

# The quadratic linking degree

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# Knot theory in a nutshell 1

Topological objects of interest are knots and links.

- A **knot** is a (closed) topological subspace of the 3-sphere  $\mathbb{S}^3$  which is homeomorphic to the circle  $\mathbb{S}^1$ .
- An **oriented knot** is a knot with a “continuous” local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.

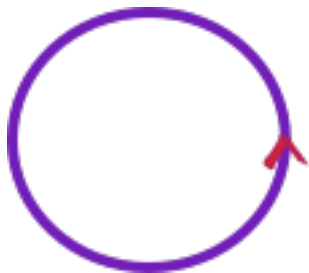


Figure: The unknot

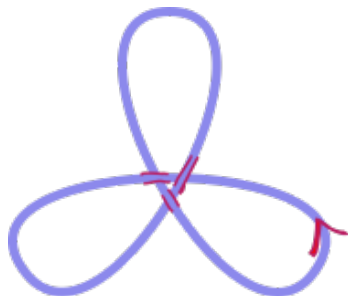


Figure: The trefoil knot

## Knot theory in a nutshell 2

- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.
- The **linking number** of an (oriented) link with two components is the number of times one of the components turns around the other component.

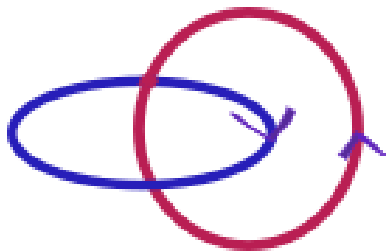


Figure: The Hopf link

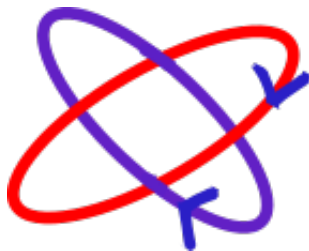
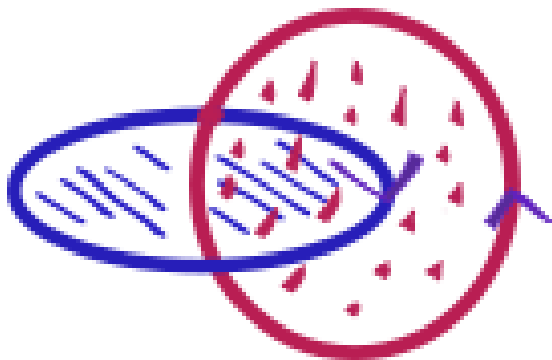
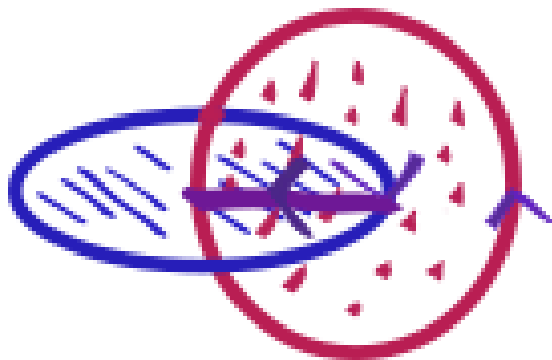


Figure: The Solomon link

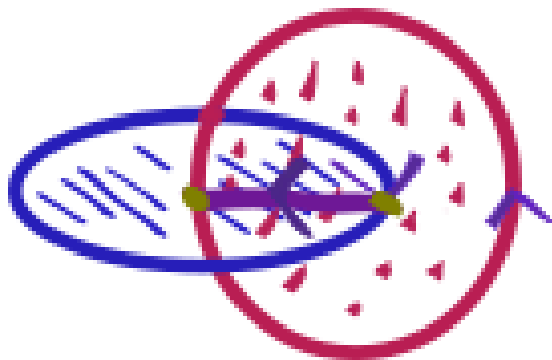
# Defining the linking number: Seifert surfaces



# Defining the linking number: intersection of $S$ . surfaces



# Defining the linking number: boundary of int. of S. surf.





# The formal definition of the linking number

Let  $L = K_1 \sqcup K_2$  be an oriented link with two components.

