University of Rennes
University of Pisa

Memoir of the internship of the 4^{th} year of the Magistère
Year 2024-2025

An introductory study of low-dimensional manifold classification

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Introduction

The classification of manifolds varies greatly with dimension, revealing a rich and diverse landscape in low-dimensional topology.

In dimension one, the picture is crystal clear: every connected smooth 1-manifold is diffeomorphic to either an interval of the real line or the circle. This complete classification reflects the simplicity of one-dimensional topology. Its proof is given for completeness of the document.

In dimension two, the structure becomes richer, yet remains entirely classifiable. Closed orientable surfaces are distinguished up to homeomorphism by their Euler characteristic, a topological invariant that not only classifies them but also determines the type of geometry they can admit. A closed orientable surface supports a spherical, Euclidean, or hyperbolic geometry depending on whether its Euler characteristic is positive, zero, or negative, respectively.

The study of 3-manifolds is far more intricate. While still within the realm of accessible classification theory, three-dimensional topology exhibits a deep and subtle richness. In contrast to dimension two, where a single homological invariant suffices for classification, dimension three requires different tools to get a satisfactory classification, combining the notion of Seifert manifolds and their Euler characteristic and Euler number.

In dimension four, the landscape shifts drastically. Here, the distinction between topological and smooth categories becomes essential: for the first time, some topological 4-manifolds admit no smooth structure, while others admit infinitely many. A central object in the study of 4-manifolds is the intersection form, a bilinear form that encapsulates crucial information about the manifold's topology.

This report was written as part of a Master's internship at the University of Pisa under the supervision of Prof. Bruno Martelli. It offers a broad, introductory survey of low-dimensional manifolds. Rather than presenting original results or in-depth technical developments, the goal is to collect and outline key ideas, constructions, and theorems that exemplify the depth and beauty of this field. Many results are stated without proof, especially when their demonstration relies on advanced techniques beyond the scope of this exposition. Readers seeking more rigorous treatments or deeper insights are encouraged to consult the references provided in the bibliography.

To fully appreciate the content of this document, a solid background in geometry and topology such as hyperbolic geometry, Riemannian geometry and algebraic topology is recommended. In particular, it would be beneficial to have followed courses such as Istituzioni di Geometria by Prof. Bruno Martelli and Elementi di Topologia Algebrica by Prof. Filippo Callegaro at the University of Pisa (respectively corresponding to Differential Topology and de Rham Cohomology I & II by Prof. Juan Souto and Algebraic Topology by Prof. Bernard Le Stum at the University of Rennes). These prerequisites are also well-covered in standard references such as Algebraic Topology by A. Hatcher [5] and Manifolds by B. Martelli [8].

This document draws inspiration from the book "An Introduction to Geometric Topology" [9] and the lecture notes "Four Manifolds" [7] by Bruno Martelli, and from the lecture notes of the 4-manifolds course given by Marco Marengon at the University of Pisa in 2025 [6]. All illustrations are taken from Prof. Martelli's book, except for Figures 2 and 3, which originate from Prof. Marengon's course notes.

Part I

One-manifolds

One-dimensional manifolds form the simplest nontrivial class of topological spaces in the theory of manifolds. Despite their simplicity, their classification provides a useful starting point for understanding how manifold structures behave across dimensions. This part presents a brief overview of their classification and serves as a preliminary step toward subsequent discussion of higher-dimensions. Thus, our objective for now is to prove the following theorem.

Theorem 1.1. Any smooth connected 1-dimensional manifold is diffeomorphic either to the circle S^1 or to some interval of real numbers.

Since any interval is diffeomorphic either to $[0, 1], (0, 1]$ or $(0, 1)$ via some map $t \mapsto a \tanh(t) + b$, the theorem implies that there are only four distinct smooth connected 1-manifolds:

Boundary Compact	Yes	No
Yes	$[0, 1]$	S^1
No	$[0, 1)$	$(0, 1)$

From now on, let M be a smooth connected 1-dimensional manifold and let I be any real interval. First, recall the definition of parametrization by arc-length.

Definition 1.2. A map $f : I \to M$ is a *parametrization by arc-length* of M if it is a diffeomorphism from I onto an open subset of M, and if for each $s \in I$ the velocity vector $df_s(1) \in T_{f(s)}M$ has unit length.

Any given local parametrization $I \to M$ can be transformed into a parametrization by arc-length by a straightforward change of variables.

Lemma 1.3. Let $f : I \to M$ and $g : J \to M$ be parametrizations by arc-length. Then the intersection $f(I) \cap g(J)$ has at most two components. If it has only one component, then f can be extended to a parametrization by arc-length of the union $f(I) \cup g(J)$. If it has two components, then M must be diffeomorphic to S^1.

Proof. Let's denote $h = g^{-1} \circ f$. The map h sends some relatively open subset of I diffeomorphically onto a relatively open subset of J. Furthermore, the derivative of h is equal to ± 1 everywhere.

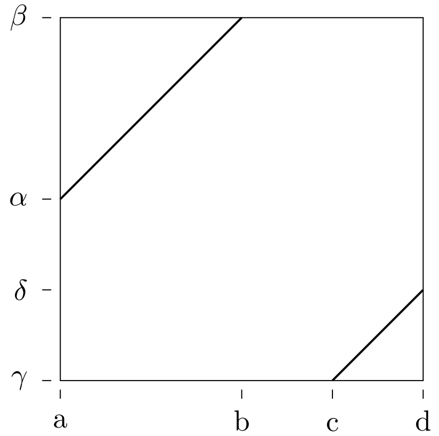
Consider the graph

$$\Gamma := \{(s, t) \in I \times J\, ;\, f(s) = g(t)\}\, .$$

Then Γ is a closed subset of $I \times J$ made up of line segments of slope ± 1 which cannot end in the interior of $I \times J$, but must extend to the boundary since h is a local diffeomorphism. There can be at most one of these segments ending on each of the four edges of the rectangle $I \times J$. Hence Γ has at most two components. Furthermore, if there are two components, the two must have the same slope. If Γ is connected, then h extends to a linear map $L : \mathbb{R} \to \mathbb{R}$. Now defining the extension:

$$F : I \cup L^{-1}(J) \to f(I) \cup g(J),$$

where $F|_I = f$ and $F|_{L^{-1}(J)} = g \circ L$, we get a parametrization by arc-length of $f(I) \cup g(J)$ as required. If Γ has two components, with slope say $+1$, they must be arranged as follows.



Translating the interval $J = (\gamma, \beta)$ if necessary, we may assume that $\gamma = c$ and $\delta = d$, so that

$$a < b \leq c < d \leq \alpha < \beta.$$

Now setting $\theta = 2\pi t/(\alpha - a)$, the required diffeomorphism $\varphi : S^1 \to M$ is defined by the formula

$$\varphi(\cos\theta, \sin\theta) = \begin{cases} f(t) & \text{for } a < t < d, \\ g(t) & \text{for } c < t < \beta. \end{cases}$$

The function is well defined since $f = g$ on $[c, d]$. The image $\varphi(S^1)$ being compact and open in M, must be the entire conncted manifold M. This proves the lemma. $\square$

Proof of the Theorem. Any parametrization by arc-length can be extended to one $f : I \to M$ which is maximal in the sense that f cannot be extended over any larger interval as a parametrization by arc-length (in particular, it is open). If M is not diffeomorphic to S^1, we will prove that f is onto, and hence is a diffeomorphism. Assume by contradiction that the open set $f(I)$ is not all of M. Then, there is a limit point x of $f(I)$ in $M \setminus f(I)$. Parametrizing a neighborhood of x by arc-length and applying the lemma, we see that f can be extended over a larger interval. This contradicts the assumption that f is maximal and hence completes the proof. $\square$

Remark 1.4. We will not provide any application of this theorem here, but numerous examples exist. In particular, it plays a key role in degree theory, where it is used, for instance, to prove Brouwer's fixed point theorem: every continuous function from the closed unit disc $\mathbb{D}^n$ to itself has at least one fixed point. A standard reference for those topics is the lecture notes of [10]. This latter well-known fact has infinitely many applications: one is explicitely used in the next part to get a classificaion of diffeomorphisms of 2-manifolds (see Definition 3.24).

Remark 1.5. We did not mention orientability since every 1-manifold has a natural orientation that is the direction in which the curve "is traveled".

Part II
Two-manifolds

This part is devoted to surfaces, that are two-dimensional differentiable manifolds. Using the Euler characteristic, one can show that any compact and connected surface is diffeomorphic to a connected sum of g tori, denoted S_g, possibly with some discs removed. Afterwards, the idea is to decompose S_g using its Euler characteristic, in a way that brings out its geometric structure. Although a surface admits many possible metrics, the Teichmüller space and its coordinates provide a rather simple expression of those different geometric structures. Finally, the Brouwer's theorem used on Thurston's compactification of Teichmüller spaces offers a good classification of self-diffeomorphisms on surfaces.

2 Surfaces

In this document, we will manipulate surfaces using various cut-and-paste tools such as boundary gluings or removal of discs and points. Detailed definitions and constructions will not be provided here and the interested reader is referred to the notes by [9]. One of the fundamental operations frequently used throughout this text is the connected sum of two manifolds: a two-step process that involves removing embedded balls from each manifold and then identifying the resulting boundary spheres via a orientation-reversing diffeomorphism.

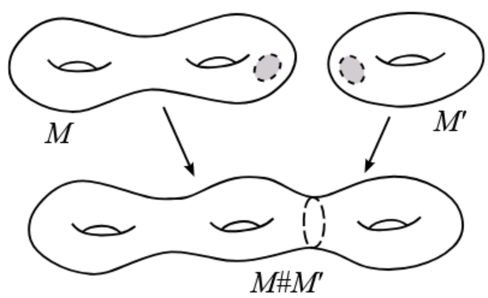


Figure 1: A connected sum of closed surfaces

We only take a short detour to examine handle decompositions, but a more complete reference is, for instance, [2].

2.1 Handle decomposition

An important tool is recalled here: the handle decomposition. The idea is to break manifolds into smaller and topologically trivial chunks that are handles.

Definition 2.1. Let M be a compact n-manifold with boundary and $0 \leqslant k \leqslant n$ be an integer. A k-handle is a manifold $D^k \times D^{n-k}$ attached to M along a diffeomorphism

$$\varphi : S^{k-1} \times D^{n-k} \to Y \subset \partial M^n.$$

The result is a new manifold

$$M' = M \cup_\varphi (D^k \times D^{n-k}).$$

Example 2.2. For instance, a 0-handle is a n-disc attached to nothing. A 1-handle is a $D^1 \times D^{n-1}$ attached to two copies of D^n.

Definition 2.3. The integer $k \geqslant 0$ of the previous definition is the *index* of the handle. The *attaching region* is $\partial D^k \times D^{n-k}$, the *attaching sphere* is $\partial D^k \times \{0\}$, the *core* is $D^k \times \{0\}$, the *belt region* is $D^k \times \partial D^{n-k}$, the *belt sphere* is $\{0\} \times \partial D^{n-k}$ and the *co-core* is $\{0\} \times D^{n-k}$.

The figure below illustrate the last definitions.

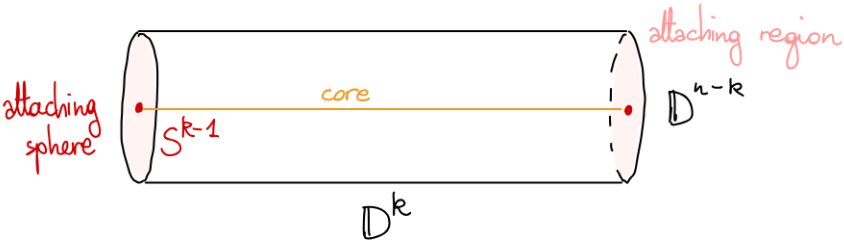


Figure 2: Illustration of attaching sphere, core and attaching region.

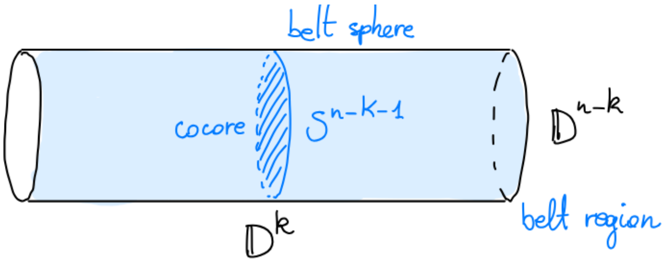


Figure 3: Illustration of belt sphere, cocore and belt region.

Definition 2.4. A *handle decomposition* of a compact manifold M (possibly with boundary) is a description of M as the result of attaching finitely many handles:

$$M = H_1 \cup_{\varphi_2} H_2 \cup_{\varphi_3} \cdots \cup_{\varphi_h} H_h,$$

where H_1 is a 0-handle, and the handle H_{j+1} is attached to the manifold

$$M_j = H_1 \cup_{\varphi_2} H_2 \cup_{\varphi_3} \cdots \cup_{\varphi_j} H_j$$

via some map φ_{j+1}.

We often omit the maps for simplicity and write

$$M = H_1 \cup \cdots \cup H_h.$$

Remark 2.5. If M is closed, the last handle H_h is necessarily an n-handle. The attaching sphere of H_{j+1} is contained in the manifold ∂M_j. Every level manifold ∂M_{j+1} is obtained from the previous one ∂M_j by surgery along the attaching sphere of H_{j+1}.

Such a decomposition is, however, not unique. In fact, we can modify a handle decomposition through a series of geometric operations, without altering its homotopy type. These operations typically include:

- *reordering handles*. Handles may always be reordered so that the lower index handles are attached first, and handles of the same index are attached simultaneously. So we can think of a decomposition as the appearing of some 0-handles, then the simultaneous attaching of some 1-handles, then of some 2-handles, and so on.

- *Turning a decomposition upside-down.* A handle decomposition of a closed manifold may be turned upside down, by reversing all arrows and interpreting every k-handle as a $(n-k)-$handle.
- *Handle sliding.* Two handles h_1, h_2 of the same index k can be slid replacing the attaching sphere of h_1^k by a new one that follows the original path of h_1^k and then loops around h_2^k.
- *Cancelling pairs of complementary handles.* Two handles of consecutive index k and $k+1$ can be cancelled if the attaching sphere of the $(k+1)-$handle intersects the core of the k-handle transverely in exactly one point.

Handle pairs in canceling position is a pair of a k-handle and a $(k+1)$-handle whose attaching regions intersect transversely in exactly one point: we can do a handle cancellation.

We have the following important theorem.

Theorem 2.6. Every compact smooth manifold may be described via some handle decomposition.

Actually, we can simplify the decomposition as described below.

Proposition 2.7. Every compact connected manifold M has a handle decomposition with a single 0-handle.

Proof. Let's start the decomposition with some 0-handles $H_1^0 \ldots, H_{i_0}^0$. If $i_0 = 1$ we are done. If not, they form a disconnected set. The addition of a k-handle with $k > 0$ does not modify the number of connected components of a manifold, except when $k = 1$ and the 1-handle is attached to distinct 0-handles. Since M is connected, there must be at least one such 1-handle. The geometric intersection of this 1-handle and one adjacent 0-handle is 1, so the pair may be canceled: the same thing is done until remains only one 0-handle. □

Remark 2.8. By turning the handle decomposition upside-down, we can prove analogously that every closed connected manifold M^n has a handle decomposition with one 0-handle and one n-handle.

Proposition 2.9. If M has a handle decomposition with n_i handles of index i then the Euler characteristic is given by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i n_i.$$

2.2 Classification and geometrisation of surfaces

The first aim of this part is to classify surfaces. Even if orientable and non-orientable surfaces are studied separately - involving connected sums of tori for the former and of projective planes for the latter case - the resulting classifications are structurally analogous.

2.2.1 The orientable case

We begin by defining the genus of a surface: an integer that, intuitively, corresponds to the number of "holes" the surface has.

Definition 2.10. Let the surface S_g be the connected sum

$$S_g = \underbrace{T\# \dots \# T}_{g}$$

of g tori $T = S^1 \times S^1$. By convention $S_0 = S^2$ is the sphere and $S_1 = T$ is the torus. The number g is the *genus* of the surface.

Figure 4: Three different ways to represent S_3.

Remark 2.11. We have $\chi(S_g) = 2 - 2g$.

Theorem 2.12. Every closed connected orientable surface is diffeomorphic to S_g for some $g \geqslant 0$.

Proof. By the previous subsection, the surface S has a handle decomposition with one 0-handle, a certain number k of 1-handles, and one 2-handle. We get by Proposotion 2.9 that $\chi(S) = 2 - k$. We prove by induction on k that $k = 2g$ is even and that S is diffeomorphic to S_g.

If $k = 0$, then S is obtained by gluing two discs (the 0- and 2-handle), and is hence a sphere. In fact, the two discs are copies of $D^2 \subset \mathbb{C}$ glued along a diffeomorphism $\phi : S^1 \to S^1$. Up to mirroring one we may suppose that ϕ is orientation-reversing, and that $\phi(z) = \overline{z}$. The resulting surface is diffeomorphic to a sphere.

If $k > 0$, then the 0-handle is a disc and the 1-handles are rectangles attached to its boundary as in Figure 5-(left). Note that since S is orientable, every rectangle is attached without a twist, otherwise it would create a Möbius strip. The 0- and 1-handles altogether form a compact surface $S' \subset S$ with only one boundary component, to which the 2-handle is attached.

Since $\partial S'$ is connected, every rectangle is linked to some other rectangle as in Figure 5-(centre). A pair of linked rectangles forms a subsurface $S'' \subset S' \subset S$ with connected boundary. If we cut S along the curve $\partial S''$ and then cap off with two discs, we perform the inverse of a connected sum.

Therefore, $S = S_1 \# S_2$, where S_1 is S'' with a disc attached, i.e., a torus, as Figure 5-(right) shows. The surface S_2 decomposes into a 0-handle, $k-2$ 1-handles, and one 2-handle. We conclude by induction on k.

□

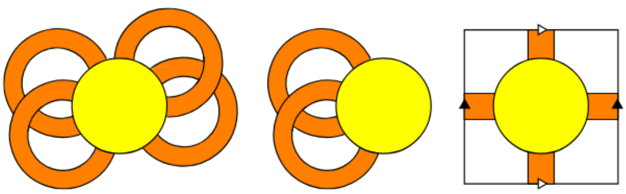


Figure 5: Illustration for the proof of Theorem 2.2.1.

We want to extend this theorem to surfaces that may not be compact or that may have boundary.

Definition 2.13. Let $g, b, p \geqslant 0$ be three natural numbers. The surface of finite type $S_{g,b,p}$ is the surface obtained from S_g by removing the interior of b discs and p points. We say that g is the genus, b the number of boundary components and p the number of punctures.

Remark 2.14. We have $\chi(S_{g,b,p}) = 2 - 2g - b - p$.

Theorem 2.15. Every compact connected orientable surface with boundary is diffeomorphic to $S_{g,b,0}$ for some $g, b \geqslant 0$.

Proof. Let S be a compact orientable surface with some b boundary components. If we glue b discs to ∂S we get a closed orientable surface, hence diffeomorphic to S_g for some $g \geqslant 0$. The original S is obtained from S_g by removing the interiors of b disjoint open discs. □

Hence, the compact connected orientable surfaces with non-negative Euler characteristic are the sphere $S^2 = S_0$ and the disc $D^2 = S_{0,1}$, which have an elliptic structure, while those with zero Euler characteristic are the annulus $A = S_{0,2}$ and the torus $T = S_1$, which have many flat structures. Note that the annulus and $S_{0,0,2}$ are homeomorphic but not diffeomorphic. In particular, there are no cusps in the elliptic and flat geometries so we do not consider surfaces with puncture when studying surface with $\chi \geqslant 0$.

