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Malliavin calculus

and applications in stochastic differential equations.

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Chapter 1

Introduction

This thesis was carried out at the University of Le Mans under the supervision of Laurent Denis and Alexandre Popier. During these three years I realized four works with my two thesis directors about the backward stochastic differential equations and the Malliavin calculus. In order of the realization it was:

- Continuity problem for BSDE and IPDE with singular terminal condition [20]. Published (2024) in Journal of Mathematical Analysis and Applications.
- Growth condition on the generator of BSDE with singular terminal value ensuring continuity up to terminal time [21]. Published (2025) in Stochastic Processes and their Applications.
- Malliavin derivative and sensitivity for optimal liquidation. Preprint (2025), submitted for publication in Stochastic Processes and their Applications.
- Malliavin calculus with respect to a Hawkes process and applications. Forthcoming paper.

1.1 Backward stochastic differential equations

In this first part of the introduction we introduce the notions of backward stochastic differential equations (BSDE in short) and the related results which are useful for the next sections.

Backward stochastic differential equations were introduced in the early 1990s, notably by Jean-Michel Bismut in [15] in the linear case and by Etienne Pardoux and Shige Peng in [70] in the general case, as a tool for solving problems in stochastic control and optimization and as an extension of the Feynman-Kac in non-linear case. Unlike traditional forward stochastic differential equations, BSDEs evolve backward in time, with their solution dependent on a terminal condition. This unique feature makes BSDEs particularly useful in fields like financial mathematics, risk management, and optimal stopping. Over the years, the theory of BSDEs has been developed and these equations have become essential in various areas of mathematical finance, stochastic processes, and optimal control.

1.1.1 BSDE driven by a Brownian motion

We consider a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ the filtration generated by a d-Brownian motion W with $d \in \mathbb{N}^*$.

Let fix a terminal time $T \in \mathbb{R}_+$ and we consider a terminal condition $\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)$ with $k \in \mathbb{N}^*$. According to the martingale representation theorem, we have a decomposition of the closed martingale

$$Y_t := \mathbb{E}[\xi \mid \mathcal{F}_t], \quad 0 \leq t \leq T.$$

Theorem 1.1.1. *Let X a $\mathbb{R}^k$-random variable. If $X \in L^2(\mathcal{F}_T, \mathbb{R}^k)$ then there exists a unique $\mathbb{R}^{k\times d}$-valued predictable process C such that $\mathbb{E}\left[\left[\int_0^T |C_s|^2 ds\right]^{\frac{p}{2}}\right] < +\infty$ and*

$$X = \mathbb{E}[X] + \int_0^T C_t dW_t.$$

Consequently

$$\mathbb{E}[X \mid \mathcal{F}_t] = \mathbb{E}[X] + \int_0^t C_s dW_s, \quad 0 \leq t \leq T.$$

Corollary 1.1.2. *There exists a unique $\mathbb{R}^{k\times d}$-valued predictable process $(Z_t)_{0\leq t\leq T}$ which is $(\mathcal{F}_t)_{0\leq t\leq T}$-predictable such that*

$$Y_t = \mathbb{E}[\xi \mid \mathcal{F}_t] = \xi - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

what we write

$$\begin{cases} dY_t &= Z_t dW_t, \quad 0 \leq t \leq T \\ Y_T &= \xi. \end{cases}$$

Remark 1.1.3. *We note $\mathbb{R}^{k\times d}$ the algebra of the matrices with k rows and d colums. Thus $Z_s dW_s$ makes sense as the matrix product between a matrix and a column vector.*

This is the simplest example of backward stochastic differential equation (BSDE) with terminal condition $Y_T = \xi$. Thus we arrive at the definition of a BSDE and associated solutions with their spaces.

Definition 1.1.4. *A BSDE is an equation*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{1.1}$$

or equivalently,

$$\begin{cases} dY_t &= -f(t, Y_t, Z_t)dt + Z_t dW_t, \quad 0 \leq t \leq T \\ Y_T &= \xi \end{cases},$$

where (Y, Z) is the unknown $\mathbb{R}^d \times \mathbb{R}^{k \times d}$-valued process, $f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longrightarrow \mathbb{R}^k$ the driver (or generator) and ξ the terminal condition are the parameters.

Definition 1.1.5. *We define the following spaces for $p > 1$.*

- *$H^p((0, T), \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued progressively measurable processes U such that*

$$\mathbb{E}\left[\left[\int_0^T |U_s|^2 ds\right]^{\frac{p}{2}}\right] < +\infty$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

- *$S^p((0, T), \mathbb{R}^d)$ is the set of $\mathbb{R}^d$-valued progressively measurable processes U with continuous paths such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |U_s|^p\right] < +\infty.$$

And also:

$$H^\infty((0, T), \mathbb{R}^d) = \bigcap_{p>1} H^p((0, T), \mathbb{R}^d), \quad S^\infty((0, T), \mathbb{R}^d) = \bigcap_{p>1} S^p((0, T), \mathbb{R}^d).$$

Definition 1.1.6. *We say that a couple of processes (Y, Z) is solution of the BSDE (1.1) if:*

- *Y is a continuous $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted $\mathbb{R}^k$-valued process.*
- *Z is a $(\mathcal{F}_t)_{0\leq t\leq T}$-predictable $\mathbb{R}^{k\times d}$-valued process with $\int_0^T |Z_t|^2 dt < +\infty$ $\mathbb{P}$-a.s.*
- *The couple (Y, Z) satisfies the equality (1.1) for any $t \in [0, T]$.*

Remark 1.1.7. *The $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted property is important. Indeed if we consider the* *null function $f = 0$ then the BSDE*

$$\begin{cases} dY_t &= 0, \quad 0 \le t \le T \\ \\ Y_T &= \xi \end{cases}$$

admits $Y = \xi$ as candidate solution but this process is not a $(\mathcal{F}_t)_{0\le t\le T}$-adapted solution if $\xi \notin \mathcal{F}_0$.

The classical assumptions we can do about the parameters (ξ, f) are integrability of ξ and Lipschitz property of f.

Assumption 1.

1. *$\xi \in L^2(\mathcal{F}_T, \mathbb{R}^k)$: ξ is $\mathcal{F}_T$-measurable and $\mathbb{E}[|\xi|^2] < +\infty$.*
2. *For any $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k\times d}$, the process $f(\cdot, y, z)$ is $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable.*
3. *The driver f is uniformly Lipschitz continuous in (y, z) with a Lipschitz constant $K \in \mathbb{R}_+^*$:*
$$\forall \omega \in \Omega, t \in [0,T], y, y' \in \mathbb{R}^k, z, z' \in \mathbb{R}^{k\times d},$$
$$|f(\omega, t, y, z) - f(\omega, t, y', z')| \le K(|y - y'| + |z - z'|).$$
4. *$f^0 := f(\cdot, 0, 0) \in H^2((0,T), \mathbb{R}^k)$: f^0 is $\mathbb{R}^k$-valued progressively measurable and*
$$\mathbb{E}\left[\int_0^T |f_t^0|^2 dt\right] = \mathbb{E}\left[\int_0^T |f(t,0,0)|^2 dt\right] < +\infty.$$

Before obtaining a general result of existence and uniqueness of a solution, we can obtain an explicit formula for the process Y if the driver f does not depend on (y, z) or if the driver f is linear with respect to (y, z) (see Proposition 4.1.1 and Proposition 4.1.2 in [87]).

Proposition 1.1.8. *Under Assumption 1 and if the driver f does not depend on (y, z): $f = f^0$, then the following BSDE*

$$Y_t = \xi + \int_t^T f_s ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T,$$

admits a unique solution $(Y, Z) \in H^2((0,T), \mathbb{R}^k) \times H^2((0,T), \mathbb{R}^{k\times d})$. Moreover we have:

- *$Y \in S^2((0,T),\mathbb{R}^k)$:*

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^2\right] < +\infty.$$

- *For any $t\in[0,T]$,*

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f_s ds \middle| \mathcal{F}_t\right].$$

Proposition 1.1.9. *We consider the following BSDE in dimension $k=1$*

$$Y_t = \xi + \int_t^T (\alpha_s Y_s + Z_s\beta_s + f_s^0)ds - \int_t^T Z_s dW_s,$$

with the unknown (Y,Z) with values in $\mathbb{R}\times\mathbb{R}^d$ and the parameters $\alpha : \Omega\times[0,T] \longrightarrow \mathbb{R}$, and $\beta : \Omega\times[0,T] \longrightarrow \mathbb{R}^d$ where $\beta_s Z_s$ is the product between a line vector and a column vector. Under Assumption 1 and if the processes α and β are $(\mathcal{F}_t)_{0\le t\le T}$-progressively measurable and bounded then this linear BSDE admits a unique solution $(Y,Z) \in S^2((0,T),\mathbb{R}) \times H^2((0,T),\mathbb{R}^d)$ where the process Y is explicitly given by

$$Y_t = \Gamma_t^{-1}\mathbb{E}\left[\Gamma_T\xi + \int_t^T \Gamma_s f_s^0 ds \middle| \mathcal{F}_t\right], \quad 0\le t\le T,$$

where Γ is the adjoint process of this linear BSDE, that is the unique solution of the SDE

$$\left\{\begin{array}{rcl} d\Gamma_t &=& \Gamma_t(\alpha_t dt + \beta_t dW_t), \quad 0\le t\le T \\ \Gamma_0 &=& 1 \end{array}\right. ,$$

or equivalently,

$$\Gamma_t = \exp\left(\int_0^t \beta_s dW_s + \int_0^t \left(\alpha_s - \frac{1}{2}|\beta_s|^2\right) ds\right), \quad 0\le t\le T,$$

where $\beta_t dW_t$ is the product between between a line vector and a column vector.

Now we can study the general BSDE (1.1) to obtain the existence and the uniqueness of the solution. Firstly we have an a priori estimate of a solution (Y,Z) for the appropriated norm (see [87, Theorem 4.2.1]).

Proposition 1.1.10. *Under Assumption 1 and if (Y,Z) is a solution of the BSDE* (1.1) *then* $Y \in S^2((0,T),\mathbb{R}^k)$ *and there exists a constant* $C = C(T,K,k,d) \in \mathbb{R}_+^*$ *such that*

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^2 + \int_0^T |Z_t|^2 dt\right] \le C\mathbb{E}\left[|\xi|^2 + \int_0^T |f_t^0|^2 dt\right].$$

Thanks to this result, we can obtain a control of the difference between two solutions of the BSDE (see [87, Theorem 4.2.3]).

Corollary 1.1.11. *If we have two sets of parameters* (ξ^1, f^1) *and* (ξ^2, f^2) *which satisfy Assumption 1, and* $(Y^1,Z^1),(Y^2,Z^2)$ *solutions of the BSDE* (1.1) *with the parameters* $(\xi^1,f^1),(\xi^2,f^2)$*, then*

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|Y_t^1 - Y_t^2|^2 + \int_0^T |Z_s^1 - Z_s^2|^2 ds\right]\\ &\le C\mathbb{E}\left[|\xi^1 - \xi^2|^2 + \int_0^T |f^1(t,Y_t^1,Z_t^1) - f^2(t,Y_t^1,Z_t^1)|^2 dt\right].\end{aligned}$$

Finally we get the desired theorem which is proven at the end of this section (see [87, Theorem 4.3.1]).

Theorem 1.1.12. *Under Assumption 1, the BSDE* (1.1) *admits a unique solution* (Y,Z) *in* $S^2((0,T),\mathbb{R}^k)\times H^2((0,T),\mathbb{R}^{k\times d})$ *in the sense of Definition 1.1.6. The uniqueness means that if* $(Y^1,Z^1),(Y^2,Z^2)$ *are two solutions in* $S^2((0,T),\mathbb{R}^k)\times H^2((0,T),\mathbb{R}^{k\times d})$ *then*

$$Y_t^1 = Y_t^2, \quad \forall t\in[0,T], \quad \mathbb{P}-a.s.,$$

$$Z_t^1 = Z_t^2, \quad dt\times d\mathbb{P}-a.s..$$

Moreover we can get a solution (Y,Z) in a space $L^p, p>1$ if the parameters (ξ,f) are also in a space L^p (see [19]).

Proposition 1.1.13. *Under Assumption 1 and if* $\xi\in L^p(\mathcal{F}_T,\mathbb{R}^k)$ *and* $f^0\in H^p((0,T),\mathbb{R}^k)$ *for some* $p\ge 2$*, then the unique solution* (Y,Z) *satisfies* $(Y,Z)\in S^p((0,T),\mathbb{R}^k)\times H^p((0,T),\mathbb{R}^{k\times d})$ *and we have the control*

$$\mathbb{E}\left[\sup_{0\le t\le T}|Y_t|^p + \left(\int_0^T |Z_t|^2 dt\right)^{\frac{p}{2}}\right] \le C\mathbb{E}\left[|\xi|^p + \left(\int_0^T |f_t^0| dt\right)^p\right].$$

Remark 1.1.14. *We also have the result for* $p\in(1,2)$ *but the proof is not the same. See Theorem 1.1.17.*

We can weaken the assumptions about the driver f to get a unique solution of the BSDE: this is enough to assume f monotone with respect to y (see [19]).

Assumption 2.

1. *The driver* $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}$ *is progressively measurable in all variables.*

2. *The driver* f *is continuous and monotone with respect to* y*: there exists* $\chi \in \mathbb{R}$ *such that*

$$\forall t \in [0,T], y, y' \in \mathbb{R}^k, z \in \mathbb{R}^{k\times d}, \quad \langle f(t,y,z) - f(t,y',z), y - y' \rangle \leq \chi |y-y'|^2.$$

3. *For any* $r \in \mathbb{R}_+^*$,

$$\sup_{|y|\leq r} |f(t,y,0) - f(t,0,0)| \in L^1(\Omega \times (0,T), \mathbb{R}).$$

4. *The driver* f *is* K*-Lipschitz in* z*:*

$$\forall t \in [0,T], y \in \mathbb{R}^k, z, z' \in \mathbb{R}^{k\times d}, \quad |f(t,y,z) - f(t,y,z')| \leq K|z-z'|.$$

5. $\xi \in L^p(\mathcal{F}_T, \mathbb{R}^k)$ *for* $p > 1$ *:* ξ *is* $\mathcal{F}_T$*-measurable and* $\mathbb{E}[|\xi|^p] < +\infty$.

6. $f^0 = f(\cdot, 0, 0) \in H^p((0,T), \mathbb{R}^k)$*:* f^0 *is* $\mathbb{R}^k$*-valued, progressively measurable and*

$$\mathbb{E}\left[\left(\int_0^T |f_t^0|^2 dt\right)^{\frac{p}{2}}\right] < +\infty.$$

The following lemma will be useful in the next parts. Indeed we will simplify our reasoning with $\chi = 0$.

Lemma 1.1.15. *Without loss of generality, we can assume* $\chi = 0$ *in the previous assumption.*

Proof. We consider

$$(\overline{Y}, \overline{Z}) = (e^{\chi(T-t)} Y_t, e^{\chi(T-t)} Z_t)_{0\leq t\leq T}.$$

Thus, by Itô formula,

$$\begin{aligned} e^{\chi(T-t)}Y_t &= \xi + \int_t^T (f(s,Y_s,Z_s)e^{\chi(T-s)} - \chi Y_s e^{\chi(T-s)})ds - \int_t^T e^{\chi(T-s)}Z_s dW_s \\ &= \xi + \int_t^T (f(s,e^{-\chi(T-s)}\overline{Y}_s, e^{-\chi(T-s)}\overline{Z}_s)e^{\chi(T-s)} - \chi\overline{Y}_s)ds - \int_t^T \overline{Z}_s dW_s \\ &= \xi + \int_t^T \overline{f}(s,\overline{Y}_s,\overline{Z}_s)ds - \int_t^T \overline{Z}_s dW_s, \quad 0 \le t \le T, \end{aligned}$$

where

$$\overline{f}(t,\overline{y},\overline{z}) = f(t,e^{-\chi(T-t)}\overline{y}, e^{-\chi(T-t)}\overline{z})e^{\chi(T-t)} - \chi\overline{y}, \quad t\in[0,T], \overline{y}\in\mathbb{R}^k, \overline{z}\in\mathbb{R}^{k\times d},$$

who satisfies the other assertions of the previous assumptions and, for any $t\in[0,T]$, $\overline{y},\overline{y}'\in\mathbb{R}^k$ and $\overline{z},\overline{z}'\in\mathbb{R}^{k\times d}$,

$$\begin{aligned} &\langle \overline{f}(t,\overline{y},\overline{z}) - \overline{f}(t,\overline{y}',\overline{z}'), \overline{y}-\overline{y}'\rangle \\ &= \langle f(t,e^{-\chi(T-t)}\overline{y}, e^{-\chi(T-t)}\overline{z})e^{-\chi(T-t)} - f(t,e^{-\chi(T-t)}\overline{y}', e^{-\chi(T-t)}\overline{z}')e^{-\chi(T-t)}, \overline{y}-\overline{y}'\rangle - \\ &\quad \chi\langle \overline{y}-\overline{y}', \overline{y}-\overline{y}'\rangle \\ &= \langle f(t,e^{-\chi(T-t)}\overline{y}, e^{-\chi(T-t)}\overline{z}) - f(t,e^{-\chi(T-t)}\overline{y}', e^{-\chi(T-t)}\overline{z}'), \overline{y}e^{-\chi(T-t)} - \overline{y}'e^{-\chi(T-t)}\rangle \\ &\quad -\chi|\overline{y}-\overline{y}'|^2 \\ &\le \chi|\overline{y}-\overline{y}'|^2 - \chi|\overline{y}-\overline{y}'|^2 \\ &= 0. \end{aligned}$$

Remark that we also have if the driver $\overline{f}$ is 0-monotone and satisfies the other assertions of the previous assumption then the driver f is χ-monotone and satisfies the same other assertions. □

Remark 1.1.16. *For $k=1$, f 0-monotone with respect to y is equivalent to f nonincreasing with respect to y.*

We get with these assumptions the following theorem which is proved in [19].

Theorem 1.1.17. *Under Assumption 2, there exists a unique solution to the BSDE* (1.1)

$$(Y,Z)\in S^p((0,T),\mathbb{R}^k)\times H^p((0,T),\mathbb{R}^{k\times d}).$$

Example 1.1.18. *In liquidation problems the following BSDE is present:*

$$Y_t = \xi - \int_t^T Y_s|Y_s|^{q-1}ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

Thus the driver is given by

$$f(y) = -y|y|^{q-1}, \quad y \in \mathbb{R},$$

which is not Lipschitz with respect to y but is 0-monotone.

A useful result when we work with solutions of BSDE is the comparison theorem (see [87, Theorem 4.4.1] and [19]).

Theorem 1.1.19. *We consider two BSDEs in dimension $k = 1$*

$$Y_t^1 = \xi^1 + \int_t^T f^1(s, Y_s^1, Z_s^1)ds - \int_t^T Z_s^1 dW_s, \quad 0 \leq t \leq T,$$

$$Y_t^2 = \xi^2 + \int_t^T f^2(s, Y_s^2, Z_s^2)ds - \int_t^T Z_s^2 dW_s, \quad 0 \leq t \leq T,$$

with $(\xi^1, f^1), (\xi^2, f^2)$ satisfying Assumption 1 or 2 and

$$\xi^1 \leq \xi^2 \ \mathbb{P} - a.s., \quad \forall y \in \mathbb{R}, z \in \mathbb{R}^d, f^1(\cdot, y, z) \leq f^2(\cdot, y, z) \ dt \times d\mathbb{P} - a.s.$$

Then

$$Y_t^1 \leq Y_t^2, \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s.$$

Remark 1.1.20. *Instead of*

$$\forall y \in \mathbb{R}, z \in \mathbb{R}^d, \quad f^1(\cdot, y, z) \leq f^2(\cdot, y, z) \ dt \times d\mathbb{P} - a.s.$$

we can suppose the same inequality but evaluated along the process (Y^2, Z^2) or (Y^1, Z^1):

$$f^1(\cdot, Y^2, Z^2) \leq f^2(\cdot, Y^2, Z^2) \quad dt \times d\mathbb{P}\text{-}a.s.$$

or

$$f^1(\cdot, Y^1, Z^1) \leq f^2(\cdot, Y^1, Z^1) \quad dt \times d\mathbb{P}\text{-}a.s.$$

In this case we get the same result.

We conclude this first section by the proof of Theorem 1.1.12 to illustrate classic arguments about the BSDE.

Proof of Theorem 1.1.12. We assume to simplify the reasonning that $d = k = 1$. Let $\alpha \in \mathbb{R}_+^*$ and $\|\cdot\|_\alpha$ be the equivalent norm on $H^2((0,T),\mathbb{R})$ defined by:

$$\|U\|_\alpha = \mathbb{E}\left[\int_0^T e^{\alpha t}|U_t|^2 dt\right], \quad U \in H^2((0,T),\mathbb{R}).$$

Let $(U,V) \in (H^2((0,T),\mathbb{R}))^2$ and the BSDE

$$Y_t = \xi + \int_t^T f(s,U_s,V_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T.$$

Thus this BSDE is like in Proposition 1.1.8: $f(\cdot,U,V) \in H^2((0,T),\mathbb{R})$ because the processes U and V are progressively measurable and

$$\begin{aligned}
\|f(\cdot,U,V)\|_{H^2((0,T),\mathbb{R})} &\le \|f(\cdot,U,V) - f^0\|_{H^2((0,T),\mathbb{R})} + \|f^0\|_{H^2((0,T),\mathbb{R})} \\
&\le K(\|U\|_{H^2((0,T),\mathbb{R})} + \|V\|_{H^2((0,T),\mathbb{R})}) + \|f^0\|_{H^2((0,T),\mathbb{R})} \\
&\le K(\|U\|_\alpha + \|V\|_\alpha + \|f^0\|_{H^2((0,T),\mathbb{R})} \\
&< +\infty.
\end{aligned}$$

Thus there exists a unique solution $(Y,Z) \in (H^2((0,T),\mathbb{R}))^2$. Moreover $Y \in S^2((0,T),\mathbb{R})$ and, for any $t \in [0,T]$,

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s,U_s,V_s)ds \middle| \mathcal{F}_t\right].$$

Let $(U',V') \in (H^2((0,T),\mathbb{R}))^2$ and (Y',Z') similarly defined from (U',V'). We consider

$$\Delta Y = Y - Y', \quad \Delta Z = Z - Z', \quad \Delta U = U - U', \quad \Delta V = V - V'.$$

Then $(\Delta Y, \Delta Z)$ satisfies the BSDE

$$\Delta Y_t = \int_t^T (f(s,U_s,V_s) - f(s,U'_s,V'_s))ds - \int_t^T \Delta Z_s dW_s, \quad 0 \le t \le T.$$

Thus, according to the Itô formula,

$$
\begin{aligned}
0 &= e^{\alpha t}(\Delta Y_t)^2 + \alpha \int_t^T e^{\alpha s}(\Delta Y_s)^2 ds + 2\int_t^T e^{\alpha s}\Delta Y_s d\Delta Y_s + \int_t^T e^{\alpha s} d\langle \Delta Y, \Delta Y\rangle_s \\
&= e^{\alpha t}(\Delta Y_t)^2 + \alpha \int_t^T e^{\alpha s}(\Delta Y_s)^2 ds - 2\int_t^T e^{\alpha s}\Delta Y_s (f(s,U_s,V_s) - f(s,U'_s,V'_s))ds \\
&\quad +2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s + \int_t^T e^{\alpha s}(\Delta Z_s)^2 ds
\end{aligned}
$$

i.e.

$$
\begin{aligned}
e^{\alpha t}(\Delta Y_t)^2 + \int_t^T e^{\alpha s}(\Delta Z_s)^2 ds &= \int_t^T e^{\alpha s}(2\Delta Y_s(f(s,U_s,V_s) - f(s,U'_s,V'_s)) - \alpha(\Delta Y_s)^2)ds \\
&\quad -2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s
\end{aligned}
$$

with, as the function f is Lipschitz continuous,

$$
\begin{aligned}
2\Delta Y_s(f(s,U_s,V_s) - f(s,U'_s,V'_s)) &\leq 2K|\Delta Y_s|(|\Delta U_s| + |\Delta V_s|) \\
&\leq \alpha|\Delta Y_s|^2 + \frac{2K}{\alpha}(|\Delta U_s| + |\Delta V_s|)^2.
\end{aligned}
$$

Thus

$$
e^{\alpha t}(\Delta Y_t)^2 + \int_t^T e^{\alpha s}(\Delta Z_s)^2 ds \leq \frac{2K}{\alpha}\int_t^T e^{\alpha s}(|\Delta U_s| + |\Delta V_s|)^2 ds - 2\int_t^T e^{\alpha s}\Delta Y_s \Delta Z_s dW_s. \quad (\star)
$$

Moreover $\int_0 e^{\alpha s}\Delta Y_s \Delta Z_s dW_s$ is a true martingale because, with the Burkholder-Davis-Gundy inequality, for any $t \in [0,T]$,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t e^{\alpha s}\Delta Y_s \Delta Z_s dW_s\right|\right] &\leq c_1 \mathbb{E}\left[\left\langle \int_0 e^{\alpha s}\Delta Y_s Z_s dW_s, \int_0 e^{\alpha s}\Delta Y_s Z_s dW_s \right\rangle_T^{\frac{1}{2}}\right] \\
&\leq c_1 \mathbb{E}\left[\left(\int_0^T e^{2\alpha s}(\Delta Y_s)^2 (\Delta Z_n)^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq c_1 e^{\alpha T}\mathbb{E}\left[\sup_{0\leq t\leq T}|\Delta Y_s|\left(\int_0^T (\Delta Z_s)^2 ds\right)^{\frac{1}{2}}\right] \\
&\leq 2c_1 e^{\alpha T}\left(\mathbb{E}\left[\sup_{0\leq t\leq T}(\Delta Y_s)^2\right] + \mathbb{E}\left[\int_0^T (\Delta Z_s)^2 ds\right]\right) \\
&< +\infty,
\end{aligned}
$$

Thus, by applying the expectation in ($\star$),

$$\begin{aligned}
&\mathbb{E}[e^{\alpha t}(\Delta Y_t)^2] + \mathbb{E}\left[\int_t^T e^{\alpha s}(\Delta Z_s)^2 ds\right] \\
&\leq \frac{2K}{\alpha}\mathbb{E}\left[\int_t^T e^{\alpha s}(|\Delta U_s| + |\Delta V_s|)^2 ds\right] \\
&\leq \frac{4K}{\alpha}\left(\mathbb{E}\left[\int_t^T e^{\alpha s}(\Delta U_s)^2 ds\right] + \mathbb{E}\left[\int_t^T e^{\alpha s}(\Delta V_s)^2 ds\right]\right).
\end{aligned}$$

In particular we directly get

$$\|\Delta Z\|_\alpha^2 \leq \frac{4K}{\alpha}(\|\Delta U\|_\alpha^2 + \|\Delta V\|_\alpha^2)$$

and

$$\sup_{0\leq t\leq T} \mathbb{E}[e^{\alpha t}(\Delta Y_t)^2] \leq \frac{4K}{\alpha}(\|\Delta U\|_\alpha^2 + \|\Delta V\|_\alpha^2).$$

Thus, according to Fubini-Tonelli theorem,

$$\begin{aligned}
\|\Delta Y\|_\alpha^2 &= \mathbb{E}\left[\int_0^T e^{\alpha s}(\Delta Y_s)^2 ds\right] \\
&= \int_0^T \mathbb{E}[e^{\alpha s}(\Delta Y_s)^2] ds \\
&\leq T \sup_{0\leq t\leq T} \mathbb{E}[e^{\alpha t}(\Delta Y_t)^2] \\
&\leq \frac{4KT}{\alpha}(\|\Delta U\|_\alpha^2 + \|\Delta V\|_\alpha^2).
\end{aligned}$$

Therefore

$$\|\Delta Y\|_\alpha^2 + \|\Delta Z\|_\alpha^2 \leq \frac{4K(1+T)}{\alpha}(\|\Delta U\|_\alpha^2 + \|\Delta V\|_\alpha^2).$$

Thus, for $\alpha > 4K(1+T)$, the map $(U,V) \longmapsto (Y,Z)$ is contracting in $(H^2((0,T),\mathbb{R}))^2$ for the norm $\|\cdot\|_\alpha$ which is a Hilbert space. Therefore this map admits a unique fix point

$$(Y,Z) \in (H^2((0,T),\mathbb{R}))^2 :$$

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T.$$

In other words (Y,Z) is a solution of the BSDE. The uniqueness is directly proven by Corollary 1.1.11. □

1.1.2 BSDE with jumps

We can consider a Poisson random measure π defined on $\Omega \times [0,T] \times \mathbb{R}$ and we assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by the random measure π and by the Brownian motion W. We also note ν the associated compensator and $\tilde{\pi} = \pi - \nu$ the associated compensated measure. We follow the assumptions and the results of [26]. We assume that there exist a nonnegative predictable process η and a kernel Q such that

$$\nu(dt, de) = Q(de)\eta(t)dt = \lambda(t, de)dt \quad \text{and} \quad \int_0^T \int_{\mathbb{R}} e^2 \nu(dt, de) < +\infty.$$

Remark 1.1.21. *In [26], this assumption is supposed to have random variables in L^2 and the chaotic decomposition. The Lévy measures do not satisfy this assumption in general. To get more general results, we can assume $\int_0^T \int_{\mathbb{R}} (e^2 \wedge 1)\nu(dt, de) < +\infty$. Nonetheless we keep this assumption of [26] because it is enough for this thesis.*

Definition 1.1.22. *A BSDE with jumps is an equation*

$$Y_t = \xi + \int_t^T f(s, Y_{s-}, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}} U_s(e)\tilde{\pi}(ds, de), \quad 0 \leq t \leq T, \quad (1.2)$$

where (Y, Z, U) is the unknown $\mathbb{R}^k \times \mathbb{R}^{k\times d} \times (\mathbb{R}^k)^{\mathbb{R}}$-valued process, and $f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \times (\mathbb{R}^k)^{\mathbb{R}} \longrightarrow \mathbb{R}^k$ the driver and ξ the terminal condition are the parameters.

Remark 1.1.23. *The notation Y_{t-} is defined by $Y_{t-} = \lim_{\substack{s\to t \\ s\leq t}} Y_s$. This is important because the process Y is càdlàg.*

Definition 1.1.24. *We define the following spaces for any $p > 1$.*

- *$S_c^p((0,T), \mathbb{R}^k)$ is the set of adapted and càdlàg processes A such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |A_t|^p\right] < +\infty.$$

- *$H_\pi^p((0,T) \times \mathbb{R}, \mathbb{R}^k)$ is the set of predictable processes A such that*

$$\mathbb{E}\left[\left(\int_0^T \int_{\mathbb{R}} |A_t(e)|^2 \nu(dt, de)\right)^{\frac{p}{2}}\right] < +\infty$$

Definition 1.1.25. *We say that a triplet of processes (Y, Z, U) is solution of the BSDE* (1.2) *if:*

- *Y is a continuous $(\mathcal{F}_t)_{0\leq t\leq T}$-adapted $\mathbb{R}^k$-valued process.*
- *Z is a $(\mathcal{F}_t)_{0\leq t\leq T}$-predictable $\mathbb{R}^{k\times d}$-valued process with $\int_0^T |Z_t|^2 dt < +\infty$ $\mathbb{P}$-a.s.*
- *U is a $(\mathcal{F}_t)_{0\leq t\leq T}$-predictable $L^2_Q(\mathbb{R})$-valued process with $\int_0^T \int_{\mathbb{R}} |U_t(e)|^2 \nu(dt,de) < +\infty$ $\mathbb{P}$-a.s.*
- *The triplet (Y,Z,U) satisfies the equality (1.2) for any $t\in[0,T]$.*

As for the BSDE without jumps, we start to study the BSDE with a simple driver thanks to the property of predictable representation.

Theorem 1.1.26. *Let M a $\mathbb{R}^k$-valued square integrable $\mathbb{F}$-martingale. Then there exists a unique $(Z,U)\in H^2((0,T),\mathbb{R}^{k\times d})\times H^2_\pi((0,T)\times\mathbb{R},\mathbb{R}^k)$ such that*

$$M_t = M_0 + \int_0^t Z_s dW_s + \int_0^t \int_{\mathbb{R}} U_s(e)\tilde{\pi}(ds,de), \quad 0\leq t\leq T.$$

In particular for any $\xi\in L^2(\mathcal{F}_T,\mathbb{R}^k)$, there exists a unique $(Z,U)\in H^2((0,T),\mathbb{R}^{k\times d})\times H^2_\pi((0,T)\times\mathbb{R},\mathbb{R}^k)$ such that

$$\xi = \mathbb{E}[\xi] + \int_0^T Z_s dW_s + \int_0^T \int_{\mathbb{R}} U_s(e)\tilde{\pi}(ds,de).$$

We deduce the same result with a driver independent of y,z,u as in the case without jumps (see [26, Proposition 3.1.2]).

