

Motivic knot theory

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- Knots and links
- The linking number

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- The quadratic linking degree
- Invariants of the quadratic linking degree

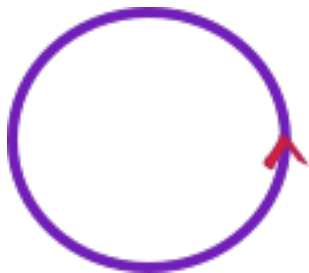


Figure: The unknot

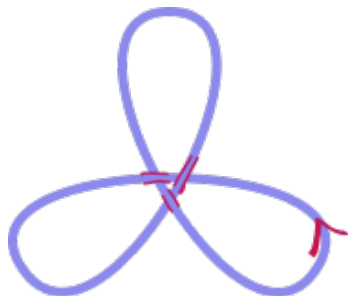


Figure: The trefoil knot

Knot theory in a nutshell

Topological objects of interest are knots and links.

- A **knot** is a (closed) topological subspace of the 3-sphere \mathbb{S}^3 which is homeomorphic to the circle \mathbb{S}^1 .
- An **oriented knot** is a knot with a “continuous” local trivialization of its tangent bundle, or equivalently of its normal bundle (the ambient space being oriented). There are two orientation classes.
- A **link** is a finite union of disjoint knots. A link is **oriented** if all its components (i.e. its knots) are oriented.

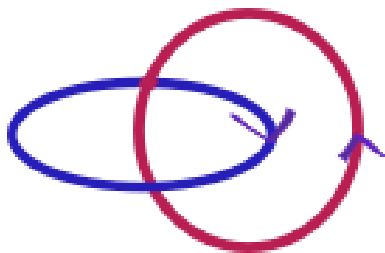


Figure: The Hopf link

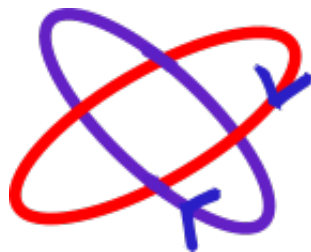
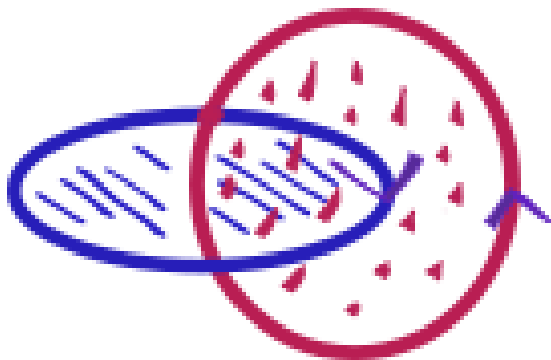


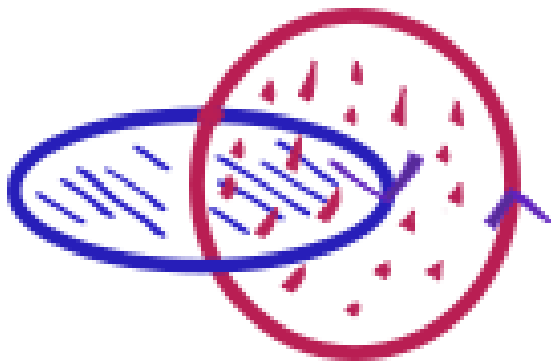
Figure: The Solomon link

The **linking number** of an oriented link with two components is the number of times one of the components turns around the other component.