We now construct hyperbolic structures on surfaces of negative Euler characteristic, starting with simpler blocks: pair-of-pants $S_{0,3,0}$, $S_{0,2,1}$, $S_{0,1,2}$ or $S_{0,0,3}$.

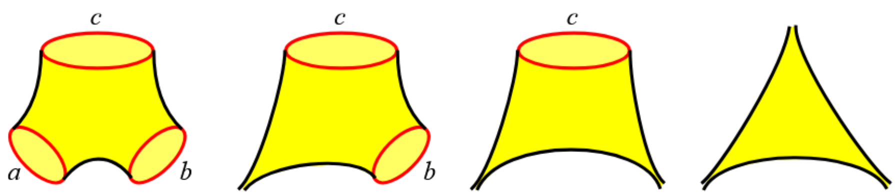


Figure 6: From left to right: $S_{0,3,0}$, $S_{0,2,1}$, $S_{0,1,2}$ and $S_{0,0,3}$

Remark 2.16. Let's observe that $\chi(S_{0,3,0}) = \chi(S_{0,2,1}) = \chi(S_{0,1,2}) = \chi(S_{0,0,3}) = -1 < 0$.

Proposition 2.17. Given three real numbers $a, b, c \geqslant 0$ there is, up to isometries, a unique complete finite-volume hyperbolic pair-of-pants with geodesic boundary curves of length a, b and c.

Sketch of proof. First, one has to prove that given three real numbers $a, b, c \geqslant 0$ there exists, up to isometries, a unique possibly degenerate -that is a, b or c can be zero- hyperbolic right-angled hexagon with three alternate sides of length a, b, and c (that is constructed by hand). To prove the uniqueness, we use the compactness to construct orthogeodesics (*i.e.*, geodesics that meet the boundary of the surface orthogonally at both endpoints) that subdivide the pair-of-pants into isometric hexagons: we use the uniqueness up to isometries of hexagons to conclude. □

Those blocks are important because they can be used to construct topologically all finite type of surfaces of negative Euler characteristic.

Proposition 2.18. If $\chi(S_{g,b,p}) < 0$ then the surface $S_{g,b,p}$ can be decomposed topologically into $-\chi(S_{g,b,p})$ possibly degenerated pairs-of-pants. In particular, $S_{g,b,p}$ admits a complete hyperbolic metric with b geodesic boundary components of arbitrary length.

Proof. If $b + p = 0$, then $g \geqslant 2$ and the surface decomposes easily in many ways, see for instance the figure below.

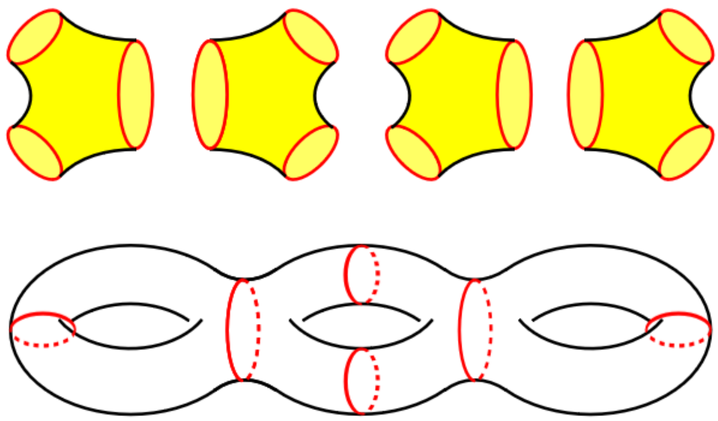


Figure 7: Since $\chi(S_3) = -4$, S_3 decomposes into 4 pair-of-pants.

If $b + p > 0$ and $\chi(S_{g,b,p}) < -1$, a decomposition for $S_{g,b,p}$ may be obtained from one of $S_{g,b-1,p}$ or $S_{g,b,p-1}$ by inserting one more (possibly degenerate) pair-of-pants.

If $\chi(S_{g,b,p}) = -1$, the surface is either a pair-of-pants, or a torus with a puncture or boundary component, which is in turn obtained by gluing two boundary components of a pair-of-pants (see one of the extremities of Figure 7). □

2.2.2 The non-orientable case

The classification of all the finite-type non-orientable surfaces is very similar to the previous work.

Definition 2.19. Let the surface S_g^{no} be the connected sum

$$S_g^{no} = \underbrace{\mathbb{RP}^2 \# \ldots \# \mathbb{RP}^2}_{g}$$

of $g \geqslant 1$ copies of the projective plane $\mathbb{RP}^2$.

Remark 2.20. We have $\chi(S_g^{no}) = 2 - g$.

Theorem 2.21. Every closed, connected, non-orientable surface is diffeomorphic to S_g^{no} for some $g \geqslant 1$.

Proof. Pick a handle decomposition of the surface S. Since it is non-orientable, at least one 1-handle is twisted and forms a Möbius strip. We have proved that S contains a Möbius strip, and we now remove it and substitute it with a disc to get a new surface S'. We have $S = S' \# \mathbb{RP}^2$ and we conclude by induction on $-\chi(S')$. □

As we did for orientable surfaces, we define $S_{g,b,p}^{no}$ the surface obtained from S_g^{no} by removing the interiors of b discs and p points. Again, compact non-orientable surfaces with non-negative Euler characteristic admit a geometric structure: the projective plane $\mathbb{RP}^2 = S_1^{no}$ has an elliptic structure while the Möbius strip and the Klein bottle, respectively $S_{1,1}^{no}$ and S_2^{no}, have flat structures.

We now construct hyperbolic structures on surfaces of negative Euler characteristic, still with pairs-of-pants.

Proposition 2.22. If $\chi(S^{no}_{g,b,p}) < 0$ then the surface $S^{no}_{g,b,p}$ can be decomposed topologically into pairs-of-pants. In particular, it admits a complete hyperbolic metric with b geodesic boundary components of arbitrary length.

2.3 Curves on surfaces

For the next sections simple but crucial concepts are the following.

Definition 2.23. A non-separating curve is a curve that does not separate the surface in two or more disconnected component. A Jordan curve is a simple closed curve. An arc of endpoint p and q in a surface S is a continue application $\alpha : [0,1] \to S$ such that $\alpha(0) = p, \quad \alpha(1) = q$ $\alpha((0,1)) \subset \mathrm{int}(S)$ and such that α is injective on $(0,1)$.

Definition 2.24. A multicurve μ in S_g is a finite set of disjoint non-trivial Jordan curves. It is said essential if it has no parallel components. The set of all simple closed multicurves is denoted by $\mathscr{M}$.

We have the following result that is intuitively obvious: one can deform any curve through an isotopy into a geodesic by "straightening" it.

Proposition 2.25. Let $g \geqslant 2$ and S_g have a hyperbolic metric. Every essential multicurve can be isotoped to a unique geodesic essential multicurve.

Another important concept is the one of earthquake that is twisted metrics on surface along some simple closed geodesics. Let m be a metric on a surface S and γ be a simple closed geodesic in S. Fix an angle $\theta \in \mathbb{R}$. Informally, a new metric m_θ on S is constructed by cutting S along γ and regluing it with a counterclockwise twist of angle θ. More formally, we have the following definition.

Definition 2.26. Let m be a complete hyperbolic, flat or elliptic metric on an oriented surface S and γ be a simple closed geodesic in S. Let the R-annulus of γ be parametrized as $S^1 \times [-R,R]$, where each slice $\{e^{it}\} \times [-R,R]$ corresponds to a geodesic segment orthogonal to γ at $\gamma(e^{it})$, parametrized by arc-length. Fix an angle $\theta \in \mathbb{R}$. Let f be a smooth function and φ a diffeomorphism such that:

$$\begin{array}{rccl} \varphi & : & S^1 \times [-R,R] & \longrightarrow & S^1 \times [-R,R] \\ & & (\mathrm{e}^{it}, s) & \longmapsto & \left(\mathrm{e}^{i(t+f(s))}, s\right), \end{array}$$

where

$$\begin{array}{rccl} f & : & [-R,R] & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \begin{cases} 0 & \text{if } t \in [-R,-R/2]\,, \\ \theta & \text{if } t \in [R/2,R]\,. \end{cases} \end{array}$$

Then the earthquake of m is a new metric m_θ that coincides with $\varphi_* m$ on the R-annulus of γ and coincides with m on the complement of $S^1 \times \left[-\frac{R}{2}, \frac{R}{2}\right]$.

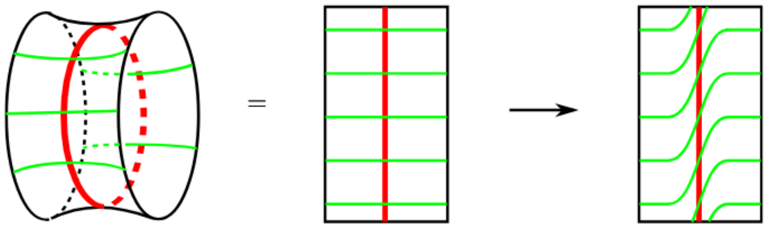


Figure 8: Earthquake

Remark 2.27. The metric tensor m_θ is well-defined and gives a complete hyperbolic, flat, or elliptic metric to S_g.

We define the Earthquake map as:

$$E_{\gamma,m} : \theta \mapsto m_\theta.$$

2.4 The mapping class group

While studying the self-diffeomorphisms of $S_{g,b,p}$, a group appears naturally: it is called the Mapping Class Group. Each element of this group has a particularly nice representative given by the Nielsen–Thurston classification theorem, which is a nonlinear analogue of the Jordan canonical form for matrices and will be stated in the next section. As such, it constitutes a fundamental component of the theory.

2.4.1 Definitions and first examples

Definition 2.28. The mapping class group of $S_{g,b,p}$ is the group

$$\mathrm{MCG}(S_{g,b,p}) = \mathrm{Diffeo}^+(S_{g,b,p})/\sim$$

where $\mathrm{Diffeo}^+(S_{g,b,p})$ indicates the group of all orientation-preserving self-diffeomorphisms of $S_{g,b,p}$ that fix pointwise the boundary. The quotient is on the equivalence relation given by: $\psi \sim \varphi$ if and only if φ and ψ are connected by an isotopy that fixes the bounadary pointwise at every level.

Remark 2.29 (Case of non-negative Euler characteristic)**.** We can compute that the groups $\mathrm{MCG}(S^2)$, $\mathrm{MCG}(S_{0,0,1})$ and $\mathrm{MCG}(D^2)$ are trivals.

The group $\mathrm{MCG}(S_{g,b,p})$ acts on $H_1(S_{g,b,p},\mathbb{Z})$ since homotopic functions induce the same maps in homology. We obtain a group homomorphism:

$$\mathrm{MCG}(S_{g,b,p}) \longrightarrow \mathrm{Aut}^+(H_1(S_{g,b,p},\mathbb{Z})) = \mathrm{Aut}^+(\mathbb{Z}^n) = \mathrm{SL}_n(\mathbb{Z})$$

with $n = 2g + \max\{b+p-1, 0\}$. Again, Aut^+ is the group of orientation-preserving automorphisms. This homomorphism is, in general, neither injective nor surjective. Its kernel is called the *Torelli group* of $S_{g,b,p}$. For instance, the Torelli group of the torus is trivial and $\mathrm{MCG}(T) \cong \mathrm{SL}_2(\mathbb{Z})$. Other such isomorphisms exist.

Proposition 2.30 (Case of zero Euler characteristic)**.** Let's denote $T = S_1$ the torus, $A = S_{0,2}$ the annulus, $D = S_{0,1,1}$ the once-punctured disk and $S = S_{0,0,2}$ the twice-punctured sphere. The following holds.

$$\mathrm{MCG}(T) \simeq \mathrm{SL}_2(\mathbb{Z}), \quad \mathrm{MCG}(D) \simeq \{0\}, \quad \mathrm{MCG}(S) \simeq \mathbb{Z}/2\mathbb{Z} \quad \text{and} \quad \mathrm{MCG}(A) \simeq \mathbb{Z}.$$

It is still unknown whether the mapping class group $\mathrm{MCG}(S_{g,b,p})$ is linear, *i.e.* isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{C})$ for some integer n.

At this stage, we limit our attention to an explicit description of some particular elements of $\mathrm{MCG}(S_{g,b,p})$, called Dehn twists.

2.4.2 Dehn twists

Dehn twists are the simplest infinite-order elements of the mapping class group. They play the role of elementary matrices in linear algebra. Strictly speaking, they have already been introduced, as they are a particular case of earthquakes.

Definition 2.31. Let γ be a nontrivial simple closed curve in the interior of $S_{g,b,p}$. Choose a tubular neighborhood of γ that is orientation-preservingly diffeomorphic to $S^1 \times [-1,1]$, such that γ corresponds to $S^1 \times \{0\}$. The Dehn twist along γ is the element $T_\gamma \in \mathrm{MCG}(S_{g,b,p})$ defined as follows.

$$\begin{array}{rcccl} T\gamma & : & S^1 \times [-1,1] & \longrightarrow & S^1 \times [-1,1] \\ & & (\mathrm{e}^{it}, s) & \longmapsto & \left(\mathrm{e}^{i(t+f(s))}, s\right), \end{array}$$

where

$$\begin{array}{rcccl} f & : & [-1,1] & \longrightarrow & \mathbb{R} \\ & & t & \longmapsto & \begin{cases} 0 & \text{if } t \in [-1,-1/2], \\ 2\pi & \text{if } t \in [1/2,1]. \end{cases} \end{array}$$

More precisely, a Dehn twist is an earthquake of angle 2π: $T_\gamma = E_{\gamma,2\pi}$. As such, $T_\gamma \in \mathrm{MCG}(S_{g,b,p})$ is well-defined and depends only on the isotopy class of γ.

Remark 2.32. Note that instead of using $+2\pi t$ we could have used $-2\pi t$. Our choice is a left twist, while the other is a right twist.

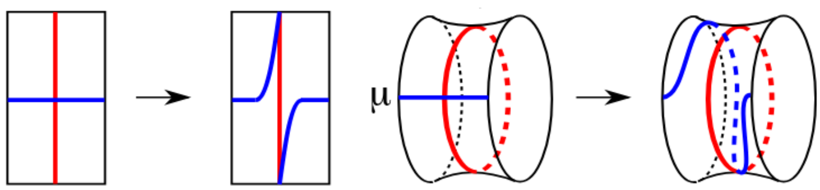


Figure 9: Dehn twist along a curve: a transverse arc μ onto an arc which makes a complete left turn.

Via the isomorphism of 2.30, the Dehn twists along the $(1,0)-$curve and the $(0,1)-$curve in T map to the matrices

$$T_m := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T_l := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Proposition 2.33 (Dehn twists on the torus)**.** The two Dehn twists T_m and T_l generate $\mathrm{MCG}(T)$.

More generally, we have the following theorem.

Theorem 2.34. For every $g, b \geqslant 0$, the group $\mathrm{MCG}(S_{g,b,0})$ is generated by Dehn twists.

Proof. First, some preliminary facts that will not be demonstrated here (see [8] for more details)

Facts. The non-separating curves in $S_{g,b,0}$ are all related, that is there is a combination of isotopies and Dehn twists transforming a Jordan curve into another. The arcs in $S_{g,b,0}$ with endpoints at p and q are all related.

Let φ be a self-diffeomorphism of $S_{g,b,0}$ fixing pointwise the boundary. We prove that φ is generated by isotopies and Dehn twists. We first consider the case $g = 0$ and proceed by induction on b. We know that $\mathrm{MCG}(S_{0,1,0})$ is trivial, so we suppose $b \geqslant 2$. Let p, q be points on distinct boundary components of $S_{0,b,0}$, and let α be an arc connecting them. All such arcs with endpoints in p and q are related by isotopy, and hence α and $\varphi(\alpha)$ are isotopic (φ fixe pointwise the boundary). Therefore, up to composing with Dehn twists and isotopies, we may suppose that φ is the identity on α, and hence also on a tubular neighbourhood of α. Cutting $S_{0,b,0}$ along α, we obtain $S_{0,b-1,0}$, with φ transformed into a self-diffeomorphism of $S_{0,b-1,0}$. By induction on b, the new φ is generated by Dehn twists and isotopies, so the original φ is as well.

We now consider the case $g > 0$, and proceed by induction on g. Let α be a non-separating Jordan curve. Since all such curves are related by diffeomorphisms, up to isotopies and Dehn twists, we may suppose that φ is the identity on α. As before, we can cut $S_{g,b}$ along α, obtaining $S_{g-1,b+2}$, and conclude by induction on g.

□

3 The Teichmüller space

3.1 Introduction

We know that S_g admits an elliptic, flat, or hyperbolic metric if and only if $g = 0, g = 1$, or $g \geqslant 2$ respectively. The elliptic metric on the two-sphere is unique up to isometries, but the flat and hyperbolic metrics on the other surfaces are not. This is the essence of the notion of the Teichmüller space.

3.1.1 Definitions

We want to define the space of all flat or hyperbolic metrics on S_g when $g \geqslant 1$.

Definition 3.1. The Teichmüller space of S_g is defined as the set

$$\mathrm{Teich}(S_g) = \{(X, \phi)\} / \sim$$

where:

- ⋆) (X, ϕ) is a structure on S_g, that is, a Riemannian surface X on which there is a complete hyperbolic *(resp. flat)* metric together with a diffeomorphism $\phi : S_g \to X$.
- ⋆) the relation $\sim$ defines an homotopy equivalence: two structures (ϕ, X) and (ψ, Y) are homotopic if there is an isometry $\varphi : X \to Y$ such that $\varphi \circ \phi$ and ψ are homotopic.

In order to introduce the notion of moduli space, we first show that $\mathrm{MCG}(S_g)$ acts on the Teichmüller space of S_g.

Proposition 3.2. A diffeomorphism $\varphi : S_g \to S_g$ transports a metric m on S_g into a new metric $\varphi_* m$ by pushing it forward.

Remark 3.3. This is what was done to define earthquakes. For a proof, see [8].

Definition 3.4. The moduli space is the quotient space

$$\mathcal{M}(S_g) = \mathrm{Teich}(S_g)/\mathrm{MCG}(S_g).$$

3.1.2 The torus case

The Teichmüller space of the torus is explicit and easy to understand.

Proposition 3.5. By sending the flat metric on T to $z \in H^2$, we get a bijection

$$\mathrm{Teich}(T) \to H^2.$$

Proof. The map is well-defined: two metrics related by an isometry isotopic to the identity produce the same z. Now, recall that $T = \mathbb{C}/\Gamma$ where Γ is a lattice isomorphic to $\mathbb{Z}^2$ with a basis (a, b). The inverse $H^2 \to \mathrm{Teich}(T)$ is constructed by identifying T with $\mathbb{C}/\langle 1, z\rangle$ sending (m, l) to $(1, z)$. □

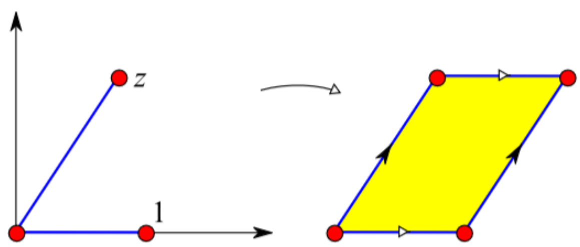


Figure 10: A $z \in \mathbb{H}^2$ determines a flat metric on the torus constructed by identifying the opposite sides of the parallelogram $0, 1, z, z+1$ which is a fundamental domain of the lattice generated by 1 and z.