Proposition 1.1.27. *If $\xi\in L^2(\mathcal{F}_T,\mathbb{R}^k)$ and $f\in H^2((0,T),\mathbb{R}^k)$ then there exists a unique*

$$(Y,Z,U)\in S^2_c((0,T),\mathbb{R}^k)\times H^2((0,T),\mathbb{R}^{k\times d})\times H^2_\pi((0,T)\times\mathbb{R},\mathbb{R}^k)$$

such that

$$Y_t = \xi + \int_t^T f_s ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}} U_s(e)\tilde{\pi}(ds,de), \quad 0\leq t\leq T. \tag{1.3}$$

The process Y is given by

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f_s ds \middle| \mathcal{F}_t\right],$$

and the processes Z,U by the predictable representation

$$\xi + \int_0^T f_s ds = \mathbb{E}\left[\xi + \int_0^T f_s ds\right] + \int_0^T Z_s dW_s + \int_0^T \int_{\mathbb{R}} U_s(e)\tilde{\pi}(ds,de).$$

As for Theorem 1.1.17, we can consider this assumptions about the parameters to get the following theorem which is proved in [55].

Assumption 3.

1. *The driver f is continuous and monotone with respect to $y \in \mathbb{R}^k$.*
2. *For every $r \in \mathbb{R}_+^*$,*

$$\sup_{|y|\leq r} |f(\cdot, y, 0, 0) - f(\cdot, 0, 0, 0)| \in L^1(\Omega \times (0, T), \mathbb{R}).$$

3. *The driver f is Lipschitz with respect to $z \in \mathbb{R}^{k\times d}$ and $u \in L^2_Q(\mathbb{R}, \mathbb{R}^k)$.*
4. *For some $p > 1$*

$$\mathbb{E}\left[|\xi|^p + \int_0^T |f(t, 0, 0, 0)|^p dt\right] < +\infty.$$

Theorem 1.1.28. *Under Assumption 3, the BSDE* (1.2) *admits a unique solution* (Y, Z, U) *in* $S^p_c((0, T), \mathbb{R}^k) \times H^p((0, T), \mathbb{R}^{k\times d}) \times H^p_\pi((0, T) \times \mathbb{R}, \mathbb{R}^k)$.

If we want to compare solutions of BSDE with jumps, we need more precise properties to get a comparison result than in this case of BSDE without jumps (see [55]).

Theorem 1.1.29. *We assume that $k = 1$, the parameters ξ, f satisfy Assumption 3 and the increments of the driver f with respect to u are controlled: for any $y \in \mathbb{R}, z \in \mathbb{R}^d, u, u' \in L^2_Q(\mathbb{R})$, there exists a predictable process*

$$\delta^{y,z,u,u'} : \Omega \times [0, T] \times \mathbb{R} \longrightarrow]-1, +\infty[$$

such that $t \longmapsto \int_{\mathbb{R}} |\delta^{y,z,u,u'}_t(e)|^2 \lambda(t, de)$ is uniformly bounded in y, z, u, u' and

$$\forall t \in [0, T], \quad f(t, y, z, u) - f(t, y, z, u') \leq \int_{\mathbb{R}} \delta^{y,z,u,u'}_t(e)(u(e) - u'(e))\lambda(t, de).$$

We also consider (ξ', f') satisfying the same assumptions. Then there exist unique solutions (Y, Z, U) and (Y', Z', U') in $S^p_c((0, T), \mathbb{R}) \times H^p((0, T), \mathbb{R}^d) \times H^p_\pi((0, T) \times \mathbb{R}, \mathbb{R})$ to the associated BSDE and if

$$\xi \geq \xi' \quad et \quad f \geq f'$$

then $Y \geq Y'$. Moreover if there exists $t_0 \in [0, T]$ such that $Y_{t_0} = Y'_{t_0}$ then $Y_t = Y'_t$ for any $t \in [t_0, T]$.

Remark 1.1.30. *The comparison principle is more difficult with jumps. Indeed the additional assumptions are important.*

Example 1.1.31. *We consider the linear BSDE*

$$Y_t = \xi - 2\int_t^T U_s ds - \int_t^T Z_s dW_s - \int_t^T U_s d\widetilde{\pi}_s, \quad 0 \le t \le T,$$

with π is generated by a Poisson process with unitary intensity. Then the associated driver satisfies Assumption 3 but for the terminal condition $\xi = \int_0^T d\pi_s$ the unique solution is given by

$$Y_t = \int_0^t d\pi_s - (T-t), \quad Z_t = 0, \quad U_t = 1, \quad 0 \le t \le T,$$

and for the terminal condition $\xi' = 0$ the unique solution is given by

$$Y' = 0, \quad Z' = 0, \quad U' = 0.$$

Thus we have $\xi \ge \xi'$ but not $Y \ge Y'$.

1.1.3 BSDE with singular terminal condition

If we try to reduce the assumption about the terminal condition ξ in dimension $k = 1$, then we can consider a singular terminal condition ξ: $\mathbb{P}(\xi = +\infty) > 0$. In this case Y is an unknown with values in $\mathbb{R}$ and we can assume for example $\xi \ge 0$ and also $f^0 \ge 0$. Furthermore we have to change the notion of solution.

Definition 1.1.32. *We say that a triplet of processes (Y, Z, U) is a solution of the BSDE* (1.2) *for $k = 1$ and a singular terminal condition ξ if there exists $p > 1$ such that*

- *$Y \in S_c^p((0,t), \mathbb{R})$ for any $t \in [0,T)$.*
- *$Z \in H^p((0,t), \mathbb{R}^d)$ for any $t \in [0,T)$.*
- *$U \in H_\pi^p((0,t), \mathbb{R})$ for any $t \in [0,T)$.*
- *For any $t \in [0,T)$ and $s \in [0,t]$,*

$$Y_s = Y_t + \int_s^t f(r, Y_r, Z_r, U_r)dr - \int_s^t Z_r dW_r - \int_s^t U_s(e)\widetilde{\pi}(ds, de).$$

- *We have the $\mathbb{P}$-almost surely convergence $\lim_{t\to T} Y_t = \xi$.*

Furthermore if we only have $\liminf_{t\to T} Y_t \ge \xi$ instead of the convergence $\lim_{t\to T} Y_t = \xi$ then we say that the couple (Y, Z) is a supersolution of the BSDE (1.1).

This notion was studied in [56] in a more general case with not necessarily ξ and f^0 nonnegative. But for the rest of this thesis we only consider the nonnegative case.

Assumption 4.

1. *The driver f satisfies Assumption 3.*
2. *The increments of the driver f with respect to $y \in \mathbb{R}_+$ are controlled:*

$$\forall t \in [0,T], y \in \mathbb{R}_+, z \in \mathbb{R}^d, u \in L^2_Q(\mathbb{R}),\ f(t,y,z,u) \leq -(q_*-1)\frac{y|y|^{q-1}}{\eta_t^{q-1}}+f(t,0,z,u),$$

   where $q > 1$, q_* the Hölder conjugate of q and η a positive process satisfying : there exists $\ell > 1$ such that

$$\mathbb{E}\left[\int_0^T \left(\eta_s^\ell + \frac{1}{\eta_s^{q-1}}\right) ds\right] < +\infty.$$

3. *$f^0 \geq 0$ and satisfies*

$$\mathbb{E}\left[\int_0^T ((T-t)^{q_*} f_t^0)^\ell dt\right] < +\infty.$$

4. *There exists $k > \max\left(\frac{\ell}{\ell-1}, 2\right)$ such that $\nu \in L^k_Q(\mathbb{R})$.*

Theorem 1.1.33. *Under Assumption 4, there exists a minimal supersolution (Y,Z,U) to the BSDE (1.2): minimal means if $(\overline{Y},\overline{Z},\overline{U})$ is a solution of the BSDE (1.2) then for any $t \in [0,T]$, $Y_t \leq \overline{Y}_t$ $\mathbb{P}$-a.s.*

Remark 1.1.34. *We ask ourselves under what conditions we have (Y,Z,U) a true solution of the BSDE (1.2). More precisely if we have $\liminf_{t\to T} Y_t = \lim_{t\to T} Y_t$ (see [76]) or $\liminf_{t\to T} Y_t = \xi$. Section 2 and Section 4 will answer to the second question.*

For example in [74], the driver is given by $f(t,y,z) = -y|y|^q, q > 0$ and the terminal condition by $\xi = g(X_T)$ with X given by a stochastic differential equation (SDE). In this case, under some assumptions about the function g and the parameters of the SDE, we have $\lim_{t\to T} Y_t = \xi$.

1.2 Malliavin calculus

In this second part, the notions of Malliavin calculus will be introduced and the related results which will be used for the next sections.

Malliavin calculus, developed by Paul Malliavin in the 1970s, extends classical differential calculus to stochastic processes, particularly in the context of Brownian motion. Initially aimed at studying the regularity of solutions to stochastic differential equations, it provides a framework for differentiating random variables with respect to the underlying probability space. Over time, Malliavin calculus has become an essential tool in areas such as stochastic control, financial mathematics, and the study of stochastic partial differential equations, enabling deeper analysis of the smoothness and sensitivity of random processes.

1.2.1 Definitions and first properties

We consider a Hilbert space H which will be $L^2((0,T),\mathbb{R}^d)$ in practice, and an isonormal Gaussian process W:

Definition 1.2.1. *We say that a process $W = (W(h))_{h\in H}$ is an isonormal Gaussian process if for any $h, g \in H$, $W(h)$ is a centered Gaussian random variable and*

$$\mathbb{E}[W(h)W(g)] = \langle h, g\rangle_H.$$

To define the Malliavin derivative, we have to define it first for smooth random variables.

Definition 1.2.2. *We say a random variable F is smooth if there exist $h_1, ..., h_n \in H$ and a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ infinitely continuously differentiable such that f and all of its partial derivatives have polynomial growth (i.e. $f \in C_p^\infty(\mathbb{R}^n)$), such that*

$$F = f(W(h_1), ..., W(h_n)).$$

We note $\mathcal{S}$ the set of such smooth functions.

Definition 1.2.3. *The Malliavin derivative of a smooth random variable*

$$F = f(W(h_1), ..., W(h_n)) \in \mathcal{S}$$

is defined by

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(W(h_1), ..., W(h_n))h_i.$$

The operator $D : \mathcal{S} \longrightarrow L^0(\Omega, H)$ satisfies the following property which will be useful to define the set of Malliavin differentiable random variables and their derivatives (see [69, Proposition 1.2.1]).

Proposition 1.2.4. *The operator D is closable from $L^p(\Omega)$ to $L^p(\Omega, H)$ for any $p > 1$: for any $(F_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$ such that*

$$F_n \underset{n\to+\infty}{\overset{L^p(\Omega)}{\longrightarrow}} 0$$

and there exists $\eta \in L^p(\Omega, H)$ such that

$$DF_n \underset{n\to+\infty}{\overset{L^p(\Omega,H)}{\longrightarrow}} \eta$$

then $\eta = D0 = 0$.

Then we can define the random variables which are Malliavin differentiable.

Definition 1.2.5. *We note $\mathbb{D}^{1,p}$ the domain of the operator D in $L^p(\Omega)$ for any $p > 1$: it is the closure of $\mathcal{S}$ with respect to the norm*

$$\|F\|_{1,p} = \left(\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_H^p]\right)^{\frac{1}{p}}.$$

We say that a random variable X is Malliavin differentiable if $X \in \mathbb{D}^{1,p}$ for $p > 1$.

We mention important properties about the Malliavin derivative: the integration by parts and the chain rule (see [69, Lemma 1.2.2] [69, Proposition 1.2.3] and [69, Proposition 1.2.4]).

Theorem 1.2.6. *Let $F, G \in \mathbb{D}^{1,p}$ and $h \in H$. Then*

$$\mathbb{E}[G\langle DF, h\rangle_H] = \mathbb{E}[FGW(h)] - \mathbb{E}[F\langle DG, h\rangle_H].$$

Theorem 1.2.7. *Let $\varphi \in C^\infty(\mathbb{R}^m)$ with bounded partial derivatives, $p > 1$ and $F_1, ..., F_m$ in $\mathbb{D}^{1,p}$. Then $\varphi(F_1, ..., F_m) \in \mathbb{D}^{1,p}$ and*

$$D(\varphi(F_1, ..., F_m)) = \sum_{i=1}^m \frac{\partial \varphi}{\partial x_i}(F_1, ..., F_m) DF_i.$$

We can reduce the assumption about the regularity of the function φ in the previous theorem.

Proposition 1.2.8. *Let* $\varphi : \mathbb{R}^m \longrightarrow \mathbb{R}$ *K-Lipschitz continuous and* $F_1, ..., F_m \in \mathbb{D}^{1,p}$. *Then* $\varphi(F_1, ..., F_m) \in \mathbb{D}^{1,p}$ *and there exist random variables* $G_1, ..., G_m$ *bounded by* K *such that*

$$D\varphi(F_1, ..., F_m) = \sum_{i=1}^{m} G_i DF_i.$$

Remark 1.2.9. *Often we have* $H = L^2((0,T), \mathbb{R}^d)$. *In this case, if* $F \in \mathbb{D}^{1,2}$ *then* DF *is a stochastic process* $(D_tF)_{0\leq t\leq T} \in L^2(\Omega \times (0,T), \mathbb{R}^d)$. *We note their coordinates* $D^iF, 1 \leq i \leq d$, *and*

$$\|F\|_{1,2}^2 = \mathbb{E}\left[|F|^2\right] + \mathbb{E}\left[\int_0^T |D_tF|^2 dt\right].$$

Moreover in practice we consider W *a Brownian motion and the associated isonormal Gaussian process defined by*

$$\forall t \in [0,T], \quad W(1_{[0,t]}) = W_t.$$

In particular we have

$$DW_t = 1_{[0,t]}.$$

In this case we have different results which will be useful for our following sections (see [69, Section 1.2.1]). We consider $(\mathcal{F}_t)_{0\leq t\leq T}$ the filtration generated by the Bownian motion W.

Proposition 1.2.10. *Let* $F \in \mathbb{D}^{1,p}$ $\mathcal{F}_t$*-measurable. Then*

$$\forall \theta \in (t,T], \quad D_\theta F = 0.$$

In particular for an adapted process $(F_t)_{0\leq t\leq T}$, *we have*

$$\forall 0 \leq t < \theta \leq T, \quad D_\theta F_t = 0.$$

Proposition 1.2.11. *Let* $(F_t)_{0\leq t\leq T}$ *a* $\mathbb{D}^{1,2}$*-valued process. Then for any* $t \in [0,T]$,

$$\int_t^T F_s ds \in \mathbb{D}^{1,2} \quad \text{and} \quad D_\theta\left(\int_t^T F_s ds\right) = \int_t^T D_\theta F_s ds, \quad 0 \leq \theta \leq T.$$

Proposition 1.2.12. *Let* $F \in \mathbb{D}^{1,2}$ *and* $t \in [0,T]$. *Then* $\mathbb{E}[F|\mathcal{F}_t] \in \mathbb{D}^{1,2}$ *and*

$$D_\theta(\mathbb{E}[F|\mathcal{F}_t]) = \mathbb{E}[D_\theta F|\mathcal{F}_t]1_{\{\theta\leq t\}}, \quad 0 \leq \theta \leq T.$$

1.2.2 Malliavin differentiability of the solution of a BSDE

We would like to get the Malliavin derivative of the solution (Y, Z) of the BSDE (1.1) under assumptions about the Malliavin differentiability of the parameters (ξ, f) and the regularity of the driver f with respect to (y, z). For this we use the results of [33] or [67].

Definition 1.2.13. *We define, for $p > 1$, $L^2_{1,p}((0,T), \mathbb{R}^k)$ as the set of the $\mathbb{R}^k$-valued process F such that $F_t \in (\mathbb{D}^{1,p})^k \quad dt$-a.s., the process $t \longmapsto DF_t$ is $L^2((0,T), \mathbb{R}^{d\times k})$-valued and admits a progressively measurable version, and*

$$\|F\|_{L^2_{1,p}} := \mathbb{E}\left[\left(\int_0^T |F_t|^2 dt\right)^{\frac{p}{2}} + \left(\int_0^T \int_0^T |D_\theta F_t|^2 d\theta dt\right)^{\frac{p}{2}}\right] < +\infty.$$

Assumption 5.

- $\xi \in (\mathbb{D}^{1,2})^k \cap L^4(\mathcal{F}_T, \mathbb{R}^k)$ *and* $D\xi \in L^2(\Omega \times (0,T), \mathbb{R}^{d\times k})$.
- $f^0 \in H^4((0,T), \mathbb{R}^k)$.
- *The driver f is continuously differentiable in $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$ with uniformly bounded and continuous partial derivative.*
- *For any $y \in \mathbb{R}^k, z \in \mathbb{R}^{k\times d}$, $f(\cdot, y, z) \in L^2_{1,2}((0,T), \mathbb{R}^k)$ with Malliavin derivative $D_\theta f$ satisfying*

$$\mathbb{E}\left[\int_0^T \int_0^T |D_\theta f(t, Y_t, Z_t)|^2 dt d\theta\right] < +\infty,$$

and, for any $t \in [0,T], (y_1, z_1), (y_2, z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$ and a.e. $\theta \in [0,T]$,

$$|D_\theta f(t, y_1, z_1) - D_\theta f(t, y_2, z_2)| \leq K_{\theta,t}(|y_1 - y_2| + |z_1 - z_2|),$$

where K_θ is an adapted nonnegative process such that

$$\mathbb{E}\left[\int_0^T \int_0^T |K_{\theta,t}|^4 dt d\theta\right] < +\infty.$$

Thanks to these assumptions we get the following theorem which is proved at the end of this section and in [33].

Theorem 1.2.14. *Under Assumption 5, the BSDE* (1.1) *admits a unique solution (Y, Z) which is $(\mathbb{D}^{1,2})^k \times (\mathbb{D}^{1,2})^{k\times d}$-valued, and for any $i \in \{1, ..., d\}$, a version of $(D^i Y, D^i Z)$ is* *given by the following linear BSDE: for any* $0 \leq t < \theta \leq T$,

$$D^i_\theta Y_t = 0, \quad D^i_\theta Z_t = 0,$$

and for any $0 \leq \theta \leq t \leq T$,

$$D^i_\theta Y_t = D^i_\theta \xi + \int_t^T (\alpha_s D^i_\theta Y_s + \beta_s D^i_\theta Z_s + F^{0,i}_{s,\theta}) ds - \int_t^T D^i_\theta Z_s dW_s, \tag{1.4}$$

where

$$\alpha_s = \nabla_y f(s, Y_s, Z_s), \quad \beta_s = \nabla_z f(s, Y_s, Z_s), \quad F^{0,i}_{s,\theta} = D^i_\theta f(s, Y_s, Z_s).$$

We can get an equality between $D_t Y_t$ and Z_t for any $t \in [0, T]$. Indeed we can consider the Markovian BSDE (in dimension one to simplify the reasoning):

$$\begin{cases} Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \\ X_t &= x_0 + \int_0^t b(s, X_s) ds + \int_0^t \sigma(s, X_s) dW_s, \quad 0 \leq t \leq T, \end{cases} \tag{1.5}$$

with parameters g, f, x_0, b, σ.

To get the Malliavin differentiability of the process (Y, Z), we have to study the Malliavin differentiabilities of the driver f and the terminal condition $\xi := g(X_T)$ (with the Malliavin differentiability of X according to the chain rule 1.2.7). This result is detailed in [67, Section 6.1].

Assumption 6.

- $x_0 \in \mathbb{R}$.
- *The (deterministic) functions* $b, \sigma : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *are continuous in* t, *continuously differentiable in* x *with bounded partial derivatives:*

$$\forall t \in [0, T], x \in \mathbb{R}, \quad \left| \frac{\partial b}{\partial x}(t, x) \right| \leq C, \quad \left| \frac{\partial \sigma}{\partial x}(t, x) \right| \leq C,$$

  and the functions $b(\cdot, 0)$ *and* $\sigma(\cdot, 0)$ *are also bounded:*

$$\forall t \in [0, T], \quad |b(t, 0)| \leq C, \quad |\sigma(t, 0)| \leq C.$$

- *The (deterministic) function* $g : \mathbb{R} \longrightarrow \mathbb{R}$ *is continuously differentiable with polynomial growth.*

- *The (deterministic) function $f : [0,T] \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ is continuous differentiable in (x, y, z) with bounded partial derivative in (y, z) uniformly in t, satisfying*

$$\mathbb{E}\left[\int_0^T |f(s,0,0,0)|^2 ds\right] < +\infty,$$

and

$$\forall t \in [0,T], x, y, z \in \mathbb{R}, \quad \left|\frac{\partial f}{\partial x}(t,x,y,z)\right| \leq C(1 + |x|^k + |y|^k + |z|^k).$$

Proposition 1.2.15. *Under Assumption 6 about the parameters b and σ, the SDE in* (1.5) *admits a unique solution X. Moreover the process X satisfies $X_t \in \mathbb{D}^{1,p}$ for any $p \geq 1$ and $t \in [0,T]$, and*

$$\forall p \geq 1, \quad \sup_{0 \leq \theta \leq T} \mathbb{E}\left[\sup_{\theta \leq s \leq T} |D_\theta X_s|^p\right] < +\infty.$$

We arrive at the essential result which is detailed in Section 2.2.1.

Corollary 1.2.16. *Under Assumption 6, the Markovian BSDE* (1.5) *admits a unique solution (Y, Z). Moreover $Y_t, Z_t \in \mathbb{D}^{1,2}$ for any $t \in [0,T]$ and*

$$D_t Y_t = Z_t \quad \mathbb{P} - a.s..$$

To conclude this section, we are going to proof Theorem 1.2.14 to illustrate the different results about the Malliavin derivative. We will use the following lemma.

Lemma 1.2.17. *We consider $Z \in H^2((0,T), \mathbb{R}^{k\times d})$ and, for $t \in [0,T]$,*

$$F = \int_t^T Z_s dW_s.$$

If $F \in (\mathbb{D}^{1,2})^k$ then $Z \in L^2((t,T), (\mathbb{D}^{1,2})^{k\times d})$ and, for any $i \in \{1, \cdots, d\}$, $d\theta \times d\mathbb{P}$-a.s.,

$$D^i_\theta F = \begin{cases} \displaystyle\int_t^T D^i_\theta Z_s dW_s & \text{if } \theta \leq t \\ \\ \displaystyle Z^i_\theta + \int_\theta^T D^i_\theta Z_s dW_s & \text{if } \theta > t \end{cases}$$

Proof of Theorem 1.2.14. We assume that $d = 1$ and as in the proof of Theorem 1.1.12

we consider the Picard iteration (Y^n, Z^n) defined by

$$Y^0 = 0, \quad Z^0 = 0,$$

and for any $n \in \mathbb{N}$

$$Y_t^{n+1} = \xi + \int_t^T f(t, Y_t^n, Z_t^n)dt - \int_t^T Z_s^{n+1} dW_s, \quad 0 \le t \le T.$$

According to Proposition 1.1.13 and Assumption 5 we have that the Picard iteration $(Y^n, Z^n) \in S^4((0,T), \mathbb{R}) \times H^4((0,T), \mathbb{R}^4)$ and

$$Y_n \xrightarrow[n\to+\infty]{S^4} Y, \quad Z_n \xrightarrow[n\to+\infty]{H^4} Z,$$

with (Y, Z) the solution of the BSDE (1.1) in $S^4((0,T), \mathbb{R}) \times H^4((0,T), \mathbb{R})$. By recursion on $n \in \mathbb{N}$ we have $(Y^n, Z^n) \in L^2((0,T), \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$. Indeed we directly have $(Y^0, Z^0) = (0,0) \in L^2((0,T), \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ and if we assume $(Y^n, Z^n) \in L^2((0,T), \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ then according to the chain rule 1.2.7 $f(t, Y_t^n, Z_t^n) \in \mathbb{D}^{1,2}$ for any $t \in [0,T]$ and

$$D_\theta f(t, Y_t^n, Z_t^n) = \frac{\partial f}{\partial y}(t, Y_t^n, Z_t^n) D_\theta Y_t^n + \frac{\partial f}{\partial z}(t, Y_t^n, Z_t^n) D_\theta Z_t^n + D_\theta f(t, Y_t^n, Z_t^n), \quad 0 \le \theta \le T.$$

Thus according to Proposition 1.2.11, $\int_t^T f(s, Y_s^n, Z_s^n)ds \in \mathbb{D}^{1,2}$ and for any $\theta \in [0,T]$

$$\begin{aligned} & D_\theta \left(\int_t^T f(s, Y_s^n, Z_s^n)ds \right) \\ & = \int_t^T D_\theta f(s, Y_s^n, Z_s^n)ds \\ & = \int_t^T \left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) D_\theta Y_s^n + \frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) D_\theta Z_s^n + D_\theta f(s, Y_s^n, Z_s^n) \right) ds. \end{aligned}$$

Moreover $\xi \in \mathbb{D}^{1,2}$ then $\xi + \int_t^T f(t, Y_s^n, Z_s^n)ds \in \mathbb{D}^{1,2}$ and for any $\theta \in [0,T]$

$$\begin{aligned} & D_\theta \left(\xi + \int_t^T f(s, Y_s^n, Z_s^n)ds \right) \\ & = D_\theta \xi + \int_t^T \left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) D_\theta Y_s^n + \frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) D_\theta Z_s^n + D_\theta f(s, Y_s^n, Z_s^n) \right) ds. \end{aligned}$$

But according to Proposition 1.1.8

$$Y_t^{n+1} = \mathbb{E}\left[\xi + \int_t^T f(s, Y_s^n, Z_s^n)ds \middle| \mathcal{F}_t\right].$$

Then according to Proposition 1.2.12 $Y_t^{n+1} \in \mathbb{D}^{1,2}$ and

$$\begin{aligned}
&D_\theta Y_t^{n+1} \\
&= \mathbb{E}\left[D_\theta\left(\xi + \int_t^T f(s, Y_s^n, Z_s^n)ds\right)\middle|\mathcal{F}_t\right]1_{\{\theta\leq t\}} \\
&= \mathbb{E}\Bigg[D_\theta\xi + \int_t^T \Bigg(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n)D_\theta Y_s^n + \frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n)D_\theta Z_s^n \\
&\qquad\qquad + D_\theta f(s, Y_s^n, Z_s^n)\Bigg)ds\Bigg|\mathcal{F}_t\Bigg]1_{\{\theta\leq t\}}.
\end{aligned}$$

Therefore

$$\int_t^T Z_s^{n+1}dW_s = \xi + \int_t^T f(s, Y_s^n, Z_s^n)ds - Y_t^{n+1} \in \mathbb{D}^{1,2}.$$

Thus according to Lemma 1.2.17 we have $Z \in L^2((0,T), \mathbb{D}^{1,2})$ and we obtain for any $\theta \in [0,t]$

$$\begin{aligned}
D_\theta Y_t^{n+1} = D_\theta\xi &+ \int_t^T \left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n)D_\theta Y_s^n + \frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n)D_\theta Z_s^n + D_\theta f(s, Y_s^n, Z_s^n)\right)ds \\
&- \int_t^T D_\theta Z_s^{n+1}dW_s,
\end{aligned}$$

and for any $\theta \in\,]t,T]$, $D_\theta Y_t^{n+1} = 0$. Let $\theta \in [0,t]$ and (Y^θ, Z^θ) the solution of the linear BSDE

$$Y_t^\theta = D_\theta\xi + \int_t^T \left(\frac{\partial f}{\partial y}(s, Y_s, Z_s)Y_s^\theta + \frac{\partial f}{\partial z}(s, Y_s, Z_s)Z_s^\theta + D_\theta f(s, Y_s, Z_s)\right)ds - \int_t^T Z_s^\theta dW_s.$$

Thus according to the a priori estimate 1.1.10

$$\left\|Y^\theta\right\|_{S^2}^2 + \left\|Z^\theta\right\|_2^2 \leq C\mathbb{E}[(D_\theta\xi)^2 + \|D_\theta f(\cdot, Y, Z)\|_2^2].$$

Then according to Assumption 5

$$\int_0^T \left(\left\|Y^\theta\right\|_{S^2}^2 + \left\|Z^\theta\right\|_2^2\right)d\theta < +\infty.$$

Furthermore according to the a priori estimate 1.1.11 with $(D_\theta Y^{n+1}, D_\theta Z^{n+1})$ and (Y^θ, Z^θ)

$$\left\| D_\theta Y^{k+1} - Y^\theta \right\|_{S^2}^2 + \left\| D_\theta Z^{k+1} - Z^\theta \right\|_2^2 \leq C \left\| \delta^{n,\theta} \right\|_2^2 = C\mathbb{E}\left[\int_\theta^T (\delta_s^{n,\theta})^2 ds \right],$$

with

$$\begin{aligned} \delta_s^{n,\theta} = & \frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) D_\theta Y_s^n - \frac{\partial f}{\partial y}(s, Y_s, Z_s) Y_s^\theta \\ & + \frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) D_\theta Z_s^n - \frac{\partial f}{\partial z}(s, Y_s, Z_s) Z_s^\theta \\ & + D_\theta f(s, Y_s^n, Z_s^n) - D^\theta f(s, Y_s, Z_s). \end{aligned}$$

Thus by convexity inequality there exists $C' \in \mathbb{R}_+^*$ such that

$$\left\| D_\theta Y^{n+1} - Y^\theta \right\|_{S^2}^2 + \left\| D_\theta Z^{n+1} - Z^\theta \right\|_2^2 \leq C' \left(A_\theta^n + B_\theta^n + C_\theta^n \right),$$

with

$$\begin{aligned} A_\theta^n = & \mathbb{E}\left[\int_\theta^T \left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) \left(D_\theta Y_s^n - Y_s^\theta \right) \right)^2 ds \right] \\ & + \mathbb{E}\left[\int_\theta^T \left(\frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) \left(D_\theta Z_s^n - Z_s^\theta \right) \right)^2 ds \right], \end{aligned}$$

$$\begin{aligned} B_\theta^n = & \mathbb{E}\left[\int_\theta^T \left(\left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial y}(s, Y_s, Z_s) \right) Y_s^\theta \right)^2 ds \right] \\ & + \mathbb{E}\left[\int_\theta^T \left(\left(\frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial z}(s, Y_s, Z_s) \right) Z_s^\theta \right)^2 ds \right]. \end{aligned}$$

and

$$C_\theta^n = \mathbb{E}\left[\int_\theta^T \left(D_\theta f(s, Y_s^n, Z_s^n) - D_\theta f(s, Y_s, Z_s) \right)^2 ds \right].$$

- For A_θ^n the partial derivative of the function f are bounded (by $M \in \mathbb{R}_+^*$), thus

$$\int_0^T A_\theta^n d\theta \leq M^2 T^2 \left\| D_\theta Y^n - Y^\theta \right\|_{S^2}^2 + M^2 T \left\| D_\theta Z^n - Z^\theta \right\|_2^2.$$

- For B_θ^n we have by continuity of the partial derivatives of the function f

$$\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial y}(s, Y_s, Z_s) \xrightarrow[n\to+\infty]{ds\times d\mathbb{P}-a.s.} 0,$$

and, as the partial derivatives are bounded, we have the domination

$$\left(\left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial y}(s, Y_s, Z_s)\right) Y_s^\theta\right)^2 \le 2M^2 \sup_{s\in[0,T]} (Y_s^\theta)^2 \in L^1(\Omega\times[0,T]\times[0,T]).$$

Thus, by dominated convergence theorem

$$\int_0^T \mathbb{E}\left[\int_\theta^T \left(\left(\frac{\partial f}{\partial y}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial y}(s, Y_s, Z_s)\right) Y_s^\theta\right)^2 ds\right] d\theta \underset{n\to+\infty}{\longrightarrow} 0.$$

In the same way

$$\left(\left(\frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial z}(s, Y_s, Z_s)\right) Z_s^\theta\right)^2 \le 2M^2 (Z_s^\theta)^2 \in L^1(\Omega\times[0,T]\times[0,T]).$$

Thus

$$\int_0^T \mathbb{E}\left[\int_\theta^T \left(\left(\frac{\partial f}{\partial z}(s, Y_s^n, Z_s^n) - \frac{\partial f}{\partial z}(s, Y_s, Z_s)\right) Z_s^\theta\right)^2 ds\right] d\theta \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore

$$\int_0^T B_\theta^n d\theta \underset{n\to+\infty}{\longrightarrow} 0.$$

- For C_θ^n we have by Assumption 5 and Proposition 1.1.13

$$\begin{aligned}
C_\theta^n &\le \mathbb{E}\left[\int_\theta^T K_{\theta,s}^2 (|Y_s^n - Y_s| + |Z_s^n - Z_s|)^2 ds\right] \\
&\le \left(\mathbb{E}\left[\int_0^T K_{\theta,s}^4 ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T (|Y_s^n - Y_s| + |Z_s^n - Z_s|)^4 ds\right]\right)^{\frac{1}{2}} \\
&\le \left(\mathbb{E}\left[\int_0^T K_{\theta,s}^4 ds\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^4 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^4 ds\right]\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus

$$\int_0^T C_\theta^n d\theta \leq \int_0^T \left(\mathbb{E}\left[\int_0^T K_{\theta,s}^4 ds\right]\right)^{\frac{1}{2}} d\theta$$
$$\times \left(\mathbb{E}\left[\int_0^T |Y_s^n - Y_s|^4 ds\right] + \mathbb{E}\left[\int_0^T |Z_s^n - Z_s|^4 ds\right]\right)^{\frac{1}{2}}$$
$$\underset{n\to+\infty}{\longrightarrow} 0.$$

We can assume that T is small enough to have

$$\alpha := \max(M^2T^2, M^2T) < 1.$$

Indeed we can consider a subdivision $(t_0, ..., t_m)$ of $[0, T]$ with a step δ such that $\alpha := \max(M^2\delta^2, M^2\delta) < 1$ and proceed on each subinterval $[t_i, t_{i+1}]$ as in the proof of Theorem 1.1.12.