Defining the linking number: Seifert surfaces

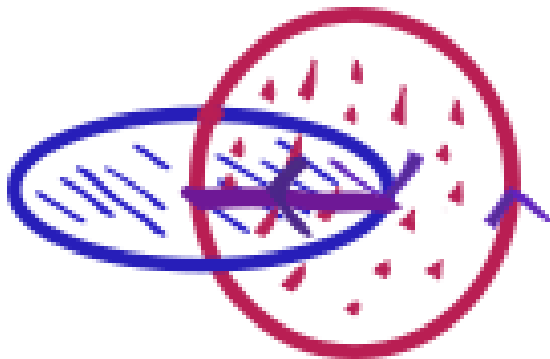


Defining the linking number: Seifert surfaces

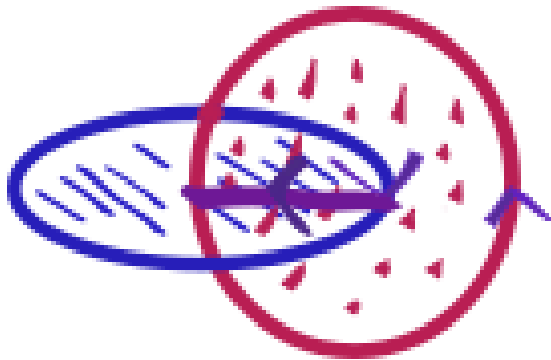


The class S_1 in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\text{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_1 is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_1 in $H^0(K_1) \subset H^0(L)$.

Defining the linking number: intersection of S . surfaces

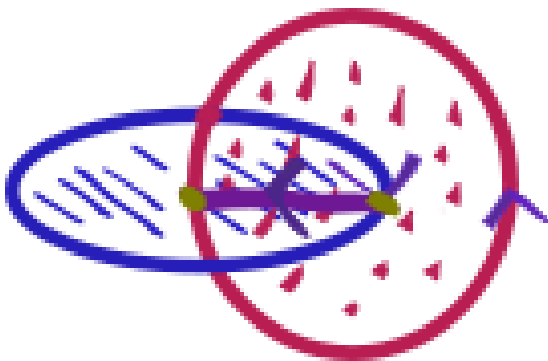


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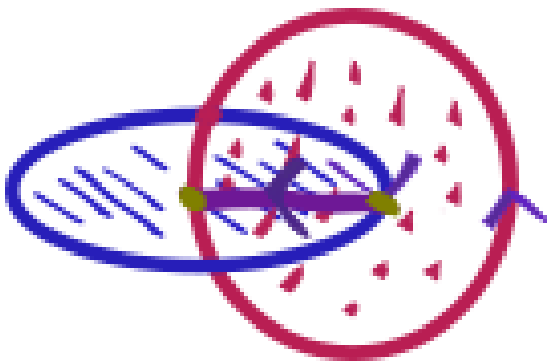


This corresponds to the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$.

Defining the linking number: boundary of int. of S. surf.

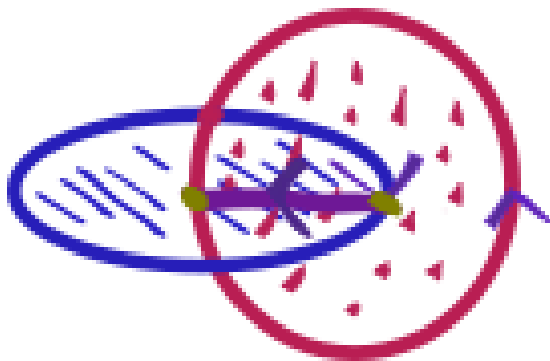


Defining the linking number: boundary of int. of S . surf.



This corresponds to $\partial(S_1 \cup S_2) \in H^1(L) \simeq H^1(Z_1) \oplus H^1(Z_2)$.

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 By comparing orientations, we get a number!

The formal definition of the linking number

Let $L = K_1 \sqcup K_2$ be an oriented link with two components.

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Oriented fundamental class and Seifert class

Let $i \in \{1, 2\}$. The class S_i in $H^1(\mathbb{S}^3 \setminus L) \simeq H_2^{\text{BM}}(\mathbb{S}^3, L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Linking class and linking number

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \rightarrow H^1(L)$. The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

When are two spaces “the same” homotopically?

Homotopic maps

Two continuous maps $f, g : X \rightarrow Y$ are homotopic if there exists a homotopy from f to g , i.e. a continuous map $H : X \times [0, 1] \rightarrow Y$ such that for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.

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Homotopy types of topological spaces

Two topological spaces X and Y have the same homotopy type if there exists a homotopy equivalence from X to Y , i.e. a couple $(i : X \rightarrow Y, j : Y \rightarrow X)$ of continuous maps such that $j \circ i$ is homotopic to the identity of X and $i \circ j$ is homotopic to the identity of Y .

