Motivic linking in the projective space

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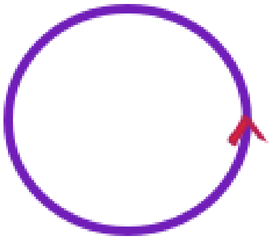
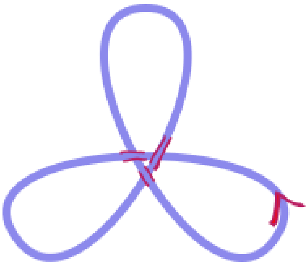
27 May 2025

Contents

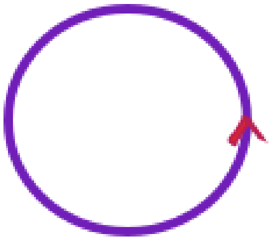


The unknot



The trefoil knot

A classical **knot** is a topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$ (+ a tameness condition e.g. smoothness).



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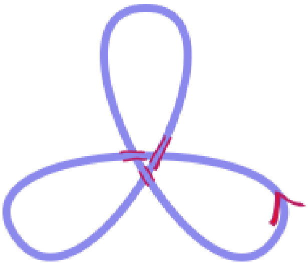
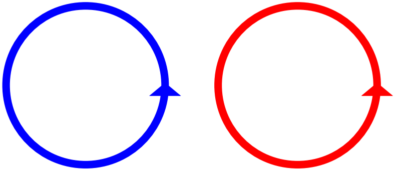
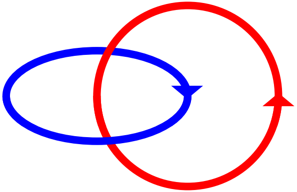


The trefoil knot

A classical **knot** is a topological subspace of the 3-sphere $\mathbb{S}^3$ which is homeomorphic to the circle $\mathbb{S}^1$ ($+$ a tameness condition e.g. smoothness). A classical **oriented knot** is a knot with a "continuous" local trivialization of its tangent bundle, or equivalently of its normal bundle.

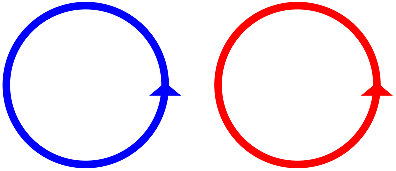


The unlink with two components

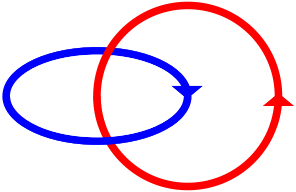


The Hopf link

A classical **link** is a finite union of disjoint knots (called components).

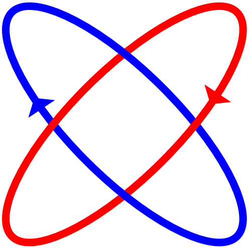


The unlink with two components
(linking number $= 0$)

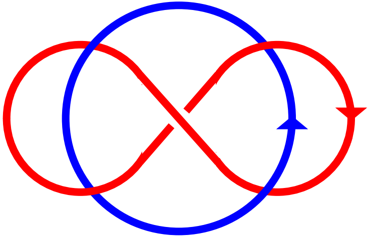


The Hopf link
(linking number $= 1$)

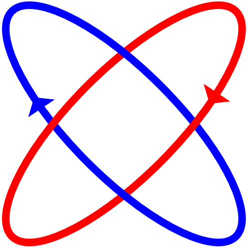
A classical **link** is a finite union of disjoint knots (called components).
The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component (the sign indicating the direction).



The Solomon link
(linking number $= 2$)



The Whitehead link
(linking number $= 0$)



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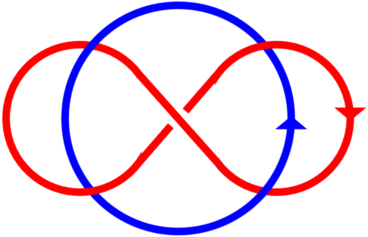

The Whitehead link
(linking number $= 0$)

The linking number is defined for A^{k-1} and B^{n-k} two disjoint oriented homologically of finite order submanifolds of an oriented n-dimensional manifold M^n as $\frac{1}{m}$ times the intersection number of C^k with B^{n-k}, where C^k is a singular chain of boundary mA^{k-1} with $m \geq 1$ an integer.
Example: $\mathbb{S}^{k-1}$ and $\mathbb{S}^{n-k}$ in $\mathbb{S}^n$, e.g. $\mathbb{S}^1$ and $\mathbb{S}^1$ in $\mathbb{S}^3$.