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## Oriented fundamental class and Seifert class

Let  $i \in \{1, 2\}$ . The class  $S_i$  in  $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\text{BM}}(\mathbb{S}^3, L)$  of Seifert surfaces of the oriented knot  $K_i$  is the unique class that is sent by the boundary map to the (oriented) fundamental class of  $K_i$  in  $H^0(K_i) \subset H^0(L)$ .

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## Linking class and linking number

The linking class of  $L$  is the image of the cup-product  $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$  by the boundary map  $\partial : H^2(\mathbb{S}^3 \setminus L) \rightarrow H^1(L)$ . The linking number of  $L = K_1 \sqcup K_2$  is the integer  $n \in \mathbb{Z}$  such that the linking class in  $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$  is equal to  $(n[\omega_{K_1}], -n[\omega_{K_2}])$  (where  $\omega_{K_i}$  is the volume form of the oriented knot  $K_i$ ).

# Homotopies in a nutshell

## Homotopic maps

Two continuous maps  $f, g : X \rightarrow Y$  are homotopic if there exists a homotopy from  $f$  to  $g$ , i.e. a continuous map  $H : X \times [0, 1] \rightarrow Y$  such that for all  $x \in X$ ,  $H(x, 0) = f(x)$  and  $H(x, 1) = g(x)$ .

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## Homotopy types of topological spaces

Two topological spaces  $X$  and  $Y$  have the same homotopy type if there exists a homotopy equivalence from  $X$  to  $Y$ , i.e. a couple  $(i : X \rightarrow Y, j : Y \rightarrow X)$  of continuous maps such that  $j \circ i$  is homotopic to the identity of  $X$  and  $i \circ j$  is homotopic to the identity of  $Y$ .

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## Important example

For all  $n \geq 1$ ,  $\mathbb{S}^n$  has the same homotopy type as  $\mathbb{R}^{n+1} \setminus \{0\}$ .

# Oriented links in algebraic geometry 1

## Link with two components

A link with two components is a couple of closed immersions

$\varphi_i : \mathbb{A}_F^2 \setminus \{0\} \rightarrow \mathbb{A}_F^4 \setminus \{0\}$  with disjoint images  $Z_i$  (where  $i \in \{1, 2\}$ ).

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## Orientation classes

Two orientations  $o_i : \nu_{Z_i} \rightarrow \mathcal{L}_i \otimes \mathcal{L}_i$  and  $o'_i : \nu_{Z_i} \rightarrow \mathcal{L}'_i \otimes \mathcal{L}'_i$  of  $Z_i$  represent the same orientation class of  $Z_i$  if there exists an isomorphism  $\psi : \mathcal{L}_i \simeq \mathcal{L}'_i$  such that  $(\psi \otimes \psi) \circ o_i = o'_i$ .

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The orientation classes of  $Z_i$  are parametrized by the elements of  $F^*/(F^*)^2$  (where  $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$ ).

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If  $F = \mathbb{Q}$  then  $F^*/(F^*)^2$  has infinitely many elements (the classes of the integers without square factors).

# The Hopf link

We fix coordinates  $x, y, z, t$  for  $\mathbb{A}_F^4$  and  $u, v$  for  $\mathbb{A}_F^2$  once and for all.

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- The orientation of the Hopf link:

$$\sigma_1 : \bar{x}^* \wedge \bar{y}^* \mapsto \mathbf{1} \otimes \mathbf{1}, \sigma_2 : \bar{z}^* \wedge \bar{t}^* \mapsto \mathbf{1} \otimes \mathbf{1}$$

# A variant of the Hopf link

- The image is the same as the Hopf link's image:

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- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1 \otimes 1$$

# Chow groups and intersection theory

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- You may know the following exact sequence where  $Y \subset X$  is closed:

$$CH_p(Y) \longrightarrow CH_p(X) \longrightarrow CH_p(X \setminus Y) \longrightarrow 0$$

It can be extended into the following long exact sequence:

$$\cdots \rightarrow A_{p+1}(X \setminus Y, -p) \rightarrow CH_p(Y) \rightarrow CH_p(X) \rightarrow CH_p(X \setminus Y) \rightarrow 0$$



# Chow-Witt groups and quadratic intersection theory

- Solution to the second problem (orientations): replace (generalised) Chow groups, a.k.a. Rost groups, with (generalised) Chow-Witt groups, a.k.a. Rost-Schmid groups; see for instance the chapter *Lectures on Chow-Witt groups* by Jean Fasel in the book *Motivic homotopy theory and refined enumerative geometry* (2020)