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Important example

For all $n \geq 1$, \mathbb{S}^n has the same homotopy type as $\mathbb{R}^{n+1} \setminus \{0\}$.

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Links in algebraic geometry

Let F be a perfect field.

Link with two components

A link with two components is a couple of closed immersions

$\varphi_i : \mathbb{A}_F^2 \setminus \{0\} \rightarrow \mathbb{A}_F^4 \setminus \{0\}$ with disjoint images Z_i (where $i \in \{1, 2\}$).

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An orientation o_i of Z_i is an isomorphism from the determinant (i.e. the maximal exterior power) of the normal sheaf $\mathcal{N}_{Z_i/\mathbb{A}_F^4 \setminus \{0\}}$ of Z_i in $\mathbb{A}_F^4 \setminus \{0\}$ to the tensor product of an invertible \mathcal{O}_{Z_i} -module \mathcal{L}_i with itself:

$$o_i : \nu_{Z_i} := \det(\mathcal{N}_{Z_i/\mathbb{A}_F^4 \setminus \{0\}}) \simeq \mathcal{L}_i \otimes \mathcal{L}_i$$

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More concretely

In our examples, an orientation of a knot will be given by the choice of a first polynomial equation f and a second polynomial equation g such that the knot corresponds to $\{f = 0, g = 0\}$.

Oriented links in algebraic geometry

Orientation classes

Two orientations $o_i : \nu_{Z_i} \rightarrow \mathcal{L}_i \otimes \mathcal{L}_i$ and $o'_i : \nu_{Z_i} \rightarrow \mathcal{L}'_i \otimes \mathcal{L}'_i$ of Z_i represent the same orientation class of Z_i if there exists an isomorphism $\psi : \mathcal{L}_i \simeq \mathcal{L}'_i$ such that $(\psi \otimes \psi) \circ o_i = o'_i$.

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Oriented link with two components

An oriented link with two components is a link with two components $(\varphi_1 : \mathbb{A}_F^2 \setminus \{0\} \rightarrow Z_1, \varphi_2 : \mathbb{A}_F^2 \setminus \{0\} \rightarrow Z_2)$ together with an orientation class \overline{o}_1 of Z_1 and an orientation class \overline{o}_2 of Z_2 .

Orientation classes in algebraic geometry

Proposition

Let $i \in \{1, 2\}$. The orientation classes of Z_i are parametrized by the elements of $F^*/(F^*)^2$ (where $(F^*)^2 = \{a \in F^*, \exists b \in F^*, a = b^2\}$).

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If $F = \mathbb{C}$ then $F^*/(F^*)^2$ has one element.

If $F = \mathbb{Q}$ then $F^*/(F^*)^2$ has infinitely many elements (the classes of the integers without square factors).

The Hopf link in algebraic geometry

We fix coordinates x, y, z, t for \mathbb{A}_F^4 and u, v for \mathbb{A}_F^2 once and for all.

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

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- The orientation of the Hopf link:

$$\sigma_1 : \bar{x}^* \wedge \bar{y}^* \mapsto \mathbf{1} \otimes \mathbf{1}, \sigma_2 : \bar{z}^* \wedge \bar{t}^* \mapsto \mathbf{1} \otimes \mathbf{1}$$

A variant of the Hopf link

- The image is the same as the image of the Hopf link:

$$\{x = y, y = 0\} \sqcup \{z = 0, at = 0\} \subset \mathbb{A}_F^4 \setminus \{0\} \text{ with } a \in F^*$$

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- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$o_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1 \otimes 1, o_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1 \otimes 1$$

Notations

- The generators of the Milnor-Witt K -theory ring of a field F are denoted $[a] \in K_1^{\text{MW}}(F)$ for every $a \in F^*$ and $\eta \in K_{-1}^{\text{MW}}(F)$.

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- We also denote by $\langle a \rangle$ the class of the symmetric bilinear form

$$\begin{cases} F \times F & \rightarrow & F \\ (x, y) & \mapsto & axy \end{cases}$$
 in $\text{GW}(F)$ and in $\text{W}(F)$. If F is of char. $\neq 2$ then

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$$\begin{cases} F & \rightarrow & F \\ x & \mapsto & ax^2. \end{cases}$$
- $\text{GW}(F)$ is made up of \mathbb{Z} -linear combinations of $\langle a \rangle$ and $\text{W}(F) = \text{GW}(F)/(\langle 1 \rangle + \langle -1 \rangle)$ is made up of sums of $\langle a \rangle$.

