k$$

By using the initial datas, we have:

$$\begin{aligned} f(x) &= \int_{-\infty}^{+\infty} \left(A_+(k) + A_-(k)\right) e^{ixk} \mathrm{d}k \\ g(x) &= ic \int_{-\infty}^{+\infty} k \left(-A_+(k) + A_-(k)\right) e^{ixk} \mathrm{d}k \end{aligned}$$

and we can try to find A_+ and A_- by using inverse Fourier transform:

$$\begin{aligned} A_+(k) + A_-(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x \\ -ickA_+(k) + ickA_-(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(x)e^{-ikx}\mathrm{d}x \end{aligned}$$

we obtain:

$$\begin{aligned} ickA_+(k) + ickA_-(k) &= \frac{ick}{2\pi}\int_{-\infty}^{+\infty} f(x)e^{-ikx}\mathrm{d}x \\ -ickA_+(k) + ickA_-(k) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} g(x)e^{-ikx}\mathrm{d}x \end{aligned}$$

If we have this hypothesis

$$\int_{-\infty}^{+\infty} g(x)\mathrm{d}x = 0$$

we obtain:

$$\begin{aligned} A_+(k) &= \frac{1}{4i\pi ck}\int_{-\infty}^{+\infty} \left(ickf(x) - g(x)\right) e^{-ikx}\mathrm{d}x \\ A_-(k) &= \frac{1}{4i\pi ck}\int_{-\infty}^{+\infty} \left(ickf(x) + g(x)\right) e^{-ikx}\mathrm{d}x \end{aligned}$$

For exemple, we can take these initial datas:

$$\begin{aligned} f(x) &= e^{-x^2} \\ g(x) &= xe^{-x^2} \end{aligned}$$

Which are two functions of the Schwartz space, and the integral of g is equal to 0.

Chapter 5

Long time behaviour of solutions

5.1 Generalities

With the solutions written on the previous way, we have (with the same hypothesis like in the previous chapter):

$$u(x;t) = \sum_{j=1}^{M} \int_{-\infty}^{+\infty} A_j(k) e^{i(kx-\omega_j(k)t)} \mathrm{d}k$$

After a long time, the solution $u(x;t)$ moves with a velocity v, and this is possible to write out $x = vt$, and we obtain $u(x;t) = u(vt;t)$, which is a function of one variable, and we can write our solution on this way:

$$u(vt;t) = \sum_{j=1}^{M} \int_{-\infty}^{+\infty} A_j(k) e^{it(vk-\omega_j(k))} \mathrm{d}k$$

which is a sum of oscillatory integrals.

By researching the critical points of the functions $k \longmapsto vk - \omega_j(k)$, we can obtain an equivalent of our solution for big values of t by using the theorem of valuation of oscillatory integrals.

With fewer calculus, this is possible to just obtain the order of decay of the solutions. The order of decay depends of the value of v, and the order of the derivative of the phase which doesn't take the value 0 at the critical point.

Look out! In this part, the variable of the integrand is not t, but k, and the parameter of the oscillatory integral is not λ, but t.

5.2 Application to Korteweg-de Vries equation

In this section, we will study the linearized Korteweg-de Vries equation, which describes the waves on shallow water surfaces:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

where c is the velocity.

5.2.1 A first kind of initial data

We take the condition $c = 1$, and this initial data: $u(x;0) = e^{-|x|}$.

The dispersion relation is $\omega - k - k^3$, which writes on this way: $\omega = k - k^3$. We find a function A like:

$$u(x;t) = \int_{-\infty}^{+\infty} A(k) e^{i(kx-(k-k^3)t)} \mathrm{d}k$$

and we have:

$$u(x;0) = \int_{-\infty}^{+\infty} A(k) e^{ikx} \mathrm{d}k$$

A computation(**cf. appendix**) shows that:

$$A(k) = \frac{1}{\pi\left(1+k^2\right)}$$

and we obtain, if our solution moves with a velocity v:

$$u(vt;t) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{e^{it(vk-(k-k^3))}}{1+k^2} \mathrm{d}k$$

We will have to study this oscillatory integral. First, we will study the critical points of the phase.

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(vk - (k-k^3)\right) = 0 \Leftrightarrow v - 1 + 3k^2 = 0$$

First case: $v > 1$. This equation doesn't have solution. The integral has a quick decay in t.

Second case: $v < 1$. We have two solutions for k:

$$k_{v\pm} = \pm\sqrt{\frac{1-v}{3}}$$

We have:

$$\frac{\mathrm{d}^2}{\mathrm{d}k^2}\left(vk - (k - k^3)\right) = 6k$$

By using the theorem (the function A verifies clearly the hypothesis of the theorem, in particuar for the integrability arround $\pm\infty$ with the operator D^T), we have:

$$u(vt;t) \underset{t\to+\infty}{\sim} e^{it\left[(v-1)\sqrt{\frac{1-v}{3}}+\left(\frac{1-v}{3}\right)^{\frac{3}{2}}\right]} \left(\frac{2}{6\sqrt{\frac{1-v}{3}}}\right)^{\frac{1}{2}} \sqrt{\pi}e^{\frac{i\pi}{4}} \frac{1}{1+\frac{1-v}{3}}\frac{1}{\sqrt{t}}$$

$$+e^{-it\left[(v-1)\sqrt{\frac{1-v}{3}}+\left(\frac{1-v}{3}\right)^{\frac{3}{2}}\right]} \left(\frac{2}{6\sqrt{\frac{1-v}{3}}}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{2}}\sqrt{\pi}e^{\frac{i\pi}{4}} \frac{1}{1+\frac{1-v}{3}}\frac{1}{\sqrt{t}}$$

$$u(vt;t) \underset{t\to+\infty}{\sim} e^{it\left[(v-1)\sqrt{\frac{1-v}{3}}+\left(\frac{1-v}{3}\right)^{\frac{3}{2}}\right]+i\frac{\pi}{4}} \sqrt{\frac{\pi}{3}} \left(\frac{2}{6\sqrt{\frac{1-v}{3}}}\right)^{-\frac{1}{4}} \frac{3}{4-v}\frac{1}{\sqrt{t}}$$

$$e^{-it\left[(v-1)\sqrt{\frac{1-v}{3}}+\left(\frac{1-v}{3}\right)^{\frac{3}{2}}\right]-i\frac{\pi}{4}} \sqrt{\frac{\pi}{3}} \left(\frac{2}{6\sqrt{\frac{1-v}{3}}}\right)^{-\frac{1}{4}} \frac{3}{4-v}\frac{1}{\sqrt{t}}$$

$$u(vt;t) \underset{t\to+\infty}{\sim} \frac{6}{4-v}\left(\frac{1-v}{3}\right)^{-\frac{1}{4}} \cos\left[\left((v-1)\sqrt{\frac{1-v}{3}}+\left(\frac{1-v}{3}\right)^{\frac{3}{2}}\right)t+\frac{\pi}{4}\right]\sqrt{\frac{\pi}{3t}}$$

and we have a decay in $t^{-\frac{1}{2}}$.

Third case: $v = 1$. The hypothesis of the theorem are verified (in particular the integrability near $\pm\infty$ with the differential operator D^T), and we need to calculate the three order derivative of the phase in order to obtain an asymptotic expansion:

$$\frac{\mathrm{d}^3}{\mathrm{d}k^3}\left(vk - (k - k^3)\right) = -6$$

By applying the theorem, we have:

$$u(vt;t) \underset{t\to+\infty}{\sim} 2\left(\frac{6}{6}\right)^{\frac{1}{2}} \Gamma\left(\frac{4}{3}\right) e^{i\frac{\pi}{6}} \frac{1}{\sqrt[3]{t}}$$

$$u(vt;t) \underset{t\to+\infty}{\sim} 2\Gamma\left(\frac{4}{3}\right) e^{i\frac{\pi}{6}} \frac{1}{\sqrt[3]{t}}$$

and we have a decay in $t^{-\frac{1}{3}}$.

Conclusion: On long time, the solutions which have a velocity which is equal to 1 will dominate the others which have differents velocities.

5.2.2 A second kind of initial data

Now, we take, in the Koteweg-de Vries equation, an unspecified value for c, and this initial data (a wave packet):

$$u(x;0) = \mathbb{1}_{\left[-\frac{2\pi n}{\underline{k}};\frac{2\pi n}{\underline{k}}\right]}(x)e^{i\underline{k}x}$$

where $n \in \mathbb{N}^*$ and $\underline{k} > 0$.

we always have our dispersion relation: $\omega = ck - k^3$, which gives us this solution (for a function A):

$$u(x;t) = \int_{-\infty}^{+\infty} A(k)e^{i(kx-(ck-k^3)t)}\mathrm{d}k$$

A simple computation (**cf. appendix**) shows that:

$$A(k) = \frac{i\sin\left(\frac{2\pi nk}{\underline{k}}\right)}{\pi(k-\underline{k})}$$

And we have:

$$u(x;t) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{e^{it(kv-(ck-k^3))}}{k-\underline{k}} \sin\left(\frac{2\pi nk}{\underline{k}}\right) \mathrm{d}k$$

When this solution moves with a velocity v, we can write out $x = vt$, giving this oscillatory integral:

$$u(vt;t) = u(x;t) = \frac{i}{\pi} \int_{-\infty}^{+\infty} \frac{e^{it(kv-(ck-k^3))}}{k-\underline{k}} \sin\left(\frac{2\pi nk}{\underline{k}}\right) \mathrm{d}k$$

If we want apply the stationnary phase method, we will have to solve this equation:

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(vk-(ck-k^3)\right) = 0 \Longleftrightarrow (v-c)+3k^2 = 0$$

First case: $v > c$. This equation doesn't have solution. The integral has a quick decay in t.
Second case: $v < c$. We have two solutions for k:

$$k_{v\pm} = \pm\sqrt{\frac{c-v}{3}}$$

By using the theorem (the function A verifies clearly the hypothesis of the theorem, in particuar for the integrability arround $\pm\infty$ with the operator D^T), we have (if $\frac{k_{v\pm}}{\underline{k}} \notin \frac{1}{2}\mathbb{Z}$, giving $A(k) \neq 0$) the decay in the order of $\frac{1}{\sqrt{t}}$.

Third case: $v = c$. We have $k_c = 0$, and, whatever the hypothesis are verified of not, we have $A(k_c) = 0$, and the decay has an order below $\frac{1}{\sqrt[3]{t}}$.

Conclusion: In this case, the solutions which move with a velocity $v < c$ will dominate on long time.

Chapter 6

Linearized Water-Waves problem

6.1 Preamble

The Water-Waves problem describes the surface of a fluid with equations. In our problem, the surface profile is named u, which is a function from $\mathbb{R} \times \mathbb{R}_+$ to $\mathbb{R}$ (one-dimensional space and time). We suppose $u(\cdot, t) \in L^1(\mathbb{R}; \mathbb{C})$ for all $t \geq 0$.