Projective linking

Another example: $\mathbb{RP}^{k-1}$ and $\mathbb{RP}^{n-k}$ in $\mathbb{RP}^n$ with $k-1$ and $n-k$ odd (hence n odd) for orientability, e.g. $\mathbb{RP}^1$ and $\mathbb{RP}^1$ in $\mathbb{RP}^3$.

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A **projective knot** is a topological subspace of the projective space $\mathbb{RP}^3$ which is homeomorphic to the projective line $\mathbb{RP}^1$ (hence to the circle $\mathbb{S}^1$).

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A **projective link** is a finite union of disjoint projective knots (called components).

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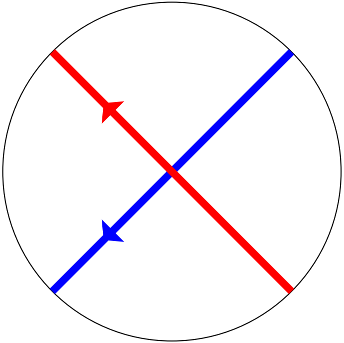
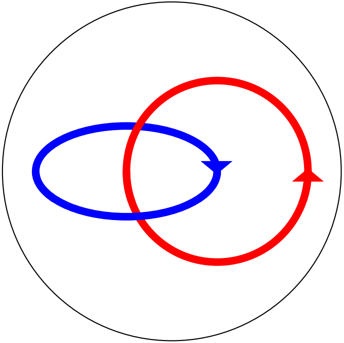
A **projective link** is a finite union of disjoint projective knots (called components).

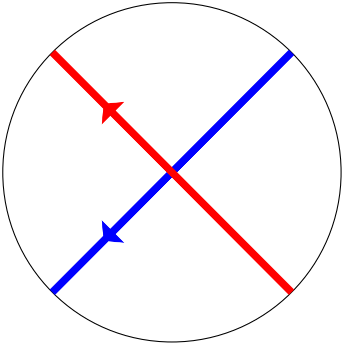
$H_1(\mathbb{RP}^3) \simeq \mathbb{Z}/2\mathbb{Z}$, thus projective knots are homologically of order 1 or 2 and the linking number is a half-integer, i.e. is of the form $\frac{l}{2}$ with $l \in \mathbb{Z}$.



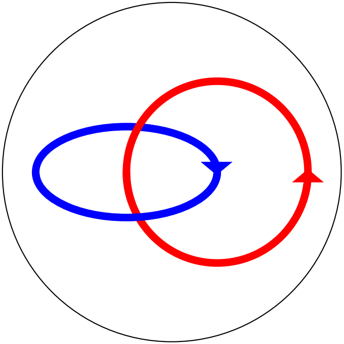
The projective Hopf link



The affine Hopf link



The projective Hopf link
(linking number $= \frac{1}{2}$)



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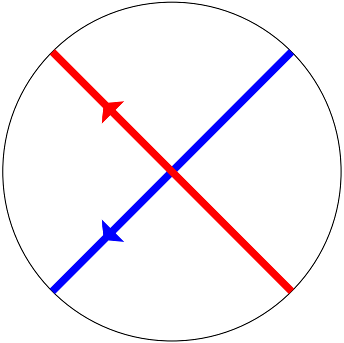

