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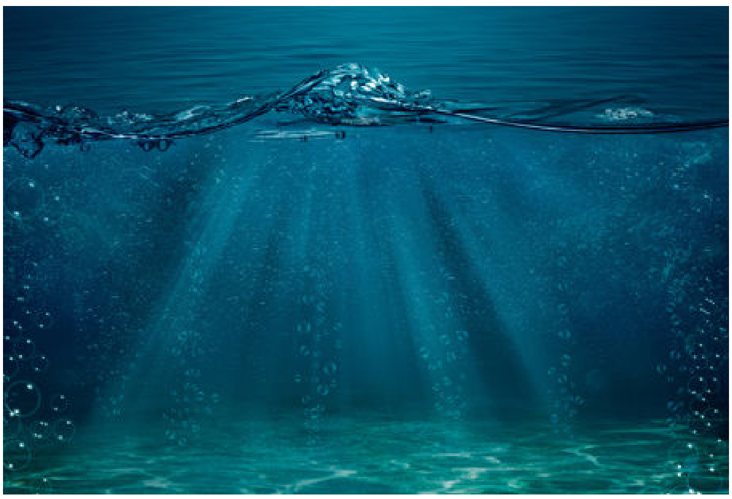
Long time behaviours for the unidimensional compressible and isentropic Navier-Stokes system



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Abstract

The goal of this internship was the study of the unidimensional Navier-Stokes system, divided in two main parts:

- The study of the linearized system, in particular the long-time behviour of solutions of the linear system, for initial conditions near an equilibrium.

- Numerical simulations over the linearized system, with qualitative study of numerical schemes, and numerical simulations over the complete non-linear system.

Key-words: Partial differential equations, fluid mechanics, linearized system, Fourier transform, residue theorem, long time decay, numerical scheme, consistency, stability.

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Chapter 1

The linearized Navier-Stokes system

1.1 Linearisation of the Navier-Stokes system

1.1.1 Generalities

We consider a fluid moving in one dimension. We note ρ the mass density field and u the velocity field of the fluid. Then, we note m the momentum density field, defined by $m = \rho u$. In one dimension, we have:

$$\rho : \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & \rho(t,x) \end{array} \qquad u : \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & u(t,x) \end{array}$$

$$m : \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & m(t,x) \end{array}$$

By supposing the flom incompressible, and neglecting the entropy variations, we obtain this system of equations:

$$\left\{ \begin{array}{ccccccc} \partial_t \rho & + & \partial_x m & = & 0 & & \\ \partial_t m & + & \partial_x \left(\frac{m^2}{\rho} \right) & = & (2\mu + \lambda)\, \partial_x^2 \left(\frac{m}{\rho} \right) & - & \partial_x p \end{array} \right.$$

with $p = P(\rho)$ is a monotically increasing function (this result is obtained by neglecting entropy variations, and P is the pressure field), and λ and μ are viscosity parameters. We will study this system, in particular the pair (ρ, m) for initial conditions near this equilibrium:

$$(\rho, m)|_{t=0} \approx (\rho*, 0)$$

where $\rho*$ is a constant density field. Now, we will linearize this system.

1.1.2 Linearization

We note $\mu_{\|} = 2\mu + \lambda$, and we introduce two new elements:

- The (reference) sound speed, defined by $c = \sqrt{P'(\rho_*)}$

- The function $\widetilde{\rho}$, defined this way:

$$\widetilde{\rho}: \begin{array}{ccc} \mathbb{R}_+ \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (t,x) & \longmapsto & \rho(t,x) - \rho_* \end{array}$$

In order to obtain the linearized system, we will only consider the 0 and 1-order terms (neglecting highest order terms, noted *hot*), by using these rules (with $p, q \in \mathbb{N}$):

- $\rho*$ is a 0-order term, and $\widetilde{\rho}$ is a 1-order term. m is a 1-order term too.

- The derivative of a p-order term remains a p-order term.

- The multiplication of a p-order term with a q-order term gives a $p+q$-order term.

Proposition 1. ***Linearized system***
If we have $\rho* = 1$, the linearized system is:

$$\left\{ \begin{array}{ccccc} \partial_t \widetilde{\rho} & + & \partial_x m & = & 0 \\ \partial_t m & + & c^2 \partial_x \widetilde{\rho} & = & \mu_{\|} \partial_x^2 m \end{array} \right.$$

Proof. We have $\rho = \rho_* + \widetilde{\rho}$. Therefore, we obtain:

$$\begin{array}{rcl} \partial_x p(t,x) & = & \partial_x P\left(\rho(t,x)\right) \\ & = & \partial_x \rho(t,x) \cdot P'\left(\rho(t,x)\right) \\ & = & \partial_x \widetilde{\rho}(t,x) P'(\rho_*) + hot \\ & = & c^2 \partial_x \widetilde{\rho}(t,x) \end{array}$$

Besides, we have:

$$\frac{1}{\rho} = \frac{1}{\rho_* + \widetilde{\rho}} = \frac{1}{\rho_*} + hot = 1 + hot$$

As a result, we have:

$$\frac{m}{\rho} = \frac{m}{\rho_*} + hot \text{ and } \frac{m^2}{\rho} = hot$$

Finally, the system can be written as:

$$\left\{ \begin{array}{ccccccccc} \partial_t \widetilde{\rho} & + & \partial_x m & = & 0 & & & & \\ \partial_t m & + & hot & = & \frac{\mu_{\|}}{\rho_*} \partial_x^2 m & + & hot & - & c^2 \partial_x \widetilde{\rho} \quad + \quad hot \end{array} \right.$$

If the terms *hot* are neglected, and $\rho_* = 1$, we obtain this:

$$\left\{ \begin{array}{ccccc} \partial_t \widetilde{\rho} & + & \partial_x m & = & 0 \\ \partial_t m & + & c^2 \partial_x & = & \mu_{\|} \partial_x^2 m \end{array} \right.$$

□

1.2 Resolution of the linear system by using the Fourier transform

If we consider the linear system of the Proposition 1, we can use the Fourier transfom for x, defined by:

$$\hat{f}(\eta) = \int_{-\infty}^{+\infty} e^{-ix\eta} f(x)\mathrm{d}x$$

An we obtain this ordinary differential equation (a system of equations):

$$\begin{cases} \partial_t \hat{\tilde{\rho}} & + & i\eta\hat{m} & = & 0 \\ \partial_t \hat{m} & + & i\eta c^2 \hat{\tilde{\rho}} & = & -\mu_{\parallel}\eta^2\hat{m} \end{cases}$$

If we note:

$$X(t,x) = \begin{bmatrix} \tilde{\rho}(t,x) \\ m(t,x) \end{bmatrix}$$

We have this ordinary differential equation in t:

$$\partial_t \hat{X}(t,\eta) = A(\eta) \cdot \hat{X}(t,\eta)$$

where:

$$A(\eta) = \begin{bmatrix} 0 & -i\eta \\ -i\eta c^2 & -\mu_{\parallel}\eta^2 \end{bmatrix}$$

The spectrum of the matrix $A(\eta)$ is given by:

$$\sigma(A(\eta)) = \left\{\lambda^+(\eta); \lambda^-(\eta)\right\}$$

where:

$$\lambda^{\mp}(\eta) = -\frac{1}{2}\mu_{\parallel}\eta^2 \pm \frac{1}{2}\sqrt{\mu_{\parallel}^2\eta^4 - 4c^2\eta^2}$$

We can obtain our solution of the Fourier transform of the linear system of equations:

$$\forall t \in \mathbb{R}_+; \hat{X}(t,\eta) = e^{tA(\eta)}\hat{X}(0,\eta)$$

Proposition 2.

$$\forall t \geq 0;$$

$$e^{tA(\eta)} = \frac{1}{\lambda^+(\eta) - \lambda^-(\eta)} \begin{bmatrix} \lambda^+(\eta) e^{\lambda^-(\eta)t} - \lambda^-(\eta) e^{\lambda^+(\eta)t} & -i\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) \\ -ic^2\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) & \lambda^+(\eta) e^{\lambda^+(\eta)t} - \lambda^-(\eta) e^{\lambda^-(\eta)t} \end{bmatrix}$$

Proof. The proof uses the Cauchy's integral formula for matrices (**cf. Chapter 3**), and the residue theorem (**cf. Chapter 3** for the computation). □

Henceforth, we will use this notation:

$$\forall (t,\eta) \in \mathbb{R}_+ \times \mathbb{R};\ \ \hat{S}(t,\eta) = e^{tA(\eta)}$$

1.3 Estimation of the linear terms

The Fourier transform gives this result:

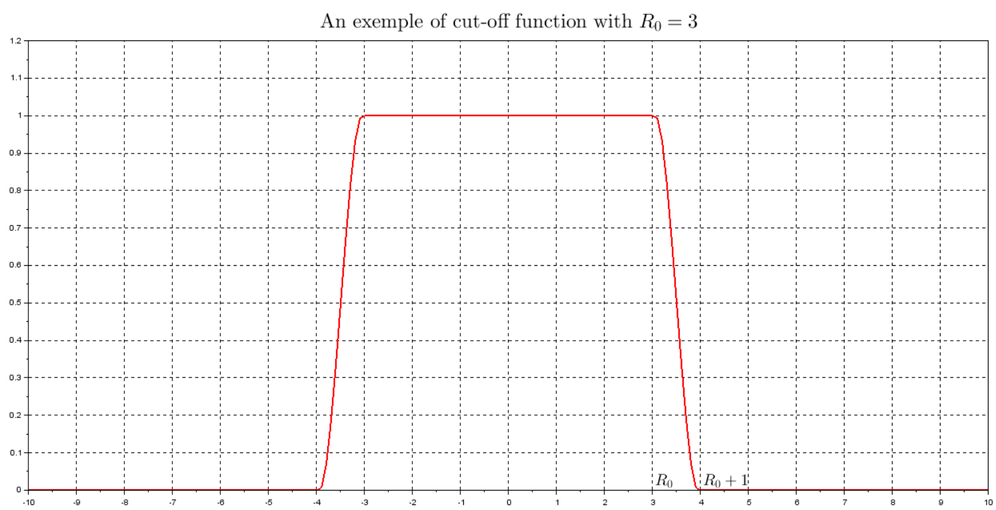
$$\begin{array}{rcccc} \widehat{X}(t,\eta) & = & e^{tA(\eta)}\widehat{X}(0,\eta) & = & \widehat{S}(t,\eta)\widehat{X}(0,\eta) \\ & = & \widehat{S^{LF}}(t,\eta)\widehat{X}(0,\eta) & + & \widehat{S^{HF}}(t,\eta)\widehat{X}(0,\eta) \end{array}$$

where:

$\widehat{S^{LF}}(t,\eta) = \chi(\eta)\widehat{S}(t,\eta)$ is the *low frequency term* and $S^{\hat{H}F}(t,\eta) = [1-\chi(\eta)]\hat{S}(t,\eta)$ is the *high frequency term.*

$\chi \in \mathcal{D}(\mathbb{R})$ is a cut-off function (a smooth function with compact support), given by:

$$\chi \equiv 1 \text{ if } |\eta| \leq R_0\ ,\ \chi \equiv 0 \text{ if } |\eta| \geq R_0 + 1 \text{ and } 0 \leq \chi \leq 1$$



1.3.1 Low frequency terms

In this subsection, we will estimate the low frequency terms in L^p-norm for two classes of the index p.

Index of estimation in [1;2]

Proposition 3. *Let $p \in [1;2]$ and $\sigma \in \mathbb{N}$:*

$$\left\|d_x^\sigma S^{LF}(t,\cdot)\right\|_{L^p} \leq C(p,\sigma)\begin{cases} 1 & \text{if} \quad 0 \leq t \leq 1 \\ t^{-\alpha(p,\sigma)} & \text{if} \quad t \geq 1 \end{cases}$$

with $C(p,\sigma) \geq 0$, and $\alpha(p,\sigma) = Min\left\{\frac{1+\sigma}{2}; \frac{1}{4}\left[1-\frac{1}{p}+\sigma\left(1-\frac{1}{2p}\right)\right]\right\}$

Proof. Let $\sigma \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \geq 0$:
We have:

$$d_x^\sigma S^{LF}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\eta}\eta^\sigma \chi(\eta) e^{tA(\eta)}\mathrm{d}\eta$$

Besides, we have:

$$\forall t \geq 0; \quad e^{tA(\eta)} = P(\eta)^{-1}\begin{bmatrix} e^{t\lambda^+(\eta)} & 0 \\ 0 & e^{t\lambda^-(\eta)} \end{bmatrix} P(\eta)$$

with $P : \mathbb{R} \longrightarrow GL_2(\mathbb{C})$ (we have to choose R_0 small enough in order to diagonalize $e^{tA(\eta)}$).

Therefore:

$$d_x^\sigma S^{LF}(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \eta^\sigma \chi(\eta) P(\eta)^{-1}\begin{bmatrix} e^{ix\eta+t\lambda^+(\eta)} & 0 \\ 0 & e^{ix\eta+t\lambda^-(\eta)} \end{bmatrix} P(\eta)\mathrm{d}\eta$$

Let $t \geq 1$. We have:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C_\sigma \left|\int_{-\infty}^{+\infty} \sqrt[3]{\chi(\eta)}\eta^\sigma e^{ix\eta}\begin{bmatrix} e^{t\lambda^+(\eta)} & 0 \\ 0 & e^{t\lambda^-(\eta)} \end{bmatrix}\mathrm{d}\eta\right|$$

But $\lambda^\pm(\eta) \underset{|\lambda|\to 0}{=} \pm ic|\lambda| - \frac{1}{2}\mu_\| \eta^2 + \mathcal{O}\left(|\lambda|^3\right)$, therefore:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C'_\sigma \left|\int_{-\infty}^{+\infty} \sqrt[3]{\chi(\eta)}\eta^\sigma e^{ix\eta - \frac{1}{2}\mu_\|\eta^2 t \pm ict\eta}\mathrm{d}\eta\right| \tag{1.1}$$

First, we study the case $t \geq 1$:
On the one hand:

$$\begin{aligned} \left|d_x^\sigma S^{LF}(t,x)\right| &\leq C''_\sigma \left|\int_{-\infty}^{+\infty} \eta^\sigma e^{i(x\pm ct)\eta} e^{-\frac{1}{2}\mu_\|\eta^2 t}\mathrm{d}\eta\right| \\ \left|d_x^\sigma S^{LF}(t,x)\right| &\leq 2C''_\sigma \int_0^{+\infty} \eta^\sigma e^{-\frac{1}{2}\mu_\|\eta^2 t}\mathrm{d}\eta \\ \left|d_x^\sigma S^{LF}(t,x)\right| &\leq \frac{C_\sigma^{(3)}}{t^{\frac{1}{2}(1+\sigma)}} \end{aligned} \tag{1.2}$$

(**cf. Chapter 3** for the computation).

On the other hand:

$$\begin{aligned}
\left|d_x^\sigma S^{LF}(t,x)\right| &\leq C_\sigma^{(4)} \left|\int_{-\infty}^{+\infty} \eta^\sigma e^{i(x\pm ct)\eta-\frac{1}{2}\mu_\parallel \eta^2 t}\mathrm{d}\eta\right| \\
\left|d_x^\sigma S^{LF}(t,x)\right| &\leq C_\sigma^{(4)} \left|\int_{-\infty}^{+\infty} \eta^\sigma e^{i(x+ct)\eta-\frac{1}{2}\mu_\parallel \eta^2 t}\mathrm{d}\eta\right| \\
&+ C_\sigma^{(4)} \left|\int_{-\infty}^{+\infty} \eta^\sigma e^{i(x-ct)\eta-\frac{1}{2}\mu_\parallel \eta^2 t}\mathrm{d}\eta\right|
\end{aligned}$$

We assume $|x \pm ct| \geq 1$. A dobble integration by parts and the triangle inequality (**cf. Chapter 3**) show that:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C_\sigma^{(4)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\}.$$
$$\int_{-\infty}^{+\infty} \left[\sigma(\sigma-1)|\eta|^{\sigma-2} + \mu_\parallel t + \mu_\parallel \sigma |\eta|^\sigma t + \mu_\parallel^2 \eta^2 t^2\right] e^{-\frac{1}{2}\mu_\parallel \eta^2 t}\mathrm{d}\eta$$

With the substitution $\nu = \frac{\eta}{\sqrt{t}}$, we obtain:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C_\sigma^{(4)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\}.$$
$$\int_{-\infty}^{+\infty} \left[\sigma(\sigma-1)|\nu|^{\sigma-2} t^{-\frac{1}{2}(\sigma-3)} + \mu_\parallel t^{\frac{1}{2}} + \mu_\parallel \sigma |\nu|^\sigma t^{-\frac{1}{2}\sigma} + \mu_\parallel^2 \nu^2 t^{\frac{1}{2}}\right] e^{-\frac{1}{2}\mu_\parallel \nu^2}\mathrm{d}\nu$$

If we only keep the factors depending on t, and consider the integrals as constants, we have:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C_\sigma^{(5)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\} \left[t^{Min\left\{0;-\frac{1}{2}(\sigma-3)\right\}} + t^{\frac{1}{2}} + t^{-\frac{1}{2}\sigma}\right]$$

hence:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq C_\sigma^{(6)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\} t^{\frac{1}{2}} \tag{1.3}$$

If we combine the equalities (1.2) and (1.3), we obtain, for $\delta \in [0;1]$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$:

$$\left|d_x^\sigma S^{LF}(t,x)\right| \leq \frac{C_\sigma^{(3)^\delta} C_\sigma^{(6)^{1-\delta}}}{t^{\frac{\delta}{2}(1+\sigma)}} t^{\frac{1}{2}(1-\delta)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\}^{1-\delta}$$

If we note $C(p,\sigma,\delta) = C_\sigma^{(3)^{\delta p}} C_\sigma^{(6)^{(1-\delta)p}}$, we obtain, for $p \in [1;2]$ and $\delta \in [0;1]$:

$$\left|d_x^\sigma S^{LF}(t,x)\right|^p \leq C(p,\sigma,\delta) t^{\frac{p}{2}-\delta p\left(1+\frac{\sigma}{2}\right)} \left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\}^{p(1-\delta)} \tag{1.4}$$

When $|x \pm ct| \leq 1$, we chosse $\delta = 1$ in the equation (1.4). Therefore:

$$\left|d_x^\sigma S^{LF}(t,x)\right|^p \leq C(p,\sigma,1)t^{-\frac{p}{2}(1+\sigma)} \tag{1.5}$$

When $|x \pm ct| \geq 1$, we have to choose $\delta \in [0;1]$ in the equation (1.4) such as:

- $2p(1-\delta) > 1 \Leftrightarrow \delta < 1 - \frac{1}{2p} \leq \frac{1}{2}$ so as to have the integrability with x
- $\frac{p}{2} - \delta p \left(1 + \frac{\sigma}{2}\right) \leq 0 \Leftrightarrow \delta \geq \frac{1}{2+\sigma}$ so as to have a decay (or a bounded solution in time t)

Therefore we can choose this value for δ (depending on p and σ):

$$\delta(p,\sigma) = \frac{1}{2}\left[\frac{1}{2+\sigma} + 1 - \frac{1}{2p}\right]$$

Besides, an argument of convexity gives the next inequality (**cf. Chapter 3** for a proof):

$$\left\{\frac{1}{(x+ct)^2} + \frac{1}{(x-ct)^2}\right\}^{p(1-\delta(p,\sigma))} \leq \frac{1}{2^{1+2p(1-\delta(p,\sigma))}}\left\{\frac{1}{|x+ct|^{2p(1-\delta(p,\sigma))}} + \frac{1}{|x-ct|^{2p(1-\delta(p,\sigma))}}\right\}$$

As a result, we have this final inequality:

$$\begin{aligned}\left|d_x^\sigma S^{LF}(t,x)\right|^p &\leq \frac{C(p,\sigma,\delta(p,\sigma))t^{\frac{p}{2}-\delta(p,\sigma)p\left(1+\frac{\sigma}{2}\right)}}{2^{1+2p(1-\delta(p,\sigma))}} \\ &\times \left\{\frac{1}{|x+ct|^{2p(1-\delta(p,\sigma))}} + \frac{1}{|x-ct|^{2p(1-\delta(p,\sigma))}}\right\}\end{aligned} \tag{1.6}$$