Proposition 3.6. The action of $\mathrm{MCG}(T)$ on $\mathrm{Teich}(T)$ is the following action of $\mathrm{SL}_2(\mathbb{Z})$ on H^2 as Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \frac{az - b}{-cz + d}.$$

Proof. The metric z assigns to T the structure $\mathbb{R}^2/\Gamma$ with $\Gamma = \langle 1, z\rangle$, and the pair (m, l) is mapped to the translations $(1, z)$. Pick $\varphi = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) = \mathrm{MCG}(T)$. Then its inverse is given by $\varphi^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

In the new metric $\varphi_* z$, the pair (m, l) is mapped to the translations $(d - cz, -b + az)$. These can be transformed, via rotation and scaling, into the pair $\left(1, \frac{az-b}{-cz+d}\right)$. □

Remark 3.7. We note in particular that $\mathrm{MCG}(T)$ acts via isometries on the hyperbolic plane H^2. The kernel is $\{\pm I\}$, thus two opposite matrices A and $-A$ act in the same way on $\mathrm{Teich}(T)$.

Recall that an orbifold is a generalization of a manifold that admits possible singularities: heuristically, for each element of a collection $\{U_i, \varphi_i\}$ corresponds a set V_i invariant under a faithful linear action of finite groups Γ_i and the charts are diffeomorphisms between V_i/Γ_i and U_i (some good properties not mentioned here must be verified to call it an atlas).

Corollary 3.8. The moduli space of T is the orbifold

$$\mathcal{M}(T) = H^2/\mathrm{PSL}_2(\mathbb{Z}).$$

3.2 Coordinates for compact surfaces and topology

Let now fix $g \geqslant 2$. We want to construct a parametrization (i.e., some coordinates) for $\text{Teich}(S_g)$ to identify it with some $\mathbb{R}^N$.

3.2.1 The Fenchel-Nielsen coordonates

The goal here is to introduce the Fenchel-Nielsen map:

$$\begin{array}{rccl} \text{FN} & : & \text{Teich}(S_g) & \longrightarrow & \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3} \\ & & m & \longmapsto & (l_1, \ldots, l_{3g-3}, \theta_1, \ldots, \theta_{3g-3}). \end{array}$$

Let $m \in \text{Teich}(S_g)$ be a hyperbolic metric. The $3g-3$ length parameters l_i are defined using the length functions: the multicurve μ has a unique geodesic representative

$$\bar{\mu} = \bar{\gamma}_1 \sqcup \cdots \sqcup \bar{\gamma}_{3g-3}$$

in the metric m (by Proposition 2.25) such that $\bar{\mu}$ decomposes S_g into geodesic pairs-of-pants. The parameters l_i are the length of the $\bar{\gamma}_i$. Note that these parameters depend only on μ.

The torsion angles θ_i are more subtle to define: the angle θ_i measures somehow the way the two geodesic pairs-of-pants are glued along the closed geodesic $\bar{\gamma}_i$. The precise definition of θ_i needs an auxiliary multicurve ν. We fix $i = 1$ for simplicity and define θ_1.

Figure 11-left shows the two geodesic pants adjacent to $\bar{\gamma}_1$. The second multicurve ν intersects these pants in four blue arcs, two of which λ, λ' intersect $\bar{\gamma}_1$: we pick one, say λ. We fix a lift $\widetilde{P} \in \mathbb{H}^2$ of $P = \bar{\gamma}_1 \cap \lambda$ and we lift all the curves incident to P: the geodesic $\bar{\gamma}_1$ lifts to a line $\tilde{\gamma}_1$ and λ lifts to a (non-geodesic) curve $\widetilde{\lambda}$ that connects two lifts $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ of the closed geodesics $\bar{\gamma}_2$ and $\bar{\gamma}_3$. See Figure 11-right (which is represented in the Poincaré disc model of $\mathbb{H}^2$).

We draw as in the figure the unique orthogeodesics connecting $\tilde{\gamma}_1$ to $\tilde{\gamma}_2$ and $\tilde{\gamma}_3$ and we denote by s_1 the signed length of the segment in $\tilde{\gamma}_1$ comprised between these two orthogeodesics, with positive sign if (as in the figure) an observer walking on an orthogeodesic towards $\tilde{\gamma}_1$ sees the other orthogeodesic on its left (here we use the orientation of S_g). Note that if we pick λ' instead of λ we find a segment of the same length s_1.

By repeating this construction for each $\bar{\gamma}_i$, we find some real numbers s_i. Finally, the torsion parameter θ_i is

$$\theta_i = \frac{2\pi s_i}{l_i}.$$

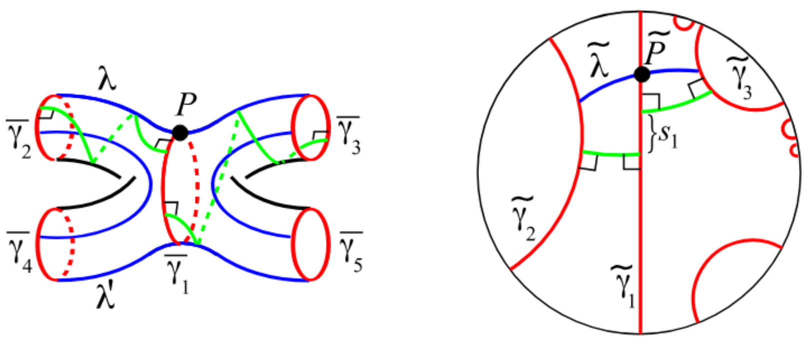


Figure 11: A closed geodesic and the two adjacent pairs-of-pants. The torsion parameter θ_1 measures the distance between the two orthogeodesics coloured in green.

Theorem 3.9. The map FN is well-defined and is a bijection.

Proof. A hyperbolic metric m' isometric to m through a diffeomorphism φ isotopic to the identity has the same parameters l_i and θ_j, since they depend only on the isotopy classes of μ and ν. Therefore, FN is well-defined.

$\star$) We prove that FN is surjective. For every vector $(l_1, \ldots, l_{3g-3}) \in \mathbb{R}^{3g-3}_{>0}$, we may use Proposition 2.17 and construct a metric on S_g by assigning to each pair-of-pants of the pants decomposition μ the (unique) hyperbolic metric with boundary lengths l_i. We obtain a metric with some arbitrary torsion angles θ, which can be changed arbitrarily by an earthquake along μ: it is easy to check that an earthquake with angles θ' changes the torsion angles from θ to $\theta + \theta'$, hence any torsion parameter can be realised and FN is surjective.

$\star$) We prove that FN is injective. If $\mathrm{FN}(m) = \mathrm{FN}(m')$, then up to acting via earthquakes we may suppose that $\mathrm{FN}(m) = \mathrm{FN}(m') = (l_1, \ldots, l_{3g-3}, 0, \ldots, 0)$. Since the torsion parameters are zero, we can find two orthogeodesics that match and project in S_g to a geodesic multicurve $\bar{\nu}$ isotopic to ν and orthogonal to $\bar{\mu}$. Therefore, $S_g \setminus (\bar{\mu} \cup \bar{\nu})$ is a tessellation of S_g into right-angled hexagons, determined by the lengths l_i. Both metrics m and m' have the same tessellation and are hence isometric, via an isometry which is isotopic to the identity.

□

Remark 3.10. We have considered only closed surfaces S_g but the arguments presented here are extendable to all surfaces $S_{g,b,p}$ of finite type with negative Euler characteristic.

3.2.2 Topology

When $g \geqslant 2$ we could use the Fenchel-Nielsen coordinates and give $\mathrm{Teich}(S_g)$ the topology of $\mathbb{R}^{6g-6}$ but for simplicity we prefer to equip the Teichmüller space with an intrinsic topology and then prove that the Fenchel-Nielsen coordinates are homeomorphisms.

Definition 3.11. We indicate by $\mathscr{S} = \mathscr{S}(S_g)$ the set of all the non-trivial Jordan curves in S_g, considered up to isotopy and orientation reversal: we say that the curves are unoriented.

Each element $\gamma \in \mathscr{S}$ induces a length function $l^\gamma : \mathrm{Teich}(S_g) \to \mathbb{R}_{>0}$ defined as follows: to each metric $m \in \mathrm{Teich}(S_g)$, the length $l^\gamma(m)$ is the length of the unique closed geodesic homotopic to γ. Note that to get a well-defined length function we may have to rescale m to have unit area.

Definition 3.12. We indicate by $\mathbb{R}^{\mathscr{S}}$ the set of all functions $\mathscr{S} \to \mathbb{R}$ and give it the product topology.

Remark 3.13. The natural map $m \in \mathrm{Teich}(S_g) \longmapsto (\gamma \mapsto l^\gamma(m)) \in \mathbb{R}^{\mathscr{S}}$ is injective and hence we may consider $\mathrm{Teich}(S_g)$ as a subset of $\mathbb{R}^{\mathscr{S}}$ and assign it the subspace topology.

Remark 3.14. Since every product of Hausdorff spaces is Hausdorff and every countable product of second-countable spaces is second-countable, the topological space $\mathbb{R}^{\mathscr{S}}$ is Hausdorff and second-countable.

In fact, in this topology, the following holds.

Proposition 3.15. The Fenchel-Nielsen map

$$\mathrm{FN} : \mathrm{Teich}(S_g) \to \mathbb{R}^{3g-3}_{>0} \times \mathbb{R}^{3g-3}$$

is a homeomorphism.

3.3 Compactification

In this next section, we introduce the compactification of the Teichmüller space. We will see that it is actually a closed disc, similarly as how $\mathbb{H}^n$ compactifies.

Definition 3.16. Let γ_1 and γ_2 be two Jordan curves on an orientable surface S. The geometric intersection $i(\gamma_1, \gamma_2)$ is defined as the minimum number of intersection points of two transverse Jordan curves (that is, each point of intersection is a transversal point) γ_1', γ_2' that are homotopic to γ_1 and γ_2, respectively.

To define the compactification, there are some steps.

i) We start with the natural projection $\pi : \mathbb{R}^{\mathscr{S}} \setminus \{0\} \to \mathbb{P}(\mathbb{R}^{\mathscr{S}})$.

ii) The injection $\mathrm{Teich}(S_g) \overset{i}{\hookrightarrow} \mathbb{R}^{\mathscr{S}}$ of Remark 3.13 gives an embedding $\pi \circ i : \mathrm{Teich}(S_g) \to \mathbb{P}(\mathbb{R}^{\mathscr{S}})$.

iii) The map $j : \gamma \in \mathscr{S} \mapsto (\eta \mapsto j(\gamma)(\eta) := i(\gamma, \eta)) \in \mathbb{R}^{\mathscr{S}}$ gives another embedding $\pi \circ j : \mathscr{S} \to \mathbb{P}(\mathbb{R}^{\mathscr{S}})$.

We can now state Thurston's compactification theorem.

Theorem 3.17. The closure $\overline{\mathrm{Teich}(S_g)}$ of $\mathrm{Teich}(S_g)$ in $\mathbb{P}(\mathbb{R}^{\mathscr{S}})$ is homeomorphic to the closed disc D^{6g-6}. Its interior is $\mathrm{Teich}(S_g)$, and its boundary sphere contains $\mathscr{S}$ as a dense subset.

We will not prove this theorem but only introduce some notions useful not only for its demonstration but also for the rest of the document. From now on, we will consider geodesics as follow.

Definition 3.18. Let M be a complete hyperbolic manifold. We call geodesic an element of the set $\mathscr{G}(M)$ of the supports of all the complete non-trivial geodesics $\mathbb{R} \to M$. The geodesic is said closed if it is the support of a closed geodesic $S^1 \to M$, and open otherwise.

Definition 3.19. Let $S_g = \mathbb{H}^2/\Gamma$ have a hyperbolic metric. A geodesic current on S_g is a locally finite Γ-invariant Borel measure on $\mathscr{G} = \mathscr{G}(\mathbb{H}^2)$. The set of all currents in S_g is indicated by $\mathscr{C}(S_g)$.

Remark 3.20. We can interpret every closed geodesic in S_g as a particular geodesic current with discrete support. In particular we get an embedding $\mathscr{S} \hookrightarrow \mathscr{C}$.

Definition 3.21. A geodesic lamination λ is a set of disjoint simple complete geodesics in a hyperbolic surface $S = \mathbb{H}^2/\Gamma$, whose union is a closed subset of S. Each geodesic (which may be closed or open) is called a leaf. Their union is the support of λ.

Example 3.22. For instance, geodesic multicurves in S and set of disjoint lines in $\mathbb{H}^2$ are laminations.

Definition 3.23. Let $\lambda \subset S$ be a geodesic lamination in a hyperbolic surface S. A complementary region (shortly, a region) is a connected component of the open complement $S \setminus \lambda$. A lamination is said to be full if every region is an ideal polygon.

We denote $\mathscr{ML}$ the set of all the measured geodesic laminations on S_g. We get the sequence of inclusions

$$\mathscr{S} \subset \mathscr{M} \subset \mathscr{ML} \subset \mathscr{C}.$$

3.4 Nielsen-Thurston classification

The main goal of this subsection is to understand how individual elements of $\mathrm{MCG}(S_g)$ look like, in the same way that the Jordan canonical form of a matrix gives us a geometric picture of what a linear transformation looks like.

Let S_g have a genus greater that 1. The mapping class group $\mathrm{MCG}(S_g)$ acts naturally on the whole space $\mathscr{C}$ of currents and in particular on the compactification $\overline{\mathrm{Teich}(S_g)} \simeq D^{6g-6}$ of the Teichmüller space.

Definition 3.24. Let $\varphi \in \mathrm{MCG}(S_g)$ be a non-trivial element. By Brouwer's fixed point theorem, φ fixes at least one point in $\overline{\mathrm{Teich}(S_g)}$. We say that it is:

1. finite order if it fixes a hyperbolic metric $m \in \mathrm{Teich}(S_g)$;
2. reducible if it fixes a multicurve $\mu \in \mathscr{M}$;
3. pseudo-Asonov in all the other cases.

3.4.1 Finite order elements

Proposition 3.25. A non-trivial element $\varphi \in \mathrm{MCG}(S_g)$ is finite order if and only if it has finite order in $\mathrm{MCG}(S_g)$.

Proof. Suppose that φ preserves the isotopy class $[m] \in \mathrm{Teich}(S_g)$ of a hyperbolic metric m on S_g. We can choose a representative for φ that fixes m. This representative is an isometry of (S_g, m). Since the isometry group of a closed hyperbolic manifold is finite, we have $\varphi^n = \mathrm{id}$ for some $n > 1$, and hence φ has finite order in $\mathrm{MCG}(S_g)$.

Conversely, let φ be an element of finite order in $\mathrm{MCG}(S_g)$. The subgroup $\langle\varphi\rangle$ generated by φ cannot act freely on $\mathrm{Teich}(S_g) \cong \mathbb{R}^{6g-6}$, otherwise it would quotient $\mathbb{R}^{6g-6}$ to an aspherical manifold (that is the universal cover of the manifold is contractible) with finite fundamental group, contradicting the fact that the fundamental group of an aspherical manifold has no torsion. Therefore, some non-trivial power of φ has a fixed point in $\mathrm{Teich}(S_g)$.

If φ has prime order, we easily conclude that φ itself has a fixed point, and we are done. However, if φ has order $p_1 \cdots p_s$ for some primes p_iWe proceed by induction on s. In fact, $\varphi' = \varphi^{p_1 \cdots p_{s-1}}$ has a fixed point $[m] \in \mathrm{Teich}(S_g)$, and is hence represented by an isometry of (S_g, m).

The isometry φ' quotients S_g to a hyperbolic orbifold, and the fixed point set $\mathrm{Fix}(\varphi')$ of φ' in $\mathrm{Teich}(S_g)$ can be naturally identified with the (suitably defined) Teichmüller space of this orbifold, which is homeomorphic to $\mathbb{R}^N$ for some $N > 0$, as in the surface case.

Since φ and φ' commute, φ acts as a mapping class on $\mathrm{Fix}(\varphi')$ with order $p_1 \cdots p_{s-1}$. By induction on s, we conclude that φ has a fixed point in $\mathrm{Teich}(S_g)$. □

Corollary 3.26. If $\varphi \in \mathrm{MCG}(S_g)$ has a finite order k, it may be represented by a diffeomorphism $\varphi : S_g \to S_g$ such that $\varphi^k = \mathrm{id}$.

Proof. The class ϕ has a representative $\phi : S_g \to S_g$ that is an isometry for some hyperbolic metric; the isometry ϕ^k is isotopic to the identity and is hence the identity since distinct isometries of a finite-volume complete hyperbolic manifold are not homotopic. □

3.4.2 Reducible elements

We must explain the terminology. If ϕ fixes a multicurve μ, one can cut S_g along μ and look at the restriction of ϕ to the resulting pieces: after extending all the theory to surfaces with boundary, one can hence study inductively each piece, and this explains the word reducible. Note that there are isometries of hyperbolic surfaces that preserve some multicurves. On the other hand, there are finite order elements that are not reducible and reducible mapping classes that are not of finite order: for instance, Dehn twists.

3.4.3 Pseudo-Asonov elements

The mapping class group $\mathrm{MCG}(S_g)$ of a surface of genus g acts on the currents space and hence on the measured geodesic laminations. In particular, the following holds.

Proposition 3.27. If $\varphi \in \mathrm{MCG}(S_g)$ fixes a non-trivial point in $\mathscr{ML}$, that is $\varphi(\mu) = \mu$ for some $\mu \in \mathscr{ML} \setminus \{0\}$, then φ is either finite order or reducible.

A pseudo-Anosov element is by definition neither finite order nor reducible. Hence, it acts freely on $\mathscr{ML} \setminus \{0\}$, but this does not prevent it from having some fixed points in $\mathbb{P}\mathscr{ML}$. In fact, we now state that there are two fixed points there, one attracting and one repelling, so that the pseudo-Asonov element looks very much like a hyperbolic isometry on the hyperbolic space.

Theorem 3.28. Let $\varphi \in \mathrm{MCG}(S_g)$ be a pseudo-Anosov element. There are two measured geodesic laminations $\mu_s, \mu_r \in \mathscr{ML}$ and a real number $\lambda > 1$ such that

$$\varphi(\mu_s) = \lambda \mu_s, \quad \varphi(\mu_r) = \frac{1}{\lambda}\mu_r.$$

The laminations μ_s, μ_r are full and they altogether fill S_g.

Note that there is a converse to the previous theorem.

Theorem 3.29. If $\varphi \in \mathrm{MCG}(S_g)$ is such that $\varphi(\mu) = \lambda\mu$ for some real number $\lambda > 1$ and full lamination $\mu \in \mathscr{ML}$, then φ is pseudo-Asonov.

For a reference of both of the previous theorems , see [8].

Part III

Three-manifolds

This part gives an idea of what we know about classification of three-manifolds. First, we will give a prime decomposition of compact oriented 3-manifolds in the idea of the well-known prime decomposition of natural numbers. Then, we introduce the notion of incompressibility which leads to that of Haken manifolds. Finally, we will scratch the surface of the concept of Seifert manifold, which is an essential tool for the geometrization of 3-manifolds.

