Motivic knot theory

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Knots and links

Topological objects of interest are knots and links.

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- An **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.

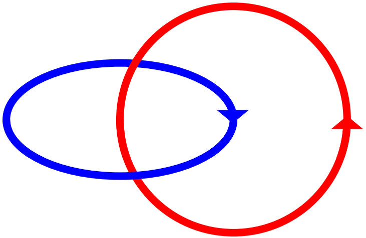


Figure: The Hopf link

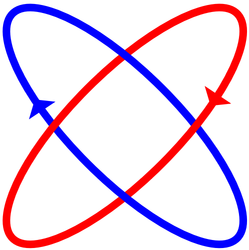
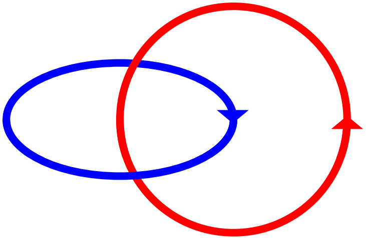


Figure: The Solomon link



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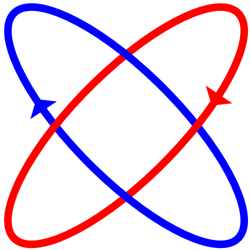

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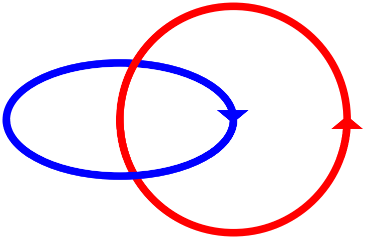
The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).



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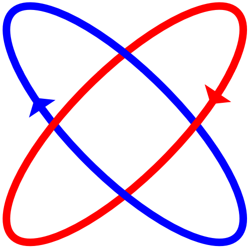
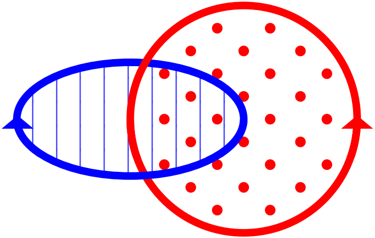


Figure: The Solomon link

The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).
The linking number is a complete invariant of oriented links with two components for link homotopy (i.e. $L = K_1 \sqcup K_2$ and $L' = K'_1 \sqcup K'_2$ are link homotopic if and only if they have the same linking number).

Defining the linking number: Seifert surfaces



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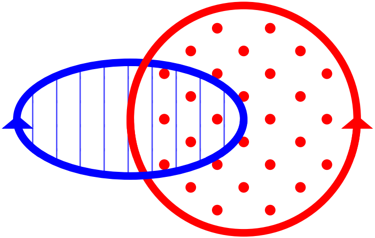
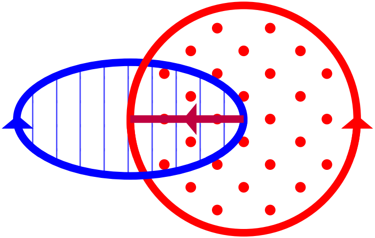


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S. surfaces



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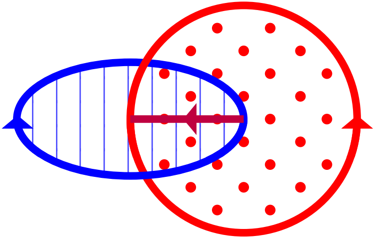
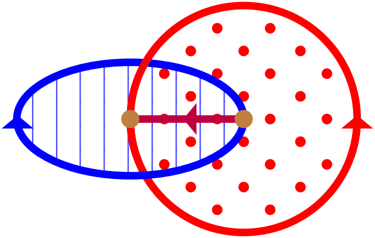


This corresponds to the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.



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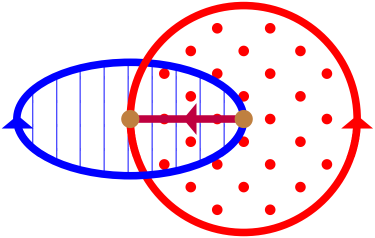

This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(K_1) \oplus H^1(K_2)$, which we call the linking class.