Let $\varepsilon \in \mathbb{R}_+^*$. Thus, according to $\int_0^T B_\theta^n d\theta, \int_0^T C_\theta^n \underset{n\to+\infty}{\longrightarrow} 0$, there exists $N \in \mathbb{N}$ such that, for any $n \geq N$,

$$\int_0^T \left(\left\|D_\theta Y^{n+1} - Y^\theta\right\|_{S^2}^2 + \left\|D_\theta Z^{n+1} - Z^\theta\right\|_2^2\right) d\theta$$
$$\leq (1-\alpha)\varepsilon + \alpha \int_0^T \left(\left\|D_\theta Y^n - Y^\theta\right\|_{S^2}^2 + \left\|D_\theta Z^n - Z^\theta\right\|_2^2\right) d\theta.$$

Therefore, by successive iterations,

$$\int_0^T \left(\left\|D_\theta Y^n - Y^\theta\right\|_{S^2}^2 + \left\|D_\theta Z^n - Z^\theta\right\|_2^2\right) d\theta$$
$$\leq (1-\alpha)\varepsilon + (1-\alpha)\varepsilon\alpha + ...$$
$$+ (1-\alpha)\varepsilon\alpha^{n-1} + \alpha^n \int_0^T \left(\left\|D_\theta Y^0 - Y^\theta\right\|_{S^2}^2 + \left\|D_\theta Z^0 - Z^\theta\right\|_2^2\right) d\theta$$
$$= \varepsilon + \alpha^n \int_0^T \left(\left\|D_\theta Y^0 - Y^\theta\right\|_{S^2}^2 + \left\|D_\theta Z^0 - Z^\theta\right\|_2^2\right) d\theta$$

Thus, as $\alpha < 1$,

$$(D_\theta Y^n, D_\theta Z^n) \underset{n\to+\infty}{\overset{S^2\times H^2}{\longrightarrow}} (Y^\theta, Z^\theta).$$

Furthermore $(Y^n, Z^n) \underset{n\to+\infty}{\overset{S^2\times H^2}{\longrightarrow}} (Y, Z)$ and the operator D is closed according to Proposition 1.2.4, thus $(Y, Z) \in L^2((0, T), \mathbb{D}^{1,2})$ and (Y^θ, Z^θ) is a version of $(D_\theta Y, D_\theta Z)$. Therefore we obtain the BSDE (1.4). □

1.3 Hawkes Processes

An overview of Hawkes processes concludes this introduction in this third part.

Hawkes processes, introduced by Alan G. Hawkes in 1971 in [44], are self-exciting point processes where past events increase the likelihood of future events. Originally developed to model aftershocks in seismic activity, they have since been applied in fields such as finance, neuroscience, and social networks. The key feature of Hawkes processes is their ability to capture event clustering and temporal dependencies, making them useful for modeling systems with memory and feedback effects. Today, they are widely used in areas like high-frequency trading and epidemiology.

1.3.1 Definitions and existence of a such process

We consider a filtered space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ and a process $(N_t)_{t\in\mathbb{R}}$. Before defining a Hawkes process, we remember the definitions of a counting process, its compensator and its intensity.

Definition 1.3.1. *We say that N is a counting process if:*

- *The process N is $\mathbb{F}$-adapted.*
- *$N_0 = 0$ and for any $t \in \mathbb{R}_+, N_t$ is finite almost surely.*
- *For any $\omega \in \Omega$, the function N is càdlàg, nondecreasing, piecewise constant and has jumps of amplitude 1.*

Definition 1.3.2. *The compensator Λ of the counting process N is the process which satisfies:*

- *The process Λ is $\mathbb{F}$-predictable.*
- *$\Lambda_0 = 0$ and for any $t \in \mathbb{R}_+, \Lambda_t \in L^1(\Omega)$.*
- *For any $\omega \in \Omega$, the function $\Lambda(\omega)$ is càdlàg and nondecreasing.*
- *The process $N - \Lambda$ is a right continuous local martingale.*

Definition 1.3.3. *We say that a process λ is the intensity of the compensator N of the counting process Λ if it is $\mathbb{F}$-predictable and*

$$\forall t \in \mathbb{R}_+, \quad \Lambda_t = \int_0^t \lambda_s ds.$$

Definition 1.3.4. *We say that the counting process N is a Hawkes process if its intensity λ satisfies*

$$\forall t \in \mathbb{R}_+, \quad \lambda_t = \lambda + \int_{(0,t)} \mu(t-s)dN_s = \lambda + \sum_{i=1}^{N_{t-}} \mu(t-T_i)$$

where $\lambda \in \mathbb{R}_+^*$, $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is integrable and $(T_i)_{i\in\mathbb{N}^*}$ the jump instants of the process N.

We consider from now a Hawkes process on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ with intensity λ.

Remark 1.3.5. *We can note that this definition is implicit because we define the Hawkes process by its intensity and its intensity by the Hawkes process. Thus the definition needs an existence theorem. We have one in [24]*

Theorem 1.3.6. *If $\int_0^{+\infty} \mu(s)ds < 1$ then the Hawkes process can be constructed as the solution (N, λ) of the equation*

$$\left\{\begin{array}{rcl} N_t &=& \int_{\mathbb{R}_+^*\times\mathbb{R}_+^*} 1_{\{u\leq t\}} 1_{\{\theta\leq\lambda_u\}} Q(du, d\theta) \\ \\ \lambda_t &=& \lambda + \int_{(0,t)} \mu(t-s)dN_s, \end{array}\right. \qquad t \in \mathbb{R}_+$$

where Q is a random Poisson measure on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ with unit intensity.

Proof. We proceed by defining its atoms. Firstly we define

$$\begin{array}{rcl} \lambda_t^0 &=& \lambda, \\ T_1 &=& \inf\left\{t \in \mathbb{R}_+^*, \quad \int_{\mathbb{R}_+^*\times\mathbb{R}_+^*} 1_{\{u\leq t\}} 1_{\{\theta\leq\lambda_u^0\}} Q(du, d\theta) > 0\right\} \\ &=& \inf\{t \in \mathbb{R}_+^*, \quad Q((0,t] \times (0,\lambda]) > 0\}. \end{array}$$

For any $\varepsilon \in \mathbb{R}_+^*, Q((0,\varepsilon] \times (0,\lambda])$ follows a Poisson distribution with intensity $\varepsilon\lambda$. Then $0 < T_1 < +\infty$ $\mathbb{P}$-almost surely. Then we define, for any $t \in (0, T_1)$,

$$\begin{array}{rcl} \lambda_t &=& \lambda_t^0 = \lambda, \\ N_t &=& 0, \\ N_{T_1} &=& \int_{\mathbb{R}_+^*\times\mathbb{R}_+^*} 1_{\{u\leq T_1\}} 1_{\{\theta\leq\lambda_u^0\}} Q(du, d\theta) = Q([0, T_1] \times [0, \lambda]). \end{array}$$

In particular T_1 is the first atom of the process N. Then we proceed by recursion. We assume that $T_1, ..., T_k$ are constructed such that they are the k first atoms of the process N in increasing order with $T_k < +\infty$ $\mathbb{P}$-a.s.: for any $j \in \{1, ..., k\}$,

$$\begin{aligned}
\lambda_t^{j-1} &= \lambda + \int_{(0,T_j)} \mu(t-s) dN_s = \lambda + \sum_{i=1}^{j-1} \mu(t-T_i) \\
T_j &= \inf\left\{ t > T_{j-1}, \quad \int_{\mathbb{R}_+^* \times \mathbb{R}_+^*} 1_{\{T_{j-1} < u \le t\}} 1_{\{\theta \le \lambda_u^{j-1}\}} Q(du, d\theta) > 0 \right\} \\
&= \inf\left\{ t > T_{j-1}, \quad \int_{(T_{j-1},t]} \int_{(0,\lambda_u^{j-1}]} Q(du, d\theta) > 0 \right\}.
\end{aligned}$$

We consider, for any $t \in \mathbb{R}_+^*$ and $\varepsilon \in \mathbb{R}_+^*$,

$$\begin{aligned}
\lambda_t^k &= \lambda + \int_{(0,T_k)} \mu(t-s) dN_s = \lambda + \sum_{i=1}^{k-1} \mu(t-T_i), \\
\mathcal{R}_\varepsilon &= \{(u,\theta) \in (T_k, T_k+\varepsilon] \times \mathbb{R}_+^*, \quad \theta \le \lambda_u^k\}.
\end{aligned}$$

Then, as the process λ^k is $\mathcal{F}_{T_k}$-measurable, we have, conditionally to T_k, $Q(\mathcal{R}_\varepsilon)$ follows a Poisson distribution with intensity $\int_{T_k}^{T_k+\varepsilon} \lambda_u^k du$. Thus

$$\mathbb{P}(Q(\mathcal{R}_\varepsilon) < +\infty) = \mathbb{E}(\mathbb{P}(Q(\mathcal{R}_\varepsilon) < +\infty \mid T_k)) = \mathbb{E}\left[\mathbb{P}\left(\int_{T_k}^{T_k+\varepsilon} \lambda_u^k du < +\infty \middle| T_k\right)\right]$$

with, conditionally to T_k,

$$\begin{aligned}
\int_{T_k}^{T_k+\varepsilon} \lambda_u^k du &= \lambda\varepsilon + \sum_{i=1}^{k} \int_{T_k}^{T_k+\varepsilon} \mu(u-T_i) du \\
&= \lambda\varepsilon + \sum_{i=1}^{k} \int_{T_k-T_i}^{T_k-T_i+\varepsilon} \mu(u) du \\
&\le \lambda\varepsilon + k \int_0^{+\infty} \mu(u) du \\
&< +\infty.
\end{aligned}$$

Therefore $0 < Q(\mathcal{R}_\varepsilon) < +\infty$ $\mathbb{P}$-almost surely. We define

$$T_{k+1} = \inf\{t > T_k, \quad Q(\mathcal{R}_{t-T_k}) > 0\} > T_k,$$

for any $t \in (T_k, T_{k+1})$,

$$N_t = N_{T_k}$$

and

$$N_{T_k} = Q(\mathcal{R}_{T_{k+1}-T_k}) = \int_{\mathbb{R}_+^* \times \mathbb{R}_+^*} 1_{\{T_k < u \le T_{k+1}\}} 1_{\{\theta \le \lambda_u^k\}} Q(du, d\theta).$$

The recurrence permits to conclude to the construction of the process (N, λ). It remains to prove that the process λ is the intensity of the process N. Indeed, as the random Poisson measure has a unit intensity, we get that

$$\begin{aligned}
\widetilde{N_t} &:= N_t - \int_0^t \lambda_u du \\
&= \int_{\mathbb{R}_+^* \times \mathbb{R}_+^*} 1_{\{u \le t\}} 1_{\{\theta \le \lambda_u\}} Q(du, d\theta) - \int_{\mathbb{R}_+^* \times \mathbb{R}_+^*} 1_{\{u \le t\}} 1_{\{\theta \le \lambda_u\}} du d\theta \\
&= \int_{\mathbb{R}_+^* \times \mathbb{R}_+^*} 1_{\{u \le t\}} 1_{\{\theta \le \lambda_u\}} \widetilde{Q}(du, d\theta)
\end{aligned}$$

is a càdlàg martingale. Thus the process λ is the intensity of the process N. Finally we have to prove that the process N does not explode in finite time, in other words that $T_k \xrightarrow[k \to +\infty]{a.s.} +\infty$. We asume by contradiction that $\mathbb{P}(\lim_{k \to +\infty} T_k < +\infty) > 0$. Thus there exist $T \in \mathbb{R}_+^*$ and $\Omega_0 \in \mathcal{F}$ such that $\mathbb{P}(\Omega_0) > 0$ and $\lim_{k \to +\infty} T_k \le T$ on Ω_0. Then, as the sequence $(T_k)_{k \in \mathbb{N}^*}$ is increasing, for any $k \in \mathbb{N}^*$,

$$\mathbb{E}[N_{T \wedge T_k}] \ge \mathbb{E}[N_{T \wedge T_k} 1_{\Omega_0}] = \mathbb{E}[N_{T_k} 1_{\Omega_0}] = k\mathbb{P}(\Omega_0).$$

In particular $\mathbb{E}[N_{T \wedge T_k}] \xrightarrow[k \to +\infty]{} +\infty$. However we have, for any $t \in \mathbb{R}_+$ and $k \in \mathbb{N}^*$,

$$\begin{aligned}
\mathbb{E}[N_{t \wedge T_k}] &= \mathbb{E}\left[\int_0^{t \wedge T_k} \lambda_u du\right] \\
&= \mathbb{E}\left[\lambda(t \wedge T_k) + \int_0^{t \wedge T_k} \int_{(0,u)} \mu(u-s) dN_s du\right] \\
&= \mathbb{E}[\lambda(t \wedge T_k)] + \mathbb{E}\left[\int_{(0,t \wedge T_k)} \int_s^{t \wedge T_k} \mu(u-s) du dN_s\right] \\
&\le \lambda t + \mathbb{E}\left[\int_{(0,t \wedge T_k]} dN_s\right] \int_0^{+\infty} \mu(u) du \\
&= \lambda t + \mathbb{E}[N_{t \wedge T_k}] \left\|\mu\right\|_1 .
\end{aligned}$$

Thus, as $\|\mu\|_1 < 1$,

$$\mathbb{E}[N_{t\wedge T_k}] \leq \frac{\lambda t}{1-\|\mu\|_1} < +\infty.$$

We arrive at a contradiction. Therefore $T_k \xrightarrow[k\to+\infty]{a.s.} +\infty$. We conclude, by using the monotone convergence theorem, that

$$\mathbb{E}[N_t] = \mathbb{E}\left[\lim_{k\to+\infty} N_{t\wedge T_k}\right] \leq \frac{\lambda t}{1-\|\mu\|_1} < +\infty.$$

□

1.3.2 Properties and examples

The first properties we can cite are about the intensity λ of the Hawkes process N which is related with the conditional increments (see [59]).

Proposition 1.3.7. *For any $t \in \mathbb{R}_+$,*

$$\begin{array}{rcl} \lim\limits_{h\to 0} \mathbb{E}\left[\dfrac{N_{t+h}-N_t}{h}\middle|\mathcal{F}_t\right] & = & \lambda_t, \\ \mathbb{P}(N_{t+h}-N_t = 0 \mid \mathcal{F}_t) & \underset{h\to 0}{=} & 1-\lambda_t h + o(h), \\ \mathbb{P}(N_{t+h}-N_t = 1 \mid \mathcal{F}_t) & \underset{h\to 0}{=} & \lambda_t h + o(h), \\ \mathbb{P}(N_{t+h}-N_t > 1 \mid \mathcal{F}_t) & \underset{h\to 0}{=} & o(h). \end{array}$$

Another useful property is about the expectation of $\lambda_t, t \in \mathbb{R}_+^*$ (see [7]).

Proposition 1.3.8. *The function* $\begin{array}{rcl} g : \mathbb{R}_+ & \longmapsto & \mathbb{R}_+ \\ t & \longmapsto & \mathbb{E}[\lambda_t] \end{array}$ *satisfies the functional equation*

$$\forall t \in \mathbb{R}_+, \quad g(t) = \lambda + \int_0^t \mu(s)g(t-s)ds = \lambda + (\mu * g)(t).$$

Proof. For any $t \in \mathbb{R}_+$, as λ is the intensity of the Hawkes process N,

$$\begin{aligned} g(t) &= \mathbb{E}[\lambda_t] = \lambda + \mathbb{E}\left[\int_{(0,t)} \mu(t-s)dN_s\right] = \lambda + \mathbb{E}\left[\int_0^t \mu(t-s)\lambda_s ds\right] \\ &= \lambda + \int_0^t \mu(t-s)\mathbb{E}[\lambda_s^*]ds = \lambda + \int_0^t \mu(s)g(t-s)ds. \end{aligned}$$

□

Remark 1.3.9. *Thanks to this proposition we can show that if $\|\mu\|_1 \geq 1$ then the process N explodes in finite time. This is developed in [7, 60].*

We conclude this section by giving examples of Hawkes processes by their function μ.

Example 1.3.10 (Exponential)**.** *We assume that, for any $t \in \mathbb{R}_+$,*

$$\mu(t) = \alpha\beta e^{-\beta t}$$

where $\alpha, \beta \in \mathbb{R}_+^*$ such that $\alpha < 1$ to get

$$\|\mu\|_1 = \alpha \int_0^{+\infty} \beta e^{-\beta t} dt = \alpha < 1.$$

In this case the associated intensity λ satisfies, for any $t \in \mathbb{R}_+$,

$$\lambda_t = \lambda + \alpha\beta \int_{(0,t]} e^{-\beta(t-s)} dN_s = \lambda + \alpha\beta e^{-\beta t} \int_{(0,t]} e^{\beta s} dN_s.$$

Thus

$$d\lambda_t = -\alpha\beta^2 e^{-\beta t} dt \int_{(0,t]} e^{\beta s} dN_s + \alpha\beta e^{-\beta t} e^{\beta t} dN_t = -\beta\lambda_t dt + \alpha\beta dN_t.$$

Example 1.3.11 (Power-law)**.** *We assume that, for any $t \in \mathbb{R}_+$,*

$$\mu(t) = \frac{\alpha\beta}{(1+\beta t)^{1+\gamma}}$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+^*$ such that $\alpha < \gamma$ to get

$$\|\mu\|_1 = \alpha \int_0^{+\infty} \frac{\beta}{(1+\beta t)^{1+\gamma}} dt = \alpha \left[-\frac{1}{\gamma(1+\beta t)^{\gamma}} \right]_0^{+\infty} = \frac{\alpha}{\gamma} < 1.$$

1.4 Presentation of results of this thesis

In the next chapters, we study numerous results on the solutions of stochastic differential equations, particularly through Malliavin calculus. We present in this introduction an abstract of the main results.

1.4.1 Limit behavior of the solution of a BSDE thanks to the Malliavin calculus

In Chapter 2 of this thesis (our article [20]) we study the limit behavior of the solution (Y, Z) of a BSDE with a singular terminal condition. In general, the solution is only the minimal supersolution: $\liminf_{t\to T} Y_t \geq \xi$ where ξ is the singular terminal condition of the BSDE (see [56]). The existence of a left limit at time T is studied in [76] and mainly depends on the generator f and not on ξ. Here we prove a theorem which gives sufficient assumptions to get the continuity at the terminal instant: a.s.

$$\liminf_{t\to T} Y_t = \xi.$$

We work within the framework of a backward stochastic differential equation of the form

$$Y_t = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s \tag{1.6}$$

with a process X solution of a forward stochastic differential equation

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

and g a deterministic function which can be equal to $+\infty$. Thus we get the following theorem (Theorem 2.2.1 in Section 2).

Theorem. *Under assumptions about the parameters b, σ, g and F (Assumptions 7, 8, 9, 10 and 11 in Section 2)), in particular*

$$F(t, x, y, z) = f(t, x, y, z) + \langle a(t, x), z\rangle, \quad (t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d,$$

and there exist $\alpha \in \mathbb{R}_+, C \in \mathbb{R}_+^*$ and $\ell \geq 1$ such that $\alpha < \frac{2(q-1)}{q+1}$ and

$$\forall (t, x, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}^d, \quad |f(t, x, 0, z) - f(t, x, 0, 0)| \leq C(1 + |x|^\ell)|z|^\alpha$$

where $q > 1$ is the growth constant of the driver with respect to $y \in \mathbb{R}_+$:

$$\forall (t, x, y, z) \in [0, T] \times \mathbb{R}^m \times \mathbb{R}_+ \times \mathbb{R}^d, \quad F(t, x, y, z) - F(t, x, 0, z) \leq -\eta(t, x)|y|^q,$$

the minimal supersolution (Y,Z) to the BSDE satisfies $\mathbb{P}$-almost surely

$$\liminf_{t\to T} Y_t = \xi. \tag{1.7}$$

In other words if the growths with respect to $y \in \mathbb{R}_+$ and with respect to $z \in \mathbb{R}^d$ are related by the coefficients q and α then the process Y satisfies a.s. Condition (1.7).

To prove this result we proceed by approaching the solution (Y,Z) by (Y^n, Z^n) solution of a BSDE with a regularized singular terminal condition and a regularized driver. Thus, by regularization, we get the Malliavin differentiability of (Y^n, Z^n) and the equality

$$D_t Y_t^n = Z_t^n, \quad 0 \le t \le T.$$

Therefore, with a kind of integration by parts, we can control the convergence of the process Y at the terminal instant T.

We generalize this result in Section 2.3.1. If the driver F now satisfies an inequality

$$\forall (t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d, \quad F(t,x,y,z) \le f(t,x,y,z) + \langle a(t,x), z\rangle,$$

then we get the same continuity property of the process Y.

Moreover we mention two applications of this theorem. Firstly in Section 2.3.3, if we consider the function u defined by

$$u(t,x) = Y_t^{t,x}, \quad (t,x) \in [0,T] \times \mathbb{R}^m,$$

where $Y^{t,x}$ is the unique minimal supersolution of the Markovian FBSDE

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}) dt + \int_t^s \sigma(r, X_r) dW_r \quad t \le s \le T, \\ Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T Z_r dW_r \quad t \le s \le T, \end{cases}$$

then the function u is a minimal viscosity solution of the associated PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}(u)(t,x) + F(t,x,u(t,x),(\nabla u \sigma)(t,x)) = 0, \quad \forall (t,x) \in [0,T] \times \mathbb{R}^m \\ \\ u(T,x) = g(x), \quad \forall x \in \mathbb{R}^m, \end{cases}$$

where $\mathcal{L}$ is the infinitesimal generator of the SDE: $\mathcal{L}(u) = \langle b, \nabla u \rangle + \frac{1}{2}\mathrm{tr}\,(\sigma\sigma^* \nabla^2 u)$. We deduce in this part the following proposition.

Proposition. *The function u is continuous in (T, x_0) for any $x_0 \in \mathbb{R}^m$:*

$$u(t,x) \underset{(t,x)\to(T,x_0)}{\longrightarrow} g(x_0).$$

Secondly in Section 2.3.2 we study the stochastic control problem where we have to minimize the functional

$$J(t,\alpha) = \mathbb{E}\left[\int_t^T (\eta_s|\alpha_s|^p + \gamma_s|\Xi_s|^p)\,ds + \xi|\Xi_T|^p \middle| \mathcal{F}_t\right]$$

over all $\alpha \in \mathcal{A}(t,x)$ where $\mathcal{A}(t,x)$ is the set of admissible controls such that Ξ satisfies the dynamics

$$\Xi_s = x + \int_t^s \alpha_u du \quad t \le s \le T, \quad \alpha \in L^1(t,\infty) \text{ a.s.}$$

where $\eta_s = \eta(s, X_s), \gamma_s = \gamma(s, X_s), 0 \le s \le T$. A minimizer of this functional is the process Ξ^* given by

$$\Xi^*_s = x \exp\left(-\int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du\right)$$

where (Y, Z) is the minimal supersolution of the BSDE

$$Y_t = \xi - \int_t^T (p-1)\frac{|Y_s|^{q-1}Y_s}{\eta_s^{q-1}}ds + \int_t^T \gamma_s ds + \int_t^T Z_s dW_s, \quad 0 \le t \le T, \tag{1.8}$$

which is of the form of the BSDE of this chapter. According to the theorem of this chapter, the process Y satisfies $\lim_{t\to T} Y_t = \xi$. Thus there is no additional cost to minimize our control problem.

1.4.2 Limit behavior of its Malliavin derivative

Chapter 3 of this thesis is the continuity of the previous application with the stochastic control problem. We consider the same functional $(J(t,\alpha))$ and BSDE (1.8) with parameters $\xi = +\infty, \eta$ and γ where $\mathbb{P}$-a.s. $\lim_{t\to T} Y_t = \xi$. We prove that the Malliavin derivative DY of the process Y exists and is continuous on $[0,T)$ but can be discontinuous at the terminal instant T. Thanks to this result we get that the solution of the associated PDE is of class C^1 on $[0,T)$ and how compute Greeks to get the sensitivity in our liquidation problem.

We assume some properties about the process η and γ in this chapter:

- The coefficient η is an Itô process:

$$\eta_t = \eta_0 + \int_0^t b_s^\eta ds + \int_0^t \sigma_s^\eta dW_s, \quad 0 \leq t \leq T,$$

  with an initial condition $\eta_0 \in \mathbb{R}$.

  1. The drift $\beta^\eta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ and the diffusion matrix $\sigma^\eta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ are progressively measurable and bounded.

  2. There exist η_* and η^* in $\mathbb{R}_+^*$ such that, a.s. for any $s \in [0,T]$,

$$0 < \eta_* \leq \eta_s < \eta^*.$$

- The process γ is a progressively measurable, non-negative and bounded: there exists $\gamma^* \in \mathbb{R}_+^*$ such that, a.s. for any $s \in [0,T]$,

$$0 \leq \gamma_s \leq \gamma^*.$$

Let us explain the difficulty. If we consider (Y^n, Z^n) the solution of the truncated BSDE

$$Y_t^n = n + \int_t^T \left(-(p-1) \frac{|Y_s^n|^{q-1} Y_s^n}{\eta_s^{q-1}} + \gamma_s \right) ds - \int_t^T Z_s^n dW_s, \quad t \in [0,T],$$

then $(Y^n, Z^n) \in L^2((0,T), \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$ and for any $0 \leq t < \theta \leq T$, $D_\theta Y_t^n = 0$, $D_\theta Z_t^n = 0$ and, for any $0 \leq \theta \leq t \leq T$,

$$D_\theta Y_t^n = \int_t^T \left(-(p-1) \frac{|Y_r^n|^{q-1}}{\eta_r^{q-1}} D_\theta Y_r^n + \frac{|Y_r^n|^{q-1} Y_r^n}{\eta_r^q} D_\theta \eta_r + D_\theta \gamma_r \right) dr - \int_t^T D_\theta Z_s^n dW_s$$

(see Proposition 3.1.4 in Chapter 3).

However if n tends to $+\infty$ in the previous BSDE then we get a linear BSDE

$$U_t = \int_t^T \left(-(p-1) \frac{|Y_r|^{q-1}}{\eta_r^{q-1}} U_r + \frac{|Y_r|^{q-1} Y_r}{\eta_r^q} D_\theta \eta_r + D_\theta \gamma_r \right) dr - \int_t^T V_s dW_s$$

with a singular generator since

$$\int_0^T \frac{|Y_r|^{q-1}}{\eta_r^{q-1}} dr = +\infty.$$

Such linear BSDEs with singular generator are studied in [52, 51] but to apply the results of these papers, the process $\frac{|Y|^{q-1}Y}{\eta^q} D_\theta \eta + D_\theta \gamma$ should be bounded. In general this property does not hold.

Thus we have to proceed in a different way. Thanks to the assumptions about the parameters, according to [42], the minimal solution Y of the BSDE can be written

$$Y_t = \frac{\eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} H_t, \quad 0 \leq t \leq T,$$

where H is the unique solution of the BSDE:

$$H_t = \int_t^T F(s, H_s) ds - \int_t^T Z_s^H dW_s, \quad 0 \leq t \leq T,$$

with a singular generator F given by:

$$\begin{aligned} F(t,h) &= [(T-t)b_t^\eta + (T-t)^p \gamma_t] \\ &\quad - (p-1)\eta_t \left[\left(1 + \frac{1}{\eta_t(T-t)} h\right) \left|1 + \frac{1}{\eta_t(T-t)} h\right|^{q-1} - 1 - q \frac{1}{\eta_t(T-t)} h \right]. \end{aligned}$$

To use the Malliavin calculus we assume that the process b^η, η and γ are in valued in $\mathbb{D}^{1,2}$ and their Malliavin derivative admit progressively measurable versions in $L^2(\Omega \times (0,T) \times (0,T))$. Thus we get the following result (Theorems 3.2.3 and 3.3.9 in Chapter 3).

Theorem. *The process Y is valued in $\mathbb{D}^{1,2}$ and, for any $0 \leq \theta \leq t < T$,*

$$D_\theta Y_t = \frac{D_\theta \eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} D_\theta H_t$$

and if for some $\varrho > 1$,

$$\sup_{\theta \in [0,T]} \mathbb{E} \left[|D_\theta \eta_T|^\varrho + \int_0^T \left(|D_\theta b_s^\eta|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho \right) ds \right] < +\infty$$

then

$$\lim_{n\to+\infty}\sup_{\theta\in[0,T]}\mathbb{E}\left[\sup_{t\in[0,T]}(T-t)^{\ell p}|D_\theta Y_t - D_\theta Y_t^n|^\ell\right]=0.$$

In particular for any $0\le\tau<T$

$$\lim_{n\to+\infty}\sup_{\theta\in[0,T]}\mathbb{E}\left[\sup_{t\in[0,\tau]}|D_\theta Y_t - D_\theta Y_t^n|^\ell\right]=0.$$

In other words there is a singularity at time T of the Malliavin derivative. But if the proceee η is deterministic then we have $\lim_{t\to T}|D_\theta Y_t|=0=D_\theta\xi$ and in this case there is no discontinuity.

1.4.3 Discontinuity with a jump term

In Chapter 4 of this thesis (our article [21]) we try to generalize the theorem of the first part about the behavior of the solution of the BSDE (1.6) by adding jumps in the BSDE. Thus our BSDE becomes

$$Y_t=\xi+\int_t^T f(s,Y_s,Z_s,U_s)ds-\int_t^T Z_s dW_s-\int_t^T U_s d\widetilde{N}_s,\quad 0\le t\le T,$$

where the jump term is driven by a simple Poisson process N with intensity λ. If we assume the same hypothesis than in the first part of this thesis, then we can conclude that there is no continuity of the solution at the terminal instant. In particular this result shows that the Malliavin derivative with jumps of [26] or [29] cannot help us in this study. In particular we do not have the integration by parts.

Theorem. *If the driver* f *is defined by*

$$f(t,x,y,z,u)=-y|y|^{q-1},\quad (t,x,y,z,u)\in[0,T]\times\mathbb{R}^4,$$

with $q\ge 2$*, and if the terminal condition is* $\xi=g(N_T)$ *where* g *is a right barrier*

$$g(x)=(+\infty)1_{\{x\ge x_0\}}+\varphi(x)1_{\{x<x_0\}},\quad x\in\mathbb{R},$$

with $\varphi:\mathbb{R}\longrightarrow\mathbb{R}_+$ *and* $x_0\in\mathbb{R}$*, then the process* Y *is not continuous at terminal instant* T *:* $\mathbb{P}$*-a.s.*

$$\lim_{t\to T}Y_t=+\infty\neq\xi.$$

To prove this result we begin by the case $q = 2$ in Section 4.2. The case $q < 2$ is shown by comparison theorem in Section 4.3.1. We consider the associated PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1), \quad (t,x) \in [0,T] \times \mathbb{R} \\ \\ u(T,x) = g(x), \quad x \in \mathbb{R}, \end{cases}$$

and approach the solution by truncation $u^K, K \in \mathbb{N}^*$. Thus the process Y is given by the solution u which is itself given by the limit of the solution u^K:

$$Y_t = u(t, N_t) = \lim_{K \to +\infty} u^K(t, N_t) = \frac{1}{T-t}, \quad t \in [0,T).$$

In particular we have $\lim_{t\to T} Y_t = \lim_{t\to T} \dfrac{1}{T-t} = +\infty \neq g(N_T)$.

In Section 4.2.3 if we add a diffusion term to the BSDE to become

$$Y_t = g(N_T) - \int_t^T Y_s|Y_s|ds - \int_t^T Z_s dW_s - \int_t^T U_s d\widetilde{N}_s$$

then the solution Y still does not satisfy the continuity property at the terminal T : $\lim_{t\to T} Y_t = +\infty \neq g(X_T)$.

But in Section 4.3.2 and in Section 4.4 we get two cases where the solution Y is continuous at the terminal instant T.

Proposition. *If* $f(t,y) = -y|y^{q-1} + f_t^0, (t,y) \in [0,T] \times \mathbb{R}$, *with* $q > 2$ *and* $f^0 \in L^1((0,T),\mathbb{R})$, *or if the function* g *is left barrier* $g(x) = (+\infty)1_{\{x \le x_0\}} + \varphi(x)1_{\{x > x_0\}}$ *then* $\lim_{t\to T} Y_t = g(X_T)$.

We conclude by the study of Euler scheme associated to this differential equation in Section 4.5:

$$\begin{cases} u'(t) - \lambda u(t) - u(t)|u(t)| = -\lambda \dfrac{1}{T-t}, \quad t \in [0,T) \\ \\ u(T) = \chi \in [0,+\infty). \end{cases}$$

We used the implicit method to define the scheme $(u_N(t_k))_{0 \le k \le N}$:

$$u_N(t_N) = \chi, \quad u_N(t_{k+1}) = u_N(t_k) - h_N f(t_k, u_N(t_k)), k \in \{0, \cdots, N\},$$

where $h_N = \dfrac{T}{N}, t_k = kh_N, k \in \{0, \cdots, N\}$ and $f(t,u) = \lambda u + u^2 - \lambda \dfrac{1}{T-t}, 0 \le t < T$.

Thus we have shown that the Euler scheme converges towards the solution $u(t) = \frac{1}{T-t}, 0 \leq t \leq T$, on all closed intervals of $[0, T)$.

Proposition. *For any* $\alpha \in (0, 1)$,

$$\max_{0 \leq k \leq \lfloor \alpha N \rfloor} \left| u_N(t_k) - \frac{1}{T - t_k} \right| \underset{N \to +\infty}{\longrightarrow} 0.$$

An interesting fact is that the limit does not depend on the value of χ. In other words, the behavior of the solution is not related to its terminal value.

