Motivic knot theory

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Contents

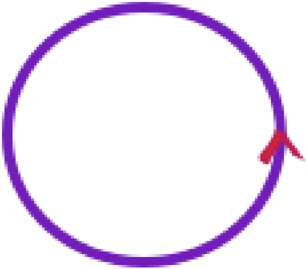


Figure: The unknot

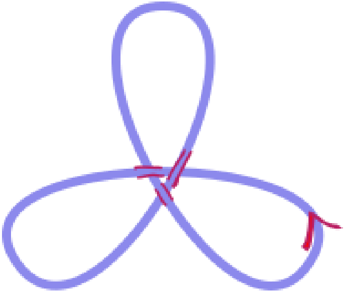


Figure: The trefoil knot

Knot theory in a nutshell

Topological objects of interest are knots and links.

- A **knot** is a (closed) topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$.
- An **oriented knot** is a knot with a “continuous” local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.

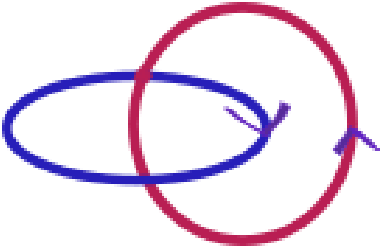


Figure: The Hopf link

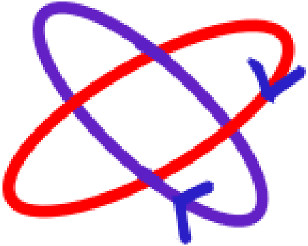
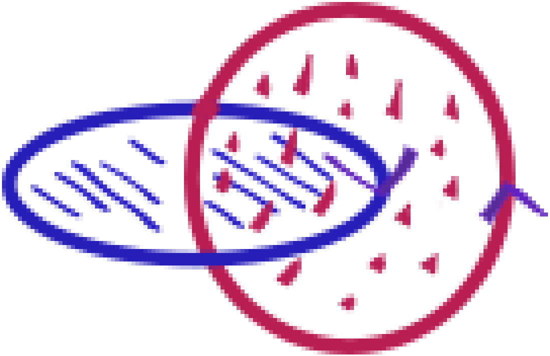


Figure: The Solomon link

The **linking number** of an (oriented) link with two components is the number of times one of the components turns around the other component.

Defining the linking number: Seifert surfaces



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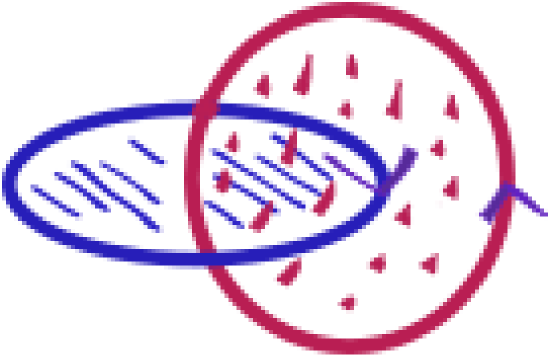
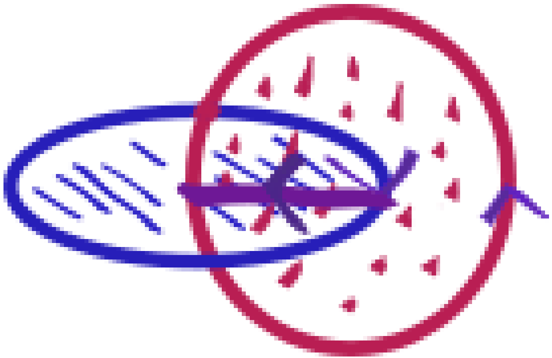


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S. surfaces



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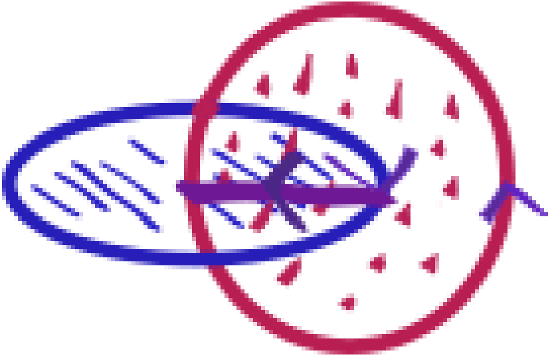
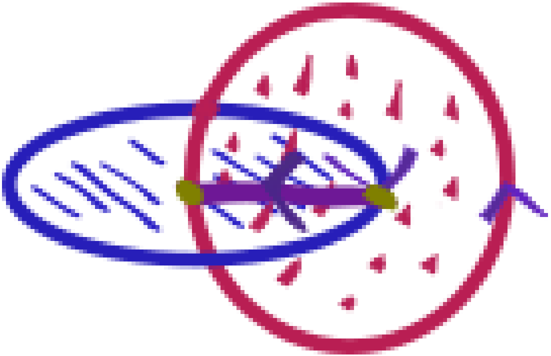