The linking number

The linking number

The linking number of L is the image of the part of the linking class which is in $H^1(K_1)$ by the composite of the morphism $i_* : H^1(L) \to H^3(\mathbb{S}^3)$ induced by the inclusion $i : L \to \mathbb{S}^3$ and of the "right-hand rule" $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$.

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The linking number does not depend on the order of the components of the oriented link, unlike the linking class.

The linking couple

The linking couple

The linking couple is the image of the linking class by the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ which is induced by the volume forms ω_{K_1} of K_1 and ω_{K_2} of K_2.

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The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

Contents

Recall that for all $n \geq 1$, $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \setminus \{0\}$.

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Link with two components

A link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ is a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$). The morphisms φ_1, φ_2 are called parametrisations of Z_1, Z_2 respectively.

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Oriented link with two components

An oriented link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ is a link with two components $(\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2)$ together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2.

Orientations

An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/\mathbb{A}^4_F\setminus\{0\}}$ of Z_i in $\mathbb{A}^4_F \setminus \{0\}$ to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/\mathbb{A}^4_F\setminus\{0\}}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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More concretely

In our examples, an orientation of a knot will be given by the choice of a first polynomial equation f and a second polynomial equation g such that the knot corresponds to $\{f = 0, g = 0\}$.

Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o'_i : \nu_{Z_i} \to \mathcal{L}'_i \otimes \mathcal{L}'_i$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}'_i$ such that $(\psi \otimes \psi) \circ o_i = o'_i$.

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Proposition

The orientation classes of Z_i are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^* \mid \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.
If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link in algebraic geometry

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

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- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1 \otimes 1$$

A variant of the Hopf link

- The image is the same as the image of the Hopf link:

$$\{x = y, y = 0\} \sqcup \{z = 0, at = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

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- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1 \otimes 1$$

Contents

Notations

- The generators of the Milnor-Witt K-theory $\mathbb{Z}$-graded ring of a field F are denoted $[a] \in K_1^{\mathrm{MW}}(F)$ for every $a \in F^*$ and $\eta \in K_{-1}^{\mathrm{MW}}(F)$.

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- We denote $\langle a \rangle := \eta[a] + 1 \in K_0^{\mathrm{MW}}(F)$ for every $a \in F^*$.
- We also denote by $\langle a \rangle$ the class of the symmetric bilinear form $\begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$ in $\mathrm{GW}(F)$ (or, abusively, in $\mathrm{W}(F)$). If the field F is of characteristic $\neq 2$ then $\langle a \rangle$ is the class in $\mathrm{GW}(F)$ of the quadratic form $\begin{cases} F & \to & F \\ x & \mapsto & ax^2 \end{cases}$ (or, abusively, in $\mathrm{W}(F)$).

Milnor-Witt K-theory and quadratic forms

- The ring $K_0^{\mathrm{MW}}(F)$ is isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ of the field F via $\langle a\rangle \in K_0^{\mathrm{MW}}(F) \leftrightarrow \langle a\rangle \in \mathrm{GW}(F)$.

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- For all $n < 0$, the abelian group $K_n^{\mathrm{MW}}(F)$ is isomorphic to the Witt group $\mathrm{W}(F)$ of the field F via $\langle a\rangle\eta^{-n} \in K_n^{\mathrm{MW}}(F) \leftrightarrow \langle a\rangle \in \mathrm{W}(F)$.

The singular complex and the Rost-Schmid complex

Classical algebraic topology

Each topological space X has a singular cochain complex:

$$\ldots \longrightarrow \mathcal{C}^i(X) \longrightarrow \mathcal{C}^{i+1}(X) \longrightarrow \ldots$$

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Motivic algebraic topology

Each smooth F-scheme X has a Rost-Schmid complex *for each integer $j \in \mathbb{Z}$ and invertible $\mathcal{O}_X$-module $\mathcal{L}$*:

$$\dots \longrightarrow \bigoplus_{p \in X^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}]$$

$$\downarrow$$

$$\bigoplus_{q \in X^{(i+1)}} K^{\mathrm{MW}}_{j-i-1}(\kappa(q)) \otimes_{\mathbb{Z}[\kappa(q)^*]} \mathbb{Z}[(\nu_q \otimes \mathcal{L}_{|q}) \setminus \{0\}] \longrightarrow \dots$$

The singular cohomology ring and the Rost-Schmid ring

Classical algebraic topology

The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X.