1.4.4 Malliavin calculus with respect to a Hawkes process

In Chapter 5, we conclude this thesis by the development of a local Malliavin calculus with respect to a Hawkes process. Our goal is to get a tool in our financial problems which are modeled with a Hawkes process. For example if a process X satisfies a SDE driven by a Hawkes process we get a absolute continuous criterion for the random variable X_T. Or we can compute Greeks with a such process.

The Hawkes process we will consider in this part is defined on the space Ω of càdlàg trajectories equipped with a σ-field $\mathcal{F}$:

$$\Omega = \left\{ \sum_{i=1}^{+\infty} i 1_{[t_i, t_{i+1})}, \quad 0 < t_1 < \cdots < t_i < \cdots \right\}$$

by, for any $\omega \in \Omega$,

$$N_t(\omega) = \sum_{s \leq t} \Delta \omega_s, \quad t \in \mathbb{R}_+.$$

The probability $\mathbb{P}$ on the filtered space $(\Omega, \mathcal{F})$ is such that the process N is a Hawkes process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose the jump instants are noted $(T_i)_{i \geq 1}$.

To get the existence of a such Hawkes process, we assume, according to [24], that the constant λ is nonnegative and the function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is positive differentiable with bounded derivative and such that $\|\mu\|_1 = \int_0^{+\infty} \mu(t) dt < 1$. The filtration is the $\mathbb{P}$-complete right continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the Hawkes process N where $T \in \mathbb{R}_+^*$ is a fixed time horizon.

The first step of our construction of a Malliavin derivative with respect to a Hawkes process is to define a directional derivative with respect to a function $m \in \mathcal{H}$ where $\mathcal{H}$ is the following Hilbert space:

$$\mathcal{H} = \left\{ m \in L^2([0,T]) \quad \int_0^T m(s)ds = 0 \right\}.$$

For this we consider a reparametrization of time τ_ε for $\varepsilon \in \mathbb{R}_+^*$ which depends on $m \in \mathcal{H}$ and such that $\tau_\varepsilon(0) = 0, \tau_\varepsilon(T) = T$ and such that the number and the order of jump times between 0 and T remain unchanged. Then we transfer this reparametrization to any configuration $\omega \in \Omega$ and any random variable $F \in L^2(\Omega)$ by

$$\mathcal{T}_\varepsilon(\omega) = \omega \circ \tau_\varepsilon, \quad \mathcal{T}_\varepsilon F = F \circ \mathcal{T}_\varepsilon.$$

Thanks to this definition we can define our first domain of differentiabilty $\mathbb{D}_m^0$ as the space vector of $F \in L^2(\Omega)$ such that the following limit exists in $L^2(\Omega)$ and is defined as its directional Malliavin derivative

$$D_m F = \frac{\partial \mathcal{T}_\varepsilon F}{\partial \varepsilon}_{|\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}.$$

However the random variables of this space are not easily manipulable. So we consider a vector subspace $\mathcal{S}$ of $\mathbb{D}_m^0$: we say that a random variable $F \in \mathcal{S}$ if

$$F = a 1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}.$$

where $a \in \mathbb{R}, d \in \mathbb{N}$ and, for any $n \in \{1, \cdots, d\}$, the function $f_n : \mathbb{R}^n \longrightarrow \mathbb{R}$ is smooth with bounded derivatives of any order. On this space we get the chain rule

$$D_m \Phi(F_1, \cdots, F_n) = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(F_1, \cdots, F_n) D_m F_n$$

where $F_1, \cdots, F_n \in \mathcal{S}$ and $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ a smooth function (see Proposition 5.1.7). Thanks this property and the absolute continuity of $\mathbb{P}^\varepsilon = \mathbb{P}\mathcal{T}_\varepsilon^{-1}$ with respect to $\mathbb{P}$ (see Proposition 5.1.8), we get the following theorem.

We define $\mathcal{E}_m$ on $\mathcal{S}$ by

$$\mathcal{E}_m(X,Y) = \mathbb{E}[D_m X D_m Y], \quad X, Y \in \mathcal{S}.$$

Theorem. *$(\mathcal{S}, \mathcal{E}_m)$ is a closable quadratic bilinear form on $L^2(\Omega)$.*

Thus we define the directional Malliavin derivative D_m on the extension $\mathbb{D}^{1,2}_m$ (see Proposition 5.1.15 and Corollary 5.1.16).

The second step is to define the Malliavin derivative $(\mathbb{D}^{1,2}, D)$ from $L^2(\Omega)$ into $L^2(\Omega;\mathcal{H})$ such that

$$\forall F \in \mathbb{D}^{1,2}, m \in \mathcal{H}, \quad D_m F = \langle DF, m\rangle_{\mathcal{H}} = \int_0^T D_t F m(t) dt.$$

For this we consider a Hilbert basis $(m_i)_{i\in\mathbb{N}}$ of the space $\mathcal{H}$, the domain

$$\mathbb{D}^{1,2} = \left\{ X \in \bigcap_{i=0}^{+\infty} \mathbb{D}^{1,2}_{m_i}, \quad \sum_{i=0}^{+\infty} \|D_{m_i} X\|^2_{L^2(\Omega)} < +\infty \right\}$$

and the bilinear form

$$\mathcal{E}(X,Y) = \sum_{i=0}^{+\infty} \mathbb{E}[D_{m_i} X D_{m_i} Y], \quad X, Y \in \mathbb{D}^{1,2}.$$

Theorem. $(\mathbb{D}^{1,2}, \mathcal{E})$ *is a local Dirichlet form which admits a carré du champ* $\Gamma[X,Y] = \langle DX, DY\rangle_{\mathcal{H}}, X, Y \in \mathbb{D}^{1,2}$ *and a gradient* D

$$DX = \sum_{i=0}^{+\infty} D_{m_i} X m_i \in L^2(\Omega, \mathcal{H}), \quad X \in \mathbb{D}^{1,2}.$$

Furthermore $\mathbb{D}^{1,2}, \mathcal{E}, \Gamma$ *and* D *do not depend on the choice of the basis* $(m_i)_{i\in\mathbb{N}}$.

Therefore we get similar properties to the directional derivative as the chain rule and the density of $\mathcal{S}$ in $\mathbb{D}^{1,2}$ (see Remark 5.2.3). Moreover now we have a Malliavin derivative, we are interested in its divergence operator $(\mathrm{Dom}(\delta), \delta)$. The main result about the divergence is its expression for a predictable process $u \in L^2(\Omega, \mathcal{H})$ in Corollary 5.2.7:

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s)) dN_s$$

where, for any $t \in [0,T]$, $\widehat{u}(t) = \int_0^t u(s) ds$ and

$$\psi(u,t) = \frac{1}{\lambda^*(t)} \int_{(0,t)} (\widehat{m}(t) - \widehat{m}(s)) \mu'(t-s) dN_s.$$

The last step of our Malliavin derivative is to get applications. The first one is a absolute continuity criterion: the image measure $F_*[\det(\Gamma[F]).\mathbb{P}]$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ (see Theorem 5.3.5); thus, conditionally to

$\Gamma[F] = (\Gamma[F_i, F_j])_{1\leq i,j\leq d} \in GL_d(\mathbb{R})$, the random vector $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$ admits a absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^d$ (see Corollary 5.3.6). This result can be used to study the law of X_T where the process X is solution of stochastic differential equation:

$$X_t = x_0 + \int_0^t f(s, X_s)ds + \int_{(0,t]} g(s, X_{s-})dN_s, \quad 0 \leq t \leq T,$$

where the parameters x_0, f and g satisfy usual assumptions (see Assumption 16). For example, in dimension $d = 1$, if, for any $(t, x) \in [0, T] \times \mathbb{R}$,

$$f(t, x + g(t, x)) - f(t, x) - \frac{\partial g}{\partial x}(t, x)f(t, x) - \frac{\partial g}{\partial t}(t, x) \neq 0$$

then, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. This is the case if the parameters f and g do not depend on $t \in [0, T]$ and if the Wronskian $W(f, g) = g' \times f - f' \times f$ satisfies $|W(f, g)| > \frac{1}{2}\|f''\|_\infty \|g\|_\infty^2$ (see Theorem 5.4.9 and Corollary 5.4.11). The second application is to compute Greeks when we consider an asset price S which satisfies a SDE driven by the Hawkes process

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{N}_t, \quad S_0 = x_0.$$

Indeed for example we get an expression of $\frac{\partial}{\partial x_0}\mathbb{E}[1_{\{N_T\geq 1\}} f(S_T)]$ with terms which can be simulated from the simulation of the Hawkes process (N, λ^*) where f is not necessarily a regular function (see (5.19) and Proposition 5.4.17).

Chapter 2

Growth condition on the generator of BSDE with singular terminal value ensuring continuity up to terminal time

In this chapter we are interested in the limit behavior of the solution of a backward stochastic differential equation (BSDE for short) with singular terminal condition. BSDEs' theory has been widely developed since more than 30 years because they are a very useful tool in two domains: stochastic optimal control and partial differential equations (PDEs for short). In the first topic, BSDEs naturally appear as the adjoint equation in the Pontryagin's maximum principle (see for example [86, Chapters 3 and 7]). The dynamics of this equation is given on a time interval $[0,T]$ in a backward way: a terminal condition is given and the equation should be solved from T to 0. Concerning their application in PDEs theory, BSDEs provide an extension of the Feynman-Kac formula to non-linear PDE and the couple forward SDE and BSDE is a method of characteristics to solve a second-order PDE (see [72, Sections 5.4 and 5.7] or [87, Chapter5]).

To obtain a solution with suitable integrability condition, it is usually assumed that the terminal condition of the BSDE, denoted ξ, is integrable, as in [72, Section 5.3]. Then a priori estimates show that the solution of the BSDE is also integrable. But a large class of PDEs doesn't satisfy such constraint. Indeed for forward reaction-diffusion PDE of the form

$$\frac{\partial u}{\partial t} - \Delta u + u^q = 0,$$

with $q > 1$, it is known that the initial value of u can be equal to $+\infty$. This property has been proved by analytic methods in [65] or by probabilistic [1] arguments in [30, 31].

1. The notion of superprocesses is used, which is completely different from our method.

Roughly speaking, the solution can blow up on a non-empty set: $\lim_{t\to 0} u(t,x) = +\infty$. Furthermore in the context of stochastic control, if a target is imposed on the final value of the state process Ξ, then a singularity appears in the related adjoint equation. The optimal liquidation problem in finance is a typical example where the mandatory liquidation constraint can be written as: $\Xi_T = 0$ and the terminal value for the BSDE is $+\infty$ almost surely (see [5, 80]).

Let us now fix some notations. In this chapter, we consider BSDE of the following form

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s,$$

where W is a d-dimensional Brownian motion, the function F is called the generator or the driver and ξ is the terminal condition. Since we impose that the solution is adapted to the underlying filtration generated by W, the solution is the couple (Y, Z). blueWe classically assume that F is continuous and monotone w.r.t. y and Lipschitz continuous w.r.t. z. When ξ is integrable, these assumptions are sufficient to obtain a solution (see [72, Section 5.3]). However to ensure the existence of a solution without integrability condition on ξ, [56, 76] show that it is sufficient to impose that there exist a positive process η and $q > 1$ such that

$$\forall y \in \mathbb{R}_+, \quad \forall (s,z) \in [0,T] \times \mathbb{R}^d, \quad F(s,y,z) - F(s,0,z) \leq -\eta_s |y|^q.$$

Then there exists a minimal solution (Y, Z) such that a.s.

$$\liminf_{t\to T} Y_t \geq \xi = Y_T. \tag{2.1}$$

This behavior is sufficient to solve the related control problem with constraint, see [56, Section 2]. The possible lack of continuity at time T is due to the singularity of ξ ; in other words if ξ is in $L^p(\Omega)$ for some $p > 1$, then a.s.

$$\lim_{t\to T} Y_t = \xi = Y_T. \tag{2.2}$$

A natural question is: if ξ is not in some L^p space, under which conditions on F or on ξ, does (2.2) hold ? This question is called the continuity problem and has been studied in [74, 76, 81, 66, 1]. This question is important and we refer to [1, Section 1.1] for the implications of this problem. The known results can be summarized as follows:

- The existence of the left limit at time T only depends on F, see [76, Theorem 3.1].
- Equality in (2.1) holds in the Markovian case and if the growth of F is sufficient fast (when y tends to ∞), see [76, Theorem 4.5]. Roughly speaking, $q > 3$ is assumed in this result.
- Going beyond the Markovian setting has been done in [66], but again under a strong growth condition on F, or in [81, 1], but for specific terminal conditions ξ.

The quadratic case ($q = 2$) nor the financial data-driven case of [4], where q is estimated around 1.6, are not included in the existing literature.

The aim of the chapter is to obtain the $\mathbb{P}$-a.s. equality

$$\liminf_{t\to T} Y_t = \xi = Y_T, \tag{2.3}$$

in the Markovian framework, without restriction on the growth of F, that is for any $q > 1$. Hence in the rest of the chapter, we consider the system

$$X_t = x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \tag{2.4}$$

$$Y_t = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{2.5}$$

with unknown stochastic processes (X, Y, Z) with values in $\mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d$ and with measurable parameters: $b : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$, $\sigma : [0, T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^{m\times d}$, $g : \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{+\infty\}$ and $F : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$.

In [74], the specific case $F(s, x, y, z) = -y|y|^{q-1}$, $q > 1$, is studied and it is proved that (2.3) holds. The arguments of the proof are not the same if $q > 3$ or not. When q is sufficiently large, an a priori estimate on Z and Hölder's inequality are used, whereas for $q \leq 3$, the Malliavin calculus is used to control the process Z, with the equality $Z_t = D_t Y_t$ and the Malliavin by-parts integration.

Here we are going to generalize this result, with two novelties. The first one of this chapter concerns the drift b. To obtain the Malliavin derivative DY of Y, it is usually assumed that σ and b have bounded derivatives w.r.t. x, which ensures that the Malliavin derivative DX of X solves a linear SDE with bounded coefficients, and thus DX verifies some strong integrability properties. These properties are then used to derive the existence of DY (see [33] or [67]). This setting is kept in [74, 76]. With the representation $Z_t = D_t Y_t$ and a by-parts integration, $\mathbb{E}(Z_t \phi(X_t))$ is transformed into $\mathbb{E}(Y_t \psi(X_t))$, where ψ depends on ϕ and on the probability density function p of X. To ensure the existence of p, a uniform ellipticity condition on σ, together with the boundedness of b and σ, are also assumed in [74, 76].

In this chapter, we do not still suppose that b is Lipschitz conitnuous in x. In contrast to the other papers, we take advantage of the uniform ellipticity condition to remove the drift with a Girsanov transformation on the SDE (2.4). Therefore we only assume that b is bounded, of class C^1, with derivatives of polynomial growth. We are aware that some works prove the existence of the Malliavin derivative for X, without the Lipschitz condition on b (see for example [68]). However the existence of DX (in a weak sense), without good integrability properties, is not sufficient to obtain the existence of DY. This is the reason why we modify the SDE in order to keep a Malliavin derivative with suitable integrability conditions. The key point is that if (2.3) holds under $\mathbb{P}$, it holds under any other probability measure equivalent to $\mathbb{P}$.

The second main novelty of this chapter concerns the generator F. The used framework is standard for BSDE, that is F is Lipschitz continuous w.r.t. z. But, to be able to manage this dependence on z, if $q \leq 3$, we suppose that:

$$|F(s,x,0,z) - F(s,x,0,0)| \leq C(1+|x|^\ell)|z|^\alpha, \quad 0 \leq \alpha < \frac{2(q-1)}{q+1} \leq 1. \tag{2.6}$$

Under this condition, we prove that (2.3) holds. If $q > 3$, we only suppose that F is Lipschitz continuous in z (as in [76]). Therefore there is an interplay between the growth of F w.r.t. y, controlled by q, and the growth w.r.t. z, controlled by α. In particular when q is close to one, f should be almost bounded in z. As far as we know, this property is new in the BSDEs' literature but also in the PDE theory.

If F is linear w.r.t. z, Condition (2.6) cannot be satisfied. But the Girsanov transformation of the SDE, used if b is not Lipschitz continuous, adds a new linear term in the generator F of the form $\langle a(t,X_t), Z_t\rangle$. Therefore this condition is incompatible with this modification. Therefore we split F into two parts: $F(t,x,y,z) = f(t,x,y,z) + \langle a(t,x), z\rangle$ and we show that it is sufficient that only the function f satisfies Condition (2.6). In other words if F is linear w.r.t. z, we don't need (2.6) to have (2.3).

Notice that our result (2.3) is not satisfied if we consider a BSDE with jumps

$$Y_t = \xi + \int_t^T F(s, X_s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbb{R}} U_s(e)\widetilde{N}(ds, de), \quad 0 \leq t \leq T,$$

where $\widetilde{N}$ is a compensated Poisson random measure. Indeed a counter-example is developped in Section 4 with a simple Poisson process N, a simple driver $F(s, x, y, z, u) = -y|y|^{q-1}, q > 1$ and a simple SDE $X = N$. In this case, there is discontinuity in T: $\lim_{t \to T} Y_t = +\infty \neq \xi$.

Breakdown of the chapter. In the next section, we present the known results on BSDE with singular terminal condition.

The main result of this chapter (Theorem 2.2.1) is stated at the beginning of Section 2.2. The rest of this chapter contains the proof of this result.

Essential points of our reasonning are based on the fact the we are in a Markovian framework, that is the randomness of the driver F at the moment t is given only through the random variable X_t and ξ is a function of X_T. First to obtain (2.3) on the non-singular set $\{\xi < +\infty\}$, we use test-processes $\varphi(X)$, where φ is a test-function with compact support in the non-singular set. Instead of Y, we study $\varphi(X)Y$. At time T, $\varphi(X_T)Y_T = \varphi(X_T)\xi$ is integrable. Moreover in the Itô's formula, the cross variation term is of the form $\nabla\varphi(X)\sigma(X)Z$. To deal with this term, especially when q is small, the Markovian setting is again crucial. Indeed we can represent the process Z_t as the Malliavin derivative of Y: $Z_t = D_t Y_t$, and we use a Malliavin integration by parts (see Proposition 2.2.5 and Corollaries 2.2.7 and 2.2.8). Let us note here that the use of test functions can be generalized to smooth Itô functionals, as in [66]. However, outside the Markovian setting, we do not know how to control the cross-variation term when q is small (see Condition (H) in [66], which requires $q > 3$).

To obtain these results, X should have a Malliavin derivative with suitable integrability conditions. Under our setting, we use the Girsanov transformation to remove the drift part of the SDE. Note that if b is Lischitz continuous, we don't need this transformation.

Since we are able to control the cross variation term, we can also deal to a linear term w.r.t. z: $\langle a, Z\rangle$. For large values of q, this linear term is superfluous and could be contained in F. But for $q < 3$, Condition (2.6) excludes linear growth and then adding this linear term $\langle a(t, x), z\rangle$ makes sense. This is the first reason why we assume the particular structure (2.13) for F. Another reason comes from the Girsanov transform. Indeed changing the probability measure to manage the drift b in the SDE, adds a linear term in the BSDE. Let us note that the same conditions are imposed on b (drift of the SDE) and a (linear part of the BSDE).

In Section 2.3, we first state a comparison result for the minimal solutions of BSDEs with singular terminal condition. A direct consequence of this comparison principle for

BSDE shows that (2.3) holds if the generator is bounded from above by a generator satisfying the assumptions of our main result.

In Part 2.3.2, we apply this continuity result to the optimal liquidation problem studied in [56]. The goal is to minimize

$$J(t,\alpha) = \mathbb{E}\left[\int_t^T (\eta_s|\alpha_s|^p + \gamma_s|\Xi_s|^p)\,ds + \xi|\Xi_T|^p \middle| \mathcal{F}_t\right],$$

among all α such that the state process Ξ is given by $d\Xi_t = \alpha_t dt$, with fixed Ξ_0. On the set $\{\xi = +\infty\}$, to have a finite cost, the terminal value Ξ_T should be equal to zero. Hence the full liquidation studied in [5] corresponds to the case $\xi = +\infty$ a.s. The value function of this control problem, together with an optimal control, are given by the solution (Y, Z) of a BSDE with terminal value ξ (see BSDE (2.28)) with the limit behavior (2.1). The lack of continuity at time T is interpreted as an extra cost due to the liquidation constraint. In the Markovian setting, we prove that there is no additional cost to minimize the control problem.

In Section 2.3.3, we also apply our result to the related PDE:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}(u)(t,x) + F(t,x,u(t,x),(\nabla u\sigma)(t,x)) = 0, \quad \forall (t,x) \in [0,T]\times\mathbb{R}^m \\ \\ u(T,x) = g(x), \quad \forall x \in \mathbb{R}^m, \end{cases} \tag{2.7}$$

where $\mathcal{L}$ is the infinitesimal generator of the SDE (2.4):

$$\mathcal{L}(u) = \langle b, \nabla u\rangle + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^*\nabla^2 u\right). \tag{2.8}$$

The singular case $\{g = +\infty\} \neq \emptyset$ has been studied in [31, 61, 65] when $F(s,x,y,z) = -y|y|^{q-1}$, but not when F depends on z. In [75], F could depend on z but only for large values of q (see [75, Theorem 2]). This paper fills the gap if (2.6) holds. In other words the minimal viscosity solution u satisfies the continuity condition

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0).$$

In the appendix 2.4, we set out all technical inequalities required in the proof of our main result.

Notations In this chapter we consider a deterministic time horizon $T \in \mathbb{R}_+^*$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a d-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ defined on the probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ the augmented filtration generated by W. For all $p \in [1, +\infty)$, we remind:

- If $p \geq 2$, $\mathbb{D}^{1,p}$ is the domain of the Malliavin derivative operator in $L^p(\Omega)$. Furthermore we note $\mathbb{D}^{1,\infty} = \bigcap_{p\geq 2} \mathbb{D}^{1,p}$. For $A \in \mathbb{D}^{1,p}$ we note $(D_\theta A)_{0\leq\theta\leq T}$ its Malliavin derivative and for X a $\mathbb{D}^{1,p}$-process we note $(D_\theta X_t)_{0\leq\theta,t\leq T}$.
- $S^p((0,T), \mathbb{R}^k)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |A_t|^p\right] < +\infty$$

  and $S^\infty((0,T), \mathbb{R}^k) = \bigcap_{p\geq 1} S^p((0,T), \mathbb{R}^k)$.
- $H^p((0,T), \mathbb{R}^k)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that

$$\mathbb{E}\left[\left(\int_0^T |A_t|^2 dt\right)^{\frac{p}{2}}\right] < +\infty$$

  and $H^\infty((0,T), \mathbb{R}^k) = \bigcap_{p\geq 1} H^p((0,T), \mathbb{R}^k)$.
- Whenever the notation $T-$ appears in the definition of a process space, we mean the set of all processes whose restrictions satisfy the respective property when $T-$ is replaced by any $T - \varepsilon$, $\varepsilon > 0$. For example,

$$S^p((0,T-), \mathbb{R}^k) = \bigcap_{\varepsilon>0} S^p((0,T-\varepsilon), \mathbb{R}^k).$$

  Moreover we say that a sequence $(F_n)_{n\in\mathbb{N}}$ converges towards $F \in S^p((0,T-), \mathbb{R}^k)$ in $S^p((0,T-), \mathbb{R}^k)$ if for any $\varepsilon > 0$, the sequence $(F_n)_{n\in\mathbb{N}}$ converges to F in $S^p((0,T-\varepsilon), \mathbb{R}^k)$.

In the rest of the chapter, C denotes a generic constant, which can depend on other coefficients, and may change from line to line.

2.1 Setting and known results

2.1.1 SDE with control of the density

To ensure existence and uniqueness of the solution X of the SDE (2.4), we suppose that the parameters b and σ satisfy the next conditions.

Assumption 7.

1. *σ is bounded and continuous on $[0,T]\times\mathbb{R}^m$ and of class C^2 with respect to x with bounded first derivatives and bounded second derivatives of $\sigma\sigma^*$:*

$$\forall i,j \in \{1,...,m\}, \quad \frac{\partial\sigma}{\partial x_i}, \frac{\partial^2(\sigma\sigma^*)}{\partial x_i \partial x_j} \in L^\infty([0,T]\times\mathbb{R}^m).$$

2. *$\sigma\sigma^*$ is uniformly λ-elliptic: there exists $\lambda > 0$ such that*

$$\forall s \in [0,T],\ \forall (x,y) \in \mathbb{R}^m \times \mathbb{R}^m, \quad \langle \sigma(s,x)\sigma^*(s,x)y, y\rangle \geq \lambda |y|^2.$$

3. *b is bounded and continuous on $[0,T]\times\mathbb{R}^m$ and of class C^1 with respect to x with polynomial growth derivatives: there exist $\ell \in [1,+\infty)$ and $C \geq 0$ such that*

$$\forall s \in [0,T], \forall x \in \mathbb{R}^m, \quad \left|\frac{\partial b}{\partial x_i}(s,x)\right| \leq C(1+|x|^\ell).$$

Under these conditions [2], according to [84, Theorem 6], the SDE (2.4) has a unique solution X in $S^\infty((0,T),\mathbb{R}^m)$. Furthermore we have according to [23]:

Proposition 2.1.1. *The process X admits a probability density p such that:*

1. *On $[\varepsilon,T]\times\mathbb{R}^m$, the density p is continuous with respect to (t,x) with a continuous derivative with respect to x.*
2. *There exists $c \in \mathbb{R}_+^*$ such that, for all $s \in (0,T]$ and $x \in \mathbb{R}^m$,*

$$\frac{1}{cs^{\frac{1}{2}}} \exp\left(-c\frac{|x-x_0|^2}{s}\right) \leq p(s,x) \leq \frac{c}{s^{\frac{1}{2}}} \exp\left(-\frac{|x-x_0|^2}{cs}\right).$$

Remark 2.1.2. *The second property is the well known Aronson's estimate (see [6, 50, 77, 82]). If the function b linearly grows with a second derivative with polynomial growth*

2. Note that boundedness of b is sufficient to get existence and uniqueness.

w.r.t. x, then the result is still true but we have to add a term $\exp(cs|x|^2)$ *in the upper bound and* $\exp(-s|x|^2/c)$ *in lower bound (see [38, Proposition 1.2]). In general, if the function σ is also with linear growth, then the Aronson estimate is not verified. However we only need a positive lower bound for the density to obtain our result. For example if X is a geometric Brownian motion, the probability density is log-normal and thus our result also holds in this case.*

2.1.2 BSDE with singular terminal condition

For the BSDE (2.5), we use classical hypotheses on the generator F.

Assumption 8.

1. *The function F is continuous and monotone with respect to y: there exists $\chi \in \mathbb{R}$ s.t.*
$$\forall (t,x,y,y',z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d,$$
$$(y-y')(F(t,x,y,z) - F(t,x,y',z)) \leq \chi |y-y'|^2.$$
2. *There exists a constant $\ell \in [1,+\infty)$ such that for all $\rho \in \mathbb{N}$, there exists $\widetilde{K}_\rho \geq 0$ such that*
$$\forall t \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d, \quad \sup_{|y|\leq \rho} |F(t,x,y,z)| \leq \widetilde{K}_\rho (1 + |x|^\ell + |z|).$$
3. *The function F is of class C^1 and uniformly Lipschitz continuous in z: there exists $K \geq 0$ such that*
$$\forall s \in [0,T], x \in \mathbb{R}^m, y \in \mathbb{R}, z, z' \in \mathbb{R}^d, \quad |F(s,x,y,z) - F(s,x,y,z')| \leq K|z-z'|.$$

Under this framework, if $\xi = g(X_T) \in L^p(\Omega)$ for some $p > 1$, then there exists a unique solution $(Y,Z) \in S^p((0,T),\mathbb{R}) \times H^p((0,T),\mathbb{R}^d)$ to the BSDE (2.5) (see [19, Theorem 4.2]).

Our goal is to deal with singular terminal condition. We consider the Markovian framework with a terminal condition
$$\xi = g(X_T)$$
where $g : \mathbb{R}^m \to \overline{\mathbb{R}}_+$ is a deterministic measurable function. We assume the next setting for g.

Assumption 9.

1. *The function* $g : \mathbb{R}^m \longrightarrow \overline{\mathbb{R}}_+$ *is measurable.*

2. *The set of singularities* $\mathcal{S} = \{x \in \mathbb{R}^m,\ g(x) = +\infty\}$ *is closed in* $\mathbb{R}^m$*.*

3. *The function* g *is continuously differentiable and Lipschitzian on each*

$$\mathcal{O}_n = \{x \in \mathbb{R}^m,\ g(x) \leq n\}$$

   for every $n \in \mathbb{N}$*.*

4. *The singular terminal condition* $\xi = g(X_T)$ *satisfies a local integrability condition: for all compact set* $\mathcal{K}$ *of* $\mathbb{R}^m \backslash \mathcal{S}$*,*

$$g(X_T)1_{\mathcal{K}}(X_T) \in L^1(\Omega, \mathcal{F}_T).$$

To construct a solution when the terminal condition ξ is not integrable, we proceed by truncation and consider the following BSDE: for any $n \in \mathbb{N}$

$$Y_t^n = \xi^n + \int_t^T F^n(s, X_s, Y_s^n, Z_s^n)ds - \int_t^T Z_s^n dW_s, \quad t \in [0, T] \tag{2.9}$$

with

$$\xi^n = \varphi_n(\xi), \quad F^n(s, x, y, z) = F(s, x, y, z) - F(s, x, 0, 0) + \varphi_n(F(s, x, 0, 0)),$$

where $(\varphi_n)_{n\in\mathbb{N}}$ a non-decreasing sequence of smooth non-decreasing functions such that

$$\forall n \in \mathbb{N}, \quad \forall u \in \mathbb{R}, \quad \varphi_n(u) = \begin{cases} u & \text{if} \quad u \leq n-1 \\ n & \text{if} \quad u \geq n+1 \end{cases}, \quad u \wedge (n-1) \leq \varphi_n(u) \leq u \wedge n. \tag{2.10}$$

Proposition 2.1.3. *Under Conditions* 7*,* 8 *and* 9*, the truncated BSDE (2.9) admits a unique solution* (Y^n, Z^n) *in* $S^p((0,T), \mathbb{R}) \times H^p((0,T), \mathbb{R}^d)$ *for all* $p \in (1, +\infty)$*. Moreover the sequence* Y^n *is non-decreasing and the process* Y^n *is bounded from above: there exists a constant* C *such that for* $m \leq n$

$$\forall t \in [0, T], \quad Y_t^m \leq Y_t^n \leq C(T+1)n.$$

Proof. Existence and uniqueness directly follows from [19, Theorem 4.2]. Indeed $0 \leq \xi^n \leq$

n, thus ξ^n is bounded. For any $s \in [0,T]$,

$$-C(1+|X_s|^\ell) \leq F^n(s,X_s,0,0) = F(s,x,0,0) \wedge n \leq n.$$

Therefore, for any $\rho > 1$,

$$\mathbb{E}\left[\int_0^T |F^n(s,X_s,0,0)|^\rho ds\right] < +\infty.$$

The driver f is Lipschitz with respect to z, continuous and Lispchitz with respect to y. For any $r \in \mathbb{R}_+^*$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \sup_{|y|\leq r} |F^n(t,X_t,y,0) - F^n(t,X_t,0,0)|dt\right] \\
&\mathbb{E}\left[\int_0^T \sup_{|y|\leq r} |F(t,X_t,y,0) - F(t,X_t,0,0)|dt\right] \\
&\leq 2\mathbb{E}\left[\int_0^T \sup_{|y|\leq r} |F(t,X_t,y,0)|dt\right] \\
&\leq 2\widetilde{K}_r\mathbb{E}\left[\int_0^T (1+|X_t|^r)dt\right] \\
&< +\infty.
\end{aligned}$$

Now standard a priori estimate on the solution of a BSDE (see [72, Theorem 5.30]) and the comparison theorem (see [72, Theorem 5.33]) imply that there exists $\widetilde{C} \geq 0$ such that if $m \leq n$, then a.s. for any $t \in [0,T]$

$$Y_t^m \leq Y_t^n \leq \widetilde{C}e^{(\chi+2\widetilde{C}^2)T}n(T+1),$$

where $\widetilde{C}$ depends on the Lipschitz constant of F w.r.t. z. This achieves the proof of this proposition. □

Since Y_t^n is a non-decreasing sequence, its limit Y_t exists. However the upper estimate on Y^n is not sufficient to ensure that Y_t is finite. But let us emphasize that for any n, $Y^0 \leq Y^n \leq Y$ with $Y^0 \in S^\infty((0,T),\mathbb{R})$.

Finally to obtain a suitable a priori estimate on Y^n, we add extra assumptions on F.

Assumption 10.

1. *The functions F satisfies:*

$$\forall s \in [0,T], x \in \mathbb{R}^m, \quad F(s,x,0,0) \geq 0.$$

2. *There exist $q > 1$, a positive process η whose inverse is of polynomial growth, $C \in \mathbb{R}_+^*$ and $\ell \in [1,+\infty)$ such that*

$$\forall s \in [0,T], x \in \mathbb{R}^m, \quad \frac{1}{\eta(s,x)} \leq C(1+|x|^\ell)$$

and

$$\forall y \in \mathbb{R}_+, t \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d, \quad F(t,x,y,z) - F(t,x,0,z) \leq -\eta(t,x)|y|^q.$$

Then we have, according to [76] applied with φ_n (controlled between $\cdot \wedge (n-1)$ and $\cdot \wedge n$) instead of with $\cdot \wedge n$, the following result.