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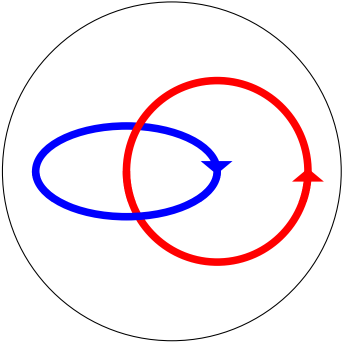
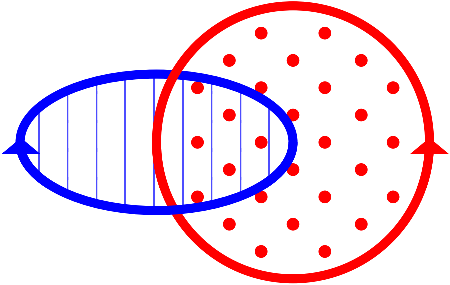


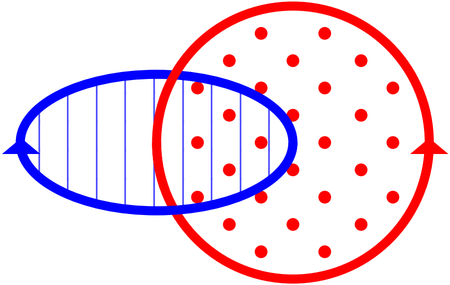
The affine Hopf link
(linking number $= 1$)

The knots in the picture on the left are homologically of order 2 whereas the knots in the picture on the right are homologically trivial (/of order 1).

The linking number of circles in $\mathbb{S}^3$: Seifert surfaces

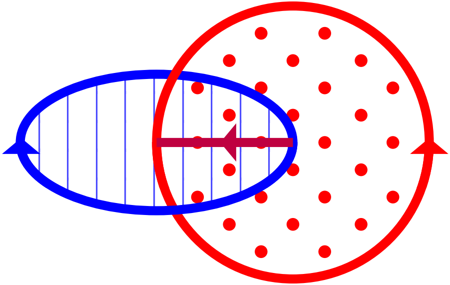


The linking number of circles in $\mathbb{S}^3$: Seifert surfaces

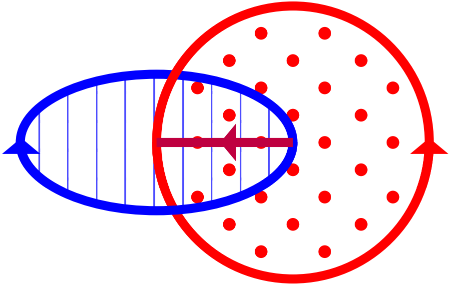


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\mathrm{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the **unique** class that is sent by the **boundary map** to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Intersection of Seifert surfaces in $\mathbb{S}^3$

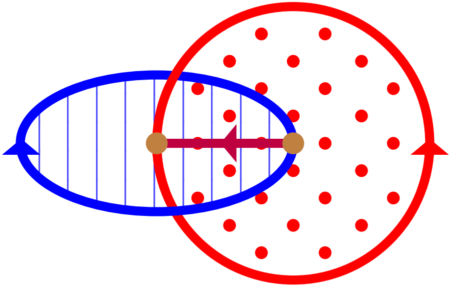


Intersection of Seifert surfaces in $\mathbb{S}^3$

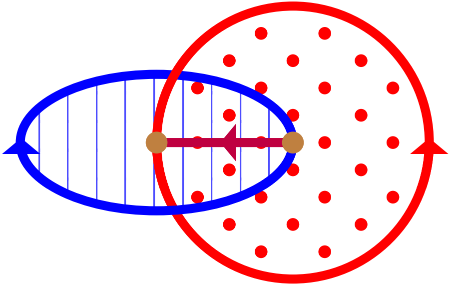


This intersection corresponds to the **cup-product** $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Boundary of the intersection of Seifert surfaces



Boundary of the intersection of Seifert surfaces



This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(K_1) \oplus H^1(K_2)$, which we call the **linking class**. Writing $\partial(S_1 \cup S_2) = (\sigma_1, \sigma_2)$, the **linking number** is $r((i_1)_*(\sigma_1)) \in \mathbb{Z}$ with $(i_1)_* : H^1(K_1) \to H^3(\mathbb{S}^3)$ induced by the inclusion.

The linking class, the linking number and the linking couple

- The **linking class** is $\partial(S_1 \cup S_2) = (\sigma_1, \sigma_2) \in H^1(K_1) \oplus H^1(K_2)$.