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The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X. The cup-product $H^i(X) \times H^{i'}(X) \to H^{i+i'}(X)$ makes $\bigoplus_{i \in \mathbb{N}_0} H^i(X)$ into a graded ring.

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Motivic algebraic topology

The i-th Rost-Schmid group $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ of X with respect to j and $\mathcal{L}$ is the i-th cohomology group of the Rost-Schmid complex of X w.r.t. j and $\mathcal{L}$. We denote $H^i(X, \underline{K}_j^{\mathrm{MW}}) := H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

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Classical algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the following long exact sequence (where ∂ is the boundary map):

$$\ldots \longrightarrow H^n(Z) \xrightarrow{i_*} H^{n+d_X-d_Z}(X) \xrightarrow{j^*} H^{n+d_X-d_Z}(U) \xrightarrow{\partial} H^{n+1}(Z) \longrightarrow$$

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Motivic algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the localization long exact sequence (where ∂ is the boundary map):

$$\dots \longrightarrow H^n(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}_{m+d_X-d_Z}^{\mathrm{MW}}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}_m^{\mathrm{MW}}\{\nu_Z\}) \longrightarrow \dots$$

Classical algebraic topology

Let $n \geq 2$ and $i \geq 0$ be integers. The singular cohomology group

$$H^i(\mathbb{S}^{n-1}) \text{ is isomorphic to } \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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Motivic algebraic topology

Let $n \geq 2$, $i \geq 0, j \in \mathbb{Z}$ be integers. The Rost-Schmid group

$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \,. \\ 0 & \text{otherwise} \end{cases}$$

In particular, $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \simeq K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$ and $H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_2) \simeq K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$. These iso. are not canonical.

Contents

Notations

- Let $L = K_1 \sqcup K_2$ be an oriented link with two comp. (in knot theory).
- Let $\mathscr{L}$ be an oriented link with two components of type $(\mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^2_F \setminus \{0\}, \mathbb{A}^4_F \setminus \{0\})$ (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i, together with orientation classes $\overline{o_i}$ (with $i \in \{1, 2\}$).

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- We denote $\nu_{Z_i} := \det(\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}})$ (with $i \in \{1, 2\}$).
- We denote $Z := Z_1 \sqcup Z_2$ and $\nu_Z := \det(\mathcal{N}_{Z/\mathbb{A}^4_F \setminus \{0\}})$.

Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Knot theory

The class S_i in $H^1(\mathbb{S}^3 \setminus L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Motivic knot theory

We define the oriented fundamental class $[o_i]$ as the unique class in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o_i}$ to the class of η in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1})$, then we define the Seifert class $\mathcal{S}_i$ as the unique class in $H^1(X \setminus Z, \underline{K}^{\mathrm{MW}}_1)$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_Z\})$.

The quadratic linking class

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$.

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Motivic knot theory

We define the quadratic linking class of $\mathscr{L}$ as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$.

The ambient quadratic linking degree

Knot theory: the linking number

The linking number of the oriented link $L = K_1 \sqcup K_2$ is the image of the part of the linking class of L which is in $H^1(K_1)$ by the composite of the morphism $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ which is induced by the inclusion $i_1 : K_1 \to \mathbb{S}^3$ and of the isomorphism $r : H^3(\mathbb{S}^3) \to \mathbb{Z}$ which corresponds to the "right-hand rule".

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Motivic knot theory: the ambient quadratic linking degree

We define the *ambient quadratic linking degree* as the image of the part of the quadratic linking class which lives over Z_1 by the composite of the morphism $(i_1)_* : H^1(Z_1, \underline{K}_0^{\mathrm{MW}}\{\nu_{Z_1}\}) \to H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}_2^{\mathrm{MW}})$ induced by the inclusion $i_1 : Z_1 \to \mathbb{A}^4_F \setminus \{0\}$ and of an isomorphism between $H^3(\mathbb{A}^4_F \setminus \{0\}, \underline{K}_2^{\mathrm{MW}})$ and $\mathrm{W}(F)$ which has been fixed once and for all (thanks to the coordinates x, y, z, t).