Now, we have to integrate the inequalities (1.5) over the domain $\{|x \pm ct| \leq 1\}$ and (1.6) over the domain $\{|x \pm ct| \geq 1\}$:

$$\begin{aligned}\left\|d_x^\sigma S^{LF}(t,x)\right\|_{L^p}^p &= \int_{-\infty}^{+\infty} \left|d_x^\sigma S^{LF}(t,x)\right|^p \mathrm{d}x \\ &= \int_{|x\pm ct|\leq 1} \left|d_x^\sigma S^{LF}(t,x)\right|^p \mathrm{d}x + \int_{|x\pm ct|\geq 1} \left|d_x^\sigma S^{LF}(t,x)\right|^p \mathrm{d}x \\ &\leq \int_{|x\pm ct|\leq 1} C(p,\sigma,1)t^{-\frac{p}{2}(1+\sigma)}\mathrm{d}x \\ &+ \frac{C(p,\sigma,\delta(p,\sigma))}{2^{1+2p(1-\delta(p,\sigma))}}\int_{|x+ct|\geq 1} \frac{t^{\frac{p}{2}-\delta(p,\sigma)p\left(1+\frac{\sigma}{2}\right)}\mathrm{d}x}{|x+ct|^{2p(1-\delta(p,\sigma))}} \qquad (1.7) \\ &+ \frac{C(p,\sigma,\delta(p,\sigma))}{2^{1+2p(1-\delta(p,\sigma))}}\int_{|x-ct|\geq 1} \frac{t^{\frac{p}{2}-\delta(p,\sigma)p\left(1+\frac{\sigma}{2}\right)}\mathrm{d}x}{|x-ct|^{2p(1-\delta(p,\sigma))}} \qquad (1.8) \\ &\leq 2C(p,\sigma,1)t^{-\frac{p}{2}(1+\sigma)} + C(p,\sigma,\delta(p,\sigma))' t^{\frac{p}{2}-p\delta(p,\sigma)\left(1+\frac{\sigma}{2}\right)}\end{aligned}$$

Remark. *The integrals* $\int_{|x\pm ct|\geq 1} \frac{\mathrm{d}x}{|x\pm ct|^{2p(1-\delta(p,\sigma))}}$ *in the expansions* (1.7) *and* (1.8) *don't depend on* t *(**cf. Chapter 3**).*

If we note:

$$\begin{aligned} C(p,\sigma)^* &:= 2^{\frac{1}{p}} Max\left\{2C(p,\sigma,1)\ ;\ C(p,\sigma,\delta(p,\sigma))'\right\}^{\frac{1}{p}} > 0 \\ \alpha(p,\sigma) &:= Min\left\{\frac{1+\sigma}{2}\ ;\ \delta(p,\sigma)\left(1+\frac{\sigma}{2}-\frac{1}{2}\right)\right\} \geq 0 \end{aligned}$$

Finally, we obtain:

$$\forall\,(p,\sigma)\in[1;2]\times\mathbb{N}, \forall t\geq 1; \qquad \left\|d_x^\sigma S^{LF}(t,x)\right\|_{L^p} \leq C(p,\sigma)^* t^{-\alpha(p,\sigma)}$$

Remark. *A simple computation shows that:*

$$\delta(p,\sigma)\left(1+\frac{\sigma}{2}\right)-\frac{1}{2} = \frac{1}{4}\left[1-\frac{1}{p}+\sigma\left(1-\frac{1}{2p}\right)\right]$$

Then, we have to study the case $t\in[0;1]$:

When $|x\pm ct|\geq 1$, we choose the parameter $\delta=0$ in the equation (1.3), and we have:

$$\begin{aligned} \left|d_x^\sigma S^{LF}(t,x)\right|^p &\leq C(p,\sigma,0)t^{\frac{p}{2}}\left\{\frac{1}{(x+ct)^2}+\frac{1}{(x-ct)^2}\right\}^p \\ &\leq C(p,\sigma,0)\left\{\frac{1}{|x+ct|}+\frac{1}{|x-ct|}\right\}^{2p} \\ &\leq \frac{C(p,\sigma,0)}{2^{1+2p}}\left\{\frac{1}{|x+ct|^{2p}}+\frac{1}{|x-ct|^{2p}}\right\} \end{aligned}$$

When $|x\pm ct|\leq 1$, we use the inequality (1.1), and we obtain :

$$\begin{aligned} \left|d_x^\sigma S^{LF}(t,x)\right| &\leq C'_\sigma\left|\int_{-\infty}^{+\infty}\sqrt[3]{\chi(\eta)}\eta^\sigma e^{ix\eta-\frac{1}{2}\mu_\parallel\eta^2 t\pm ict\eta}\mathrm{d}\eta\right| (1.1) \\ &\leq C'_\sigma\int_{-\infty}^{+\infty}\sqrt[3]{\chi(\eta)\,|\eta|^\sigma}\mathrm{d}\eta \\ &\leq C_\sigma^{(7)} \end{aligned}$$

Then, we have:

$$\begin{aligned} \int_{-\infty}^{+\infty}\left|d_x^\sigma S^{LF}(t,x)\right|^p\mathrm{d}x &= \int_{|x\pm ct|\leq 1}\left|d_x^\sigma S^{LF}(t,x)\right|^p\mathrm{d}x \\ &+ \int_{|x\pm ct|\geq 1}\left|d_x^\sigma S^{LF}(t,x)\right|^p\mathrm{d}x \\ &\leq 4C_\sigma^{(7)^p}+\frac{C(p,\sigma,0)}{2^{1+2p}}\int_{|x+ct|\geq 1}\frac{\mathrm{d}x}{|x+ct|^{2p}} \\ &+ \frac{C(p,\sigma,0)}{2^{1+2p}}\int_{|x-ct|\geq 1}\frac{\mathrm{d}x}{|x-ct|^{2p}} \end{aligned}$$

As a result, we obtain:

$$\forall t\in[0;1]\,;\quad \left\|d_x^\sigma S^{LF}(t,\cdot)\right\|_{L^p} \leq C_{\sigma,p}$$

where $C_{\sigma,p}$ is a constant independant of t.

Finally, if we note:

$$C(p,\sigma) := Max\left\{C(p,\sigma)^* \; ; \; C_{\sigma,p}\right\}$$

We obtain the main result of the proposition. □

Index of estimation higher than 2

Proposition 4. *Let* $p \in [2;+\infty]$ *and* $\sigma \in \mathbb{N}$*:*

$$\left\|d_x^\sigma S^{LF}(t,\cdot)\right\|_{L^p} \leq C_p^{(\sigma)} \begin{cases} 1 & \text{if} \quad 0 \leq t \leq 1 \\ t^{-\frac{1}{2}\left(1-\frac{1}{p}+\sigma\right)} & \text{if} \quad t \geq 1 \end{cases}$$

with $C_p^{(\sigma)} \geq 0$.

Proof. Let $p \in [2;+\infty]$ and $\sigma \in \mathbb{N}$. We have:

$$X^{LF}(t,x) = \mathcal{F}^{-1}\left(\widehat{X^{LF}}(t,\cdot)\right) = \mathcal{F}^{-1}\left(\widehat{S^{LF}}(t,\cdot)\,\widehat{X}(0,\cdot)\right)$$

Therefore, we obtain:

$$d_x^\sigma X^{LF}(t,x) = d_x^\sigma\left[\mathcal{F}^{-1}\widehat{S^{LF}}(t,\cdot) * X_0\right](x) = \left[\left\{d_x^\sigma \mathcal{F}^{-1}\left(\widehat{S^{LF}}(t,\cdot)\right)\right\} * X_0\right](x)$$

With the Young's inequality, we obtain:

$$\left\|d_x^\sigma X^{LF}(t,\cdot)\right\|_{L^r} \leq \left\|d_x^\sigma \mathcal{F}^{-1}\left(\widehat{S^{LF}}(t,\cdot)\right)\right\|_{L^p} \|X_0\|_{L^q}$$

where $1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}$.

We have to estimate this factor:

$$\left\|d_x^\sigma \mathcal{F}^{-1}\left(\widehat{S^{LF}}(t,\cdot)\right)\right\|_{L^p} = \left\|d_x^\sigma S^{LF}(t,\cdot)\right\|_{L^p}$$

Let $t \geq 1$:

By using a duality property of the Fourier transform (**cf. Chapter 3**), we obtain:

$$\left\|d_x^\sigma \mathcal{F}^{-1}\left(S^{\hat{L}F}(t,\cdot)\right)\right\|_{L^p} \leq A_p^{(\sigma)} \left\|\eta^\sigma \widehat{S^{LF}}(t,\cdot)\right\|_{L^{p'}}$$

where $\frac{1}{p}+\frac{1}{p'} = 1$ and $A_p^{(\sigma)} \geq 0$.

Besides,

$$\left\|\widehat{d_x^\sigma S^{LF}}(t,\cdot)\right\|_{L^{p'}} = \left\|\chi(\cdot)\eta^\sigma e^{tA(\eta)}\right\|_{L^{p'}}$$

.

The diagonalization property gives:

$$A(\eta) = P(\eta)^{-1} \begin{bmatrix} \lambda^{+}(\eta) & 0 \\ 0 & \lambda^{-}(\eta) \end{bmatrix} P(\eta)$$

where $P : \mathbb{R} \longrightarrow GL_2(\mathbb{C})$

hence:

$$e^{tA(\eta)} = P(\eta)^{-1} e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} P(\eta)$$

Let $t \geq 1$:

$$\begin{aligned} \left\| \widehat{d_x^{\sigma} S^{LF}}(t, \cdot) \right\|_{L^{p'}} &= \left\| \chi \eta^{\sigma} P^{-1} e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} P \right\|_{L^{p'}} \\ \left\| \widehat{d_x^{\sigma} S^{LF}}(t, \cdot) \right\|_{L^{p'}} &= \left\| \sqrt[3]{\chi} P^{-1} \sqrt[3]{\chi} \eta^{\sigma} e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} \sqrt[3]{\chi} P \right\|_{L^{p'}} \end{aligned}$$

By using the Hölder's inequality, we obtain:

$$\left\| \widehat{d_x^{\sigma} S^{LF}}(t, \cdot) \right\|_{L^{p'}} \leq \left\| \sqrt[3]{\chi} P^{-1} \right\|_{L^{\infty}} \left\| \sqrt[3]{\chi} P \right\|_{L^{\infty}} \left\| \sqrt[3]{\chi} \eta^{\sigma} e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} \right\|_{L^{p'}}$$

If we note:

$$C_* = \left\| \sqrt[3]{\chi} P^{-1} \right\|_{L^{\infty}} \left\| \sqrt[3]{\chi} P \right\|_{L^{\infty}}$$

we obtain this inequality:

$$\left\| \widehat{d_x^{\sigma} S^{LF}}(t, \cdot) \right\|_{L^{p'}} \leq C_* \left\| \sqrt[3]{\chi} \eta^{\sigma} e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} \right\|_{L^{p'}}$$

but:

$$e^{tdiag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)} = diag\left(\lambda^{+}(\eta), \lambda^{-}(\eta)\right)$$

and

$$\lambda^{\pm}(\eta) \underset{|\lambda| \longrightarrow 0}{=} -\frac{1}{2} \mu_{\|} \eta^2 \pm ic\eta + \mathcal{O}\left(\eta^3\right)$$

Therefore:

$$\left\| \sqrt[3]{\chi} \eta^{\sigma} e^{tdiag\left(\lambda^{+}(\cdot), \lambda^{-}(\cdot)\right)} \right\|_{L^{p'}}^{p'} \leq \widetilde{A_{p'}^{(\sigma)}} \int_{-\infty}^{+\infty} |\eta|^{p'\sigma} e^{-\frac{p'}{2} \mu_{\|} \eta^2 t} \mathrm{d}\eta \quad [\star]$$

In the integral $[\star]$, we make the variable substitution:

$$\eta = \left(\frac{2}{p' \mu_{\|} t} \right)^{\frac{1}{2}} \nu$$

We have:

$$|\eta|^{p'\sigma}e^{-\frac{p'\mu_{\|}t}{2}\eta^2}\mathrm{d}\eta = \left(\frac{2}{p'\mu_{\|}t}\right)^{\frac{1}{2}(p'\sigma+1)}|\nu|^{\sigma}e^{-\nu^2}\mathrm{d}\nu$$

Therefore:

$$\begin{aligned}\int_{-\infty}^{+\infty}|\eta|^{p'\sigma}e^{-\frac{p'\mu_{\|}t}{2}\eta^2}\mathrm{d}\eta &= \left(\frac{2}{p'\mu_{\|}t}\right)^{\frac{1}{2}(p'\sigma+1)}\int_{-\infty}^{+\infty}|\nu|^{p'\sigma}e^{-\nu^2}\mathrm{d}\nu \\ \int_{-\infty}^{+\infty}|\eta|^{p'\sigma}e^{-\frac{p'\mu_{\|}t}{2}\eta^2}\mathrm{d}\eta &= C_{p'}^{(\sigma)}t^{-\frac{1}{2}(p'\sigma+1)}\end{aligned}$$

where:

$$C_{p'}^{(\sigma)} = \left(\frac{2}{p'\mu_{\|}}\right)^{\frac{1}{2}(p'\sigma+1)}\int_{-\infty}^{+\infty}|\nu|^{p'\sigma}e^{-\nu^2}\mathrm{d}\nu$$

But we have $\frac{1}{p}+\frac{1}{p'}=1$ (and, as a result, $p'=\frac{p}{p-1}$), and, if we note:

$$\widetilde{C_p^{(\sigma)}} = C_{\frac{p}{p-1}}^{(\sigma)}{}^{\frac{p-1}{p}}$$

We have:

$$\forall t \geq 1;\;\; \left\|d_x^{\sigma}S^{LF}(t,\cdot)\right\|_{L^p} \leq A_p^{(\sigma)}C_*A_{p'}^{(\sigma)}{}^{\frac{1}{p'}}\,\widetilde{C_p^{(\sigma)}}\cdot t^{-\frac{1}{2}\left(1-\frac{1}{p}+\sigma\right)}$$

Let $t\in[0;1]$:

$$\exists \widetilde{C_p^{(\sigma)}}' \geq 0 :\;\; \left\|d_x^{\sigma}S^{LF}(t,\cdot)\right\|_{L^p} \leq \widetilde{C_p^{(\sigma)}}'$$

where the constant $\widetilde{C_p^{(\sigma)}}'$ is independant of t.

If we note:

$$C_p^{(\sigma)} = Max\left\{A_p^{(\sigma)}C_*A_{p'}^{(\sigma)}{}^{\frac{1}{p'}}\,\widetilde{C_p^{(\sigma)}}\;;\;\widetilde{C_p^{(\sigma)}}'\right\}$$

we have the inequality of the proposition.

□

1.3.2 High frequency terms

In this subsection, we will study high frequency terms.

Some formulas and properties

Recall.

$$\forall\,(t,\eta)\in\mathbb{R}_+\times\mathbb{R};$$

$$\widehat{S}\,(t,\eta)=\begin{bmatrix}\frac{\lambda^+(\eta)e^{\lambda^-(\eta)t}-\lambda^-(\eta)e^{\lambda^+(\eta)t}}{\lambda^+(\eta)-\lambda^-(\eta)} & -i\eta\left(\frac{e^{\lambda^+(\eta)t}-e^{\lambda^-(\eta)t}}{\lambda^+(\eta)-\lambda^-(\eta)}\right)\\ -ic^2\eta\left(\frac{e^{\lambda^+(\eta)t}-e^{\lambda^-(\eta)t}}{\lambda^+(\eta)-\lambda^-(\eta)}\right) & \frac{\lambda^+(\eta)e^{\lambda^+(\eta)t}-\lambda^-(\eta)e^{\lambda^-(\eta)t}}{\lambda^+(\eta)-\lambda^-(\eta)}\end{bmatrix}$$

where $\lambda^{\pm}(\eta)=-\frac{1}{2}\mu_{\|}\eta^2\pm\frac{1}{2}\sqrt{\mu_{\|}^2\eta^4-4c^2\eta^2}$

Definition 1. *We introduce these notations:*

Let $r>0$ *and* $t\geq 0$*:*

$$\begin{aligned}A(t,r)&=\frac{1}{2i\pi}\int_{S^+\sqcup S^-}\frac{e^{tz}}{p(r,z)}\mathrm{d}z\\ B(t,r)&=\partial_t A(t,r)+\mu_{\|}r^2A(t,r)\\ D(t,r)&=e^{-\mu_{\|}r^2t}\int_0^t e^{\mu_{\|}r^2s}A(s,r)\mathrm{d}s\end{aligned}$$

where:

$$\begin{aligned}S^+&=\partial\mathbb{D}\left(-\frac{c^2}{\mu_{\|}};\frac{c^2}{2\mu_{\|}}\right)\\ S^-&=\partial\mathbb{D}\left(-\mu_{\|}r^2+\frac{c^2}{\mu_{\|}};\frac{c^2}{2\mu_{\|}}\right)\\ p(r,z)&=z^2+\mu_{\|}r^2z+c^2r^2\end{aligned}$$

Proposition 5.

$$\forall\,(t,\eta)\in\mathbb{R}_+\times\mathbb{R};$$

$$\widehat{S}\,(t,\eta)=\begin{bmatrix}B(t,|\eta|) & -iA(t,|\eta|)\eta\\ -ic^2A(t,|\eta|)\eta & e^{-\mu_{\|}\eta^2t}-c^2\eta^2D(t,|\eta|)\end{bmatrix}$$

Proof. First, as $p(|\eta|,z)=(z-\lambda^+(\eta))\,(z-\lambda^-(\eta))$, we have:

$$\begin{aligned}A(t,|\eta|)&=\frac{1}{2i\pi}\int_{S^+}\frac{e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z\\ &+\frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z\end{aligned}$$

but we have these asyptotic expansions:

$$\begin{aligned}
\lambda^{+}(\eta) &\underset{|\eta|\to+\infty}{=} -\frac{c^2}{\mu_{\|}} + \mathcal{O}\left(\frac{1}{|\eta|^2}\right) \\
\lambda^{-}(\eta) &\underset{|\eta|\to+\infty}{=} -\mu_{\|}|\eta|^2 + \frac{c^2}{\mu_{\|}} + \mathcal{O}\left(\frac{1}{|\eta|^2}\right)
\end{aligned}$$

Therefore, $A(t,|\eta|)$ is an Cauchy's integral formula for $\lambda^{\pm}(\eta)$. As a result, we obtain:

$$A(t,|\eta|) = \frac{e^{\lambda^{+}(\eta)t} - e^{\lambda^{-}(\eta)t}}{\lambda^{+}(\eta) - \lambda^{-}(\eta)}$$

Besides, as $\mu_{\|}\eta^2 = -\left(\lambda^{+}(\eta) + \lambda^{-}(\eta)\right)$ we have:

$$\begin{aligned}
B(t,|\eta|) &= \partial_t A(t,|\eta|) + \mu_{\|}\eta^2 A(t,|\eta|) \\
B(t,|\eta|) &= \frac{\lambda^{+}(\eta)\, e^{\lambda^{+}(\eta)t} - \lambda^{-}(\eta)\, e^{\lambda^{-}(\eta)t}}{\lambda^{+}(\eta) - \lambda^{-}(\eta)} \\
&- \left(\lambda^{+}(\eta) + \lambda^{-}(\eta)\right) \frac{e^{\lambda^{+}(\eta)t} - e^{\lambda^{-}(\eta)t}}{\lambda^{+}(\eta) - \lambda^{-}(\eta)} \\
B(t,|\eta|) &= \frac{\lambda^{+}(\eta)\, e^{\lambda^{-}(\eta)t} - \lambda^{-}(\eta)\, e^{\lambda^{+}(\eta)t}}{\lambda^{+}(\eta) - \lambda^{-}(\eta)}
\end{aligned}$$

As a result, we have:

$$\begin{aligned}
\widehat{S^{1,1}}(t,\eta) &= B(t,|\eta|) \\
\widehat{S^{2,1}}(t,\eta) &= -ic^2 A(t,|\eta|)\eta \\
\widehat{S^{1,2}}(t,\eta) &= -iA(t,|\eta|)\eta
\end{aligned}$$

We will stuty $\widehat{S^{2,2}}(t,\eta)$ by using the mapping D. We konw that

$$\begin{aligned}
\partial_t \widehat{S}(t,\eta) &= A(\eta) \\
\text{with } A(\eta) &= \begin{bmatrix} 0 & -i\eta \\ -ic^2\eta & -\mu_{\|}\eta^2 \end{bmatrix}
\end{aligned}$$

by identifying the coefficients, we obtain:

$$\partial_t \widehat{S^{2,2}}(t,\eta) = -i\eta c^2 \widehat{S^{1,2}}(t,\eta) - \mu_{\|}\widehat{S^{2,2}}(t,\eta)$$

Therefore, the mapping $t \longmapsto \widehat{S^{2,2}}(t,\eta)$ is the solution of this ordinary differential equation:

$$\left\{\begin{aligned}
\partial_t \widehat{S^{2,2}}(t,\eta) &= -c^2\eta^2 A(t,|\eta|) - \mu_{\|}\eta^2 \widehat{S^{2,2}}(t,\eta) \\
\widehat{S^{2,2}}(0,\eta) &= 1(\text{ for } \widehat{S}(0,\eta) = I_2)
\end{aligned}\right.$$

