École normale supérieure of Rennes
University of Pisa

First memoir of the internship of the 4th year of the Magistère
Year 2024-2025

Hyperbolic geometry

Presented by : Clara Genes
Supervisor : Bruno Martelli

Table des matières

In this document, we will give an introduction to hyperbolic space and, more generally, hyperbolic manifolds. Except if specified, (M, g) will always denote a Riemannian manifold, that is a differential manifold M with a metric tensor that is positive definite at every point. Let's first enunciate without a proof the Hopf-Rinow's theorem that will be used at different moments of this document. This theorem was proved during the course *Istituzioni di Geometria* that I followed in 2023 and a proof can be found in Manifolds by B. Martelli.

Theorem (Hopf-Rinow)**.** Let M be a connected Riemannian manifold. The following are equivalent :

1. M is geodesically complete (that is, every geodesic can be extended on $\mathbb{R}$ entirely) ;
2. M is complete.

The major part of this document is largely inspired by the book An introduction to Geometric Topology by B. Martelli.

1 Hyperbolic space

Note $\mathfrak{X}(M)$ the space of vector fields on M, denote $[X, Y]$ the Lie bracket of $X, Y \in \mathfrak{X}(M)$ and ∇ the Levi-Civita connection of (M, g). Recall that the *Riemann tensor* R is given by

$$\begin{array}{rcccl} R & : & \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) & \longrightarrow & \mathfrak{X}(M) \\ & & (X, Y, Z) & \longmapsto & R(X, Y)Z \coloneqq \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \end{array}$$

Let $p \in M$ be a point and $W \subset T_pM$ be a 2-dimensional vector space. Recall that the *sectional curvature* K in (u, v) generating W is given by

$$K(W) = K(u, v) = \frac{g(R(u, v)v, u)}{g(u, u)g(v, v) - g(u, v)^2}.$$

We have the fondamental following definition.

Definition 1.1. A Riemannian manifold (M, g) has a *constant sectional curvature* K if the sectional curvature of every 2-dimensional vector space $W \subset T_pM$ at every point $p \in M$ is always K, that is $K(W)$ is independent of $p \in M$ and of $W \subset T_pM$.

Remark 1.2. By rescaling the metric, we may transform every Riemannian manifold with constant sectional curvature K into one with constant sectional curvature $1, 0, -1$.

In every dimension $n \geq 2$ there exists a unique simply connected complete Riemannian manifold with sectional curvature $1, 0, -1$. These are respectively the sphere S^n, the Euclidean space $\mathbb{R}^n$ and the hyperbolic space $\mathbb{H}^n$: we are going to prove this assertion in the case of the hyperbolic space but similar ideas are used to prove it for the Euclidean space and the sphere.

In this section, we will introduce three models of the hyperbolic space : the hyperboloid, the Poincaré disk model (also called the conform disk model) and the Poincaré half-plane model. These models will give some good properties of the hyperbolic space.

1.1 The hyperboloid

As the sphere $\mathbb{S}^n$ is the set of all points with norm 1 in $\mathbb{R}^{n+1}$ with the Euclidean scalar product, the hypobolic space $\mathbb{H}^n$ can be defined as a subset of all points of norm -1 in $\mathbb{R}^{n+1}$ with the Lorentzian scalar product.

Definition 1.3. The Lorentzian scalar product on $\mathbb{R}^{n+1}$ is given by

$$\langle x, y \rangle := \sum_{i=1}^{n} x_i y_i - x_{n+1} y_{n+1}.$$

The hyperboloid model I^n is defined as the set

$$I^n = \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1, x_{n+1} > 0\}.$$

We can observe that the Lorentzian scalar product has signature $(n, 1)$ and that the hyperboloid is connected.

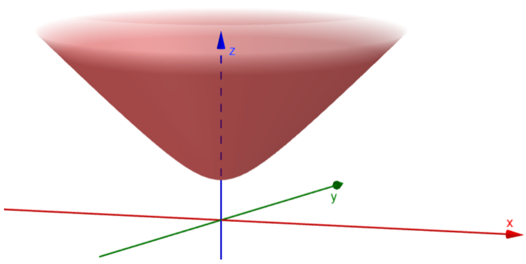

Figure 1. The upper sheet I^3 of the hyperboloid defined by the equation $\langle x, x \rangle = -1$ in dimension 3 : first model of the hyperbolic space.

Proposition 1.4. The hyperboloid I^n is a Riemannian manifold.

Proof. Let set $\mathbb{R}^{n+1}_+ = \{x \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$. As it is an open set of $\mathbb{R}^{n+1}$, it is a submanifold. Let f be the function

$$\begin{array}{rccc} f & : & \mathbb{R}^{n+1}_+ & \longrightarrow & \mathbb{R} \\ & & x & \longmapsto & \langle x, x \rangle. \end{array}$$

Then, for every $x, y \in \mathbb{R}^{n+1}_+$, holds the equality

$$\mathrm{d}f_x(y) = 2\langle x, y \rangle.$$

Hence, for all $x \in I^n$, the differential $\mathrm{d}f_x$ is a surjection and thus $I^n = f^{-1}(\{-1\})$ is a submanifold of codimension 1.

In addition, the tangent space $T_x I^n$ at $x \in I^n$ is the hyperplan orthogonal to x for the Lorentzian scalar product :

$$T_x I^n = \ker \mathrm{d}f_x = \{y \in T_x \mathbb{R}^{n+1}_+ = \mathbb{R}^{n+1} \mid \langle x, y \rangle = 0\} = x^\perp.$$

Since $\langle x, x \rangle = -1$ and $\langle x, y \rangle = 0$ for every $y \in x^\perp$, the restriction of the scalar product on $x^\perp$ is positive definite and hence defines a metric tensor on I^n. □

The hyperboloid I^n is a first model of the hyperbolic space $\mathbb{H}^n$. We will now introduce some material to prove that $\mathbb{H}^n$ is complete, simply connected and has constant curvature -1.

Definition 1.5. A k-dimensional subspace of I^n is the intersection of a $(k+1)-$dimensional vector subspace of $\mathbb{R}^{n+1}$ with I^n, when it is not empty. In particular, a 1-subspace is a line and a $(n-1)$-subspace is an hyperplane.