The quadratic linking degree couple

The linking couple

The linking couple is the image of the linking class by the isomorphism $h_1 \oplus h_2 : H^1(K_1) \oplus H^1(K_2) \to \mathbb{Z} \oplus \mathbb{Z}$ which is induced by the volume forms ω_{K_1} of K_1 and ω_{K_2} of K_2.

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Motivic knot theory

We define the quadratic linking degree couple of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism
$H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}) \to$
$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \oplus H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

This isomorphism between $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}})$ and $\mathrm{W}(F)$ has been fixed once and for all (thanks to the coordinates u, v).

The Hopf link

Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrisation of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

The (amb.) quadratic linking degree (cpl.) of the Hopf link

Or. fund. cl.	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert cl.	$\langle x \rangle \otimes \overline{y}^*$	|	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $(i_1)_*$	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$		
Apply ∂	$-\eta^2 \otimes (\overline{x}^* \wedge \overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$		
Amb. qld.	-1		
Quad. lk. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Qld. couple	-1	$\oplus$	1

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

- The parametrisation is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}]$ $\quad [o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$

since $\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad$ since $\begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$

$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$ $\quad \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$
$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The ambient quadratic linking degree of the variant is $-\langle a \rangle$.
The quadratic linking degree couple of the variant is $(-\langle a \rangle, 1)$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}^4_F \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$.

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- The parametrisation is $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u, v)$ and $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u, -v)$.

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- The orientation is the following:

$$o_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, o_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1.

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- Its ambient quadratic linking degree is $-1 \in \mathrm{W}(F)$.

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- If we change its orientations and its parametrisations then we get as ambient quadratic linking degree $\langle a \rangle \in \mathrm{W}(F)$ for some $a \in F^*$.

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- If we change its orientations and its parametrisations then we get as quadratic linking degree couple $(\langle b \rangle, \langle c \rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ for some $b, c \in F^*$.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.

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- In motivic knot theory, we define the image of the Solomon link as:

$$\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrisation is $\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u^2 - v^2, 2uv)$ and $\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u^2 + v^2, -2uv)$.
- The orientation is the following:

$$o_1 : \overline{z - x^2 + y^2}^* \wedge \overline{t - 2xy}^* \mapsto 1, o_2 : \overline{z + x^2 - y^2}^* \wedge \overline{t + 2xy}^* \mapsto 1$$

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- We want invariants of the quadratic linking degree!

Changing orientations and parametrisations

Proposition (Prop. 5.11 p.120, Prop. 5.18 p.128 and Prop. 5.20 p.129)

Let $\mathscr{L}$ be an oriented link with two components of ambient quadratic linking degree $\alpha \in \mathrm{W}(F)$ and of quadratic linking degree couple $(\beta, \gamma) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$. If $\mathscr{L}'$ is obtained from $\mathscr{L}$ by changing orientations and parametrisations then the ambient quadratic linking degree of $\mathscr{L}'$ is equal to $\langle a \rangle \alpha$ for some $a \in F^*$ and the quadratic linking degree couple of $\mathscr{L}'$ is equal to $(\langle b \rangle \beta, \langle c \rangle \gamma)$ for some $b, c \in F^*$.

Case $F = \mathbb{R}$

The absolute value of an element of $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in \mathbb{R}^*$. This gives an invariant of the qld.

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General case

The rank modulo 2 of an element of $\mathrm{W}(F)$ is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$. This gives an invariant of the quadratic linking degree.

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General case (see Lemma-Def. 5.27 p.136 and Thm 5.28 p.138)

- $\Sigma_2 : \begin{cases} \mathrm{W}(F) & \to \quad \mathrm{W}(F)/(1) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto \quad \sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases}$ (if $n < 2$, it sends $\sum_{i=1}^{n} \langle a_i \rangle$ to 0)

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- $\Sigma_2(\langle a \rangle \sum_{i=1}^{n} \langle a_i \rangle) = \sum_{1 \leq i < j \leq n} \langle a^2 a_i a_j \rangle = \Sigma_2(\sum_{i=1}^{n} \langle a_i \rangle)$

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- This gives an invariant of the quadratic linking degree.

Binary links (which show that Σ_2 is interesting)

Here, F is a perfect field of characteristic different from 2 and $a \in F^*$.