4 Prime decomposition

4.1 Irreducibility

In this section, we will call disc and ball respectively the closed discs $D = D^2$ and $B = D^3$. In dimension 3, we can freely remove and add balls without affecting much the topology of the manifold. In particular, by removing the interior of a ball from S^3 we get another ball B, and by attaching a ball to B we get S^3 back.

Definition 4.1. Let M be a connected and oriented 3-manifold with (possibly empty) boundary. The manifold M is irreducible if every sphere $S \subset \text{int}(M)$ bounds a ball.

Intuitively, an irreducible manifold does not contain any "air bubble". The first natural 3-manifold to look at is $\mathbb{R}^3$: we would like to prove that it is irreducible. To do that, we need some Morse theory.

Definition 4.2. Let $s \subset \mathbb{R}^3$ be a closed surface and $f(x, y, z) = z$ the height function. It is a Morse function for S if $f|_S$ has finitely many critical points and if at each critical point the Hessian of $f|_S$ is non-singular.
The critical point is, as usual, said to be a local minimum, saddle or local maximum according to the signature of the Hessian. These critical points have index 0,1 and 2 respectively.

Theorem 4.3 (Alexander's Theorem)**.** The space $\mathbb{R}^3$ is irreducible.

Proof. Let $S \subset \mathbb{R}^3$ be a 2-sphere. Up to a small rotation we suppose that the height function $f|_S$ is a Morse function, and after a further small rotation we may suppose that the k critical points of $f|_S$ stay at distinct heights $z_1 < \cdots < z_k$. Pick a regular value $u_i \in (z_i, z_{i+1})$ for every $i = 1, \ldots, k-1$. The horizontal plane P at height u_i intersects S transversely into circles. Starting from the innermost ones, we cut S along these circles and cap them off by adding pairs of discs as in Figure 12. The resulting surface is disjoint from P.

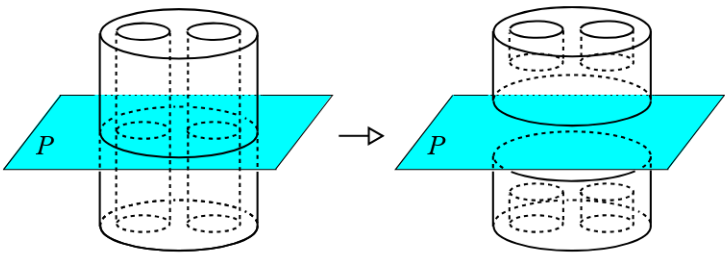


Figure 12: The plane P intersects S into circles. Cutting S along the circles and capping them off by adding pairs of discs we get a resulting surface that does not intersect P anymore.

This operation decomposes a sphere into two spheres. If we do this for every $i = 1, \dots, k-1$ we end up with many spheres of the types shown in Figure 13, that clearly bound balls in $\mathbb{R}^3$.

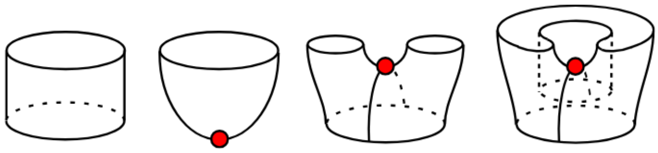


Figure 13: After capping off at each u_i we end up with many spheres of these basic types, which bound balls.

Now we reverse the process and undo all the cuts: we prove inductively that at each backward step we have a set of spheres bounding balls.In fact, the reverse operation replace two spheres S_1, S_2 bounding balls B_1, B_2 with one sphere S. If the interiors of B_1 and B_2 are disjoint, then S bounds the ball $B_1 \cup B_2$. If they are not disjoint, then one is contained in the other, say $B_1 \subset B_2$ and S bounds the ball $B_2 \setminus \mathrm{int}(B_1)$. □

Corollary 4.4. Let $p : M \to N$ be a covering of 3-manifolds. If M is irreducible then N also is. Hence, elliptic, flat, hyperbolic 3-manifolds are irreducible.

We can deduce that every sphere in S^3 bounds a ball on both sides. The situation in higher dimensions is much more problematic: it is still unknown whether every smooth 3-sphere in $\mathbb{R}^4$ bounds a smooth 4-disc.

4.2 The decomposition

If $p \in \mathbb{N}$ is prime, every decomposition in natural numbers $p = a \cdot b$ implies that $a = 1$ or $b = 1$. We can define a prime manifold analogously.

Definition 4.5. A connected sum $M \# N$ is trivial if either M or N is a sphere.

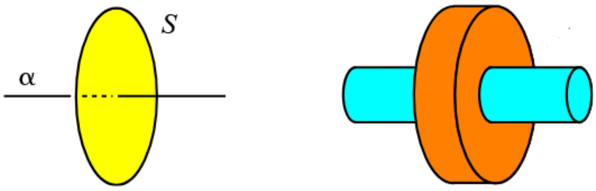
Definition 4.6. A connected 3-manifold M is prime if every connected sum $M = M_1 \# M_2$ is trivial.

For oriented manifolds, being prime is equivalent to being irreducible, with a single exception.

Proposition 4.7. Every oriented 3-manifold $M \neq S^2 \times S^1$ is prime if and only if it is irreducible.

Proof. The inverse operation of a connected sum $M = M_1 \# M_2$ consists of cutting along a separating sphere $S \subset M$ and then capping off the two resulting manifolds N_1, N_2 with balls. The capped N_i is S^3 if and only if N_i is a ball. Therefore the connected sum is trivial if and only if S bounds a ball on one side. Therefore M is prime if and only if every separating sphere $S \subset M$ bounds a ball.

If M is irreducible, then it is clearly prime. If M is prime and not irreducible, there is a non-separating sphere $S \subset M$. There is a simple closed curve $\alpha \subset M$ intersecting S transversely in one point. Pick two tubular neighborhoods of S and α as in the figure below: their union is a manifold N with a boundary sphere $\partial N = S'$. The sphere S' is separating and M is prime, hence S' bounds a ball B on the other side and $M = N \cup B$.



We now prove that $M = S^2 \times S^1$. We embed $S \cup \alpha$ naturally in $S^2 \times S^1$ as $S = S^2 \times \{y\}$ and $\alpha = \{x\} \times S^1$. Decompose $S^2 = D \cup D'$ into two discs and $S^1 = I \cup I'$ into two intervals. The manifold N also embeds as $S^2 \times I \cup D \times S^1$ and its complement $B = D' \times I'$ is a ball. Therefore $M = S^2 \times S^1$. □

Notwithstanding the previous proposition, we have the following.

Proposition 4.8. The manifold $S^2 \times S^1$ is prime.

Proof. Let $S \subset S^2 \times S^1$ be a separating sphere: we must prove that it bounds a ball. It separates $S^2 \times S^1$ into two manifolds M and N, and on fundamental groups we get $\mathbb{Z} = \pi_1(M) * \pi_1(N)$. This implies easily that either $\pi_1(M)$ or $\pi_1(N)$ must be trivial: suppose the first.

Since M is simply connected, a copy M' of M lifts to the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$. We identify $S^2 \times \mathbb{R} = \mathbb{R}^3 \setminus \{0\}$. Then M' now lies in $\mathbb{R}^3$, and $\partial M' = S^2$ implies that M' is a ball by Alexander's theorem. □

We now state that the connected sum operation on oriented three-manifolds behaves like the product of natural numbers: every object decomposes uniquely into prime factors.

Theorem 4.9 (Prime decomposition)**.** Every compact oriented 3-manifold M with (possibly empty) boundary decomposes into prime manifolds:

$$M = M_1 \# \ldots \# M_k.$$

This list of prime factors is unique up to permutations and adding or removing copies of S^3.

Sketch of proof of the existence. For the existence, if M contains a non-separating sphere, then the proof of Proposition 4.7 shows that $M = M' \# (S^2 \times S^1)$. Since $H_1(M) = H_1(M') \oplus \mathbb{Z}$, up to factoring finitely many copies of $S^2 \times S^1$ (that is prime, by Proposition 4.8), we may suppose that every sphere in M is separating.

If M is prime we are done. If not, it decomposes as $M = M_1 \# M_2$. We keep decomposing each factor until all factors are prime: this process must end, because a decomposition $M = M_1 \# \cdots \# M_k$ gives rise to a system of $k-1$ spheres, where k cannot be arbitrarily big (which is a fact that require a proof). □

5 Incompressible surfaces

5.1 Incompressibility

Throughout all this section M denotes a compact orientable 3-manifold with (possibly empty) boundary and $S \subset M$ a properly embedded surface.

Definition 5.1. The surface S is *∂-parallel* if it is obtained by slightly pushing (done by a translation) inside M the interior of a compact surface $S' \subset \partial M$, possibly with boundary.
We say that:

- a sphere is essential if it does not bound a ball;
- a disc is essential if it is not ∂-parallel.

The manifold M is:

- irreducible if it does not contain essential spheres;
- ∂-irreducible if it does not contain essential discs.

Definition 5.2. A compressing disc for the surface S is a disc $D \subset M$ with $\partial D = D \cap S$, such that ∂D does not bound a disc in S. If $\chi(S) \leqslant 0$, S is said to be compressible if it has a compressing disc, and incompressible otherwise (see Figure 14). The operation of compression is illustrated in Figure 15.

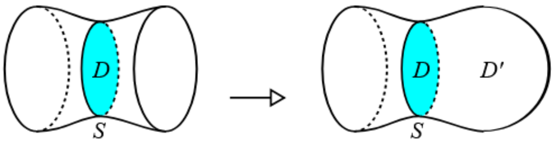


Figure 14: A surface S is incompressible if the existence of a disc D implies the existence of another disc $D' \subset S$ as illustrated. If in addition M is irreducible, the two discs D and D' form a sphere which bounds a ball, and hence by substituting D' with D we get two isotopic surfaces.

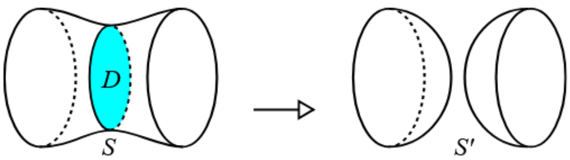


Figure 15: We can surger a surface S along a disc D with $\partial D = D \cap S$. The operation consists of removing an annular tubular neighbourhood of ∂D in S and adding two parallel copies of D. We get a new surface S'.

Definition 5.3. Let $S \subset M$ be a properly embedded, orientable surface in a 3-manifold M. A ∂-compressing disc for S is a disc D such that $\partial D = \alpha \cup \beta$, where $\alpha \subset S$ and $\beta \subset \partial M$. We also require that there are no sub-disc $D' \subset S$ with $\partial D' = \alpha \cup \beta'$ and $\beta' \subset \partial S$. If $\chi(S) \leq 0$, S is said to be ∂-compressible if it admits a ∂-compressing disc, and ∂-incompressible otherwise (see Figure 16).
The operation of ∂-compression is illustrated in Figure 17.

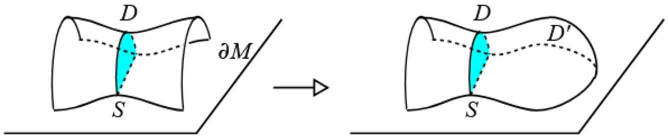


Figure 16: A surface S is ∂-incompressible if the existence of a disc D implies the existence of another disc $D' \subset S$ as illustrated. If in addition M is ∂-irreducible, the two discs D and D' form a disc which is ∂-parallel, and hence by substituting D' with D we get two isotopic surfaces.

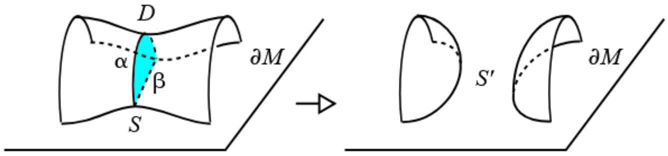


Figure 17: We can surger a surface S along a disc D touching the boundary in a segment. The result is a new properly embedded surface S'.

The ∂-compression transforms a surface S into a surface $S' \subset M$ simpler than S.

Proposition 5.4. The surface S' may have one or two components S'_i, and $\chi(S'_i) > \chi(S)$ for each component.

Proof. We have $\chi(S') = \chi(S) + 1$. If S' has only one component, the result is immediate. Suppose instead that $S' = S'_1 \sqcup S'_2$. Since α did not bound a disc in S, no S'_i is a disc; hence $\chi(S'_i) \leq 0$. This implies $\chi(S'_i) > \chi(S)$ for $i = 1, 2$. □

Proposition 5.5. There are no incompressible surfaces in $\mathbb{R}^3$.

Proof. Let S be a surface in $\mathbb{R}^3$. As in the proof of Alexander's Theorem, we find that S transforms into spheres after surgerying along discs. Therefore S compresses somewhere. □

A direct consequence of this proposition if the following.

Corollary 5.6. There are incompressible surfaces neither in S^3 nor in the ball B.

5.2 Haken manifolds

There are two classes of irreducible closed three-manifolds: those that contain incompressible surfaces, and those that do not. We already know that $\mathbb{R}^3$, S^3 and B are in the second class. The manifolds belonging to the first class are called Haken manifolds and are somehow easier to study.

Definition 5.7. A Haken manifold is a compact, connected and oriented 3-manifold with (possibly empty) boundary, which is irreducible, ∂-irreducible, and contains an incompressible and $\partial-$incompressible surface.

Proposition 5.8. Every boundary component X of a Haken manifold M has negative Euler characteristic and is incompressible.

Proof. No component X of ∂M is a sphere: if it were so, it would bound a ball B and we would have $M = B$, contradicting corollary 5.6. Hence $\chi(X) \leq 0$ and X is incompressible because M is ∂-irreducible. □

There are plenty of Haken manifolds.

Proposition 5.9. Let M be oriented, compact, irreducible, and ∂-irreducible.

⋆) If $H_2(M, \partial M; \mathbb{Z}) \neq \{e\}$, then M is Haken.

⋆) If $\partial M \neq \emptyset$ and $M \neq B$, then M is Haken.

Proof. The second item is a consequence of the first one. The first come from the fact that every non-trivial homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$ is represented by a disjoint union of incompressible and ∂-incompressible oriented surfaces. □

To understand better those Haken manifolds, we can cut them into smaller pieces that are simpler to study: the procedure to cut an Haken manifold is called a hierarchy.

Definition 5.10. A *hierarchy* for a Haken 3-manifold M is a sequence of 3-manifolds

$$M = M_0 \xrightarrow{S_0} M_1 \xrightarrow{S_1} M_2 \xrightarrow{S_2} \dots \xrightarrow{S_{h-1}} M_h$$

where each M_{i+1} is obtained by cutting M_i along a properly embedded (possibly disconnected) surface $S_i \subset M_i$, such that the following hold.

- Every component of S_i is an incompressible and ∂-incompressible surface or an essential disc, for all i.
- The final manifold M_h consists of balls.

The number h is the *height* of the hierarchy.

The hierarchy is not unique: the height is not defined as the minimum number of steps necessary to decompose the manifold in ball. In particular, one can prove the following.

Theorem 5.11. Every Haken manifold has a hierarchy of height 3.

We can use hierarchy to prove theorems on Haken manifolds, for example we state the following.

Proposition 5.12. A Haken manifold has infinite fundamental group. In particular, elliptic 3-manifolds are not Haken.

Remark 5.13. Flat 3-manifolds are always Haken but hyperbolic 3-manifolds may or may not be Haken.

Staying within the framework of decomposing manifolds to better understand their structure, one can further decompose any irreducible and ∂-irreducible 3-manifold along a canonical collection of set. The resulting pieces are either simple or belong to a special class known as Seifert manifolds. We introduce this class in the next section.

6 Seifert manifolds

Seifert manifolds are important because they appear naturally not only decomposing 3-manifolds but also studying their geometry. In fact, of the eight three-dimensional geometries, six of them are realised precisely by this class of manifolds.

6.1 Dehn-filling

If a 3-manifold has a spherical boundary component, we can cap it off with a 3-ball, much like placing a cap on an open bottle. However, if it has a torus boundary component, there is no canonical way to do so. This is precisely where Dehn-filling comes into play. Let M be a 3-manifold having a toric boundary component, that is, ∂M contains a torus T.

Definition 6.1. A Dehn-filling of M along T is the operation of gluing a solid torus $D \times S^1$ to M via a diffeomorphism $\phi : \partial D \times S^1 \to T$. The result of this operation is a new manifold denoted M^{fill}.

Remark 6.2. The closed curve $\partial D \times \{pt\}$ is glued to some Jordan curve $\gamma \subset T$. We say that the Dehn-filling kills the curve γ since it is exactly happening in the fundamental group: $\pi_1(M^{fill}) = \pi_1(M)/N(\gamma)$ where $N(\gamma)$ is the smallest normal subgroup of $\pi_1(M)$ containing γ.

Definition 6.3. A slope on a torus T is the isotopy class of an unoriented homotopically non-trivial Jordan curve. If we fix a basis (m, l) of $H_1(T; \mathbb{Z}) = \pi_1(T)$, every slope may be written as $\gamma = \pm(pm + ql)$ for some coprime pair (p, q).

Remark 6.4. The definition gives a 1-1 correspondence sending an element γ of the set of slopes to a rational point $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$. In particular, if T is a boundary component of M, every $\frac{p}{q}$ determines a Dehn-filling of M that kills the corresponding γ: from now, we will refer to a $(p, q)-$Dehn-filling.

6.2 Circle bundles

We now introduce a class of 3-manifolds: the orientable circle bundles over some compact surface. First, recall that if M is a smooth manifold with boundary, its double is obtained by gluing two copies of M together along their common boundary: the double is $M \times \{0, 1\}/\sim$ where $(x, 0) \sim (x, 1)$ for all $x \in \partial M$. Moreover, as expected, a circle bundle is a fiber bundle where the fiber is the circle S^1.

Definition 6.5. Let S be a compact connected surface. As every connected manifold, it has a unique orientable line bundle $S \times I$ or $S \tilde{\times} I$ depending on whether S is orientable or not. We denote by

$$S \times S^1 \quad \text{or} \quad S \tilde{\times} S^1$$

respectively the double along the boundary of

$$S \times I \quad \text{and} \quad S \tilde{\times} I.$$

If we do not know whether S is orientable or not, we use the symbols $S \overset{(\sim)}{\times} I$ and $S \overset{(\sim)}{\times} S^1$ to denote these objects. The second one is an orientable circle bundle over S, called the trivial one.

If the base surface S has non-empty boundary, every bundle over S is a 3-manifold with boundary where the boundary consists of tori (one fibering above each circle in ∂S) because the torus is the unique orientable surface that fibres over S^1. The following property says that there is essentially only one bundle over S.

Proposition 6.6. If $\partial S \neq \emptyset$, the orientable circle bundles on S are all isomorphic.

If the base surface is closed (in particular has empty boundary), we can parametrise the oriented circle bundles over the surface thanks to an interger called the Euler number.

Let S be a compact surface with non-empty boundary. Pick $M = S \overset{(\sim)}{\times} S^1$ and fix an orientation for M. Since a section is determined by its image, we denote by S the zero-section of M. Every boundary component T of M is an oriented torus, which contains two natural unoriented simple closed curves: the boundary $m = T \cap \partial S$ of the section S, and the fiber l of the bundle. If oriented, the curves m and l form a basis (m, l) for $H_1(T, \mathbb{Z})$. We choose the orientation such that (m, l) forms a positively oriented basis: there is a unique choice up to reversing both m and l.