By solving this differential equation, we obtain:

$$\begin{aligned}
\widehat{S^{2,2}}(t,\eta) &= e^{-\mu_{\|}\eta^2 t} - c^2\eta^2 \int_0^t e^{-\mu_{\|}\eta(t-s)} A(s,|\eta|)\mathrm{d}s \\
\widehat{S^{2,2}}(t,\eta) &= e^{-\mu_{\|}\eta^2 t} - c^2\eta^2 e^{\mu_{\|}\eta^2 t} \int_0^t e^{-\mu_{\|}\eta^2 s} A(s,|\eta|)\mathrm{d}s \\
\widehat{S^{2,2}}(t,\eta) &= e^{-\mu_{\|}\eta^2 t} - c^2\eta^2 D(t,|\eta|)
\end{aligned}$$

□

Estimation of A,B and D

Proposition 6. *We have:*

$$\begin{aligned} A(t,r) &= \sum_{k=0}^{+\infty} A_k\left(t,r\right) r^{-2k-2} \\ B(t,r) &= e^{-\frac{c^2 t}{\mu_\parallel}} + \sum_{k=0}^{+\infty} B_k\left(t,r\right) r^{-2k-2} \\ D(t,r) &= \sum_{k=0}^{+\infty} D_k\left(t,r\right) r^{-2k-4} \end{aligned}$$

and $\forall k \in \mathbb{N}$;

$$\left|A_k\left(t,r\right)\right|, \left|B_k\left(t,r\right)\right|, \left|D_k\left(t,r\right)\right| \leq C\left[e^{-\frac{c^2 t}{2\mu_\parallel}} + e^{-\frac{\mu_\parallel r^2 t}{2}}\right] r_0^k$$

where C *and* r_0 *are two constants which not depend on* k*,*r *and* t*.*

Proof. Let $r = |\eta| >> 1$

$$\begin{aligned} A(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{e^{tz}\mathrm{d}z}{p(r,z)} + \frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}\mathrm{d}z}{p(r,z)} \\ A(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2 r^2} + \frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2 r^2} \\ A(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{e^{tz}\mathrm{d}z}{z^2+r^2[\mu_\parallel z + c^2]} + \frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2 r^2} \\ A(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{e^{tz}r^{-2}[\mu_\parallel z + c^2]^{-1}\mathrm{d}z}{\left(\frac{z}{r}\right)^2[\mu_\parallel z + c^2]^{-1}+1} + \frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2 r^2} \\ A(t,r) &\overset{|z|<<r}{=} \frac{1}{2i\pi}\int_{S^+} e^{tz}r^{-2}[\mu_\parallel z + c^2]^{-1}\sum_{k=0}^{+\infty}(-1)^k[\mu_\parallel z + c^2]^{-k}z^{2k}r^{-2k}\mathrm{d}z \\ &+ \frac{1}{2i\pi}\int_{S^-}\frac{e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2 r^2} \\ A(t,r) &= \sum_{k=0}^{+\infty} A_k\left(t,r\right) r^{-2k-2} \end{aligned}$$

with

$$A_0\left(t,r\right) = \frac{1}{2i\pi}\int_{S^+} e^{tz}[\mu_\parallel z + c^2]^{-1}\mathrm{d}z + \frac{1}{2i\pi}\int_{S^-}\frac{r^2 e^{tz}\mathrm{d}z}{z[z+\mu_\parallel r^2]+c^2 r^2}$$

and, $\forall k \in \mathbb{N}^*$;

$$A_k\left(t,r\right) = \frac{(-1)^k}{2i\pi}\int_{S^+} e^{tz}[\mu_\parallel z + c^2]^{-(k+1)} z^{2k}\mathrm{d}z$$

As a result, by using a property of line integrals (**cf. Chapter 3**), we can make these estimations:

$$
\begin{aligned}
|A_0(t,r)| &\leq C_0 e^{-\frac{c^2}{\mu_\parallel}t} + C_0^- r^2 e^{-\frac{\mu_\parallel r^2}{2}t} \underbrace{\underset{z\in S^-}{Sup}\left|\frac{1}{z[z+\mu_\parallel r^2]+c^2r^2}\right|}_{\leq r_0^- r^{-2}} \\
|A_0(t,r)| &\leq \left(C_0 + C_0^- r_0^-\right)\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]
\end{aligned}
$$

and, $\forall k \in \mathbb{N}^*$:

$$
\begin{aligned}
|A_k(t,r)| &\leq C_0^+ \underbrace{\underset{z\in S^+}{Sup}\left|\frac{z^2}{\mu_\parallel z + c^2}\right|}_{\leq r_0^+}{}^k e^{-\frac{c^2}{2\mu_\parallel}t} \\
|A_k(t,r)| &\leq C_0^+ {r_0^+}^k e^{-\frac{c^2}{2\mu_\parallel}t} \\
|A_k(t,r)| &\leq C_0^+ {r_0^+}^k e^{-\frac{c^2}{2\mu_\parallel}t}
\end{aligned}
$$

As a result, $\forall k \in \mathbb{N}$;

$$
|A_k(t,r)| \leq C^{(A)}\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right] {r_0^{(A)}}^k
$$

where:

$$
\begin{aligned}
C^{(A)} &= Max\left\{C_0 + C_0^- r_0^- \ ; \ C_0^+\right\} \\
r_0^{(A)} &= Max\left\{r_0^+ \ ; \ 1\right\}
\end{aligned}
$$

Besides, we will study the mapping B:

$$
\begin{aligned}
B(t,r) &= \partial_t A(t,r) + \mu_\parallel r^2 A(t,r) \\
B(t,r) &= \frac{1}{2i\pi}\int_{S^+\sqcup S^-}\frac{ze^{tz}}{p(r,z)}\mathrm{d}z + \frac{\mu_\parallel r^2}{2i\pi}\int_{S^+\sqcup S^-}\frac{e^{tz}}{p(r,z)}\mathrm{d}z \\
B(t,r) &= \frac{1}{2i\pi}\int_{S^+\sqcup S^-}\frac{ze^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2r^2} + \frac{1}{2i\pi}\int_{S^+\sqcup S^-}\frac{\mu_\parallel r^2 e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2r^2} \\
B(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{(z+\mu_\parallel r^2)e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2r^2} + \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\parallel r^2)e^{tz}\mathrm{d}z}{z^2+\mu_\parallel r^2 z + c^2r^2}
\end{aligned}
$$

$$
\begin{aligned}
B(t,r) &= \frac{1}{2i\pi}\int_{S^+}\frac{r^2[\mu_\| z+c^2]^{-1}(z+\mu_\| r^2)e^{tz}}{\left(\frac{z}{r}\right)^2[\mu_\| z+c^2]^{-1}+1}\mathrm{d}z \\
&+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z^2+\mu_\| r^2 z+c^2r^2} \\
B(t,r) &= \frac{1}{2i\pi}\int_{S^+} r^{-2}(\mu_\| z+c^2)^{-1}(z+\mu_\| r^2)e^{tz}\sum_{k=0}^{+\infty}(-1)^k[\mu_\| z+c^2]^{-k}\left(\frac{z}{r}\right)^{2k}\mathrm{d}z \\
&+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z^2+\mu_\| r^2 z+c^2r^2} \\
B(t,r) &= \sum_{k=0}^{+\infty}\left[\frac{(-1)^k}{2i\pi}\int_{S^+}[\mu_\| z+c^2]^{-(k+1)}[z+\mu_\| r^2]z^{2k}e^{tz}\mathrm{d}z\right]r^{-2k-2} \\
&+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z^2+\mu_\| r^2 z+c^2r^2}
\end{aligned}
$$

We study the term $(k=0)$ of the first sum, and we have:

$$
\frac{1}{2i\pi}\int_{S^+}[\mu_\| z+c^2]^{-1}[z+\mu_\| r^2]e^{tz}\mathrm{d}z r^{-2} = \left[1-\left(\frac{c}{\mu_\|}\right)^2\right]e^{-\frac{c^2}{\mu_\|}t} \tag{1.9}
$$

Therefore, we obtain:

$$
\begin{aligned}
B(t,r) &= e^{-\frac{c^2}{\mu_\|}t}-\left(\frac{c}{\mu_\|}\right)^2 e^{-\frac{c^2}{\mu_\|}t} \\
&+ \sum_{k=1}^{+\infty}\left[\frac{(-1)^k}{2i\pi}\int_{S^+}[\mu_\| z+c^2]^{-(k+1)}[z+\mu_\| r^2]z^{2k}e^{tz}\mathrm{d}z\right]r^{-2k-2} \\
&+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z[z+\mu_\| r^2]+c^2r^2}
\end{aligned}
$$

$$\begin{aligned} B(t,r) &= e^{-\frac{c^2}{\mu_\|}t} - \left(\frac{c}{\mu_\| r}\right)^2 e^{-\frac{c^2}{\mu_\|}t} \\ &+ \sum_{k=0}^{+\infty}\left[\frac{(-1)^{k+1}}{2i\pi}\int_{S^+}\frac{z+\mu_\| r^2}{[\mu_\| z + c^2]^{k+2}} z^{2(k+1)} e^{tz}\mathrm{d}z\right] r^{-2k-4} \\ &+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z[z+\mu_\| r^2]+c^2r^2} \end{aligned} \tag{1.10}$$

Remarks. .

- *The computation of the line* (1.9)*is made by using the Cauchy's integral formula.*

- *An index change* $k \longmapsto k-1$ *is made on this sum, giving the term at the line* (1.10)

Finally, we obtain:

$$B(t,r) = e^{-\frac{c^2}{\mu_\|}t} + \sum_{k=0}^{+\infty} B_k\,(t,r)\, r^{-2k-2}$$

with

$$\begin{aligned} B_0\,(t,r) &= -\left(\frac{c}{\mu_\|}\right)^2 e^{-\frac{c^2}{\mu_\|}t} - \frac{1}{2i\pi}\int_{S^+}[\mu_\| z + c^2]^{-2} z^2 r^{-2}[z+\mu_\| r^2]e^{tz}\mathrm{d}z \\ &+ \frac{1}{2i\pi}\int_{S^-}\frac{(z+\mu_\| r^2)e^{tz}\mathrm{d}z}{z[z+\mu_\| r^2]+c^2r^2} r^2 \end{aligned}$$

and $\forall k \in \mathbb{N}^*$;

$$B_k\,(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}[\mu_\| z + c^2]^{-(k+2)} r^{-2}[z+\mu_\| r^2] z^{2k+2} e^{tz}\mathrm{d}z$$

We will again makes estimations, over B:

$$
\begin{aligned}
|B_0(t,r)| &\leq \left|\frac{c}{\mu_\parallel}\right|^2 e^{-\frac{c^2}{\mu_\parallel}t} + \underline{\alpha_0} \underbrace{\underset{z\in S^+}{Sup}\left|\frac{z}{\mu_\parallel z + c^2}\right|^2}_{\leq \underline{\alpha_0}'} \underbrace{\left|z+\mu_\parallel r^2\right|}_{\leq \underline{\alpha_0}'' r^2} \frac{1}{r^2} e^{-\frac{c^2}{2\mu_\parallel}t} \\
&+ \underline{C_0}^- r^2 e^{-\frac{\mu_\parallel r^2}{2}t} \underbrace{\underset{z\in S^-}{Sup}\left|\frac{z+\mu_\parallel r^2}{z[z+\mu_\parallel r^2]+c^2r^2}\right|}_{\leq \underline{r_0}^- r^{-2}} \\
|B_0(t,r)| &\leq \left[\left|\frac{c}{\mu_\parallel}\right|^2 + \underline{\alpha_0 \alpha_0}' \underline{\alpha_0}'' + \underline{C_0}^- \underline{r_0}^-\right]\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]
\end{aligned}
$$

Besides, for $k \in \mathbb{N}^*$, we obtain:

$$
\begin{aligned}
|B_k(t,r)| &\leq \underline{C_0}^+ \underbrace{\underset{z\in S^+}{Sup}\left|\frac{z}{\mu_\parallel z + c^2}\right|^2}_{\leq \underline{\alpha_0}^+} \cdot \underbrace{\underset{z\in S^+}{Sup}\left|\frac{z^2}{\mu_\parallel z + c^2}\right|^k}_{\leq \underline{r_0}^+} \frac{1}{r^2} \underbrace{\underset{z\in S^+}{Sup}\left|z+\mu_\parallel r^2\right|}_{\leq \underline{r_0}' r^2} e^{-\frac{c^2}{2\mu_\parallel}t} \\
|B_k(t,r)| &\leq \underline{C_0}^+ \underline{\alpha_0}^+ \underline{r_0}' \underline{r_0}^{+^k} e^{-\frac{c^2}{2\mu_\parallel}t}
\end{aligned}
$$

Finally:

$$
\forall k \in \mathbb{N}; \quad |B_k(t,r)| \leq C_0^{(B)} \left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right] r_0^{(B)^k}
$$

where:

$$
\begin{aligned}
C_0^{(B)} &= Max\left\{\underline{C_0}^+ \underline{\alpha_0}^+ \underline{r_0}' \; ; \; \left|\frac{c}{\mu_\parallel}\right|^2 + \underline{\alpha_0 \alpha_0}' \underline{\alpha_0}'' + \underline{C_0}^- \underline{r_0}^-\right\} \\
r_0^{(B)} &= Max\left\{\underline{r_0}^+ \; ; \; 1\right\}
\end{aligned}
$$

Finally, we will study the mapping D:

$$
\begin{aligned}
D(t,r) &= e^{-\mu_\parallel r^2 t}\int_0^t e^{\mu_\parallel r^2 s} A\left(s,r\right)\mathrm{d}s\\
D(t,r) &= e^{-\mu_\parallel r^2 t}\int_0^t e^{\mu_\parallel r^2 s}\sum_{k=0}^{+\infty} A_k\left(s,r\right) r^{-2k-2}\mathrm{d}s\\
D(t,r) &= \sum_{k=0}^{+\infty}\left[e^{-\mu_\parallel r^2 t} r^2\int_0^t e^{\mu_\parallel r^2 s} A_k\left(s,r\right)\mathrm{d}s\right] r^{-2k-4}\\
D(t,r) &= \sum_{k=0}^{+\infty} D_k\left(t,r\right) r^{-2k-4}
\end{aligned}
$$

with $\forall k\in\mathbb{N}$;

$$
D_k\left(t,r\right) = e^{-\mu_\parallel r^2 t} r^2\int_0^t e^{\mu_\parallel r^2 s} A_k\left(s,r\right)\mathrm{d}s
$$

We will estimate the terms D_k:

$$
\begin{aligned}
\left|D_k\left(t,r\right)\right| &= e^{-\mu_\parallel r^2 t} r^2\left|\int_0^t e^{\mu_\parallel r^2 s} A_k\left(s,r\right)\mathrm{d}s\right|\\
\left|D_k\left(t,r\right)\right| &\leq e^{-\mu_\parallel r^2 t} r^2\int_0^t e^{\mu_\parallel r^2 s}\left|A_k\left(s,r\right)\right|\mathrm{d}s\\
\left|D_k\left(t,r\right)\right| &\leq e^{-\mu_\parallel r^2 t} r^2 C_0^{(A)} r_0^{(A)^k}\int_0^t e^{\mu_\parallel r^2 s-\frac{c^2}{2\mu_\parallel}s}+e^{\mu_\parallel r^2 s-\frac{\mu_\parallel r^2}{2}s}\mathrm{d}s\\
\left|D_k\left(t,r\right)\right| &\leq C_0^{(A)} e^{-\mu_\parallel r^2 t} r^2 r_0^{(A)^k}\int_0^t e^{\left(\mu_\parallel r^2-\frac{c^2}{2\mu_\parallel}\right)s}+e^{\frac{c^2}{2\mu_\parallel}s}\mathrm{d}s\\
\left|D_k\left(t,r\right)\right| &\leq C_0^{(A)} e^{-\mu_\parallel r^2 t} r^2 r_0^{(A)^k}\left[\frac{e^{\left(\mu_\parallel r^2-\frac{c^2}{2\mu_\parallel}\right)s-1}}{\mu_\parallel r^2-\frac{c^2}{2\mu_\parallel}}+\frac{e^{\frac{\mu_\parallel r^2}{2}t-1}}{\frac{\mu_\parallel r^2}{2}}\right]\\
\left|D_k\left(t,r\right)\right| &\overset{\mu_\parallel r^2-\frac{c^2}{2\mu_\parallel}\leq \underline{\underline{r_0}} r^{-2}}{\leq} C_0^{(A)} r^2 r_0^{(A)^k}\left[\underline{\underline{r_0}} r^{-2}\left(e^{-\frac{c^2}{2\mu_\parallel}t}-e^{-\mu_\parallel r^2 t}\right)\right.\\
&\quad\left.+\frac{2}{\mu_\parallel r^2}\left(e^{-\frac{\mu_\parallel r^2}{2}t}-e^{-\mu_\parallel r^2 t}\right)\right]\\
\left|D_k\left(t,r\right)\right| &\leq C_0^{(A)} r^2 r_0^{(A)^k}\left(\underline{\underline{r_0}}+\frac{2}{\mu_\parallel}\right) r^{-2}\left[e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel r^2}{2}t}\right]\\
\left|D_k\left(t,r\right)\right| &\leq C_0^{(D)}\left[e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel r^2}{2}t}\right] r_0^{(D)^k}
\end{aligned}
$$

with:

$$\begin{aligned} C_0^{(D)} &= C_0^{(A)}\left(\underline{\underline{r_0}} + \frac{2}{\mu_\parallel}\right) \\ r_0^{(d)} &= r_0^{(A)} \end{aligned}$$

As a result, we obtain these estimations:

$$\forall k \in \mathbb{N};\ \ |A_k\left(t,r\right)|, |B_k\left(t,r\right)|, |D_k\left(t,r\right)| \ \leq\ C\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right] r_0^k$$

where:

$$\begin{aligned} C &= Max\left\{C_0^{(A)}\ ;\ C_0^{(B)}\ ;\ C_0^{(D)}\right\} \\ r_0 &= Max\left\{r_0^{(A)}\ ;\ r_0^{(B)}\ ;\ r_0^{(D)}\right\} \end{aligned}$$

C and r_0 are two constants, which don't depend on k, r and t. □

Remark. *The estmations allow switch series and integrals*

By summation, we obtain:

$$\begin{aligned} |A(t,r)| &\leq \sum_{k=0}^{+\infty} |A_k\left(t,r\right)| r^{-2k-2} \\ &\leq C\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]\frac{r^{-2}}{1-\frac{r_0}{r^2}} \\ |B(t,r)| &\leq e^{-\frac{c^2}{\mu_\parallel}t} + \sum_{k=0}^{+\infty} |B_k\left(t,r\right)| r^{-2k-2} \\ &\leq e^{-\frac{c^2}{\mu_\parallel}t} + C\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]\frac{r^{-2}}{1-\frac{r_0}{r^2}} \\ |D(t,r)| &\leq \sum_{k=0}^{+\infty} |D_k\left(t,r\right)| r^{-2k-4} \\ &\leq C\left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]\frac{r^{-4}}{1-\frac{r_0}{r^2}} \end{aligned}$$

Estimation of the derivatives of A, B and D

Proposition 7. *We have,* $\forall j \in \mathbb{N}$;

$$\begin{aligned}
\left|\partial_r^j A(t,r)\right| &\leq C_j \left[e^{-\frac{c^2}{4\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{4}t}\right] r^{-2-j} \\
\left|\partial_r^j B(t,r)\right| &\leq C_j \left[e^{-\frac{c^2}{4\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{4}t}\right] r^{-2-j} \\
\left|\partial_r^j B(t,r)\right| &\leq C_j \left[e^{-\frac{c^2}{4\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{4}t}\right] r^{-4-j}
\end{aligned}$$

Proof. We will make an estimation of $\partial_r^j A(t,r)$, $\partial_r^j B(t,r)$ and $\partial_r^j D(t,r)$. We assume we can switch derivation and series (wich is correct, dur to the estimation).