Proposition 1.6. A non-trivial complete geodesic in $\mathbb{H}^n$ is a line run at constant speed. Concretely, let $p \in \mathbb{H}^n$ be a point and $v \in T_p\mathbb{H}^n$ a unit vector. The geodesic γ exiting from p with velocity v is for all $t \in \mathbb{R}$ defined by

$$\gamma(t) = \cosh(t) \cdot p + \sinh(t) \cdot v.$$

Proof. Let $p \in I^n$ be a point, $v \in T_p\mathbb{H}^n$ a unit vector and γ the geodesic exiting from p with velocity v. The plane $W \subset \mathbb{R}^{n+1}$ generated by p (seen as a vector starting at the origin) and v is a 2-dimensional vector space and hence intersects I^n into a line $L = I^n \cap W$ containing p (seen as a point) and tangent to v. Noticing that $\mathbb{R}^{n+1} = W \oplus W^\perp$, we set the reflection r_L to be :

$$r_L|_W = \mathrm{id}|_W, \qquad r_L|_{W^\perp} = -\mathrm{id}|_{W^\perp}.$$

This isometry fixes W, so p and v and hence γ (since r_L is an isometry and the geodesic is unique). Thus,

$$\forall t \in \mathbb{R}, \quad \gamma(t) \in W.$$

By definition of the geodesic, it has constant speed and the support of γ is included in I^n, so it is included in L which is a line : non-trivial geodesics are lines run at constant speed.

We now consider the curve

$$\alpha(t) = \cosh t \cdot p + \sinh t \cdot v$$

for all $t \in \mathbb{R}$. We have $\alpha(0) = p$ and $\alpha'(0) = v$. It remains to prove that α parametrizes L with unit speed (hence by unicity, $\alpha = \gamma$). From

$$\langle \alpha(t), \alpha(t) \rangle = \cosh^2(t)\langle p, p \rangle + 2\cosh(t)\sinh(t)\langle p, v \rangle + \sinh^2(t)\langle v, v \rangle = -1,$$

we deduce that α parametrizes L (it is by definition contained in W). Now, the velocity is

$$\|\alpha'(t)\| = \|\sinh(t) \cdot p + \cosh(t) \cdot v\| \equiv 1.$$

It concludes. □

Corollary 1.7. The space $\mathbb{H}^n$ is complete.

Proof. The previous proposition proves that the geodesics are completes (defined on $\mathbb{R}$), hence by Hopf-Rinow's theorem, the space is complete. □

Remark 1.8. The distance between two points can easily be calculated thanks to the proposition 1.6 : let p, q be points of I^n, then $\cosh(\mathrm{d}(p,q)) = -\langle p, q\rangle$. This can be showed taking the geodesic γ starting from p at time 0 and arriving at q at time t_0 and setting $v := \gamma'(0)$.

1.2 The Poincaré disk

We now introduction a second model : the Poincaré disk.

Definition 1.9. The Poincaré disk is given by

$$D^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$$

with the Euclidean norm but with the metric tensor g^D at $x \in D^n$ given by

$$g_x^D = \left(\frac{2}{1 - \|x\|^2}\right)^2 \cdot g_x^E$$

where g^E is the Euclidean metric tensor on $D^n \subset \mathbb{R}^n$, that is the metric tensor represented by the identity matrix.

Why this metric tensor ? First, one can observe that the metric tensor character of g^E induce one on g^D. Second, we identify $\mathbb{R}^n$ with $\{x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0\}$. Let's consider the projection

$$\begin{array}{rccc} p : & I^n & \longrightarrow & D^n \\ & (x_1, \ldots, x_{n+1}) & \longmapsto & \dfrac{1}{x_{n+1} + 1}(x_1, \ldots, x_n) \end{array}$$

that is a diffeomorphism transporting the metric tensor on I^n to some metric tensor on D^n. The inverse of p is

$$\begin{array}{rccc} q : & D^n & \longrightarrow & I^n \\ & (x_1, \ldots, x_n) & \longmapsto & \dfrac{1}{1 - \|x\|^2}(2x_1, \ldots, 2x_n, 1 + \|x\|^2) \end{array}$$

and the multiplication by the factor $2/(1 - \|x\|^2)$ appears clearly. We also call the Poincaré disk as the conformal model because it is a model where the metric differs from the Euclidean one only by multiplication by a positive scalar that depends smoothly on x.

Definition 1.10. The k-subspaces in D^n are the images of k-subspaces of I^n by the projection p.

Example 1.11. In $\mathbb{R}^3$, the intersection of I^2 with the subspaces $\mathbb{R}^2 \times \{0\}$ gives an hyperbole. The image of this hyperbole by the projection p is a diameter of D^2. Hence, every diameters are 1-subspaces of D^2.

Proposition 1.12. The k-subspaces in D^n are the intersections of D^n with k-spheres and k-planes of $\mathbb{R}^n$ orthogonal to ∂D^n.

Proof. Since every k-subspace is an intersection of hyperplanes, we can restrict the proof to $k = n-1$. A hyperplan in I^n is an intersection $I^n \cap v^\perp$ for some $v \in \mathbb{R}^{n+1}$ spacelike vector (*i.e.*$\langle v, v\rangle > 0$).

1. If v has its last coordinate equal to zero, then for all $x \in I^n$, we have $x \in v^\perp$ if and only if $p(x) \in v^\perp$. Hence $p(I^n \cap v^\perp) = D^n \cap v^\perp$ which is a hyperplan orthogonal to ∂D^n.

2. Otherwise, up to rescaling and rotating around x_{n+1}, we may suppose $v = (\alpha, 0, \dots, 0, 1)$ with $\alpha > 1$ to get a spacelike vector. The hyperplan of I^n given by the orthogonal of v is

$$I^n \cap v^\perp = \{x \in \mathbb{R}^{n+1} \,|\, x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = -1\,,\, x_{n+1} = \alpha x_1\}.$$

We now want to find the projection of this set on D^n. The following are equivalent :

- A point y of $\mathbb{R}^n$ is in D^n ;
- There is a point $x \in I^n$ such that $y = p(x)$;
- There is an $x \in I^n$ such that $y = \dfrac{1}{x_{n+1}+1}(x_1, \dots, x_n)$;
- There is an $x \in I^n$ such that $\|y\|^2 = \dfrac{x_{n+1}-1}{x_{n+1}+1}$ and $y_1 = \dfrac{x_1}{1+x_{n+1}}$.