Proposition 2.1.4. *Under Conditions* 7, 8, 9 *and* 10, *the sequence $(Y^n, Z^n)_{n\in\mathbb{N}}$ converges to (Y,Z) in $S^\infty((0,T-),\mathbb{R}) \times H^\infty((0,T-),\mathbb{R}^d)$. The limit (Y,Z) is the minimal supersolution to the BSDE (2.5) on $[0,T[$ in the sense that:*

1. *The couple (Y,Z) belongs to $S^\infty((0,T-),\mathbb{R}) \times H^\infty((0,T-),\mathbb{R}^d)$.*
2. *The process Y is nonnegative.*
3. *For all $0 \leq s \leq t < T$,*

$$Y_s = Y_t + \int_s^t F(r, X_r, Y_r, Z_r)dr - \int_s^t Z_r dW_r.$$

4. *The process Y satisfies the supercondition* (2.3) *on the left at $t = T$: a.s.*

$$\liminf_{t\to T} Y_t \geq \xi.$$

5. *The process (Y,Z) is minimal: if $(\widetilde{Y}, \widetilde{Z})$ satisfies the four previous points, then a.s. for any t, $Y_t \leq \widetilde{Y}_t$.*

Proof. We check that the assumptions of this section holds to apply [76, Theorem 2.6].

- $\xi = g(X_T)$ and $F(t,X_t,0,0)$ are nonnegative and $\mathbb{P}(\xi = +\infty) > 0$.
- The driver F is continuous and monotone with respect to y.

- For all $\rho > 0$,

$$\begin{aligned}
&\sup_{|y|\leq\rho} |F(t,X_t,y,0) - F(t,X_t,0,0)| \\
&= \sup_{|y|\leq\rho} |f(t,X_t,y,0) - f(t,X_t,0,0)| \leq 2 \sup_{|y|\leq\rho} |F(t,X_t,y,0)| \\
&\quad \leq 2K_\rho(1+|X_t|^p) \leq 2K_\rho \left(1 + \sup_{0\leq s\leq T} |X_s|^p\right) \in L^1((0,T)\times\Omega).
\end{aligned}$$

- The driver F is Lipschitz continuous with respect to z.
- The driver F does not depend on a jump term.
- The growth of the driver F with respect to y satisfies for all $y \in \mathbb{R}_+$,

$$\forall y \in \mathbb{R}_+, t \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d, \quad F(t,x,y,z) - F(t,x,0,z) \leq -\eta(t,x)|y|^q.$$

- As the inverse of the process η is of polynomial growth, $X \in S^\infty((0,T),\mathbb{R}^m)$ and according to Assumption 8, we have, writing q^* the Hölder conjugate of q,

$$\mathbb{E}\left[\left(\int_0^T \left(\frac{1}{(q-1)\eta(s,X_s)}\right)^{\frac{1}{q-1}} + (T-s)^{q_*} F(s,X_s,0,0)\right)^\ell ds\right] < +\infty.$$

We deduce the result according to [76, Theorem 2.6]. □

A key step to obtain this result is the existence of a suitable a priori estimate on Y^n.

Proposition 2.1.5. *Under Conditions* 7 *to* 10*, for any $r > 1$, there exists a constant K_r depending on $r > 1$ (and the constants in our assumptions) such that a.s. for any $t \in [0,T]$ and $n \geq 1$:*

$$\begin{aligned}
Y_t^n \leq \frac{1}{(T-t+\frac{1}{n^{q-1}})^{q_*}} \Bigg\{ \frac{1}{n^{q-1}} + K_r \Bigg[\mathbb{E}\Bigg(\int_t^T \Bigg(\left(\frac{q_*-1}{\eta(s,X_s)}\right)^{q_*-1} \\
+ (T-s+1)^{q_*} F(s,X_s,0,0)\Bigg)^r ds \Bigg| \mathcal{F}_t\Bigg)\Bigg]^{\frac{1}{r}} \Bigg\}
\end{aligned} \tag{2.11}$$

where q_* is the Hölder conjugate of q: $\dfrac{1}{q} + \dfrac{1}{q_*} = 1$.

The proof of this proposition is set out in the appendix. As a consequence, the process Y satisfies on $[0,T)$:

$$0 \leq Y_t^n \leq Y_t \leq \frac{K_r}{(T-t)^{q_*}} \left[\mathbb{E} \left(\int_t^T \left(\left(\frac{q_* - 1}{\eta(s, X_s)} \right)^{q_* - 1} \right. \right. \right.$$

$$\left. \left. \left. + (T - s + 1)^{q_*} F(s, X_s, 0, 0) \right)^r ds \middle| \mathcal{F}_t \right) \right]^{\frac{1}{r}}. \qquad (2.12)$$

Remark 2.1.6. *The non-negativity condition in* 10 *on* $F(s, x, 0, 0)$ *can be relaxed. Then the minimal supersolution* Y *is only bounded from below by* $Y^0 \in S^\infty((0, T), \mathbb{R})$*. We give details in Remark 2.3.4.*

As mentioned in the introduction, our aim is to prove that (2.3) holds: a.s. $\liminf_{t \to T} Y_t = \xi$. Note that on the event $\{\xi = +\infty\}$, we directly have $\lim_{t \to T} Y_t = \liminf_{t \to T} Y_t = \xi = +\infty$.

2.2 Main result and its proof

Taking into account:

- the fact that (2.6) rules out the linear case,
- the transformation of the SDE (2.4) when b is not Lipschitz continuous (see Girsanov transformation 2.2.1),

we suppose that the driver of the BSDE (2.5) can be written:

$$F(t, X_t, Y_t, Z_t) = f(t, X_t, Y_t, Z_t) + \langle a(t, X_t), Z_t \rangle \qquad (2.13)$$

where $f : [0, T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and $a : [0, T] \times \mathbb{R}^m \to \mathbb{R}^d$ are measurable deterministic functions. We assume that a is bounded and f satisfies Condition 8 and 10. Hence these two assumptions are also verified by F. Therefore one can easily check that Propositions 2.1.3, 2.1.4, 2.1.5 are still valid. Moreover to obtain the Malliavin differentiability of (Y^n, Z^n), we impose that F is of class C^1 w.r.t. (x, y, z) with derivatives of polynomial growth. In details:

Assumption 11.

1. *The function* f *is of class* C^1 *with respect to* y *and the partial derivative* $\dfrac{\partial f}{\partial y}$ *is locally uniformly bounded:*

$$\forall M \in \mathbb{R}_+^*, \quad \exists C_M \in \mathbb{R}_+^*,$$

$$\forall y \in [-M,M], s \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d,$$
$$\left|\frac{\partial f}{\partial y}(s,x,y,z)\right| \leq C_M.$$

2. *The function f is of class C^1 with respect to x and the partial derivative are locally uniformly polynomial growth with respect to y:*

$$\exists \ell \in [1,+\infty), \quad \forall M \in \mathbb{R}_+^*, \quad \exists C_M \in \mathbb{R}_+^*, \quad \forall y \in [-M,M], s \in [0,T], x \in \mathbb{R}^m, z \in \mathbb{R}^d,$$
$$\left|\frac{\partial f}{\partial x_i}(s,x,y,z)\right| \leq C_M(1+|x|^\ell+|z|^\ell).$$

3. *The function a verifies the same condition as b in Assumption 7, namely a is bounded and of class C^1 with respect to x and the partial derivatives of the function a are polynomial growth: there exists $C \in \mathbb{R}_+^*$ such that*

$$\exists \ell \in [1,+\infty), \quad \forall s \in [0,T], x \in \mathbb{R}^m, \quad \left|\frac{\partial a}{\partial x_i}(s,x)\right| \leq C(1+|x|^\ell).$$

Remark that since f is Lipschitz continuous w.r.t. z (Assumption 8), its partial derivative w.r.t. z is bounded. Moreover we can choose a constant ℓ as the maximum between the quoted constants in Assumptions 7, 8, 10 and 11. Let us state our main result.

Theorem 2.2.1. *Assume that the generator F admits the structure given by* (2.13) *and that Assumptions* 7 *to* 11 *hold. If for $q \leq 3$ there exist $0 \leq \alpha < \dfrac{2(q-1)}{q+1}$, $C \in \mathbb{R}_+^*$ and $\ell \geq 1$ such that*

$$\forall (s,x,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R}^d, \quad |f(s,x,0,z) - f(s,x,0,0)| \leq C(1+|x|^\ell)|z|^\alpha, \tag{2.14}$$

then the minimal supersolution (Y,Z) to the BSDE (2.5) *built in Proposition 2.1.4 satisfies $\mathbb{P}$-almost surely: $\liminf_{t\to T} Y_t = \xi$.*

Remark 2.2.2. *The growth assumption of f w.r.t. z is new and never appears in the existing literature. Note that for $q > 3$, Condition* (2.14) *holds with $\alpha = 1$ from Assumption 8, as in [76]. Let us again emphasize that this growth assumption does not concern the linear part $\langle a, z \rangle$ of F.*

The rest of this section concerns the proof of this result. Evoke that (Y^n, Z^n) is the solution of the BSDE (2.9) and satisfies the properties of Proposition 2.1.3, whereas the limit (Y, Z) is given in Proposition 2.1.4.

2.2.1 Malliavin differentiability of the couple (Y^n, Z^n)

Our main tool is the Malliavin calculus. As explained in the introduction, the Malliavin differentiability of (Y, Z) relies on the differentialbility of X. If b is Lipschitz continuous (or of class C^1 with bounded derivative), we could directly apply Proposition 2.2.3 below to the SDE (2.4) and keep the initial system (2.4)-(2.5).

With our more general setting, we can apply [68, Theorem 3.3], which states that X has a Malliavin derivative $D_\theta X$. But the integrability properties of $D_\theta X$ may be lost and these properties are crucial to obtain the Malliavin derivative of Y^n (see Proposition 2.2.5).

A transformation of the system (2.4)-(2.5)

To circumvent this issue, we modify the system using Girsanov's transformation. From Assumption 7, σ is uniformly elliptic thus $(\sigma\sigma^*)(s, x)$ is invertible for any $s \in [0, T], x \in \mathbb{R}^m$. The SDE (2.4) can be written:

$$\begin{aligned} X_t &= x_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \\ &= x_0 + \int_0^t \sigma(s, X_s) \Big[dW_s + \sigma^*(s, X_s)(\sigma(s, X_s)\sigma^*(s, X_s))^{-1}b(s, X_s)ds\Big] \\ &= x_0 + \int_0^t \sigma(s, X_s)d\widetilde{W}_s. \end{aligned}$$

Moreover $\sigma^*(\sigma\sigma^*)^{-1}b$ is bounded. Hence according to the Girsanov theorem, there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for any $t \in [0, T]$

$$\widetilde{W}_t = W_t + \int_0^t (\sigma^*(\sigma\sigma^*)^{-1})(s, X_s)b(s, X_s)ds$$

defines a $\mathbb{Q}$-Brownian motion. Since $\mathbb{Q}$ and $\mathbb{P}$ are equivalent, $\mathbb{P}$-a.s. convergence is equivalent to a $\mathbb{Q}$-a.s. convergence. In other words (2.3) can be proved $\mathbb{Q}$ or $\mathbb{P}$ almost surely.

Let us emphasize that W and $\widetilde{W}$ generate the same filtration. Indeed, since X is the solution of (2.4), X and $\widetilde{W}$ are adapted to the filtration $\mathbb{F}^W$ of W. By the strong uniqueness of the solution of the SDE $dX = \sigma(\cdot, X)d\widetilde{W}$, X is also $\mathbb{F}^{\widetilde{W}}$ measurable, which means that

$$W_t = \widetilde{W}_t - \int_0^t (\sigma^*(\sigma\sigma^*)^{-1})(s, X_s)b(s, X_s)ds$$

is also measurable w.r.t. $\mathbb{F}^{\widetilde{W}}$. Hence the filtrations coincide.

Concerning the BSDE (2.5), we obtain

$$\begin{aligned}
dY_t &= -F(t, X_t, Y_t, Z_t)dt + \langle(\sigma^*(\sigma\sigma^*)^{-1})(t, X_t)b(t, X_t), Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle \\
&= -f(t, X_t, Y_t, Z_t)dt - \langle\underbrace{a(t, X_t) - (\sigma^*(\sigma\sigma^*)^{-1})(t, X_t)b(t, X_t)}_{=:\widetilde{a}(t,X_t)}, Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle \\
&= -f(t, X_t, Y_t, Z_t)dt - \langle\widetilde{a}(t, X_t), Z_t\rangle dt + \langle Z_t, d\widetilde{W}_t\rangle.
\end{aligned}$$

The term $\widetilde{a} = a - \sigma^*(\sigma\sigma^*)^{-1}b$ satisfies the same assumptions as a and b: $\widetilde{a}$ is bounded and of class C^1 with polynomial growth derivatives.

Hence applying the Girsanov theorem leads to

$$X_t = x_0 + \int_0^t \sigma(s, X_s)d\widetilde{W}_s \tag{2.15}$$

$$\begin{aligned}
Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds + \int_t^T \langle\widetilde{a}(s, X_s), Z_s\rangle ds - \int_t^T \langle Z_s, d\widetilde{W}_s\rangle \\
&= g(X_T) + \int_t^T \widetilde{F}(s, X_s, Y_s, Z_s)ds - \int_t^T \langle Z_s, d\widetilde{W}_s\rangle
\end{aligned} \tag{2.16}$$

with $\widetilde{a} = a - \sigma^*(\sigma\sigma^*)^{-1}b$. Thus, even if it means considering $\mathbb{Q}, \widetilde{W}, \widetilde{a}$ instead of $\mathbb{P}, W, a$, we can assume

$$b = 0.$$

Now we can compute the Malliavin derivative of X. Indeed under Conditions (7) on σ, according to [69, Theorem 2.2.1], we have:

Proposition 2.2.3. *The SDE* (2.15) *admits a unique solution X in $S^\infty((0, T), \mathbb{R}^m)$ such that:*

1. *For all $t \in [0, T]$, $X_t^i \in \mathbb{D}^{1,\infty}$ and for all $p \in [1, +\infty[$,*

$$\sup_{0\le\theta\le t} \mathbb{E}\left[\sup_{\theta\le s\le T} |D_\theta X_s^i|^p\right] < +\infty. \tag{2.17}$$

2. The process DX^i satisfies the linear SDE

$$D_\theta X_t^i = \sigma_i(\theta, X_\theta) + \sum_{j=1}^{d} \sum_{k=1}^{m} \int_\theta^t \frac{\partial \sigma_i^j}{\partial x_k}(s, X_s) D_\theta X_s^k dW_s^j, \quad 0 \le \theta \le t \le T,$$

where we note $\sigma = (\sigma_i^j)_{1 \le i \le m, 1 \le j \le d}$, and $D_\theta X_t^i = 0, \quad 0 \le t < \theta \le T$.

Remark 2.2.4. *When b is of class C^1 but with bounded derivatives, the previous result holds for the SDE* (2.4) *with*

$$D_\theta X_t^i = \sigma_i(\theta, X_\theta) + \sum_{k=1}^{m} \int_\theta^t \frac{\partial b_i}{\partial x_k}(s, X_s) D_\theta X_s^k ds + \sum_{j=1}^{d} \sum_{k=1}^{m} \int_\theta^t \frac{\partial \sigma_i^j}{\partial x_k}(s, X_s) D_\theta X_s^k dW_s^j,$$

without needing to assume $b = 0$ by considering $\mathbb{Q}, \widetilde{W}, \widetilde{a}$.

Differentiability for the BSDE

We show the Malliavin differentiability of the couple (Y^n, Z^n), due to [67, Theorem 5.1 and Application 6.1]. The next result is based on Proposition 2.2.3, in particular on the estimates (2.17) on $D_\theta X$.

Proposition 2.2.5. *Under Conditions* 7 *to* 11*, the solution (Y^n, Z^n) of the truncated BSDE (2.9) is in $L^2([0,T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$. Moreover for all $0 \le t < \theta \le T$, $D_\theta Y_t^n = 0$, $D_\theta Z_t^n = 0$ and, for all $0 \le \theta \le t \le T$,*

$$\begin{aligned} D_\theta Y_t^n = D_\theta \xi^n - \int_t^T D_\theta Z_s^n dW_s + \int_t^T \Bigg(& \frac{\partial F^n}{\partial y}(s, X_s, Y_s^n, Z_s^n) D_\theta Y_s^n \\ & + \sum_{i=1}^{d} \frac{\partial F^n}{\partial z_i}(s, X_s, Y_s^n, Z_s^n) D_\theta Z_s^{i,n} + D_\theta F^n(s, X_s, Y_s^n, Z_s^n) \Bigg) ds. \end{aligned} \tag{2.18}$$

Proof. Note that $0 \le Y_t^n \le Cn(T+1)$. Indeed Proposition 2.1.3 holds if a is replaced by $\widetilde{a}$ (only the constant C is modified). Thus we can assume that the driver F^n admits a bounded partial derivative w.r.t. y. Indeed with assumptions 11, we can consider a function $\widetilde{f}$ with bounded partial derivative w.r.t. y and which coincides with the function f for $y \in [0, Cn(T+1)]$. Replacing f by $\widetilde{f}$ in (2.9) leads to the same solution (Y^n, Z^n). In this case we deduce the assumptions of the application 6.1 of [67]: (A_1) due to 7.1-2,

$(A_2)(i)$ due to 9.3 and $(A_2)(ii)$ due to 8.3, 11 and because

$$\mathbb{E}\left[\int_0^T |F^n(t, X_t, 0, 0)|^2 dt\right] \leq \mathbb{E}\left[\int_0^T |f(t, X_t, 0, 0) \wedge n|^2 dt\right] \leq Tn^2 < +\infty.$$

□

Then we deduce the representation of Z^n as the Malliavin derivative of Y^n:

Corollary 2.2.6. *We have for any $t \in [0, T]$, $D_t Y_t^n = Z_t^n$.*

Proof. According to [64, Lemma 2.4]:

$$D_\theta Y_t^n = \nabla Y_t^n (\nabla X_\theta)^{-1} \sigma(\theta, X_\theta) 1_{\{\theta \leq t\}},$$

where $\nabla X, \nabla Y^n, \nabla Z^n$ are the notations of the variational equation associated to the FBSDE on $[t, T]$

$$\begin{cases} X_s^{t,x} &= x + \int_t^s \sigma(r, X_r) dW_r \\ Y_s^{n,t,x} &= g_n(X_T^{t,x}) + \int_s^T F^n(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) dr - \int_s^T Z_r^{n,t,x} dW_r, \end{cases}$$

with $g_n = \varphi_n \circ g$. In other words for each $i \in \{1, \dots, m\}$, $(\nabla_i X, \nabla_i Y^n, \nabla_i Z^n)$ are the solutions of the FBSDE

$$\begin{cases} \nabla_i X_s &= e_i + \sum_{j=1}^d \int_t^s \nabla_x \sigma^j(r, X_r^{t,x}) \nabla_i X_r dW_r^j \\ \nabla_i Y_s^n &= \nabla_x g_n(X_T) \nabla_i X_T + \int_s^T \Big(\nabla_x F^n(r, X_r, Y_r^n, Z_r^n) \\ & \quad + \frac{\partial F^n}{\partial y}(r, X_r, Y_r^n, Z_r^n) + \langle \nabla_z F^n(r, X_r, Y_r^n, Z_r^n), \nabla_i Z_r^n \rangle \Big) dr - \int_s^T \nabla_i Z_r^n dW_r, \end{cases}$$

where $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$. Note that $\nabla_x g_n$ and $\nabla_x F_r^n$ make sense because we have truncated with a smooth function φ_n, and note that ∇X satisfies a linear SDE with initial condition $\nabla X_t = I_m$, thus $\nabla X_s \in GL_m(\mathbb{R})$ for any $s \in [t, T]$, $\mathbb{P}$-a.s. and $(\nabla X_s)^{-1}$ makes sense. Moreover, according to [87, Lemma 5.2.3]:

$$Z_t^n = \nabla Y_t^n (\nabla X_t)^{-1} \sigma(t, X_t).$$

Therefore we deduce the desired result. □

We also deduce the following results from [74, Proposition 16]. The difference with [74] is the dependence with respect to z of the driver F but the proof uses the same arguments. In particular due to the fact that we are in a Markovian framework, the property that $Y_t^n = u^n(t, X_t)$ with a deterministic function u^n (solution of the partial differential equation associated to the truncated BSDE) holds (see [33, Theorem 4.1]).

Corollary 2.2.7. *For all $\varphi \in C_c^2(\mathbb{R}^m)$ (set of functions of class C^2 with compact support), there exists a measurable function $\psi : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ such that, for all $t \in [0,T]$,*

$$\mathbb{E}[|Y_t^n \psi(t, X_t)|] < +\infty, \quad \mathbb{E}[\langle Z_t^n, \nabla\varphi(X_t)\sigma(t, X_t)\rangle] = -\mathbb{E}[Y_t^n \psi(t, X_t)]$$

and the function ψ is given by the formula

$$\begin{aligned}\psi(t,x) = &\sum_{i=1}^d (\nabla\varphi\sigma)_i(t,x) \frac{div(p\sigma_i)(t,x)}{p(t,x)} + tr\Big(\nabla^2\varphi(x)(\sigma\sigma^*)(t,x)\Big) \\ &+ \sum_{i=1}^d \langle \nabla\varphi(x), ((\nabla\sigma_i)\sigma_i)(t,x)\rangle,\end{aligned}$$

with $p(t,\cdot)$ the density of X_t.

Corollary 2.2.8. *For all $\varphi \in C_c^2(\mathbb{R}^m)$, there exists a measurable function $\overline{\psi} : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ such that, for all $t \in [0,T]$,*

$$\mathbb{E}[|Y_t^n \overline{\psi}(t, X_t)|] < +\infty, \quad \mathbb{E}\left[\varphi(X_t)\langle a(t,X_t), Z_t^n\rangle\right] = -\mathbb{E}\left[Y_t^n \overline{\psi}(t, X_t)\right]$$

and the function $\overline{\psi}$ is given by

$$\begin{aligned}\overline{\psi}(t,x) = \sum_{i=1}^d \Bigg(&\varphi(x) a_i(t,x) \frac{div(p\sigma_i)(t,x)}{p(t,x)} + a_i(t,x)\langle \nabla\varphi(x), \sigma_i(t,x)\rangle \\ &+ \varphi(x)\langle \nabla a_i(t,x), \sigma_i(t,x)\rangle\Bigg).\end{aligned}$$

From now on and in the rest of this part 2.2, we work either with (2.4)-(2.5) under $\mathbb{P}$ or with (2.15)-(2.16) under $\mathbb{Q}$, with the same notations, so that the statements of Corollaries 2.2.7 and 2.2.8 hold.

2.2.2 Central equation

To study the limit behavior of the process Y at time T, we consider the term $\varphi(X_t)Y_t^n$ for every function φ regular with support included in the complementary of the singular set $\mathcal{S}$ and we study the behavior at time T of this term. We suppose w.l.o.g. that f satisfies Conditions 8 with $\chi = 0$, that is f is non-increasing w.r.t. y (see [56, Remark 1] or Lemma 1.1.15).

First we use Itô's formula and the previous corollaries to deduce:

Proposition 2.2.9. *Under Conditions* 7 *to* 11*, for all* $\varphi \in C_c^2(\mathbb{R}^m)$*, we have for any* n *and any* t

$$\begin{aligned} &\mathbb{E}[\varphi(X_T)Y_T^n] - \mathbb{E}[\varphi(X_t)Y_t^n] + \mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right] \\ &= \mathbb{E}\left[\int_t^T Y_s^n \overline{\Psi}(s,X_s)ds\right] - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right] \\ &\quad - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right], \end{aligned} \tag{2.19}$$

with

$$\overline{\Psi}(t,x) = \Psi(t,x) + \overline{\psi}(t,x) = \mathcal{L}(\varphi)(s,x) - \psi(t,x) + \overline{\psi}(t,x),$$

ψ *and* $\overline{\psi}$ *being given in Corollaries 2.2.7 and 2.2.8, and* $\mathcal{L}$ *by* (2.8) *with* $b = 0$.

Proof. We have, thanks to the Itô formula,

$$\begin{aligned} Y_T^n\varphi(X_T) &= Y_t^n\varphi(X_t) + \int_t^T Y_s^n \mathcal{L}(\varphi)(s,X_s)ds + \sum_{i=1}^m\sum_{j=1}^d \int_t^T Y_s^n \frac{\partial \varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)dW_s^j \\ &\quad - \int_t^T \varphi(X_s)F^n(s,X_s,Y_s^n,Z_s^n)ds + \int_t^T \varphi(X_s)Z_s^n dW_s \\ &\quad + \sum_{i=1}^m\sum_{j=1}^d \int_t^T Z_s^{j,n}\frac{\partial \varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)ds. \end{aligned}$$

But the appearing stochastic integrals are true martingales because $\dfrac{\partial \varphi}{\partial x_i}, \varphi$ and σ are bounded and $(Y^n, Z^n) \in S^2((0,T),\mathbb{R})\times H^2((0,T),\mathbb{R}^d)$. Thus, by applying the expectation,

$$\begin{aligned}
\mathbb{E}[Y^n_T \varphi(X_T)] &= \mathbb{E}[Y^n_t\varphi(X_t)] + \mathbb{E}\left[\int_t^T Y^n_s \mathcal{L}(\varphi)(s,X_s)ds\right] \\
&\quad -\mathbb{E}\left[\int_t^T \varphi(X_s)F^n(s,X_s,Y^n_s,Z^n_s)ds\right] \\
&\quad +\sum_{i=1}^m\sum_{j=1}^d \mathbb{E}\left[\int_t^T Z^{j,n}_s \frac{\partial \varphi}{\partial x_i}(X_s)\sigma_{i,j}(s,X_s)ds\right].
\end{aligned}$$

Furthermore, due to the Fubini theorem and to Corollary 2.2.7,

$$\begin{aligned}
\mathbb{E}[Y^n_T \varphi(X_T)] &= \mathbb{E}[Y^n_t\varphi(X_t)] + \mathbb{E}\left[\int_t^T Y^n_s \mathcal{L}(\varphi)(s,X_s)ds\right] \\
&\quad -\mathbb{E}\left[\int_t^T \varphi(X_s)F^n(s,X_s,Y^n_s,Z^n_s)ds\right] - \mathbb{E}\left[\int_t^T Y^n_s \psi(s,X_s)ds\right].
\end{aligned}$$

We arrive at the equality

$$\begin{aligned}
\mathbb{E}[Y^n_T \varphi(X_T)] &= \mathbb{E}[Y^n_t\varphi(X_t)] - \mathbb{E}\left[\int_t^T \varphi(X_s)F^n(s,X_s,Y^n_s,Z^n_s)ds\right] \\
&\quad +\mathbb{E}\left[\int_t^T Y^n_s \Psi(s,X_s)ds\right],
\end{aligned}$$

with $\Psi(s,x) = \mathcal{L}(\varphi)(s,x) - \psi(s,x)$. But we also have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T \varphi(X_s)F^n(s,X_s,Y^n_s,Z^n_s)ds\right] \\
&= \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y^n_s,Z^n_s) - f(s,X_s,0,Z^n_s))ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z^n_s) - f(s,X_s,0,0))ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T \varphi(X_s)\langle a(s,X_s), Z^n_s\rangle ds\right] \\
&\quad + \mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right],
\end{aligned}$$

with, according to Corollary 2.2.8 and the Fubini theorem,

$$\mathbb{E}\left[\int_t^T \varphi(X_s)\langle a(s,X_s), Z_s^n\rangle ds\right] = -\mathbb{E}\left[\int_t^T Y_s^n \overline{\psi}(s,X_s)ds\right].$$

Therefore

$$\begin{aligned}
&\mathbb{E}[\varphi(X_T)Y_T^n] - \mathbb{E}[\varphi(X_t)Y_t^n] + \mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right] \\
&= \mathbb{E}\left[\int_t^T Y_s^n \overline{\Psi}(s,X_s)ds\right] - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right] \\
&\quad - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right],
\end{aligned}$$

with $\overline{\Psi} = \Psi + \overline{\psi}$, which achieves the proof. □

2.2.3 Control of the different terms in the central equation

The set $\mathcal{S}$ is closed (Assumption 9), so $\mathcal{S}^c = \{x \in \mathbb{R}^m,\ g(x) < +\infty\}$ is open. We consider any $\zeta \in C_c^2(\mathbb{R}^m)$ such that $0 \leq \zeta \leq 1,\ \zeta_{|\mathcal{S}} = 0$ and $\varphi = \zeta^\beta$ with

$$\beta \geq 2q_* = \frac{2q}{q-1} > \frac{q}{q-1} > 1. \tag{2.20}$$

This power β will be usefull when we will have to differentiate the function φ, in particular in Lemma 2.4.5.

Thanks to the study of the behavior of different terms in the central equation (2.19) with the function $\varphi = \zeta^\beta$, we can pass to the limit as n tends to ∞ in Equation (2.19).

Proposition 2.2.10. *Under conditions of Theorem 2.2.1, as n tends to $+\infty$, we obtain: for any $t \in (0,T]$*

$$\begin{aligned}
&\mathbb{E}[\varphi(X_T)\xi] - \mathbb{E}[\varphi(X_t)Y_t] + \mathbb{E}\left[\int_t^T \varphi(X_s)f(s,X_s,0,0)ds\right] \\
&= \mathbb{E}\left[\int_t^T Y_s \overline{\Psi}(s,X_s)ds\right] - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s,Z_s) - f(s,X_s,0,Z_s))ds\right] \\
&\quad - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))ds\right].
\end{aligned} \tag{2.21}$$

In this equation, all terms are finite.

Proof. We are going to study each term of the central equation (2.19) given in Proposition 2.2.9:

1. For the terms $\mathbb{E}[\varphi(X_T)Y_T^n]$ and $\mathbb{E}[\varphi(X_t)Y_t^n]$, convergence is obtained by monotone convergence theorem

$$\mathbb{E}[\varphi(X_T)Y_T^n] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_T)\xi], \quad \mathbb{E}[\varphi(X_t)Y_t^n] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}[\varphi(X_t)Y_t].$$

Indeed if $\mathcal{K}$ is the support of φ,

$$0 \le \varphi(X_T)\xi = \varphi(X_T)g(X_T)1_{\{X_T\in\mathcal{K}\}} \le g(X_T)1_{\mathcal{K}}(X_T) \in L^1(\Omega),$$

thanks to Assumption 9, and also $\varphi(X_t)Y_t \in L^1(\Omega)$ because φ is bounded and $Y \in \mathcal{S}^\infty((0,T-),\mathbb{R})$ (see Proposition 2.1.4).

2. For the term $\mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right]$, by construction of the φ_n and monotone convergence,

$$\mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right] \underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left[\int_t^T \varphi(X_s)f(s,X_s,0,0)ds\right].$$

Indeed

$$0 \le \mathbb{E}\left[\int_t^T \varphi(X_s)|f(s,X_s,0,0)|ds\right] \le CT\left(1+\mathbb{E}\left[\sup_{0\le s\le T}|X_s|^\ell\right]\right) < +\infty.$$

3. For the term $\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n)-f(s,X_s,0,0))ds\right]$, we have

$$\begin{aligned}&|\varphi(X_s)(f(s,X_s,0,Z_s^n)-f(s,X_s,0,0))| \\ &\quad = \varphi(X_s)\frac{|f(s,X_s,0,Z_s^n)-f(s,X_s,0,0)|}{|Z_s^n|^\alpha}1_{\{|Z_s^n|\neq 0\}}|Z_s^n|^\alpha.\end{aligned}$$

with, by Condition (2.14) on the function f

$$\frac{|f(s,X_s,0,Z_s^n)-f(s,X_s,0,0)|}{|Z_s^n|^\alpha}1_{\{|Z_s^n|\neq 0\}} \le C(1+|X_s|^\ell).$$

Since φ is bounded, we obtain:

$$|\varphi(X_s)(f(s,X_s,0,Z^n_s) - f(s,X_s,0,0))| \leq C(1+|X_s|^\ell)|Z^n_s|^\alpha. \tag{2.22}$$

Thus, according to Proposition 2.2.3 and Lemma 2.4.4 in the appendix, we deduce that there exist $C \geq 0$ and $\nu > 0$ such that

$$\mathbb{E}\left(\int_0^T |\varphi(X_s)(f(s,X_s,0,Z^n_s) - f(s,X_s,0,0))|^{1+\frac{\nu}{2}}\right) ds \leq C.$$

Indeed we have by Hölder inequality with $p = \dfrac{1+\nu}{1+\frac{\nu}{2}} > 1$

$$\begin{aligned}
&\mathbb{E}\left(\int_0^T |\varphi(X_s)(f(s,X_s,0,Z^n_s) - f(s,X_s,0,0))|^{1+\frac{\nu}{2}} ds\right) \\
&\leq C\mathbb{E}\left(\int_0^T (1+|X_s|^\ell)^{1+\frac{\nu}{2}}|Z^n_s|^{\alpha(1+\frac{\nu}{2})} ds\right) \\
&\leq C\left(\mathbb{E}\left[\int_0^T (1+|X_s|^\ell)^{(1+\frac{\nu}{2})p_*} ds\right]\right)^{\frac{1}{p_*}} \left(\mathbb{E}\left[\int_0^T |Z^n_s|^{\alpha(1+\nu)} ds\right]\right)^{\frac{1}{p}} \\
&\leq C.
\end{aligned}$$

The sequence of processes $(\varphi(X)(f(\cdot,X,0,Z^n) - f(\cdot,X,0,0)))_{n\in\mathbb{N}}$ is bounded in $L^{1+\frac{\nu}{2}}(\Omega \times [0,T])$.