Milnor-Witt K -theory and quadratic forms

Theorem

The ring $K_0^{\text{MW}}(F)$ is isomorphic to the Grothendieck-Witt ring $\text{GW}(F)$ of the field F via $\langle a \rangle \in K_0^{\text{MW}}(F) \leftrightarrow \langle a \rangle \in \text{GW}(F)$.

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Theorem

For all $n < 0$, the abelian group $K_n^{\text{MW}}(F)$ is isomorphic to the Witt group $W(F)$ of the field F via $\langle a \rangle \eta^{-n} \in K_n^{\text{MW}}(F) \leftrightarrow \langle a \rangle \in W(F)$.

The singular complex and the Rost-Schmid complex

Classical algebraic topology

Each topological space X has a singular cochain complex:

$$\dots \longrightarrow C^i(X) \longrightarrow C^{i+1}(X) \longrightarrow \dots$$

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Motivic algebraic topology

Each smooth F -scheme X has a Rost-Schmid complex for each integer $j \in \mathbb{Z}$ and invertible \mathcal{O}_X -module \mathcal{L} :

$$\begin{array}{c} \dots \longrightarrow \bigoplus_{p \in X^{(i)}} K_{j-i}^{\text{MW}}(\kappa(p)) \otimes_{\mathbb{Z}[\kappa(p)^*]} \mathbb{Z}[(\nu_p \otimes \mathcal{L}|_p) \setminus \{0\}] \\ \downarrow \\ \bigoplus_{q \in X^{(i+1)}} K_{j-i-1}^{\text{MW}}(\kappa(q)) \otimes_{\mathbb{Z}[\kappa(q)^*]} \mathbb{Z}[(\nu_q \otimes \mathcal{L}|_q) \setminus \{0\}] \longrightarrow \dots \end{array}$$

The singular cohomology ring and the Rost-Schmid ring

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The i -th cohomology group $H^i(X)$ of X is the i -th cohomology group of the singular cochain complex of X . The cup-product $H^i(X) \times H^{i'}(X) \rightarrow H^{i+i'}(X)$ makes $\bigoplus_{i \in \mathbb{N}_0} H^i(X)$ into a graded ring.

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Motivic algebraic topology

The i -th Rost-Schmid group $H^i(X, \underline{K}_j^{\text{MW}}\{\mathcal{L}\})$ of X with respect to j and \mathcal{L} is the i -th cohomology group of the Rost-Schmid complex of X w.r.t. j and \mathcal{L} . We denote $H^i(X, \underline{K}_j^{\text{MW}}) := H^i(X, \underline{K}_j^{\text{MW}}\{\mathcal{O}_X\})$.

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Classical algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the following long exact sequence (where ∂ is the boundary map):

$$\dots \longrightarrow H^n(Z) \xrightarrow{i_*} H^{n+d_X-d_Z}(X) \xrightarrow{j^*} H^{n+d_X-d_Z}(U) \xrightarrow{\partial} H^{n+1}(Z) \longrightarrow \dots$$

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Motivic algebraic topology

Let (Z, i, X, j, U) be a boundary triple. We have the localization long exact sequence (where ∂ is the boundary map):

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Classical algebraic topology

Let $n \geq 2$ and $i \geq 0$ be integers. The singular cohomology group

$$H^i(\mathbb{S}^{n-1}) \text{ is isomorphic to } \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = n - 1. \\ 0 & \text{otherwise} \end{cases}$$

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In particular, $H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \simeq K_{-2}^{\text{MW}}(F) \simeq W(F)$. We can fix such an isomorphism, but it is not canonical.