Hence, as y is a point of D^n and p is a surjection, such a x always exists. In particular, as a point $x \in I^n$ is orthogonal to v if and only if $x_{n+1} = \alpha x_1$, we get the equivalence

$$\left[y \in D^n \text{ and } \|y\|^2 = -1 + 2\alpha y_1\right] \Longleftrightarrow \left[\exists x \in I^n, y = p(x) \text{ and } x \in v^\perp\right]$$

and finally

$$p(I^n \cap v^\perp) = S^{n-1}_{eucl}((\alpha, 0 \dots, 0), \sqrt{\alpha^2 - 1}) \cap D^n$$

. The $(n-1)$-sphere is orthogonal to ∂D^n.

□

1.3 The half-space

We introduce here a third model, that is also a conformal model.

Definition 1.13. A diffeomorphism $f : M \to N$ between two oriented Riemannian manifolds is conformal (resp. anticonformal) if for any $p \in M$ the differential $\mathrm{d}f_p$is the product of a scalar $\lambda_p > 0$ and an isometry that preserves (resp. inverts) the orientation. The scalar must depend smoothly on p.

Remark 1.14. In a conformal model (that is a model based on a conformal metric), the lengths of vectors change from the Euclidean ones by multiplication by $\sqrt{\lambda_p}$ but the angles formed by two adjacent vectors are preserved.

Let first define some special geometric transformation : inversions.

Definition 1.15. Let $S = S(x_0, r)$ be the sphere in $\mathbb{R}^n$ centered in x_0 and with radius r. The inversion along S is the map

$$\begin{array}{cccc} \varphi & : & \mathbb{R}^n \backslash \{x_0\} & \longrightarrow & \mathbb{R}^n \backslash \{x_0\} \\ & & x & \longmapsto & x_0 + r^2 \dfrac{x - x_0}{\|x - x_0\|^2}. \end{array}$$

As the map can be extended continuously to $\mathbb{R}^n \cup \{\infty\}$ setting $\varphi(x_0) = \infty$ and $\varphi(\infty) = x_0$, we can define φ on a sphere that is identified with $\mathbb{R}^n \cup \{\infty\}$ through the stereographic projection. Note that for all $x \in \mathbb{R}^n$ we have

$$\|\varphi(x) - x_0\| \|x - x_0\| = r^2.$$

Intuitively (see Figure 2. below), the closer x is to the center x_0, the closer $\varphi(x)$ is to ∞ and vice-versa. Moreover, from the definition of φ we can see that every $x \neq x_0$ is co-linear to $\varphi(x)$. Thus in 2-dimension, if we imagine a line as a circle of radius infinite (or containing the point ∞), we can see that an inversion sends a circle S on a circle S' and, in particular, if $x_0 \in S$, then $\infty \in S'$ and S' is a line and vice-versa.

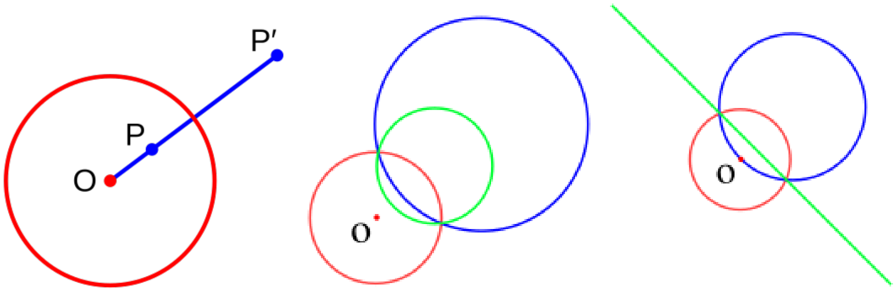

Figure 2. The inversion through a sphere of center O and radius r moves P to P' so that $OP \times OP' = r^2$ (left).It transforms a 2-sphere (blue) in a 2-sphere (green) if the blue sphere doesn't meet O (center), or in a 1-plan (green) if it meets it (right).

In 3-dimension, we have the same intuition with spheres and planes. The following proposition is the generalization of this intuition.

Proposition 1.16. The following holds :

1. Every inversion is a smooth and anticoformal map.
2. Inversions send k-spheres and k-planes to k-spheres and k-planes.

Proof. 1. Up to conjugating with translations and dilatations we may suppose $x_0 = 0$ and $r = 1$ and thus

$$\varphi(x) = \frac{x}{\|x\|^2}.$$

Algebraically, a map is anticonformal if at every point its Jacobian is a product of a positive scalar and an orthogonal matrix with negative determinant. Let's compute the Jacobian of φ in x :

$$\frac{\partial \varphi_i}{\partial x_j} = \frac{\delta_{ij}\|x\|^2 - 2x_i x_j}{\|x\|^4} \qquad \text{implies} \qquad Jac(\varphi)_x = \frac{1}{\|x\|^2}\left(I_n - \frac{2}{\|x\|^2}xx^T\right) =: \frac{1}{k}Q.$$

The matrix Q is symmetric and is such that

$$QQ^T = I_n - \frac{4}{k}xx^T + \frac{4}{k^2}kxx^T = I_n.$$

In addition, since the eigenvalues of xx^T are $\{k, 0, \dots, 0\}$, the eigenvalues of Q are $\{-1, 1, \dots, 1\}$ and so the determinant of Q is -1. Hence, the matrix $Jac(\varphi)_x$ is the product of the positive scalar $1/\|x\|^2$ and an orthogonal matrix of negative determinant.
2. This proof is not really relevant for the subject as it is globally a Euclidean geometric proof in dimension 2. A proof can be found in section 37 of Geometry : Euclid and Beyond by Robin Hartshorne. □

We can now introduce the half-space model.