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- To a smooth  $F$ -scheme  $Y$ , an integer  $j \in \mathbb{Z}$  and an invertible  $\mathcal{O}_Y$ -module  $\mathcal{L}$  we associate the corresponding Rost-Schmid complex
$$\bigoplus_{i \geq 0} \bigoplus_{p \text{ point of codim } i \text{ in } Y} K_{j-i}^{\text{MW}}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}|_p) \setminus \{0\}].$$

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- The  $i$ -th cohomology group, called Rost-Schmid group, is denoted  $H^i(Y, \underline{K}_j^{\text{MW}}\{\mathcal{L}\})$ . If  $j = i$  then we call  $H^i(Y, \underline{K}_i^{\text{MW}}\{\mathcal{L}\})$  the  $i$ -th Chow-Witt group of  $Y$  twisted by  $\mathcal{L}$  and denote it  $\widetilde{CH}^i(Y, \mathcal{L})$ . We have a canonical morphism  $\widetilde{CH}^i(Y, \mathcal{L}) \rightarrow CH^i(Y)$ .

# Intersection product

We have an intersection product

$$\cdot : H^i(Y, \underline{K}_j^{\text{MW}}) \times H^{i'}(Y, \underline{K}_{j'}^{\text{MW}}) \rightarrow H^{i+i'}(Y, \underline{K}_{j+j'}^{\text{MW}})$$

which makes  $\bigoplus_{i \in \mathbb{N}_0, j \in \mathbb{Z}} H^i(Y, \underline{K}_j^{\text{MW}})$  into a graded  $K_0^{\text{MW}}(F)$ -algebra.

In particular, we have  $\cdot : \widetilde{CH}^i(Y) \times \widetilde{CH}^{i'}(Y) \rightarrow \widetilde{CH}^{i+i'}(Y)$  which makes  $\bigoplus_{i \in \mathbb{N}_0} \widetilde{CH}^i(Y)$  into a graded  $K_0^{\text{MW}}(F)$ -algebra (the Chow-Witt ring).



# Milnor-Witt $K$ -theory and quadratic forms

The generators of the Milnor-Witt  $K$ -theory ring of a field  $F$  are denoted  $[a] \in K_1^{\text{MW}}(F)$  for every  $a \in F^*$  and  $\eta \in K_{-1}^{\text{MW}}(F)$ . We denote  $\langle a \rangle = \eta[a] + 1 \in K_0^{\text{MW}}(F)$  for every  $a \in F^*$ .

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The (commutative) ring with unit  $K_0^{\text{MW}}(F)$  is isomorphic to the Grothendieck-Witt ring  $\text{GW}(F)$  of  $F$  via  $\langle a \rangle \in K_0^{\text{MW}}(F) \leftrightarrow \langle a \rangle \in \text{GW}(F)$ . For all  $n < 0$ , the abelian group  $K_n^{\text{MW}}(F)$  is isomorphic to the Witt group  $W(F)$  of  $F$  via  $\langle a \rangle \eta^{-n} \in K_n^{\text{MW}}(F) \leftrightarrow \langle a \rangle \in W(F)$ .

For all  $a \in F^*$ ,  $\langle a \rangle$  is the equivalence class of the symmetric bilinear form  $\begin{cases} F \times F & \rightarrow & F \\ (x, y) & \mapsto & axy \end{cases}$  or, if  $F$  is of characteristic  $\neq 2$ , of the quadratic form  $\begin{cases} F & \rightarrow & F \\ x & \mapsto & ax^2 \end{cases}$ .  $\text{GW}(F)$  is made up of  $\mathbb{Z}$ -linear combinations of  $\langle a \rangle$  and  $W(F) = \text{GW}(F)/(\langle 1 \rangle + \langle -1 \rangle)$  is made up of sums of  $\langle a \rangle$ .