This corresponds to the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.



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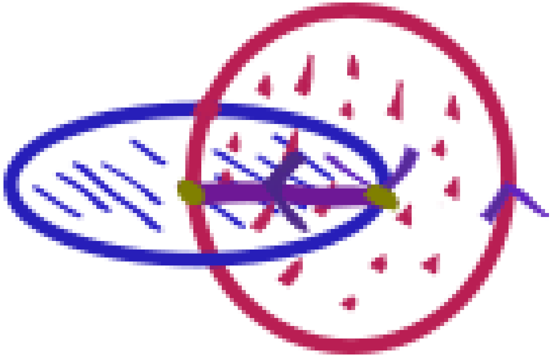

This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(Z_1) \oplus H^1(Z_2)$.

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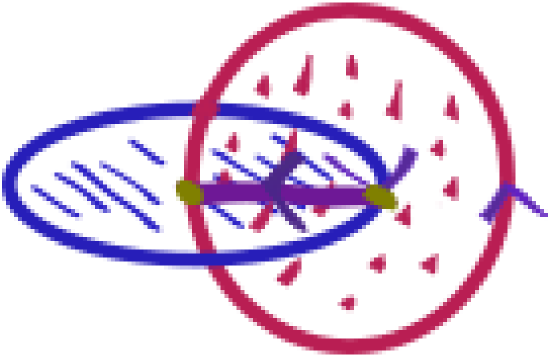

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By comparing orientations, we get a number!

The formal definition of the linking number

Let $L = K_1 \sqcup K_2$ be an oriented link with two components.

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Oriented fundamental class and Seifert class

Let $i \in \{1, 2\}$. The class S_i in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{BM}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Linking class and linking number

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$. The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

When are two spaces "the same" homotopically?

Homotopic maps

Two continuous maps $f, g : X \to Y$ are homotopic if there exists a homotopy from f to g, i.e. a continuous map $H : X \times [0,1] \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

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Homotopy types of topological spaces

Two topological spaces X and Y have the same homotopy type if there exists a homotopy equivalence from X to Y, i.e. a couple $(i : X \to Y, j : Y \to X)$ of continuous maps such that $j \circ i$ is homotopic to the identity of X and $i \circ j$ is homotopic to the identity of Y.

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Important example

For all $n \geq 1$, $\mathbb{S}^n$ has the same homotopy type as $\mathbb{R}^{n+1} \setminus \{0\}$.

Contents

Links in algebraic geometry

Let F be a perfect field.

Link with two components

A link with two components is a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$).

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An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}}$ of Z_i in $\mathbb{A}^4_F \setminus \{0\}$ to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/\mathbb{A}^4_F \setminus \{0\}}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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More concretely

In our examples, an orientation of a knot will be given by the choice of a first polynomial equation f and a second polynomial equation g such that the knot corresponds to $\{f = 0, g = 0\}$.

Oriented links in algebraic geometry

Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o_i' : \nu_{Z_i} \to \mathcal{L}_i' \otimes \mathcal{L}_i'$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}_i'$ such that $(\psi \otimes \psi) \circ o_i = o_i'$.

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Oriented link with two components

An oriented link with two components is a link with two components $(\varphi_1 : \mathbb{A}^2_F \setminus \{0\} \to Z_1, \varphi_2 : \mathbb{A}^2_F \setminus \{0\} \to Z_2)$ together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2.