We have:

$$\begin{aligned}
A(t,r) &= \sum_{k=0}^{+\infty} A_k(t,r)\, r^{-2k-2} \\
\partial_r^j A(t,r) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j} \binom{j}{l} \partial_r^l A_k(t,r)\, r^{-2k-2-(j-l)}\alpha_{l,j} \\
\partial_r^j A(t,r) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j} \binom{j}{l} \alpha_{l,j}\partial_r^l A_k(t,r)\, r^l r^{-2k-2-j} \\
A_k(t,r) &= \frac{(-1)^k}{2i\pi}\int_{S^+} e^{tz}[\mu_\parallel z + c^2]^{-(k+1)} z^{2k}\mathrm{d}z \\
&\quad \left(+\frac{1}{2i\pi}\int_{S^-}\frac{r^2 e^{tz}\mathrm{d}z}{z^2+[\mu_\parallel z + c^2]r^2} \text{ if } k=0\right)
\end{aligned}$$

Let $d(r) = -\mu_\parallel r^2 + \frac{c^2}{\mu_\parallel}$. If $l > 0$, as we integrate over a compact, we can switch derivation and integration:

$$\begin{aligned}
\partial_r^l A_k(t,r) &= \frac{(-c^2)^k}{2i\pi}\partial_r^l \int_0^{2\pi} e^{t\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]}\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{-(k+1)} \\
&\quad \underbrace{\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}+\mu_\parallel r^2\right]}_{=\frac{c^2}{\mu_\parallel}+\frac{c^2}{2\mu_\parallel}e^{i\theta}}^{-(k+1)} r^{4k+2} i\frac{c^2}{2\mu_\parallel}e^{i\theta}\mathrm{d}\theta
\end{aligned}$$

By using the general Leibniz rule (**cf. Chapter 3**), we obtain:

$$\begin{array}{rcl}\partial_r^l A_k(t,r) & = & \dfrac{(-c^2)^k}{2i\pi}\displaystyle\int_0^{2\pi}\sum_{k_1+k_2+k_3=l}\frac{l!}{k_1!k_2!k_3!}\partial_r^{k_1}\left[e^{t\left(d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right)}\right] \\ & \cdot & \partial_r^{k_2}\left[\left(d(r)+\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\right)^{-(k+1)}\right]\partial_r^{k_3}\left[r^{4k+2}\right] \\ & \ddots & \left[\dfrac{c^2}{\mu_\parallel}+\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{-(k+1)} i\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\mathrm{d}\theta \\ \partial_r^l A_k(t,r) & = & \dfrac{(-c^2)^k}{2i\pi}\displaystyle\int_0^{2\pi}\sum_{k_1+k_2+k_3=l}\frac{l!}{k_1!k_2!k_3!}P_{k_1}(r)e^{t\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]} \\ & \cdot & \dfrac{Q_{k_2}(r)R_{k_3}(r)r^{4k+2-k_3}}{\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{1+k+k_2}}\left[\dfrac{c^2}{\mu_\parallel}+\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{-(k+1)} i\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\mathrm{d}\theta\end{array}$$

where P_{k_1}, Q_{k_2} and R_{k_3} are three polynomials and:

$$\begin{array}{rcl} deg\left(P_{k_1}\right) & = & k_1 \\ deg\left(Q_{k_2}\right) & = & k_2 \\ deg\left(R_{k_3}\right) & = & k_3 \end{array}$$

$$\begin{array}{rcl}\partial_r^l A_k(t,r) & = & \dfrac{(-c^2)^k}{2i\pi}\displaystyle\int_0^{2\pi}\sum_{k_1+k_2+k_3=l}\frac{l!}{k_1!k_2!k_3!}T_{k_1,k_2,k_3}(r)e^{t\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]} \\ & \cdot & \underbrace{\left[d(r)+\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{-(1+k_1+k_2)}}_{[1]}\underbrace{r^{4k+2-2k_3}}_{[2]}\left[\dfrac{c^2}{\mu_\parallel}+\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\right]^{-(k+1)} \\ & \cdot & i\dfrac{c^2}{2\mu_\parallel}e^{i\theta}\mathrm{d}\theta\end{array}$$

where T_{k_1,k_2,k_3} is a polynomial, and:

$$deg\left(T_{k_1,k_2,k_3}\right) \leq l = k_1+k_2+k_3$$

as

$$e^{t\left[d(r)+\frac{c^2}{2\mu_\parallel}e^{i\theta}\right]} \underset{r\longrightarrow+\infty}{=} \mathcal{O}\left(\underbrace{r^{-2k-2k_1}}_{[3]}e^{-\frac{\mu_\parallel r^2}{2}t}\right)$$

Remark. *A similar computation shows that*

$$\left|\frac{1}{2i\pi}\int_{S^-}\frac{r^2e^{tz}\mathrm{d}z}{z^2+[\mu_\parallel z+c^2]r^2}\right| \leq C_l r_0 e^{-\frac{\mu_\parallel r^2}{2}t}r^{-l} \tag{1.11}$$

we obtain:

$$\begin{aligned}\left|\sum_{l=0}^{j}\binom{l}{j}\alpha_{l,j}\partial_r^l A_k(t,r)\,r^l\right| &\leq \sum_{l=0}^{j}\binom{l}{j}|\alpha_{l,j}|\left|\partial_r^l A_k(t,r)\right| r^l \\ &\leq \underbrace{\sum_{l=0}^{j}\binom{l}{j}|\alpha_{l,j}|\,C_l r_0^k}_{:=C_j^{(A)}}\left[e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel r^2}{2}t}\right] \\ &\times \underbrace{r^l r^{\overbrace{l}^{T_{k_1,k_2,k_3}}\overbrace{-2-2k-2k_2}^{[1]}\overbrace{+4k+2-2k_3}^{[2]}\overbrace{-2k-2k_1}^{[3]}}}_{=1}\end{aligned}$$

Therefore, by summation, we have:

$$\left|\partial_r^j A(t,r)\right| \leq C_j^{(A)}\left[e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel r^2}{2}t}\right]\sum_{k=0}^{+\infty}\left(\frac{r_0}{r^2}\right)^k r^{-2-j}$$

Besides, we have:

$$B(t,r) = e^{-\frac{c^2}{\mu_\parallel}t}+\sum_{k=0}^{+\infty}B_k(t,r)\,r^{-2k-2}$$

Let $j \in \mathbb{N}^*$:

$$\begin{aligned}\partial_r^j B(t,r) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j}\binom{l}{j}\alpha'_{l,j}\partial_r^l B_k(t,r)\,r^{-(j-l)}r^{-2k-2} \\ &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j}\binom{l}{j}\alpha'_{l,j}\partial_r^l B_k(t,r)\,r^l r^{-2k-2-j}\end{aligned}$$

By a similar reasoning as A (**cf. Chapter 3**), we show that:

$$\left|\partial_r^l B_k(t,r)\right| \leq C_l' r_0'^k \left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right] r^{-l}$$

Therefore:

$$\left|\sum_{l=0}^{j}\binom{l}{j}\alpha_{l,j}\partial_r^l A_k(t,r)\, r^l\right| \leq \underbrace{\sum_{l=0}^{j}\binom{l}{j}\left|\alpha_{l,j}'\right| C_l'}_{:=C_j^{(B)}} r_0'^k \left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right]$$

As a result, a summation gives:

$$\left|\partial_r^j B(t,r)\right| \leq C_j^{(B)} \left[e^{-\frac{c^2}{2\mu_\parallel}t} + e^{-\frac{\mu_\parallel r^2}{2}t}\right] \sum_{k=0}^{+\infty}\left(\frac{r_0}{r^2}\right)^k r^{-2-j}$$

Finally, we have to estimate the derivatives of D.
Let $j \in \mathbb{N}$:

$$\begin{aligned}
\partial_r^j D(t,r) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j}\binom{j}{l}\alpha_{l,j}''\partial_r^l D_k(t,r)\, r^{-2k-4-j+l} \\
\partial_r^j D(t,r) &= \sum_{k=0}^{+\infty}\sum_{l=0}^{j}\binom{j}{l}\alpha_{l,j}''\partial_r^l D_k(t,r)\, r^l r^{-2k-4-j}
\end{aligned}$$

with, $\forall k \in \mathbb{N}$:

$$\begin{aligned}
D_k(t,r) &= e^{-\mu_\parallel r^2 t} r^2 \int_0^t e^{\mu_\parallel r^2 s} A_k(s,r)\,\mathrm{d}s \\
D_k(t,r) &= r^2 \int_0^t e^{\mu_\parallel r^2 (s-t)} A_k(s,r)\,\mathrm{d}s
\end{aligned}$$

By using the general Leibniz rule, we obtain:

$$\partial_r^l D_k(t,r) = \sum_{k_1+k_2+k_3=l}\frac{l!}{k_1!k_2!k_3!}\beta_{k_1} r^{2-k_1}\int_0^t P_{k_2}(r(s-t))\, e^{\mu_\parallel r^2(s-t)}\partial_r^{k_3} A_k(s,r)\,\mathrm{d}s$$

where:

$$\begin{aligned}
deg(P_{k_2}) &= k_2 \\
\beta_{k_1} &= 0 \text{ if } k_1 > 2
\end{aligned}$$

As a result, we have:

$$
\begin{aligned}
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq \sum_{k_1+k_2+k_3=l} \frac{l!}{k_1!k_2!k_3!}\beta_{k_1} r^{2-k_1} \int_0^t \left|P_{k_2}\left(r(s-t)\right)\right| e^{\mu_\| r^2(s-t)} \\
&\cdot \left|\partial_r^{k_3} A_k\left(s,r\right)\right| \mathrm{d}s \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq \sum_{k_1+k_2+k_3=l} \frac{l!}{k_1!k_2!k_3!}\beta_{k_1} r^{2-k_1} \int_0^t \gamma_{k_2} r^{k_2} \left|s-t\right|^{k_2} e^{\mu_\| r^2(s-t)} \\
&\cdot C_{k_3} r_0^k \left[e^{-\frac{c^2}{2\mu_\|}s} + e^{-\frac{\mu_\| r^2}{2}s}\right] r^{-l}\mathrm{d}s \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq \sum_{k_1+k_2+k_3=l} \frac{l!}{k_1!k_2!k_3!}\beta_{k_1} r^{2-k_1} \int_0^t \gamma'_{k_2} e^{\frac{\mu_\| r^2(s-t)}{2}} C_{k_3} r_0^k \\
&\cdot \left[e^{-\frac{c^2}{2\mu_\|}s} + e^{-\frac{\mu_\| r^2}{2}s}\right] r^{-l}\mathrm{d}s \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq e^{-\frac{-\mu_\| r^2}{2}t} \underbrace{\sum_{k_1+k_2+k_3=l} \frac{l!}{k_1!k_2!k_3!}\beta_{k_1}\gamma'_{k_2} C_{k_3} r^2}_{:=C''_l} \\
&\cdot \int_0^t e^{\frac{\mu_\| r^2}{2}t}\left[e^{-\frac{c^2}{4\mu_\|}s} + e^{-\frac{\mu_\| r^2}{4}s}\right]\mathrm{d}s r_0^k r^{-l} \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq C''_l r_0^k r^{2-l} e^{-\frac{\mu_\| r^2}{2}t} \int_0^t e^{\left[\frac{\mu_\| r^2}{2} - \frac{c^2}{4\mu_\|}\right]s} + e^{\frac{\mu_\| r^2}{4}s}\mathrm{d}s \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq C''_l r_0^k r^{2-l} e^{-\frac{\mu_\| r^2}{2}t} \left[\frac{e^{\left[\frac{\mu_\| r^2}{2} - \frac{c^2}{4\mu_\|}\right]t-1}}{\frac{\mu_\| r^2}{2} - \frac{c^2}{4\mu_\|}} + \frac{e^{\frac{\mu_\| r^2}{4}t}-1}{\frac{\mu_\| r^2}{4}}\right] \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq C''_l r_0^k r^{2-l} \left[\underline{\underline{r_0}} r^{-2}\left(e^{-\frac{c^2}{4\mu_\|}t} - e^{-\frac{\mu_\| r^2}{2}t}\right)\right. \\
&+ \left.\frac{4}{\mu_\| r^2}\left(e^{-\frac{\mu_\| r^2}{4}t} - e^{-\frac{\mu_\| r^2 t}{2}}\right)\right] \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq C''_l r_0^k r^{-l}\left(\underline{\underline{r_0}} + \frac{4}{\mu_\|}\right)\left[e^{-\frac{c^2}{4\mu_\|}t} + e^{-\frac{\mu_\| r^2}{4}t}\right]
\end{aligned}
$$

By making a summation, we obtain:

$$\begin{aligned}
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq \sum_{k=0}^{+\infty} \underbrace{\sum_{l=0}^{j} \binom{j}{j} \left|\alpha_{l,j}''\right| C_l'' \left(\underline{\underline{r_0}} + \frac{4}{\mu_\|}\right)}_{:=C_j^{(D)}} r_0^k r^{-l} \left[e^{-\frac{c^2}{4\mu_\|}t} - e^{-\frac{\mu_\| r^2}{4}t}\right] \\
&\quad . \quad r^l r^{-2k-4-j} \\
\left|\partial_r^l D_k\left(t,r\right)\right| &\leq C_j^{(D)} \left[e^{-\frac{c^2}{4\mu_\|}t} - e^{-\frac{\mu_\| r^2}{4}t}\right] \sum_{k=0}^{+\infty} \left(\frac{r_0}{r^2}\right)^k r^{-4-j}
\end{aligned}$$

Finally, we have the inequalities of the statement of the proposition if we note:

$$C_j = Max\left\{C_j^{(A)} \ ; \ C_j^{(B)} \ ; \ C_j^{(D)}\right\} \underset{r>r_0}{Sup} \left[\sum_{k=0}^{+\infty} \left(\frac{r_0}{r^2}\right)^k\right]$$

□

Now, we will proove the exponential decay (in t) for the high frequency terms.

Proposition 8. ***Estimation of High frequancy terms***

$$\exists C > 0; \forall p \in [1; +\infty]\, ; \forall X_0 \in L^p\left(\mathbb{R}\right);$$

$$\left\|S^{HF}\left(t,\cdot\right) * X_0\right\|_{L^p} \leq Ce^{-bt} \left\|X_0\right\|_{L^p}$$

where $p = \frac{1}{2} Min\left\{\frac{c^2}{4\mu_\|} \ ; \ \frac{\mu_\| R_0^2}{4}\right\}$ *and* $t > 0$.

Proof. We will apply the Marcikiewic multiplier theorem (**cf. Chapter 3**) to the functions $\eta \longmapsto e^{bt}\widehat{S^{HF}}\left(t,\eta\right)$. We have to verify the estimations of the theorem. We have:

$$\widehat{S^{HF}}\left(t,\eta\right) = \left[1-\chi(\eta)\right] \begin{bmatrix} B(t,|\eta|) & -iA(t,|\eta|)\eta \\ -ic^2A(t,|\eta|)\eta & e^{-\mu_\|\eta^2 t} - c^2\eta^2 D(t,|\eta|) \end{bmatrix}$$

As a result, we have $\widehat{S^{HF}}\left(t,\eta\right) \equiv 0$ when $|\eta| \leq R_0$. Let $\eta \in \mathbb{R}$ with $|\eta| \geq R_0$.
Besides, we have:

$$\begin{aligned}
\partial_\eta\left[e^{bt}\widehat{S^{HF}}\left(t,\eta\right)\right] &= -\chi(\eta)\widehat{S}\left(t,\eta\right)e^{bt} \\
&+ \left[1-\chi(\eta)\right] \begin{bmatrix} \partial_r B(t,|\eta|) & -iA(t,|\eta|) - i\partial_r A(t,|\eta|)\eta \\ -ic^2A(t,|\eta|) - ic^2\partial_r A(t,|\eta|)\eta & -2\mu_\| t\eta e^{-\mu_\|\eta^2 t} - 2c^2D(t,|\eta|) - c^2\eta^2\partial_r D(t,|\eta|) \end{bmatrix} e^{bt} \\
\partial_\eta^2\left[e^{bt}\widehat{S^{HF}}\left(t,\eta\right)\right] &= -\chi''(\eta)\widehat{S}\left(t,\eta\right)e^{bt} - 2\chi'(\eta)\partial_\eta\widehat{S}\left(t,\eta\right)e^{bt} \\
&+ \left[1-\chi(\eta)\right]e^{bt} \begin{bmatrix} \partial_r^2 B(t,|\eta|) & -2\partial_r A(t,|\eta|) - i\partial_r^2 A(t,|\eta|)\eta \\ -2ic^2\partial_r A(t,|\eta|) - ic^2\partial_r^2 A(t,|\eta|)\eta & -2\mu_\| t\left(1-2\mu_\| t\eta^2\right)e^{-\mu_\|\eta^2 t} - \left(2c^2 + 4c^2\eta\partial_r + c^2\eta^2\partial_r\right)D(t,|\eta|) \end{bmatrix}
\end{aligned}$$

Theeby, by using the estimates of the propositions 6 and 7, we obtain these estimations:

$$
\begin{aligned}
\left|e^{bt}\widehat{S^{HF}}(t,\eta)\right| &\leq C_0 \left|1-\chi(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel\eta^2}{2}t}\right\} \\
&\times \left[|\eta|^{-2}+|\eta|^{-1}+e^{-\mu_\parallel\eta^2 t}\right] \\
\left|\partial_\eta\left[e^{bt}\widehat{S^{HF}}(t,\eta)\right]\right| &\leq C_1\left|\chi'(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel\eta^2}{2}t}\right\} \\
&\times \left[|\eta|^{-2}+|\eta|^{-1}+e^{-\mu_\parallel\eta^2 t}\right] \\
&+ C_1'\left|1-\chi(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{4\mu_\parallel}t}+e^{-\frac{\mu_\parallel\eta^2}{4}t}\right\} \\
&\times \left[|\eta|^{-3}+|\eta|^{-2}+t|\eta|e^{-\mu_\parallel\eta^2 t}\right] \\
\left|\partial_\eta^2\left[e^{bt}\widehat{S^{HF}}(t,\eta)\right]\right| &\leq C_2\left|\chi''(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{2\mu_\parallel}t}+e^{-\frac{\mu_\parallel\eta^2}{2}t}\right\} \\
&\times \left[|\eta|^{-2}+|\eta|^{-1}+e^{-\mu_\parallel\eta^2 t}\right] \\
&+ C_2'\left|\chi'(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{4\mu_\parallel}t}+e^{-\frac{\mu_\parallel\eta^2}{4}t}\right\} \\
&\times \left[|\eta|^{-3}+|\eta|^{-2}+t|\eta|e^{-\mu_\parallel\eta^2 t}\right] \\
&+ C_2''\left|1-\chi(\eta)\right| e^{bt}\left\{e^{-\frac{c^2}{4\mu_\parallel}t}+e-\frac{\mu_\parallel\eta^2}{4}t\right\} \\
&\times \left[|\eta|^{-4}+|\eta|^{-3}+t\left(1+y\eta^2\right)e^{-\mu_\parallel\eta^2 t}\right]
\end{aligned}
$$

If we choose $b = \frac{1}{2} Min \left\{ \frac{c^2}{4\mu_\|} \ ; \ \frac{\mu_\| R_0^2}{4} \right\}$, we obtain:

$$\begin{array}{rcl} \left| e^{bt} \widehat{S^{HF}}(t,\eta) \right| & \leq & \underline{C_0} |\eta|^{-1} \\ \left| \partial_\eta \left[e^{bt} \widehat{S^{HF}}(t,\eta) \right] \right| & \leq & \underline{C_1} |\eta|^{-2} \\ \left| \partial_\eta \left[e^{bt} \widehat{S^{HF}}(t,\eta) \right] \right| & \leq & \underline{C_2} |\eta|^{-3} \end{array}$$

As a result, the hypothesis of the multiplier theorem is verified, and we obtain, for $p \in [1; +\infty]$ and $X_0 \in L^p(\mathbb{R})$:

$$\left\| \left[e^{bt} S^{HF}(t,\cdot) \right] * X_0 \right\|_{L^p} \leq C \left\| X_0 \right\|_{L^p}$$

which gives the result.