Let  $n \geq 2$  be an integer,  $i \in \mathbb{N}_0, j \in \mathbb{Z}$ . The Rost-Schmid group  $H^i(\mathbb{A}_F^n \setminus \{0\}, \underline{K}_j^{\text{MW}})$  is isomorphic to

$$\begin{cases} K_j^{\text{MW}}(F) & \text{if } i = 0 \\ K_{j-n}^{\text{MW}}(F) & \text{if } i = n - 1. \\ 0 & \text{otherwise} \end{cases}$$



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This is similar to the fact in classical homotopy theory that  $H^i(\mathbb{S}^{n-1})$  is

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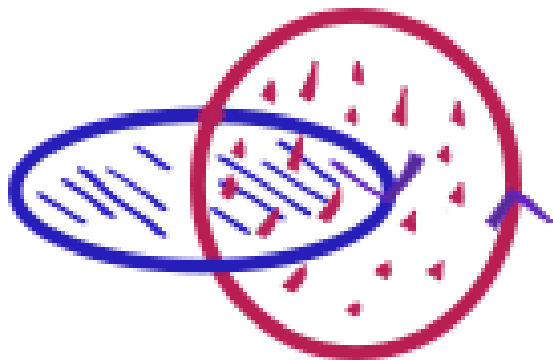
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In particular,  $H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \simeq K_{-2}^{\text{MW}}(F)$ . We can fix such an isomorphism, but it is not canonical.

# The linking number and the quadratic linking degree

Let  $L = K_1 \sqcup K_2$  be an oriented link (in knot theory) and  $\mathcal{L}$  be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions  $\varphi_i : \mathbb{A}_F^2 \setminus \{0\} \rightarrow \mathbb{A}_F^4 \setminus \{0\}$  with disjoint images  $Z_i$  and orientation classes  $\overline{o}_i$ . We denote  $Z := Z_1 \sqcup Z_2$ .

## Step 1 in a picture: Seifert surfaces



# Step 1

## Knot theory

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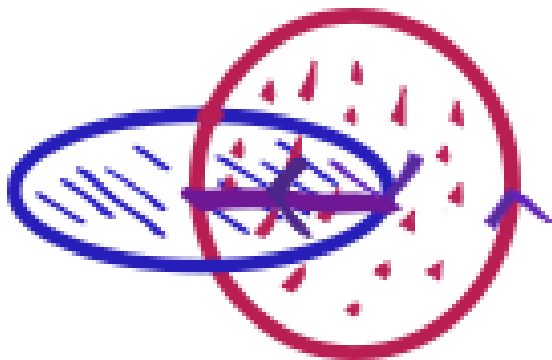
## Knot theory

The class  $S_i$  in  $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\text{BM}}(\mathbb{S}^3, L)$  of Seifert surfaces of the oriented knot  $K_i$  is the unique class that is sent by the boundary map to the (oriented) fundamental class of  $K_i$  in  $H_1(K_i) \simeq H^0(K_i) \subset H^0(L)$ .

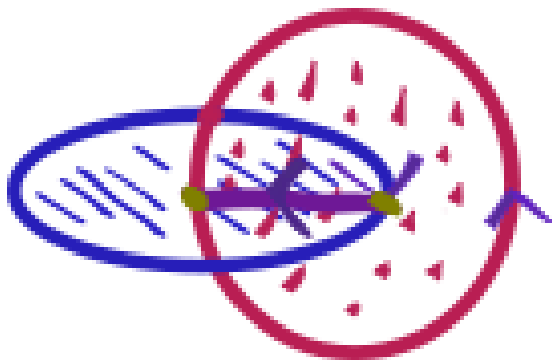
## Motivic knot theory

We define an analogue  $[o_i] \in H^0(Z_i, \underline{K}_{-1}^{\text{MW}}\{\nu_{Z_i}\})$  of the oriented fundamental class of each oriented component of  $\mathcal{L}$  then we define the Seifert class  $\mathcal{S}_i$  as the unique class in  $H^1(X \setminus Z, \underline{K}_1^{\text{MW}})$  that is sent by the boundary map to the oriented fundamental class  $[o_i] \in H^0(Z, \underline{K}_{-1}^{\text{MW}}\{\nu_Z\})$ .

## Step 2 in two pictures: intersection of Seifert surfaces



## Step 2 in two pictures: boundary of int. of S. surfaces





## Step 2

### Knot theory

The linking class of  $L$  is the image of the cup-product  $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$  by the boundary map  $\partial : H^2(\mathbb{S}^3 \setminus L) \rightarrow H^1(L)$ .