We suppose the fluid irrotational, giving the existence of a function $\Phi \in \mathcal{C}^{2,3}(\mathbb{R} \times \mathbb{R}_+; \mathbb{C})$ like:

$$u = \frac{\partial \Phi}{\partial t}$$

Besides, we consider that $\Phi(\,\cdot\,; 0) \in L^1(\mathbb{R}; \mathbb{C})$.

In this chapter, we will study the linearized Water-Waves problem, and its equations leads to this main equation:

$$\partial_t^2 u + \frac{1}{\mu}\mathcal{G}_0 u = 0 \qquad [1]$$

Where μ is the square of the relation between the depth and the wavelength, and $\mathcal{G}_0$ is the Dirichlet-Neumann operator, which has an explicit definition with its Fourier transform:

$$\frac{1}{\mu}\widehat{\mathcal{G}_0 f}(\xi) = \frac{|\xi| \tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}}\hat{f}(\xi)$$

We give these initial datas too:

$$\left\{ \begin{array}{rcl} u(x;0) & = & u_0(x) \\ \partial_t u(x;0) & = & v_0(x) \end{array} \right.$$

6.2 Research of solutions

If we take the equation [1] (cf. previous page), and we apply the Fourier transform to it, we obtain:

$$\widehat{\partial_t^2 u} + \frac{1}{\mu}\widehat{\mathcal{G}_0 u} = 0$$

which leads to:

$$\partial_t^2 \widehat{u} + \frac{|\xi| \tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}} \widehat{u} = 0$$

and the initial datas become:

$$\left\{ \begin{array}{rcl} \widehat{u}(\xi;0) & = & \widehat{u_0}(\xi) \\ \partial_t \widehat{u}(\xi;0) & = & \widehat{v_0}(\xi) \end{array} \right.$$

Remark. In order to apply the Fourier transform to the initial datas, we assume they are in $L^1(\mathbb{R};\mathbb{C})$, and as smooth as we can switch integral and derivation from t.

The equation with Fourier transform becomes:

$$\partial_t^2 \widehat{u} + \frac{|\xi| \tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}} \widehat{u} = 0$$

which is an ordinary differential equation in t. For this reason, there exist two functions A and B , depending of ξ, like:

$$\forall(\xi;t) \in \mathbb{R} \times \mathbb{R}_+; \;\; \widehat{u}(\xi;t) = A(\xi)e^{i\omega(\xi)t} + B(\xi)e^{-i\omega(\xi)t}$$

where:

$$\omega(\xi) = \left(\frac{|\xi| \tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}} \right)^{\frac{1}{2}} sgn(\xi)$$

Remark. The function *sgn* is the sign function, which takes the value -1 if $\xi < 0$, 0 if $\xi = 0$ and 1 if $\xi > 0$.

6.2.1 Study of Fourier transform of solutions

We can give an explicit expansion of A and B in function of $\widehat{u_0}$ and $\widehat{v_0}$ by using the initial datas.

First, we have:

$$\left\{ \begin{array}{rcl} \widehat{u}(\xi;0) & = & A(\xi) + B(\xi) \\ \partial_t \widehat{u}(\xi;0) & = & i\omega(\xi)A(\xi) - i\omega(\xi)B(\xi) \end{array} \right.$$

And, by solving this linear system of two equations with two unknowns, we have:

$$\left\{ \begin{array}{rcl} A(\xi) & = & \frac{1}{2}\widehat{u_0}(\xi) - \frac{1}{2}i\frac{\widehat{v_0}(\xi)}{\omega(\xi)} \\ B(\xi) & = & \frac{1}{2}\widehat{u_0}(\xi) + \frac{1}{2}i\frac{\widehat{v_0}(\xi)}{\omega(\xi)} \end{array} \right.$$

If an asymptotic expansion of ω near 0 is:

$$\omega(\xi) \underset{\xi \to 0}{\sim} \xi$$

And the functions A and B could not be defined when $\xi = 0$, but, according to the equation [1], we have:

$$\partial_t^2 u + \frac{1}{\mu}\mathcal{G}_0 \cdot u = 0$$

When we take the fourier transform, we have:

$$\partial_t^2 \widehat{u} = -\frac{1}{\mu}\widehat{\mathcal{G}_0 u} = -\omega(\xi)^2 \widehat{u}$$

According to the hypothesis concerning Φ (in particular the integrability, and, as a result, the possibility to apply it he Fourier transform), we obtain:

$$\partial_t^2 \widehat{u}(\xi;t) = -\omega(\xi)^2 \partial_t \widehat{\Phi}(\xi;t)$$

When we integrate this equality, for exemple betweenn $+\infty$ and ξ ($\widehat{u}$ and $\widehat{\Phi}$ tend to 0 near $+\infty$, **cf. appendix**), we obtain:

$$\partial_t \widehat{u}(\xi;t) = -\omega(\xi)^2 \widehat{\Phi}(\xi;t)$$

When $t = 0$, we have:

$$v_0(\xi) = -\omega(\xi)^2 \widehat{\Phi}(\xi;0).$$

According to the hypothesis of integrability of $\Phi(\cdot;0)$, its Fourier transform is defined.

With this result, we obtain:

$$\begin{cases} A(\xi) & = & \frac{1}{2}\widehat{u_0}(\xi) + \frac{1}{2} i\omega(\xi)\widehat{\Phi}(\xi;0) \\ B(\xi) & = & \frac{1}{2}\widehat{u_0}(\xi) - \frac{1}{2} i\omega(\xi)\widehat{\Phi}(\xi;0) \end{cases}$$

And, if $u_0, \Phi(\cdot;0) \in L^1(\mathbb{R};\mathbb{C})$, its Fourier transform is continuous, and the functions A and B are defines when $\xi = 0$, like the function ω

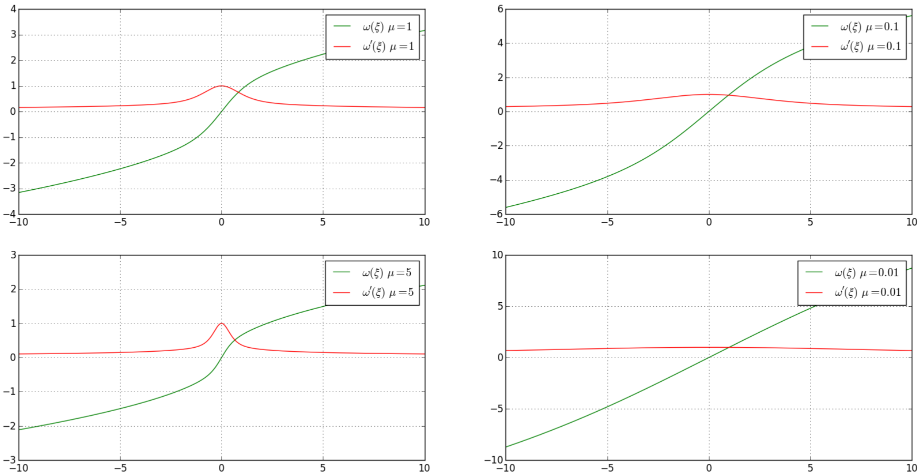
6.2.2 The function ω

Our function ω is defined on this way:

$$\omega(\xi) = \left(\frac{|\xi| \tanh\left(\sqrt{\mu}|\xi|\right)}{\sqrt{\mu}} \right)^{\frac{1}{2}} sgn(\xi) = \left(\frac{\xi \tanh\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}} \right)^{\frac{1}{2}} sgn(\xi)$$

These graphics show the function ω for differents values of μ.

Phase of the oscillatory integral linked to the water wave problem



With these graphics, we observe the equation $\frac{d\omega}{d\xi}(\xi) - v$ has two solutions equidistants from the y-axis when $0 < v < 1$, 0 is the only solution when $v = 1$, and there exists any slution for the others values of v. These informations are importants for the study of long-time behaviour of solutions, by using the stationnary phase method.

Property. The function ω is odd and $\mathcal{C}^3$-class on $\mathbb{R}$, and its derivative is strictly decreasing on $\mathbb{R}_+^*$, and equal to 1 when $\xi = 0$. We have $\omega'(\xi) \underset{\xi \to +\infty}{\longrightarrow} 0$.

Proof. This proof is very specialised, and, for this reason, given in the **appendix** □

6.2.3 Study of long time behaviour of the solutions

In order to reduce notations, we will write A and B, even if we know their explicit expansions.

If we know the Fourier transform of the solution u (in space), we can obtain the solution by applying the inverse Fourier transform to this function, giving us:

$$u(x;t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left(A(\xi) e^{i\omega(\xi)} + B(\xi) e^{-i\omega(\xi)} \right) e^{ix\xi} d\xi$$

When the solution, after a long (long, long...) moment, moves with a limit velocity v, we have $x = vt$ and:

$$u(vt;t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} A(\xi) e^{i[\omega(\xi)+v\xi]t} + B(\xi) e^{i[-\omega(\xi)+v\xi]t} d\xi$$

and we have to study an oscillatory integral when $t \to +\infty$

First case:

If $v = \pm 1$, we have a solution to the equation $\frac{d}{d\xi}(\pm\omega(\xi) + v\xi) = \pm\omega'(\xi) + 1 = 0$, which is $\xi = 0$. If we apply the stationnary phase method, we have: $\frac{d^2}{d\xi^2}(\pm\omega(\xi) + v\xi) = \frac{d^2\omega}{d\xi^2}(0) = 0$. For this reason, wwe need to look the third order derivative of ω. This function is $\mathcal{C}^3$-class near 0, and, according to the Taylor-Young formula, we can calculate its Taylor approximation to the third order.