□

1.3.3 Middle frequency terms

In order to estimate of the Low frequency term, we have to consider R_0 small enough, whereas to estimate the high frequency term, we have to consider R_0 large enough. As a result, we have to make a new division of the function $\widehat{S}(t,\eta)$ on this way (by adding a **middle frequency term**):

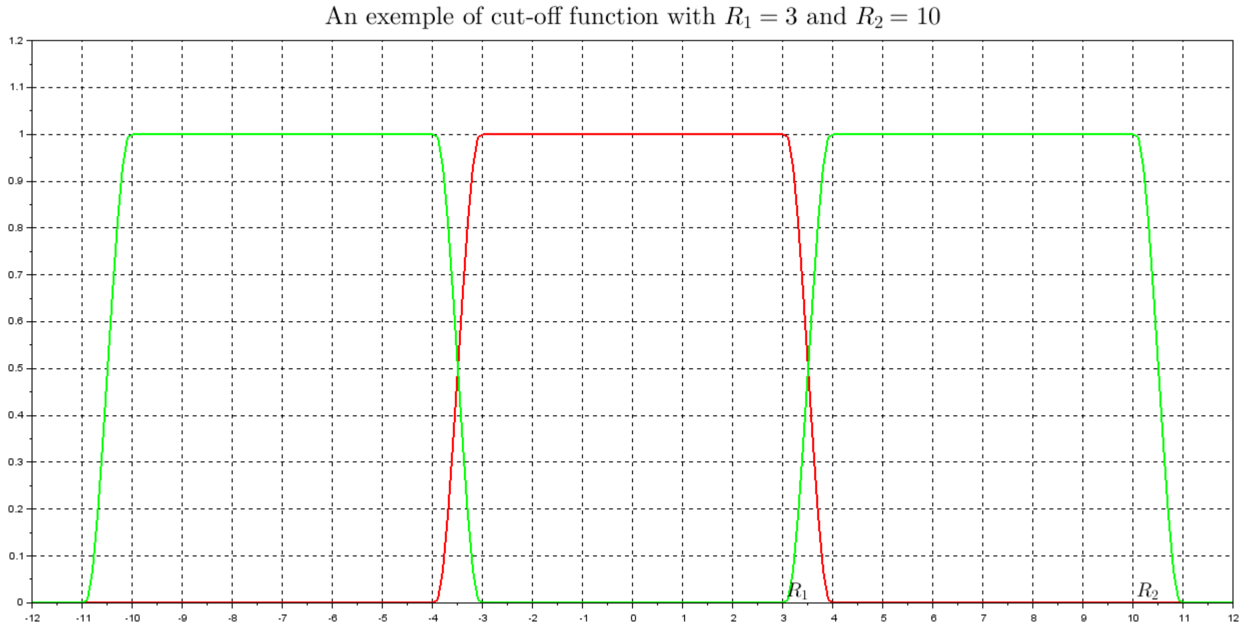
$$\widehat{X}(t,\eta) = \widehat{S^{LF}}(t,\eta)\,\widehat{X}(0,\eta) + \widehat{S^{HF}}(t,\eta)\,\widehat{X}(0,\eta) + \widehat{S^{HF}}(t,\eta)\,\widehat{X}(0,\eta)$$

with:

$$\begin{array}{rcl} \widehat{S^{LF}}(t,\eta) & = & \chi_1(\eta)\widehat{S}(t,\eta) \\ \widehat{S^{MF}}(t,\eta) & = & \chi_2(\eta)\widehat{S}(t,\eta) \\ \widehat{S^{HF}}(t,\eta) & = & \left[1 - \chi_1(\eta) - \chi_2(\eta)\right]\widehat{S}(t,\eta) \end{array}$$

where χ_1 and χ_2 are two cut-off functions, with:

$$\begin{array}{rcl} Supp(\chi_1) & = & [-R_1 - 1; R_1 + 1] \\ Supp(\chi_2) & = & [-R_2 - 1; -R_1] \sqcup [R_1; R_2 + 1] \end{array}$$

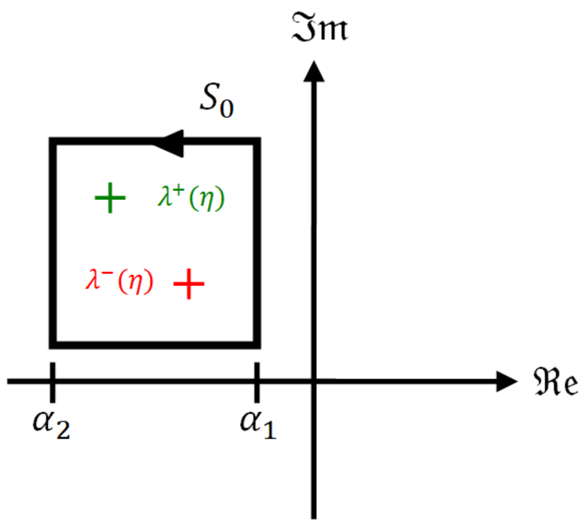


Proposition 9. *Decay of the middle frequency term.*

$$\exists \alpha_1 > 0 : \quad \left\| \widehat{S^{MF}}(t, \cdot) * X_0 \right\|_{L^p} \leq C_p e^{-\alpha_1 t} \left\| X_0 \right\|_{L^p}$$

for $p \in [1; +\infty]$, $C_p > 0$ *and* $X_0 \in L^p(\mathbb{R})$.

Proof. As $\widehat{S^{MF}}(t, \eta) = \chi_1(\eta) e^{tA(\eta)}$, and $\Re(\lambda^{\pm}(\eta)) < 0$ for $R_1 \leq |\eta| \leq R_2$, we can choose a complex contour S_0 on this way (for $\alpha_1, \alpha_2 > 0$):



As we have:

$$\widehat{S^{MF}}(t,\eta) = \frac{\chi_2(\eta)}{2i\pi}\int_{S_0} e^{tz}\left(zI - A(\eta)\right)^{-1}\mathrm{d}z$$

Thereby, we obtain:

$$\left|\widehat{S^{MF}}(t,\eta)\right| \leq C_0\left|\chi_2(\eta)\right| \underset{z\in S_0}{Sup}\left|\left(zI - A(\eta)\right)^{-1}\right| e^{-\alpha_1 t}$$

As the mapping:

$$\begin{array}{ccc} S_0 & \longrightarrow & \mathbb{R}_+ \\ z & \longmapsto & \left|\left(zI - A(\eta)\right)^{-1}\right| \end{array}$$

is continuous over the compact S_0 (indeed, $\forall z \in S_0;\ \ zI - A(\eta) \in GL_2(\mathbb{C})$), this mapping in bounded over S_0. Therefore, we have:

$$\underset{z\in S_0}{Sup}\left|\left(zI - A(\eta)\right)^{-1}\right| < +\infty$$

As a result, by applying the Marcinkiewic multiplier theorem to the function $\eta \longmapsto S^{MF}(t,\eta)\, e^{\alpha_1 t}$, we obtain the property. □

1.3.4 Conclusion

As a conclusion, we can sum up the long time behaviour of the linear terms of the Navier-Stokes system in this table:

Frequency	Kind of decay
Low	Inverse polynomial
Middle	Exponential
Low	Exponential

As a result, on long time, the Low-frequency terms will dominate... only if we consider linear terms.

Chapter 2

Numerical simulations

In this chapter, we will study the numerical properties of the schemes of numerical resolution of the Navier-Stokes system (in one dimension). Besides, we will make numerical simulations over the complete system.

2.1 Numerical simulations over the linearized system

2.1.1 Introduction

We consider the linearized system of Navier-Stokes:

$$\left\{ \begin{array}{ccccc} \partial_t \tilde{\rho} & + & \partial_x m & = & 0 \\ \partial_t m & + & c^2 \partial_x \tilde{\rho} & = & \mu_{\|} \partial_x^2 m \end{array} \right. \quad \text{for } (t,x) \in [0;T] \times [0;L] \tag{2.1}$$

We consider, for $(N;J) \in \mathbb{N}^{*2}$, these partitions:

$$\begin{array}{lll} (t_n)_{n \in [\![0;N]\!]} & \text{a partition of the interval} & [0;T] \\ (x_j)_{j \in [\![0;J]\!]} & \text{a partition of the interval} & [0;L] \end{array}$$

We note:

$$\begin{array}{lll} h_t & \text{the step of discretization for} & t \\ h_x & \text{the step of discretization for} & x \end{array}$$

Besides, we have to verify these conditions:

$$\forall t \in [0;T]; \quad \begin{aligned} \widetilde{\rho}(t,0) &= \widetilde{\rho}(t,L) = 0 \\ m(t,0) &= m(t,L) = 0 \end{aligned} \tag{2.2}$$

We will note, for $(n;j) \in [\![0;N]\!] \times [\![0;J]\!]$, $R_j^{[n]}$ and $m_j^{[n]}$ respectively the approximation of $\widetilde{\rho}(t_n, x_j)$ and $m(t_n, x_j)$. Therefore, we have:

$$\forall n \in [\![0;N]\!], \quad \begin{aligned} R_0^{[n]} &= R_J^{[n]} = 0 \\ m_0^{[n]} &= m_J^{[n]} = 0 \end{aligned}$$

We consider these vectors too:

$$R^{[n]} = \begin{bmatrix} R_1^{[n]} \\ \vdots \\ R_{J-1}^{[n]} \end{bmatrix} \quad \text{and} \quad m^{[n]} = \begin{bmatrix} m_1^{[n]} \\ \vdots \\ m_{J-1}^{[n]} \end{bmatrix}$$

We will condider this implicit scheme, based on approximation of derivative formulas (**cf. Chapter 3**):

$\forall j \in [\![1; J-1]\!]$;

$$\begin{aligned} \frac{1}{h_t}\left[R_j^{[n+1]} - R_j^{[n]}\right] + \frac{m_{j+1}^{[n+1]} - m_{j-1}^{[n+1]}}{2h_x} &= 0 \\ \frac{1}{h_t}\left[m_j^{[n+1]} - m_j^{[n]}\right] + c^2\frac{R_{j+1}^{[n+1]} - R_{j-1}^{[n+1]}}{2h_x} &= \mu_\parallel \frac{m_{j-1}^{[n+1]} - 2m_j^{[n+1]} + m_{j+1}^{[n+1]}}{h_x^2} \end{aligned} \tag{2.3}$$

and we note:

$$X^{[n]} = \begin{bmatrix} R^{[n]} \\ m^{[n]} \end{bmatrix} \quad \in \quad \mathbb{R}^{2(J-1)}$$

As a result, if we condider these matrices:

$$D = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & (0) & \\ & \ddots & \ddots & \ddots & \\ & (0) & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & (0) & \\ & \ddots & \ddots & \ddots & \\ & (0) & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

and:

$$A = \begin{bmatrix} 0 & \frac{1}{2h_x} D \\ \frac{c^2}{2h_x} D & \frac{\mu_{\|}}{h_x^2} M \end{bmatrix}$$

By using the scheme (2.3), we obtain this explicit scheme with matrices:

$$\begin{cases} X^{[0]} & \text{is known} \\ \forall n \in \mathbb{N}^*; \quad X^{[n+1]} & = & (I_2 + h_t A)^{-1} X^{[n]} \end{cases}$$

2.1.2 The hyperbolic case

In this subsection, we will study the case $\mu_{\|} = 0$. So as to study the stability of the scheme, we will use the Fourier-Von Neumann method (**cf. Chapter 3**).

Stability

Proposition 10. *The scheme in the hyperbolic case is unconditionally stable*

Proof. By using the scheme (2.3) with $\mu_{\|} = 0$, we have, by appliying the Fourier-Von Neumann method:

$$\begin{aligned} \overset{\vee}{R}^{[n+1]}(x) &= \overset{\vee}{R}^{[n]}(x) - \frac{h_t}{2h_x}\left(e^{2i\pi x} - e^{-2i\pi x}\right)\overset{\vee}{m}^{[n+1]}(x) \\ \overset{\vee}{m}^{[n+1]}(x) &= \overset{\vee}{m}^{[n]}(x) - \frac{c^2 h_t}{2h_x}\left(e^{2i\pi x} - e^{-2i\pi x}\right)\overset{\vee}{R}^{[n+1]}(x) \end{aligned}$$

If we note:

$$\overset{\vee}{X}^{[n]}(x) = \begin{bmatrix} \overset{\vee}{R}^{[n]}(x) \\ \overset{\vee}{m}^{[n]}(x) \end{bmatrix}$$

we obtain:

$$\begin{aligned} \overset{\vee}{A}(x)\overset{\vee}{X}^{[n+1]}(x) &= \overset{\vee}{X}^{[n]}(x) \\ \text{where } \overset{\vee}{A}(x) &= \begin{bmatrix} 1 & i\frac{h_t}{h_x}\sin(2\pi x) \\ i\frac{c^2 h_t}{h_x}\sin(2\pi x) & 1 \end{bmatrix} \end{aligned}$$

Therefore, we have:

$$\begin{aligned} \chi_{\overset{\vee}{A}(x)} &= \begin{vmatrix} X-1 & -\frac{i h_t \sin(2\pi x)}{h_x} \\ -\frac{i c^2 h_t \sin(2\pi x)}{h_x} & X-1 \end{vmatrix} \\ \chi_{\overset{\vee}{A}(x)} &= (X-1)^2 + \left(\frac{c h_t \sin(2\pi x)}{h_x}\right)^2 \end{aligned}$$

As a result, if $\lambda \in \sigma\left(\overset{\vee}{A}(x)\right)$:, we obtain:

$$\begin{aligned} \lambda &= 1 \pm i \left| \frac{ch_t \sin(2\pi x)}{h_x} \right| \\ |\lambda|^2 &= 1 + \left(\frac{ch_t \sin(2\pi x)}{h_x} \right)^2 \\ |\lambda| &\geq 1 \end{aligned}$$

Thereby, we have:

$$\begin{aligned} \overset{\vee}{X}^{[n+1]}(x) &= \overset{\vee}{A}(x)^{-1} \overset{\vee}{X}^{[n]}(x) \\ \text{with} \quad \sigma\left(\overset{\vee}{A}(x)^{-1}\right) &\subset \overline{\mathbb{D}} \end{aligned}$$

As a result, we obtain, $\forall n \in \mathbb{N}$;

$$\begin{aligned} \left\| X^{[n]} \right\|_{l^2(\mathbb{Z})} &= \left\| \overset{\vee}{X}^{[n]} \right\|_{L^2(\mathbb{T})} \\ &\leq \left\| \overset{\vee}{X}^{[0]} \right\|_{L^2(\mathbb{T})} \\ &= \left\| X^{[0]} \right\|_{l^2(\mathbb{Z})} \end{aligned}$$

which gives the unconditionnal stability

□

Consistency

As we work with the norms in $l^2(\mathbb{Z}) \longleftrightarrow L^2(\mathbb{T})$ so as to study the stability, we have to work with the same topology in order to study the consistency.

Definition 2. ***Consistency error***
We define the consistency error at the time t_n on this way:

$$\begin{aligned}
\varepsilon^{(n)} &= \left\| \frac{\widetilde{\rho}(x_j, t_{n+1}) - \widetilde{\rho}(x_j, t_n)}{h_t} + \frac{m(x_{j+1}, t_{n+1}) - m(x_{j-1}, t_{n+1})}{2h_x} \right\|_{l_j^2(\mathbb{Z})} \\
&+ \left\| \frac{m(x_j, t_{n+1}) - m(x_j, t_n)}{h_t} + c^2 \frac{\widetilde{\rho}(x_{j+1}, t_{n+1}) - \widetilde{\rho}(x_{j-1}, t_{n+1})}{2h_x} \right\|_{l_j^2(\mathbb{Z})}
\end{aligned}$$

Proposition 11. *The consistency error verifies this estimation:*

$$\varepsilon^{(n)} = \mathcal{O}\left(h_t + h_x^2\right)$$

Proof. According to the conditions (2.2), we can suppose $\widetilde{\rho}, m \in \mathcal{C}^3_{c,x}(\mathbb{R}) \cap \mathcal{C}^2_t(\mathbb{R}) \cap L^\infty_t(\mathbb{R})$ ($\mathcal{C}^3-$class functions in x with compact support and $\mathcal{C}^2-$class functions in t bounded. Then, we can suppose $\partial_t^2 m, \partial_t^2 \widetilde{\rho} \in L^\infty_t(\mathbb{R})$ and $\partial_x^3 m, \partial_x^3 \widetilde{\rho} \in L^\infty_t(\mathbb{R})$

Let $j \in \mathbb{Z}$. We have $x_{j\pm1} = x_j \pm h_x$, $t_{n\pm1} = t_n \pm 1$ and, by using the Taylor's formulas, we obtain:

$$\begin{aligned}
&\frac{\widetilde{\rho}(x_j, t_{n+1}) - \widetilde{\rho}(x_j, t_n)}{h_t} + \frac{m(x_{j+1}, t_{n+1}) - m(x_{j-1}, t_{n+1})}{2h_x} \\
&= \underbrace{\partial_t \widetilde{\rho}(x_j, t_{n+1}) + \partial_x m(x_j, t_{n+1})}_{=0(\text{equation})} - \frac{1}{2} h_t \partial_t^2 \widetilde{\rho}(x_j, \tau_n) + \frac{1}{6} h_x^2 \partial_x^3 m(x_j + \zeta_j, t_{n+1}) \\
&= -\frac{1}{2} h_t \partial_t^2 \widetilde{\rho}(x_j, \tau_n) + \frac{1}{6} h_x^2 \partial_x^3 m(x_j + \zeta_j, t_{n+1}) \\
&\qquad \text{with } \tau_n \in [t_n, t_{n+1}] \text{ and } \zeta_j \in [-h_x, h_x]
\end{aligned}$$

As a result:

$$\begin{aligned}
&\left\| \frac{\widetilde{\rho}\left(x_j, t_{n+1}\right) - \widetilde{\rho}\left(x_j, t_n\right)}{h_t} + \frac{m\left(x_{j+1}, t_{n+1}\right) - m\left(x_{j-1}, t_{n+1}\right)}{2h_x} \right\|_{l_j^2(\mathbb{Z})} \\
&\leq \frac{1}{2} h_t \left\| \partial_t^2 \widetilde{\rho}(x_j, \tau_n) \right\|_{l_j^2(\mathbb{Z})} + \frac{1}{6} h_x^2 \left\| \partial_x^3 m(x_j + \zeta_j, t_{n+1}) \right\|_{l_j^2(\mathbb{Z})} \\
&\leq \frac{1}{2} h_t \left\| \partial_t^2 \widetilde{\rho}(x_j, t) \right\|_{l_j^2(\mathbb{Z}), L_t^\infty(\mathbb{R})} + \frac{1}{6} h_x^2 \left\| \partial_x^3 m(x_j + \zeta_j, t) \right\|_{l_j^2(\mathbb{Z}), L_t^\infty(\mathbb{R})}
\end{aligned}$$

Besides, for $j \in \mathbb{Z}$:

$$\begin{aligned}
&\frac{m\left(x_j, t_{n+1}\right) - m\left(x_j, t_n\right)}{h_t} + c^2 \frac{\widetilde{\rho}\left(x_{j+1}, t_{n+1}\right) - \widetilde{\rho}\left(x_{j-1}, t_{n+1}\right)}{2h_x} \\
&= \underbrace{\partial_t m\left(x_j, t_{n+1}\right) + c^2 \partial_x \widetilde{\rho}(x_j, t_{n+1})}_{=0 (\text{equation})} - \frac{1}{2} h_t \partial_t^2 m(x_j, \nu_n) + \frac{c^2}{6} h_x^2 \partial_x^3 \widetilde{\rho}(x_j + \omega_j, t_{n+1}) \\
&= -\frac{1}{2} h_t \partial_t^2 m(x_j, \nu_n) + \frac{c^2}{6} h_x^2 \partial_x^3 \widetilde{\rho}(x_j + \omega_j, t_{n+1}) \\
&\text{with } \nu_n \in [t_n, t_{n+1}] \text{ and } \omega_j \in [-h_x, h_x]
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\left\| \frac{m\left(x_j, t_{n+1}\right) - m\left(x_j, t_n\right)}{h_t} + c^2 \frac{\widetilde{\rho}\left(x_{j+1}, t_{n+1}\right) - \widetilde{\rho}\left(x_{j-1}, t_{n+1}\right)}{2h_x} \right\|_{l_j^2(\mathbb{Z})} \\
&\leq \frac{1}{2} h_t \left\| \partial_t^2 m(x_j, \tau_n) \right\|_{l_j^2(\mathbb{Z})} + \frac{c^2}{6} h_x^2 \left\| \partial_x^3 \widetilde{\rho}(x_j + \zeta_j, t_{n+1}) \right\|_{l_j^2(\mathbb{Z})} \\
&\leq \frac{1}{2} h_t \left\| \partial_t^2 m(x_j, t) \right\|_{l_j^2(\mathbb{Z}), L_t^\infty(\mathbb{R})} + \frac{c^2}{6} h_x^2 \left\| \partial_x^3 \widetilde{\rho}(x_j + \zeta_j, t) \right\|_{l_j^2(\mathbb{Z}), L_t^\infty(\mathbb{R})}
\end{aligned}$$

As a result, we have:

$$\varepsilon^{(n)} = \mathcal{O}\left(h_t + h_x^2\right)$$

□

Convergence

Proposition 12. *The scheme is convergent on this order:* $\mathcal{O}\left(h_t + h_x^2\right)$

Proof. The scheme is stable and consistent. As a result, according to the Lax equivalence theorem, this scheme is also convergent, and the order of convergence is the same as the order of consistency. □

2.1.3 The parabolic case

Now, we will consider the case $c = 0$, and $\mu_{\|} \geq 0$, which gives a parabolic equation (like the heat equation).