## Step 2

### Knot theory

The linking class of  $L$  is the image of the cup-product  $\mathcal{S}_1 \cup \mathcal{S}_2 \in H^2(\mathbb{S}^3 \setminus L)$  by the boundary map  $\partial : H^2(\mathbb{S}^3 \setminus L) \rightarrow H^1(L)$ .

### Motivic knot theory

We define the quadratic linking class of  $\mathcal{L}$  as the image of the intersection product  $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\text{MW}})$  by the boundary map  $\partial : H^2(X \setminus Z, \underline{K}_2^{\text{MW}}) \rightarrow H^1(Z, \underline{K}_0^{\text{MW}}\{\nu_Z\})$ .

## Step 3

### Knot theory

The linking number of  $L = K_1 \sqcup K_2$  is the integer  $n \in \mathbb{Z}$  such that the linking class in  $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$  is equal to  $(n[\omega_{K_1}], -n[\omega_{K_2}])$  (where  $\omega_{K_i}$  is the volume form of the oriented knot  $K_i$ ).

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### Motivic knot theory

We define the quadratic linking degree of  $\mathcal{L}$  as the image of the quadratic linking class of  $\mathcal{L}$  by the isomorphism

$$H^1(Z, \underline{K}_0^{\text{MW}} \{ \nu_Z \}) \rightarrow H^1(Z, \underline{K}_0^{\text{MW}}) \rightarrow H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \oplus H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \rightarrow W(F) \oplus W(F).$$

We fixed an isomorphism  $H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \rightarrow K_{-2}^{\text{MW}}(F)$  once and for all and there is a canonical isomorphism  $K_{-2}^{\text{MW}}(F) \rightarrow W(F)$ .

# The Hopf link

Recall that we fixed coordinates  $x, y, z, t$  for  $\mathbb{A}_F^4$  and  $u, v$  for  $\mathbb{A}_F^2$ .

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

- The parametrization of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$\sigma_1 : \bar{x}^* \wedge \bar{y}^* \mapsto 1, \sigma_2 : \bar{z}^* \wedge \bar{t}^* \mapsto 1$$

# The quadratic linking degree of the Hopf link

Or. fund. classes	$\eta \otimes (\bar{x}^* \wedge \bar{y}^*)$		$\eta \otimes (\bar{z}^* \wedge \bar{t}^*)$
Seifert classes	$\langle x \rangle \otimes \bar{y}^*$		$\langle z \rangle \otimes \bar{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\bar{t}^* \wedge \bar{y}^*)$		
Quad. link. class	$-\langle z \rangle \eta \otimes (\bar{t}^* \wedge \bar{x}^* \wedge \bar{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\bar{y}^* \wedge \bar{z}^* \wedge \bar{t}^*)$
Apply $\tilde{o}_1 \oplus \tilde{o}_2$	$-\langle z \rangle \eta \otimes \bar{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \bar{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \bar{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \bar{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\bar{u}^* \wedge \bar{v}^*)$	$\oplus$	$\eta^2 \otimes (\bar{u}^* \wedge \bar{v}^*)$
Quad. link. degree	$-1$	$\oplus$	$1$

## A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}_F^4 \setminus \{0\} \text{ with } a \in F^*$$

- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$\alpha_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, \alpha_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

# The quadratic linking degree of a variant of the Hopf link

$$[o_1^{var}] = \eta \otimes \overline{x - y}^* \wedge \overline{y}^* = [o_1^{Hopf}] \quad [o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$$

$$\text{since } \begin{pmatrix} x - y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{since } \begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$$

$$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$$

$$\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$

$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The quadratic linking degree is  $(-\langle a \rangle, 1)$ .



## Fact

Let  $\mathcal{L}$  be an oriented link with two components of quadratic linking degree  $(d_1, d_2) \in W(F) \oplus W(F)$ . Let  $a = (a_1, a_2)$  be a couple of elements of  $F^*$  and  $\mathcal{L}_a$  be the link obtained from  $\mathcal{L}$  by changing the orientation  $o_1$  into  $o_1 \circ (\times a_1)$  and the orientation  $o_2$  into  $o_2 \circ (\times a_2)$ . Then  $\text{Qlc}_{\mathcal{L}_a} = \langle a_1 a_2 \rangle \text{Qlc}_{\mathcal{L}}$  and  $\text{Qld}_{\mathcal{L}_a} = (\langle a_2 \rangle d_1, \langle a_1 \rangle d_2)$ .