A simple computation (**cf. appendix**) gives:

$$\omega(\xi) = \xi - \frac{1}{6}\mu\xi^3 + \underset{\xi \to 0}{o}\left(\xi^4\right)$$

By identifying with the Taylor's formula (and the uniqueness of the approximation), we have:

$$\frac{\mathrm{d}^3\omega}{\mathrm{d}\xi^3}(0) = -\mu$$

If we apply the stationnary phase method, and we add the hypothesis $\left(D^T\right)(A)$ and $\left(D^T\right)(B)$ are integrible near $\pm\infty$, $A(0) \neq 0$ and $B(0) \neq 0$, we have:

If $v = 1$:

$$u(vt;t) \underset{t \to +\infty}{\sim} 2A(0)\left(\frac{6}{-\mu}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) e^{\frac{i\pi}{6}} \frac{1}{\sqrt[3]{t}}$$

$$u(vt;t) \underset{t \to +\infty}{\sim} 2A(0)\left(\frac{6}{\mu}\right)^{\frac{1}{3}} \left(e^{-i\pi}\right)^{\frac{1}{3}} \Gamma\left(\frac{4}{3}\right) e^{\frac{i\pi}{6}} \frac{1}{\sqrt[3]{t}}$$

$$u(vt;t) \underset{t \to +\infty}{\sim} 2A(0)\Gamma\left(\frac{4}{3}\right) e^{-\frac{i\pi}{6}} \frac{6^{\frac{1}{3}}}{\sqrt[3]{\mu t}}$$

If $v = -1$, a similar computation shows that:

$$u(vt;t) \underset{t\to+\infty}{\sim} 2B(0)\Gamma\left(\frac{4}{3}\right)e^{-\frac{i\pi}{6}}\frac{6^{\frac{1}{3}}}{\sqrt[3]{\mu t}}$$

Second case:

If $0 < v < 1$, according to the table of variations of ω' (cf. p. 73), there exists $z_v > 0$ like $\omega_0'(z_v) = v$ (the function ω_0, and, as a result, ω, is invertible from $\mathbb{R}_+^*$ to $]0;1[$). We have:

$$\omega_0'(z_v) = \omega_0'\left(\frac{\sqrt{\mu}z_v}{\sqrt{\mu}}\right) = \omega'\left(\frac{z_v}{\sqrt{\mu}}\right) = v$$

A s a result, we obtain:

$$\omega'\left(\frac{(\omega_0')^{-1}(v)}{\sqrt{\mu}}\right) = v$$

We write out: $z_v = (\omega_0')^{-1}(v)$. By convention, we consider $z_v > 0$, and we have: $\omega_0'(z_v) = \omega_0'(-z_v) = v$. As a result, $\frac{z_v}{\sqrt{\mu}}$ and $-\frac{z_v}{\sqrt{\mu}}$ are two solutions of the equation $\frac{\mathrm{d}}{\mathrm{d}\xi}(-\omega(\xi) + v\xi) = 0$, which is associated to the function B in the oscillatory integral.

If we consider the same hypothesis concerning A and B, and we add the hypothesis u_0 and v_0 are real-valued, we have, by using the stationnary phase method:

$$u(vt;t) \underset{t\to+\infty}{\sim} 2e^{i\left(-\omega\left(\frac{z_v}{\sqrt{\mu}}\right)+\frac{vz_v}{\sqrt{\mu}}\right)}B\left(\frac{z_v}{\sqrt{\mu}}\right)\left(\frac{2}{\omega''\left(\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)e^{\frac{i\pi}{4}}\frac{1}{\sqrt{t}}$$

$$+2e^{-i\left(-\omega\left(\frac{z_v}{\sqrt{\mu}}\right)+\frac{vz_v}{\sqrt{\mu}}\right)}B\left(-\frac{z_v}{\sqrt{\mu}}\right)\left(\frac{2}{\omega''\left(-\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}}\Gamma\left(\frac{3}{2}\right)e^{\frac{i\pi}{4}}\frac{1}{\sqrt{t}}$$

Besides, we have:

$\omega''(\xi) = \sqrt{\mu}\omega_0''(\sqrt{\mu}\xi)$ and $\Gamma\left(\frac{3}{2}\right) = \frac{\sqrt{\pi}}{2}$

The function B is computed on this way:

$$B(\xi) = \frac{1}{2}\widehat{u_0}(\xi) + \frac{1}{2}i\frac{\widehat{v_0}(\xi)}{\omega(\xi)}$$

and we obtain:

$$B(\xi) = \frac{1}{2}\widehat{u_0}(-\xi) + \frac{1}{2}i\frac{\widehat{v_0}(-\xi)}{\omega(-\xi)} = \frac{1}{2}\widehat{u_0}(-\xi) - \frac{1}{2}i\frac{\widehat{v_0}(-\xi)}{\omega(\xi)}$$

Moreover, we have:

$$\widehat{u_0}(-\xi) = \int_{-\infty}^{+\infty} u_0(x)e^{i\xi x}\mathrm{d}x = \int_{-\infty}^{+\infty} \overline{f(x)e^{-i\xi x}}\mathrm{d}x = \overline{\widehat{u_0}(\xi)}$$

A similar computation shows that:

$$\widehat{v_0}(-\xi) = \overline{\widehat{v_0}(\xi)}$$

As a result, we have: $B\left(-\frac{z_v}{\sqrt{\mu}}\right) = \overline{B\left(\frac{z_v}{\sqrt{\mu}}\right)}$
We have too:

$$\left(\frac{2}{\omega''\left(\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\left(\sqrt{\mu}\omega_0''(z_v)\right)^{\frac{1}{2}}} = \frac{\sqrt{2}}{\sqrt{\mu}\left|\omega_0''(z_v)\right|}e^{-\frac{i\pi}{2}}$$

And:

$$\left(\frac{2}{\omega''\left(-\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}} = \left(\frac{2}{-\omega''\left(\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}} = e^{-\frac{i\pi}{2}}\left(\frac{2}{\omega''\left(-\frac{z_v}{\sqrt{\mu}}\right)}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{\mu}\left|\omega_0''(z_v)\right|}e^{-i\pi}$$

Finally, we have:

$$u(vt;t) \underset{t\to+\infty}{\sim} \sqrt{2\pi}\left[e^{\frac{i}{\sqrt{\mu}}(-\omega_0(z_v)+vz_v)}B\left(\frac{z_v}{\sqrt{\mu}}\right)e^{-\frac{i\pi}{4}} + e^{-\frac{i}{\sqrt{\mu}}(-\omega_0(z_v)+vz_v)}\overline{B\left(\frac{z_v}{\sqrt{\mu}}\right)}e^{-\frac{5i\pi}{4}}\right]\frac{1}{\sqrt{t}}$$

$$u(vt;t) \underset{t\to+\infty}{\sim} \sqrt{2\pi}e^{-\frac{3i\pi}{4}}\left[e^{\frac{i}{\sqrt{\mu}}(-\omega_0(z_v)+vz_v)}B\left(\frac{z_v}{\sqrt{\mu}}\right)e^{-\frac{i\pi}{2}} + e^{-\frac{i}{\sqrt{\mu}}(-\omega_0(z_v)+vz_v)}\overline{B\left(\frac{z_v}{\sqrt{\mu}}\right)}e^{-\frac{i\pi}{2}}\right]\frac{1}{\sqrt{t}}$$

Finally, we have this asymptotic expansion:

$$u(vt;t) \underset{t\to+\infty}{\sim} 2\sqrt{2\pi}e^{-\frac{3i\pi}{4}}\mathfrak{Re}\left[e^{\frac{i}{\sqrt{\mu}}(-\omega_0(z_v)+vz_v)}B\left(\frac{z_v}{\sqrt{\mu}}\right)e^{-\frac{i\pi}{2}}\right]\frac{1}{\sqrt{t}}$$

If $-1 < v < 0$, the equation $\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\omega(\xi) + v\xi\right) = 0$ has two solutions, who are $\frac{z_v}{\sqrt{\mu}}$ and $-\frac{z_v}{\sqrt{\mu}}$, and a similar computation shows that:

$$u(vt;t) \underset{t\to+\infty}{\sim} 2\sqrt{2\pi}e^{-\frac{3i\pi}{4}}\mathfrak{Re}\left[e^{\frac{i}{\sqrt{\mu}}(\omega_0(z_v)+vz_v)}A\left(\frac{z_v}{\sqrt{\mu}}\right)e^{-\frac{i\pi}{2}}\right]\frac{1}{\sqrt{t}}$$

Remark. When $\mu \to 0$ (i.e for low frequencies), we have $\frac{z_v}{\sqrt{\mu}} \to +\infty$, and, if A is built with Fourier transforms, L^1, we will have $A\left(\frac{z_v}{\sqrt{\mu}}\right) \underset{\mu\to 0}{\longrightarrow} 0$ (moreover, a simple computation shows that $\frac{1}{\omega\left(\frac{z_v}{\sqrt{\mu}}\right)} \underset{\mu\to 0}{\longrightarrow} 0$), and the height of the solutions will tend toward 0.

Third case:

If we have $v \leq -1$ or $v \geq 1$, the both equations $\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\omega(\xi) + v\xi\right) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}\xi}\left(-\omega(\xi) + v\xi\right) = 0$ don't have solution, and, if we consider the same hypothesis concerning the functions A and B like in the previous parts, the solutions for these velocities have a quick decay.

Conclusion:

On long times, the solutions which move with a velocity of 1 or -1 will dominate, like for the linearized Korteweg-de Vries equation. Besides, according to the formula given at the end of the subsubsection "First case", the solutions will have a taller height when $\mu \to 0$, and, as a result, for low frequencies.

6.3 Confirmation of the results

6.3.1 Comparison with other results

If we study the publication of Benoît Mésognon-Gireau, the document gives an estimation of solutions of the linearized Water-Waves problem, for a solution $u_0 \in \mathcal{S}(\mathbb{R})$:

There exists $C > 0$, independant of μ, such that, for all $\mu > 0$ and $t > 0$:

$$\|u(\cdot;t)\|_{L^\infty} \leq C \left[\frac{\|u_0\|_{L^1}}{\mu^{\frac{3}{4}} \left(1 + \frac{1}{\sqrt{\mu}}\right)^{\frac{1}{3}}} + \frac{\|u_0\|_{H^1} + \|x \mapsto x\partial_x u_0(x)\|_{L^2}}{\left(1 + \frac{t}{\sqrt{\mu}}\right)^{\frac{1}{2}}} \right]$$

And:

$$\|u(\cdot;t)\|_{L^\infty} \leq C \left[\frac{\|x \mapsto x u_0(x)\|_{L^2}}{\mu^{\frac{3}{4}} \left(1 + \frac{t}{\sqrt{\mu}}\right)^{\frac{1}{3}}} + \frac{\|u_0\|_{H^1} + \|x \mapsto x\partial_x u_0(x)\|_{L^2}}{\left(1 + \frac{t}{\sqrt{\mu}}\right)^{\frac{1}{2}}} \right]$$

Where:

$$\|u_0\|_{H^1} = \|u_0\|_{L^2} + \|u_0'\|_{L^2}$$

is the norm corresponding with the Sobolev space H^1 (which is complete, and, as a result, an Hilbertian space), where the function u_0' can be considered with the weak form (by using the distributions).

The first term shows a decay for the velocities of 1 or -1, whereas the second term shows the decay for the velocities in the domain $]-1;0[\cup]0;1[$.

Remark. In the scientific document, we give an estimation of the function $t \longmapsto u(\cdot;t)$, whereas in our study, we estimate the function $t \longmapsto u(vt;t)$ for differents values of v.