Suppose now that S has only one boundary component and let M^{fill} be obtained by the $(1, q)$-Dehn-filling of M. Let $\hat{S}$ be the closed surface obtained by capping S with a disc.

Proposition 6.7. The circle bundle $M \to S$ extends to a circle bundle $M^{fill} \to \hat{S}$. Every oriented circle bundle on $\hat{S}$ is obtained in this way, and distinct values of q yield vector bundles that are not orientation-preservingly isomorphic.

Proof. The meridian of the torus is $m' = m + ql$. The fibre l has geometric intersection 1 with m', and is hence a longitude for the torus. We may represent the torus as $D \times S^1$ with $m' = S^1 \times \{y\}$ and $l = \{x\} \times S^1$. The circle bundle $M \to S$ extends to a circle bundle $M^{fill} \to \hat{S}$ with $\hat{S} = S \cup D$.

Every closed circle bundle $N \to \hat{S}$ arises in this way: the bundle above a disc $D \subset \hat{S}$ is the trivial $D \times S^1$, and if we remove it we get $M \to S$ back. The number q is intrinsically determined: the meridian m does not depend on the section of $M \to S$ and the equality $m' = m + ql$ determines q. Therefore, distinct values of q yield non-isomorphic bundles. □

Definition 6.8. The integer q of the previous proposition is called the Euler number of the circle bundle.

This integer measures how "twisted" the bundle is. In the case of the tangent bundle of a smooth manifold, it generalizes the classical notion of Euler characteristic.

Corollary 6.9. For every $e \in \mathbb{Z}$ and every closed surface S, there is a unique oriented circle bundle over S with Euler number e.

Remark 6.10. A change of orientation for M transforms e into $-e$. An oriented circle bundle over a closed surface is trivial if and only if $e = 0$, if and only if the bundle has a non-zero section. Hence, the Euler number can be interpreted as an obstruction for having a section.

6.3 Seifert manifolds

A particular case of Dehn-fillings of trivial bundles over surfaces with boundary are the Seifert manifold.

Let M be the oriented bundle $S \overset{(\sim)}{\times} S^1$ over a compact connected surface with boundary and let S denote the zero-section. Let $T_1, \ldots, T_k$ be the boundary tori of M. As before, on each T_i we choose an orientation for the meridian $m_i = T_i \cap \partial S$ and for the fiber l_i of the bundle so that the basis (m_i, l_i) for $H_1(T_i;\mathbb{Z})$ is positively oriented.

Definition 6.11. A $(p_i, q_i)-$Dehn-filling on T_i is said fiber-parallel if $p_i = 0$.

Definition 6.12. A Seifert manifold is any 3-manifold N obtained from M by Dehn-filling some $h \leqslant k$ boundary tori in a non-fiber parallel way: $p_i \neq 0$ for all $i = 1, ..., k$.

Remark 6.13. The Seifert manifold is closed if $h = k$, and has $k-h$ boundary tori otherwise. Moreover, it is not important to know which h tori are filled. In fact, every permutation of the boundary tori is realised by a self-diffeomorphism of M preserving pairs $\pm(m_i, l_i)$.

The pairs (p_i, q_i) are determined up to sign, so we can always suppose $p_i > 0$ and we fully encode the Seifert manifold N using the following notation:

$$N = (\widehat{S}, (p_1, q_1), \ldots, (p_h, q_h)), \tag{6.1}$$

where $\widehat{S}$ is S with h boundary components capped off.

Let (p, q) be two coprime integers with $p > 0$.

Definition 6.14. A *standard fibered solid torus* with coefficients (p, q) is the solid torus

$$D \times [0,1]/_{\psi}$$

where $\psi : D \times \{0\} \to D \times \{1\}$ is a rotation of angle $2\pi\frac{q}{p}$.

The fibration into vertical segments $\{\mathrm{pt}\} \times [0,1]$ extends to a fibration into circles of the solid torus. The central fibre, obtained by identifying the endpoints of $\{0\} \times [0,1]$, is the core of the solid torus, and every non-central fibre winds p times around the core of M.

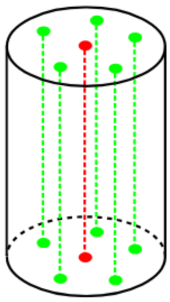


Figure 18: A standard fibered solid torus with $p = 5$. Every non-central fibre (green) winds 5 times along the core (red).

Definition 6.15. A Seifert fibration is a partition of a compact oriented 3-manifold N with (possibly empty) boundary into circles, such that every circle has a fibered neighbourhood diffeomorphic to a standard fibered solid torus.

If we denote S the topological space obtained from N by quotienting circles to points, the map $N \to S$ is in fact what we call a Seifert fibration. The notation

$$N = (S, (p_1, q_1), \ldots, (p_h, q_h)) \tag{6.2}$$

defines a Seifert fibration $N \to S$ and a Seifert manifold N. We can now extend the definition of Euler number.

Definition 6.16. We define the Euler number of the fibration 6.2 to be the rational number

$$e(N) := \sum_{i=1}^{h} \frac{q_i}{p_i}.$$

Remark 6.17. The Euler number is only defined modulo $\mathbb{Z}$ when N has boundary.

Proposition 6.18. The universal cover of a closed Seifert manifold is shown in Table 1.

	$\chi > 0$	$\chi = 0$	$\chi < 0$
$e = 0$	$S^2 \times \mathbb{R}$	$\mathbb{R}^3$	$\mathbb{R}^3$
$e \neq 0$	S^3	$\mathbb{R}^3$	$\mathbb{R}^3$

Table 1. The universal cover of a closed Seifert manifold depends on its invariants e and χ.

Proof. The universal cover of a circle bundle over a surface S with $\chi(S) \leqslant 0$ is a line bundle over the universal cover $\mathbb{R}^2$ of S. The line bundle is trivial since $\mathbb{R}^2$ is contractible and we get $\mathbb{R}^3$. □

7 The eight geometries

Previously, it has been established that every compact, orientable 3-manifold admits a unique decomposition as a connected sum of prime manifolds. This reduces the study of closed 3-manifolds to the classification of prime 3-manifolds.

In parallel, in Part II, we proved Poincaré's Uniformization Theorem, stating that every connected, simply connected surface can be uniformized, meaning that it can be given one of three canonical geometries: the spherical geometry, the Euclidean geometry or the hyperbolic geometry.

This theorem admits a three-dimensional analogue, known as the Geometrization Conjecture, formulated by William Thurston in 1976 and proven by Grigori Perelman in 2003. More precisely, we have the following statement.

Theorem 7.1 (Thurston's conjecture)**.** Every oriented prime closed 3-manifold can be cut along tori, so that the interior of each of the resulting manifolds has a geometric structure with finite volume.

Note that this conjecture implies other, the most famous being the Poincaré conjecture. The signification of "having a geometric structure" as well as the list of the possible geometries is the heart of this section. First, some vocabulary.

7.1 Surface bundles

Recall that a fiber bundle is a structure (E, B, π, F) where E, B and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying the following local triviality condition: for every $x \in B$, there is an open neighborhood $U \subset B$ such that there is a homeomorphism $\varphi : \pi^{-1}(U) \to U \times F$ in such a way that π agrees with the natural projection onto the first factor $p_1 : U \times F \to U$. Namely, the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times F \\ {\scriptstyle \pi}\downarrow & \swarrow_{p_1} & \\ U & & \end{array}$$

The space F is called the fiber of (E, B, π, F).

Definition 7.2. A surface bundle over S^1 is a fiber bundle $M \to S^1$ where M is a compact and orientable 3-manifold (possibly with boundary), and the fiber Σ is a connected, compact, orientable surface.

If M has boundary, then so does Σ, and ∂M consists of tori fibering over S^1.

Definition 7.3. Let Σ be a non-orientable surface, with orientable double cover $p : \widetilde{\Sigma} \to \Sigma$ and deck transformation τ that gives $\Sigma = \widetilde{\Sigma}/\tau$. We have

$$\Sigma \mathbin{\widetilde{\times}} (-1, 1) = (\widetilde{\Sigma} \times (-1, 1))/_{(\tau,\iota)}$$

with $\iota(x) = -x$. A local semi-bundle is the map $\Sigma \mathbin{\widetilde{\times}} (-1, 1) \to [0, 1)$ that sends (p, x) to $|x|$. The fiber over 0 is Σ. The one over a point $x \in (0, 1)$ is $\widetilde{\Sigma}$. A semi-bundle $M \to [-1, 1]$ is a map which is a local semi-bundle when restricted to $[-1, 1 - \varepsilon)$ and $(\varepsilon, 1]$. The fiber over ± 1 is Σ and the fiber over a point $x \in (-1, 1)$ is $\widetilde{\Sigma}$.

A connected Riemannian manifold M is **homogeneous** if for every $p, q \in M$ there is an isometry of M sending p to q, and is **isotropic** if at each point p every isometry of T_pM is realized by an isometry of M. It is easy to prove that a complete isotropic manifold is also homogeneous and has constant sectional curvature: the fundamental examples of isotropic spaces are $\mathbb{S}^n$, $\mathbb{R}^n$, and $\mathbb{H}^n$ that are the three geometries used for the classification in dimension 2.

Definition 7.4. We define the following notions.

- ⋆) A manifold X is a geometric model if it is a smooth and simply connected manifold on which acts freely and transitively a discrete subgroup $\Gamma < G$ of a Lie group G and whose stabilizers are compact. Such a geometry is said maximal if G is the biggest group that acts on X this way.
- ⋆) A geometric structure on a manifold M is a diffeomorphism from M to some X/Γ where X is a geometric model.

Thurston showed that there exists 8 geometric models X in dimension 3 that are maximal and for which there exists a compact manifold with geometric structure X. The eight models are the following homogeneous simply-connected complete Riemannian 3-manifolds:

$$\mathbb{S}^3, \quad \mathbb{R}^3, \quad \mathbb{H}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \text{Nil}, \quad \widetilde{\text{SL}_2}, \quad \text{Sol}. \tag{7.1}$$

The first three manifolds are isotropic and have constant sectional curvature; the other five are not.

Definition 7.5. Let X be one of the eight model manifolds listed in 7.1. We say that a Riemannian 3-manifold M has a geometric structure modelled on X if M is locally isometric to X, that is, if every point $p \in M$ has an open neighbourhood isometric to some open set in X.

Remark 7.6. This definition implies in particular that M is locally homogeneous. If M is complete, then $M = X/\Gamma$ for some discrete group $\Gamma < \mathrm{Isom}(X)$ acting freely.

7.2 Elliptic three-manifolds

Let's start with elliptic 3-manifolds, that are manifolds modelled on S^3.

Theorem 7.7. A closed 3-manifold admits an elliptic metric if and only if it is a Seifert manifold with $e \neq 0$ and $\chi > 0$.

In this situation, the idea is that the manifold M is elliptic and hence $M = S^3/\Gamma$. Either Γ is a particular cyclic group implying that M is a lens space, or, up to conjugaison, Γ preserves the Hopf fibration of S^3 which descends to a Seifert fibration. Thus, the universal cover of such a manifold is S^3 and hence $\chi > 0$ and $e \neq 0$ by Table 1.

Remark 7.8. Every elliptic 3-manifold is orientable.

7.3 Flat three-manifolds

Let's continue with flat 3-manifolds, that are manifolds modelled on $\mathbb{R}^3$.

Theorem 7.9. A closed orientable 3-manifold admits a flat metric if and only if it is a Seifert manifold with $e = \chi = 0$.

In this situation, the manifold M is flat and thus $M = \mathbb{R}^3/\Gamma$ for some crystallograpic group $\Gamma < \mathrm{Isom}(\mathbb{R}^3)$ acting freely. Every element in Γ is either a translation or a rotation. Hence, one can consider the exact sequence

$$0 \longrightarrow H \longrightarrow \Gamma \longrightarrow r(\Gamma) \longrightarrow 0$$

where $H \triangleleft \Gamma$ is the translation subgroup and $r(\Gamma) < \mathrm{SO}(3)$ is the image of Γ by the homomorphism that sends every isometry to its rotational part. The group $r(\Gamma)$ is finite and one has to prove that Γ preserves a foliation of $\mathbb{R}^3$ into parallel lines that projects to a Seifert structure on M. By Bieberbach's theorem (stating that every closed flat n-manifold is finitely covered by a flat n-torus), the idea is to show that $\chi(S) = 0$ and $e = 0$ if and only if M is covered by the 3-torus.

For the other implication, one has to study every possible Seifert manifold with $\chi = e = 0$ and construct a flat metric for each of them.

7.4 Product geometries

7.4.1 $S^2 \times \mathbb{R}$

We give $S^2 \times \mathbb{R}$ the product metric. There are very few manifolds modelled on $S^2 \times \mathbb{R}$. In particular, thanks to the fact that its sectional curvature is not constant, we have

$$\mathrm{Isom}(S^2 \times \mathbb{R}) = \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R}).$$

Theorem 7.10. An orientable manifold admits a finite-volume $S^2 \times \mathbb{R}$ geometry if and only if it is a closed Seifert manifold with $e > 0$ and $\chi > 0$.

Proof. The possible closed Seifert manifolds with $e = 0$ and $\chi > 0$ are only $S^2 \times S^1$ and $\mathbb{RP}^2 \tilde{\times} S^1$. They are diffeomorphic to $(S^2 \times \mathbb{R})/\Gamma$ where Γ is generated respectively by

$$\{(\mathrm{id}, \tau)\} \quad \text{and} \quad \{(\iota, r), (\iota, r')\}$$

where τ is any translation, ι is the antipodal map, and r, r' are reflections with respect to distinct points in $\mathbb{R}$.

Conversely, pick an orientable manifold $M = (S^2 \times \mathbb{R})/\Gamma$ where $\Gamma < \mathrm{Isom}(S^2) \times \mathrm{Isom}(\mathbb{R})$. Then, Γ preserves the foliation into spheres $S^2 \times \{x\}$, which descends into a foliation into spheres and/or projective planes for M. Therefore M decomposes into orientable interval bundles $S^2 \times I$ and $\mathbb{RP}^2 \tilde{\times} I$, and is hence either $S^2 \times S^1$ or $\mathbb{RP}^2 \tilde{\times} S^1$.

□

7.4.2 $\mathbb{H}^2 \times \mathbb{R}$

We give $\mathbb{H}^2 \times \mathbb{R}$ the product metric. Again,

$$\mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) = \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R}).$$

Before giving the principal theorem in this case, let's consider the following exact sequence:

$$0 \to \mathrm{Isom}(\mathbb{R}) \to \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R}) \xrightarrow{p} \mathrm{Isom}(\mathbb{H}^2) \to 0.$$

The following proposition holds.

Proposition 7.11. A discrete group $\Gamma < \mathrm{Isom}(\mathbb{H}^2 \times \mathbb{R})$ is cofinite, that is $\mathbb{H}^2 \times \mathbb{R}/\Gamma$ has finite volume, if and only if both $p(\Gamma)$ and $\Gamma \cap \ker(p)$ are discrete and cofinite.

The proof of this proposition relies on notions from hyperbolic geometry, including horocycles and fundamental domains. While it appears in this subsection, it will later be adapted to various other geometric models.

Theorem 7.12. The interior of a compact orientable manifold admits a finite-volume complete $\mathbb{H}^2 \times \mathbb{R}$ geometry if and only if it is a Seifert manifold with $\chi < 0$ and either $\partial M \neq \emptyset$ or $e = 0$.

To prove the direct implication of this theorem, one has to write $\mathrm{int}(M) = (\mathbb{H}^2 \times \mathbb{R})/\Gamma$ with Γ cofinite: by the previous proposition, the group $\Gamma \cap \ker(p)$ quotients every line $\{x\} \times \mathbb{R}$ to a circle in M, giving a Seifert fibration $M \to S$ onto $S = \mathbb{H}^2/p(\Gamma)$ which has finite-area. We have $\chi(S) < 0$, and either $e(M) = 0$ or $\partial M \neq \emptyset$ because $\mathbb{H}^2 \times \{y\}$ projects to a section for $M \to S$.

For the other implication, we consider a section $M \to S$ (that exists by hypothesis) which is the fibre of a bundle $M \to O$ over a 1-orbifold O. We may write $S = \mathbb{H}^2/\Gamma$ identifying $\pi_1(S)$ with Γ and analogously consider $\pi_1(O)$ inside $\mathrm{Isom}(\mathbb{R})$. This would give an injective map $\pi_1(M) \to \mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{R})$ which has a discrete image that acts freely on $\mathbb{H}^2 \times \mathbb{R}$, inducing a $\mathbb{H}^2 \times \mathbb{R}$ structure on M.

7.5 Nil

The Heisenberg group in dimension 3 consists of all matrices of type:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x, y, z \in \mathbb{R},$$

together with the usual multiplicative operation on matrices. This is a nilpotent but non-abelian Lie group also called Nil which is diffeomorphic to $\mathbb{R}^3$: we identify Nil with $\mathbb{R}^3$ using the coordinates (x, y, z). There is a Lie group exact sequence

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Nil} \longrightarrow \mathbb{R}^2 \longrightarrow 0$$

where $\mathbb{R} = [\mathrm{Nil}, \mathrm{Nil}]$ is the centre of Nil and consists of all matrices with $x = y = 0$. Therefore Nil is a line bundle over $\mathbb{R}^2$.

First of all, one has to find and study its isometry group Isom(Nil). We will not do it here, but we state that the following sequence is exact:

$$0 \to \mathbb{R} \to \mathrm{Isom}^+(\mathrm{Nil}) \xrightarrow{p} \mathrm{Isom}(\mathbb{R}) \to 0,$$

where the map p assigns to each isometry of Nil its induced action on the vertical direction. The following proposition states that, in this context, cocompact groups $\Gamma < \mathrm{Isom}(\mathbb{R}^2)$, that is groups Γ such that $\mathbb{R}^2/\Gamma$ is compact, do not lift.

Proposition 7.13. Let $\Gamma < \mathrm{Isom}(\mathbb{R}^2)$ be a discrete and cocompact group. There is no homomorphism $f : \Gamma \to \mathrm{Isom}^+(\mathrm{Nil})$ such that $p \circ f = \mathrm{id}$.

Futhermore, Proposition 7.11 holds for $\Gamma < \mathrm{Isom}^+(\mathrm{Nil})$. Now, we get the following theorem.

Theorem 7.14. The interior of a compact orientable manifold admits a finite-volume complete Nil geometry if and only if it is a closed Seifert manifold with $\chi = 0$ and $e \neq 0$.