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Similarly, changes of parametrizations of the link can only multiply each component of the quadratic linking degree by elements of the form  $\langle a \rangle$  with  $a \in F^*$  (and do not change the quadratic linking class).

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We want invariants of the quadratic linking degree. (Similarly to the absolute value of the linking number in knot theory.)

# Invariants by multiplication by $\langle a \rangle$ for all $a \in F^*$

## Case $F = \mathbb{R}$

If  $F = \mathbb{R}$ , the absolute value of an element of  $W(\mathbb{R}) \simeq \mathbb{Z}$  is invariant by multiplication by  $\langle a \rangle$  for all  $a \in F^*$ .

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## General case

The rank modulo 2 is invariant by multiplication by  $\langle a \rangle$  for all  $a \in F^*$ .

## Definition

Let  $d \in W(F)$ . There exists a unique sequence of abelian groups  $Q_{d,k}$  and of elements  $\Sigma_k(d) \in Q_{d,k}$ , where  $k$  ranges over the nonnegative even integers, such that:

- $Q_{d,0} = W(F)$  and  $\Sigma_0(d) = 1 \in Q_{d,0}$ ;
- for each positive even integer  $k$ ,  $Q_{d,k}$  is the quotient group  $Q_{d,k-2}/(\Sigma_{k-2}(d))$ ;
- for each positive even integer  $k$ ,

$$\Sigma_k(d) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \langle \prod_{1 \leq j \leq k} a_{i_j} \rangle \in Q_{d,k} \text{ whenever}$$

$$d = \sum_{i=1}^n \langle a_i \rangle \in W(F).$$

## General case

The  $\Sigma_k$  are invariant by multiplication by  $\langle a \rangle$  for all  $a \in F^*$ .

- $\Sigma_2 : \begin{cases} W(F) & \rightarrow W(F)/(1) \\ \sum_{i=1}^n \langle a_i \rangle & \mapsto \sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases}$  (if  $n < 2$ , it sends  $\sum_{i=1}^n \langle a_i \rangle$  to 0)

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- $\Sigma_4 : \begin{cases} W(F) & \rightarrow \bigcup_{d \in W(F)} (W(F)/(1))/(\Sigma_2(d)) \\ \sum_{i=1}^n \langle a_i \rangle & \mapsto \sum_{1 \leq i < j < k < l \leq n} \langle a_i a_j a_k a_l \rangle \end{cases}$

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- We only want to compare  $\Sigma_4(d)$  and  $\Sigma_4(d')$  if  $\Sigma_2(d) = \Sigma_2(d')$ .

## Another Hopf link

From now on,  $F$  is a perfect field of characteristic different from 2. Recall that we fixed coordinates  $x, y, z, t$  for  $\mathbb{A}_F^4$  and  $u, v$  for  $\mathbb{A}_F^2$ .

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

But the change of coordinates  $x' = z - x$ ,  $y' = t - y$ ,  $z' = z + x$ ,  $t' = t + y$  would give  $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$ .

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- The orientation is the following:

$$\mathfrak{o}_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, \mathfrak{o}_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by  $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$  in  $\mathbb{S}_\varepsilon^3 = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$  for  $\varepsilon$  small enough and has linking number 1.

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- Its quadratic linking degree is  $(\langle 1 \rangle, \langle -1 \rangle) = (1, -1) \in W(F) \oplus W(F)$ .



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- If  $F = \mathbb{R}$ , the absolute value of each component is 1.
- The rank modulo 2 of each component is 1.
- For every positive even integer  $k$ , the image by  $\Sigma_k$  of each component is 0.