Definition 1.17. Let the half-space model be the space

$$H^n = \{x \in \mathbb{R}^n \,|\, x_n > 0\}$$

obtained from the disk model D^n by the inversion in $\mathbb{R}^n$ with center $C = (0, \ldots, 0, -1)$ and radius $r = \sqrt{2}$

The definition makes sense because, in $\mathbb{R}^n$, the disk model is an n-sphere that passes through the center C : its inversion is an n-plane. The boundary ∂D^n that is the $(n-1)-$sphere $\mathbb{S}^{n-1}$ (which passes through C) is send to ∂H^n that is the $(n-1)$-plane $\{x_n = 0\} \cup \{\infty\}$. It is shown in dimension 2 in the following figure.

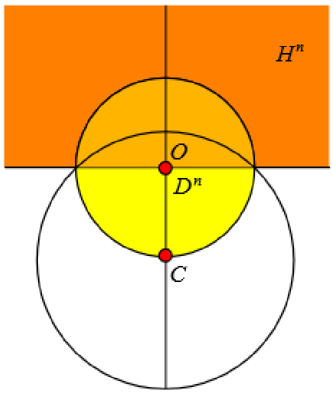

Figure 3. The inversion along the sphere with center $C = (0, -1)$ and radius $r = \sqrt{2}$ transforms the Poincaré disk D^2 into the half-space model H^2.

Proposition 1.18. The hyperbolic space $\mathbb{H}^n$ is simply connected

Proof. There are homeomorphisms between each of the three models of $\mathbb{H}^n$ and an open ball which is simply connected. □

Definition 1.19. The k-subspaces of H^n are the inversion of the one in D^n.

We have in particular the following proposition.

Proposition 1.20. The k-subspaces of H^n are the k-planes and k-spheres in $\mathbb{R}^n$ orthogonal to ∂H^n.

Proof. The orthogonality comes from the fact that the inversion is an anticonformal map and hence preserves angles. □

As it was done for the disk, let give a look at the metric tensor of the half-space model. Recall that the metric tensor of the disk model is $g_x^D = \left(\frac{2}{1-\|x\|^2}\right) \cdot g_x^E$ at every point x of the disk.

Proposition 1.21. The metric tensor on H^n at a point x is given by

$$g_x^H = \frac{1}{x_n^2} \cdot g_x^E$$

with $g^E$the Euclidean metric tensor on $H^n \subset \mathbb{R}^n$.

Proof. The inversion $\varphi : D^n \to H^n$ that defines H^n is the anticonformal map

$$\begin{aligned}\varphi(x_1, \ldots, x_n) &= C + r^2 \frac{x - C}{\|x - C\|^2} = (0, \ldots, 0, -1) + 2 \frac{(x_1, \ldots, x_{n-1}, x_n + 1)}{\|(x_1, \ldots, x_{n-1}, x_n + 1)\|^2} \\ &= \frac{(2x_1, \ldots, 2x_{n-1}, -\|x\|^2 + 1)}{\|x\|^2 + 2x_n + 1}\end{aligned}$$

where the norm is the Euclidean one. As seen in the proof of proposition 1.16, the Jacobian can be written as the multiplication of the scalar dilatation of the inversion $r^2/\|x - C\|^2$ that is here

$$\frac{1}{\beta} := \frac{2}{\|x\|^2 + 2x_n + 1}$$

by an orthogonal matrix $Q = \{(q_{ij})\} := \left\{\left(\frac{1}{\beta}\frac{\partial \varphi_i}{\partial x^j}\right)\right\}$. Hence, the metric tensor on H^n is given by the pullback

$$g_x^H = (\varphi^* g_x^D) = \left(\frac{2}{1 - \|x\|^2}\right)^2 \cdot (\varphi^* g_x^E).$$

where the pullback is

$$(\varphi^* g_x^E)_{ij} = \sum_{k,l=1}^{n} g_{x,kl}^E \frac{\partial \varphi_k}{\partial x^i} \frac{\partial \varphi_l}{\partial x^j} = \sum_{k,l=1}^{n} \delta_{kl} \frac{\partial \varphi_k}{\partial x^i} \frac{\partial \varphi_l}{\partial x^j} = \sum_{k=1}^{n} \frac{\partial \varphi_k}{\partial x^i} \frac{\partial \varphi_k}{\partial x^j} = \beta \sum_{k=1}^{n} q_{ki} q_{kj} = \beta \delta_{ij} = \beta^2 g_{x,ij}^E$$

Thus,

$$g_x^H = \left(\frac{2}{1 - \|x\|^2}\right)^2 \cdot \left(\frac{\|x\|^2 + 2x_n + 1}{2}\right)^2 g_x^E = \frac{1}{\varphi_n(x)^2} g_x^E.$$

□

Let's have a look at the geodesics.

Definition 1.22. A vertical geodesic in the half-space model H^n is a geodesic that remains constant in the horizontal directions $(x_1, \ldots, x_{n-1})$ and varies only in the vertical direction $x_n > 0$. Its general form is

$$\gamma(t) = (x_1, \ldots, x_{n-1}, y(t)),$$

where the first $n - 1$ coordinates are constants and $y(t)$ is always strictly positive.

Actually, we have the following proposition.

Proposition 1.23. The vertical geodesic in H^n passing through the point $(x_1, \ldots, x_{n-1}, 1)$ at time $t = 0$ pointing upward with unit speed is

$$\gamma(t) = (x_1, \ldots, x_{n-1}, \mathrm{e}^t).$$

Proof. Given the initial point $(x_1, \dots, x_{n-1}, 1)$ and the initial speed 1 (with the metric g^H), there exist a unique vertical geodesic satisfying those initial data. The function γ satisfies

$$\begin{cases} \gamma(0) = (x_1, \dots, x_{n-1}, 1) \\ \|\gamma'(t)\|_H = \ \|(0, \dots, 0, \mathrm{e}^t)\|_H = \ \dfrac{1}{\mathrm{e}^t}\|(0, \dots, 0, \mathrm{e}^t)\|_E = \ 1. \end{cases}$$