Orientation classes in algebraic geometry

Proposition

Let $i \in \{1, 2\}$. The orientation classes of Z_i are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.
If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link in algebraic geometry

We fix coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$ once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

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- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1 \otimes 1$$

A variant of the Hopf link

- The image is the same as the image of the Hopf link:

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$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1 \otimes 1$$

Chow groups and intersection theory

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- You may know the following exact sequence where $Y \subset X$ is closed:

$$CH_p(Y) \longrightarrow CH_p(X) \longrightarrow CH_p(X \setminus Y) \longrightarrow 0$$

It can be extended into the following long exact sequence:

$$\cdots \to A_{p+1}(X \setminus Y, -p) \to CH_p(Y) \to CH_p(X) \to CH_p(X \setminus Y) \to 0$$

Chow-Witt groups and quadratic intersection theory

- Solution to the second problem (orientations): replace (generalised) Chow groups, a.k.a. Rost groups, with (generalised) Chow-Witt groups, a.k.a. Rost-Schmid groups; see for instance the chapter *Lectures on Chow-Witt groups* by Jean Fasel in the book *Motivic homotopy theory and refined enumerative geometry* (2020)

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- In cohomological notation, instead of considering the Rost complexes

$$\dots \longrightarrow \bigoplus_{p \in Y^{(i)}} K^{\mathrm{M}}_{j-i}(\kappa(p)) \longrightarrow \bigoplus_{q \in Y^{(i+1)}} K^{\mathrm{M}}_{j-i-1}(\kappa(q)) \longrightarrow \dots$$

(for each $j \in \mathbb{Z}$) whose cohomology groups are the Rost groups $A^i(Y, j)$ (the i-th Chow group $CH^i(Y)$ when $i = j$), we consider

$$\dots \longrightarrow \bigoplus_{p \in Y^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}]$$

$$\downarrow$$

$$\bigoplus_{q \in Y^{(i+1)}} K^{\mathrm{MW}}_{j-i-1}(\kappa(q)) \otimes_{\mathbb{Z}[\kappa(q)^*]} \mathbb{Z}[(\nu_q \otimes \mathcal{L}_{|q}) \setminus \{0\}] \longrightarrow \dots$$

Milnor-Witt K-theory

Definition

The Milnor-Witt K-theory ring associated to F, denoted $K_*^{\mathrm{MW}}(F)$, is the $\mathbb{Z}$-graded ring with unit generated by the elements $[a]$ of degree 1, for $a \in F^*$, and the element η of degree -1, subject to the relations:

- $[ab] = [a] + [b] + \eta[a][b]$ for all $a, b \in F^*$
- $[a][1-a] = 0$ for all $a \in F \setminus \{0,1\}$ (Steinberg relation)
- $\eta[a] = [a]\eta$ for all $a \in F^*$
- $\eta(\eta[-1] + 2) = 0$

The Milnor K-theory ring associated to F is $K_*^{\mathrm{M}}(F) = K_*^{\mathrm{MW}}(F)/(\eta)$.

Notations

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- We also denote by $\langle a \rangle$ the class of the symmetric bilinear form $\begin{cases} F \times F & \to & F \\ (x, y) & \mapsto & axy \end{cases}$ in $\mathrm{GW}(F)$ and in $\mathrm{W}(F)$. If F is of char. $\neq 2$ then $\langle a \rangle$ is the class of the quadratic form $\begin{cases} F & \to & F \\ x & \mapsto & ax^2 \end{cases}$.

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- $\mathrm{GW}(F)$ is made up of $\mathbb{Z}$-linear combinations of $\langle a\rangle$ and $\mathrm{W}(F) = \mathrm{GW}(F)/(\langle 1\rangle + \langle -1\rangle)$ is made up of sums of $\langle a\rangle$.

Milnor-Witt K-theory and quadratic forms

Theorem

The ring $K_0^{\mathrm{MW}}(F)$ is isomorphic to the Grothendieck-Witt ring $\mathrm{GW}(F)$ of the field F via $\langle a \rangle \in K_0^{\mathrm{MW}}(F) \leftrightarrow \langle a \rangle \in \mathrm{GW}(F)$.

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Theorem

For all $n < 0$, the abelian group $K_n^{\mathrm{MW}}(F)$ is isomorphic to the Witt group $\mathrm{W}(F)$ of the field F via $\langle a\rangle\eta^{-n} \in K_n^{\mathrm{MW}}(F) \leftrightarrow \langle a\rangle \in \mathrm{W}(F)$.