The linking number and the quadratic linking degree

- Let $L = K_1 \sqcup K_2$ be an oriented link (in knot theory).
- Let \mathcal{L} be an oriented link with two components (in motivic knot theory), i.e. a couple of closed immersions $\varphi_i : \mathbb{A}_F^2 \setminus \{0\} \rightarrow \mathbb{A}_F^4 \setminus \{0\}$ with disjoint images Z_i and orientation classes \bar{o}_i (with $i \in \{1, 2\}$).
- We denote $Z := Z_1 \sqcup Z_2$ and $\nu_Z := \det(\mathcal{N}_{Z/\mathbb{A}_F^4 \setminus \{0\}})$.

Step 1: oriented fundamental classes and Seifert classes

Let $i \in \{1, 2\}$.

Knot theory

The class S_i in $H^1(\mathbb{S}^3 \setminus L)$ of Seifert surfaces of the oriented knot K_i is the unique class that is sent by the boundary map to the (oriented) fundamental class of K_i in $H^0(K_i) \subset H^0(L)$.

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Motivic knot theory

We define the oriented fundamental class $[o_i]$ as the unique class in $H^0(Z_i, \underline{K}_{-1}^{\text{MW}}\{\nu_{Z_i}\})$ that is sent by \tilde{o}_i to the class of η in $H^0(Z_i, \underline{K}_{-1}^{\text{MW}})$, then we define the Seifert class \mathcal{S}_i as the unique class in $H^1(X \setminus Z, \underline{K}_1^{\text{MW}})$ that is sent by the boundary map ∂ to the oriented fundamental class $[o_i] \in H^0(Z, \underline{K}_{-1}^{\text{MW}}\{\nu_Z\})$.

Step 2: the quadratic linking class

Knot theory

The linking class of L is the image of the cup-product $S_1 \cup S_2 \in H^2(\mathbb{S}^3 \setminus L)$ by the boundary map $\partial : H^2(\mathbb{S}^3 \setminus L) \rightarrow H^1(L)$.

Step 2: the quadratic linking class

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Motivic knot theory

We define the quadratic linking class of \mathcal{L} as the image of the intersection product $\mathcal{S}_1 \cdot \mathcal{S}_2 \in H^2(X \setminus Z, \underline{K}_2^{\text{MW}})$ by the boundary map $\partial : H^2(X \setminus Z, \underline{K}_2^{\text{MW}}) \rightarrow H^1(Z, \underline{K}_0^{\text{MW}}\{\nu_Z\})$.

Step 3: the quadratic linking degree

Knot theory

The linking number of $L = K_1 \sqcup K_2$ is the integer $n \in \mathbb{Z}$ such that the linking class in $H^1(L) = \mathbb{Z}[\omega_{K_1}] \oplus \mathbb{Z}[\omega_{K_2}]$ is equal to $(n[\omega_{K_1}], -n[\omega_{K_2}])$ (where ω_{K_i} is the volume form of the oriented knot K_i).

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Motivic knot theory

We define the quadratic linking degree of \mathcal{L} as the image of the quadratic linking class of \mathcal{L} by the isomorphism

$$H^1(Z, \underline{K}_0^{\text{MW}} \{ \nu_Z \}) \rightarrow H^1(Z, \underline{K}_0^{\text{MW}}) \rightarrow H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \oplus H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \rightarrow W(F) \oplus W(F).$$

We fixed an isomorphism $H^1(\mathbb{A}_F^2 \setminus \{0\}, \underline{K}_0^{\text{MW}}) \rightarrow K_{-2}^{\text{MW}}(F)$ once and for all and there is a canonical isomorphism $K_{-2}^{\text{MW}}(F) \rightarrow W(F)$.

The Hopf link

Recall that we fixed coordinates x, y, z, t for \mathbb{A}_F^4 and u, v for \mathbb{A}_F^2 .

- The image of the Hopf link:

$$\{x = 0, y = 0\} \sqcup \{z = 0, t = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

- The parametrization of the Hopf link:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation of the Hopf link:

$$\sigma_1 : \bar{x}^* \wedge \bar{y}^* \mapsto 1, \sigma_2 : \bar{z}^* \wedge \bar{t}^* \mapsto 1$$