Stability

Proposition 13. *The scheme in the parabolic case in unconditionnaly stable*

Proof. By applying the Fourier-Von Neuman method over the scheme, according to the scheme (2.2) in the case $c = 0$, we have:

$$\begin{aligned}
\frac{1}{h_t}\left[\overset{\vee}{R}^{[n+1]}(x) - \overset{\vee}{R}^{[n]}(x)\right] &= -\frac{i\sin(2\pi x)}{h_x}\overset{\vee}{m}^{[n+1]}(x) \\
\frac{1}{h_t}\left[\overset{\vee}{m}^{[n+1]}(x) - \overset{\vee}{m}^{[n]}(x)\right] &= -\frac{2\mu_{\|}}{h_x^2}\left(1 - \cos(2\pi x)\right)\overset{\vee}{m}^{[n+1]}(x)
\end{aligned}$$

And we obtain:

$$\begin{aligned}
\overset{\vee}{B}(x)\overset{\vee}{X}^{[n+1]}(x) &= \overset{\vee}{X}^{[n]}(x) \\
\text{where } \overset{\vee}{B}(x) &= \begin{bmatrix} 1 & \frac{ih_t\sin(2\pi x)}{h_x} \\ 0 & 1 + \frac{2\mu_{\|}h_t(1-\cos(2\pi x))}{h_x^2} \end{bmatrix}
\end{aligned}$$

We clearly have:

$$\forall \lambda \in \sigma\left(\overset{\vee}{B}(x)\right); \quad |\lambda| \geq 1$$

By a same reasonning as the hyperbolic case, we show that, as $\sigma\left(\overset{\vee}{B}(x)^{-1}\right) \subset \overline{\mathbb{D}}$, we have:

$$\forall n \in \mathbb{N}; \quad \left\|X^{[n]}\right\|_{l^2(\mathbb{Z})} \leq \left\|X^{[0]}\right\|_{l^2(\mathbb{Z})}$$

which gives the unconditionnal stability.

□

Consistency

Definition 3. *Consistency error*
We define the consistency error at the time t_n on this way:

$$\begin{aligned}
\overset{\sim}{\varepsilon}^{(n)} &= \left\|\frac{\tilde{\rho}\left(x_j, t_{n+1}\right) - \tilde{\rho}\left(x_j, t_n\right)}{h_t} + \frac{m\left(x_{j+1}, t_{n+1}\right) - m\left(x_{j-1}, t_{n+1}\right)}{2h_x}\right\|_{l_j^2(\mathbb{Z})} \\
&+ \left\|\frac{m(x_j, t_{n+1}) - m(x_j, t_n)}{h_t} + \mu_{\|}\frac{-m(x_{j-1}, t_{n+1}) + 2m(x_j, t_{n+1}) - m(x_{j+1}, t_{n+1})}{h_x^2}\right\|_{l_j^2(\mathbb{Z})}
\end{aligned}$$

Proposition 14. *The consistency error verifies this estimation:*

$$\overset{\sim}{\varepsilon}^{(n)} = \mathcal{O}\left(h_t + h_x^2\right)$$

Proof. We make the same hypothesis over m and $\overset{\sim}{\rho}$ than for the hyperbolic csae.

The first term can be bounded on the same way than for the hyperbolic case (this is the same term !)

We study the second term. Let $j \in \mathbb{Z}$:

$$\begin{aligned}
&\frac{m(x_j,t_{n+1}) - m(x_j,t_n)}{h_t} + \mu_{\|}\frac{-m(x_{j-1},t_{n+1}) + 2m(x_j,t_{n+1}) - m(x_{j+1},t_{n+1})}{h_x^2} \\
&= \underbrace{\partial_t m(x_j,t_{n+1}) - \mu_{\|}\partial_x^2 m(x_j,t_{n+1})}_{=0 \text{ (equation)}} - \frac{1}{2}h_t\partial_t^2 m(x_j,\kappa_n) - \frac{1}{12}h_x^2\partial_x^4 m(x_j+\sigma_j,t_{n+1}) \\
&= -\frac{1}{2}h_t\partial_t^2 m(x_j,\kappa_n) - \frac{1}{12}h_x^2\partial_x^4 m(x_j+\sigma_j,t_{n+1}) \\
&\quad \text{with } \kappa_n \in [t_n,t_{n+1}], \sigma_j \in [-h_x,h_x]
\end{aligned}$$

Therefore:

$$\begin{aligned}
&\left\| \frac{m(x_j,t_{n+1}) - m(x_j,t_n)}{h_t} + \mu_{\|} \frac{-m(x_{j-1},t_{n+1}) + 2m(x_j,t_{n+1}) - m(x_{j+1},t_{n+1})}{h_x^2} \right\|_{l_j^2(\mathbb{Z})} \\
&\qquad \leq \frac{1}{2} h_t \left\| \partial_t^2 \tilde{\rho}(x_j,\kappa_n) \right\|_{l_j^2(\mathbb{Z})} + \frac{1}{12} h_x^2 \left\| \partial_x^3 m(x_j+\sigma_j,t_{n+1}) \right\|_{l_j^2(\mathbb{Z})} \\
&\qquad \leq \frac{1}{2} h_t \left\| \partial_t^2 \tilde{\rho}(x_j,t) \right\|_{l_j^2(\mathbb{Z}),L_t^\infty(\mathbb{R})} + \frac{1}{12} h_x^2 \left\| \partial_x^3 m(x_j+\sigma_j,t) \right\|_{l_j^2(\mathbb{Z}),L_t^\infty(\mathbb{R})}
\end{aligned}$$

As a result, we have:

$$\tilde{\varepsilon}^{(n)} = \mathcal{O}\left(h_t + h_x^2\right)$$

□

Convergence

Proposition 15. *The scheme is convergent on this order:* $\mathcal{O}\left(h_t + h_x^2\right)$

Proof. The argument is the same as the previous case. We just have to use the Lax equivalence theorem, the consistency and the stability of the scheme in the parabolic case. □

2.1.4 Numerical simulations over the linearized system

Numerical simulations have been made with *Scilab*. The programme has been modified by creating "periodic" conditions (by assuming the values over the tw sides of the figure are the same).

Hyperbolic case

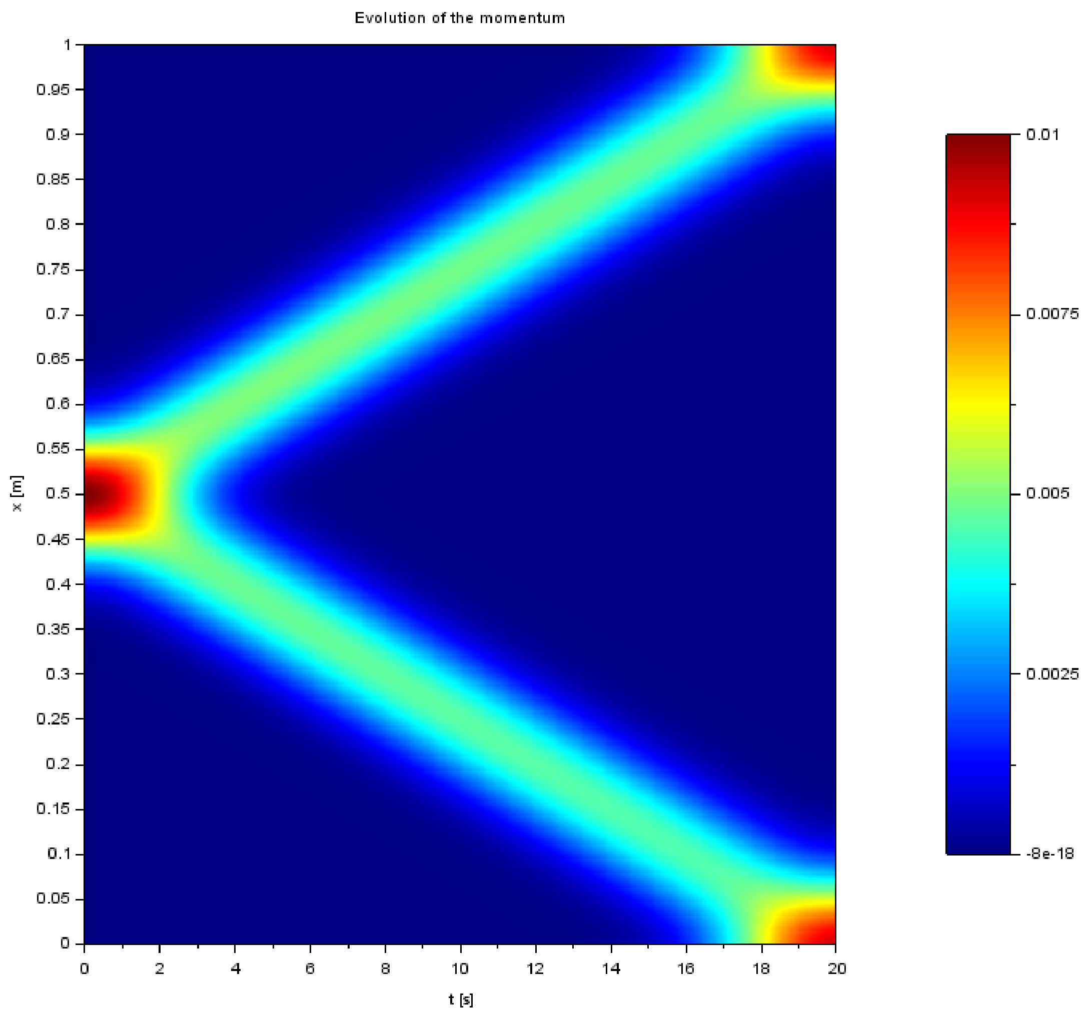


Figure 2.1: This figure was created with $N = J = 500$, $\mu_{||} = 0 Pa.s$, $c = 2.5 \times 10^{-2} m.s^{-1}$, $L = 1m$ and $T = 20s$. With a gaussian initial datum, we can observe a transport with the speed c on both sides. This is the **Wave equation**. The red zones on the right side correspond to the periodic contitions (the two parts are joined again).

Parabolic case

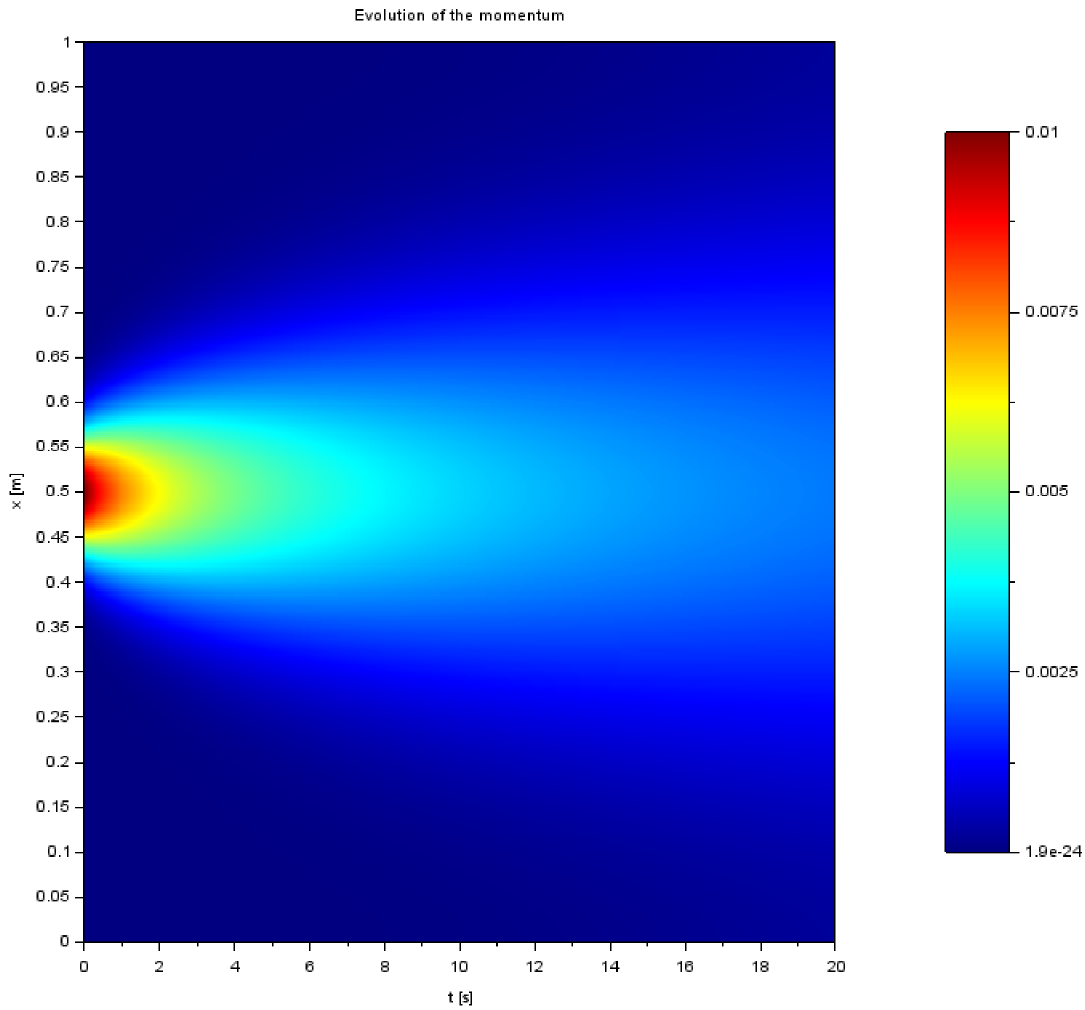


Figure 2.2: This figure was created with $N = J = 500$, $\mu_{\parallel} = 10^{-3} Pa.s$ ($\approx$ Water), $c = 0 m.s^{-1}$, $L = 1m$ and $T = 20s$. With a gaussian initial datum, we can observe a diffusion. This is the **Heat equation**.

Crossed case

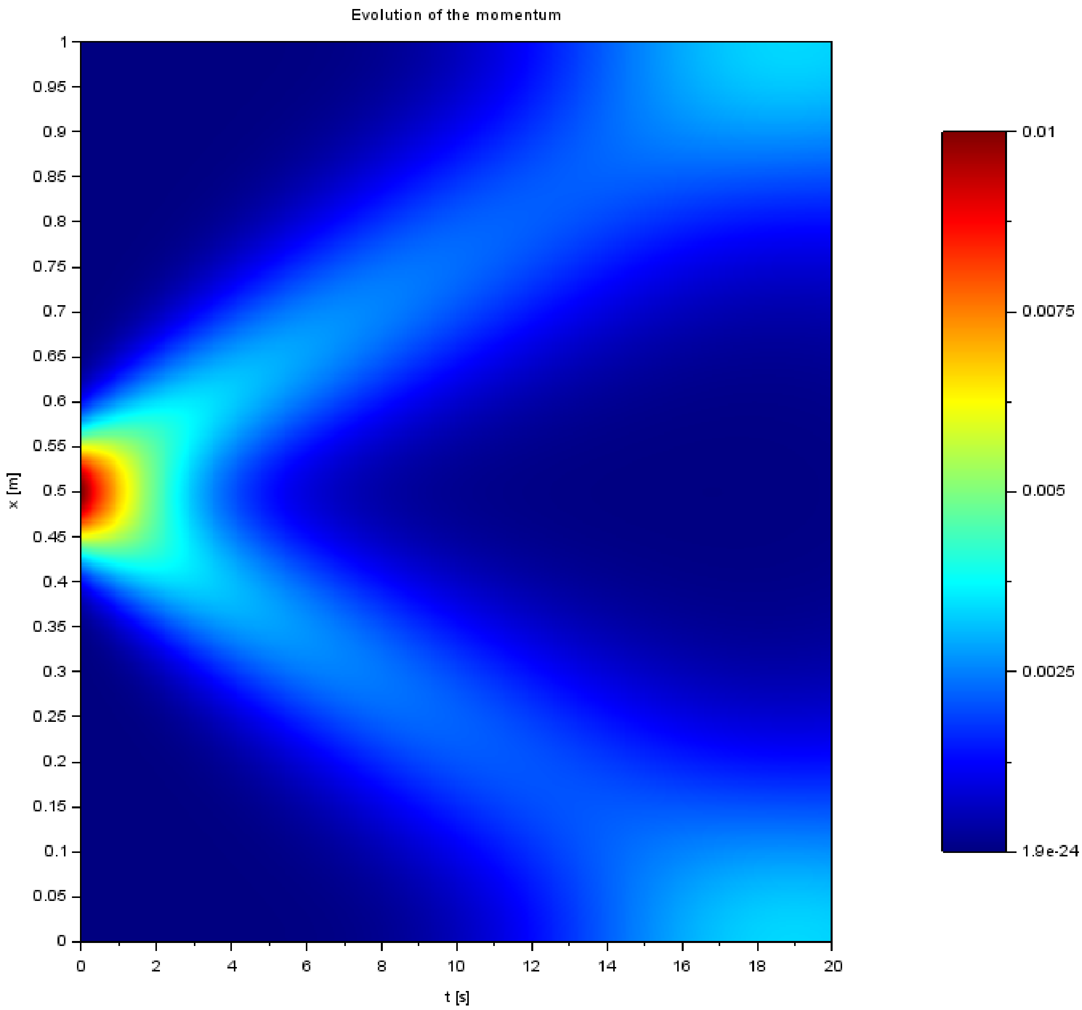


Figure 2.3: This figure was created with $N = J = 500$, $\mu_{\parallel} = 2 \times 10^{-3} Pa.s$ ($\approx$ Milk), $c = 2.5 \times 10^{-2} m.s^{-1}$, $L = 1m$ and $T = 20s$. With a gaussian initial datum, we can observe both transport and diffusion. The clear blue zones on the right side correspond to the perdiodic conditions (the two parts are joined again).

2.1.5 Study of the decay with two norms

In the proposition 4, we have established these results:

$$\|S(t,\cdot)\|_{L^2} \underset{t \longrightarrow +\infty}{=} \mathcal{O}\left(\frac{1}{\sqrt{t}}\right)$$
$$\|S(t,\cdot)\|_{L^\infty} \underset{t \longrightarrow +\infty}{=} \mathcal{O}\left(\frac{1}{t^{\frac{1}{4}}}\right)$$

Remark. *In reality, only the low-frequency terms have this kind of behaviour. The other terms have an exponential decay*

We will illustrate these results with this exemple:

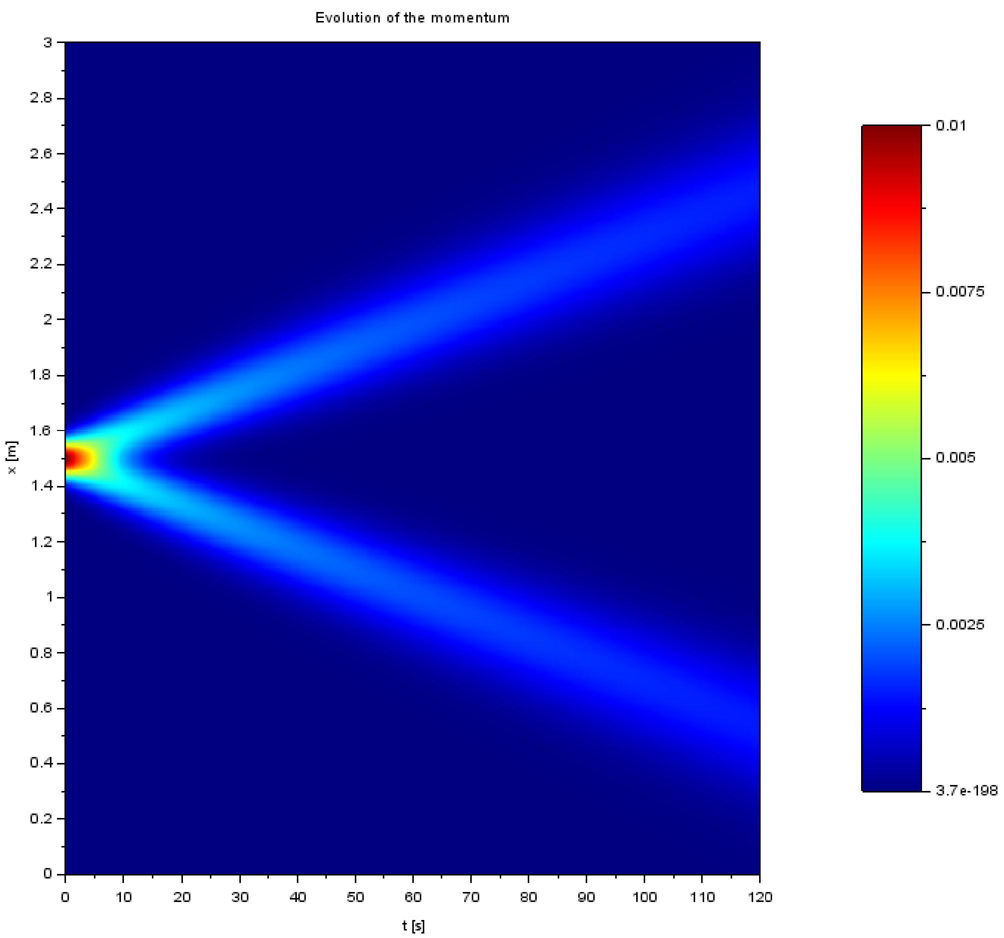


Figure 2.4: Figure made with $L = 3m$, $T = 120s$, $c = 8 \times 10^{-3} m.s^{-1}$, $\mu_{\|} = 2 \times 10^{-4} Pa.s$ and $N = J = 500$, with gaussian initial datum