6.3.2 Numerical simulations

In order to confirm our results, we make numerical simulations with Python. In order to obtain some results, some graphs were made, representing the function $\ln\|u(\cdot;t)\|_{L^\infty} = f(\ln(t))$, for $\ln(t) \in [5.6; 6.9]$(**cf. appendix**). In order to approximate prefect straight lines, higher values of t, and, as a result, of $\ln(t)$, are chosen. After a long time, we have (theorically), this asymptotic expansion:

$$\|u(\cdot;t)\|_{L^\infty} \underset{t\to+\infty}{\sim} Bt^a$$

with $B > 0$. As a result, we obtain, when t is enought big:

$$\ln\|u(\cdot;t)\|_{L^\infty} = a\ln(t) + b$$

by writing out: $b = \ln(B)$. We compute a and b by using a linear regression.

First scenario: 2012

We choose these initial datas:

$$\begin{cases} u_0(x) &= xe^{-x^2} \\ v_0(x) &= 0 \end{cases}$$

We have:

$$\widehat{u_0}(0) = \int_{-\infty}^{+\infty} xe^{-x^2}\mathrm{d}x = 0$$

For this reason, we have $A(0) = 0$, and the lower decay of the solutions is not in the order of $\frac{1}{t^{\frac{1}{3}}}$, but in the order of $\frac{1}{t^{\frac{1}{2}}}$. These are the values of the linear regressions for some values of μ:

μ	a	b	R^2
0.005	-0.4617	1.1022	0.9997
0.006	-0.4839	1.1670	0.9998
0.007	-0.5021	1.2148	0.9999
0.008	-0.5158	1.2400	0.9999
0.009	-0.5272	1.2567	0.9999
0.01	-0.5371	1.2684	0.9999

We obtain this average value of a:

$$\overline{a} = -0.5046$$

with this uncertainty (with 95 % of trust):

$$U(a) = 0.023$$

Conclusion: This value is correct, and is consistent with the theory.

Second scenario: Deep Impact

We choose now these initial datas:

$$\begin{cases} u_0(x) & = & e^{-x^2} \\ v_0(x) & = & 0 \end{cases}$$

giving:

$$\widehat{u_0}(0) = \int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi} \neq 0$$

For this reason, we have $A(0) \neq 0$ in the asymptotic expansion, and, theorically, a decay in the order of $\frac{1}{t^{\frac{1}{3}}}$. These are the values of the linear regressions for some values of μ:

μ	a	b	R^2
0.005	-0.33	1.3299	≥ 0.99
0.006	-0.33	1.3515	≥ 0.99
0.007	-0.33	1.2459	≥ 0.99
0.008	-0.34	1.2900	≥ 0.99
0.009	-0.34	1.2778	≥ 0.99
0.01	-0.33	1.1473	≥ 0.99

We obtain this average value of a:

$$\overline{a} = -0.33 \approx -\frac{1}{3}$$

with this uncertainty (with 95 % of trust):

$$U(a) = 4.2 \times 10^{-3}$$

Conclusion: This value is correct too, and consistent with the theory.

Part III

Further informations

Chapter 7

Appendix

7.1 The differential operator

Return

Property. If we have this hypothesis:

$$\frac{f(t)g(t)}{i\lambda\Phi'(t)} \underset{t\to a,b}{\longrightarrow} 0$$

We have $\langle Df|g\rangle = \langle f|D^T g\rangle$, where:

$$\langle f|g\rangle = \int_a^b f(t)g(t)\mathrm{d}t$$

is the common dot product on $L^2(]a;b[;\mathbb{R})$.

Proof. Let $a < A < B < b$: we have, by using an integration by parts:

$$\int_A^B Df(t)g(t)\mathrm{d}t = \int_A^B \frac{1}{i\lambda\Phi'(t)}f'(t)g(t)\mathrm{d}t$$

$$= \underbrace{\left[\frac{f(t)g(t)}{i\lambda\Phi'(t)}\right]_{t=A}^{t=B}}_{=0 \text{ (hypothesis)}} - \int_A^B f(t)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{g(t)}{i\lambda\Phi'(t)}\right)\mathrm{d}t$$

$$= \int_A^B f(t)\left(D^T\right)g(t)\mathrm{d}t$$

and we conclude when $(A;B) \to (a;b)$ □

7.2 The quick decay for some oscillatory integrals

Return N integrations by parts show that:

$$\int_a^b e^{i\lambda\Phi(t)}g(t)\mathrm{d}t = \sum_{j=0}^{N-1}\left[\frac{e^{i\lambda\Phi(b)}\left(D^T\right)^j(g)(b)}{i\lambda\Phi'(b)} - \frac{e^{i\lambda\Phi(a)}\left(D^T\right)^j(g)(a)}{i\lambda\Phi'(a)}\right]$$

$$+\int_a^b e^{i\lambda\Phi(t)}\left(D^T\right)^N(g)(t)\mathrm{d}t$$

But if g has a compact support, which is a closed interval in $]a;b[$, an open interval, necessarily, the lower and upper bounds of the support of g are respectively strictly higher and upper than a and b. For this reason, we have g and its derivatives which takes the value 0 at the points a and b, and we finally obtain:

$$\int_a^b e^{i\lambda\Phi(t)}g(t)\mathrm{d}t = \int_a^b e^{i\lambda\Phi(t)}\left(D^T\right)^N(g)(t)\mathrm{d}t$$

7.3 The integral outside the critical point

Return

We will show, by using a mathematical induction, for all $n \in [\![0;N-1]\!]$, the property $\mathcal{P}_n$:

$$\int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}g(t)\mathrm{d}t = \frac{e^{i\lambda\Phi(t_0+2\varepsilon)}}{i\lambda\Phi'(t_0+2\varepsilon)}\sum_{j=0}^{n-1}\left(D^T\right)^j(g)(t_0+2\varepsilon) + \int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}\left(D^T\right)^n(g)(t)\mathrm{d}t$$

Base case: $\mathcal{P}_0$ is true. Indeed, we have:

$$\int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}\mathrm{d}t = \int_{t_0+2\varepsilon}^{+\infty} e^{i\lambda\Phi(t)}\left(D^T\right)^0(g)(t)\mathrm{d}t$$

Inductive step: Assume $\mathcal{P}_n$ holds (for some unspecified value of n). We have the formula given at the begining of this subsection. Besides, for $T > 0$, we have:

$$\int_{t_0+2\varepsilon}^{T} e^{i\lambda\Phi(t)}\left(D^T\right)^n(g)(t)\mathrm{d}t = \int_{t_0}^{T} i\lambda\Phi'(t)e^{i\lambda\Phi(t)}\frac{\left(D^T\right)^n(g)(t)}{i\lambda\Phi'(t)}\mathrm{d}t$$

$$= \left[\frac{e^{i\lambda\Phi(t)}\left(D^T\right)^n(g)(t)}{i\lambda\Phi'(t)}\right]_{t=t_0+2\varepsilon}^{t=T} + \int_{t_0+2\varepsilon}^{T} e^{i\lambda\Phi(t)}\left(D^T\right)^{n+1}(g)(t)$$

When $T \to +\infty$, by using the hypothsis, we obtain finally the formula for the rank $n+1$, and the property $\mathcal{P}_{n+1}$ holds.

7.4 The second integrand in the Lemma 2

Return

By using a mathematical induction, we will show this property, named $\mathcal{P}_N$ for all $N \in \mathbb{N}$:

$$\left(D^T\right)^N \left(t^l \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right) = \frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l}\, t^j \left(\frac{\mathrm{d}}{dt}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]$$

Base case: $\mathcal{P}_0$ is true. Indeed, we have:

$$\left(D^T\right)^0 \left(t^l \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right) = t^l C_{0,0,l} \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right) \text{ with } C_{0,0,l}=1$$

Inductive step: Assume $\mathcal{P}_N$ holds (for some unspecified value of N). We have:

$$\begin{aligned}
\left(D^T\right)^{N+1} \left(t^l \eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right) &= D^T \left(\frac{1}{\lambda^N} t^{l-kN} \sum_{j=0}^{N} C_{N,j,l} t^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right) \\
&= -\frac{1}{i\lambda^{N+1}} \frac{\mathrm{d}}{\mathrm{d}t} \left(t^{l+1-k(N+1)} \sum_{j=0}^{p} C_{N,j,l} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right) \\
&= -\frac{1}{i\lambda^{N+1}} \left[(l+1-k(N+1)) t^{l-k(N+1)} \sum_{j=0}^{N} C_{N,j,l} t^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&\quad -\frac{1}{i\lambda^{N+1}} \left[t^{l+1-k(N+1)} \sum_{j=1}^{N} C_{N,j,l}\, j t^{j-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&\quad -\frac{1}{i\lambda^{N+1}} \left[t^{l+1-k(N+1)} \sum_{j=0}^{N} C_{N,j,l} t^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j+1} \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&= -\frac{1}{i\lambda^{N+1}} \left[(l+1-k(N+1)) t^{l-k(N+1)} \sum_{j=0}^{N} C_{N,j,l} t^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&\quad -\frac{1}{i\lambda^{N+1}} \left[t^{l-k(N+1)} \sum_{j=1}^{N} C_{N,j,l}\, j t^{j} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&\quad -\frac{1}{i\lambda^{N+1}} \left[t^{l-k(N+1)} \sum_{j=0}^{N} C_{N,j,l} t^{j+1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{j+1} \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&= -\frac{1}{i\lambda^{N+1}} \left[(l+1-k(N+1)) t^{l-k(N+1)} \sum_{j=0}^{N} C_{N,j,l} t^j \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right] \\
&\quad -\frac{1}{i\lambda^{N+1}} \left[t^{l-k(N+1)} \sum_{j=1}^{N} C_{N,j,l}\, j t^{j} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right]
\end{aligned}$$

$$-\frac{1}{i\lambda^{N+1}}\left[t^{l-k(N+1)}\sum_{j=1}^{N+1}C_{N,j-1,l}t^j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j\left[\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right]$$

$$=\frac{1}{\lambda^{N+1}}t^{l-k(N+1)}\left[i(l+1-k(N+1))C_{N,0,l}t^0\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^0t^l\left(\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right)\right]$$

$$+\frac{1}{\lambda^{N+1}}t^{l-k(N+1)}\left[\sum_{j=1}^{N}i\left[(l+1+j-k(N+1))C_{N,j,l+C_{N,j-1,l}}\right]t^j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j\left[t^l\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right]$$

$$+\frac{1}{\lambda^{N+1}}t^{l-k(N+1)}\left[iC_{N,N,l}t^{N+1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{N+1}\left[t^l\eta(t)\left(1-\mu_{\varepsilon,0}(t)\right)\right]\right]$$

with:

$$\begin{array}{rrcl} & C_{N+1,0,l} & = & iC_{N,0,l} \\ \forall j\in[\![1;N]\!]; & C_{N+1,j,l} & = & i(l+1+j-k(N+1))C_{N,j,l}+iC_{N,j-1,l} \\ & C_{N+1,N+1,l} & = & iC_{N,N,l} \end{array}$$

We finally have our formula at the rank $N+1$ and the property $\mathcal{P}_{N+1}$ holds.