The linking class, the linking number and the linking couple

- The **linking class** is $\partial(S_1 \cup S_2) = (\sigma_1, \sigma_2) \in H^1(K_1) \oplus H^1(K_2)$.
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- The **linking couple** is the couple of integers $(h_1(\sigma_1), h_2(\sigma_2))$ with $h_i : H^1(K_i) \simeq \mathbb{Z}$ induced by the volume form ω_{K_i} of K_i (which is induced by the orientation of K_i).

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Important fact

The linking couple is equal to $(\pm n, \pm n)$ with n the linking number.

The linking class, the linking number and the linking couple

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Important fact

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$r((i_2)_*(\sigma_2))$ is the opposite of the linking number and $(i_1)_*, (i_2)_*$ are surjective group morphisms.

Contents

Links in algebraic geometry

Let F be a perfect field and X be a smooth finite-type irred. F-scheme.

Link with two components

A link with two components in X is a couple of disjoint smooth finite-type irreducible closed F-subschemes Z_1 and Z_2 of X such that:

- Z_1 and Z_2 have the same codimension c in X;
- $H^{c-1}(X, \underline{K}^{MW}_{j_1+c}) = 0$ and $H^{c}(X, \underline{K}^{MW}_{j_1+c}) = 0$ for some $j_1 \leq 0$;
- $H^{c-1}(X, \underline{K}^{MW}_{j_2+c}) = 0$ and $H^{c}(X, \underline{K}^{MW}_{j_2+c}) = 0$ for some $j_2 \leq 0$.

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Example: $Z_1 \simeq \mathbb{P}^1_F$ and $Z_2 \simeq \mathbb{P}^1_F$ disjoint closed F-subschemes of $X = \mathbb{P}^3_F$.

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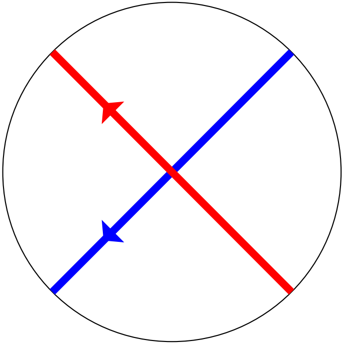
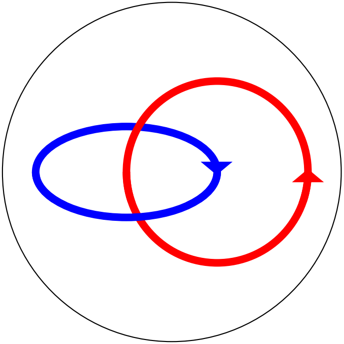
$H^2(\mathbb{P}^3_F, \underline{K}^{MW}_0) = 0$ so in this case every knot is homologically trivial ($\neq$ for the projective knots we described earlier; $H^2(\mathbb{RP}^3) \simeq \mathbb{Z}/2\mathbb{Z}$).



The projective Hopf link



The affine Hopf link

$$x = 0, y = 0$$
$$z = 0, t = 0$$

$$x^2 = y^2 + z^2, t = 0$$
$$x^2 = (z - x)^2 + t^2, y = 0$$

Oriented links in algebraic geometry

An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/X}$ of Z_i in X to the tensor product of an invertible $\mathcal{O}_{Z_i}$-module $\mathcal{L}_i$ with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/X}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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Orientation classes

Two orientations $o_i : \nu_{Z_i} \to \mathcal{L}_i \otimes \mathcal{L}_i$ and $o_i' : \nu_{Z_i} \to \mathcal{L}_i' \otimes \mathcal{L}_i'$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}_i'$ such that $(\psi \otimes \psi) \circ o_i = o_i'$.

The link (Z_1, Z_2) together with an orientation class $\overline{o_1}$ of Z_1 and an orientation class $\overline{o_2}$ of Z_2 is an oriented link with two components.