The quadratic linking degree of the Hopf link

Or. fund. classes	$\eta \otimes (\bar{x}^* \wedge \bar{y}^*)$		$\eta \otimes (\bar{z}^* \wedge \bar{t}^*)$
Seifert classes	$\langle x \rangle \otimes \bar{y}^*$		$\langle z \rangle \otimes \bar{t}^*$
Apply int. prod.	$\langle xz \rangle \otimes (\bar{t}^* \wedge \bar{y}^*)$		
Quad. link. class	$-\langle z \rangle \eta \otimes (\bar{t}^* \wedge \bar{x}^* \wedge \bar{y}^*)$	\oplus	$\langle x \rangle \eta \otimes (\bar{y}^* \wedge \bar{z}^* \wedge \bar{t}^*)$
Apply $\tilde{o}_1 \oplus \tilde{o}_2$	$-\langle z \rangle \eta \otimes \bar{t}^*$	\oplus	$\langle x \rangle \eta \otimes \bar{y}^*$
Apply $\varphi_1^* \oplus \varphi_2^*$	$-\langle u \rangle \eta \otimes \bar{v}^*$	\oplus	$\langle u \rangle \eta \otimes \bar{v}^*$
Apply $\partial \oplus \partial$	$-\eta^2 \otimes (\bar{u}^* \wedge \bar{v}^*)$	\oplus	$\eta^2 \otimes (\bar{u}^* \wedge \bar{v}^*)$
Quad. link. degree	-1	\oplus	1

A variant of the Hopf link

- The image is the same as the Hopf link's image:

$$\{x = y, y = 0\} \sqcup \{z = 0, a \times t = 0\} \subset \mathbb{A}_F^4 \setminus \{0\} \text{ with } a \in F^*$$

- The parametrization is the same:

$$\varphi_1 : (x, y, z, t) \leftrightarrow (0, 0, u, v), \varphi_2 : (x, y, z, t) \leftrightarrow (u, v, 0, 0)$$

- The orientation is different:

$$\sigma_1 : \overline{x - y}^* \wedge \overline{y}^* \mapsto 1, \sigma_2 : \overline{z}^* \wedge \overline{at}^* \mapsto 1$$

The quadratic linking degree of a variant of the Hopf link

$$[o_1^{var}] = \eta \otimes \overline{x-y}^* \wedge \overline{y}^* = [o_1^{Hopf}] \quad [o_2^{var}] = \eta \otimes \overline{z}^* \wedge \overline{at}^* = \langle a \rangle [o_2^{Hopf}]$$

$$\text{since } \begin{pmatrix} x-y \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{since } \begin{pmatrix} z \\ at \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} z \\ t \end{pmatrix}$$

$$\mathcal{S}_1^{var} = \mathcal{S}_1^{Hopf}$$

$$\mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_2^{Hopf}$$

$$\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var} = \langle a \rangle \mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf}$$

$$\partial(\mathcal{S}_1^{var} \cdot \mathcal{S}_2^{var}) = \langle a \rangle \partial(\mathcal{S}_1^{Hopf} \cdot \mathcal{S}_2^{Hopf})$$

The quadratic linking degree of the variant is $(-\langle a \rangle, 1)$.

Another Hopf link

From now on, F is a perfect field of characteristic different from 2. Recall that we fixed coordinates x, y, z, t for \mathbb{A}_F^4 and u, v for \mathbb{A}_F^2 .

- The image is different from the Hopf link we saw before:

$$\{z = x, t = y\} \sqcup \{z = -x, t = -y\} \subset \mathbb{A}_F^4 \setminus \{0\}$$

But the change of coordinates $x' = z - x$, $y' = t - y$, $z' = z + x$, $t' = t + y$ would give $\{x' = 0, y' = 0\} \sqcup \{z' = 0, t' = 0\} \subset \mathbb{A}_F^4 \setminus \{0\}$.

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$$\mathfrak{o}_1 : \overline{z - x}^* \wedge \overline{t - y}^* \mapsto 1, \mathfrak{o}_2 : \overline{z + x}^* \wedge \overline{t + y}^* \mapsto 1$$

- This Hopf link is an analogue of the Hopf link in knot theory! In knot theory, the Hopf link is given by $\{z = x, t = y\} \sqcup \{z = -x, t = -y\}$ in $\mathbb{S}_\varepsilon^3 = \{(x, y, z, t) \in \mathbb{R}^4, x^2 + y^2 + z^2 + t^2 = \varepsilon^2\}$ for ε small enough and has linking number 1 (i.e. linking class $(1, -1)$).