7.5 A first property of the Gamma function

Return

Let $T>0$. With an integration by parts, we obtain:

$$\int_0^T t^{\frac{1}{k}-1}e^{-t}\mathrm{d}t=\left[kt^{\frac{1}{k}}e^{-t}\right]_{t=0}^{t=T}+k\int_0^t t^{\frac{1+k}{k}-1}e^{-t}\mathrm{d}t$$

When $T\to+\infty$, we obtain:

$$\Gamma\left(\frac{1}{k}\right)=k\Gamma\left(\frac{k+1}{k}\right)$$

which gives our property.

7.6 A second property of the Gamma function

Return

We have:

$$\Gamma\left(\frac{3}{2}\right)=\int_0^{+\infty}t^{\frac{1}{2}}e^{-t}\mathrm{d}t$$

Let $T>0$. We write out:

$$\Gamma_T:=\int_0^T t^{\frac{1}{2}}e^{-t}\mathrm{d}t$$

We make the variable change $t = u^2$, and have these properties:

$$\begin{aligned} dt &= 2udu \\ t^{\frac{1}{2}}e^{-t}dt &= 2u^2e^{-u^2}du \end{aligned}$$

We have, by using an integration by parts:

$$\Gamma_T = 2\int_0^{\sqrt{T}} u^2 e^{-u^2}\mathrm{d}u = \left[-ue^{-u^2}\right]_{u=0}^{u=\sqrt{T}} + \int_0^{\sqrt{T}} e^{-u^2}\mathrm{d}u$$

Finally, when $T \to +\infty$, we obtain:

$$\Gamma\left(\frac{3}{2}\right) = \int_0^{+\infty} e^{-u^2}\mathrm{d}u$$

If we admit the equality

$$\int_0^{+\infty} e^{-u^2}\mathrm{d}u = \frac{\sqrt{\pi}}{2}$$

we have our property.

7.7 A property of the Airy function

Return

We have:

$$Ai(x) = C\sqrt{x}\int_{-\infty}^{+\infty} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}\mathrm{d}u$$

Let $T > 0$. By using an integration by parts, we have:

$$\begin{aligned} C\sqrt{x}\int_{-T}^{T} e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}\mathrm{d}u &= C\sqrt{x}\int_{-T}^{T} \frac{ix^{\frac{3}{2}}\left(u^2+1\right)e^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}}{ix^{\frac{3}{2}}\left(u^2+1\right)}\mathrm{d}u \\ &= \left[\frac{C}{x}\frac{e^{ix^{\frac{3}{2}}}\left(\frac{1}{3}u^3+u\right)}{i\left(u^2+1\right)}\right] + \frac{C}{x}\int_{-T}^{T}\frac{2ue^{ix^{\frac{3}{2}}\left(\frac{1}{3}u^3+u\right)}}{i\left(u^2+1\right)}\mathrm{d}u \end{aligned}$$

When $T \to +\infty$, we obtain the property.

7.8 The first initial data in the KDV equation

Return

If we have:

$$u(x;0) = \int_{-\infty}^{+\infty} A(k)e^{ikx}\mathrm{d}k$$

$2\pi A$ is the inverse Fourier transform of $x \longmapsto e^{-|x|}$, and, in order to calculate A, we have to calculate the Fourier transform of the initial data:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|-ikx}\mathrm{d}x$$

We have:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|-ikx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|}\left[\cos(kx) - i\sin(kx)\right]\mathrm{d}x$$

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|}\cos(kx)\mathrm{d}x - i\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|}\sin(kx)\mathrm{d}x$$

The integrand of the second integral is odd. For this reason, the second integral is equal to 0, giving:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-|x|}\cos(kx)\mathrm{d}x = \frac{1}{\pi}\int_{0}^{+\infty} e^{-x} cos(kx)\mathrm{d}x$$

because the integrand is even.
Then, let $T > 0$. An integration by parts shows that:

$$\int_0^T e^{-x}\cos(kx)\mathrm{d}x = \left[-e^{-x}\cos(kx)\right]_{t=0}^{t=T} - k\int_0^T e^{-x}\sin(kx)\mathrm{d}x$$

and we have, by using a second integration by parts:

$$\int_0^T e^{-x}\sin(kx)\mathrm{d}x = \left[-e^{-x}\sin(kx)\right]_{t=0}^{t=T} + k\int_0^T e^{-x}\cos(kx)\mathrm{d}x$$

giving this result:

$$\int_0^T e^{-x}\cos(kx)\mathrm{d}x = 1 - e^{-T}\cos(kt) - k\left(-e^{-T}\cos(kT) + k\int_0^T e^{-x}\cos(kx)\mathrm{d}x\right)$$

Finally, when $T \longrightarrow +\infty$, we have:

$$\int_0^{+\infty} e^{-x}\cos(kx)\mathrm{d}x = \frac{1}{1+k^2}$$

which gives:

$$A(k) = \frac{1}{\pi\left(1+k^2\right)}$$

7.9 The second initial data in the KDV equation

Return

We have:

$$u(x;0) = \int_{-\infty}^{+\infty} A(k)e^{ikx}\mathrm{d}k = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi A(k)e^{ikx}\mathrm{d}k$$

The function $x \longmapsto u(x;0)$ is the invere Fourier transform of the function A. In order to compute A, we will have to compute the Fourier transform of the function $x \longmapsto \frac{1}{2\pi}u(x;0)$:

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(x;0)e^{-ikx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathbb{1}_{\left[-\frac{2\pi n}{\underline{k}};\frac{2\pi n}{\underline{k}}\right]}(x)e^{i(\underline{k}-k)x}\mathrm{d}x$$

$$A(k) = \frac{1}{2\pi}\int_{-\frac{2\pi n}{\underline{k}}}^{\frac{2\pi n}{\underline{k}}} e^{i(\underline{k}-k)x}\mathrm{d}x = \frac{1}{2\pi}\left[\frac{e^{i(\underline{k}-k)x}}{\underline{k}-k}\right]_{x=-\frac{2\pi n}{\underline{k}}}^{x=\frac{2\pi n}{\underline{k}}}$$

$$A(k) = \frac{1}{2\pi(\underline{k}-k)}\left[e^{-\frac{2\pi nk}{\underline{k}}} - e^{\frac{2\pi nk}{\underline{k}}}\right] = \frac{i\sin\left(\frac{2\pi nk}{\underline{k}}\right)}{\pi(k-\underline{k})}$$

7.10 A property of the Fourier transform

Return

Let $f \in L^1(\mathbb{R};\mathbb{C})$. We have:

$$\widehat{f}(\xi) = \int_{-\infty}^{+\infty} f(x)e^{-i\xi x}\mathrm{d}x$$

Besides, we have:

$$\widehat{f} = -\int_{-\infty}^{+\infty} f(x)e^{-i\xi x}e^{-i\pi}\mathrm{d}x = -\int_{-\infty}^{+\infty} f(x)e^{-i\xi\left(x+\frac{\pi}{\xi}\right)}\mathrm{d}x$$

If we make the variable change $y = x + \frac{\pi}{\xi}$, we have:

$$f(x)e^{-i\xi\left(x+\frac{\pi}{\xi}\right)}\mathrm{d}x = f\left(y - \frac{\pi}{\xi}\right)e^{-i\xi y}\mathrm{d}y$$

and we obtain this integral:

$$\widehat{f}(\xi) = -\int_{-\infty}^{+\infty} f\left(y - \frac{\pi}{\xi}\right)e^{-i\xi y}\mathrm{d}y$$

and, as a result, this equality:

$$2\widehat{f}(\xi) = \int_{-\infty}^{+\infty} \left[f(y) - f\left(y - \frac{\pi}{\xi}\right)\right]e^{-i\xi y}\mathrm{d}y$$

$$\widehat{f}(\xi) = \frac{1}{2}\int_{-\infty}^{+\infty} \left[f(y) - f\left(y - \frac{\pi}{\xi}\right)\right]e^{-i\xi y}\mathrm{d}y$$

Finally, we can boud from above $\widehat{f}(\xi)$ on this way:

$$\left|\widehat{f}(\xi)\right| = \frac{1}{2}\left|\int_{-\infty}^{+\infty} \left[f(y) - f\left(y - \frac{\pi}{\xi}\right)\right]e^{-i\xi y}\mathrm{d}y\right| \leq \frac{1}{2}\underbrace{\left\|y \longmapsto f(y) - f\left(y - \frac{\pi}{\xi}\right)\right\|_{L^1}}_{\underset{\xi\to+\infty}{\longrightarrow} 0}$$

7.11 The function ω

Return

Th function ω is defined by:

$$\forall \xi \in \mathbb{R} : \quad \omega(\xi) = \left(\frac{\xi \tanh\left(\sqrt{\mu}\xi\right)}{\sqrt{\mu}} \right)^{\frac{1}{2}} sgn(\xi)$$

This function is clearly odd on $\mathbb{R}$.

In order to simplify our raisonings, we will consider this function:

$$\omega_0(\xi) = (\xi \tanh(\xi))^{\frac{1}{2}} \, sgn(\xi)$$

And we have:

$$\omega(\xi) = \frac{1}{\sqrt{\mu}} \omega_0\left(\sqrt{\mu}\xi\right)$$

7.11.1 The $\mathcal{C}^3$-class function

First, we have:

$$\omega_0(\xi) \underset{\xi \to 0}{\sim} \xi$$

This function has a linear apprximation, and is $\mathcal{C}^1$-class near 0, and, as a result, on $\mathbb{R}$, and its derivative is given by:

$$\omega_0'(\xi) = \frac{\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi \tanh(\xi)\right)}{2\left(\xi \tanh(\xi)\right)} sgn(\xi) = \frac{\tanh(\xi) + \xi\left(1 - \tanh^2(\xi)\right)}{2\omega_0(\xi)}$$

with $\omega_0'(0) = 0$, giving $\omega'(0) = 0$ too.