For the proof, the first implication is similar to the previous subsection: writing $\mathrm{int}(M) = \mathrm{Nil}/\Gamma$ with Γ cofinite, we get a Seifert fibration $M \to S$ over a finite-area orbifold $S = \mathbb{R}^2/p(\Gamma)$. Finite-area flat orbifolds have $\chi(S) = 0$ and are closed, since there are no cusps in flat geometry. Hence, M is closed. We must have $e \neq 0$, otherwise (up to passing to a finite-index subgroup) we would get $M = T \times S^1$, contradicting Proposition 7.13. To prove the converse implication, we must take a such Seifert manifold, denoted $M = (S, (p_1, q_1) \dots, (p_h, q_h))$. We fix any flat structure on S and get an injection $\pi_1(S) \hookrightarrow \mathrm{Isom}^+(\mathbb{R}^2)$. From that, the idea is to lift this map to get an injection $\pi_1(M) \hookrightarrow \mathrm{Isom}^+(\mathrm{Nil})$ such that the following diagram commutes:

$$\begin{array}{ccc} \pi_1(M) & \dashrightarrow & \mathrm{Isom}^+(\mathrm{Nil}) \\ \downarrow & & \downarrow \\ \pi_1(S) & \hookrightarrow & \mathrm{Isom}^+(\mathbb{R}^2) \end{array}$$

7.6 $\widetilde{\mathrm{SL}}_2$

Recall that the tangent bundle TM of a Riemannian manifold M has a natural Riamannian structure and that the unit tangent bundle $UM \subset TM$ consists of all unitary tangent vectors and inherits a Riemannian structure as well. Focusing on the case $M = \mathbb{H}^2$ where we represent $\mathbb{H}^2$ as the upper-half-plane model H^2, we get $T\mathbb{H}^2 = H^2 \times \mathbb{C}$ and $U\mathbb{H}^2 = H^2 \times S^1$.

Let $\widetilde{\mathrm{SL}}_2$ be the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Being the universal cover of a Lie group, it is itself a Lie group and we have the following exact sequence:

$$\widetilde{\mathrm{SL}_2} \longrightarrow \mathrm{SL}_2(\mathbb{R}) \longrightarrow \mathrm{PSL}_2(\mathbb{R}) = \mathrm{Isom}^+(\mathbb{H}^2).$$

The group $\mathrm{PSL}_2(\mathbb{R})$ acts freely and transitively on the unit tangent bundle $U\mathbb{H}^2$, so we can identify $\mathrm{PSL}_2(\mathbb{R})$ with $U\mathbb{H}^2$. With this identification, $\mathrm{PSL}_2(\mathbb{R})$ inherits a left-invariant Riemannian metric, which lifts to a left-invariant Riemannian metric on $\widetilde{\mathrm{SL}}_2$: we may identify the latter with $\mathbb{H}^2 \times \mathbb{R}$.

Again, one has to find and study the isometry group of $\widetilde{\mathrm{SL}}_2$ which will not be done here. We only state that the following sequence is exact:

$$0 \to \mathbb{R} \to \mathrm{Isom}(\widetilde{\mathrm{SL}}_2) \xrightarrow{p} \mathrm{Isom}(\mathbb{H}^2) \to 0.$$

Proposition 7.11 holds for $\Gamma < \mathrm{Isom}^+(\widetilde{\mathrm{SL}}_2)$ and Proposition 7.13 holds for $\Gamma < \mathrm{Isom}(\mathbb{H}^2)$ and $f : \Gamma \to \mathrm{Isom}^+(\widetilde{\mathrm{SL}}_2)$.

Proposition 7.15. The interior of a compact orientable manifold admits a finite-volume complete $\widetilde{\mathrm{SL}}_2$ geometry if and only if it is a Seifert manifold with $\chi < 0$ and either $\partial M \neq \emptyset$ or $e \neq 0$.

Similarly to the two previous theorems, the idea of the first implication is the following: writing $\mathrm{int}(M)$ as $\widetilde{\mathrm{SL}}_2/\Gamma$ with Γ cofinite, we get a Seifert fibration $M \to S$ over the finite-area orbifold $S = \mathbb{H}^2/p(\Gamma)$. If M is closed, we get $e \neq 0$: otherwise, up to taking a finite-index subgroup, we would have $M = S \times S^1$. The converse implication use exactly the same idea as for the Nil geometry.

7.7 Sol

The Sol geometry is the least symmetric one among the eight. Hence, before defining it we must introduce some new concepts.

7.7.1 Asonov monodromy

Proposition 7.16. Every surface bundle over S^1 is constructed by taking $\Sigma \times [0, 1]$ and gluing $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ via an orientation-preserving diffeomorphism ψ.

Proof. Such a gluing clearly gives rise to a surface bundle over S^1. Conversely, cutting a surface bundle over S^1 along a fiber yields a surface bundle over the interval, which is diffeomorphic to the product $\Sigma \times [0, 1]$. ☐

The diffeomorphism ψ is called the monodromy of the surface bundle denoted M_ψ.

Remark 7.17. Since isotopic gluings produce diffeomorphic manifolds, the three-manifold M_ψ depends only of the class of ψ in the mapping class group $\mathrm{MCG}(\Sigma)$ of Σ. More than that, it actually depends only on its conjugacy class.

Definition 7.18. A torus bundle is a surface bundle $M \to S^1$ which fiber is a torus T.

Every matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ defines a torus bundle M_A with monodromy A. To understand when M_A is a Seifert manifold, we have the following proposition.

Proposition 7.19. Let $M = M_A$ be a torus bundle with monodromy $A \neq \pm I$. The following holds:

- if $|\mathrm{tr}A| < 2$ then M is a Seifert manifold with $e = 0$ and $\chi = 0$,
- if $|\mathrm{tr}A| = 2$ then M is a Seifert manifold with $e \neq 0$ and $\chi = 0$,
- if $|\mathrm{tr}A| > 2$ then M is not a Seifert manifold.

When $|\mathrm{tr}A| > 2$, we say that the monodromy A is Anosov.

Every semi-bundle is doubly covered by a canonical bundle. A torus bundle is said to be of Anosov type if its monodromy is Anosov. A torus semi-bundle is of Anosov type if its double covering is.

7.7.2 Sol geometry

The Sol group has a bundle structure, but with a one-dimensional basis: it is a $\mathbb{R}^2$-bundle over $\mathbb{R}$. Again, the geometry is fully governed by a Lie group Sol which is the space $\mathbb{R}^3$ equipped with the following operation:

$$(x,y,z)\cdot(x',y',z') = \left(x+e^{-z}x',\ y+e^{z}y',\ z+z'\right).$$

The subgroup $\mathbb{R}^2$ consisting of all elements $(x,y,0)$ is the center of Sol, and by setting $p(x,y,z)=z$, we get an exact sequence:

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \mathrm{Sol} \xrightarrow{\ p\ } \mathbb{R} \longrightarrow 0.$$

Therefore, Sol is a plane bundle over $\mathbb{R}$. We define a metric on Sol by assigning the scalar product

$$\begin{pmatrix} e^{2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

at the point (x,y,z). This metric is left-invariant, and every plane $\{z=k\}$ is isometric to the Euclidean $\mathbb{R}^2$. This is the geometry with the smallest amount of symmetries.

Again, we will not study its isometry group. Nevertheless, now we can understand the following theorem.

Theorem 7.20. The interior of a compact orientable manifold admits a finite-volume complete Sol geometry if and only if it is a torus (semi-)bundle of Anosov type.

7.8 Summary

If we sum up all the theorems stated, we get the following.

Theorem 7.21. A closed orientable 3-manifold has a geometric structure modelled on one of the following six geometries

$$\mathbb{S}^3,\ \mathbb{R}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ \mathrm{Nil},\ \widetilde{\mathrm{SL}_2}$$

if and only if it is a Seifert manifold of the appropriate commensurability class, as prescribed by Table 2. It has a Sol geometric structure if and only if it is a torus (semi-)bundle of Anosov type.

	$\chi>0$	$\chi=0$	$\chi<0$
$e=0$	$S^2\times\mathbb{R}$	$\mathbb{R}^3$	$\mathbb{H}^2\times\mathbb{R}$
$e\neq 0$	S^3	Nil	$\widetilde{\mathrm{SL}_2}$

Table 2. The closed manifolds modelled on six geometries are precisely the six commensurable classes of Seifert manifolds, distinguished by the numbers e and χ.

As expected, the geometries are mutually exclusive: recall that two manifolds are said commensurable if there is a manifold that covers both of them with finite degrees.

Proposition 7.22. Two closed 3-manifolds admitting different geometries are not diffeomorphic, and not even commensurable.

The main goal of this part was to give some mathematical culture about 3-manifolds. A lot of technical proof were skipped and a lot of choices were made. In fact, we did not mentioned the construction of three-manifolds or the Mostow rigidity theorem, for instance. Complementary information can be found in references such as [9], [12] or [11].

Part IV
Four-manifolds

Recall that a topological manifold is a second countable, Hausdorff topological space locally homeomorphic to $\{x \in \mathbb{R}^n \mid x^n \geqslant 0\}$ where the boundary corresponds to $\{x^n = 0\}$. For $n \leqslant 3$, a smooth structure on a manifold exists on every topological manifold: the notions coincide ($n = 2$: Radò, 1925, $n = 3$: Moise, 1952). This is no longer true in four-dimensional spaces. For instance, smooth manifolds admit an essentially unique triangulation but not all topological manifolds are triangulable (Casson, 1990). We will first introduce the h-cobordism theorem, whose proof rely on handle decomposition and then give some idea of classifications for topological and smooth manifolds thanks to the intersection forms. This part is largely inspired by lectures notes from Bruno Martelli [7] and Marco Marengon [6].

8 Handles and h-cobordism

Let's denote CAT the category of manifolds that we consider: CAT = C^∞ or TOP. Recall from sub-section 2.1 that every compact smooth manifold may be described via some handle decomposition.

Theorem 8.1. A TOP-handle decomposition of a TOP-4-manifold M exists if, and only if, W is smoothable.

Idea of the proof. The reverse implication is clear: if the manifold is smooth then it admits a C^∞ handle decomposition that is in particular a TOP handle decomposition. For the direct implication, any homeomorphic embedding of smooth 3-manifolds is uniquely smoothable, so a handle decomposition of a topological 4-manifold determines a smooth structure. □

Since we will use handle decompositions, for the rest of this section we only consider smooth manifolds.

8.1 Revisiting Handle Decomposition

Handle decomposition was already introduced in the sub-section 2.1. We give a closer look in dimension 4.

a) One-handles Let M be a connected compact smooth 4-manifold, possibly with boundary. The manifold M has a handle decomposition with one 0-handle and at most one 4 handle. The boundary of the 0-handle is S^3. Every 1-handle is attached to S^3, more specifically it glues $D^3 \times D^1$ to two 3-discs in S^3 along $D^3 \times S^0$. We can therefore encode each 1-handle by drawing couples of 3-discs in S^3.

Remark 8.2. Recall that the boundary-connected-sum is defined by two embeddings $\phi : D^{n-1} \to \partial M$ and $\psi : D^{n-1} \to \partial N$ and by the construction $M \cup_{\psi\circ\phi} N$ denoted $M \natural N$.

Definition 8.3. Let's denote $\#_k M$ the connected sum of k copies of M setting $\#_0 M^n = S^n$. Analogously, let $\natural_k M^n$ be the $\partial-$connected sum of k copies of M setting $\natural_0 M^n = D^n$.

Using only 0- and 1-handles, we can actually construct few manifolds.

Proposition 8.4. Let a connected 4-manifold M decompose into 0-handle and 1-handles. Then, $M = \natural_k(D^3 \times S^1)$ and $\partial M = \#_k(S^2 \times S^1)$ for some $k \geqslant 0$.

Proof. First, by proposition 2.7, we can suppose there is one 0-handle only. Now we claim that if M^n has connected boundary and N^n obtained from M by attaching a 1-handle, then $N = M\natural(D^{n-1} \times S^1)$. This would prove the proposition. To prove the claim, observe that the 1-handle is attached along two $(n-1)$-discs $D_1, D_2 \subset \partial M$. The Cerf Lemma claims that there is a unique orientation-preserving embedding between a disc D^n and any connected oriented manifold M^n, up to self homeomorphisms of M. Hence, these two discs D_1 and D_2 are contained in a bigger $(n-1)$-disc $D \subset \partial M$. If we cut N along D we get M and $D^{n-1} \times S^1$ as required. □

b) Two-handles By definition, a 2-handle is the product $D^2 \times D^2$ attached along the solid torus $S^1 \times D^2$. Different 2-handles are attached along disjoint solid tori. The attaching sphere is a circle of the form $S^1 \times \{0\}$ which can be depicted as a circle embedded in S^3.

However, the gluing of the solid torus $D^2 \times S^1$ is not fully determined by the image of the attaching circle alone. Attaching such a torus requires choosing a framing of the normal bundle that means fixing two independent sections of the normal bundle that form a basis at each point. In practice, one section suffices: the second is then determined up to homotopy and up to sign, but these do not affect the way the 2-handle is attached.

c) Three- and four-handles To know whether we can attach 3- and 4-handles to close up a manifold, we use the following.

Lemma 8.5. Let a 4-manifold N decompose into 0-, 1-, and 2-handles. It is possible to get a closed manifold by attaching 3- and 4-handles to N if and only if $\partial N \cong \#_k(S^2 \times S^1)$ for some $k \geqslant 0$.

Proof. Handles of index 3 and 4 may be turned upside down. They become 1- and 0-handles. Therefore, together they form a manifold as in Proposition 8.4, whose boundary is diffeomorphic to $\#_k(S^2 \times S^1)$. □

There are many ways to attach those 3- and 4-handles to close up the manifold, but they luckily all lead to the same one, thanks to the following.

Proposition 8.6 (Laudenbach-Poenaru)**.** If M and M' are two closed manifolds obtained by attaching 3- and 4-handles to the same manifold N with boundary, then $M \cong M'$.

d) Without 1- and 3-handles Many interesting manifolds admit a decomposition without 1- and 3-handles. That is, we have one 0-handle, some 2-handles, and maybe one 4-handle to close the manifold up. A manifold having a decomposition of this type is necessarily simply connected; conversely, we still do not know if every closed simply connected 4-manifold may be described in this way:

Does every simply connected compact 4-manifold have a decomposition without 1-handles?

8.2 h-cobordism theorem

Recall the notation $-M$ for an oriented manifold M with the opposite orientation.

Definition 8.7. A cobordism W from M^n to N^n that are compact without boundary oriented n-manifolds is a compact oriented $(n+1)-$manifold with boundary $\partial W = (-M) \cup N$. If such a W exists, M and N are called CAT-cobordant.

Remark 8.8. Since $\partial W = (-M) \cup N = -(-N) \cup (-M)$, W is also a cobordism from $-N$ to $-M$: this is called the *upside down* cobordism. Note that we haven't changed the orientation of W.

Remark 8.9. We got a natural cobordism $I \times M$ from M to M, called the *identity cobordism.*

Definition 8.10. A cobordism W from M to N is a h-cobordism if both the inclusions $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

Let's first state the h-cobordism theorem in high dimension and give the idea of its proof.

Theorem 8.11 (h-cobordism, Smale 1960s)**.** Let W^{n+1} be a CAT-cobordism from M to N with $\pi_1(W) = \pi_1(M) = \pi_1(N) = 1$ and $H_*(W, M; \mathbb{Z}) = 0$. If $n \geqslant 5$, there exists a CAT-isomorphism $W \cong I \times M$ which is the identity on $M \to \{0\} \times M$. Thus,

$$M \underset{\text{CAT}}{\cong} N.$$

Idea of the proof. We consider only the case CAT $= \mathcal{C}^\infty$ for simplicity: we can use handle decomposition without worrying that the smooth techniques go through (tangent bundles, transversality,...). The TOP version is done by Kirby-Siebenmann.

Pick a handle decomposition for W with starting point M. The goal is to modify it removing handles to arrive at the identity cobordism $I \times M$. Without loss of generality, we can suppose that W:

1. has no 0-handles. In fact, connctedness is completely determined by 0- and 1- handles: every 0-handle is connected to another 0-handle or to M by a 1-handle, necessarily geometrically complementary. We cancel them.

2. has no 1-handles. In fact, simply-connctedness is completely determined by 1- and 2-handles. A 1-handle gives a loop γ, which can be pushed in M_2: $\gamma = \partial D^2$ with $D^2 \hookrightarrow W^{n+1}$ (Whitney embedding). Without loss of generality, we can suppose that γ is neither in an attaching sphere not in an attaching belt since belt spheres have dimension $n-3$, they miss 2-dimensional objects in M_i. Using the fact that $\pi_1(W) = 1$, we can use a $2-$handle to cancel the geometrically complementary 1-handle given by γ.

3. has no n- and $(n+1)-$handles. We can use the two previous points on the upside down cobordism.

The handles define a handle chain complex by $C_k \coloneqq \mathbb{Z}\langle k - \text{handles}\rangle$ and by:

$$\begin{array}{cccl} \partial_k & : & C_k & \longrightarrow & C_{k-1} \\ & & h^k & \longmapsto & \sum_{h^{k-1}} \#(A^{k-1} \cap B^{n+1-k})\, h^{k-1}. \end{array}$$

where the number in the sum is the algebraic intersection of the attaching sphere and the belt sphere in M_{k-1}. This construction gives an homology $H_*(C_k)$ is isomorphic to the singular holomogy of (W, M) (the proof of this fact is the same as the one for cellular homology). Since, by hypothesis, $H_*(W, M; \mathbb{Z}) = 0$, each k-handle is paired with either a $(k-1)$- or $(k+1)$-handle and they are algebraically complementary. If we can make them geometrically complementary, then we can cancel them all and win.

Facts. M_k is simply connected if $n \geqslant 4$ and $M_k \setminus (A^k \cup B^{n-k})$ is simply connected if $n \geqslant 5$.

Using the fact, if we consider $x, y \in A^k \cap B^{n-k}$ of opposite signs, the path γ from x to y passing through A^k and from y to x passing through B^{n-k} is homotopically trivial. By Whitney's embedding theorem (here $n \geqslant 2 \cdot \dim D^2 + 1$), we can assume $D^2 \subseteq M_k$ embedded. Hence, we can define an isotopy of A^k supported in a neighbourhood of D^2 that removes the two intersections. □

In 1982, Freedman showed that the h-cobordism holds in CAT= TOP in dimension 4. The theorem is a powerfull tool to the classification of manifolds.

Theorem 8.12 (Topological h-cobordism in dimension 4)**.** Let $W^5 : M^4 \to N^4$ be TOP-cobordism with $\pi_1(W) = \pi_1(M) = \pi_1(N) = 1$ and $H_*(W, M; \mathbb{Z}) = 0$. Then, there exists an isomorphism $W \cong_{TOP} I \times M$ which is the identity on $M \to \{0\} \times M$. Thus,

$$M \underset{\text{TOP}}{\cong} N.$$

Sketch of proof. The beginning is the same as in dimension $n \geqslant 5$:

$\star$) Take a TOP-handle decomposition of W rel M, which exists by a theorem of Quinn,

$\star$) Rearrange handle and cancel all 0-, 1- , 4- , and 5-handles: W consists of only 2- and 3- handles.

$\star$) The handle complex is $0 \to C_3 \xrightarrow{\partial_3} C_2 \to 0$, and since $H_*(W, M; \mathbb{Z}) = 0$, the map ∂_3 is an isomorphism: after change of basis (achieved by handleslides) 2- and 3-handles are paired-up.

$\star$) h^2 and h^3 are algebraically complementary, but now the Whitney trick fails.

Again, we consider γ a path from x to y on A and from y to x on B. The 4-manifold after the 2-handles and before the 3-handles M_2 is still simply connected: there exists an immersed disc D in M_2 with boundary γ. There might have problems due to the fact that the dimension is strictly smaller than 5:

$\star$) D might intersect $A \cup B$: we know that $M_2 \setminus A$ and $M_2 \setminus B$ are simply connected, so by a theorem from Freeman, there exist spheres that are geometrically dual to A and B. After some "Casson moves" we get a knew collection of A' and B' such that $\pi_1(M_2 \setminus (A' \cup B')) = 1$.