We have made linear regressions by using logarithmic scale:

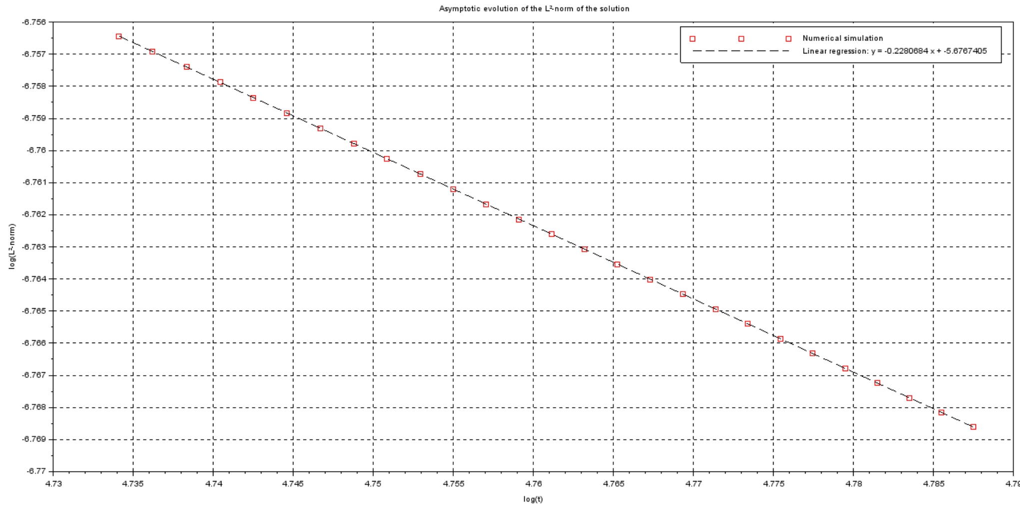


Figure 2.5: The case of the L^2-norm (energy), ploted from $t = 114s$ to $t = 120s$

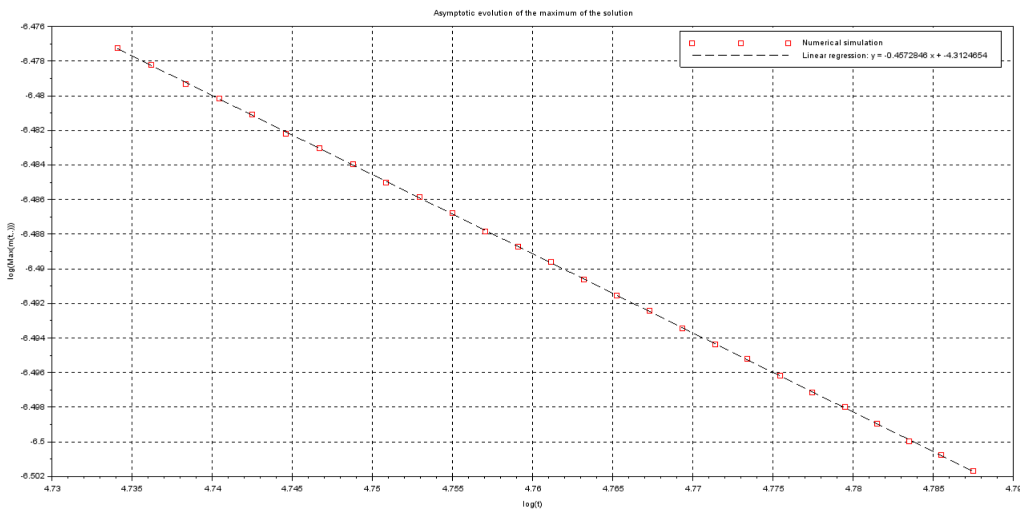


Figure 2.6: The case of the $L^\infty-$norm (maximum), ploted from $t = 114s$ to $t = 120s$

We can explain the slower decay observed with the numerical simulations: In reality, the proposition 4 gives an **asymptotic** decay, in particulat there exists a constant as a factor, which allow a slower decay for "short" times. However, we obtain consistent results (a faster decay when the index of the norm increases).

2.2 Numerical simulations over the complete system

Now, we will study the complete system of Navier-Stokes:

$$\left\{\begin{array}{ccccccc} \partial_t \rho & + & \partial_x m & = & 0 & & \\ \partial_t m & + & \partial_x \left(\frac{m^2}{\rho}\right) & = & \mu_{\parallel} \partial_x^2 \left(\frac{m}{\rho}\right) & - & \partial_x p \end{array}\right.$$

We won't study consistency or stability in this case, but, according to the system, we can suggest this scheme, based on the Euler's method:

$$\left\{\begin{array}{ccccc} \frac{1}{h_t}\left[R_j^{[n+1]} - R_j^{[n]}\right] & + & \frac{1}{2h_x}\left[m_{j+1}^{[n]} - m_{j-1}^{[n]}\right] & = & 0 \\ \frac{1}{h_t}\left[m_j^{[n+1]} - m_j^{[n]}\right] & + & \frac{1}{2h_x}\left[\frac{{m_{j+1}^{[n]}}^2}{R_{j+1}^{[n]}} - \frac{{m_{j-1}^{[n]}}^2}{R_{j-1}^{[n]}}\right] & = & \frac{\mu_{\parallel}}{h_x^2}\left[\frac{m_{j+1}^{[n]}}{R_{j+1}^{[n]}} - 2\frac{m_j^{[n]}}{R_j^{[n]}} + \frac{m_{j-1}^{[n]}}{R_{j-1}^{[n]}}\right] \\ & & & - & \frac{1}{2h_x}\left[P\left(R_{j+1}^{[n]}\right) - P\left(R_{j-1}^{[n]}\right)\right] \end{array}\right.$$

We can simplify it by using matrices:

$$\begin{aligned} R^{[n+1]} &= R^{[n]} - \frac{h_t}{2h_x} D m^{[n]} \\ m^{[n+1]} &= m^{[n]} - \frac{h_t}{2h_x} D\left(m^{[n]}\,.^2\,./R^{[n]}\right) - \frac{\mu_{\parallel} h_t}{h_x^2} M\left(m^{[n]}\,./R^{[n]}\right) - \frac{h_t}{2h_x} D \cdot P\left(R^{[n]}\right) \end{aligned}$$

where D and M are defined p. 38 and:

$$m^{[n]}\ .^2\ ./R^{[n]} = \begin{bmatrix} \frac{{m_1^{[n]}}^2}{R_1^{[n]}} \\ \vdots \\ \frac{{m_{J-1}^{[n]}}^2}{R_{J-1}^{[n]}} \end{bmatrix} \quad ; \quad \frac{m^{[n]}}{R^{[n]}} = \begin{bmatrix} \frac{m_1^{[n]}}{R^{[n]}} \\ \vdots \\ \frac{m_{J-1}^{[n]}}{R_{J-1}^{[n]}} \end{bmatrix}$$

$$\text{and} \quad P\left(R^{[n]}\right) = \begin{bmatrix} P\left(R_1^{[n]}\right) \\ \vdots \\ P\left(R_{J-1}^{[n]}\right) \end{bmatrix}$$

Now, we will study an exemple of resolution of the non-linear system. We consider these datums, and compare with the linear system.

$$\begin{aligned} P(\rho) &= 5 \times 10^{-5} \rho^{1.4} \\ \rho_* &= 1 Kg.m^{-3} \\ \mu_{\|} &= 2 \times 10^{-4} Pa.s \\ \widetilde{\rho}(0, x) &= 10^{-2} e^{-\frac{10^3}{5}\left(x - \frac{L}{2}\right)^2} \\ m(0, x) &= \widetilde{\rho}(0, x) \end{aligned}$$

A simple computation shows that:

$$\begin{aligned} c &= \sqrt{P'(\rho_*)} \\ c &= \sqrt{7 \times 10^{-5}} m.s^{-1} \\ c &= 8.3 \times 10^{-3} m.s^{-1} \end{aligned}$$

Besides, we take $J = 300$ and $N = 1500$ for the two simultions.

The two next figures compare the resolutions of linear and complete system.

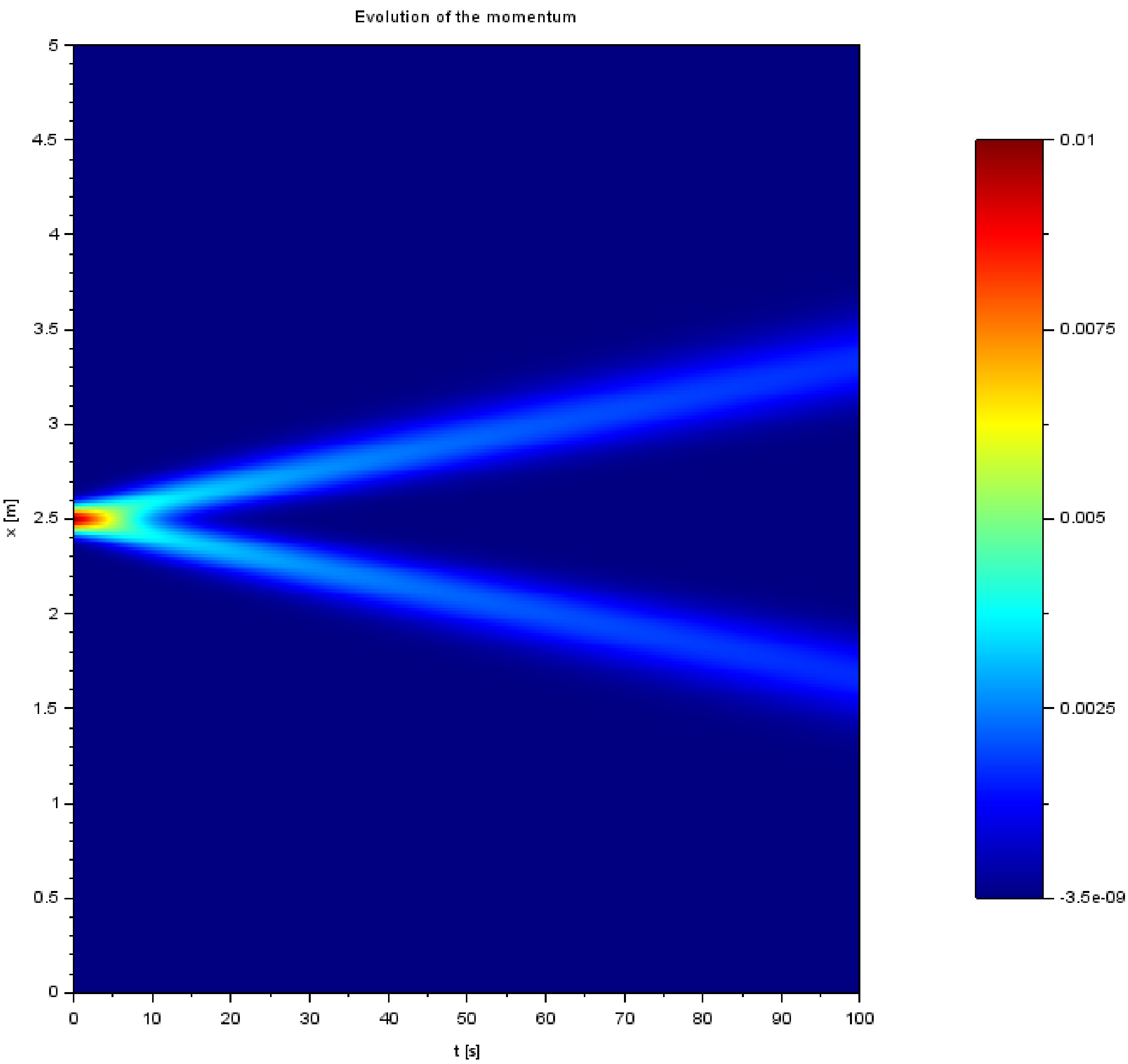


Figure 2.7: The resolution of the linear system gives this result

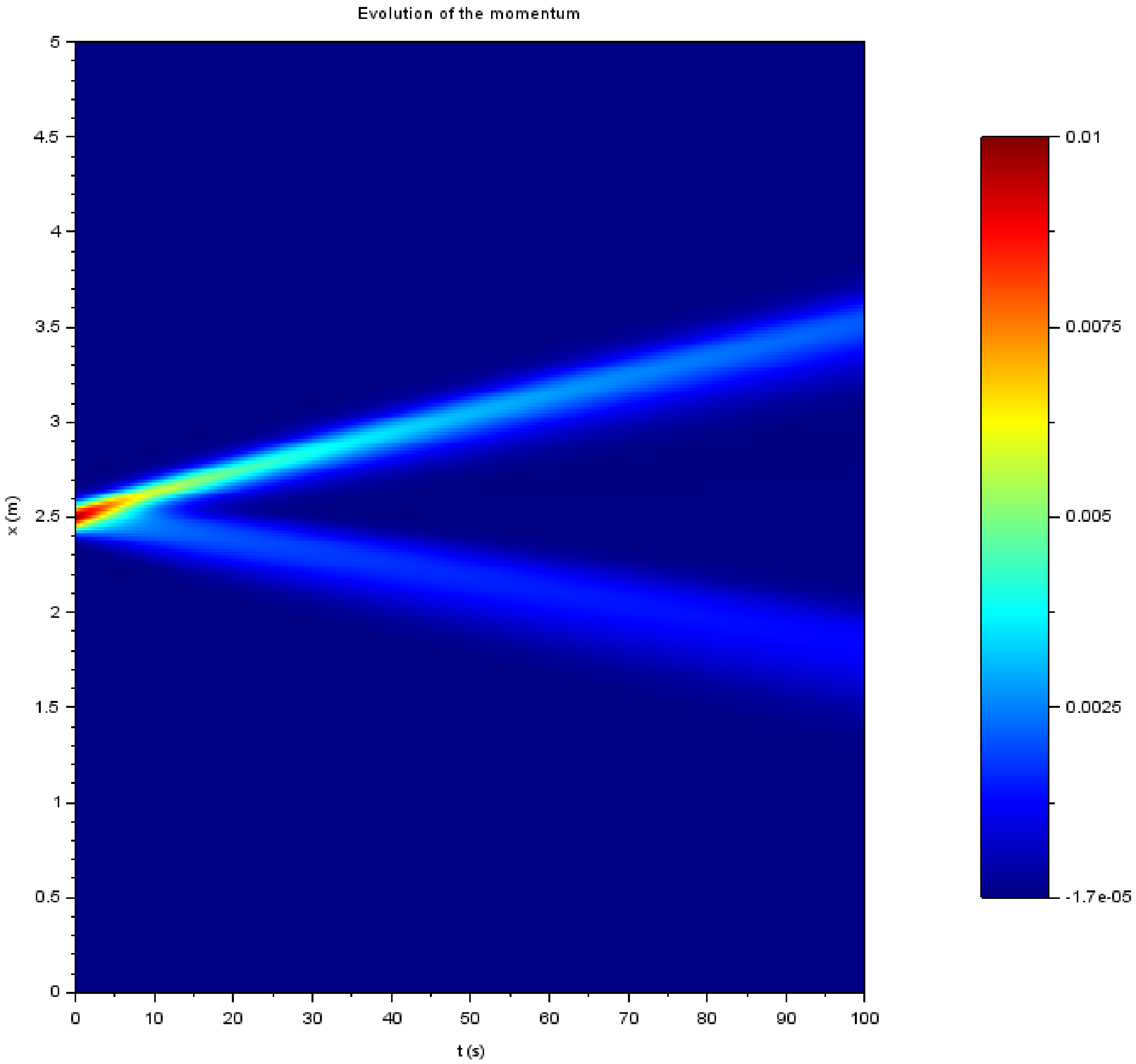


Figure 2.8: The resolution of the complete system gives this second result

We can explain this phenomenon by two reasons:

- A first reason concerns the numerical scheme which is not stable or consistent, and gives wrong results. Indeed, any work has been done in order to establish consistency or stability of our scheme.

- The second possible reason is the non-linear terms (which are note studied in this intership) which can't be neglected.

Chapter 3

Additionnal informations

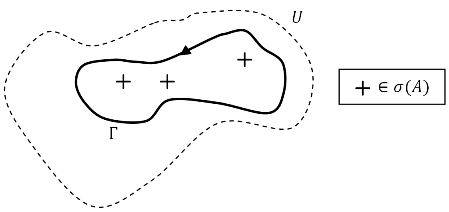
3.1 Chapter 1

3.1.1 The Cauchy's integral formula for matrices

Return

Theorem 1. ***Cauchy's integral formula for matrices***
Let $U \subset \mathbb{C}$ an openset and $f : U \longrightarrow \mathbb{C}$ an holomorphic function. Let $\Gamma \subset U$ a simple loop, and $A \in \mathcal{M}_n(\mathbb{C})$ with $\sigma(A) \subset Int(\Gamma)$ (interior). We have:

$$f(A) = \frac{1}{2i\pi} \int_\Gamma f(z)\,(zI_n - A)^{-1}\,\mathrm{d}z$$



3.1.2 The end of the proof using the residue theorem

Return

In our situation, we will use the Cauchy's integral formula by using the holomorphic function $f_t : z \longmapsto e^{tz}$ (for $t \geq 0$). If we note $I = I_2$, we have:

$$e^{tA(\eta)} = \frac{1}{2i\pi} \int_\Gamma e^{tz}\,(zI - A(\eta))^{-1}\,\mathrm{d}z$$

Where Γ is a simple loop containing $\lambda^{\pm}(\eta)$ in the complex plane $\mathbb{C}$. We note $\widetilde{A}_z(\eta) = zI - A(\eta)$.

$$\widetilde{A}_z(\eta) = \begin{bmatrix} z & i\eta \\ i\eta c^2 & z + \mu_{\|}\eta^2 \end{bmatrix}$$

Therefore, we have:

$$\widetilde{A}_z(\eta)^{-1} = \frac{1}{det\left(\widetilde{A}_z(\eta)\right)} Adj\left[\widetilde{A}_z(\eta)\right]^T$$

where Adj means the adjugate matrix:

$$\widetilde{A}_z(\eta)^{-1} = \frac{1}{z^2 - \mu_\| \eta^2 z + c^2\eta^2}\begin{bmatrix} z+\mu_\|\eta^2 & -i\eta c^2 \\ -i\eta & z \end{bmatrix}^T$$

$$\widetilde{A}_z(\eta)^{-1} = \frac{1}{z^2 - \mu_\| \eta^2 z + c^2\eta^2}\begin{bmatrix} z+\mu_\|\eta^2 & -i\eta \\ -i\eta c^2 & z \end{bmatrix}$$

$$\widetilde{A}_z(\eta)^{-1} = \frac{1}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\begin{bmatrix} z+\mu_\|\eta^2 & -i\eta \\ -i\eta c^2 & z \end{bmatrix}$$

Therefore;

$$\frac{1}{2i\pi}\int_\Gamma e^{tz}\widetilde{A}_z(\eta)^{-1}\,\mathrm{d}z = \frac{1}{2i\pi}\begin{bmatrix} \int_\Gamma \frac{\left(z+\mu_\|\eta^2\right)e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z & \int_\Gamma \frac{-i\eta e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z \\ \int_\Gamma \frac{-ic^2\eta e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z & \int_\Gamma \frac{z e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z \end{bmatrix}$$

We want to compute, for $j \in [\![0;1]\!]$;

$$I_j(t) = \int_\Gamma \frac{z^j e^{tz}}{(z-\lambda^+(\eta))(z-\lambda^-(\eta))}\mathrm{d}z$$

We have:

$$\frac{z^j e^{tz}}{z^2+\mu_\|\eta^2 z + c^2\eta^2} \underset{z\to\lambda^+(\eta)}{\sim} \frac{\lambda^+(\eta)^j e^{t\lambda^+(\eta)}}{(\lambda^+(\eta)-\lambda^-(\eta))(z-\lambda^+(\eta))}$$

and:

$$\frac{z^j e^{tz}}{z^2+\mu_\|\eta^2 z + c^2\eta^2} \underset{z\to\lambda^-(\eta)}{\sim} \frac{-\lambda^-(\eta)^j e^{t\lambda^-(\eta)}}{(\lambda^+(\eta)-\lambda^-(\eta))(z-\lambda^-(\eta))}$$

The residue theorem gives:

$$\left[e^{tA(\eta)}\right]_{1,1} = \frac{\lambda^+(\eta)e^{\lambda^+(\eta)t} - \lambda^-(\eta)e^{\lambda^-(\eta)t} + \mu_\|\eta^2\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right)}{\lambda^+(\eta)-\lambda^-(\eta)}$$

$$\left[e^{tA(\eta)}\right]_{1,2} = \frac{-i\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right)}{\lambda^+(\eta)-\lambda^-(\eta)}$$

$$\left[e^{tA(\eta)}\right]_{2,1} = \frac{-ic^2\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right)}{\lambda^+(\eta)-\lambda^-(\eta)}$$

$$\left[e^{tA(\eta)}\right]_{2,2} = \frac{\lambda^+(\eta)e^{\lambda^+(\eta)t} - \lambda^-(\eta)e^{\lambda^-(\eta)t}}{\lambda^+(\eta)-\lambda^-(\eta)}$$

Besides, we have $\mu_{\|}\eta^2 = -\left(\lambda^+(\eta) + \lambda^-(\eta)\right)$.