Besides, the function ω_0' is differentiable on $\mathbb{R}^*$, and we can compute its derivative (for $\xi \neq 0$):

$$\omega_0''(\xi) = \frac{sgn(\xi)}{2\xi \tanh(\xi)} \left[\left(1 - \tanh^2(\xi) + 1 - \tanh^2(\xi) + 2\xi \tanh^3(\xi) - 2\xi \tanh(\xi)\right) \left(\xi \tanh(\xi)\right)^{\frac{1}{2}} \right]$$

$$- \frac{sgn(\xi)}{2\xi \tanh(\xi)} \left[\frac{1}{2\left(\xi \tanh(\xi)\right)^{\frac{1}{2}}} \left(\tanh(\xi) + \xi\left(1 - \tanh^2(\xi)\right)^2 \right) \right]$$

$$\omega_0''(\xi) = \frac{sgn(\xi)}{2\left(\xi \tanh(\xi)\right)^{\frac{3}{2}}} \left[2\left(1 - \tanh^2(\xi) + x \tanh^3(\xi) - \xi \tanh(\xi)\right) \xi \tanh(\xi) \right]$$

$$- \frac{sgn(\xi)}{2\left(\xi \tanh(\xi)\right)^{\frac{3}{2}}} \left[\frac{1}{2} \left(\tanh(\xi) + \xi - \xi \tanh^2(\xi) \right)^2 \right]$$

$$\omega_0''(\xi) = \frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\left[4\xi\tanh(\xi) - 4\xi\tanh^3(\xi) + 4\xi^2\tanh^4(\xi) - 4\xi^2\tanh^2(\xi)\right]$$

$$-\frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\left(\tanh(\xi) + x - x\tanh^2(\xi)\right)^2$$

$$\omega_0''(\xi) = \frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\left[4\xi\tanh(\xi) - 4\xi\tanh^3(\xi) + 4\xi^2\tanh^4(\xi) - 4\xi^2\tanh^2(\xi)\right]$$

$$-\frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\left[\tanh^2(\xi) + \xi^2 + \xi^2\tanh^2(\xi) + 2\xi\tanh(\xi) - 2\xi\tanh^3(\xi) - 2\xi^2\tanh^2(\xi)\right]$$

$$\omega_0''(\xi) = \frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\left[2\xi\tanh(\xi) - 2\xi\tanh^3(\xi) + 3\xi^2\tanh^4(\xi) - 2\xi^2\tanh^2(\xi) - \tanh^2(\xi) - \xi^2\right]$$

$$\omega_0''(\xi) = \frac{sgn(\xi)}{4\left(\xi\tanh(\xi)\right)^{\frac{3}{2}}}\underbrace{\left[-2\xi\tanh^3(\xi) - 2\xi^2\tanh^2(\xi) + 3\xi^2\tanh^4(\xi) - \left(\xi - \tanh(\xi)\right)^2\right]}_{:=\delta(\xi)}$$

When $\xi \to 0$, we have $\left(\xi\tanh(\xi)\right)^{\frac{3}{2}} = \mathcal{O}\left(\xi^3\right)$, and, if we use a Taylor approximation, we obtain $\delta(\xi) = \mathcal{O}\left(\xi^4\right)$, and, as a result, we have:

$$\omega_0''(\xi) \underset{\xi\to 0}{=} \mathcal{O}\left(\xi\right)$$

We conclude the function ω_0'' is continuous near 0, and ω_0 is $\mathcal{C}^2$-class on $\mathbb{R}$, and, as a result, ω too. Besides, ω_0'' has a lonear approximation, and, as a result, she is $\mathcal{C}^1$-class, and ω_0 (and ω) is $\mathcal{C}^3$-class on $\mathbb{R}$.

7.11.2 The variations of the derivative

We will study the variations of the derivative function of ω_0 on $\mathbb{R}_+^*$. Let $\xi > 0$. We have:

$$\omega_0''(\xi) = \frac{\delta(\xi)}{4\left(\xi \tanh(\xi)\right)^{\frac{3}{2}}}$$

If we want to know the sign of ω_0'', we need only to know the sign ofδ (the denominator has a positive sign when $\xi > 0$).

We have:

$$\delta(\xi) = -2\xi \tanh^3(\xi) - 2\xi^2 \tanh^2(\xi) + 3\xi^2 \tanh^4(\xi) - (\xi - \tanh(\xi))^2$$

$$\delta(\xi) = -2\xi \tanh^3(\xi) - 2\xi^2 \tanh^2(\xi) + 3\xi^2 \tanh^4(\xi) - \xi^2 - +2\xi \tanh(\xi) - \tanh^2(\xi)$$

By using the inequality $\tanh(\xi) \leq 1$, we obtain:

$$\delta(\xi) \leq -2\xi \tanh^3(\xi) - 2\xi^2 \tanh^2(\xi) + 3\xi^2 \tanh^2(\xi) - \xi^2 - +2\xi \tanh(\xi) - \tanh^2(\xi)$$

$$\delta(\xi) \leq -2\xi \tanh^3(\xi) + \xi^2 \tanh^2(\xi) - \xi^2 - +2\xi \tanh(\xi) - \tanh^2(\xi)$$

$$\delta(\xi) \leq -2\xi \left(1 - \tanh^2(\xi)\right) - \tanh^2(\xi) + \xi^2 \left(\tanh^2(\xi) - 1\right)$$

By using the inequality $|ab| \leq \frac{1}{2}\left(a^2 + b^2\right)$ for $a, b \in \mathbb{R}$ (the Swiss army knife od euclidian spaces...), we have:

$$\delta(\xi) \leq \left(\xi^2 + \tanh^2(\xi)\right)\left(1 - \tanh^2(\xi)\right) - \tanh^2(\xi) + \xi^2 \left(\tanh^2(\xi) - 1\right)$$

When we simplify, we obtain:

$$\delta(\xi) \leq \tanh^2(\xi)\left(1 - \tanh^2(\xi)\right) - \tanh^2(\xi)$$

and finally:

$$\delta(\xi) \leq -\tanh^4(\xi)$$

which leads to $\delta(\xi) < 0$ because $\xi > 0$.

We have $\forall \xi \in \mathbb{R}$; $\omega''(\xi) = \sqrt{\mu}\omega_0''(\sqrt{\mu}\xi)$.

Besides, if ω is odd, ω' is even. A s a resule, if we know the behaviour of ω' on $\mathbb{R}_+$, we know it on $\mathbb{R}$.

7.11.3 The limit of ω'

We have $\forall \xi \in \mathbb{R}$; $\omega'(\xi) = \omega_0'(\sqrt{\mu}\xi)$, and we write ω_0' on this way (for $\xi > 0$):

$$\omega_0'(\xi) = \frac{\tanh(\xi)}{2\left(\xi \tanh(\xi)\right)^{\frac{1}{2}}} + \frac{\xi\left(1 - \tanh^2(\xi)\right)}{2\left(\xi \tanh(\xi)\right)^{\frac{1}{2}}}$$

The first term decreases toward 0 when $\xi \to +\infty$, and, by noticing this property:

$$1 - \tanh^2(\xi) = 1 - \left(\frac{e^{\xi} - e^{-\xi}}{e^{\xi} - e^{-\xi}}\right)^2 = \frac{e^{2\xi} + 2 + e^{-2\xi} - e^{2\xi} + 2 - e^{-2\xi}}{e^{2\xi} + 2 + e^{-2\xi}}$$

$$1 - \tanh^2(\xi) = \frac{4}{e^{2\xi} + 2 + e^{-2\xi}}$$

We have:

$$\frac{\xi\left(1 - \tanh^2(\xi)\right)}{2\left(\xi \tanh(\xi)\right)^{\frac{1}{2}}} \underset{\xi \to +\infty}{\sim} \frac{1}{2}\sqrt{\xi}e^{-2\xi}$$

We obtain the limit of ω_0' near $+\infty$, and, as a result, this is the same limit for ω' (which is 0).

7.11.4 Table of variations of ω'

ξ	$-\infty$		0		$+\infty$
Sign of ω''		+	0	$-$	
Variations of ω'			1		
		$\nearrow$		$\searrow$	
	0				0

7.12 The third order Taylor's approximation of ω

Return

We have:

$$\omega_0(\xi) = sgn(\xi)\left(\xi \tanh(\xi)\right)^{\frac{1}{2}}$$

By using the third order Taylor's approximation of tanh, which is $\tanh(\xi) = \xi - \frac{1}{3}\xi^3 + \underset{\xi\to 0}{o}\left(\xi^4\right)$, we have:

$$\omega_0(\xi) = sgn(\xi)\left[\xi\left(\xi - \frac{1}{3}\xi^3 + \underset{\xi\to 0}{o}\left(\xi^4\right)\right)^{\frac{1}{2}}\right]$$

$$\omega_0(\xi) = sgn(\xi)\left[\xi^2\left(1 - \frac{1}{3}\xi^2 + \underset{\xi\to 0}{o}\left(\xi^3\right)\right)^{\frac{1}{2}}\right]$$

$$\omega_0(\xi) = sgn(\xi)|\xi|\left(1 - \frac{1}{3}\xi^2 + \underset{\xi\to 0}{o}\left(\xi^3\right)\right)^{\frac{1}{2}}$$

By using the Taylor's approximation $(1+\xi)^\alpha = 1 + \alpha\xi + \underset{\xi\to 0}{o}(\xi)$, we obtain:

$$\omega_0(\xi) = \xi\left(1 - \frac{1}{6}\xi^2 + \underset{\xi\to 0}{o}\left(\xi^3\right)\right)$$

$$\omega_0(\xi) = \xi - \frac{1}{6}\xi^3 + \underset{\xi\to 0}{o}\left(\xi^4\right)$$

And, if $\omega(\xi) = \frac{1}{\sqrt{\mu}}\omega_0(\sqrt{\mu}\xi)$, we obtain this formula:

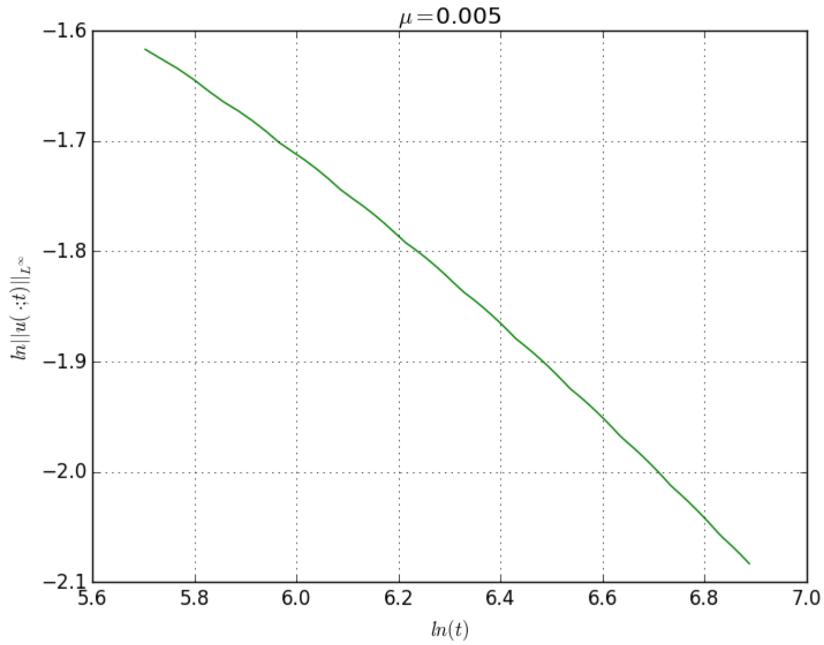
$$\omega(\xi) = \xi - \frac{1}{6}\mu\xi^3 + \underset{\xi\to 0}{o}\left(\xi^4\right)$$