$\star$) D might have a wrong framing: there exists an immersed sphere T_D algebrically dual to D such that $T_D \cdot T_D = \pm 1$, we take $D \cup T_D$ and we get a new disc with the right framing.

$\star$) D might have double points: near a double point, there are two branches. Connect them with a loop γ' (that goes from the double point to itself). If $\gamma' = \partial D'$, an embedded disc in the complement of $(A \cup B \cup D)$, then apply the Whitney trick and get rid of the double point (using kink).

However, in finding D' we run in the same 3-problem as before.

1) Making the complement of D simply connected (so there exist D' immersed). Fact: the complement of D has perfect π_1.

2) Finding a framing of D': can be fixed with an algebraically dual sphere to D.

3) Double points of D': we push the problem to D''.

By iterating the procedure, we construct an infinite object called a Casson handle C. We know that this Casson handle is homotopically equivalent and homeomorphic to $D^2 \times \mathbb{R}^2$ hence is a genuine topological Whitney disc and can be used to cancel interaction between A and B. ☐

9 Topological 4-manifold

Much topology in dimension 4 is controlled by an important object, which is absent in lower dimensions: the intersection form.

9.1 Intersection forms

On a simply connected 4-manifold X, all the homology is concentrated in the free second homology group $H_2(X, \mathbb{Z})$. In fact, the homology groups $H_k(X; Z)$ are trivial for $k \notin \{0, 2, 4\}$. For $k \notin \{0, 4\}$, these homology groups are canonically identified with $\mathbb{Z}$. The only remaining homology group carries a symmetric bilinear form called *intersection pairing.*

Definition 9.1. If X is a compact oriented topological 4-manifold, its intersection form is

$$\begin{array}{rcrcl} Q_X & : & H^2(X, \partial X; \mathbb{Z}) \times H^2(X, \partial X; \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ & & (\alpha, \beta) & \longmapsto & \langle \alpha \cup \beta, X \rangle := (\alpha \cup \beta)([X]). \end{array}$$

where $[X] \in H_4(X, \mathbb{Z})$ is the fundamental class associated to the orientation.

Remark 9.2. Using properties of the embedding, we can extend the definition to the case where X is not connected and/or not compact. In the case where X is not orientable, there still exists a definition of Q_X over $\mathbb{Z}/2\mathbb{Z}$.

Remark 9.3. Changing the orientation replaces the intersection form by its negative: $Q_{-X} = -Q_X$.

A geometric way to see the intersection form is given by the following.

Theorem 9.4. Let X be a compact oriented smooth manifold and let α, β be elements of $H_2(X, \partial X)$. If $[\Sigma_\alpha], [\Sigma_\beta] \in H_2(X)$ are the Poincaré duals, then

$$Q_X(\alpha, \beta) = \#(\Sigma_\alpha \pitchfork \Sigma_\beta)$$

Proof. We can represent α by a 2-form η_α supported in a neighbourhood of Σ_α. In coordinates, if $\Sigma_\alpha = \{x = y = 0\}$, then η_α can be chosen as

$$\eta_\alpha = f(x, y) \mathrm{d}x \wedge \mathrm{d}y$$

where $f(\cdot, \cdot)$ is a bump function near 0 with integral on $\mathbb{R}^2$ equal to 1. Analogously, choose a similar η_β for β and we get

$$\int_X \alpha \cup \beta = \sum_{p \in \Sigma_\alpha \cap \Sigma_\beta} \int_{\nu(p)} f(x, y) \cdot f(z, w) \cdot (\pm \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}w) = \sum_p \mathrm{sgn}(p) = \#(\Sigma_\alpha \pitchfork \Sigma_\beta).$$

The sign in the integral depending on the sign of the intersection. □

It follows from the definition some properties of the intersection form.

Proposition 9.5. The intersection form is symmetric and vanishes on the torsion part of $H^2(X, \partial X; \mathbb{Z})$.

Remark 9.6. The determinant of Q_X is well-defined.

Example 9.7. Let's give some examples.

- Q_{S^4}: since $H^2(S^4) = 0$, the intersection form is trivial;
- $Q_{\mathbb{CP}^2} = (1)$. In fact, since $H_2(\mathbb{CP}^2) = \mathbb{Z}$, any projective line represents a generator. By Bezout's theorem, any two lines intersect in exacty ont point, thus the matrix representing $Q_{\mathbb{CP}^2}$ is [1] ;

- $Q_{\mathbb{CP}^2\#\overline{\mathbb{CP}^2}} = Q_{\mathbb{CP}^2} \oplus -Q_{\mathbb{CP}^2} = [1] \oplus [-1] = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$;
- $Q_{S^2\times S^2} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Indeed, $H_2(S^2 \times S^2; \mathbb{Z}) = \mathbb{Z}[\{pt\} \times S^2] \oplus \mathbb{Z}[S^2 \times \{pt\}]$. Hence, the generators are represented by fibers of the projections onto the factors. As fibers of the same projection do not intersect and fibers of different projections intersect in exactly one point, the intersection form is represented by the hyperbolic plane.

The intersection form has some invariant.

Definition 9.8. For a compact oriented topological 4-manifold X, we define

- The rank as $\operatorname{rk} Q_X = \operatorname{rk}_{\mathbb{Z}}(H^2(X, \partial X; \mathbb{Z}))$, the second Betti number.
- The signature $\sigma(X) = \sigma(Q_X \otimes_{\mathbb{Z}} \mathbb{R})$, which is equal to the number of positive eigenvalues minus the number of negative eigenvalues. We also define $b_2^{\pm}$ as the rank of the maximal $(\pm)-$definite subspace. Hence, the signature is $b_2^+ - b_2^-$.
- the parity: Q_X is even if $Q_X(x, x) \equiv 0 \pmod 2$ for all $x \in H_2$ and is odd otherwise.

Remark 9.9. Q_X is even if, and only if, all diagonal entries of the associated matrix are even.

Remark 9.10. We can define the same notions of rank, signature and parity for any symmetric bilinear form $Q : \Lambda \times \Lambda \to \mathbb{Z}$ by replacing H_2 by Λ. The first is the rank of Λ as a module over the intergers and the second is well defined thanks to Sylvester's law of inertia.

Example 9.11. The intersection forms $Q_{\mathbb{CP}^2}$ and $Q_{S^2\times S^2}$ are ever while $Q_{\mathbb{CP}^2\#\overline{\mathbb{CP}^2}}$ is odd.

Definition 9.12. Let $\Lambda = \mathbb{Z}^n$, and $Q : \Lambda \times \Lambda \to \mathbb{Z}$ be a symmetric bilinear form. Then Q is called unimodular is $x \in \Lambda \mapsto Q(x, \cdot) \in \Lambda^*$ is an isomorphism.

Remark 9.13. Given a basis b for Λ and its dual basis b^* for Λ^*, the matrices associated to Q and L are the same. It follows that Q is unimodular if and only if its determinant is ± 1.

Actually, there was a reason why we gave the definition only for intersection forms and not for generic bilinear forms: it is stated without proof in the following theorem.

Theorem 9.14. Every symmetric bilinear form is the intersection form of some compact simply connected 4-manifold with boundary.

9.2 The quadratic form E_8

There are finitely many isomorphism types of unimodular forms on each rank r. Every form of rank $r \leqslant 8$ is constructed by summing the elementary forms $[1], [-1], H$, where $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, except the following form of rank $r = 8$.

$$E_8 := \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & & 1 \\ & & & & 1 & 2 & 1 & \\ & & & & & 1 & 2 & 1 \\ & & & & 1 & & & 2 \end{pmatrix}.$$

Proposition 9.15. The matrix E_8 is even, definite, positive. Its principal determinants are respectively 2,3,4,5,6,7,8,1 where $1 = \det(E_8)$, and its signature is $\sigma(E_8) = 8$.

Proof. These are calculus except for the first assertion that holds thanks to Remark 9.9 □

Remark 9.16. The only even forms we may obtain with $[1], [-1]$, and H are nH where $nH = H \oplus \cdots \oplus H$ if $n > 0$, $nH = (-H) \oplus \cdots \oplus (-H)$ if $n < 0$ and $0H$ is the zero rank (empty) form. All the nH have signature zero: therefore E_8 is a new form. By using E_8 and H, we may construct more even forms.

9.3 Freedman's theorems

The main results of Freedman presented here are highly non-trivial and are thus only stated. They furnish a beautiful and simple description of the world of topological 4-manifolds.

9.3.1 Homology spheres

We introduce here some notions that are essential to the comprehension of Freedman's results: homology spheres.

Definition 9.17. A homology sphere is a closed n-manifold N having the same $\mathbb{Z}-$homology as a sphere S^n. That is, having $H_i(N, \mathbb{Z}) = \{e\}$ for all $i = 1, ..., n-1$.

Proposition 9.18. Let M be an oriented compact simply connected 4-manifold with boundary such that ∂M is connected. The intersection form Q on $H_2(M, Z)$ is unimodular if and only if ∂M is a homology sphere.

Proof. Consider the exact sequence (over the integers)

$$H_3(M, \partial M) \xrightarrow{\partial} H_2(\partial M) \xrightarrow{i} H_2(M) \xrightarrow{j} H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i} H_1(M)$$

Since M is simply connected, the first and last modules are trivial. Therefore we have

$$0 \xrightarrow{\partial} H_2(\partial M) \xrightarrow{i} H_2(M) \xrightarrow{j} H_2(M, \partial M) \xrightarrow{\partial} H_1(\partial M) \xrightarrow{i} 0$$

The module $H_2(M, \partial M)$ is canonically identified with $H^2(M)$, which can in turn be identified with the dual module $H_2(M)^*$ since there is no torsion (because M is simply connected). The map j can therefore be interpreted as a map

$$j : H_2(M) \to H_2(M)^*$$

and this map is in fact the adjunction of the intersection form on $H_2(M)$. By definition, the intersection form is unimodular if and only if j is an isomorphism. This holds if and only if both $H_2(\partial M)$ and $H_1(\partial M)$ vanish, as required.

□

A particular case of the proposition is the following.

Corollary 9.19. The boundary of a contractible 4-manifold is a homology sphere.

There are actually plenty of contractible smooth 4-manifolds. Note that we need 1-handles to construct them: by using 0- and 2-handles we only get D^4 (since the Euler characteristic must be 1).

Remark 9.20. The only smooth contractible manifold currently known bounded by S^3 is D^4: whether this is the only one or not, the question is equivalent to the still open smooth 4-dimensional Poincaré conjecture:

Let M^4 be smooth and homotopy equivalent to S^4. Is M diffeomorphic to S^4?

9.3.2 Freedman's theorems

The previous sub-section raised natural questions:

⋆) Which intersection forms are realized by 4-manifolds?

⋆) Are 4-manifolds completely determined by their intersection forms?

Freedman's theorem can be considered as the first thunderbolt that struck 4-manifold theory by replying those questions.

Theorem 9.21 (Freedman)**.** Every homology sphere bounds a topological contractible 4-manifold.

Sketch of proof. Let Σ be a homology 3-sphere. Take $\Sigma \times [0,1]$.

1) By doing some topological surgery, transform $\Sigma \times [0,1]$ into a manifold S with the same boundary and homology, but simply connected.

2) Take countably many copies $S_1, S_2, \ldots, S_k \ldots$ of S, glue them altogether and compactify with one point. The resulting object is clearly contractible. Much less clearly, it is a topological manifold: the cone point has indeed a neighborhood homeomorphic to a 4-ball.

□

Remark 9.22. This theorem is not valid in the smooth category. For instance, Poincaré homology sphere does not bound any smooth contractible 4-manifold.

Corollary 9.23 (Freedman)**.** Every symmetric unimodular bilinear form is the intersection form of a simply connected closed topological 4-manifold.

Proof. By Theorem 9.14 every symmetric unimodular bilinear form Q is the intersection form of a simply connected 4-manifold M with boundary. By Proposition 9.18 the boundary ∂M is a homology sphere. It therefore bounds a contractible topological manifold N. Glue M and N together. Since N is contractible, the resulting manifold has the same fundamental group and 2-homology as M. □

Again, this result is far from being true in the smooth setting. This leads to plenty of topological 4-manifolds having no smooth structure.

Theorem 9.24. Every even *(resp. odd)* symmetric unimodular bilinear form is the intersection form of precisely one *(resp. two)* simply connected topological closed 4-manifold, up to homeomorphism.

Example 9.25. The form [1] is odd. There are therefore two topological manifolds with this form. One is $\mathbb{CP}^2$, while the other one is denoted by $*\mathbb{CP}^2$. Both such manifolds can be constructed as follows: take a knot. Its boundary is a homology sphere, which can be closed up via a contractible topological manifold. The trivial knot is homeomorphic to $\mathbb{CP}^2$ while the trefoil knot gives $*\mathbb{CP}^2$.

Example 9.26. The form E_8 is even. There is only one topological closed, simply connected manifold with form E_8 and is named the E_8-manifold. It does not admit a smooth structure. One can prove it by absurd using Rokhlin's theorem: "If a smooth compact 4-manifold has a spin structure, then the signature of its intersection form is divisible by 16". We will neither explain what a spin is nor give more details about this result, the curious reader can see lectures notes of [6] or [3] for further information.

As a corollary of Freedman's theorem, we have the following result that responds to Remark 9.20 in the topological category.

Corollary 9.27 (Poincaré conjecture in dimension 4)**.** A closed topological manifold homotopically equivalent to S^4 is homeomorphic to S^4.

10 Smooth 4-manifolds

For the smooth category, some important theorems about 4-manifolds were stated and proved before 1970. In particular, there are Whitehead's theorem, which says that the homotopy type of a closed simply connected 4-manifold is entirely determined by its intersection form; Wall's theorem, which says that two simply-connected smooth manifolds become diffeomorphic after summing with some copies of $S^2 \times S^2$ and Rohlin theorem, which says that an oriented 4-manifold with zero signature bounds a 5-manifold.

10.1 The Whitehead theorem

A way to prove Whitehead's theorem is to use a famous construction determining various homotopy groups of spheres.

10.1.1 The Thom-Pontryagin construction and wedge product of spheres

Except if specified, X denote a smooth closed manifold.

Definition 10.1. A framed submanifold $Y^k \subset X^{m+k}$ of X is a smooth submanifold equipped with a trivialization of the normal bundle (that is, a framing on the normal bundle), *i.e.* it is a smooth submanifold with an identification of N_Y with $Y \times \mathbb{R}^m$.

Definition 10.2. A cobordism of two framed k-manifolds Y_0, Y_1 of X is a properly embedded framed $(k+1)-$manifold $Z \subset X \times [0,1]$, whose intersection with $X \times \{0\}$ and $X \times \{1\}$ coincides with $Y_0 \times \{0\}$ and $Y_1 \times \{1\}$ as framed manifolds.

Fix a point $p \in S^m$ and let $f : X^{m+k} \to S^m$ be a smooth map which is transverse to p. Its counterimage $f^{-1}(p)$ is a smooth submanifold $Y^k \subset X$. Take a small disc D^m around p. Over this disc, the map looks like a projection $Y \times D^m \to D^m$ and this equips the manifold Y with a framing.

Proposition 10.3 (Thom-Pontrjagin construction)**.** This operation defines a bijection

$$[X^{m+k}, S^m] \longrightarrow \Omega_k^{framed}(X)$$

between the set $[X^{m+k}, S^m]$ of maps from X^{m+k} to S^m seen up to homotopy, and the set $\Omega_k^{framed}(X)$ of k-dimensional framed submanifolds in X seen up to cobordism.

Proof. We prove that the function

$$\Psi : [X^{m+k}, S^m] \longrightarrow \Omega_k^{\text{framed}}(X^{m+k})$$

introduced above is well-defined. Given f, the trivialization on $Y = \Psi(f)$ is well-defined only up to homotopy; however, homotopic trivializations are easily seen to be cobordant, so this is not a problem. Let f_0 and f_1 be two functions, both transverse to p, linked by a homotopy $F : X \times I \to S^m$. They define two framed manifolds Y_1 and Y_2. We can perturb F so that it is also transverse to p. The preimage $F^{-1}(p)$ is thus a framed manifold $Z \subset X \times I$ which connects Y_1 and Y_2: these are thus cobordant, as required.

We define an inverse

$$\Phi : \Omega_k^{\text{framed}}(X^{m+k}) \longrightarrow [X^{m+k}, S^m]$$

as follows. Let $Y^k \subset X^{m+k}$ be a framed manifold. A tubular neighborhood is identified with $Y^k \times D^m$. Let $D^m \to S^m$ be the surjective map which sends 0 to p and collapses ∂D^m to the antipodal point q. By projecting $Y^k \times D^m$ onto its second factor we get

$$Y^k \times D^m \longrightarrow D^m \longrightarrow S^m.$$

Extend this map to the whole of X by sending every point in $X^{k+m} \setminus (Y^k \times D^m)$ to q. We get a map $X \to S^m$, as required. If Y^k changes by cobordism, the resulting map changes by homotopy. This defines Φ.

The map $\Psi \circ \Phi$ is clearly the identity. We prove that $\Phi \circ \Psi$ also is. A map f_0 induces a framed $Y = \Psi(f_0)$, which in turn induces another map $f_1 = \Phi(Y)$. The maps f_0 and f_1 coincide (up to homotopy) on a fixed tubular neighborhood $Y \times D^m$, which is sent to a disc $D \subset S^m$, and may differ a lot on the complement $X \setminus (Y \times D^m)$. However, such a complement is sent by both f_0 and f_1 to the complementary disc $S^m \setminus \text{int}(D)$. Two maps with values in a disc are homotopic (relative to their boundary), and hence we are done. □

The following generalization of Pontryagin-Thom construction will be needed in the proof of Whitehead's theorem. Let $\vee_h S^2$ be a wedge product of h spheres. Fix points $p_1, \ldots, p_h$ in distinct spheres, disjoint from the vertex v of the wedge. Let f be a continuous map

$$f : X^{m+k} \longrightarrow \vee_h S^2$$

which is everywhere smooth except at $f^{-1}(v)$, and is transverse to $p_1, \ldots, p_h$. The counterimages $f^{-1}(p_1), \ldots, f^{-1}(p_h)$ define h disjoint (not necessarily connected) framed k-submanifolds of X. Futhermore, one can prove the following.

Proposition 10.4. This operation defines an isomorphism of groups

$$\pi_3(\vee_h S^2) \longrightarrow S(h, \mathbb{Z}) \cong \mathbb{Z}^{\frac{(h+1)h}{2}}$$

where $S(h, \mathbb{Z})$ is the group of all symmetric integer matrices of rank h.

10.1.2 The Whitehead theorem

Let's first state some prerequisites to the main proof of this sub-section.

Theorem 10.5 (Whitehead, homology)**.** Let $f : X \to Y$ be a continuous map between simply connected CW-complexes. It is a homotopy equivalence if and only if it induces isomorphisms $f_* : H_n(X, \mathbb{Z}) \to H_n(Y, \mathbb{Z})$ on all homology groups.

Theorem 10.6 (Hurewicz)**.** Let X be a conncted CW-complex. If $\pi_1(X) = \cdots = \pi_n(X) = 0$ for some $n \geqslant 1$, then $f_{n+1} : \pi_n(X) \to H_n(X;\mathbb{Z})$ is an isomorphism.

Corollary 10.7. If X is a simply connected 4-manifold, every element in $H_2(X^4, \mathbb{Z})$ can be realized as an immersed sphere.