$Z_i \simeq \mathbb{P}^1_F$ of degree d in $\mathbb{P}^3_F$

We have the two Euler sequences:

$$0 \longrightarrow \Omega^1_{Z_i/F} \longrightarrow \mathcal{O}_{Z_i}(-1) \oplus \mathcal{O}_{Z_i}(-1) \longrightarrow \mathcal{O}_{Z_i} \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_{\mathbb{P}^3_F/F} \longrightarrow \mathcal{O}_{\mathbb{P}^3_F}(-1)^{\oplus 4} \longrightarrow \mathcal{O}_{\mathbb{P}^3_F} \longrightarrow 0$$

as well as the short exact sequence

$$0 \longrightarrow T_{Z_i/F} \longrightarrow (T_{\mathbb{P}^3_F/F})_{|Z_i} \longrightarrow \mathcal{N}_{Z_i/\mathbb{P}^3_F} \longrightarrow 0$$

$$\nu_{Z_i} := \det(\mathcal{N}_{Z_i/X}) \simeq \mathcal{O}_{Z_i}(-2) \otimes \mathcal{O}_{Z_i}(4d) \simeq \mathcal{O}_{Z_i}(2d-1) \otimes \mathcal{O}_{Z_i}(2d-1)$$

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For Z_i in the projective Hopf link, $d = 1$ thus $\nu_{Z_i} \simeq \mathcal{O}_{Z_i}(1) \otimes \mathcal{O}_{Z_i}(1)$.

$Z_i \simeq \mathbb{P}^1_F$ of degree d in $\mathbb{P}^3_F$

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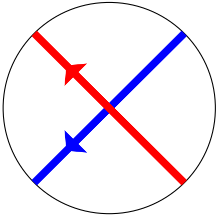
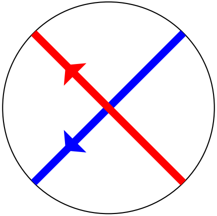
For Z_i in the projective Hopf link, $d = 1$ thus $\nu_{Z_i} \simeq \mathcal{O}_{Z_i}(1) \otimes \mathcal{O}_{Z_i}(1)$.
For Z_i in the affine Hopf link, $d = 2$ thus $\nu_{Z_i} \simeq \mathcal{O}_{Z_i}(3) \otimes \mathcal{O}_{Z_i}(3)$.



$\varphi_1 : \mathbb{P}^1_F \to \mathbb{P}^3_F$ which sends $[u : v]$ to $[0 : 0 : u : v]$ maps $U_u := \{u \neq 0\}$ to $Z_1 \cap \{z \neq 0\}$ and $U_v := \{v \neq 0\}$ to $Z_1 \cap \{t \neq 0\}$. We choose $\overline{o_1}$ to be the orientation class which is given on $Z_1 \cap \{z \neq 0\}$ by $\overline{\frac{x}{z}}^* \wedge \overline{\frac{y}{z}}^* \mapsto 1 \otimes 1$ and on $Z_1 \cap \{t \neq 0\}$ by $\overline{\frac{x}{t}}^* \wedge \overline{\frac{y}{t}}^* \mapsto 1 \otimes 1$.



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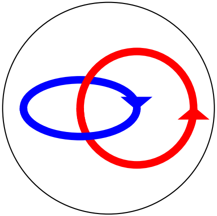
$\varphi_2 : \mathbb{P}^1_F \to \mathbb{P}^3_F$ sends $[u : v]$ to $[u : v : 0 : 0]$ and we choose $\overline{o_2}$ to be the orientation class which is given on $Z_2 \cap \{x \neq 0\}$ by $\overline{\frac{z}{x}}^* \wedge \overline{\frac{t}{x}}^* \mapsto 1 \otimes 1$ and on $Z_2 \cap \{y \neq 0\}$ by $\overline{\frac{z}{y}}^* \wedge \overline{\frac{t}{y}}^* \mapsto 1 \otimes 1$.



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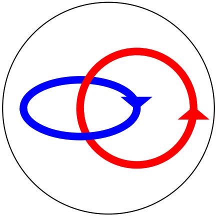

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Oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Definition

- We define the **oriented fundamental class** $[o_i]_{j_i}$ with respect to $j_i \leq 0$ as the unique class in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_{Z_i}\})$ that is sent by $\widetilde{o}_i$ to the class of η^{-j_i} in $H^0(Z_i, \underline{K}_{j_i}^{\mathrm{MW}})$.