The singular complex and the Rost-Schmid complex

Classical algebraic topology

Each topological space X has a singular cochain complex:

$$\dots \longrightarrow \mathcal{C}^i(X) \longrightarrow \mathcal{C}^{i+1}(X) \longrightarrow \dots$$

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Motivic algebraic topology

Each smooth F-scheme X has a Rost-Schmid complex *for each integer $j \in \mathbb{Z}$ and invertible $\mathcal{O}_X$-module $\mathcal{L}$*:

$$\dots \longrightarrow \bigoplus_{p \in X^{(i)}} K^{\mathrm{MW}}_{j-i}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}_{|p}) \setminus \{0\}]$$

$$\downarrow$$

$$\bigoplus_{q \in X^{(i+1)}} K^{\mathrm{MW}}_{j-i-1}(\kappa(q)) \otimes_{\mathbb{Z}[\kappa(q)^*]} \mathbb{Z}[(\nu_q \otimes \mathcal{L}_{|q}) \setminus \{0\}] \longrightarrow \dots$$

The singular cohomology ring and the Rost-Schmid ring

Classical algebraic topology

The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X.

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The i-th cohomology group $H^i(X)$ of X is the i-th cohomology group of the singular cochain complex of X. The cup-product $H^i(X) \times H^{i'}(X) \to H^{i+i'}(X)$ makes $\bigoplus_{i\in\mathbb{N}_0} H^i(X)$ into a graded ring.

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Motivic algebraic topology

The i-th Rost-Schmid group $H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{L}\})$ of X with respect to j and $\mathcal{L}$ is the i-th cohomology group of the Rost-Schmid complex of X w.r.t. j and $\mathcal{L}$. We denote $H^i(X, \underline{K}_j^{\mathrm{MW}}) := H^i(X, \underline{K}_j^{\mathrm{MW}}\{\mathcal{O}_X\})$.

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In particular, the intersection product makes $\bigoplus_{i \in \mathbb{N}_0} \widetilde{CH}^i(Y)$ into a graded $K_0^{\mathrm{MW}}(F)$-algebra (the Chow-Witt ring; where $\widetilde{CH}^i(Y) = H^i(X, \underline{K}_i^{\mathrm{MW}})$).

Classical algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the following long exact sequence (where ∂ is the boundary map):

$$\ldots \longrightarrow H^n(Z) \xrightarrow{i_*} H^{n+d_X-d_Z}(X) \xrightarrow{j^*} H^{n+d_X-d_Z}(U) \xrightarrow{\partial} H^{n+1}(Z) \longrightarrow$$

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Motivic algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the localization long exact sequence (where ∂ is the boundary map):

$$\dots \longrightarrow H^n(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \xrightarrow{i_*} H^{n+d_X-d_Z}(X, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{j^*}$$

$$\xrightarrow{j^*} H^{n+d_X-d_Z}(U, \underline{K}^{\mathrm{MW}}_{m+d_X-d_Z}) \xrightarrow{\partial} H^{n+1}(Z, \underline{K}^{\mathrm{MW}}_m\{\nu_Z\}) \longrightarrow \dots$$

Classical algebraic topology

Let $n \geq 2$ and $i \geq 0$ be integers. The singular cohomology group

$$H^i(\mathbb{S}^{n-1}) \text{ is isomorphic to } \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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Let $n \geq 2$, $i \geq 0, j \in \mathbb{Z}$ be integers. The Rost-Schmid group

$$H^i(\mathbb{A}^n_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_j) \text{ is isomorphic to } \begin{cases} K^{\mathrm{MW}}_j(F) & \text{if } i = 0 \\ K^{\mathrm{MW}}_{j-n}(F) & \text{if } i = n-1 \\ 0 & \text{otherwise} \end{cases}.$$

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In particular, $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}^{\mathrm{MW}}_0) \simeq K^{\mathrm{MW}}_{-2}(F) \simeq \mathrm{W}(F)$. We can fix such an isomorphism, but it is not canonical.

The linking number and the quadratic linking degree

- Let $L = K_1 \sqcup K_2$ be an oriented link (in knot theory).
- Let $\mathscr{L}$ be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}^2_F \setminus \{0\} \to \mathbb{A}^4_F \setminus \{0\}$ with disjoint images Z_i and orientation classes $\overline{o_i}$ (with $i \in \{1, 2\}$).
- We denote $Z := Z_1 \sqcup Z_2$ and $\nu_Z := \det(\mathcal{N}_{Z/\mathbb{A}^4_F \setminus \{0\}})$.