Proof. By Hurewicz Theorem the map $\pi_2(X) \to H_2(X;\mathbb{Z})$ is an isomorphism. Therefore every element in $H_2(X;\mathbb{Z})$ is realized as a map $f : S^2 \to X$ which can be perturbed to an immersion. □

Theorem 10.8 (Whitehead)**.** Let M and N be two closed smooth oriented simply connected 4-manifolds. They are homotopically equivalent if and only if their intersection forms Q_M and Q_N are isomorphic.

Proof. Two homotopically equivalent manifolds have the same cohomology ring and thus the same intersection form. Conversely, let M and N have isomorphic intersection forms. Let $\dot{M}$ be M with the interior of a 4-disc removed. The only non-trivial homologies of $\dot{M}$ are $H_0 = \mathbb{Z}$ and $H_2 = \mathbb{Z}^h$. Since M is simply connected, Hurewicz theorem guarantees that every element in H_2 is represented by an immersed sphere. In particular, a basis $\{\alpha_1, \ldots, \alpha_h\}$ is represented by immersions $f_i : S^2 \to M$ with $i = 1, \ldots, h$. We can form a bouquet of these immersions (we homotope them so that they all touch a fixed point in M) and get a map

$$f : \vee_h S^2 \to \dot{M}.$$

This map induces isomorphisms on homologies H_0 and H_2, and is thus a homotopy equivalence by Whitehead's Homology theorem. The manifold M is obtained from $\dot{M}$ by attaching a 4-cell. The attaching map translates via the homotopy equivalence to an attaching map $\psi : \partial D^4 \to \vee_h S^2$, well-defined up to homotopy. The homotopy equivalence extends to an equivalence between M and the CW-complex $\vee_h S^2 \cup_\psi D^4$. The map ψ defines an element in $\pi_3(\vee_h S^2)$. By proposition 10.4, the map ψ is determined up to homotopy by the corresponding matrix Q. It remains to show that Q represents the intersection form Q_M . Following Thom-Pontryagin construction, take a point p_i in each 2-sphere. Let F_i be an oriented surface properly embedded in D^4 bounding a link defined by the p_i. By collapsing ∂F_i to a point we get an oriented surface $\overline{F_i}$ in $\vee_h S^2 \cup_\psi D^4$ and hence a homology element in $H_2(\vee_h S^2 \cup_\psi D^4) \cong H_2(M)$. The homology elements we get are dual to $\alpha_1, \ldots, \alpha_h$, thus they form a basis. The way they intersect (transversely) in D^4 transports to M : therefore Q represents Q_M. □

10.2 Wall theorem

We know from Whitehead theorem that two closed simply connected smooth oriented 4-manifolds with isomorphic intersection forms are homotopy equivalent. Actually, we know from Freedman theorem that they are homeomorphic. Moreover, a theorem of Wall shows that they become diffeomorphic after summing with some copies of $S^2 \times S^2$. A tool for the proof is the following theorem.

Theorem 10.9. Two closed oriented 4-manifolds are cobordant if and only if they have the same signature.

Recall now that $Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2} \neq Q_{S^2 \times S^2}$: in terms of invariants the first is odd while the second is even. We anticipate Serre theorem 11.6 to say that they are not isomorphic. They are both S^2-bundles over S^2 but they are different since their intersection forms are different. We denote $S^2 \widetilde{\times} S^2 := \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$.

Remark 10.10. One can prove that the $\mathbb{R}^k$-bundles over S^h up to isomorphism are in $1-1$ correspondance with $\pi_{h-1}(SO(k))$. Since $\pi_1(SO(3)) = \mathbb{Z}_2$, then $S^2 \tilde{\times} S^2$ and $S^2 \times S^2$ are the only two $\mathbb{R}^3-$bundles over S^2.

Lemma 10.11. Let N^5 be obtained by adding a $2-$handle to $\mathbb{D}^5$. Then ∂N^5 is either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$.

Proof. The attaching sphere of the 2-handle is a loop, and all loops are isotopic in S^4. Represent D^5 as $D^2 \times D^3$ and take $S^1 \times \{0\}$ as a loop. Attach the handle along $S^1 \times D^3$. The result is the attaching of two copies of $D^2 \times D^3$ which extends to a D^3-fibering over S^2. Its boundary is a S^2-fibering over S^2. It remains to use Remark 10.10 to conclude. □

Theorem 10.12 (Wall)**.** Let M^4 and N^4 be two closed simply connected smooth oriented 4-manifolds with isomorphic intersection forms. There is a natural number h such that $M^4 \#_h (S^2 \times S^2)$ is diffeomorphic to $N^4 \#_h (S^2 \times S^2)$.

Proof. Since they have the same signature, the two manifolds are cobordant. We thus get a five-dimensional cobordism W^5 with $\partial W = M \cup \overline{N}$. Take a handle decomposition of this cobordism. As in the proof of the h-cobordism, we can modify the handle decomposition to end up with 2- and 3-handles only. Let Z^4 be the level manifold between 2- and 3-handles. We show that the attaching of a 2-handle changes the level manifold by a connected sum with either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. Every (five-dimensional) 2-handle is attached along a loop in the (four-dimensional) level manifold. This four-dimensional level manifold is simply connected, and hence the loop is isotopic to the trivial one. Therefore the loop is contained in a 4-disc, and the level manifold is changed via a connected sum with the manifold of Lemma 10.11, which is indeed either $S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. Therefore we get

$$Z \cong M \#_h (S^2 \times S^2) \#_k (S^2 \tilde{\times} S^2) \cong N \#_l (S^2 \times S^2) \#_m (S^2 \tilde{\times} S^2).$$

□

10.3 Cobordism groups

In this section, we will only give some ideas of the proof of two important results: every closed 3-manifold bounds a 4-manifold, and every closed 4-manifold of zero signature bounds a 5-manifold. The steps are the following:

(1) Embed M^n in $\mathbb{R}^{n+k}$, or equivalently in S^{n+k};

(2) If T is the tubolar neighborhood of M^n in S^{n+k} and $\dot{T}$ its interior, then find a section of M in ∂T which is homologically trivial in the complement $S^{n+k} \setminus \dot{T}$;

(3) The section bounds a cycle: try to represent it by a manifold.

The third step works if $k \leqslant 3$, as the following states.

Lemma 10.13. Let Z^{n+k} be a compact smooth manifold and $M^n \subset \partial Z^{n+k}$ a closed oriented connected submanifold which is homologically trivial, i.e., $[M^n] = 0$ in $H_n(Z^{n+k}, \mathbb{Z})$. If $k \leqslant 3$, there is a properly embedded smooth oriented submanifold $W^{n+1} \subset Z^{n+k}$ such that $\partial W^{n+1} = M^n$.

The second step works if $k \leqslant 2$, as the following states.

Theorem 10.14. Every oriented connected smooth manifold $M^n \subset \mathbb{R}^{n+2}$ bounds a smooth oriented Seifert manifold $W^{n+1} \subset \mathbb{R}^{n+2}$.

We are left with the point (1) of our program. If we can embed M^n in $\mathbb{R}^{n+2}$, we are done. However, Whitney's theorem only provides embeddings in the much larger $\mathbb{R}^{2n}$. When $n = 3$, we get an embedding of a 3-manifold in $\mathbb{R}^6$ instead of the required $\mathbb{R}^5$.

Whitney's theorem provides an immersion of M^n in $\mathbb{R}^{2n-1}$ and even if there are manifolds that do not embed in $\mathbb{R}^{2n-1}$, up to cobordism, they do.

Theorem 10.15. Every closed oriented manifold M^n is cobordant to a closed oriented manifold embedded in $\mathbb{R}^{2n-1}$.

We are now ready to enunciate the following.

Corollary 10.16. Every oriented 3-manifold bounds an oriented 4-manifold. (That is, $\Omega_3 = 0$.)

Proof. A closed oriented 3-manifold is cobordant to a closed oriented 3-manifold embedded in $\mathbb{R}^5$. Such a manifold bounds an oriented 4-manifold by Theorem 10.14. □

Let's first enunciate the following.

Theorem 10.17 (Cobordism, Rokhlin)**.** Let W be smooth, closed, and oriented. Then $W = \partial M$ for some compact, smooth, oriented 5-manifold if and only if $\sigma(W) = 0$. That is, the signature $\sigma : \Omega_4 \to \mathbb{Z}$ is an isomorphism.

The main idea is that all the arguments used in the 3-dimensional case may be adapted to the 4-dimensional one up to summing up with some copies of $\mathbb{CP}^2$.

11 Classification of intersection forms

The classification of all forms that arise as intersection forms of smooth 4- manifolds is not yet complete. However, much is known.

Theorem 11.1. Given two compact topological manifolds X_1 and X_2, then $Q_{X_1 \# X_2} \cong Q_{x_1} \oplus Q_{X_2}$.

Proof. Removing a ball B^4 and gluing along a sphere S^3 does not change the second homology (and hence the intersection form). □

Remark 11.2. In CAT= TOP, the converse (for simply connected manifolds) holds: if $\pi_1(X) = \{e\}$ and $Q_X \cong Q_1 \oplus Q_2$, then there exist two topological manifolds X_1, X_2 such that $Q_{x_i} \cong Q_i$ and $X \cong_{TOP} X_1 \# X_2$.

The converse of Theorem 11.1 does not hold in CAT $= C^\infty$: for instance, one can compute Q_{K3}. But we have the following.

Theorem 11.3 (Freedman-Taylor)**.** Let X^4 be smooth, compact and simply connected. Suppose that $Q_X \cong Q_1 \oplus Q_2$, then there exist two smooth manifolds X_1, X_2 such that $Q_{x_i} \cong Q_i$ and $X \cong_{C^\infty} X_1 \cup_Y X_2$, where Y is a $\mathbb{Z}HS^3$, *i.e.* a 3-manifold with the same $\mathbb{Z}$-homology as S^3.

The study of intersection forms is divided in two categories of bilinear forms: definite and indefinite.

Definition 11.4. A bilinear symmectric unimodular form Q is indefinite if it has both positive and negative eigenvalues. It is definite otherwise.

In number theory, Meyer's lemma state that if the equation $Q(x) = 0$ has a non-zero real solution, then it has a non-zero rational solution. Up to multiplication by the denominators, an integral solution x may also be found. This result is usually deduced from the Hasse–Minkowski theorem (which was proved later) and from the following statement: "A rational quadratic form in five or more variables represents zero over the field $\mathbb{Q}_p$ of p-adic numbers for all p". We state it properly.

Lemma 11.5 (Meyer's lemma)**.** Let $Q : \Lambda \times \Lambda \to \mathbb{Z}$ be a bilinear symmectric unimodular form. If it is indefinite, then there exists an element x_0 in Λ such that $Q(x_0, x_0) = 0$.

Meyer's lemma, that is admitted, is used to prove the following classification result (already mentioned above Remark 10.10).

Theorem 11.6 (Serre)**.** Two symmetric bilinear unimodular forms Q, Q' both indefinite are such that the following holds.

$$Q \cong Q' \quad \text{if and only if} \quad \begin{cases} \mathrm{rk} Q = \mathrm{rk} Q' \\ \sigma(Q) = \sigma(Q') \\ \text{have same parity} \end{cases}$$

$\star$) If Q is odd, then $Q \cong a^+[1] \oplus a^-[-1]$ with $a^\pm = \frac{\mathrm{rk}Q \pm \sigma(Q)}{2}$.

$\star$) If Q is even, then $Q \cong b \cdot E_8 \oplus c \cdot H$ with $b = \frac{-\sigma(Q)}{8}$ and $c = \frac{\mathrm{rk}(Q) - |\sigma(Q)|}{2}$.

Remark 11.7. To give a sense to the theorem, we first have to check that $a^\pm, b$ and c are well defined (*i.e.*, are natural numbers).

Remark 11.8. Note that $\mathrm{rk}(Q) \equiv \sigma(Q) \pmod 2$ if $\mathrm{Ann}(Q) = \{0\}$. Thus, in Serre theorem, $a^\pm$ and c are always natural numbers.

11.1 The odd case

Given a submodule $B \subset A$ and a symmetric bilinear form Q, we define the Q-orthogonal $B^\perp$ as usual. In general, we cannot split A as $B \oplus B^\perp$ as we often do with vector spaces. In fact, we can split precisely when $Q|_B$ is unimodular, as the following shows.

Lemma 11.9. We have $A = B \oplus B^\perp$ if and only if $Q|_B$ is unimodular. If this holds, we have $Q = Q|_B \oplus Q|_{B^\perp}$.

Proof. We always have $B \cap B^\perp = \{0\}$. We show that $B + B^\perp = A$ if and only if $Q|_B$ is unimodular. Indeed, if $B \oplus B^\perp = A$, then $Q = Q|_B \oplus Q|_{B^\perp}$ and $\det Q = \det Q|_B \cdot \det Q|_{B^\perp}$, so necessarily we must have $\det Q|_B = \pm 1$. Conversely, suppose $Q|_B$ is unimodular. We want to show that $B + B^\perp = A$. Let $x \in A$ be any element. We will show that it lies in $B + B^\perp$. The unimodular form Q defines an adjoint

$x^* \in A^*$ by $x^*(a) = Q(x, a)$. The restriction $x^*|_B$ is an element of B^*. By the unimodularity of $Q|_B$, such an element is dual to some $x_B \in B$. That is, we have

$$Q(x_B, b) = Q(x, b) \quad \text{for all } b \in B.$$

Write $x = x_B + (x - x_B)$. Since $Q(x - x_B, b) = 0$ for all $b \in B$, we have $x - x_B \in B^\perp$, and we are done. ☐

Proposition 11.10 (Odd Serre)**.** Let Q be an odd indefinite unimodular form. Then, we have $Q \cong a^+[1] \oplus a^-[-1]$ with $a^\pm = \frac{\mathrm{rk} Q \pm \sigma(Q)}{2}$.

Proof. Let Q be defined over some free module A. We prove our assertion by induction on $\dim A$. By Meyer's lemma, there exists an element $v \in A$ with $Q(v, v) = 0$. We may suppose that v is primitive.

There exists an element w such that $Q(v, w) = 1$: it suffices to complete v to a basis for A and take $w = v^*$ in a dual basis.

Consider the submodule B generated by v and w. Then we have $Q|_B \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & x \end{smallmatrix}\right)$ for some integer x. Thus, $\det Q|_B = -1$, and Lemma 11.9 gives $Q = Q|_B \oplus Q|_{B^\perp}$.

We want x to be odd. If x is even, then $Q|_B$ is even, and thus $Q|_{B^\perp}$ must be odd. Therefore, there exists an odd element $u \in B^\perp$, and by substituting w with $w + u$, we obtain an odd integer x.

It is now easy to construct a change of basis that transforms

$$\begin{pmatrix} 0 & 1 \\ 1 & x \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and finally into} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This in particular proves our assertion when $\dim A = 2$.

If $\dim A > 2$, we argue by induction. We have $Q \cong [1] \oplus [-1] \oplus Q|_{B^\perp}$. Both $[1] \oplus Q|_{B^\perp}$ and $[-1] \oplus Q|_{B^\perp}$ are odd. One of these is indefinite. By induction, it is isomorphic to $m[1] \oplus n[-1]$, and we are done. ☐

11.2 The even case

Definition 11.11. Given a symmetric bilinear unimodular form (Λ, Q), an element $w \in \Lambda$ is said to be characteristic if for all $y \in \Lambda$, we get $Q(w, y) \equiv Q(y, y) \pmod 2$. The element is said to be ordinary otherwise.

Remark 11.12. In some sense, a characteristic element controls the parity of all elements in Λ.

Example 11.13. The trivial element $0 \in \Lambda$ is characteristic if, and only if, Q is even.

Lemma 11.14 (Van der Blij)**.** We have $Q(w, w) = \sigma(Q) \pmod 8$.

Proof. First, take two characteristic elements w, w'. Thus $Q(w - w', z)$ is even for all $z \in A$. Since Q is unimodular, this implies that $w - w' = 2v$ for some $v \in A$. Therefore,

$$\begin{aligned} Q(w', w') = Q(w - 2v, w - 2v) &= Q(w, w) - 4Q(v, w) + 4Q(v, v) \\ &= Q(w, w) + 4(Q(v, v) - Q(v, w)). \end{aligned}$$

Since $Q(v, v) - Q(v, w)$ is even, we get

$$Q(w, w) \equiv Q(w', w') \pmod 8.$$

Every characteristic element thus yields the same number in $\mathbb{Z}_8$. It remains to prove that it is the same number determined by $\sigma(Q)$. There are two cases:

(1) If Q is odd and indefinite, we have $Q \cong m[1] \oplus n[-1]$ by Serre's theorem. Take $w = (1, \ldots, 1)$. We get $Q(w, w) = m - n = \sigma(Q)$.

(2) In all other cases, the form $Q' = Q \oplus [1] \oplus [-1]$ is odd and indefinite. If w is a characteristic element for Q, then $w' = (w, 1, 1)$ is characteristic for Q'.

By the previous point, the theorem holds for w'. Since $Q(w, w) = Q(w', w')$ and $\sigma(Q) = \sigma(Q')$, it also holds for w.

□

Corollary 11.15. If Q is an even form, then $\sigma(Q)$ is divisible by 8.

Proof. The trivial element is characteristic. □

Example 11.16. The hyperbolic form $H = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is even, has rank 2 and signature 0.

Corollary 11.17. Let Q be an even indefinite unimodular form. Then $Q \cong b \cdot E_8 \oplus c \cdot H$ with $b = \frac{-\sigma(Q)}{8}$ and $c = \frac{\mathrm{rk}(Q) - |\sigma(Q)|}{2}$.

Example 11.18 (K3)**.** A $K3$ space is simply connected and has $c_1 = 0$: it is even, indefinite, of rank 22 and signature -16. It follows that the intersection form is $Q_{K3} = 2(-E_8) \oplus 3H$.

Remark 11.19. When we look at definite forms, we get too many of them. For instance, if Q is even and has rank $\mathrm{rk}(Q) = 35$ we get more than 80 millions of them, when $\mathrm{rk}(Q) = 40$, we get more than 10^{51} of them. When Q is odd, it is even worse!

11.3 The Kirby-Siebenmann invariant

To get a more precise statement than Theorem 9.14, we introduce a new invariant.

Definition 11.20. Given a closed, connected, topological 4-manifold, there exists a constant $ks(X)$ in $\mathbb{Z}/2\mathbb{Z}$ such that

⋆) $ks(X \# Y) = ks(X) + ks(Y)$;

⋆) if X admits a smooth structure, then $ks(X) = 0$.

Remark 11.21. There exists manifold without smooth structure that has Kirby-Siebenmann constant zero.

This definition gives an obstruction to endowing a manifold with a smooth structure (in reality, it is with a piecewise-linear structure, but in dimension 4 it is equivalent). The existence of such a constant must be demonstrated. We admit it here to enunciate the following theorem by Freedman in 1982.

Theorem 11.22. Let Q be a bilinear symmetric unimodular form. Then

i) Q is even $\implies$ there is a unique topological manifold X with $Q_X \cong Q$;

ii) Q is odd $\implies$ there are exactly two topological manifolds with $Q_{X_i} \cong Q$ distinguished by their ks invariant.

While 1- and 2-dimensional manifolds are now fully classified, understood, and geometrized, manifolds in dimensions 3, 4 and higher still pose many deep and open questions. This brief and selective document necessarily involved choices, leaving out many fascinating aspects of the topic. We therefore warmly encourage the interested reader to consult other references such as [4] for 3-manifolds, [3] for 4-manifolds and [1] for both.

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