Hence this sequence is uniformly integrable and we can deduce that for any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any n

$$\mathbb{E}\left[\int_{T-\delta_0}^T |\varphi(X_s)(f(s,X_s,0,Z^n_s) - f(s,X_s,0,0))| ds\right] \leq \varepsilon.$$

Furthermore, again with Lemma 2.4.4 the same arguments prove:

$$\mathbb{E}\left[\int_0^T \varphi(X_s)|f(s,X_s,0,Z_s) - f(s,X_s,0,0)| ds\right] \leq C\mathbb{E}\left[\int_0^T |Z_s|^\alpha ds\right] < +\infty.$$

Then there exists $\delta_1 \in (0,\delta_0]$ such that

$$\mathbb{E}\left[\int_{T-\delta_1}^T |\varphi(X_s)(f(s,X_s,0,Z_s) - f(s,X_s,0,0))| ds\right] \leq \varepsilon.$$

Now for any $p > 1$, the sequence $(Z^n)_{n\in\mathbb{N}}$ converges in $H^p((0, T - \delta_1), \mathbb{R}^d)$ to Z (Proposition 2.1.4). Therefore since f is a Lipschitz continuous function w.r.t. z,

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^{T-\delta_1} \varphi(X_s)(f(s, X_s, 0, Z^n_s) - f(s, X_s, 0, 0))ds\right] = \mathbb{E}\left[\int_0^{T-\delta_1} \varphi(X_s)(f(s, X_s, 0, Z_s) - f(s, X_s, 0, 0))ds\right].$$

Hence there exists $N \in \mathbb{N}$ such that for any $n \geq N$

$$\Bigg|\mathbb{E}\left[\int_0^{T-\delta_1} \varphi(X_s)(f(s, X_s, 0, Z^n_s) - f(s, X_s, 0, 0))ds\right] - \mathbb{E}\left[\int_0^{T-\delta_1} \varphi(X_s)(f(s, X_s, 0, Z_s) - f(s, X_s, 0, 0))ds\right]\Bigg| \leq \varepsilon.$$

With $f(s, X_s, 0, 0) = f^0(s)$, we deduce that

$$\Bigg|\mathbb{E}\left[\int_0^{T} \varphi(X_s)(f(s, X_s, 0, Z^n_s) - f^0(s))ds\right] - \mathbb{E}\left[\int_0^{T} \varphi(X_s)(f(s, X_s, 0, Z_s) - f^0(s))ds\right]\Bigg| \leq 3\varepsilon$$

Thus

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^{T} \varphi(X_s)(f(s, X_s, 0, Z^n_s) - f^0(s))ds\right] = \mathbb{E}\left[\int_0^{T} \varphi(X_s)(f(s, X_s, 0, Z_s) - f^0(s))ds\right].$$

4. For the term $\mathbb{E}\left[\int_t^T Y^n_s \overline{\Psi}(s, X_s)ds\right]$, we have,

$$Y^n_s \overline{\Psi}(s, X_s) = \left(\eta(s, X_s)^{\frac{1}{q}} Y^n_s \varphi(X_s)^{\frac{1}{q}}\right) \times \left(\overline{\Psi}(s, X_s)\eta(s, X_s)^{-\frac{1}{q}} \varphi(X_s)^{-\frac{1}{q}} 1_{\varphi(X_s)>0}\right).$$

Thus, by Hölder's inequality,

$$\mathbb{E}\left[\int_t^T \left|Y_s^n\overline{\Psi}(s,X_s)\right| ds\right] \le \left(\mathbb{E}\left[\int_t^T \eta(s,X_s)(Y_s^n)^q\varphi(X_s)ds\right]\right)^{\frac{1}{q}} \left(\mathbb{E}\left[\int_t^T \Gamma(s,X_s)ds\right]\right)^{\frac{1}{q*}}, \tag{2.23}$$

with

$$\Gamma(s,x) = |\overline{\Psi}(s,x)|^{q_*}\eta(s,x)^{-\frac{1}{q-1}}\varphi(x)^{-\frac{1}{q-1}}1_{\{\varphi(x)>0\}}. \tag{2.24}$$

Given that the function Ψ involves the density p of the process X which has a singularity in $t=0$, we consider $\varepsilon > 0$ and verifiy that $\Gamma(\cdot, X(\cdot)) \in L^1([\varepsilon,T]\times\Omega)$. The fact to consider $[\varepsilon,T]$ is not a problem because we study the behavior at time T, that is when t tends to T. We have, according to Lemma 2.4.5 in the appendix, for any $t \ge \varepsilon$,

$$\mathbb{E}\left[\int_t^T \Gamma(s,X_s)ds\right] < +\infty.$$

According to the previous points,

$$\begin{aligned}&\mathbb{E}[\varphi(X_T)Y_T^n] - \mathbb{E}[\varphi(X_t)Y_t^n] + \mathbb{E}\left[\int_t^T \varphi(X_s)\varphi_n(f(s,X_s,0,0))ds\right]\\ &+ \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,0,Z_s^n) - f(s,X_s,0,0))ds\right]\end{aligned}$$

is the term of a convergent sequence. Moreover from our equation of interest (2.19), it is equal to

$$-\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right] + \mathbb{E}\left[\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right].$$

Hence there exists a constant C such that

$$\left(\mathbb{E}\left[\int_t^T \Gamma(s,X_s)ds\right]\right)^{\frac{1}{q*}} \le C$$

and

$$-\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right] + \mathbb{E}\left[\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right] \le C.$$

From the second assumption of 10,

$$\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s,X_s,Y_s^n,Z_s^n) - f(s,X_s,0,Z_s^n))ds\right] \leq -\mathbb{E}\left[\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right].$$

Hence for any n and t

$$\mathbb{E}\left[\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right] + \mathbb{E}\left[\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right] \leq C.$$

Let's introduce some notations to understand the behavior of sequences

$$u_n = \mathbb{E}\left[\int_t^T \varphi(X_s)\eta(s,X_s)(Y_s^n)^q ds\right], \quad v_n = \mathbb{E}\left[\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right].$$

We have $u_n + v_n \leq C$, and, by the inequality (2.23) and our choice of C, $|v_n| \leq C u_n^{\frac{1}{q}}$. Thus $u_n \leq C - v_n \leq C + u_n^{\frac{1}{q}}$, i.e. $u_n - u_n^{\frac{1}{q}} \leq C$. But, noting $h_q(x) = x - x^{\frac{1}{q}}$, the set $\left\{x \in \mathbb{R}_+, x - x^{\frac{1}{q}} \leq C\right\} = h_q^{-1}([0,C])$ is bounded. Indeed $\lim_{+\infty} h_q = +\infty$ and $[0,C]$ is compact. Thus $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence by $C \in \mathbb{R}_+^*$. Therefore the sequence

$$\gamma_n = \varphi(X)^{\frac{1}{q}}\eta(\cdot,X)^{\frac{1}{q}}Y^n$$

is bounded in $L^q([0,T]\times\Omega)$ and nondecreasing. Thus we have the convergence of γ_n to γ in $L^q([0,T]\times\Omega)$. But we also have the almost sure convergence: a.s. for all $s \in [0,T]$,

$$\gamma_n = \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s^n \underset{n\to+\infty}{\longrightarrow} \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s.$$

Thus, by limit uniqueness, $\gamma(s) = \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s$. For any $t \geq \varepsilon$,

$$\mathbb{E}\left[\int_t^T Y_s^n\overline{\Psi}(s,X_s)ds\right] = \mathbb{E}\left[\int_t^T \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s^n \frac{\overline{\Psi}(s,X_s)}{\varphi(X_s))^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}}1_{\{\varphi(X_s)>0\}}ds\right]$$

$$\underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left[\int_t^T \varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}Y_s \frac{\overline{\Psi}(s,X_s)}{\varphi(X_s)^{\frac{1}{q}}\eta(s,X_s)^{\frac{1}{q}}}1_{\{\varphi(X_s)>0\}}ds\right] = \mathbb{E}\left[\int_t^T Y_s\overline{\Psi}(s,X_s)ds\right]$$

because

$$\frac{\overline{\Psi}(s,X_s)}{\varphi(X_s)^{\frac{1}{q}}\eta(s)^{\frac{1}{q}}}1_{\{\varphi(X_s)>0\}} \in L^{q_*}([\varepsilon,T]\times\Omega).$$

Indeed from Lemma 2.4.5

$$\left| \frac{\overline{\Psi}(s, X_s)}{\varphi(X_s)^{\frac{1}{q}} \eta(s)^{\frac{1}{q}}} \mathbf{1}_{\{\varphi(X_s)>0\}} \right|^{q_*} = \Gamma(s, X_s) \in L^1([\varepsilon, T] \times \Omega).$$

For the remaining term $\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, Y_s^n, Z_s^n) - f(s, X_s, 0, Z_s^n))ds\right]$, f is supposed to be non-increasing with respect to y (see the beginning of Section 2.2.2) and, by construction of the processes Y^n, the sequence $(Y^n)_{n\in\mathbb{N}^*}$ increasingly converges to Y. Thus the monotone convergence theorem gives us the convergence

$$\begin{aligned} &\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, Y_s^n, Z_s^n) - f(s, X_s, 0, Z_s^n))ds\right] \\ &\underset{n\to+\infty}{\longrightarrow} \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, Y_s, Z_s) - f(s, X_s, 0, Z_s))ds\right]. \end{aligned}$$

Coming back to the central equation (2.19), we also deduce the convergence of the sequence

$$\begin{aligned} &\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, Y_s^n, Z_s^n) - f(s, X_s, 0, Z_s^n))ds\right] \\ &\underset{n\to+\infty}{\longrightarrow} -\mathbb{E}[\varphi(X_T)\xi] + \mathbb{E}[\varphi(X_t)Y_t] - \mathbb{E}\left[\int_t^T \varphi(X_s) f(s, X_s, 0, 0)ds\right] \\ &\qquad + \mathbb{E}\left[\int_t^T Y_s \overline{\Psi}(s, X_s)ds\right] \\ &\qquad - \mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, 0, Z_s) - f(s, X_s, 0, 0))ds\right] \end{aligned}$$

where the limit is finite due to the previous estimates, which achieves the proof of this proposition. □

Remark 2.2.11. *The Malliavin calculus allows us to control each linear term which involves $Z_t^n = D_t Y_t^n$. Even if $a = 0$, Z^n appears in the cross variation. For the term with the increment of f with respect to z*

$$\mathbb{E}\left[\int_t^T \varphi(X_s)(f(s, X_s, 0, Z_s^n) - f(s, X_s, 0, 0))ds\right],$$

even if we could linearize it (see Section 2.4.1 and the proof of Proposition 2.1.5 in the Appendix) and get $\langle l_s^n, Z_s^n \rangle$, we don't have an expression of the Malliavin derivative of l^n.

This is the reason why we add this regularity condition (2.14)*.*

2.2.4 Conclusion about the BSDE

The different terms in Equation (2.21) of Proposition 2.2.10 are integrable. Thus, when t tends to T, we obtain

$$\mathbb{E}[\varphi(X_T)\xi] = \lim_{t\to T}\mathbb{E}[\varphi(X_t)Y_t].$$

So, according to the Fatou lemma and Proposition 2.1.4,

$$\mathbb{E}[\varphi(X_T)\xi] \geq \mathbb{E}\left[\liminf_{t\to T}\varphi(X_t)Y_t\right] = \mathbb{E}\left[\varphi(X_T)\liminf_{t\to T}Y_t\right] \geq \mathbb{E}[\varphi(X_T)\xi].$$

Therefore inequalities above are equalities. Thus for every function φ whose support is included in $\{\xi < +\infty\}$, we have

$$\varphi(X_T)\liminf_{t\to T}Y_t = \xi\varphi(X_T).$$

Thus, on $\{\xi < +\infty\}$, $\liminf_{t\to T} Y_t = \xi$. Since we already know that this equality holds on $\{\xi = +\infty\}$, this achieves the proof of Theorem 2.2.1.

2.3 Corollaries and applications

2.3.1 General comparison theorem and generalization of Theorem 2.2.1

We can consider two singular terminal conditions ξ_1 and ξ_2 and two drivers $F_1, F_2 : [0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ which satisfy comparison inequalities. Thus we obtain this following corollary about the associated minimal solutions.

Proposition 2.3.1. *We assume that the Asumptions* 8 *and* 10 *hold for the two different drivers* F_1, F_2 *instead of the driver* F*. We also suppose that the terminal conditions are non-negative and that the following inequalities hold:*

$$\xi_1 \leq \xi_2, \tag{2.25}$$

$$\forall (t,x,y,z)\in[0,T]\times\mathbb{R}^m\times\mathbb{R}\times\mathbb{R}^d, \quad F_1(t,x,y,z)\leq F_2(t,x,y,z). \tag{2.26}$$

Thus the minimal solutions $(Y^1, Z^1), (Y^2, Z^2)$ *(in the sense of Proposition 2.1.4), associated to the BSDE* (2.5) *with respectively the paramaters* $(F_1, \xi_1), (F_2, \xi_2)$*, satisfy the comparison principle: a.s.*

$$\forall t \in [0, T], \quad Y_t^1 \leq Y_t^2. \tag{2.27}$$

Proof. Let us consider $(Y^{1,n}, Z^{1,n})$ the solution of the BSDE with terminal condition $\xi_1^n = \varphi_n(\xi_1)$ and generator

$$F_1^n(s, x, y, z) = F_1(s, x, y, z) - F_1(s, x, 0, 0) + \varphi_n(F_1(s, x, 0, 0))$$

where the functions (φ_n) are given by (2.10). For all $\varepsilon > 0$ and $0 \leq t \leq T - \varepsilon < T$, we have

$$\begin{aligned} Y_t^2 - Y_t^{1,n} = Y_{T-\varepsilon}^2 - Y_{T-\varepsilon}^{1,n} &+ \int_t^{T-\varepsilon} F_2(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^2, Z_s^2) ds \\ &+ \int_t^{T-\varepsilon} F_1(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^{1,n}, Z_s^{1,n}) ds \\ &+ \int_t^{T-\varepsilon} F_1(s, X_s, 0, 0) - \varphi_n(F_1(s, X_s, 0, 0)) ds - \int_t^{T-\varepsilon} Z_s^2 - Z_s^{1,n} dW_s. \end{aligned}$$

Note that we can split the next term:

$$\begin{aligned} &\int_t^{T-\varepsilon} F_1(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^{1,n}, Z_s^{1,n}) ds \\ &= \int_t^{T-\varepsilon} F_1(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^{1,n}, Z_s^2) ds \\ &\quad + \int_t^{T-\varepsilon} F_1(s, X_s, Y_s^{1,n}, Z_s^2) - F_1(s, X_s, Y_s^{1,n}, Z_s^{1,n}) ds. \end{aligned}$$

and use a classical linearization trick (see among other [87, Theorem 4.2.3])

$$\begin{aligned} &\int_t^{T-\varepsilon} F_1(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^{1,n}, Z_s^{1,n}) ds \\ &= \int_t^{T-\varepsilon} \alpha_s^n (Y_s^2 - Y_s^{1,n}) ds + \int_t^{T-\varepsilon} \beta_s^n (Z_s^2) - Z_s^{1,n}) ds. \end{aligned}$$

From our assumption 8, β^n is a bounded process (by K, uniformly in n) and α^n is bounded from above (by χ, uniformly in n). Using the expression of the solution of a linear BSDE

([87, Proposition 4.2.1]), we obtain

$$\begin{aligned} Y_t^2 - Y_t^{1,n} = \mathbb{E}\Bigg[& (Y_{T-\varepsilon}^2 - Y_{T-\varepsilon}^{1,n})\Gamma^n(t, T-\varepsilon) \\ & + \int_t^{T-\varepsilon} \left(F_2(s, X_s, Y_s^2, Z_s^2) - F_1(s, X_s, Y_s^2, Z_s^2) \right) \Gamma^n(t,s) ds \Bigg| \mathcal{F}_t \Bigg] \\ & + \mathbb{E}\left[\int_t^{T-\varepsilon} \left(F_1(s, X_s, 0, 0) - \varphi_n(F_1(s, X_s, 0, 0))\right) \Gamma^n(t,s) ds \Bigg| \mathcal{F}_t \right] \end{aligned}$$

with

$$\Gamma^n(t,s) = \exp\left(\int_t^s \left(\alpha_u^n - \frac{1}{2}(\beta_u^n)^2 \right) du + \int_t^s \beta_u^n dW_u \right).$$

From the definition of φ_n, the last term is non-negative. From our assumption on F_1 and F_2, the last but one term is also non-negative. Hence

$$Y_t^2 - Y_t^{1,n} \geq \mathbb{E}\left[(Y_{T-\varepsilon}^2 - Y_{T-\varepsilon}^{1,n})\Gamma^n(t, T-\varepsilon) \Big| \mathcal{F}_t \right].$$

From Proposition 2.1.3, we know that

$$Y_{T-\varepsilon}^2 - Y_{T-\varepsilon}^{1,n} \geq -Cn(T+1)$$

since Y^2 is non-negative. Γ^n is non-negative and bounded from above by

$$\zeta^n(t,s) = \exp\left(-\int_t^s \frac{1}{2}(\beta_u^n)^2 du + \int_t^s \beta_u^n dW_u \right)$$

which belongs to any $S^p((t,T),\mathbb{R})$, $p > 1$ (see [79]). Hence we can use Fatou's lemma to deduce that

$$Y_t^2 - Y_t^{1,n} \geq \mathbb{E}\left[\liminf_{\varepsilon\to 0} (Y_{T-\varepsilon}^2 - Y_{T-\varepsilon}^{1,n})\Gamma^n(t,T) \Big| \mathcal{F}_t \right] \geq \mathbb{E}\left[(\xi_2 - \varphi_n(\xi_1))\Gamma^n(t,T) \Big| \mathcal{F}_t \right] \geq 0.$$

We used the fact that a.s.

$$\liminf_{t\to T} Y_t^2 \geq \xi_2, \quad \lim_{t\to T} Y_t^{1,n} = \varphi_n(\xi_1).$$

Hence a.s. $Y^2 - Y^{1,n} \geq 0$. Since this inequality holds for all n, the same holds for $Y^2 - Y^1$, which achieves the proof.

□

Remark 2.3.2. *The Markovian setting is not used here. This result holds if the generators are defined on* $[0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d$ *and if the generators are singular in the sense of [56]: Condition 8-2 is replaced by: for any* $\rho \geq 0$

$$\mathbb{E}\left[\sup_{|y|\leq\rho} |F_i(t,\omega,y,z) - F_i(t,\omega,0,z)|\right] < +\infty, \quad \mathbb{E}\int_0^T (T-s)^{q_*} F_i(s,\omega,0,0)ds < +\infty.$$

The continuity at time T *is also not necessary to compare the minimal solutions. Hence we do not need all assumptions of Theorem 2.2.1.*

With the previous proposition, we can generalize Theorem 2.2.1. We consider a driver $F : [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ which satisfies the following conditions.

Assumption 12. *The function* F *satisfies Conditions 8 and 10 and there exists* f *and* a *such that*

$$\forall (t,x,y,z) \in [0,T] \times \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^d, \quad F(t,x,y,z) \leq f(t,x,y,z) + \langle a(t,x), z\rangle,$$

where the functions f *and* a *verify Assumptions 8, 10 and 11. Moreover* (2.14) *holds for* f*.*

We can apply Proposition 2.1.4 for the BSDE with driver F and obtain a minimal supersolution (Y^F, Z^F).

Corollary 2.3.3. *Under Assumptions 7, 9 and 12 we have the limit behavior:* $\mathbb{P}$*-almost surely*

$$\liminf_{t\to T} Y_t^F = \xi.$$

Proof. We have, according to Assumption 12 and Proposition 2.3.1, a.s. for any $t \in [0,T]$ $Y_t^F \leq Y_t$. Then $\xi \leq \liminf_{t\to T} Y_t^F \leq \liminf_{t\to T} Y_t = \xi$. The proof is complete. □

Remark 2.3.4. *Instead to suppose* $g \geq 0$ *and* $F(s,x,0,0) \geq 0$*, we can assume that for some* $\rho > 1$*,*

$$\xi^- = (g(X_T))^- \in L^\rho(\Omega), \quad (F(s,X_s,0,0))^- \in L^\rho((0,T)\times\Omega).$$

Indeed we have by Itô's formula

$$(Y_t^n)^- \leq \xi^- - \int_t^T (F(s,X_s,Y_s^n,Z_s^n) - F(s,X_s,0,0))\mathbf{1}_{\{Y_s^n \leq 0\}} ds + \int_t^T F(s,X_s,0,0)^- ds \\ - \int_t^T Z_s^n \mathbf{1}_{\{Y_s^n \leq 0\}} dW_s.$$

Then for any n,

$$(Y_t^n)^- \leq \mathbb{E}\left[\xi^- \mathcal{E}(t,T) + \int_t^T F(s,X_s,0,0)^- \mathcal{E}(t,s) ds \middle| \mathcal{F}_t \right] =: \hat{Y}_t,$$

with $\mathcal{E}(t,T) = \exp\left(\chi(T-t) + K(W_T - W_t) - K^2 \frac{T-t)}{2}\right)$. *Thus* $\hat{Y}_t \in S^{\rho-\varepsilon}((0,T),\mathbb{R})$ *for any* $\varepsilon > 0$ *and* $-\hat{Y}_t \leq Y_t^n \leq Y_t$. *Furthermore we have* $\lim_{t\to T} Y_t^- = \xi^-$, $\liminf_{t\to T} Y_t \geq \xi$. *So we can limit our study to* Y_t^+.

2.3.2 Application to optimal liquidation problem

With the notations of the papers [5, 56] in the Brownian case without jumps, we consider the Markovian BSDE

$$Y_t = \xi - \int_t^T (p-1)\frac{|Y_s|^{q-1}Y_s}{\eta_s^{q-1}} ds + \int_t^T \gamma_s ds + \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{2.28}$$

where $\eta_s = \eta(s,X_s)$ and $\gamma_s = \gamma(s,X_s)$. X still denotes the solution of the SDE (2.4) with Condition 7. Thus, with the assumptions of [5, 56], the driver of this BSDE satisfies Assumption 12. So we have a unique minimal supersolution (Y,Z).

From here we consider the stochastic control problem to minimize the functional

$$J(t,\alpha) = \mathbb{E}\left[\int_t^T (\eta_s|\alpha_s|^p + \gamma_s|\Xi_s|^p)\, ds + \xi|\Xi_T|^p \middle| \mathcal{F}_t\right]$$

over all $\alpha \in \mathcal{A}(t,x)$ where $\mathcal{A}(t,x)$ is the set of admissible controls such that Ξ satisfy the dynamics

$$\Xi_s = x + \int_t^s \alpha_u du \quad t \leq s \leq T, \quad \alpha \in L^1(t,\infty) \text{ a.s.}$$

Note that there is an implicit constraint on Ξ_T: when $\xi = +\infty$, to obtain a finite cost, Ξ_T must be equal to zero. The mandatory liquidation corresponds to the case $\xi = +\infty$ a.s. and is studied in [5].

In [56], it is proved that a minimizer of the functional J is the process Ξ^* given by

$$\Xi^*_s = x \exp\left(- \int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du \right)$$

where (Y, Z) is the minimal supersolution of the BSDE (2.28). Moreover the value function of this control problem is given by $v(t, x) = |x|^p Y_t$.

Evoke that a.s. $\lim_{t\to T} Y_t \geq \xi$ (see Proposition 2.1.4; existence of the limit is given by [76]). This condition is sufficient to solve the control problem. But it means that the value function only satisfies: a.s.

$$\lim_{t\to T} v(t, x) \geq |x|^p \xi.$$

Hence there could be an extra cost, due to the liquidation constrain (or the terminal singularity of the BSDE).

Now if 7, 9 and 12 hold, then Corollary 2.3.3 shows that a.s.

$$\lim_{t\to T} Y_t = \liminf_{t\to T} Y_t = \xi \qquad \text{and} \qquad \lim_{t\to T} v(t, x) = |x|^p \xi.$$

Therefore there is no additional cost to minimize our control problem. This result was already proved in [76] but for large values on q, that is small values on p. Our result states that the equality also holds for any $p > 1$.

Remark 2.3.5. *The Malliavin calculus has been used to control Z and establish the continuity property of Y. In that application to mathematical finance, it could be used to analyze sensitivity with respect to the parameters η and γ. So, it would be natural to study the Malliavin derivability of Y and the convergence of $D.Y^n$ to $D.Y$. This will be the object of further researches.*

2.3.3 Application to partial differential equation

We consider the partial differential equation (2.7) associated to the FBSDE (2.4)-(2.5). where the operator $\mathcal{L}$ is given by (2.8). Thus, according to the article [75], we consider the function

$$u(t, x) = Y_t^{t,x} \quad (t, x) \in [0, T] \times \mathbb{R}^m,$$

where $Y^{t,x}$ is the unique minimal super-solution of the Markovian FBSDE

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dt + \int_t^s \sigma(r, X_r)dW_r & t \le s \le T, \\ Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T F(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr - \int_s^T Z_r dW_r & t \le s \le T. \end{cases}$$

We still assume that 7, 9 and 12 hold (plus some regularity condition w.r.t. t; see Assumption (C) of [75]). According to [75, Theorem 1], the function u is deterministic and a minimal viscosity solution of the PDE (2.7) among all nonnegative solutions satisfying (2.7), with the terminal constraint

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \ge g(x_0).$$

From [75, Theorem 2], if q is sufficiently large (q is defined in Assumption 10), continuity holds:

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0).$$

Roughly speaking the condition is $q > 3$ (see [75, Remark 2]). Here we prove that under our setting, for any $q > 1$, the previous equality holds a.s. The arguments are almost the same as in [75], together with the arguments of the proof of Theorem 2.2.1. Since F depends on the gradient of u, to our best knowledge, this result does not exist in the PDE literature.

2.4 Appendix

2.4.1 A priori estimates of the solution

Proof of Proposition 2.1.5. For $z \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$ we denote by z_i its $i-$th coordinate and ε_i^- the map from $\mathbb{R}^d$ into $\mathbb{R}^d$ such that $(\varepsilon_i^- z)_j = z_j$ if $j \neq i$ and $(\varepsilon_i^-)_j = 0$. Now we set $\varepsilon_0^- z = z$ and define the $\mathbb{R}^d$-valued process l^n by $\forall i \in \{1, \cdots, d\}, \ \forall t \in [0,T]$

$$l_i^n(t) = \frac{f(t, X_t, 0, \varepsilon_0^- \circ \cdots \circ \varepsilon_{i-1}^- Z_t^n) - f(t, X_t, 0, \varepsilon_0^- \circ \cdots \circ \varepsilon_i^- Z_t^n)}{Z_{i,t}^n} 1_{\{Z_{i,t}^n \neq 0\}},$$

so that due to the Lipschitz hypothesis on h (see Assumption 8), l^n is a bounded process and

$$\langle l_t^n, Z_t^n \rangle = f(t, X_t, 0, Z_t^n) - f(t, X_t, 0, 0).$$

The main idea is to linearize the backward equation satisfied by (Y^n, Z^n). To this end, we introduce the process
$$\gamma_t^n = l_t^n + a(t, X_t).$$
Then there exists a constant C which does not depend on n such that $|\gamma_t^n| \leq C$. We consider the driver

$$H^n(t, y, z) = \kappa_t^n - \frac{q_*}{T - t + \frac{1}{n^{q-1}}} y + \langle \gamma_t^n, z \rangle,$$

with
$$\kappa_t^n = \left(\frac{q_* - 1}{\eta(t, X_t)} \right)^{q_*-1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} + \varphi_n(f(t, X_t, 0, 0))$$
and denote by $(\mathcal{Y}^n, \mathcal{Z}^n)$ the solution of the BSDE on $[0, T]$ with driver H^n and terminal condition $\mathcal{Y}_T^n = n$.

Hence $(\mathcal{Y}^n, \mathcal{Z}^n)$ is solution of a linear BSDE, so we have

$$\mathcal{Y}_t^n = \mathbb{E}\left(\Gamma_{t,T}^n n + \int_t^T \Gamma_{t,s}^n \kappa_s^n ds \middle| \mathcal{F}_t \right) = \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} \left[\frac{1}{n^{q-1}} + \mathbb{E}\left(\int_t^T V_{t,s}^n (T - s + \frac{1}{n^{q-1}})^{q_*} \kappa_s^n ds \middle| \mathcal{F}_t \right) \right]$$

where for $t \leq s \leq T$

$$\Gamma_{t,s}^n = \exp\left(- \int_t^s \frac{q_*}{T - u + \frac{1}{n^{q-1}}} du \right) V_{t,s}^n = \left(\frac{T - s + \frac{1}{n^{q-1}}}{T - t + \frac{1}{n^{q-1}}} \right)^{q_*} V_{t,s}^n, \qquad V_{t,s}^n = 1 + \int_t^s V_{t,u}^n \gamma_u^n dW_u.$$

Moreover since $Y_t^n \geq 0$ a.s.

$$\begin{aligned} F^n(t, X_t, Y_t^n, Z_t^n) &= f(t, X_t, Y_t^n, Z_s^n) - f(t, X_t, 0, Z_s^n) + F^n(t, X_t, 0, Z_t^n) \\ &\leq -\eta(t, X_t)(Y_t^n)^q + F^n(t, X_t, 0, Z_t^n) \end{aligned}$$

and, according to the expressions of $F^n, l_t^n, \gamma_t^n, \kappa_t^n$ and H^n,

$$\begin{aligned} F^n(t, X_t, 0, Z_t^n) &= f(t, X_t, 0, Z_t^n) + \langle a(t, X_t), Z_t^n \rangle - f(t, X_t, 0, 0) + \varphi_n(f(t, X_t, 0, 0)) \\ &= \langle l_t^n, Z_t^n \rangle + \langle a(t, X_t), Z_t^n \rangle + \varphi_n(f(t, X_t, 0, 0)) = \langle \gamma_t^n, Z_t^n \rangle + \varphi_n(f(t, X_t, 0, 0)) \\ &= \langle \gamma_t^n, Z_t^n \rangle + \kappa_t^n - \left(\frac{q_* - 1}{\eta(t, X_t)} \right)^{q_*-1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} \\ &= H^n(t, Y_t^n, Z_t^n) + \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y_t^n - \left(\frac{q_* - 1}{\eta(t, X_t)} \right)^{q_*-1} \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}}. \end{aligned}$$

It follows that, with $\beta_t = \dfrac{q_* - 1}{\eta(t, X_t)}$,

$$\begin{aligned} F^n(t, X_t, Y^n_t, Z^n_t) &\leq H^n(t, Y^n_t, Z^n_t) - \eta(t, X_t)(Y^n_t)^q - \frac{\beta_t^{q_*-1}}{(T - t + \frac{1}{n^{q-1}})^{q_*}} + \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y^n_t \\ &\leq H^n(t, Y^n_t, Z^n_t), \end{aligned}$$

the last inequality comes from the Young inequality since

$$\begin{aligned} &\eta(t, X_t)(Y^n_t)^q + \frac{\beta_t^{q_*-1}}{(T - t + \frac{1}{n^{q-1}})^{q_*}} - \frac{q_*}{T - t + \frac{1}{n^{q-1}}} Y^n_t \\ &\quad = \eta(t, X_t) q \left(\frac{(Y^n_t)^q}{q} + \frac{1}{q_*} \left(\frac{q_* - 1}{(T - t + \frac{1}{n^{q-1}})\eta(t, X_t)} \right)^{q_*} - (q_* - 1) \frac{Y^n_t}{(T - t + \frac{1}{n^{q-1}})\eta(t, X_t)} \right) \\ &\quad \geq 0. \end{aligned}$$

The comparison theorem implies $Y^n_t \leq \mathcal{Y}^n_t$ for all $t \in [0, T]$, that is:

$$Y^n_t \leq \frac{1}{(T - t + \frac{1}{n^{q-1}})^{q_*}} \left[\frac{1}{n^{q-1}} + \mathbb{E}\left(\int_t^T V^n_{t,s} (T - s + \frac{1}{n^{q-1}})^{q_*} \kappa^n_s ds \middle| \mathcal{F}_t \right) \right]. \tag{2.29}$$

Recall that $V^n_{t,\cdot}$ belongs to $H^\varrho((0, T), \mathbb{R})$ for $\varrho \geq 1$ and there exists a constant K_ϱ such that a.s. for any n: $\mathbb{E}\left(\int_t^T (V^n_{t,s})^\varrho ds \middle| \mathcal{F}_t \right) \leq C_\varrho$. The process $((T - t + \frac{1}{n^{q-1}})^{q_*} \kappa^n_t)_{\ 0 \leq t \leq T}$ belongs to $H^r((0, T), \mathbb{R})$. Therefore by Hölder inequality we obtain for any $n \geq 1$

$$\begin{aligned} \mathbb{E}\left(\int_t^T V^n_{t,s} (T - s + \frac{1}{n^{q-1}})^{q_*} \kappa^n_s ds \middle| \mathcal{F}_t \right) &\leq K_r \mathbb{E}\left(\int_t^T ((T - s + \frac{1}{n^{q-1}})^{q_*} \kappa^n_s)^r ds \middle| \mathcal{F}_t \right)^{\frac{1}{r}} \\ \leq K_r \mathbb{E}\left(\int_t^T \left(\left(\frac{q_* - 1}{\eta(s, X_s)} \right)^{q_*-1} + (T - s + 1)^{q_*} (F(s, X_s, 0, 0)) \right)^r ds \middle| \mathcal{F}_t \right)^{\frac{1}{r}} \end{aligned}$$

The last inequality comes from the very definition of κ^n. Thus we obtain the upper bound in (2.11).