Therefore;

$$\begin{aligned}
& \lambda^+(\eta)\, e^{\lambda^+(\eta)t} - \lambda^-(\eta)\, e^{\lambda^-(\eta)t} + \mu_{\|}\eta^2 \left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) \\
= \; & \lambda^+(\eta)\, e^{\lambda^+(\eta)t} - \lambda^-(\eta)\, e^{\lambda^-(\eta)t} - \left(\lambda^+(\eta) + \lambda^-(\eta)\right)\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) \\
= \; & \lambda^+(\eta)\, e^{\lambda^+(\eta)t} - \lambda^-(\eta)\, e^{\lambda^-(\eta)t} - \lambda^+(\eta)\, e^{\lambda^+(\eta)t} + \lambda^+(\eta) \\
& -\lambda^-(\eta)\, e^{\lambda^+(\eta)t} + \lambda^-(\eta)\, e^{\lambda^-(\eta)t} \\
= \; & \lambda^+(\eta)\, e^{\lambda^-(\eta)t} - \lambda^-(\eta)\, e^{\lambda^+(\eta)t}
\end{aligned}$$

Finally, we obtain:

$$e^{tA(\eta)} = \begin{bmatrix} \lambda^+(\eta)\, e^{\lambda^-(\eta)t} - \lambda^-(\eta)\, e^{\lambda^+(\eta)t} & -i\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) \\ -ic^2\eta\left(e^{\lambda^+(\eta)t} - e^{\lambda^-(\eta)t}\right) & \lambda^+(\eta)\, e^{\lambda^+(\eta)t} - \lambda^-(\eta)\, e^{\lambda^-(\eta)t} \end{bmatrix}$$

3.1.3 The computation of the integral

Return

In the integral

$$2C''_\sigma \int_0^{+\infty} \eta^\sigma e^{-\frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta$$

We will use this substitution: $\eta = \frac{\nu}{\sqrt{t}}$. Therefore:

$$2C''_\sigma \int_0^{+\infty} \eta^\sigma e^{-\frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta = \frac{2C''_\sigma}{\sqrt{t}} t^{-\frac{\sigma}{2}} \int_0^{+\infty} \nu^\sigma e^{-\frac{1}{2}\mu_\parallel \nu^2} \mathrm{d}\nu = \frac{C^{(3)}_\sigma}{t^{\frac{1}{2}(1+\sigma)}}$$

where:

$$C^{(3)}_\sigma = 2C''_\sigma \int_0^{+\infty} \nu^\sigma e^{-\frac{1}{2}\mu_\parallel \nu^2} \mathrm{d}\nu$$

3.1.4 The two integrations by parts and the triangle inequality

Return

Assume $|x \pm ct| \geq 1$:

$$\int_{-\infty}^{+\infty} \eta^\sigma e^{i(x\pm ct)\eta - \frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta \overset{IBP_1}{=} \frac{-1}{x \pm ct} \int_{-\infty}^{+\infty} e^{i(x\pm ct)\eta} \left[\sigma \eta^{\sigma-1} - \mu_\parallel \eta t\right] e^{-\frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta$$

$$\overset{IBP_2}{=} \frac{1}{(x \pm ct)^2} \int_{-\infty}^{+\infty} e^{i(x\pm ct)\eta} \left[\sigma(\sigma-1)\eta^{\sigma-2} - \mu_\parallel t - \mu_\parallel \eta t \left(\sigma \eta^{\sigma-1} - \mu_\parallel \eta t\right)\right] e^{-\frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta$$

Finally, the triangle inequality over the third integral gives:

$$\left| \int_{-\infty}^{+\infty} \eta^\sigma e^{i(x\pm ct)\eta - \frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta \right| \leq$$

$$\int_{-\infty}^{+\infty} \left[\sigma(\sigma-1)|\eta|^{\sigma-2} + \mu_\parallel t + \mu_\parallel \sigma |\eta|^\sigma t + \mu_\parallel^2 \eta^2 t^2\right] e^{-\frac{1}{2}\mu_\parallel \eta^2 t} \mathrm{d}\eta$$

3.1.5 The inequality using convexity

Return

Lemma 1. *Let* $p \in [1;2]$ *and* $\sigma \in \mathbb{N}$. *Assume that* $|x \pm ct| \geq 1$*:*

$$\left\{\frac{1}{(x+ct)^2}+\frac{1}{(x-ct)^2}\right\}^{p(1-\delta(p,\sigma))} \leq \frac{1}{2^{1+2p(1-\delta(p,\sigma))}}\left\{\frac{1}{|x+ct|^{2p(1-\delta(p,\sigma))}}+\frac{1}{|x-ct|^{2p(1-\delta(p,\sigma))}}\right\}$$

Proof. $\delta(p,\sigma)$ is chosen such as $2p(1-\delta(p,\sigma))>1$. As a result, the mapping $f: v \longmapsto v^{2p(1-\delta(p,\sigma))}$ is a convex function, and satisfies this property:

$$\forall (a;b) \in \mathbb{R}^2, \forall \zeta \in [0;1]\,;\quad f(\zeta a+(1-\zeta)b) \leq \zeta f(a)+(1-\zeta)f(b)$$

In particular, we have:

$$f(a+b) = f\left(\frac{2a+2b}{2}\right) \leq \frac{1}{2}\left[f(2a)+f(2b)\right] \tag{3.1}$$

Besides, we have this inequality:

$$\frac{1}{(x+ct)^2}+\frac{1}{(x-ct)^2} \leq \left(\frac{1}{|x+ct|}+\frac{1}{|x-ct|}\right)^2$$

In the inequality (3.1), we choose $a=\frac{1}{|x+ct|}$ and $b=\frac{1}{|x-ct|}$, giving this:

$$\left\{\frac{1}{(x+ct)^2}+\frac{1}{(x-ct)^2}\right\}^{p(1-\delta(p,\sigma))} \leq \left(\frac{1}{|x+ct|}+\frac{1}{|x-ct|}\right)^{2p(1-\delta(p,\sigma))}$$

$$\begin{aligned}\left\{\frac{1}{(x+ct)^2}+\frac{1}{(x-ct)^2}\right\}^{p(1-\delta(p,\sigma))} &\leq f\left(\frac{1}{|x+ct|}+\frac{1}{|x-ct|}\right)\\ &\leq \frac{1}{2}\left[f\left(\frac{2}{|x+ct|}\right)+f\left(\frac{2}{|x-ct|}\right)\right]\end{aligned}$$

A simple computation gives the result of the lemma. □

3.1.6 The integrals which don't depend on time

Return

We will only show the result for the integral:

$$\int_{|x+ct|\geq 1} \frac{\mathrm{d}x}{|x+ct|^{2p(1-\delta(p,\sigma))}}$$

The computation over the other integral is made on the same way.

We make the substitution $y = x + ct$, and we obtain:

$$\begin{aligned}
\int_{|x+ct|\geq 1} \frac{\mathrm{d}x}{|x+ct|^{2p(1-\delta(p,\sigma))}} &= \int_{|y|\geq 1} \frac{\mathrm{d}y}{|y|^{2p(1-\delta(p,\sigma))}} \\
&= 2\int_{1}^{+\infty} \frac{\mathrm{d}y}{y^{2p(1-\delta(p,\sigma))}} \\
&= \frac{2}{2p(1-\delta(p,\sigma))-1}
\end{aligned}$$

As a result, we have:

$$C(p,\sigma,\delta(p,\sigma))' = \frac{4}{(2p(1-\delta(p,\sigma))-1)\, 2^{1+2p(1-\delta(p,\sigma))}}$$

3.1.7 The duality property of the Fourier transform

Return

Proposition 16. ***Duality property of the Fourier transform***
Let $p \in [2; +\infty]$

$$\mathcal{F} : L^{p'}(\mathbb{R}) \xrightarrow{\sim} L^{p}(\mathbb{R})$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In particular, let $f \in L^{p}(\mathbb{R})$*:*

$$\exists C_f \geq 0 : \quad \|f\|_{L^p} \leq C_f \left\|\hat{f}\right\|_{L^{p'}}$$

3.1.8 The property of line integrals

Return

Proposition 17. ***Property of estimation of a line integal***
Let $U \subset \mathbb{C}$ an openset and $f : U \longrightarrow \mathbb{C}$ an holomorphic function. Let $\Gamma \subset U$ a contour. We have this property of estimation:

$$\exists C_\Gamma \geq 0 : \quad \left| \int_\Gamma f(z) \mathrm{d}z \right| \leq C_\Gamma \underset{z \in \Gamma}{Sup} |f(z)|$$

Proof. We can consider this parametrization for the contour Γ:

$$\begin{array}{cccc} \gamma : & [0;1] & \longrightarrow & U \\ & s & \longmapsto & \gamma(s) \end{array}$$

where γ is a contiuously differentiable mapping. The line integral is given by:

$$\int_\Gamma f(z) \mathrm{d}z = \int_0^1 f(\gamma(s)) \gamma'(s) \mathrm{d}s$$

As a result, we have:

$$\begin{aligned} \left| \int_\Gamma f(z) \mathrm{d}z \right| &= \left| \int_0^1 f(\gamma(s)) \gamma'(s) \mathrm{d}s \right| \\ &\leq \int_0^1 \left| f(\gamma(s)) \gamma'(s) \right| \mathrm{d}s \\ &\overset{\text{Hölder's inequality}}{\leq} \int_0^1 \left| \gamma'(s) \right| \mathrm{d}s \cdot \underset{s \in [0;1]}{Sup} \left| f(\gamma(s)) \right| \\ &\leq \int_0^1 \left| \gamma'(s) \right| \mathrm{d}s \cdot \underset{z \in \Gamma}{Sup} |f(z)| \end{aligned}$$

and we obtain our inequality by choosing

$$C_\Gamma = \int_0^1 \left| \gamma'(s) \right| \mathrm{d}s$$

We can notice as the parametrization of Γ is not unic, the constant C_Γ is not unic too. ☐

3.1.9 The general Leibniz rule

Return

Proposition 18. ***General Leibniz rule***
Let $f_1, \cdots, f_n$ with $n \in \mathbb{N}^*$ functions of class $\mathcal{C}^p$, for $p \in \mathbb{N}$. We have this formula:

$$(f_1 \cdots f_n)^{(p)} = \sum_{k_1+\cdots+k_n=p} \frac{p!}{k_1! \cdots k_n!} f_1^{(k_1)} \cdots f_n^{(k_n)}$$

Proof. We will make an induction on n.

At the step $n = 0$, the formula is clear.

We assume the property at the stp $n \in \mathbb{N}^*$. We have:

$$\begin{aligned}
(f_1 \cdots f_{n+1})^{(p)} &= (f_1 \cdots f_n f_{n+1})^{(p)} \\
&= \sum_{j=0}^{p} \binom{p}{j} (f_1 \cdots f_{n+1})^{(j)} f_{n+1}^{(p-j)} \\
&\overset{\text{Induction}}{=} \sum_{j=0}^{p} \sum_{\substack{0 \leq k_1, \cdots, k_n \leq p \\ k_1 + \cdots k_n = j}} \frac{j!}{k_1! \cdots k_n!} \frac{p!}{j!(p-j)!} \\
&\times f_1^{k_1} \cdots f_n^{(k_n)} f_{n+1}^{(p-j)} \\
&\underset{j=p-k_{n+1}}{\overset{\text{Index Change}}{=}} \sum_{k_{n+1}=0}^{p} \sum_{\substack{0 \leq k_1, \cdots, k_n \leq p \\ k_1 + \cdots k_n = p - k_{n+1}}} \frac{p!}{k_1! \cdots k_n! k_{n+1}!} \\
&\times f_1^{(k_1)} \cdots f_n^{(k_n)} f_{n+1}^{(k_{n+1})} \\
&= \sum_{\substack{0 \leq k_1, \cdots, k_n, k_{n+1} \leq p \\ k_1 + \cdots k_n + k_{n+1} = p}} \frac{p!}{k_1! \cdots k_n! k_{n+1}!} \\
&\times f_1^{(k_1)} \cdots f_n^{(k_n)} f_{n+1}^{(k_{n+1})}
\end{aligned}$$

The property holds at the step $n + 1$, which completes the proof. □

3.1.10 The estimation of the derivative of B

Return

We have these formulas:

$$B_0(t,r) = -\left(\frac{c^2}{\mu_\|}\right)e^{-\frac{c^2}{\mu_\|}t} - \frac{1}{2i\pi}\int_{S^+}[\mu_\| z + c^2]^{-2} z^2 \frac{1}{r^2}[z+\mu_\| r^2]e^{tz}\mathrm{d}z$$

$$B_k(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}[\mu_\| z + c^2]^{-(k+2)} \underbrace{\frac{1}{r^2}[z+\mu_\| r^2]}_{[r]} z^{2k+2}e^{tz}\mathrm{d}z$$

Onnly the factors in $[r]$ depend on r. Let $l \in \mathbb{N}^*$ and $k \in \mathbb{N}^*$:

$$\partial_r^l B_k(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}\frac{z^{2k+2}e^{tz}}{[\mu_\| z + c^2]^{k+2}}\sum_{m=0}^{l}\binom{l}{m}\partial_r^m[z+\mu_\| r^2]\partial_r^{l-m}\left[\frac{1}{r^2}\right]\mathrm{d}z$$

$$\partial_r^l B_k(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}\frac{z^{2k+2}e^{tz}}{[\mu_\| z + c^2]^{k+2}}\left\{[z+\mu_\| r^2]\partial_r^l\left[\frac{1}{r^2}\right] + 2l\mu_\| r\partial_r^{l-1}\left[\frac{1}{r^2}\right]\right.$$
$$\left.\frac{l(l-1)}{2}\partial_r^{l-2}\left[\frac{1}{r^2}\right]\right\}\mathrm{d}z$$

$$\partial_r^l B_k(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}\frac{z^{2k+2}e^{tz}}{[\mu_\| z + c^2]^{k+2}}\left\{[z+\mu_\| r^2]\frac{\Delta_l^{(0)}}{r^{2+l}} + 2l\mu_\| r\frac{\Delta_l^{(1)}}{r^{2+l-1}}\right.$$
$$\left.+\frac{l(l-1)}{2}\frac{\Delta_l^{(2)}}{r^{2+l-2}}\right\}\mathrm{d}z$$

$$\partial_r^l B_k(t,r) = \frac{(-1)^{k+1}}{2i\pi}\int_{S^+}\frac{z^{2k+2}e^{tz}}{[\mu_\| z + c^2]^{k+2}}\left\{\frac{\Delta_l^{(0)}z}{r^{2+l}} + \left(\Delta_l^{(0)}\mu_\| + 2l\mu_\|\Delta_l^{(1)}\right.\right.$$
$$\left.\left.+\frac{l(l-1)}{2}\Delta_l^{(2)}\right)\frac{1}{r^l}\right\}\mathrm{d}z$$

As a result, the property over the line integrals gives:

$$\left|\partial_r^l B_k(t,r)\right| \le C_l' r_0'^k e^{-\frac{c^2}{2\mu_\|}t} r^{-l}$$

Besides, a similar computation shows that:

$$\left|\partial_r^l B_0(t,r)\right| \le C_l' r_0'^k \left[e^{-\frac{c^2}{2\mu_\|}t} + e^{-\frac{\mu_\| r^2}{2}t}\right] r^{-l}$$

Which gives the result.

3.1.11 The Marcinkiewic multiplier theorem

Return

Theorem 2. ***Marcinkiewic multiplier theorem***
Assume $\widehat{f} \in L^\infty \cap \mathcal{C}^2(\mathbb{R}^*)$, with these two estimations:

$$\left|\widehat{f}^{(\alpha)}(\eta)\right| \leq C' \left\{ \begin{array}{ll} |\eta|^{-\alpha+\sigma_1}, & |\eta| \leq R, \alpha = 1 \\ |\eta|^{-\alpha-\sigma_2}, & |\eta| \geq R, \alpha \in [\![0;2]\!] \end{array} \right.$$

where $\sigma_1, \sigma_2 > 0$.

$\widehat{f}$ *is continuous at* 0 *and:*

$$f = m_1 + m_2\delta_0$$

where $m_1 \in L^1(\mathbb{R})$ *and* $\|m_1\|_{L^1} \leq C(C')$ *and* m_2 *is the constant:*

$$m_2 = \frac{1}{2\pi} \underset{|\eta|\to+\infty}{lim} \widehat{f}(\eta)$$

Besides, $\widehat{f}$ *is a strong* L^p-*multiplier, for* $p \in [1;+\infty]$*:*

$$\forall g \in L^p(\mathbb{R})\,; \quad \|f * g\|_{L^p} \leq C\,\|g\|_{L^p} \text{ for } p \in [1;+\infty]$$

and C *only depends on* m_2 *and* C'.

3.2 Chapter 2

3.2.1 The formulas for approximations of derivatives

Return

Proposition 19. ***Approximation of derivatives***
Let f *a* $\mathcal{C}^2-$*class function. We have:*

$$\begin{aligned} f'(x) &= \frac{f(x+h)-f(x)}{h} + \mathcal{O}(h) \\ f'(x) &= \frac{f(x+h)-f(x-h)}{2h} + \mathcal{O}\left(h^2\right) \\ f''(x) &= \frac{f(x+h)-2f(x)+f(x-h)}{h^2} + \mathcal{O}\left(h^2\right) \end{aligned}$$

Proof. By using the Taylor's formulas, we have:

$$f'(x+h) = f(x) + hf'(x) + \mathcal{O}\left(h^2\right)$$

which immediately gives the first formula.

Besides, we have, at the second order:

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f^{(3)}(x) + \mathcal{O}\left(h^3\right) && (3.2) \\
f(x-h) &= f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f^{(3)}(x) + \mathcal{O}\left(h^3\right) && (3.3)
\end{aligned}
$$

The equality $\frac{(3.2)-\ (3.3)}{2h}$ gives the second formula, and the equality $\frac{(3.2)+\ (3.3)}{2h}$ gives the third formula. □

3.2.2 The Fourier-Von Neumann method

Return

If we have a vector $(u_j)_{j\in[\![1;J-1]\!]}$, we can extend this vector to a sequence $(u_j)_{j\in\mathbb{Z}} \in l^2(\mathbb{Z})$ (by giving the value 0 to the other terms). Then, we consider a function $\overset{\vee}{u} \in L^2(\mathbb{T})$ (periodic, with $\mathbb{T} = \frac{\mathbb{R}}{2\pi\mathbb{Z}}$). As a result, we can write this function as a Fourier serie, and $(u_j)_{j\in\mathbb{Z}}$ is the sequence of its Fourier coefficients, giving:

$$
\forall j \in \mathbb{Z};\quad u_j \quad = \quad \int_0^1 \overset{\vee}{u}(x) e^{2i\pi jx} \mathrm{d}x
$$

Therefore, the Parseval's theorem gives:

$$
\|(u_j)\|_{l^2(\mathbb{Z})} \quad = \quad \left\|\overset{\vee}{u}\right\|_{L^2(\mathbb{T})}
$$

and study the stability for $(u_j)_{j\in\mathbb{Z}}$ is equivalent to study the stability for $\overset{\vee}{u}$.
An important consequence is this relation:

$$
\begin{aligned}
u_{j+1} &= \int_0^1 \overset{\vee}{u}(x) e^{2i\pi(j+1)x} \mathrm{d}x \\
&= \int_0^1 \overset{\vee}{u}(x) e^{2i\pi x} e^{2i\pi jx} \mathrm{d}x
\end{aligned}
$$

as a result, we have:

$$
\begin{aligned}
u_j &\longleftrightarrow \overset{\vee}{u}(x) \\
u_{j+1} &\longleftrightarrow \overset{\vee}{u}(x) e^{2i\pi x}
\end{aligned}
$$

Conclusion

This internship offered to me several opportunities:

1. On the one hand, to discover new mthematic tools and ways of reasoning, such as the Cauchy's integral formula for matrices, new properties of the Fourier transform, the Marcinkiewic multiplier theorem (and the reasoning to use it).

2. On the other hand, to perform in numerical analysis (study of schemes and programming).

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