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- We define the **Seifert class** $\mathcal{S}_{o_i, j_i}$ with respect to j_i as the unique class in $H^{c-1}(X \setminus Z, \underline{K}_{j_i+c}^{\mathrm{MW}})$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i]_{j_i} \in H^0(Z, \underline{K}_{j_i}^{\mathrm{MW}}\{\nu_Z\})$.

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The assumptions $H^{c-1}(X, \underline{K}_{j_i+c}^{\mathrm{MW}}) = 0$ and $H^c(X, \underline{K}_{j_i+c}^{\mathrm{MW}}) = 0$ made earlier are there to ensure the unicity and the existence resp. of the Seifert class.

The (ambient) quadratic linking class / degree

The quadratic linking class

We define the **quadratic linking class** with respect to (j_1, j_2) as the image of the intersection product $\mathcal{S}_{o_1,j_1} \cdot \mathcal{S}_{o_2,j_2}$ by the boundary map $\partial : H^{2c-2}(X \setminus Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c}) \to H^{c-1}(Z, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_Z\})$.

The **quadratic linking degree** (couple) is the image of the quadratic linking class by an isomorphism.

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The ambient quadratic linking class

We define the **ambient quadratic linking class** with respect to (j_1, j_2) as the image of the part of the quadratic linking class which is in $H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\})$ by the morphism $(i_1)_* : H^{c-1}(Z_1, \underline{K}^{\mathrm{MW}}_{j_1+j_2+c}\{\nu_{Z_1}\}) \to H^{2c-1}(X, \underline{K}^{\mathrm{MW}}_{j_1+j_2+2c})$.

The **ambient quadratic linking degree** is the image of the ambient quadratic linking class by an isomorphism.

The projective Hopf link

To make computations easier, we choose a "nice" $D \simeq \mathbb{A}^3_F$ inside $\mathbb{P}^3_F$. We let $h := x + y + z + t \in F[x, y, z, t]$ and $D := \{h \neq 0\}$ in X and in D we denote $x' := \frac{x}{h}$, $y' := \frac{y}{h}$ and $z' := \frac{z}{h}$ (so that $\frac{t}{h} = 1 - x' - y' - z'$).

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- the oriented fundamental class $[o_1] \in H^0(Z_1, \underline{K}^{MW}_{-2}\{\nu_{Z_1}\})$ is represented by the cycle $\eta^2 \otimes (\overline{x'}^* \wedge \overline{y'}^*)$;
- the oriented fundamental class $[o_2] \in H^0(Z_2, \underline{K}^{MW}_{-2}\{\nu_{Z_2}\})$ is represented by the cycle $\eta^2 \otimes (\overline{z'}^* \wedge \overline{1 - x' - y' - z'}^*)$;

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- the oriented fundamental class $[o_2] \in H^0(Z_2, \underline{K}^{MW}_{-2}\{\nu_{Z_2}\})$ is represented by the cycle $\eta^2 \otimes (\overline{z'}^* \wedge \overline{1 - x' - y' - z'}^*)$;
- the Seifert class $\mathcal{S}_1 \in H^1(X \setminus Z, \underline{K}^{MW}_0)$ is represented by the cycle $\eta\langle x'\rangle \otimes \overline{y'}^*$;
- the Seifert class $\mathcal{S}_2 \in H^1(X \setminus Z, \underline{K}^{MW}_0)$ is represented by the cycle $\eta\langle z'\rangle \otimes \overline{1 - x' - y' - z'}^*$;

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- the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_0^{\mathrm{MW}})$ is represented by the cycle $\eta^2\langle x'z'\rangle \otimes (\overline{1 - x' - y' - z'}^* \wedge \overline{y'}^*)$;

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- the quadratic linking class, i.e. $\partial(\mathcal{S}_1 \cdot \mathcal{S}_2) \in H^1(Z, \underline{K}_{-2}^{\mathrm{MW}}\{\nu_Z\})$, is represented by the cycle $\eta^3\langle -1\rangle \otimes (\overline{1 - x' - y' - z'}^* \wedge \overline{x'}^* \wedge \overline{y'}^*) \oplus \eta^3 \otimes (\overline{y'}^* \wedge \overline{z'}^* \wedge \overline{1 - x' - y' - z'}^*)$ (the first term being over $[0:0:1:0]$ and the second term being over $[1:0:0:0]$);

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- the quadratic linking degree is $(-1, 1) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ (i.e. $(x \mapsto -x^2, x \mapsto x^2)$).