Step 1: oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Knot theory

The class S_i in $H^1(\mathbb{S}^3 \setminus L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Motivic knot theory

We define the oriented fundamental class $[o_i]$ as the unique class in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o_i}$ to the class of η in $H^0(Z_i, \underline{K}^{\mathrm{MW}}_{-1})$, then we define the Seifert class $\mathcal{S}_i$ as the unique class in $H^1(X \setminus Z, \underline{K}^{\mathrm{MW}}_1)$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}^{\mathrm{MW}}_{-1}\{\nu_Z\})$.

Step 2: the quadratic linking class

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \to H^1(L)$.

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Motivic knot theory

We define the quadratic linking class of $\mathscr{L}$ as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\mathrm{MW}}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\})$.

Step 3: the quadratic linking degree

Knot theory

The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

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Motivic knot theory

We define the quadratic linking degree of $\mathscr{L}$ as the image of the quadratic linking class of $\mathscr{L}$ by the isomorphism
$H^1(Z, \underline{K}_0^{\mathrm{MW}}\{\nu_Z\}) \to H^1(Z, \underline{K}_0^{\mathrm{MW}}) \to$
$H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \oplus H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to \mathrm{W}(F) \oplus \mathrm{W}(F)$.

We fixed an isomorphism $H^1(\mathbb{A}^2_F \setminus \{0\}, \underline{K}_0^{\mathrm{MW}}) \to K_{-2}^{\mathrm{MW}}(F)$ once and for all and there is a canonical isomorphism $K_{-2}^{\mathrm{MW}}(F) \to \mathrm{W}(F)$.

The Hopf link

Recall that we fixed coordinates x, y, z, t for $\mathbb{A}^4_F$ and u, v for $\mathbb{A}^2_F$.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}^4_F \setminus \{0\}$$

- The parametrization of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$o_1 : \overline{x}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{t}^* \mapsto 1$$

The quadratic linking degree of the Hopf link

Or. fund. classes	$\eta \otimes (\overline{x}^* \wedge \overline{y}^*)$	|	$\eta \otimes (\overline{z}^* \wedge \overline{t}^*)$
Seifert classes	$\langle x \rangle \otimes \overline{y}^*$	|	$\langle z \rangle \otimes \overline{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\overline{t}^* \wedge \overline{y}^*)$		
Quad. link. class	$-\langle z \rangle \eta \otimes (\overline{t}^* \wedge \overline{x}^* \wedge \overline{y}^*)$	$\oplus$	$\langle x \rangle \eta \otimes (\overline{y}^* \wedge \overline{z}^* \wedge \overline{t}^*)$
Apply $\widetilde{o_1} \oplus \widetilde{o_2}$	$-\langle z \rangle \eta \otimes \overline{t}^*$	$\oplus$	$\langle x \rangle \eta \otimes \overline{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \overline{v}^*$	$\oplus$	$\langle u \rangle \eta \otimes \overline{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$	$\oplus$	$\eta^2 \otimes (\overline{u}^* \wedge \overline{v}^*)$
Quad. link. degree	-1	$\oplus$	1

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}^4_F \setminus \{0\} \text{ with } a \in F^*$$

- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}]$ $[o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$

since $\begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ since $\begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$

$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$ $\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$
$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The quadratic linking degree of the variant is $(-\langle a \rangle, 1)$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for $\mathbb{A}_F^4$ and u, v for $\mathbb{A}_F^2$.

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

  But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$.

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- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}^3_\varepsilon = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1 (i.e. linking class $(1, -1)$).

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- If we change its orientations and its parametrizations then we get $(\langle a\rangle, \langle b\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ with $a, b \in F^*$.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in $\mathbb{S}^3_\varepsilon$ for ε small enough and has linking number 2.

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- We want a means of saying that $(\langle a\rangle + \langle a\rangle, \langle b\rangle + \langle b\rangle)$ is "fundamentally different" from $(\langle c\rangle, \langle d\rangle)$ for all $a, b, c, d \in F^*$ (the Solomon link is "more" different from the Hopf link than the variants of the Hopf link are different from the Hopf link).

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- More generally, we want to compute quantities from the quadratic linking degree which are invariant by changes of orientations and changes of parametrizations of the oriented link.