# The Solomon link

- In knot theory, the Solomon link is given by  $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$  in  $\mathbb{S}_\varepsilon^3$  for  $\varepsilon$  small enough and has linking number 2.
- In motivic knot theory, the image of the Solomon link is:

$$\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

- The parametrization is  $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u^2 - v^2, 2uv)$  and  $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u^2 + v^2, -2uv)$ .
- The orientation is the following:

$$o_1 : \overline{z - x^2 + y^2}^* \wedge \overline{t - 2xy}^* \mapsto 1, o_2 : \overline{z + x^2 - y^2}^* \wedge \overline{t + 2xy}^* \mapsto 1$$

- Its quadratic linking degree is  $(\langle 1 \rangle + \langle 1 \rangle, \langle -1 \rangle + \langle -1 \rangle) = (2, -2) \in W(F) \oplus W(F)$ .

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- The rank modulo 2 of each component is 0.
- For every positive even integer  $k$ , the image by  $\Sigma_k$  of each component is 0.
- More generally, we have analogues of the torus links  $T(2, 2n)$  (of linking number  $n$ ); the quadratic linking degree of  $T(2, 2n)$  is  $(n, -n) \in W(F) \oplus W(F)$ , which gives  $n$  as absolute value if  $F = \mathbb{R}$ ,  $n$  modulo 2 as rank modulo 2, and 0 for the  $\Sigma_k$ .

# Binary links

- The image of the binary link  $B_a$  with  $a \in F^* \setminus \{-1\}$ :

$$\{f_1 = 0, g_1 = 0\} \sqcup \{f_2 = 0, g_2 = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

with  $f_1 = t - ((1 + a)x - y)y$ ,  $g_1 = z - x(x - y)$ ,  
 $f_2 = t + ((1 + a)x - y)y$ ,  $g_2 = z + x(x - y)$ .

- The parametrization of the binary link  $B_a$ :

$$\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, ((1 + a)u - v)v, u(u - v))$$

$$\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -((1 + a)u - v)v, -u(u - v))$$

- The orientation of the binary link  $B_a$ :

$$\alpha_1 : \overline{f_1}^* \wedge \overline{g_1}^* \mapsto 1, \alpha_2 : \overline{f_2}^* \wedge \overline{g_2}^* \mapsto 1$$

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$		$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$		$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+ \langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1 + \langle a \rangle) \eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1 + \langle a \rangle) \eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. lk. deg.	$1 + \langle a \rangle$	$\oplus$	$-(1 + \langle a \rangle)$

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$		$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$		$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+ \langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
...	...		
Apply $\partial \oplus \partial$	$(1 + \langle a \rangle) \eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$-(1 + \langle a \rangle) \eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. lk. deg.	$1 + \langle a \rangle$	$\oplus$	$-(1 + \langle a \rangle)$

- If we change its orientations and its parametrizations then we get  $(\langle a \rangle + \langle b \rangle, \langle ca \rangle + \langle cb \rangle) \in W(F) \oplus W(F)$  with  $a, b, c \in F^*$  such that  $a + b \neq 0$ . The rank modulo 2 of each component is 0.

Or. fund. cyc.	$\eta \otimes (\overline{f_1}^* \wedge \overline{g_1}^*)$		$\eta \otimes (\overline{f_2}^* \wedge \overline{g_2}^*)$
Seifert divisors	$\langle f_1 \rangle \otimes \overline{g_1}^*$		$\langle f_2 \rangle \otimes \overline{g_2}^*$
Apply inter. prod.	$\langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x - y)$ $+ \langle f_1 f_2 \rangle \otimes (\overline{g_2}^* \wedge \overline{g_1}^*) \cdot (z, x)$		
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- If  $F = \mathbb{R}$ , the absolute value of each component is  $\begin{cases} 2 & \text{if } a > 0 \\ 0 & \text{if } a < 0 \end{cases}$ .
- $\Sigma_2$  of each component is  $\langle a \rangle \in W(F)/(1)$ . For instance, if  $F = \mathbb{Q}$ ,  $\Sigma_2$  distinguishes between all the  $B_p$  with  $p$  prime numbers.  $\Sigma_4 = 0$  etc.



Everything new I presented can be found in my preprint “The quadratic linking degree”:

- HAL: Clémentine Lemarié--Rieusset. THE QUADRATIC LINKING DEGREE. 2022. ⟨hal-03821736⟩
- arXiv: Clémentine Lemarié--Rieusset. The quadratic linking degree. arXiv:2210.11048 [math.AG]

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**Thanks for your attention!**