□

Now we have that

$$\frac{1}{\eta(s, x)} \leq C(1 + |x|^\ell), \quad F(s, x, 0, 0) \leq C(1 + |x|^\ell).$$

Then for any $r > 1$

$$\mathbb{E}\left(\int_t^T \left(\left(\frac{q_* - 1}{\eta(s, X_s)} \right)^{q_*-1} + (T-s+1)^{q_*} F(s, X_s, 0, 0) \right)^r ds \middle| \mathcal{F}_t \right)$$
$$\leq C\mathbb{E}\left(\int_t^T (1 + |X_s|^{r(q_*-1)\ell}) ds \middle| \mathcal{F}_t \right) < +\infty.$$

Hence in (2.11), we can choose any $r > 1$. Also note that for any $\eta > 0$ and $r > 1$

$$\mathbb{E}\left(\int_0^T (T-s)^{-1+\eta} \left(\left(\frac{q_* - 1}{\eta(s, X_s)} \right)^{q_*-1} + (T-s+1)^{q_*} F(s, X_s, 0, 0) \right)^r ds \right) < +\infty. \quad (2.30)$$

Passing though the limit in Estimate (2.11) we deduce (2.12) for $0 \leq t < T$ and $n \geq 1$.

Remark 2.4.1. *When F does not depend on Z, then we can choose $r = K_r = 1$.*

In the sequel let us denote by Ξ the process

$$\Xi_t = \frac{K_r^r}{T-t} \mathbb{E}\left(\int_t^T \left(\left(\frac{q_* - 1}{\eta(s, X_s)} \right)^{q_*-1} + (T-s+1)^{q_*} F(s, X_s, 0, 0) \right)^r ds \middle| \mathcal{F}_t \right).$$

Thus Estimate (2.12) can be written: for all $0 \leq t < T$

$$0 \leq Y_t^n \leq Y_t \leq \frac{1}{(T-t)^{q_* - \frac{1}{r}}} \Xi_t^{\frac{1}{r}}. \quad (2.31)$$

Lemma 2.4.2. *We have for any $\eta > 0$, $\mathbb{E}\left(\int_0^T (T-s)^{-1+\eta} \Xi_s ds \right) < +\infty$.*

Proof. Note that

$$\begin{aligned}
\mathbb{E}(\Xi_s) &= K_r^r (T-s)^{-1} \int_s^T \mathbb{E}\left(\left(\frac{q_* - 1}{\eta(u, X_u)} \right)^{q_*-1} + (T-u+1)^{q_*} F(u, X_u, 0, 0) \right)^r du \\
&= K_r^r (T-s)^{-1} \int_0^T \theta_u \mathbf{1}_{\{u \geq s\}} du
\end{aligned}$$

with the continuous deterministic function

$$\theta_u = \mathbb{E}\left(\left(\frac{q_* - 1}{\eta(u, X_u)} \right)^{q_*-1} + (T-u+1)^{q_*} F(u, X_u, 0, 0) \right)^r \leq C(1 + \mathbb{E}|X_u|^{r(q_*-1)\ell}).$$

Thus for any $s \in [0,T)$

$$\mathbb{E}(\Xi_s) \leq K_r^\tau \sup_{u\in[0,T]} \theta_u = \Theta < +\infty. \tag{2.32}$$

Hence by Fubini's theorem

$$\mathbb{E}\left(\int_0^T (T-s)^{-1+\eta}\Xi_s ds\right) \leq \Theta\int_0^T (T-s)^{-1+\eta} ds = \frac{\Theta}{\eta}T^\eta.$$

This achieves the proof of the lemma. □

Let us now derive an a priori estimate on Z^n.

Lemma 2.4.3. *For $r > 1, \eta > 0$ and $\varpi = 2q_* - \frac{2}{r} + \frac{2\eta}{r} = \frac{2}{q-1} + 2\left(1-\frac{1}{r}\right) + \frac{2\eta}{r}$, there exists a constant C s.t. for any n*

$$\mathbb{E}\left(\left(\int_0^T (T-s)^{\varpi}|Z_s^n|^2 ds\right)^{\frac{r}{2}}\right) \leq C.$$

The same estimate also holds for Z.

Proof. For $\eta > 0$ and $r > 1$, let us define

$$\delta = rq_* - 1 + \eta > 0.$$

We define $c(r) = \frac{r((r-1)\wedge 1)}{2}$ and we apply Itô's formula to $(T-t)^\delta (Y_t^n)^r$ (see [19, Corollary 2.3]). Evoke that Y^n is non-negative. We fix $\varepsilon > 0$ and $\tau = T-\varepsilon$ in the sequel. Hence we have for $0 \leq t \leq \tau$:

$$\begin{aligned}(T-t)^\delta (Y_t^n)^r \leq\ & \varepsilon^\delta (Y_{T-\varepsilon}^n)^r + \int_t^\tau \delta(T-s)^{\delta-1}(Y_s^n)^r ds \\ & + r\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1}\mathbf{1}_{\{Y_s^n\neq 0\}} F(s,X_s,Y_s^n,Z_s^n) ds \\ & - r\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1}\mathbf{1}_{\{Y_s^n\neq 0\}} Z_s^n dW_s \\ & - c(r)\int_t^\tau (T-s)^\delta (Y_s^n)^{r-2}|Z_s^n|^2\mathbf{1}_{\{Y_s^n\neq 0\}} ds. \end{aligned} \tag{2.33}$$

The monotonicity condition implies that

$$\begin{aligned}&\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} 1_{\{Y_s^n\neq 0\}} F(s,X_s,Y_s^n,Z_s^n)ds \\ &\leq \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} 1_{\{Y_s^n\neq 0\}} F(s,X_s,0,Z_s^n)ds\end{aligned}$$

and we use the regularity Lipschitz condition w.r.t. z to obtain:

$$\begin{aligned}&\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} 1_{\{Y_s^n\neq 0\}} F(s,X_s,0,Z_s^n)ds \\ &\leq K\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1}|Z_s^n|ds + \int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} F(s,X_s,0,0)ds.\end{aligned}$$

Young's inequality leads to:

$$\begin{aligned}&rK\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1}|Z_s^n|ds \\ &\leq \frac{K^2r^2}{2c(r)}\int_t^\tau (T-s)^\delta (Y_s^n)^r ds + \frac{c(r)}{2}\int_t^\tau (T-s)^\delta (Y_s^n)^{r-2}|Z_s^n|^2 1_{\{Y_s^n\neq 0\}}ds\end{aligned}$$

and

$$\begin{aligned}&r\int_t^\tau (T-s)^\delta (Y_s^n)^{r-1} F(s,X_s,0,0)ds \\ &\leq (r-1)\int_t^\tau (T-s)^\delta (Y_s^n)^r ds + \int_t^\tau (T-s)^\delta (F(s,X_s,0,0))^r ds.\end{aligned}$$

Finally all local martingales involved above in (2.33) are true martingales. Hence taking the expectation and using the convexity of $x \mapsto |x|^r$ we have:

$$\begin{aligned}&\sup_{t\in[0,\tau]} \mathbb{E}\left((T-t)^\delta (Y_t^n)^r\right) + \frac{c(r)}{2}\mathbb{E}\left(\int_t^\tau (T-s)^\delta (Y_s^n)^{r-2}|Z_s^n|^2 1_{\{Y_s^n\neq 0\}}ds\right) \quad (2.34)\\ &\leq \varepsilon^\delta \mathbb{E}((Y_{T-\varepsilon}^n)^r) + \mathbb{E}\left(\int_t^\tau \delta(T-s)^{\delta-1}(Y_s^n)^r ds\right) \\ &\quad + \left(2\frac{K^2r^2}{2c(r)} + (r-1)\right)\mathbb{E}\left(\int_t^\tau (T-s)^\delta (Y_s^n)^r ds\right) \\ &\quad + \mathbb{E}\left(\int_t^\tau (T-s)^\delta (F(s,X_s,0,0))^r ds\right).\end{aligned}$$

Let us emphasize that this inequality holds with (Y, Z) instead of (Y^n, Z^n) since $\tau = T - \varepsilon < T$.

Using (2.31), the second term on the right-hand side can be controlled as follows:

$$\begin{aligned}\mathbb{E}\left(\int_0^\tau (T-s)^{\delta-1}(Y_s^n)^r ds\right) &\leq \mathbb{E}\left(\int_0^T (T-s)^{\delta-1}\frac{1}{(T-s)^{rq_*-1}}\Xi_s ds\right)\\ &= \mathbb{E}\left(\int_0^T (T-s)^{-1+\eta}\Xi_s ds\right) < +\infty.\end{aligned}$$

The third one satisfies the same estimate:

$$\mathbb{E}\left(\int_0^\tau (T-s)^{\delta}(Y_s^n)^r ds\right) \leq \mathbb{E}\left(\int_0^T (T-s)^{\eta}\Xi_s ds\right) < +\infty.$$

And the last term does not depend on (Y^n, Z^n) and is bounded.

For the first term, since Y^n is bounded by $Cn(T+1)$, we immediately obtain

$$\lim_{\varepsilon\to 0} \varepsilon^\delta \mathbb{E}((Y^n_{T-\varepsilon})^r) = 0.$$

If Y^n is replaced by Y, from (2.32) we have

$$\varepsilon^\delta \mathbb{E}((Y_{T-\varepsilon})^r) \leq \varepsilon^\delta \frac{1}{\varepsilon^{q_* r-1}}\mathbb{E}(\Xi_{T-\varepsilon}) \leq \Theta \varepsilon^{\delta - q_* r+1} = \Theta\varepsilon^\eta$$

and the limit is again equal to zero.

Therefore we can let ε go to zero in (2.34) and we can replace every τ by T:

$$\begin{aligned}&\sup_{t\in[0,T]} \mathbb{E}\left((T-t)^\delta (Y_t^n)^r\right) + \frac{c(r)}{2}\mathbb{E}\left(\int_t^T (T-s)^\delta (Y_s^n)^{r-2}|Z_s^n|^2 \mathbf{1}_{\{Y_s^n\neq 0\}} ds\right)\\ &\leq \mathbb{E}\left(\int_t^T \delta(T-s)^{\delta-1}(Y_s^n)^r ds\right)\\ &\quad + \left(2\frac{K^2r^2}{2c(r)} + (r-1)\right)\mathbb{E}\left(\int_t^T (T-s)^\delta (Y_s^n)^r ds\right)\\ &+\mathbb{E}\left(\int_t^T (T-s)^\delta (F(s,X_s,0,0))^r ds\right).\end{aligned}$$

The same inequality holds with (Y, Z).

Next, by standard arguments, we can control the quantity $\mathbb{E}\left(\sup_{t\in[0,T]}(T-t)^\delta (Y_t^n)^r\right)$ by the same right-hand side (up to some multiplicative constant). Hence there exists C s.t.

for any n:

$$\mathbb{E}\left(\sup_{t\in[0,T]}(T-t)^\delta (Y_t^n)^r\right) + \frac{c(r)}{2}\mathbb{E}\left(\int_t^T (T-s)^\delta (Y_s^n)^{r-2}|Z_s^n|^2 1_{\{Y_s^n\neq 0\}} ds\right) \leq C. \quad (2.35)$$

Then if $r \geq 2$, we use (2.35) with $r = 2$ and the result follows immediately with $\delta = 2q_* - 1 + \eta$. If $1 < r < 2$, the conclusion is more tricky. Let us define $M = \sup_{t\in[0,T]}(T-t)^{\frac{\delta}{r}} Y_t^n$ and:

$$\begin{aligned}
\mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}}|Z_s^n|^2 ds\right)^{\frac{r}{2}}\right) &= \mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}} 1_{\{Y_s^n\neq 0\}}|Z_s^n|^2 ds\right)^{\frac{r}{2}}\right) \\
&= \mathbb{E}\left(\left(\int_0^T (T-s)^{2\frac{\delta}{r}} (Y_s^n)^{2-r} (Y_s^n)^{r-2} 1_{Y_s^n\neq 0}|Z_s^n|^2 ds\right)^{\frac{r}{2}}\right) \\
&\leq \mathbb{E}\left(M^{\frac{(2-r)r}{2}} \left(\int_0^T (T-s)^{\delta} (Y_s^n)^{r-2} 1_{Y_s^n\neq 0}|Z_s^n|^2 ds\right)^{\frac{r}{2}}\right) \\
&\leq \left(\mathbb{E}\left(M^r\right)\right)^{\frac{2-r}{2}} \left(\mathbb{E}\left(\int_0^T (T-s)^{\delta} (Y_s^n)^{r-2} 1_{\{Y_s^n\neq 0\}}|Z_s^n|^2 ds\right)\right)^{\frac{r}{2}} \\
&\leq \frac{2-r}{2}\mathbb{E}\left(M^r\right) + \frac{r}{2}\mathbb{E}\left(\int_0^T (T-s)^{\delta} (Y_s^n)^{r-2} 1_{\{Y_s^n\neq 0\}}|Z_s^n|^2 ds\right) < +\infty.
\end{aligned}$$

where we have used Hölder's and Young's inequality with $\frac{2-r}{2} + \frac{r}{2} = 1$. This achieves the proof of the lemma with $\varpi = \dfrac{2\delta}{r}$. □

Evoke that $0 \leq \alpha \leq 1$.

Lemma 2.4.4. *If $\alpha < \dfrac{2(q-1)}{q+1}$ for $q \leq 3$ and $\alpha \leq 1$ for $q > 3$, there exist $\nu > 0$ and $C > 0$ such that*

$$\forall n \in \mathbb{N}, \quad \mathbb{E}\left[\int_0^T |Z_s^n|^{\alpha(1+\nu)} ds\right] \leq C.$$

The same estimate holds for Z.

Proof. For $r > 1$, $\alpha \leq 1$ and $\nu > 0$, suppose that $\beta = \alpha(1+\nu) < r \wedge 2$. Then:

$$\begin{aligned}
\int_t^T |Z_s^n|^{\alpha(1+\nu)} ds &= \int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2}}} (T-s)^{\frac{\varpi\beta}{2}} |Z_s^n|^\beta ds \\
&\leq \left(\int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds \right)^{\frac{2-\beta}{2}} \left(\int_t^T (T-s)^\varpi |Z_s^n|^2 ds \right)^{\frac{\beta}{2}} \\
&\leq \frac{r-\beta}{r} \left(\int_t^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds \right)^{\frac{(2-\beta)r}{2(r-\beta)}} + \frac{\beta}{r} \left(\int_t^T (T-s)^\varpi |Z_s^n|^2 ds \right)^{\frac{r}{2}}
\end{aligned}$$

with Young's inequality for $\dfrac{\beta}{r} + \dfrac{r-\beta}{r} = 1$. Therefore

$$\mathbb{E}\left(\int_0^T |Z_s^n|^{\alpha(1+\nu)} ds \right) \leq \frac{r-\beta}{r} \left(\int_0^T \frac{1}{(T-s)^{\frac{\varpi\beta}{2-\beta}}} ds \right)^{\frac{(2-\beta)r}{2(r-\beta)}} + \frac{\beta}{r} \mathbb{E} \left(\int_0^T (T-s)^\varpi |Z_s^n|^2 ds \right)^{\frac{r}{2}}.$$

From Lemma 2.4.3, the last term is bounded uniformly in n if for some $\eta > 0$, $\varpi = \dfrac{2}{q-1} + 2\left(1 - \dfrac{1}{r}\right) + \dfrac{2\eta}{r}$. And the first integral is finite if and only if

$$\frac{\varpi\beta}{2-\beta} = \frac{\beta}{2-\beta} \left[\frac{2}{q-1} + 2\left(1-\frac{1}{r}\right) + \frac{2\eta}{r} \right] < 1.$$

But $\alpha < 2\dfrac{q-1}{q+1}$, then $(q+1)\alpha = 2\alpha + \alpha(q-1) < 2(q-1)$, thus $2\alpha < (2-\alpha)(q-1)$, that is $\dfrac{2\alpha}{(q-1)(2-\alpha)} < 1$. Now we can choose $\nu_0 > 0$ such that for any $0 < \nu \leq \nu_0$

$$\frac{\beta}{2-\beta} \frac{2}{q-1} \leq (1+\nu_0) \left(\frac{2-\alpha}{2-\alpha-\alpha\nu_0} \right) \frac{2\alpha}{(q-1)(2-\alpha)} < 1$$

and

$$0 \leq \frac{\alpha}{2-\alpha} \leq \frac{\beta}{2-\beta} = \frac{\alpha}{2-\alpha-\alpha\nu}(1+\nu) \leq \frac{\alpha}{2-\alpha-\alpha\nu_0}(1+\nu_0).$$

We can fix $r > 1$, $0 < \nu < \nu_0$ and $\eta > 0$ such that $r > (1+\nu) \geq (1+\nu)\alpha$ and

$$\frac{\varpi\beta}{2-\beta} = \frac{2\beta}{(q-1)(2-\beta)} + 2\left(1-\frac{1}{r}\right)\frac{\beta}{2-\beta} + \frac{2\eta}{r}\frac{\beta}{2-\beta} < 1.$$

Thus all integrals are finite and the conclusion holds. □

2.4.2 Control in the central equation

Lemma 2.4.5. *For* Γ *given by* (2.24)*, we have:* $\mathbb{E}\left[\int_t^T \Gamma(s,X_s)ds\right] < +\infty$.

Proof. Evoke that from (2.20), $\beta \geq 2q_* = \dfrac{2q}{q-1}$. Moreover the compact support of ζ is denoted by $\mathcal{K}$. Now

$$\begin{aligned}
\Gamma(s,X_s) &\leq C\left(|\Psi(s,X_s)|^{q_*} + |\overline{\psi}(s,X_s)|^{q_*}\right)\left(1+|X_s|^{\frac{\ell}{q-1}}\right)\varphi(X_s)^{-\frac{1}{q-1}} \\
&\leq C\varphi(X_s)^{-\frac{2}{q-1}}|\mathrm{tr}\left(\nabla^2\varphi(X_s)\sigma\sigma^*(s,X_s)\right)|^{2q_*} \\
&\quad + C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d\left|(\nabla\varphi\sigma)_i(s,X_s)\frac{\mathrm{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*} \\
&\quad + C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|\langle\nabla\varphi(X_s),((\nabla\sigma_i)\sigma_i)(s,X_s)\rangle|^{2q_*} \\
&\quad + C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d\left|\varphi(X_s)a_i(s,X_s)\frac{\mathrm{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*} \\
&\quad + C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|a_i(s,X_s)|^{2q_*}|\langle\nabla\varphi(X_s),\sigma_i(s,X_s)\rangle|^{2q_*} \\
&\quad + C\varphi(X_s)^{-\frac{2}{q-1}}\sum_{i=1}^d|\varphi(X_s)|^{2q_*}|\langle\nabla a_i(s,X_s),\sigma_i(s,X_s)\rangle|^{2q_*} + C(1+|X_s|^{\frac{2\ell}{q-1}}) \\
&=: C(A_s^1+A_s^2+A_s^3+A_s^4+A_s^5+A_s^6+A_s^7).
\end{aligned}$$

Evoke that $\varphi = \zeta^\beta$. For $A_s^1 = \varphi(X_s)^{-\frac{2}{q-1}}|\mathrm{tr}\left(\nabla^2\varphi(X_s)\sigma\sigma^*(s,X_s)\right)|^{2q_*}$, we have

$$\begin{aligned}
(\nabla^2\varphi(x))_{ij} &= \frac{\partial^2(\zeta^\beta)}{\partial x_i\partial x_j}(x) = \frac{\partial}{\partial x_i}\left(\frac{\partial(\zeta^\beta)}{\partial x_j}(x)\right) = \frac{\partial}{\partial x_i}\left(\beta\frac{\partial\zeta}{\partial x_j}(x)\zeta^{\beta-1}(x)\right) \\
&= \beta\frac{\partial^2\zeta}{\partial x_i\partial x_j}(x)\zeta^{\beta-1}(x) + \beta(\beta-1)\frac{\partial\zeta}{\partial x_i}(x)\frac{\partial\zeta}{\partial x_j}(x)\zeta^{\beta-2}(x) \\
&= \underbrace{\left(\beta\frac{\partial^2\zeta}{\partial x_i\partial x_j}(x)\zeta(x) + \beta(\beta-1)\frac{\partial\zeta}{\partial x_i}(x)\frac{\partial\zeta}{\partial x_j}(x)\right)}_{=:\widetilde{\zeta}_{ij}(x)}\zeta^{\beta-2}(x),
\end{aligned}$$

with $\widetilde{\zeta}$ bounded. Thus

$$A_s^1 \leq C\zeta(X_s)^{-\frac{2\beta}{q-1}+(\beta-2)2q_*}|\sigma(s,X_s)|^{4q_*} \leq C\zeta(X_s)^{2(\beta-2q_*)}\left(|X_s|^{4q_*}+C\right) \leq C(|X_s|^{4q_*}+C)$$

because $\beta \geq 2q_*$ from (2.20). Therefore

$$\mathbb{E}\left[\int_t^T A_s^1 ds\right] \leq C\left(\mathbb{E}\left[\sup_{0\leq s\leq T} |X_s|^{4q_*}\right] + 1\right) < +\infty.$$

For $A_s^2 = \varphi(X_s)^{-\frac{2}{q-1}} \sum_{i=1}^d \left|(\nabla\varphi\sigma)_i(s,X_s)\frac{\operatorname{div}(p\sigma_i)(s,X_s)}{p(s,X_s)}\right|^{2q_*}$, we have

$$\begin{aligned}
A_s^2 &\leq C\varphi(X_s)^{-\frac{2}{q-1}} \sum_{i=1}^d |\sigma(s,X_s)|^{2q_*} |\nabla\varphi(X_s)|^{2q_*} \frac{|\operatorname{div}(p\sigma_i)(s,X_s)|^{2q_*}}{p(s,x)^{2q_*}} \\
&\leq C\varphi(X_s)^{-\frac{2}{q-1}} \left(\sum_{i=1}^d |\operatorname{div}(p\sigma_i)(s,X_s)|^{2q_*}\right) \frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}} |\nabla\varphi(X_s)|^{q_*} \\
&= C\zeta(X_s)^{2\left(\beta - \frac{q}{q-1}\right)} \left(\sum_{i=1}^d |\operatorname{div}(p\sigma_i)(s,X_s)|^{2q_*}\right) \frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}} |\nabla\zeta(X_s)|^{2q_*} \\
&= C\zeta(X_s)^{2\left(\beta - \frac{q}{q-1}\right)} \left(\sum_{i=1}^d |\operatorname{div}(p\sigma_i)(s,X_s)|^{2q_*}\right) \frac{|\sigma(s,X_s)|^{2q_*}}{p(s,X_s)^{2q_*}} |\nabla\zeta(X_s)|^{2q_*} 1_{\{X_s\in\mathcal{K}\}} \leq C_{\mathcal{K}},
\end{aligned}$$

because $\beta > q_* = \dfrac{q}{q-1}$, all functions are regular and we can control $\dfrac{1}{p}$ due to property 2 of Propositon 2.1.1. Thus $\mathbb{E}\left[\int_t^T A_s^2 ds\right] < +\infty$. The study of terms A_3 to A_6 is similar and A_7 is direct. □

Chapter 3

Malliavin derivative convergence

Optimal liquidation is an important and challenging topic in mathematical finance. There is a huge literature on this subject ; the reference [43] is interesting to have an overall view (see also the references therein). Here we consider the stochastic extension of the Almgren & Chriss model, first exposed in [3]. Namely for some $p > 1$, we consider the stochastic control problem to minimize the functional

$$J(t,\alpha) = \mathbb{E}\left[\int_t^T \left(\eta_s|\alpha_s|^p + \gamma_s|\Xi_s|^p\right) ds \middle| \mathcal{F}_t\right]$$

over all $\alpha \in \mathcal{A}(t,x)$ where $\mathcal{A}(t,x)$ is the set of admissible controls such that Ξ satisfy the dynamics

$$\Xi_s = x + \int_t^s \alpha_u du \quad t \leq s \leq T, \quad \alpha \in L^1(t,\infty) \text{ a.s.}$$

together with the terminal constraint $\Xi_T = 0$ a.s. (mandatory liquidation). The positive process η is a measure of the liquidity of the market ; the quantity $\eta|\alpha|^p$ corresponds to the immediate impact of the trading α on the market price (see [43, Chapter 3.2]). The non-negative process γ is a parameter of the model and the quantity $\gamma|\Xi|^p$ is a risk measure on the size of the portfolio.

This control problem and the link with backward stochastic differential equations (BSDE in abbreviate form) has been studied in [5]. It is proved that a minimizer of the functional J is the process Ξ^* given by

$$\Xi_s^* = x \exp\left(-\int_t^s \left(\frac{Y_u}{\eta_u}\right)^{q-1} du\right) \tag{3.1}$$

where (Y,Z) is the minimal solution of the BSDE

$$Y_t = \xi - \int_t^T (p-1)\frac{|Y_s|^{q-1}Y_s}{\eta_s^{q-1}} ds + \int_t^T \gamma_s ds - \int_t^T Z_s dW_s, \tag{3.2}$$

with singular terminal condition $\xi = +\infty$ a.s. Here and in the rest of the chapter, q is the Hölder conjugate of p. Moreover the value function of this control problem is given by $v(t,x) = |x|^p Y_t$. Remark that due to the singular terminal condition, the standard notion of solution for BSDE has to be adapted (see Proposition 3.1.3 below).

Another way to solve this control problem is the use of the Hamilton-Jacobi-Bellman equation. In [41], it is proved that there exists a smooth solution v of the related partial differential equation (PDE in short). Both approaches are connected through the link: $Y_s = v(s, X_s)$, where X is the underlying forward process.

The aim of the chapter is to study the Malliavin differentiability of the solution Y. Our motivations are pluralist:

- From the theoretical point of view, this question naturally appears in Chapter 2 (our work [21]). We studied the continuity at time T of Y. To obtain our result, we used the Malliavin calculus on the approximating sequence Y^n (see Proposition 3.1.4 below). We left the Malliavin differentiability of Y and the convergence of the Malliavin derivative DY^n out.
- It is also well-known that Malliavin derivatives (and the related divergence operator and integration by parts) can be used to analyze the sensitivity of the optimal state w.r.t. the parameters (see among many others [35, 39]). In our context, since the optimal state is given by (3.1), the Malliavin derivative of Y plays a crucial role.
- In [41] the existence of a solution $v \in C([0,T); D(\mathcal{L}))$ is proved. In dimension one, it implies that $v \in C^2$ (see [41, Remark 2.5]). Nonetheless the gradient of this solution and its behavior at time T is not studied. Formally this gradient is related to the Malliavin derivative by the formula: $\partial_x v(s, X_s)\sigma(s, X_s) = D_s Y_s$, σ being the volatility matrix of the forward process X.

In this work, we prove that Y has a Malliavin derivative DY and that the sequence of Malliavin derivatives DY^n converges to DY on $[0,T)$. We also study the behavior at time T of DY and show that there is a singularity at time T: roughly speaking, $D_\theta Y_t$ tends to $+\infty$ when t tends to T. Applications to PDE or sensitivity analysis are then provided. To the best of our knowledges, all these results are completely new.

We are well aware that some parts of the proofs are quite heavy due to the handling of the function $y \mapsto -y|y|^{q-1}$ and its derivatives. The arguments would be much more easier in the quadratic case, that is for $q = 2$. Nonetheless we also know that in practice the case $q = 2$ is too restrictive. In [4] the authors analyze a large data set from the Citigroup US equity trading desks and show that $q = 1.6$ is a good estimate of this parameter. This is the reason why we keep this general setting.

Breakdown of the chapter. In Section 3.1, we evoke the known results concerning the BSDE (3.2), its approximating sequence Y^n and why the convergence of the sequence of derivatives DY^n cannot be done with the same arguments. In few words, the proof that Y^n converges to Y is heavily based on the non-linearity of the BSDE (3.2). But the equation satisfied by DY^n is linear. In Section 3.2, the asymptotic expansion of Y allows us to show the differentiability of Y and the behavior of its derivative. The next section 3.3 studies the convergence of DY^n to DY. For the liquidation problem such convergence seems quite incidental and the reader interested by the applications can skip it. In Section 3.4, we explain how to apply the results to the gradient of the related PDE and to the sensitivity. In the last section, for the sake of completeness, we give the proof for the asymptotic behavior of Y, which is the key tool of this chapter. If most of the arguments can be found in [42], we simplify them and give a better estimate of the parameters.

Notations In this chapter we consider a deterministic time horizon $T \in \mathbb{R}_+^*$, a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a d-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ defined on the probability space and $(\mathcal{F}_t)_{t\in[0,T]}$ the augmented filtration generated by W. For all $\rho \in (1, +\infty)$, we remind:

- $\mathbb{D}^{1,\rho}$ is the domain of the Malliavin derivative operator in $L^\rho(\Omega)$. Furthermore we note $\mathbb{D}^{1,\infty} = \bigcap\limits_{\rho\geq 2} \mathbb{D}^{1,\rho}$. For $A \in \mathbb{D}^{1,\rho}$ we note $(D_\theta A)_{0\leq\theta\leq T}$ its Malliavin derivative and for X a $\mathbb{D}^{1,\rho}$-process we note $(D_\theta X_t)_{0\leq\theta,t\leq T}$.
- $S^\rho((0,T),\mathbb{R}^k)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |A_t|^\rho\right] < +\infty$$

  and $S^\infty((0,T),\mathbb{R}^k) = \bigcap_{\rho>1} S^\rho((0,T),\mathbb{R}^k)$.
- $H^\rho((0,T),\mathbb{R}^k)$ is the space of stochastic progressively measurable processes $(A_t)_{0\leq t\leq T}$ with values in $\mathbb{R}^k$ such that

$$\mathbb{E}\left[\left(\int_0^T |A_t|^2 dt\right)^{\frac{\rho}{2}}\right] < +\infty$$

  and $H^\infty((0,T),\mathbb{R}^k) = \bigcap_{\rho>1} H^\rho((0,T),\mathbb{R}^k)$.

- Whenever the notation $T-$ appears in the definition of a process space, we mean the set of all processes whose restrictions satisfy the respective property when $T-$ is replaced by any $T-\varepsilon$, $\varepsilon > 0$. For example,

$$S^\rho((0,T-),\mathbb{R}^k) = \bigcap_{\varepsilon>0} S^\rho((0,T-\varepsilon),\mathbb{R}^k).$$

  Moreover we say that a sequence $(F_n)_{n\in\mathbb{N}}$ converges in $S^\rho((0,T-),\mathbb{R}^k)$ to F in $S^\rho((0,T-),\mathbb{R}^k)$ if for any $\varepsilon > 0$, the sequence $(F_n)_{n\in\mathbb{N}}$ converges to F in $S^\rho((0,T-\varepsilon),\mathbb{R}^k)$.

In the rest of the chapter, C denotes a generic constant, which can depend on other coefficients, and may change from line to line.

3.1 Setting and known results

From now on, we fix $q > 1$ and p still denotes its Hölder conjugate. We suppose that

Assumption 13.

1. *The coefficient η is an Itô process:*

$$\eta_t = \eta_0 + \int_0^t b^\eta_s ds + \int_0^t \sigma^\eta_s dW_s, \quad 0 \le t \le T,$$

   with an initial condition $\eta_0 \in \mathbb{R}$.

   (a) The drift $b^\eta : \Omega \times [0,T] \longrightarrow \mathbb{R}$ and the diffusion matrix $\sigma^\eta : \Omega \times [0,T] \longrightarrow \mathbb{R}^d$ are progressively measurable and bounded.

   (b) There exist η_* and η^* in $\mathbb{R}^*_+$ such that, a.s. for any $s \in [0,T]$,

$$0 < \eta_* \le \eta_s < \eta^*.$$

2. *The process γ is a progressively measurable, non-negative and bounded: there exists $\gamma^* \in \mathbb{R}^*_+$ such that, a.s. for any $s \in [0,T]$,*

$$0 \le \gamma_s \le \gamma^*.$$

In Section 3.4, Example 3.4.2 provides a case where Assumption 13 holds.