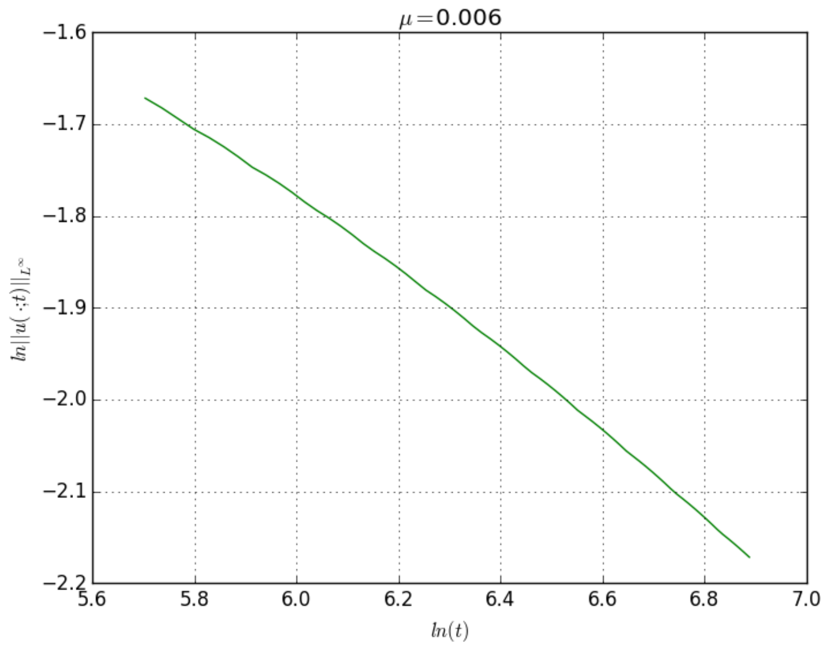
.

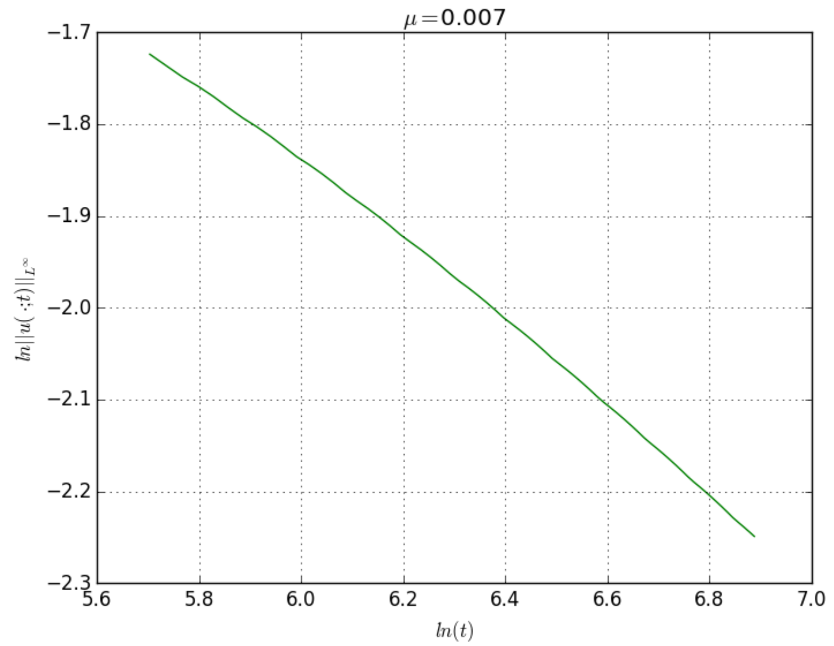
7.13 Graphs of numerical simulations

Return

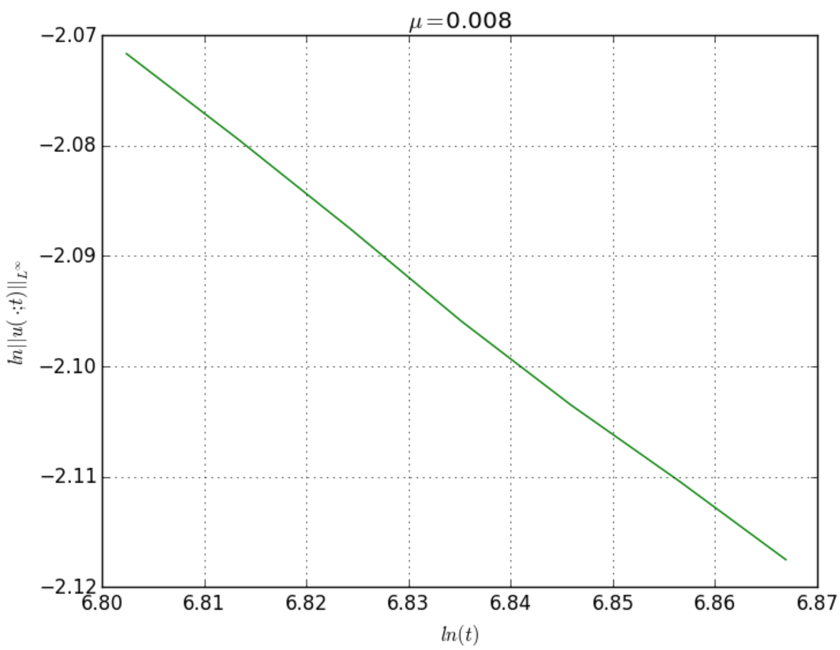
7.13.1 First scenario: 2012



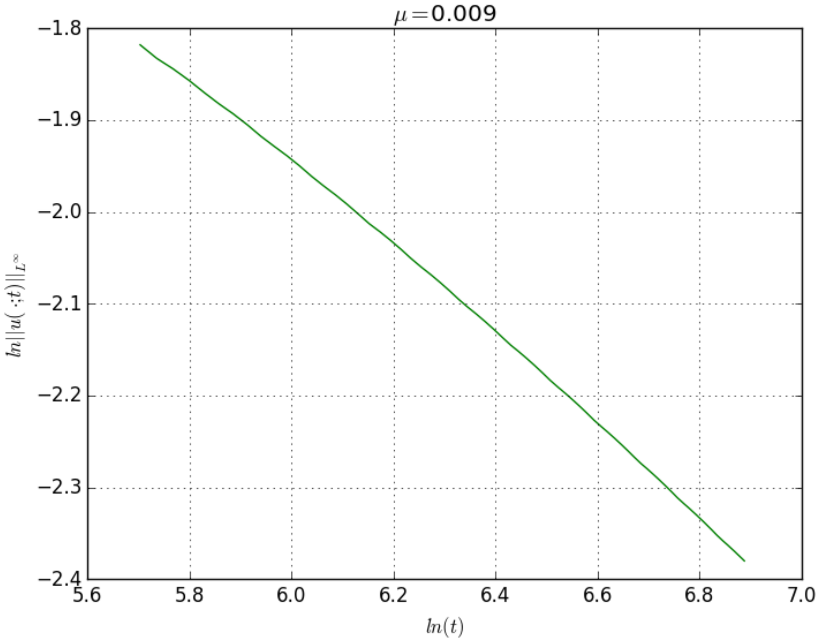


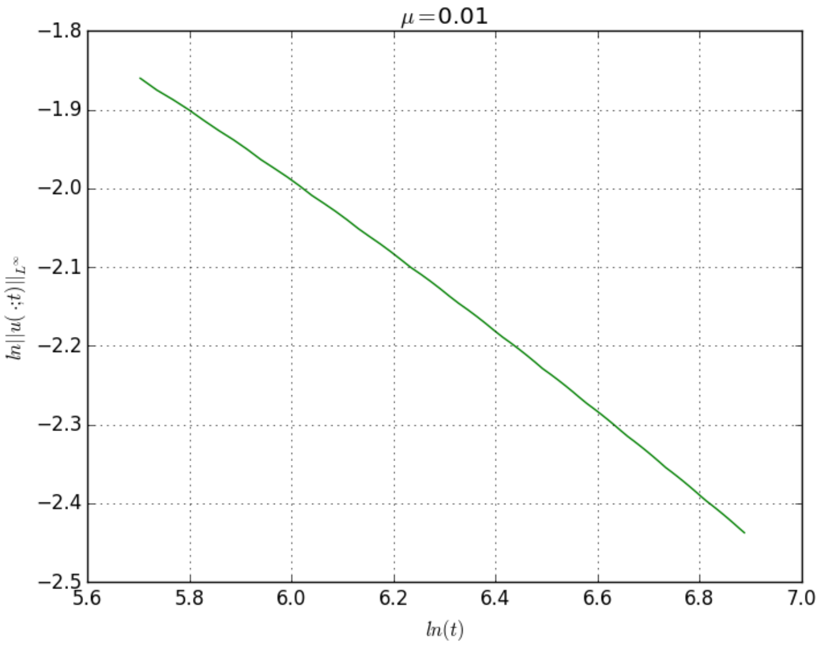


$\mu = 0.007$
$ln\|u(\cdot;t)\|_{L^\infty}$
$ln(t)$
−1.7
−1.8
−1.9
−2.0
−2.1
−2.2
−2.3
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0