It remains to prove that γ is a geodesic. The proposition 1.21 gives the metric of the half-space model. Hence the Christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2}\sum_m g_x^{H,im}\left(\frac{\partial g^H_{x,mj}}{\partial x^k} + \frac{\partial g^H_{x,mk}}{\partial x^j} - \frac{\partial g^H_{x,jk}}{\partial x^m}\right)$$

are reduced to

$$\begin{cases} \Gamma^n_{ii} = \dfrac{1}{x_n} & \text{for all } i = 1, \dots, n-1 \\ \Gamma^i_{ni} = \Gamma^i_{in} = -\dfrac{1}{x_n} & \text{for all } i = 1, \dots, n \\ \Gamma^k_{ij} = 0 & \text{otherwise} \end{cases}$$

Moreover, the derivatives of γ are $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_i = 0$ if $i \neq n$ and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_n = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma_n = \mathrm{e}^t$. Thus, the formula of the geodesic becomes

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma_i(t) + \displaystyle\sum_{j,k}\Gamma^i_{jk}\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k = 0 + \Gamma^i_{nn}\mathrm{e}^{2t} = 0 & \text{if } i \neq n; \\ \dfrac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma_n(t) + \displaystyle\sum_{j,k}\Gamma^n_{jk}\frac{\mathrm{d}}{\mathrm{d}t}\gamma_j\frac{\mathrm{d}}{\mathrm{d}t}\gamma_k = \mathrm{e}^t - \Gamma^n_{nn}\mathrm{e}^{2t} = \mathrm{e}^t - \mathrm{e}^t = 0. \end{cases}$$

Hence the function satisfies the geodesic equation and is the unique vertical geodesic passing through the initial data given. □

The hyperbolic tangent and its derivative are, for ervy $t \in \mathbb{R}$:

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{\mathrm{e}^t + \mathrm{e}^{-t}}, \qquad \tanh'(t) = 1 - \tanh^2(t).$$

Proposition 1.24. The geodesic in D^n passing through the origin at time $t = 0$ and pointing towards $x \in \mathbb{S}^{n-1}$ at unit speed is

$$\gamma(t) = \frac{\mathrm{e}^t - 1}{\mathrm{e}^t + 1} \cdot x = \tanh\left(\frac{t}{2}\right) \cdot x.$$

Proof. The function γ verifies $\gamma(0) = 0$, and

$$\|\gamma'(t)\|_D = \left(\frac{2}{1 - \|\gamma(t)\|_E^2}\right)^2 \|\gamma'(t)\|_E = \left(\frac{2}{2\tanh'\frac{t}{2}}\right)^2\left(\tanh'\frac{t}{2}\right)^2 = 1.$$

We can make the same proof as the previous one or more simply observe that we can suppose that $x = (0, \dots, 0, 1)$ and obtain the curve γ from the vertical line in H^n through inversion. □

We obtain in particular the expression of the exponential map that is defined by the following definition.

Definition 1.25. Let p be a point of a smooth manifold M and v a vector of T_pM. There is a unique geodesic γ_v such that $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. We call I_v the domain of the geodesic γ_v and V the subset of the tangent bundle

$$V = \{v \in TM \,|\, 1 \in I_v\}.$$

The exponential map is defined by

$$\begin{array}{rcccc} \exp & : & V & \longrightarrow & M \\ & & v & \longmapsto & \gamma_v(1). \end{array}$$

Remark 1.26. Let's recall that $\exp_p : v \in T_pM \mapsto \gamma_v(1) \in M$ is a local diffeomorphism at the origin since its differential is the identity. Let's call the *invectivity radius* $\mathrm{inj}_p(M) > 0$ of M at a point p of M the supremum of all $r > 0$ such that $\exp_p|_{B(0,r)}$ is a diffeomorphism onto its image.

Corollary 1.27. The exponential map $\exp_0 : T_0D^n \to D^n$ at the origin is the diffeomorphism

$$\exp_0(x) = \frac{\mathrm{e}^{\|x\|} - 1}{\mathrm{e}^{\|x\|} + 1} \cdot \frac{x}{\|x\|} = \left(\tanh \frac{\|x\|}{2}\right) \cdot \frac{x}{\|x\|},$$

with the Euclidean norm.

Proof. Let x be a point of T_0D^n, the vector $v = x/\|x\|$ is in $\mathbb{S}^{n-1}$ and the previous proposition gives the geodesic $\gamma_v(t\|x\|)$. The proposition results from the definition of the exponential map. □

It remains to prove that the hyperbolic space has constant sectional curvature -1. The area of a disc is necessary to obtain a proof.

Proposition 1.28. The disc of radius r in $\mathbb{H}^2$ has area

$$A(r) = \pi\left(\mathrm{e}^{r/2} - \mathrm{e}^{-r/2}\right)^2 = 4\pi \sinh^2 \frac{r}{2} = 2\pi\left(\cosh r - 1\right).$$

Proof. The volume form is

$$\omega = \sqrt{\det g^D} \;\cdot\; \mathrm{d}x_1 \cdots \mathrm{d}x_n,$$

where the square-root means the square-root of the determinant of the matrix associated. Let $D(r)$ be a disc in $\mathbb{H}^n$ of radius r. If we center it in 0 in the disc model, its Euclidean radius is $\tanh \frac{r}{2}$ by the previous observations. Thus,

$$\begin{aligned} A(r) &= \int_{D(r)} \sqrt{\det g^D} \;\cdot\; \mathrm{d}x_1\mathrm{d}x_2 = \int_{D(r)} \left(\frac{2}{1 - x^2 - y^2}\right)^2 \mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\int_0^{\tanh r/2} \left(\frac{r}{1 - s^2}\right)^2 s \cdot \mathrm{d}s\mathrm{d}\theta = 2\pi\left[\frac{2}{1 - s^2}\right]_0^{\tanh r/2} \\ &= 4\pi \sinh^2\left(\frac{r}{2}\right). \end{aligned}$$

□

Corollary 1.29. The hyperbolic space $\mathbb{H}^n$ has sectional curvature -1.