The isomorphism which gives the QLD from the QLC is the composite:

$$H^1(Z_1, \underline{K}_{-2}^{\mathrm{MW}}\{\nu_{Z_1}\}) \oplus H^1(Z_2, \underline{K}_{-2}^{\mathrm{MW}}\{\nu_{Z_2}\}) \simeq H^1(Z_1, \underline{K}_{-2}^{\mathrm{MW}}) \oplus H^1(Z_2, \underline{K}_{-2}^{\mathrm{MW}})$$
$$\simeq H^1(\mathbb{P}^1_F, \underline{K}_{-2}^{\mathrm{MW}}) \oplus H^1(\mathbb{P}^1_F, \underline{K}_{-2}^{\mathrm{MW}}) \simeq \mathrm{W}(F) \oplus \mathrm{W}(F)$$

Quadratic linking degrees

- The affine Hopf link: $(\langle 1\rangle + \langle 3\rangle, \langle -2\rangle(\langle 1\rangle + \langle 3\rangle)) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ (i.e. $((x,y) \mapsto x^2 + 3y^2, (x,y) \mapsto -2x^2 - 6y^2)$).

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- A projective conic $(\{t = 0, xz = y^2\})$ and a projective line "inside" it $(\{y = 0, z = x\})$: $(\langle -1\rangle + \langle -1\rangle, \langle -5\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ (i.e. $((x,y) \mapsto -x^2 - y^2, x \mapsto 5x^2)$).

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- A projective conic $(\{t = 0, xz = y^2\})$ and a projective line "across" it $(\{t = x, z = -x\})$: $(0, \langle -1\rangle) \in \mathrm{W}(F) \oplus \mathrm{W}(F)$ (i.e. $(0 \mapsto 0, x \mapsto -x^2)$).

Other interesting contexts for quadratic linking degrees

- $\mathbb{P}^n_F \sqcup \mathbb{P}^n_F \subset \mathbb{P}^{2n+1}_F$ with $n \geq 1$ odd (and $j_1, j_2 \leq -2$; $\mathrm{W}(F)$);

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- $\mathbb{A}^n_F \setminus \{0\} \sqcup Q_n \to \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ with $n \geq 3$ ($\mathrm{W}(F)$ or $\mathrm{GW}(F)$ or $K_1^{\mathrm{MW}}(F)$);
- $Q_2 \sqcup Q_2 \to \mathbb{A}^4_F \setminus \{0\}$ ($\mathrm{W}(F)$ or $\mathrm{GW}(F)$ or $K_1^{\mathrm{MW}}(F)$);
- $Q_n \sqcup Q_n \to \mathbb{A}^{n+\lfloor \frac{n}{2} \rfloor+1}_F \setminus \{0\}$ with $n \geq 3$ ($\mathrm{W}(F)$ or $\mathrm{GW}(F)$ or $K_1^{\mathrm{MW}}(F)$);
- $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1}$ with $n \geq 2$ ($\mathrm{W}(F)$ or $\mathrm{GW}(F)$ or $K_1^{\mathrm{MW}}(F)$).

In the cases $Q_n \sqcup Q_n \to Q_{n+\lfloor \frac{n}{2} \rfloor+1} = X$ with $n \in \{2, 3, 4\}$, the conditions $H^c(X, \underline{K}^{\mathrm{MW}}_{j_1+c}) = 0$ and $H^c(X, \underline{K}^{\mathrm{MW}}_{j_2+c}) = 0$ (which are there to ensure the existence of Seifert classes) are not verified (but there are some nice examples there).

Thanks for your attention!