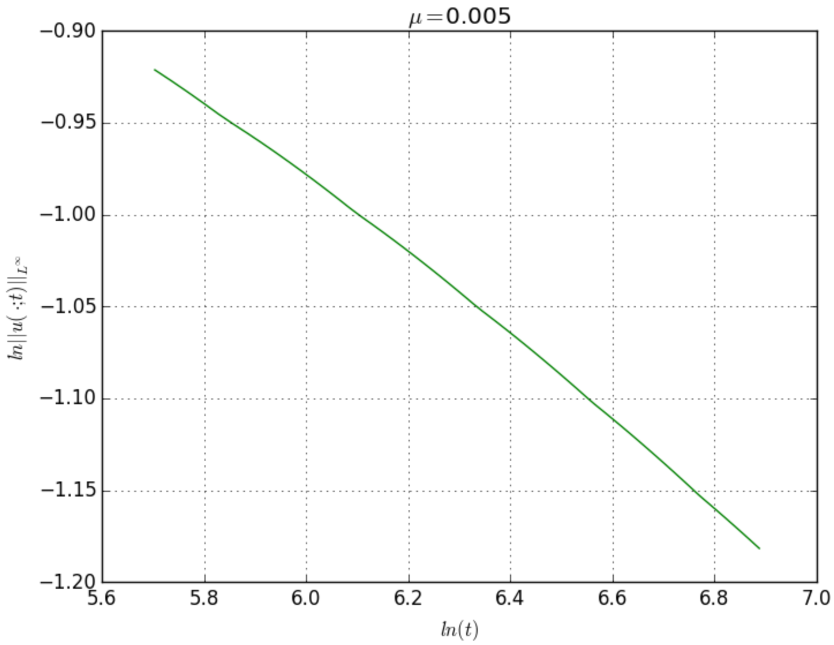


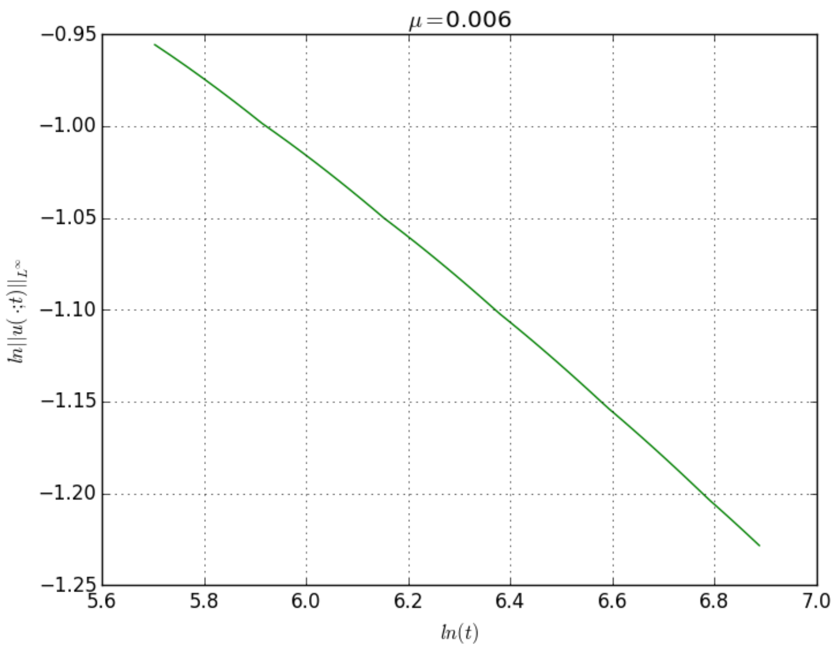
$\mu = 0.008$
$ln\|u(\cdot;t)\|_{L^\infty}$
$ln(t)$
−2.07
−2.08
−2.09
−2.10
−2.11
−2.12
6.80
6.81
6.82
6.83
6.84
6.85
6.86
6.87

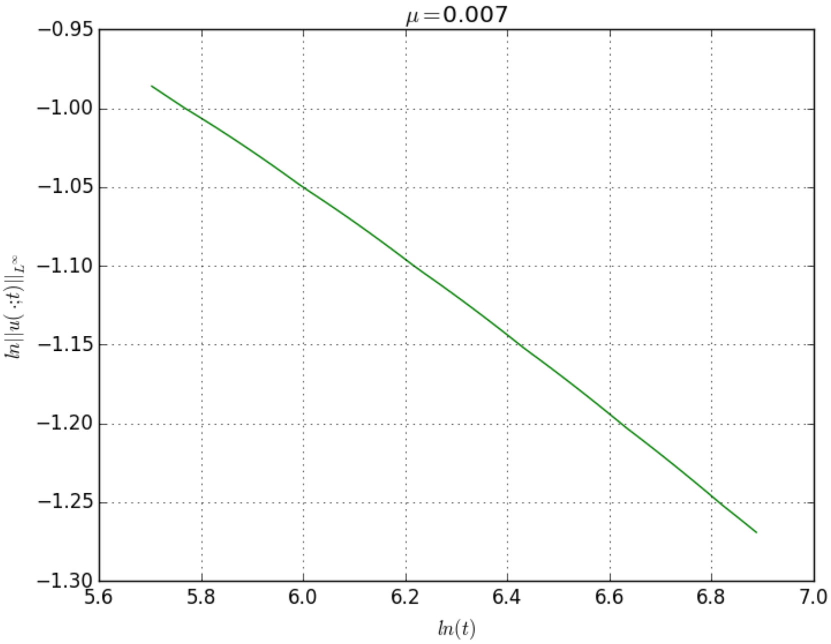

$\mu = 0.009$
$ln||u(\cdot;t)||_{L^\infty}$
$ln(t)$
−1.8
−1.9
−2.0
−2.1
−2.2
−2.3
−2.4
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0

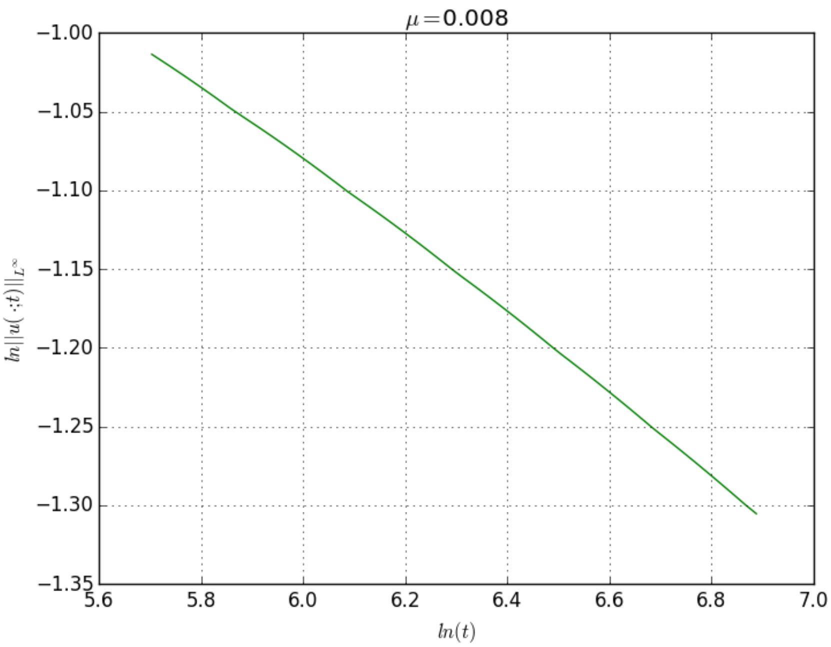

$\mu = 0.01$
$ln||u(\cdot;t)||_{L^\infty}$
$ln(t)$
−1.8
−1.9
−2.0
−2.1
−2.2
−2.3
−2.4
−2.5
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0

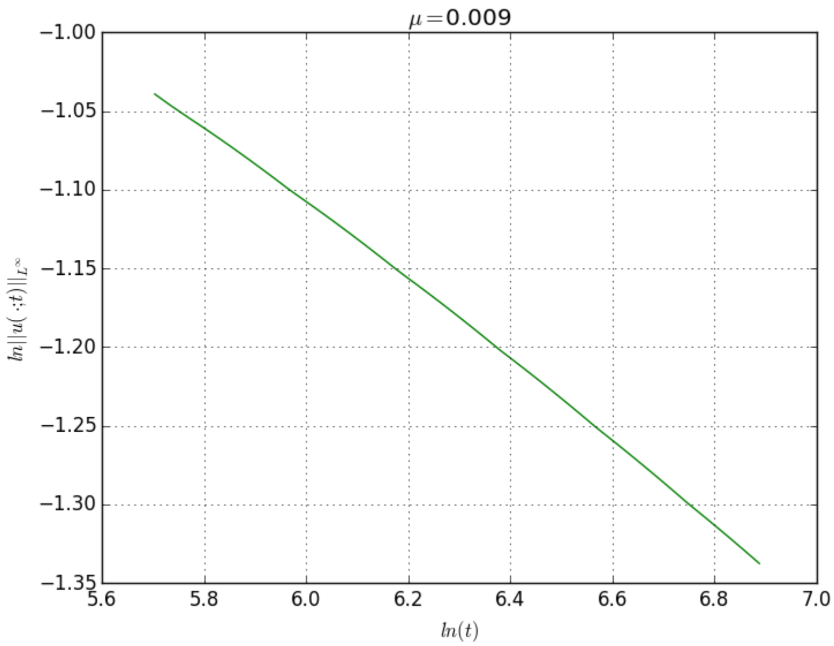
7.13.2 Second scenario: Deep Impact

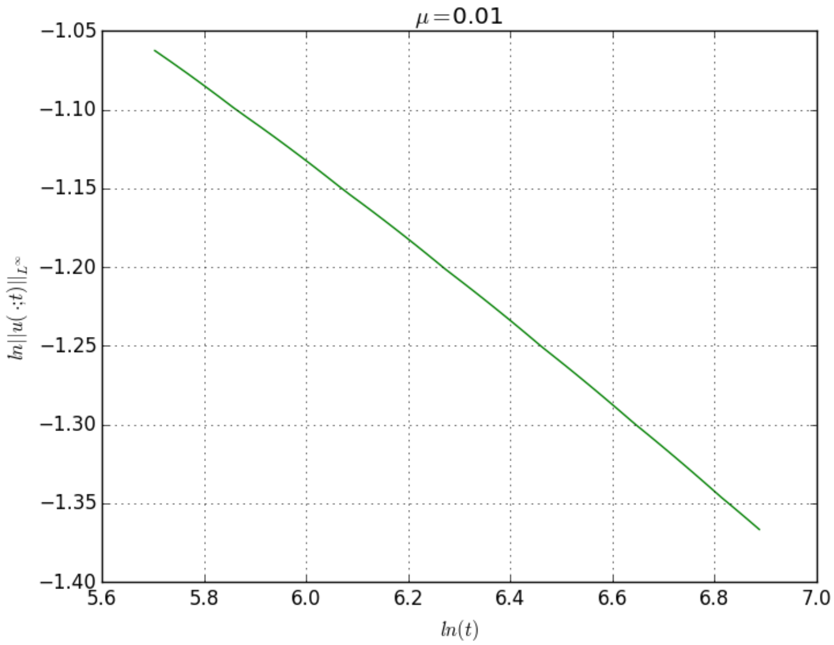





μ=0.007
−0.95
−1.00
−1.05
−1.10
−1.15
−1.20
−1.25
−1.30
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0
ln(t)
ln||u(·;t)||_{L∞}


μ=0.008
−1.00
−1.05
−1.10
−1.15
−1.20
−1.25
−1.30
−1.35
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0
ln(t)
ln||u(·;t)||_{L∞}


$\mu = 0.009$
$ln||u(\cdot;t)||_{L^\infty}$
$ln(t)$
−1.00
−1.05
−1.10
−1.15
−1.20
−1.25
−1.30
−1.35
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0


$\mu = 0.01$
$ln||u(\cdot;t)||_{L^\infty}$
$ln(t)$
−1.05
−1.10
−1.15
−1.20
−1.25
−1.30
−1.35
−1.40
5.6
5.8
6.0
6.2
6.4
6.6
6.8
7.0

Chapter 8

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