Proof. For a ball $B(p,\varepsilon)$, the following formula gives a connexion between its area and the gaussian curvature of a surface :

$$Area(B(p,\varepsilon)) = \pi\varepsilon^2 - \frac{\pi\varepsilon^4}{12}K + O(\varepsilon^4).$$

Observe first that in $\mathbb{H}^2$ we can already conclude :

$$A(r) = 2\pi(\cosh r - 1) = 2\pi\left(\frac{r^2}{2} + \frac{r^4}{4!} + o(r^4)\right) = \pi r^2 + \frac{\pi r^4}{12} + o(r^4),$$

which gives $K = -1$. In the more general case, let's pick $p \in \mathbb{H}^n$ and $W \subset T_p\mathbb{H}^n$ a 2-dimensional subspace. The image $\exp_p(W)$ is diffeomorph to the hyperbolic plan $\mathbb{H}^2$ and hence has constant sectional curvature -1. □

2 Hyperbolic manifolds

Once proved the fact that the hyperbolic space is a complete simply connected Riemmanian manifold with constant sectional curvature -1, an interesting fact is to understand how are made the manifolds that have this kind of properties.

2.1 The group of isometries

2.1.1 Isometries and coverings

Definition 2.1. A diffeomorphism $f : M \to N$ between two Riemannian manifolds (M, g) and (N, h) is an isometry if it preserves that scalar product :

$$\forall p \in M, \forall(v,w) \in T_pM, \quad g(v,w) = h(\mathrm{d}f_p(v), \mathrm{d}f_p(w)).$$

The map f is said to be a local isometry if every point $p \in M$ has an open neighbourhood U such that $f|_U$ is an isometry onto its image.

Let's introduce some properties that will be usefull for the second subsection.

Theorem 2.2. Let $f, g : M \to N$ be two isometries (resp. local isometries) between two connected Riemannian manifolds. If there is a point $p \in M$ such that $f(p) = g(p)$ and $\mathrm{d}f_p = \mathrm{d}g_p$, then $f = g$ everywhere.

Proof. The proof consists of demonstrate that the set $S = \{p \in M \mid f(p) = g(p), \mathrm{d}f_p = \mathrm{d}g_p\}$ is open and closed. The fact that S is closed is obvisous since the equality conditions are closed. Remains to prove that every $p \in S$ has an open neighbourhood $V_p \subset S$. By the remark 1.26, there is an open neighbourhood $U_p \subset T_pM$ of the origin where $\exp_p$ is a diffeomorphism onto its image. Let's show that the open set $V_p \coloneqq \exp_p(U_p)$ is entirely contained in S. First, recall that p is contained in V_p since it is the image of 0. A point $q \in V_p$ is the image of a vector $v \in U_p$ hence $q = \exp_p(v) = \gamma_v(1)$. The maps f and g are isometries (resp. local isometries) and hence send geodesics to geodesics : here $f \circ \gamma_v$ and $g \circ \gamma_v$ start from

$$f \circ \gamma_v(0) = f(p) = g(p) = g \circ \gamma_v(0)$$

with the velocity
$$\mathrm{d}f_{\gamma_v(0)} \circ \gamma_v'(0) = \mathrm{d}f_p(v) = \mathrm{d}g_p(v) = \mathrm{d}g_{\gamma_v(0)} \circ \gamma_v'(0).$$
Thus they are the same geodesic : $f(q) = g(q)$ and so $f|_{V_p} = g|_{V_p}$ thus their differential also coincide and finally $V_p \subset S$. □

Corollary 2.3. Let $U, V \subset \mathbb{H}^n$ be two connected open subsets, and let $f : U \to V$ be a local isometry. Then f extends uniquely to a global isometry $\widetilde{f} : \mathbb{H}^n \to \mathbb{H}^n$.

Sketch of the proof. The map f is necesseraly differentiable and analytic. Linking $x \in \mathbb{H}^n$ to $y \in U$ by a geodesic (possible since $\mathbb{H}^n$ is complete), we can extend f along the geodesic to a map defined on open neighbors of x that preserves the metric. The unicity of the extension comes from the previous proposition. □

Proposition 2.4. Let G act on a Hausdorff connected space X. The following are equivalent :

1. G acts freely and properly discontinously ;
2. the quotient X/G is Hausdorff and $X \to X/G$ is a covering.

Sketch of the proof. Let's call π the map $\pi : X \to X/G$

1. ⇒ 2. Let's take x and y in two differents orbits and let's consider U_x, U_y open neighbors of x, y relatively compacts. Applying the definition of properly discontinuous to the adherence of $U_x \cup U_y$, the open U_x intersects only a finite numbers of translated of U_y : note $a_1, \dots, a_n \in G$ the elements such that $U_x \cap a_i U_y \neq \emptyset$. As X is Hausdorff, there exists open neighbors U_x^i, U_y^i of x, y such that $U_x^i \cap a_i U_y^i = \emptyset$. Consider $U = U_x \bigcap_i U_x^i$ and $V = U_y \bigcap_i U_y^i$. By consrtuction, U and V are open neighbors that do not intersect any translated : the quotient is Hausdorff. The quotient X/G as the quotient topology and π is a continuous surjection. By both hypothesis on the action, the map π is locally an injection. Hence, for every $y \in X/G$ there is an $x \in X$ such that $\pi(x) = y$ and an open neighbor U of x such that $\pi|_U$ is an homeomorphism. Thus, π is a covering.

2. ⇒ 1. The quotient is Hausdorff : if x, y are from different orbits, there are two open disjoint sets U, V such that $x \in U$ and $y \in V$. Thus, for every couple $(x, y) \in X \times X$, there are U, V such that $g \cdot U \cap V \neq \emptyset$ The action is properly discontinuous : let K be a compact subset of X. A cover of $K \times K$ is a finite number of open sets $U \times V$ verifying the previous property. The action is free : for every $x \in X$, for every $g \in G \backslash \{e\}$ we have $g \cdot x \neq x$ by the fact that π is a covering (comes from the definition and the injectivity of the restrictions).

□

Remark 2.5. With similar arguments, we can show that if 1. is valid, the group X/G inherits an unique structure of manifold such that the quotient map $\pi : X \to X/G$ is a local diffeomorphism.