- The image of the binary link B_a is:

$$\{f_1 = 0, g_1 = 0\} \sqcup \{f_2 = 0, g_2 = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

with $f_1 = t - ((1+a)x - y)y$, $g_1 = z - x(x-y)$,
$f_2 = t + ((1+a)x - y)y$, $g_2 = z + x(x-y)$.

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$f_2 = t + ((1+a)x - y)y$, $g_2 = z + x(x-y)$.

- The parametrisation of the binary link B_a is:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (u, v, u(u-v), ((1+a)u - v)v)$$

$$\varphi_2 : (x, y, z, t) \leftrightarrow (u, v, -u(u-v), -((1+a)u - v)v)$$

- The orientation of the binary link B_a is:

$$o_1 : \overline{f_1}^* \wedge \overline{g_1}^* \mapsto 1, o_2 : \overline{f_2}^* \wedge \overline{g_2}^* \mapsto 1$$

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- If $p \neq q$ are prime numbers then $\langle p\rangle \in \mathrm{W}(\mathbb{Q})/(1)$ corresponds to $1 \in \mathrm{W}(\mathbb{Z}/p\mathbb{Z}) \subset \bigoplus_{r \text{ prime}} \mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ and $\langle q\rangle \in \mathrm{W}(\mathbb{Q})/(1)$ corresponds to $1 \in \mathrm{W}(\mathbb{Z}/q\mathbb{Z}) \subset \bigoplus_{r \text{ prime}} \mathrm{W}(\mathbb{Z}/r\mathbb{Z})$ hence $\langle p\rangle \neq \langle q\rangle$ in $\mathrm{W}(\mathbb{Q})/(1)$ hence the invariant induced by Σ_2 distinguishes between infinitely many oriented links (the binary links B_r with r ranging the prime numbers).

General case (see Lemma-Def. 5.27 p.136 and Thm 5.28 p.138)

- $$\Sigma_4 : \begin{cases} \mathrm{W}(F) & \to \bigcup_{d \in \mathrm{W}(F)} (\mathrm{W}(F)/(1))/(\Sigma_2(d)) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto \sum_{1 \leq i < j < k < l \leq n} \langle a_i a_j a_k a_l \rangle \end{cases}$$

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- Similarly, for all $m \in \mathbb{N}$, there is Σ_{2m} which gives an invariant of the quadratic linking degree.

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- Under these assumptions, we define an oriented fundamental class $[o_i] \in H^0(Z_i, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_{Z_i}\}) \subset H^0(Z, \underline{K}^{\mathrm{MW}}_{j_i}\{\nu_Z\})$ and a Seifert class $\mathcal{S}_i \in H^{c-1}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_i+c})$ for each $i \in \{1, 2\}$, as well as a quadratic linking class in $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

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- We took a particular interest in smooth models of motivic spheres.
- A smooth model of the motivic sphere $S^i \wedge \mathbb{G}_m^{\wedge j}$ is a smooth F-scheme which has the $\mathbb{A}^1$-homotopy type of $S^i \wedge \mathbb{G}_m^{\wedge j}$.

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In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor +1} = X$ with $n \in \{2,3,4\}$, the only conditions which are not verified are the ones which are there to ensure the existence of Seifert classes ($H^c(X, \underline{K}^{MW}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{MW}_{j_2+c}) = 0$).

In these settings, the ambient quadratic linking degree is in $\mathrm{W}(F)$ or in $\mathrm{GW}(F)$ and each component of the quadratic linking degree couple is either in the zero group, in $\mathrm{W}(F)$, in $\mathrm{GW}(F)$ or in $K_1^{\mathrm{MW}}(F)$.

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- In the general case, the rank, and:
- $\Sigma_k(\sum_{i=1}^{n}\varepsilon_i\langle a_i\rangle)=\sum_{1\leq i_1<\cdots<i_k\leq n}(\prod_{1\leq l\leq k}\varepsilon_{i_l})\langle\prod_{1\leq j\leq k}a_{i_j}\rangle$ with $k\geq 2$ even, where $\Sigma_k:\mathrm{GW}(F)\to\mathrm{GW}(F)$ (see Lemma-Def. 5.32 p.141 and Thm 5.33 p.142).

Thank you for your attention!