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- Had we used the change of coordinates above and our first Hopf link to define the parametrizations and the orientations of this Hopf link, we would have had the same quadratic linking degree as for our first Hopf link (i.e. $(-1, 1) \in W(F) \oplus W(F)$).

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- If we change its orientations and its parametrizations then we get $(\langle a \rangle, \langle b \rangle) \in W(F) \oplus W(F)$ with $a, b \in F^*$.

The Solomon link

- In knot theory, the Solomon link is given by $\{z = x^2 - y^2, t = 2xy\} \sqcup \{z = -x^2 + y^2, t = -2xy\}$ in \mathbb{S}_ε^3 for ε small enough and has linking number 2.

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- The orientation is the following:

$$o_1 : \overline{z - x^2 + y^2}^* \wedge \overline{t - 2xy}^* \mapsto 1, o_2 : \overline{z + x^2 - y^2}^* \wedge \overline{t + 2xy}^* \mapsto 1$$

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- We want a means of saying that $(\langle a \rangle + \langle a \rangle, \langle b \rangle + \langle b \rangle)$ is “fundamentally different” from $(\langle c \rangle, \langle d \rangle)$ for all $a, b, c, d \in F^*$ (the Solomon link is “more” different from the Hopf link than the variants of the Hopf link are different from the Hopf link).

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- More generally, we want to compute quantities from the quadratic linking degree which are invariant by changes of orientations and changes of parametrizations of the oriented link.

Proposition

Let \mathcal{L} be an oriented link with two components of quadratic linking degree $(d_1, d_2) \in W(F) \oplus W(F)$. If \mathcal{L}' is obtained from \mathcal{L} by changing orientations and parametrisations (isomorphisms with $\mathbb{A}_F^2 \setminus \{0\}$) then the quadratic linking degree of \mathcal{L}' is equal to $(\langle a \rangle d_1, \langle b \rangle d_2)$ for some $a, b \in F^*$.

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Case $F = \mathbb{R}$

If $F = \mathbb{R}$, the absolute value of an element of $W(\mathbb{R}) \simeq \mathbb{Z}$ is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$, thus $(|d_1|, |d_2|)$ is invariant.

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General case

The rank modulo 2 is invariant by multiplication by $\langle a \rangle$ for all $a \in F^*$.

$$\bullet \Sigma_2 : \begin{cases} W(F) & \rightarrow W(F)/(1) \\ \sum_{i=1}^n \langle a_i \rangle & \mapsto \sum_{1 \leq i < j \leq n} \langle a_i a_j \rangle \end{cases} \text{ (if } n < 2, \text{ it sends } \sum_{i=1}^n \langle a_i \rangle \text{ to } 0)$$

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- Etc. for Σ_{2m} with $m \in \mathbb{N}$

Everything new I presented up until now can be found in my preprint “The quadratic linking degree”:

- HAL: Clémentine Lemarié--Rieusset. THE QUADRATIC LINKING DEGREE. 2022. ⟨hal-03821736⟩
- arXiv: Clémentine Lemarié--Rieusset. The quadratic linking degree. arXiv:2210.11048 [math.AG]

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- $H^{c-1}(X, \underline{K}_{j_1+c}^{\text{MW}}) = 0$, $H^{c-1}(X, \underline{K}_{j_2+c}^{\text{MW}}) = 0$, $H^c(X, \underline{K}_{j_1+c}^{\text{MW}}) = 0$ and $H^c(X, \underline{K}_{j_2+c}^{\text{MW}}) = 0$ for some $j_1, j_2 \leq 0$.

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- One family of examples is: $\mathbb{A}_F^{n+1} \setminus \{0\} \sqcup \mathbb{A}_F^{n+1} \setminus \{0\} \subset \mathbb{A}_F^{2n+2} \setminus \{0\}$ with $n \geq 1$ and $j_1, j_2 \leq 0$.

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- Another family of examples is: $\mathbb{P}_F^n \sqcup \mathbb{P}_F^n \subset \mathbb{P}_F^{2n+1}$ with $n \geq 1$ odd and $j_1, j_2 \leq -2$.