Lemma 2.6. Consider M a Riemannian manifold. Let Γ be a subgroup of $\mathrm{Isom}(M)$ that acts freely and properly discontinuously on M. There is a unique Riemannian structure on the manifold M/Γ such that the covering $\pi : M \to M/\Gamma$ is a local isometry.

Proof. The fact that M/Γ is a manifold come from remark 2.5. Let $U \subset M/\Gamma$ be a well-covered set : we have $\pi^{-1}(U) = \sqcup_{i \in I} U_i$ and $\pi|_{U_i}$ an homeomorphism for every $i \in I$. Pick an $i \in I$ and transport along π the smooth and Riemannian structure of $U_i \subset M$ induced by the one on U (induced itself by the one on M). The resulting structure on U does not depend on the chosen i because the open sets $\{U_i\}_{i\in I}$ are related by isometries of Γ that preserve the metric. The uniqueness of the Riemannian structure comes from the fact that we define it from the one on M that makes π a local isometry. $\square$

Proposition 2.7. Let $f : M \to N$ be a local isometry. The following hold :

1. If f is a covering, then M is complete if and only if N is complete.
2. If M is complete, then f is a covering.

Proof.

1. Since f is a local isometry, every geodesic in M projects to a geodesic in N. If f is also a covering, the converse holds : every geodesic in N can be lifted to a geodesic in M. Hence, every geodesics in M can be extended to $\mathbb{R}$ is and only if every geodesic in N can and we can conclude using Hopf-Rinow's theorem.
2. Since M is complete, every geodesic in N can be lifted to a geodesic in M. Let's show that the ball $B = B(p, \mathrm{inj}_p N)$ is a well covered open set for all $p \in N$, that is

$$f^{-1}(B(p, \mathrm{inj}_p N) = \bigsqcup_{q\in f^{-1}(p)} B(q, \mathrm{inj}_p N).$$

Let's do that by double inclusion.

- $\subset$ Given a point $r \in f^{-1}(B)$, the geodesic in B connecting $f(r)$ to p lifts to a geodesic connecting r to some $q \in f^{-1}(p)$;
- $\supset$ For every $q \in f^{-1}(p)$, the map f sends geodesics exiting from q to ones exiting from p : it sends isometrically $B(q, \mathrm{inj}_p(N))$ onto B.

$\square$

2.1.2 Isometries in dimensions two and three

Consider the Riemann sphere $S = \mathbb{C} \cup \{\infty\}$ homeomorphic to $\mathbb{S}^2$. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on S as follows : a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$ determines the Möbius transformation :

$$z \mapsto \frac{az+b}{cz+d},$$

which is an orientation-preserving diffeomorphism of S and the Möbius anti-transformation :

$$z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d},$$

which is an orientation-reversing diffeomorphism of S.

Remark 2.8. The composition of two anti-transformations is a Möbius transformation.

Definition 2.9. These two types of maps together form a group $\mathrm{Conf}(S)$.

Let's now consider the following :

- the half-plane $H^2 \subset \mathbb{C}$ as $H^2 = \{z \in \mathbb{C} \,|\, \Im m(z) > 0\}$;
- the half-space $H^3 \subset \mathbb{C} \times \mathbb{R} = \{(z,t) \,|\, z \in \mathbb{C}, t \in \mathbb{R}\}$ as $H^3 = \{(z,t) \,|\, t > 0\}$;

With those identifications, let's denote $\text{Conf}(H^2)$ the subgroup of $\text{Conf}(S)$ consisting of all maps that preserve H^2 and let's identify ∂H^3 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$.

Proposition 2.10. The following hold :

1. $\text{Isom}(H^2) = \text{Conf}(H^2)$ and in particular $\text{Isom}^+(H^2) = \text{PSL}_2(\mathbb{R})$;
2. $\text{Isom}(H^3) = \text{Conf}(S)$ and in particular $\text{Isom}^+(H^3) = \text{PSL}_2(\mathbb{C})$.

Sketch of the proof. Let's first observe that the group $\text{Isom}(H^n)$ is generated by inversions and reflections since it is generated by the reflections along hyperplanes introduced in the proof of proposition 1.6.

1. Both groups are generated by inversions along circles and reflections along lines orthogonal to $\partial H^2 = \mathbb{R}$. In particular, $\text{Isom}^+(H^2) = \text{PSL}_2(\mathbb{R})$ since $\text{Isom}^+(H^2)$ is a subgroup of $\text{Isom}(H^2)$ of index two and the Möbius transformations in $\text{Conf}(H^2)$ is a subgroup of index two naturally isomorphic to $\text{PSL}_2(\mathbb{R})$.
2. The group $\text{Isom}(H^3)$ is generated by inversions along spheres and reflections along planes orthogonal to ∂H^3. Their traces are inversions along circles and reflections along lines in S. These generate $\text{Conf}(S)$. In this case, the Möbius transformations in $\text{Conf}(S)$ is a subgroup of index two naturally isomorphic to $\text{PSL}_2(\mathbb{C})$.

$\square$

2.2 Unicity up to isometry

Definition 2.11. A hyperbolic manifold is a connected Riemannian n-manifold that may be covered by open sets isometric to open sets of $\mathbb{H}^n$.

Remark 2.12. A hyperbolic manifold has constant sectional curvature -1. The cover by open balls together with their isometries form a special case of atlas (where every open set inherits of the hyperbolic geometry) : hence, the cover can always be chosen countable.

Theorem 2.13. Every complete simply connected hyperbolic manifold M is isometric to $\mathbb{H}^n$.