Proposition

Let $\mathscr{L}$ be an oriented link with two components of quadratic linking degree $(d_1, d_2) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$. If $\mathscr{L}'$ is obtained from $\mathscr{L}$ by changing orientations and parametrisations (isomorphisms with $\mathbb{A}^2_F \setminus \{0\}$) then the quadratic linking degree of $\mathscr{L}'$ is equal to $(\langle a \rangle d_1, \langle b \rangle d_2)$ for some $a, b \in F^*$.

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Case $F = \mathbb{R}$

If $F = \mathbb{R}$, the absolute value of an element of $\mathrm{W}(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$, thus $(|d_1|, |d_2|)$ is invariant.

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General case

The rank modulo 2 is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

- $\Sigma_2 : \begin{cases} \mathrm{W}(F) & \rightarrow & \mathrm{W}(F)/(1) \\ \sum_{i=1}^{n} \langle a_i \rangle & \mapsto & \sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases}$ (if $n < 2$, it sends $\sum_{i=1}^{n} \langle a_i \rangle$ to 0)

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- Etc. for Σ_{2m} with $m \in \mathbb{N}$

Everything new I presented up until now can be found in my preprint “The quadratic linking degree”:

- HAL: Clémentine Lemarié−−Rieusset. THE QUADRATIC LINKING DEGREE. 2022. ⟨hal-03821736⟩
- arXiv: Clémentine Lemarié−−Rieusset. The quadratic linking degree. arXiv:2210.11048 [math.AG]

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- One family of examples is: $\mathbb{A}^{n+1}_F \setminus \{0\} \sqcup \mathbb{A}^{n+1}_F \setminus \{0\} \subset \mathbb{A}^{2n+2}_F \setminus \{0\}$ with $n \geq 1$ and $j_1, j_2 \leq 0$.

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- Another family of examples is: $\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \subset \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd and $j_1, j_2 \leq -2$.

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Motivic spheres

For all $i, j \in \mathbb{Z}$, we denote by S^i the i-th smash-product of S^1 and we call the smash-product $S^i \wedge \mathbb{G}_m^{\wedge j}$ (in the stable homotopy category) a motivic sphere.

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The projective line $\mathbb{P}^1_F$ is a smooth model of $S^1 \wedge \mathbb{G}_m$.

- $Q_{2n} := \mathrm{Spec}(F[x_1, \ldots, x_n, y_1, \ldots, y_n, z]/(\sum_{i=1}^{n} x_i y_i - z(1+z)))$
- Q_{2n} is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge n}$
- $Q_{2n+1} := \mathrm{Spec}(F[x_1, \ldots, x_{n+1}, y_1, \ldots, y_{n+1}]/(\sum_{i=1}^{n+1} x_i y_i - 1))$
- Q_{2n+1} is a smooth model of $S^n \wedge \mathbb{G}_m^{\wedge(n+1)}$
- **Which closed immersions of smooth models of motivic spheres have a quadratic linking class?**

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- $\mathbb{A}^2_F \setminus \{0\} \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$;
- $\mathbb{A}^n_F \setminus \{0\} \sqcup Q_n \to \mathbb{A}_F^{n+\lfloor \frac{n}{2} \rfloor +1} \setminus \{0\}$ with $n \geq 3$;

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- $Q_n \sqcup Q_n \to \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ with $n \geq 2$;
- $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1}$ with $n \geq 5$.

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In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1} = X$ with $n \in \{2, 3, 4\}$, the only conditions which are not verified are the ones which are there to ensure the existence of Seifert classes: $H^c(X, \underline{K}^{MW}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{MW}_{j_2+c}) = 0$.

Depending on $j_1, j_2 \leq 0$, the quadratic linking class lives in a group isomorphic to $\mathrm{W}(F) \oplus \mathrm{W}(F)$, $\mathrm{GW}(F) \oplus \mathrm{GW}(F)$, $K_1^{\mathrm{MW}}(F) \oplus K_1^{\mathrm{MW}}(F)$ (or $\mathrm{W}(F)$, $\mathrm{GW}(F)$, $K_1^{\mathrm{MW}}(F)$).

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To get the quadratic linking degree from the quadratic linking class, apply the isomorphism $H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c})$ induced by the orientation classes, then the isomorphism induced by the parametrisation of Z, then (if you have one) the explicit isomorphism between the direct sum of the Rost-Schmid groups of the schemes you are considering and a well-known group ($\mathrm{W}(F) \oplus \mathrm{W}(F)$ etc.).