Proof. Pick a point $x \in M$ and choose an isometry $D : U \to V$ between an open ball $U \subset M$ containing x and an open ball $V \subset \mathbb{H}^n$. We show that D extends (uniquely) to an isometry $D : M \to \mathbb{H}^n$. The idea is the same as the one in 2.3 : extend the isometry along a curve that join a point $x \in U$ and a point $y \in M \backslash U$.
For every $y \in M$, choose an arc

$$\begin{array}{cccc} \alpha & : & [0,1] & \longrightarrow & M \\ & & 0 & \longmapsto & x \\ & & 1 & \longmapsto & y \end{array}$$

that always exists by simple-connectedness. Since M is supposed to be hyperbolic, it is locally isometric to $\mathbb{H}^n$: set $\{U_i\}_{i\in\mathbb{N}}$ the open cover of M such that for each $i \in \mathbb{N}$ there exists an isometry $D_i : U_i \to V_i$ with V_i open balls of $\mathbb{H}^n$. By compactness of $[0,1]$, there is a partition $0 = t_0 < \cdots < t_k = 1$ such that $\alpha([t_i, t_{i+1}]) \subset U_i$ for every $i \in \mathbb{N}$. We may suppose that $U_0 \subset U$ and $D_0 = D|_{U_0}$.

Inductively on $i \in \mathbb{N}$, we want to modify D_i so that D_{i-1} and D_i coincide on the component C of $U_{i-1} \cap U_i$ containing $\alpha(t_i)$ as shown in Figure 3.

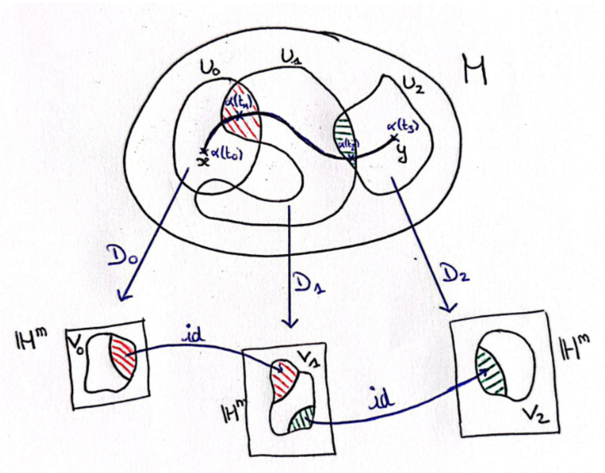

Figure 3.

To do so, note that

$$D_{i-1} \circ D_i^{-1} : D_i(C) \to D_{i-1}(C)$$

is an isometry of open connected sets in $\mathbb{H}^n$ and hence extends to an isometry $\widetilde{D}_i$ of $\mathbb{H}^n$ (Corollary 2.3). Then considering $D'_i \coloneqq \widetilde{D}_i \circ D_i$ we have to isometries D_{i-1} and D'_i that coincide on C. Finally we define $D(y) = D_{k-1}(y)$.

Remains to prove that $D(y)$ does not depend of the partition chosen of $[0,1]$: if β is another such path, the simple connectedness of M furnishes an homotopy connecting α and β. The image of the homotopy is compact and hence covered by finitely many open balls U_i that are isometric to some $V_i \subset \mathbb{H}^n$ via some isometries D_i. By the Lebesgue number theorem, there is a number $N > 0$ such that in the grid of $[0,1]^2$ made of $\frac{1}{N^2}$ squares, the image of each square is entirely contained in at least one U_i. We now modify the maps D_i as above inductively on the grid so that they all glue up and show that $D(y)$ does not depend on the path.

The resulting map $D : M \to \mathbb{H}^n$ is a local isometry by construction : since M is complete, by the proposition 2.7 D is a covering. Thus, it is a global isometry since $\mathbb{H}^n$ is simply connected. □

Remark 2.14. The isometry constructed in the proof above is called a *developing map*. The same proof shows that every complete simply connected flat (or elliptic) n-manifold is isometric to $\mathbb{R}^n$ (or $\mathbb{S}^n$).

This theorem determines the unique complete simply connected hyperbolic manifold up to isometry. Let's now look at complete hyperbolic manifolds with arbitrary fundamental group.

Definition 2.15. Let $p : \widetilde{X} \to X$ be a covering map between two path-connected topological spaces. A deck trasformation or automorphism for p is a homeomorphism $f : \widetilde{X} \to \widetilde{X}$ such that $p \circ f = p$. The deck transformations form a group $\mathrm{Aut}(p)$ called the deck transformation group of p.

Theorem 2.16. Every complete hyperbolic n-manifold M is isometric to $\mathbb{H}^n/\Gamma$ for some subgroup Γ of $\mathrm{Isom}(\mathbb{H}^n)$ acting freely and properly discontinuously.

Proof. The universal covering $\widetilde{M}$ of M inherits a Riemannian structure that is complete (by the proposition 2.7) hyperbolic (because $\widetilde{M}$ inherits the metric from M) and simply connected (by definition). By the previous theorem, $\widetilde{M}$ is isometric to $\mathbb{H}^n$. The deck transformations Γ of the covering $\mathbb{H}^n \to M$ are necessarily local isometries because the deck transformations of the universal covering are local diffeomorphisms and thus preserve the local structure. Therefore they are global isometries since they are bijections that are locally isometries. The group Γ acts freely and properly discontinuously and thus we conclude that $M = \mathbb{H}^n/\Gamma$ from the lemma 2.6. □

Remark 2.17. Note that Γ is isomorphic to the fundamental group $\pi_1(M)$. Hence, the fundamental group of a complete hyperbolic manifold has no torsion since a subgroup of $\text{Isom}(H^n)$ that acts freely and properly discontinuously has no torsion.

There is no classification of simply connected non-complete hyperbolic manifolds. However, the first part of the proof of the theorem 2.13 still applies and provides the following proposition.

Proposition 2.18. Let M be a non-complete simply connected hyperbolic n-manifold. There is a local isometry $D : M \to \mathbb{H}^n$ which is unique up to post-composing with a element of $\text{Isom}(\mathbb{H}^n)$.

Proof. Construct D as in the proof of the theorem 2.13 : the completeness of M was used only to apply the proposition 2.7 to show that the developing map was a covering.

As a local isometry, by definition the map D is determined by its first order behaviour at any point $p \in M$: it is unique up to post-composing with an isometry of $\mathbb{H}^n$. □

Bibliography

Martelli, B. (2023). *An Introduction to Geometric Topology.* Independently published, 3rd edition.

Martelli, B. (2024). Manifolds. `https://people.dm.unipi.it/martelli/didattica/matematica/2024/Manifolds.pdf`. [Online ; Accessed on 01-12-2024].