Remark 3.1.1. *In particular the process η and γ satisfy*

$$\mathbb{E}\left[\int_0^T \left(\frac{1}{\eta_s^{q-1}} + \eta_s\right) ds\right] < +\infty, \quad \mathbb{E}\left[\int_0^T \gamma_s ds\right] < +\infty.$$

Under this framework, if $\xi \in L^\rho(\Omega)$ for some $\rho > 1$, then there exists a unique solution $(Y, Z) \in S^\rho((0,T), \mathbb{R}) \times H^\rho((0,T), \mathbb{R}^d)$ to the BSDE (3.2) (see [19, Theorem 4.2]). When $\xi = +\infty$, following [5, 56], we proceed by truncation and consider the following BSDE: for any $n \in \mathbb{N}$

$$Y_t^n = n + \int_t^T \left(-(p-1)\frac{|Y_s^n|^{q-1}Y_s^n}{\eta_s^{q-1}} + \gamma_s\right) ds - \int_t^T Z_s^n dW_s, \quad t \in [0,T]. \tag{3.3}$$

Lemma 3.1.2. *Under Conditions* 13*, the truncated BSDE (3.3) admits a unique solution (Y^n, Z^n) in $S^\rho((0,T), \mathbb{R}) \times H^\rho((0,T), \mathbb{R}^d)$ for all $\rho \in (1, +\infty)$. Moreover the sequence Y_n is non-decreasing and the process Y^n is bounded from above: for $m \leq n$*

$$\forall t \in [0,T], \quad 0 \leq Y_t^m \leq Y_t^n \leq n + \mathbb{E}\left[\int_t^T \gamma_s ds \middle| \mathcal{F}_t\right].$$

Proof. Existence and uniqueness directly follows from [19, Theorem 4.2]. Now standard a priori estimate on the solution of a BSDE (see [72, Theorem 5.30]) and the comparison theorem (see [72, Theorem 5.33]) imply that Y^n is non-negative and the desired estimation. This achieves the proof of this proposition. □

Since Y_t^n is a non-decreasing sequence, its limit Y_t exists. However the upper estimate on Y^n is not sufficient to ensure that Y_t is finite. According to [56] or Chapter 2 (our work [21]), we have:

Proposition 3.1.3. *Under Conditions* 13*, the sequence (Y^n, Z^n) converges to (Y, Z) in $S^\infty((0,T-), \mathbb{R}) \times H^\infty((0,T-), \mathbb{R}^d)$. The limit (Y, Z) is the minimal supersolution to the BSDE* (3.2) *on $[0,T[$ in the sense that:*

1. *The couple (Y, Z) belongs to $S^\infty((0,T-), \mathbb{R}) \times H^\infty((0,T-), \mathbb{R}^d)$.*
2. *The process Y is non negative.*
3. *For all $0 \leq s \leq t < T$,*

$$Y_s = Y_t + \int_s^t \left(-(p-1)\frac{|Y_r|^{q-1}Y_r}{\eta_s^{q-1}} + \gamma_r\right) dr - \int_s^t Z_r dW_r.$$

4. *The process Y satisfies a.s. $\lim_{t\to T} Y_t = +\infty$.*
5. *The process (Y,Z) is minimal: if $(\widetilde{Y},\widetilde{Z})$ satisfies the four previous points, then a.s. for any t, $Y_t \le \widetilde{Y}_t$.*

Moreover for any $t \in [0,T]$ and $n \ge 1$:

$$Y_t^n \le \frac{1}{(T-t+\frac{1}{n^{q-1}})^p}\left\{\frac{1}{n^{q-1}} + \mathbb{E}\left[\int_t^T (\eta_s + (T-s+1)^p\gamma_s)\, ds \middle| \mathcal{F}_t\right]\right\}. \tag{3.4}$$

As a consequence, the process Y satisfies: for all $0 \le t < T$

$$0 \le Y_t^n \le Y_t \le \frac{1}{(T-t)^p}\mathbb{E}\left[\int_t^T (\eta_s + (T-s+1)^p\gamma_s)\, ds \middle| \mathcal{F}_t\right]. \tag{3.5}$$

We show the Malliavin differentiability of the couple (Y^n, Z^n), due to [67, Theorem 5.1 and Application 6.1]. We assume that:

Assumption 14. *The processes b^η, η, γ are in valued in $\mathbb{D}^{1,2}$, their Malliavin derivatives $Db^\eta, D\eta, D\gamma$ admit progressively measurable versions which are in $L^2(\Omega \times [0,T] \times [0,T])$.*

Proposition 3.1.4. *Under Conditions 13 and 14, the solution (Y^n, Z^n) of the truncated BSDE (3.3) is in $L^2([0,T], \mathbb{D}^{1,2} \times \mathbb{D}^{1,2})$. Moreover for all $0 \le t < \theta \le T$, $D_\theta Y_t^n = 0$, $D_\theta Z_t^n = 0$ and, for all $0 \le \theta \le t \le T$,*

$$D_\theta Y_t^n = \int_t^T \left(-(p-1)\frac{|Y_r^n|^{q-1}}{\eta_r^{q-1}} D_\theta Y_r^n + \frac{|Y_r^n|^{q-1}Y_r^n}{\eta_r^q} D_\theta \eta_r + D_\theta \gamma_r\right) dr - \int_t^T D_\theta Z_s^n dW_s. \tag{3.6}$$

Proof. Under our setting, we can directly apply 2.2.5 ([21, Proposition 6]). □

In our setting, the approximating sequence Y^n has a Malliavin derivative DY^n, which satisfies the linear BSDE given in Proposition 3.1.4. Very natural questions are: does the minimal solution Y have also a Malliavin derivative DY and do we have the convergence of DY^n to DY? Evoke that to obtain the convergence of Y^n, the a priori estimate (3.4) together with the monotonicity of the sequence Y^n are crucial. We do not have similar tools for the Malliavin derivative. And we derive the a priori estimate using the non-linearity of the BSDE (3.2). Hence it is not possible to directly pass to the limit in the linear BSDE (3.6).

Formally if we pass to the limit in (3.6), we should have a linear BSDE of the form:

$$U_t = \int_t^T \left(-(p-1)\frac{|Y_r|^{q-1}}{\eta_r^{q-1}} U_r + \frac{|Y_r|^{q-1} Y_r}{\eta_r^q} D_\theta \eta_r + D_\theta \gamma_r \right) dr - \int_t^T V_s dW_s$$

with a singular generator since a.s.

$$\int_0^T \frac{|Y_r|^{q-1}}{\eta_r^{q-1}} dr = +\infty.$$

This property comes from the liquidation condition $\Xi_T^* = 0$ a.s. and Equation (3.1). In [52, 51], such linear BSDEs with singular generator are studied. Nonetheless to apply the results of these papers, the process $\frac{|Y|^{q-1}Y}{\eta^q} D_\theta \eta + D_\theta \gamma$ should be bounded. In general this property does not hold.

However in the liquidation problem, that is for the BSDE (3.2) when $\xi = +\infty$ a.s., we can prove the Malliavin differentiability of Y. Let us evoke the results of [42] thanks to Assumption 13. The minimal solution Y of (3.2) can be written: for all $0 \leq t < T$

$$Y_t = \frac{\eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} H_t. \tag{3.7}$$

The process H is the unique solution of the BSDE:

$$H_t = \int_t^T F(s, H_s) ds - \int_t^T Z_s^H dW_s, \quad 0 \leq t \leq T \tag{3.8}$$

with generator F given by: for all $0 \leq t < T$

$$\begin{aligned} F(t,h) = &[(T-t)b_t^\eta + (T-t)^p \gamma_t] \\ &- (p-1)\eta_t \left[\left(1 + \frac{1}{\eta_t (T-t)} h\right) \left|1 + \frac{1}{\eta_t (T-t)} h\right|^{q-1} - 1 - q\frac{1}{\eta_t(T-t)} h \right]. \end{aligned} \tag{3.9}$$

This generator F is singular at time T, in the sense that for $h \neq 0$, a.s.

$$\int_0^T |F(t,h)| dt = +\infty.$$

Therefore existence and uniqueness of H needs to be explained. Here we prove that there exist two constants $\delta > 0$ and $R \geq 0$, depending only on q, T, η and γ, such that on the time interval $[0, T]$, H satisfies a.s. for any $t \in [T-\delta, T]$, $|H_t| \leq R(T-t)^2$, H is bounded on $[0, T]$, and the solution (H, Z^H) is obtained by a Picard iteration argument in the space of adapted processes

$$\mathcal{H}^\delta = \{H \in L^\infty(\Omega; C([T-\delta, T]; \mathbb{R})), \ \|H\|_{\mathcal{H}^\delta} < +\infty\}$$

endowed with the weighted norm

$$\|H\|_{\mathcal{H}^\delta} = \left\| \sup_{t \in [T-\delta, T]} (T-t)^{-2} H_t \right\|.$$

See the appendix for more details. Let us give the value of the constants R and δ:

$$R = \|b^\eta\|_\infty + \frac{2}{p+1}\gamma^\star, \quad L = \frac{qR}{\eta_\star} 2^{|q-2|}, \quad \delta = \min\left(1, T, \frac{1}{2L}, \frac{\eta_\star}{2R}\right). \tag{3.10}$$

The main results of this paper, Theorems 3.2.3 and 3.3.9, are summarized in the following statement:

Theorem 3.1.5. *The process Y is valued in $\mathbb{D}^{1,2}$ and, for any $0 \leq \theta \leq t < T$,*

$$D_\theta Y_t = \frac{D_\theta \eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} D_\theta H_t$$

and if for some $\varrho > 1$,

$$\sup_{\theta \in [0,T]} \mathbb{E}\left[|D_\theta \eta_T|^\varrho + \int_0^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho \right) ds \right] < +\infty$$

then

$$\lim_{n \to +\infty} \sup_{\theta \in [0,T]} \mathbb{E}\left[\sup_{t \in [0,T]} (T-t)^{\ell p} |D_\theta Y_t - D_\theta Y^n_t|^\ell \right] = 0.$$

In particular for any $0 \leq \tau < T$

$$\lim_{n \to +\infty} \sup_{\theta \in [0,T]} \mathbb{E}\left[\sup_{t \in [0,\tau]} |D_\theta Y_t - D_\theta Y^n_t|^\ell \right] = 0.$$

3.2 Malliavin derivative for the minimal solution and its behaviour at time T

Evoke that the generator F is given by (3.9) and we define on the set $[0,T)\times\mathbb{R}\times[\eta_\star,\eta^\star]$ the function G by:

$$G(t,h,\eta)=(p-1)\eta\left[\left(1+\frac{1}{\eta(T-t)}h\right)\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-1-q\frac{1}{\eta(T-t)}h\right].$$

Thus we have

$$F(t,h)=(T-t)b_t^\eta+(T-t)^p\gamma_t-G(t,h,\eta_t),\quad 0\le t<T, h\in\mathbb{R}.$$

On the set $[0,T)\times\mathbb{R}\times[\eta_\star,\eta^\star]$ we have:

$$\frac{\partial G}{\partial h}(t,h,\eta)=\frac{p}{(T-t)}\left(\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-1\right)\tag{3.11}$$

$$\begin{aligned}\frac{\partial G}{\partial \eta}(t,h,\eta)&=(p-1)\left[\left(1+\frac{1}{\eta(T-t)}h\right)\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-1-q\frac{1}{\eta(T-t)}h\right]\\&\quad-\frac{ph}{\eta(T-t)}\left(\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-1\right)\\&=p\left(1+\frac{1}{\eta(T-t)}h\right)\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}\\&\quad-\left(1+\frac{1}{\eta(T-t)}h\right)\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-p+1\\&\quad-\underbrace{(p-1)q}_{=p}\frac{1}{\eta(T-t)}h-\frac{ph}{\eta(T-t)}\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}+\frac{ph}{\eta(T-t)}\\&=-\left(1+\frac{1}{\eta(T-t)}h\right)\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}+1+p\left(\left|1+\frac{1}{\eta(T-t)}h\right|^{q-1}-1\right).\end{aligned}\tag{3.12}$$

In the case where $q=p=2$, we get the easier expression

$$G(t,h,\eta)=\frac{h^2}{\eta(T-t)^2},\quad \frac{\partial G}{\partial h}(t,h,\eta)=\frac{2h}{\eta(T-t)^2},\quad \frac{\partial G}{\partial \eta}(t,h,\eta)=-\frac{h^2}{\eta^2(T-t)^2}.$$

A key point in the proof of [42, Theorem 23] is a.s. for any $T-\delta \leq t \leq T$, $|H_t| \leq R(T-t)^2$. Therefore these derivatives are bounded when we replace (h,η) by the processes (H_t, η_t) for $t \in [T-\delta, T]$.

From time to time, we use the next representation of G and their derivatives:

$$\begin{aligned} G(t,h,\eta) &= \frac{qh^2}{\eta(T-t)^2}\int_0^1 \left|1+a\frac{1}{\eta(T-t)}h\right|^{q-2} \operatorname{sign}\left(1+a\frac{1}{\eta(T-t)}h\right)(1-a)da \\ \frac{\partial G}{\partial h}(t,h,\eta) &= \frac{qh}{\eta(T-t)^2}\int_0^1 \left|1+a\frac{1}{\eta(T-t)}h\right|^{q-2} \operatorname{sign}\left(1+a\frac{1}{\eta(T-t)}h\right)da \qquad (3.13) \\ \frac{\partial G}{\partial \eta}(t,h,\eta) &= -\frac{qh^2}{\eta^2(T-t)^2}\int_0^1 \left|1+a\frac{1}{\eta(T-t)}h\right|^{q-2} \operatorname{sign}\left(1+a\frac{1}{\eta(T-t)}h\right)ada. \qquad (3.14) \end{aligned}$$

These representations may be not well-defined when $q<2$ if $1+a\frac{1}{\eta(T-t)}h=0$. However in the construction of the solution H, δ is chosen such that

$$\frac{|H_t|}{\eta_t(T-t)} \leq \frac{1}{2}. \qquad (3.15)$$

Hence we can use these versions with $h = H_t$ on the whole time interval $[T-\delta, T]$.

Lemma 3.2.1. *Under the previous assumptions, the process H has a Malliavin derivative, such that $D_\theta H_t = 0$ if $t<\theta$ and for any $0 \leq \theta \leq t \leq T$*

$$\begin{aligned} D_\theta H_t = &\int_t^T \left[(T-s)D_\theta b^\eta_s + (T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial \eta}(s,H_s,\eta_s)D_\theta \eta_s - \frac{\partial G}{\partial h}(s,H_s,\eta_s)D_\theta H_s\right] ds \\ &- \int_t^T D_\theta Z^H_s dW_s. \qquad (3.16) \end{aligned}$$

Proof. It is an adaptation of the proof of [33, Proposition 5.4]. The solution (H, Z^H) is the limit in $\mathcal{H}^\delta$ of the sequence $(H^k, Z^{H,k})$ unique solution in $\mathcal{H}^\delta$ of

$$H^k_t = \int_t^T F(s, H^{k-1}_s)ds - \int_t^T Z^{H,k}_s dW_s$$

with $(H^0, Z^{H,0}) = (0,0)$. Moreover for any k and $t \in [T-\delta, T]$, $|H^k_t| \leq R(T-t)^2$ and $|H^k_t| \leq C$. By recursion we prove that H^k has a Malliavin derivative DH^k such that for

$\theta \leq t \leq T$ and $t \geq T - \delta$:

$$D_\theta H^k_t = \int_t^T \Big[(T-s) D_\theta b^\eta_s + (T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial \eta}(s, H^{k-1}_s, \eta_s) D_\theta \eta_s \\ - \frac{\partial G}{\partial h}(s, H^{k-1}_s, \eta_s) D_\theta H^{k-1}_s \Big] ds - \int_t^T D_\theta Z^{H,k}_s dW_s.$$

The processes $\dfrac{\partial G}{\partial \eta}(s, H^k_s, \eta_s)$ and $\dfrac{\partial G}{\partial h}(s, H^k_s, \eta_s)$ are bounded, uniformly w.r.t. k. Indeed using the representations (3.13) and (3.14) we obtain that for $T-\delta \leq s \leq T$

$$\begin{aligned} \left| \frac{\partial G}{\partial h}(s, H^k_s, \eta_s) \right| &\leq \frac{q |H_s|^k}{\eta_s (T-s)^2} \int_0^1 \left| 1 + a \frac{1}{\eta_s (T-s)} H^k_s \right|^{q-2} da \\ &\leq \frac{qR}{\eta_\star} \int_0^1 \left| 1 + a \frac{1}{\eta_s (T-s)} H^k_s \right|^{q-2} da \\ &\leq \frac{qR}{\eta_\star} \int_0^1 \left(1 + a \frac{1}{\eta_s (T-s)} |H^k_s| \right)^{q-2} da \\ &\leq \frac{qR}{\eta_\star} \int_0^1 \left(1 + \frac{a}{2} \right)^{q-2} da \\ &\leq \frac{qR}{\eta_\star} 2^{|q-2|}, \end{aligned}$$

where we used that $|H^k_t| \leq R(T-t)^2$ and that Equation (3.15) also holds for H^k from the choice of δ given by (3.10). Similarly

$$\begin{aligned} \left| \frac{\partial G}{\partial \eta}(s, H^k_s, \eta_s) \right| &\leq \frac{q (H^k_s)^2}{\eta_s^2 (T-s)^2} \int_0^1 \left| 1 + a \frac{1}{\eta_s (T-s)} H^k_s \right|^{q-2} a da \\ &\leq \frac{q R^2 (T-s)^2}{\eta_\star^2} \int_0^1 \left(1 + \frac{a}{2} \right)^{q-2} a da \\ &\leq \frac{q R^2 (T-s)^2}{\eta_\star^2} 2^{|q-2|}. \end{aligned}$$

Hence using classical a priori estimates for BSDE (see [72, Section 5.3.1]), we prove that the sequence $D_\theta H^k$ converges to the solution of (3.16) on $[T-\delta, T]$. Moreover we can also use classical arguments on $[0, T-\delta]$ ([33] or [67]) thanks to the expression of H on $[0, T-\delta]$ given by (3.7) (see end of the proof of Proposition 3.5.1 for the properties of F

on $[0, T-\delta]$). Fianlly we get

$$\mathbb{E}\left[\int_0^T \int_0^T |D_\theta H_t|^2 d\theta dt\right] < +\infty.$$

□

Note that for $q = 2$, the BSDE (3.16) becomes

$$D_\theta H_t = \int_t^T \left[(T-s) D_\theta b_s^\eta + (T-s)^2 D_\theta \gamma_s + \frac{H_s^2}{\eta_s^2 (T-s)^2} D_\theta \eta_s - \frac{2H_s}{\eta_s (T-s)^2} D_\theta H_s\right] ds \\ - \int_t^T D_\theta Z_s^H dW_s.$$

Lemma 3.2.2. *There exists C such that the Malliavin derivative of H satisfies: a.s. for any $0 \leq \theta \leq t \leq T$*

$$|D_\theta H_t| \leq C(T-t)\mathbb{E}\left[\int_t^T \left(|D_\theta b_s^\eta| + |D_\theta \gamma_s| + |D_\theta \eta_s|\right) ds \middle| \mathcal{F}_t\right]. \tag{3.17}$$

Proof. Evoke that $|H_t| \leq R(T-t)^2$ and remark that, by solution of the linear BSDE (3.16),

$$D_\theta H_t = \mathbb{E}\left[\int_t^T \left[(T-s) D_\theta b_s^\eta + (T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) D_\theta \eta_s\right] \Gamma_{t,s} ds \middle| \mathcal{F}_t\right] \tag{3.18}$$

where

$$\Gamma_{t,s} = \exp\left(-\int_t^s \frac{\partial G}{\partial h}(u, H_u, \eta_u) du\right). \tag{3.19}$$

We already know that on $[T-\delta, T]$:

$$\left|\frac{\partial G}{\partial h}(s, H_s, \eta_s)\right| \leq \frac{qR}{\eta_\star} 2^{|q-2|}, \quad \left|\frac{\partial G}{\partial \eta}(s, H_s, \eta_s)\right| \leq \frac{qR^2}{\eta_\star^2} 2^{|q-2|}(T-s)^2.$$

If $0 \leq \theta \leq T$ and $(T-\delta) \vee \theta \leq t \leq T$, the result directly follows from (3.18). If

$0 \leq \theta \leq t \leq T-\delta$,

$$\begin{aligned} D_\theta H_t &= \mathbb{E}\left[D_\theta H_{T-\delta} \Gamma_{t,T-\delta} \Big| \mathcal{F}_t \right] \\ &\quad + \mathbb{E}\left[\int_t^{T-\delta} \left[(T-s) D_\theta b^\eta_s + (T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) D_\theta \eta_s \right] \Gamma_{t,s} ds \Big| \mathcal{F}_t \right] \end{aligned}$$

and $\left| \frac{\partial G}{\partial h}(\cdot, H, \eta) \right|$ and $\left| \frac{\partial G}{\partial \eta}(\cdot, H, \eta) \right|$ are bounded processes on $[0, T-\delta]$. Hence there exists a constant C such that

$$\begin{aligned} |D_\theta H_t| &\leq C \mathbb{E}\left[|D_\theta H_{T-\delta}| \Big| \mathcal{F}_t \right] + \mathbb{E}\left[\int_t^{T-\delta} \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s| \right) ds \Big| \mathcal{F}_t \right] \\ &\leq C \mathbb{E}\Big[C\delta \int_{T-\delta}^T \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s| \right) ds \\ &\qquad + \int_t^{T-\delta} \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s| \right) ds \Big| \mathcal{F}_t \Big] \\ &\leq C \mathbb{E}\left[\int_t^T \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s| \right) ds \Big| \mathcal{F}_t \right] \\ &= \frac{C}{\delta}(T-t) \mathbb{E}\left[\int_t^T \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s| \right) ds \Big| \mathcal{F}_t \right]. \end{aligned}$$

□

From Equation (3.7) and the previous lemma, we deduce the following result.

Theorem 3.2.3. *The process Y is valued in $\mathbb{D}^{1,2}$ and, for any $0 \leq \theta \leq t < T$,*

$$D_\theta Y_t = \frac{D_\theta \eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} D_\theta H_t. \tag{3.20}$$

From Equation (3.17) and Assumption 14, we deduce that:

$$|D_\theta H_t| \leq C(T-t)^{3/2} \mathbb{E}\left[\left(\int_t^T \left(|D_\theta b^\eta_s|^2 + |D_\theta \gamma_s|^2 + |D_\theta \eta_s|^2 \right) ds \right)^{1/2} \Big| \mathcal{F}_t \right].$$

Thus close to T, the leading term in $D_\theta Y_t$ is $\frac{D_\theta \eta_t}{(T-t)^{p-1}}$, and if $\lim_{t \to T} D_\theta \eta_t = D_\theta \eta_T$ a.s. (for example if $\eta_t = \eta(t, X_t)$ with X solution of SDE (3.41), see Section 3.4.1) then, on the set $\{D_\theta \eta_T \neq 0\}$,

$$\lim_{t\to T} |D_\theta Y_t| = +\infty.$$

Therefore there is a singularity at time T for the Malliavin derivative, even in this quite simple case where $\xi = +\infty$ a.s. And we expect that a similar singular behavior holds in the general setting studied in [56, 21] when ξ can be equal to $+\infty$ with a positive probability.

Remark 3.2.4. *However if η is deterministic, then $D_\theta \eta = 0$ and $D_\theta Y_t = \dfrac{1}{(T-t)^p} D_\theta H_t$ and from* (3.18)

$$|D_\theta H_t| \leq \mathbb{E}\left[\int_t^T (T-s)^p |D_\theta \gamma_s| \Gamma_{t,s} ds \middle| \mathcal{F}_t\right] \leq C(T-t)^p \mathbb{E}\left[\int_t^T |D_\theta \gamma_s| ds \middle| \mathcal{F}_t\right].$$

Then a.s.

$$\lim_{t\to T} |D_\theta Y_t| = 0.$$

Thus in this case we do not have singularity of the derivative.

3.3 Convergence of the Malliavin derivatives

Evoke that Y_t^n converges a.s. to Y_t given by (3.7). Moreover Y^n and Y belong to $\mathbb{D}^{1,2}$ (Proposition 3.1.4 and Theorem 3.2.3) and thus a natural question is: does the sequence DY^n converge to DY ? The goal of this section is to give an answer to this question.

Let us define the process $\mathcal{H}^n$ by:

$$\mathcal{H}_t^n = \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p Y_t^n - \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)\eta_t, \quad 0 \leq t \leq T,$$

to get

$$Y_t^n = \frac{1}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}}\eta_t + \frac{1}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p}}\mathcal{H}_t^n. \tag{3.21}$$

Thus

$$\begin{aligned}\mathcal{H}^n_T &= \left(\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p Y^n_T - \left(\frac{\eta^\star}{n}\right)^{q-1}\eta_T \\ &= \left(\frac{\eta^\star}{n}\right)^{q} Y^n_T - \left(\frac{\eta^\star}{n}\right)^{q-1}\eta_T \\ &= \left(\frac{\eta^\star}{n}\right)^{q} n - \left(\frac{\eta^\star}{n}\right)^{q}\frac{n}{\eta^\star}\eta_T.\end{aligned}$$

The process $\mathcal{H}^n$ is the solution of the BSDE

$$\mathcal{H}^n_t = (\eta^\star)^q\left(1-\frac{\eta_T}{\eta^\star}\right)\frac{1}{n^{q-1}} + \int_t^T F\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s\right)ds - \int_t^T \mathcal{Z}^n_s dW_s. \tag{3.22}$$

Note that the terminal value is non-negative:

$$\mathcal{H}^n_T = (\eta^\star)^q\left(1-\frac{\eta_T}{\eta^\star}\right)\frac{1}{n^{q-1}} \geq 0.$$

Let us start with an estimate on $\mathcal{H}^n$. With an abuse of notations, we still denote the constant δ in the next statement. Indeed we can always take the minimum between the constant δ coming from the existence of the process H and the constant δ coming from this lemma.

Lemma 3.3.1. *There exist $\delta > 0$, $n_0 \in \mathbb{N}$ and two positive constants C_1 and C_2 s.t. for any $n \geq n_0$ and $T-\delta \leq t \leq T$:*

$$-C_1\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^2 \leq \mathcal{H}^n_t \leq C_2\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right).$$

Moreover there exists C such that for any n, on $[0,T-\delta]$, $|\mathcal{H}^n_t| \leq C$.

Proof. We already know that

$$0 \leq \mathcal{H}^n_T = (\eta^\star)^q\left(1-\frac{\eta_T}{\eta^\star}\right)\frac{1}{n^{q-1}} \leq \frac{(\eta^\star)^q}{n^{q-1}}.$$

Hence the estimate holds at time T for any $C_1 \geq 0$ and $C_2 \geq \eta^\star$.

Now if $V_t = C_2\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)$, then $-dV_t = C_2 dt$ and

$$F\left(t-\left(\frac{\eta^\star}{n}\right)^{q-1}, V_t\right) = \left[\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right) b_t^\eta + \left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p \gamma_t\right] - (p-1)\eta_t\left[\left(1+\frac{C_2}{\eta_t}\right)^q - 1 - \frac{qC_2}{\eta_t}\right].$$

Hence $F\left(t-\left(\frac{\eta^\star}{n}\right)^{q-1}, V_t\right) \leq C_2$ if for any t and n:

$$(p-1)\eta_t\left[\left(1+\frac{C_2}{\eta_t}\right)^q - 1 - \frac{qC_2}{\eta_t}\right] + C_2 \geq \left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right) b_t^\eta + \left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p \gamma_t.$$

In particular it holds if

$$\begin{aligned}
&(p-1)\left[\left(1+\frac{C_2}{\eta_t}\right)^q - 1 - \frac{qC_2}{\eta_t}\right] + \frac{C_2}{\eta_t} \\
&\quad = (p-1)\left[\left(1+\frac{C_2}{\eta_t}\right)^q - 1 - \frac{C_2}{\eta_t}\right] \\
&\quad \geq \frac{1}{\eta_\star}(T+(\eta^\star)^{q-1})\left[\|b^\eta\|_\infty + (T+(\eta^\star)^{q-1})^p\gamma^\star\right],
\end{aligned}$$

which is equivalent to

$$\left(1+\frac{C_2}{\eta_t}\right)^q - 1 - \frac{C_2}{\eta_t} \geq \frac{q-1}{\eta_\star}(T+(\eta^\star)^{q-1})\left[\|b^\eta\|_\infty + (T+(\eta^\star)^{q-1})^p\gamma^\star\right].$$

The function $\psi : x \mapsto (1+x)^q - 1 - x$ is continuous and increasing on $[0,+\infty[$, with $\psi(0) = 0$ and $\psi(\infty) = \infty$. Hence it is sufficient to take

$$C_2 \geq \eta^\star\left[\psi^{-1}\left(\frac{q-1}{\eta_\star}(T+(\eta^\star)^{q-1})\left[\|b^\eta\|_\infty + (T+(\eta^\star)^{q-1})^p\gamma^\star\right]\right) \vee 1\right],$$

to obtain, by the comparison result for BSDE, that a.s. for any t, $\mathcal{H}_t^n \leq V_t$. Remark that for $q=2$, C_2 can be explicitly obtained by solving a second degree equation.

To get the lower estimate, we similarly proceed, by defining

$$U_t = -C_1\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^2.$$

This quantity satisfies $-dU_t = -2C_1 \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) dt$ and

$$\begin{aligned} F\left(t - \left(\frac{\eta^\star}{n}\right)^q, U_t\right) &= \left[\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) b_t^\eta + \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p \gamma_t\right] \\ &\quad - (p-1)\eta_t \left[(1-\chi_t)\left|1-\chi_t\right|^{q-1} - 1 + q\chi_t\right] \\ &\geq -2\eta_t \chi_t \end{aligned} \tag{3.23}$$

where

$$\chi_t = \frac{C_1}{\eta_t}\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) \geq 0.$$

Note that for some $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$C_1\left(\delta + \left(\frac{\eta^\star}{n_0}\right)^{q-1}\right) \leq \frac{\eta_\star}{2}, \tag{3.24}$$

the quantity χ_t satisfies: $\chi_t \leq 1/2$. Moreover let us recall that if $1 - \chi_t > 0$, then

$$(1-\chi_t)\left|1-\chi_t\right|^{q-1} - 1 + q\chi_t = q(q-1)\chi_t^2 \int_0^1 \left|1 - a\chi_t\right|^{q-2}(1-a)da.$$

Then the previous inequality (3.23) holds if

$$q\eta_t\chi_t^2 \int_0^1 \left|1 - a\chi_t\right|^{q-2}(1-a)da - 2\eta_t\chi_t \leq -\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)\left[\|b^\eta\|_\infty + (T + (\eta^\star)^q)^{p-1}\gamma^\star\right],$$

which can be written as:

$$(C_1)^2 \frac{\zeta_t}{\eta_t}\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) - 2C_1 + K \leq 0,$$

with

$$\zeta_t = q\int_0^1 \left|1 - a\chi_t\right|^{q-2}(1-a)da, \quad K = \left[\|b^\eta\|_\infty + (T + (\eta^\star)^q)^{p-1}\gamma^\star\right].$$

Note that for $q = 2$, $\zeta_t = 1$, which simplifies the discussion here after. First under (3.24), for any $T - \delta \leq t \leq T$ and $n \geq n_0$

$$\zeta_\star = \frac{q}{2}\left(\frac{1}{2^{q-2}} \wedge 1\right) \leq \zeta_t = q\int_0^1 \left|1 - a\chi_t\right|^{q-2}(1-a)da \leq \frac{q}{2}\left(\frac{1}{2^{q-2}} \vee 1\right) = \zeta^\star.$$

Now if

$$\delta + \left(\frac{\eta^\star}{n_0}\right)^{q-1} \leq \frac{\eta_\star}{K\zeta^\star}, \tag{3.25}$$

then for any $T-\delta \leq t \leq T$ and $n \geq n_0$

$$\frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right) \leq 1.$$

Thus the desired estimate holds if

$$\frac{\eta_t}{\zeta_t \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)} \left(1 - \sqrt{1 - \frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}\right) \leq C_1 \tag{3.26}$$

and

$$C_1 \leq \frac{\eta_t}{\zeta_t \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)} \left(1 + \sqrt{1 - \frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}\right). \tag{3.27}$$

For the upper bound, remark that for $T-\delta \leq t \leq T$ and $n \geq n_0$

$$\begin{aligned} &\frac{\eta_t}{\zeta_t \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)} \left(1 + \sqrt{1 - \frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}\right) \\ &\qquad \geq \frac{\eta_\star}{\zeta^\star \left(\delta + \left(\frac{\eta^\star}{n_0}\right)^{q-1}\right)} \left(1 + \sqrt{1 - \frac{\zeta^\star}{\eta_\star} K \left(\delta + \left(\frac{\eta^\star}{n_0}\right)^{q-1} \right)}\right). \end{aligned}$$

And the lower bound satisfies:

$$\begin{aligned} &\frac{\eta_t}{\zeta_t \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)} \left(1 - \sqrt{1 - \frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}\right) \\ &\quad = \frac{K}{1 + \sqrt{1 - \frac{\zeta_t}{\eta_t} K \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}} \leq \frac{K}{1 + \sqrt{1 - \frac{\eta^\star}{\eta_\star} \left(\delta + \left(\frac{\eta^\star}{n_0}\right)^{q-1} \right)}}. \end{aligned}$$

Therefore there exists $\delta > 0$, $n_0 \in \mathbb{N}$ and $C_1 > 0$ such that Conditions (3.24),(3.25), (3.26) and (3.27) hold. Then by comparison principle for BSDEs, for any $T-\delta \leq t \leq T$ and $n \geq n_0$

$$\mathcal{H}^n_t \geq -C_1 \left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)^2.$$

The last statement of the lemma comes from (3.5): on $[0, T-\delta]$

$$0 \leq Y_t^n \leq Y_t \leq \frac{1}{\delta^p}\left(\frac{1}{\eta_\star} + (T+1)^p \gamma^\star\right)$$

and from the very definition (3.21) of $\mathcal{H}^n$:

$$|\mathcal{H}_t^n| \leq \left(T + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p Y_t^n + \eta^\star.$$

This achieves the proof of the lemma. □

Since Y_t^n converges a.s. to Y_t for any t, we already know that $\mathcal{H}_t^n$ converges to H_t. The next result is a convergence result with weights.

Lemma 3.3.2. *We have a.s.*

$$\lim_{n\to+\infty} \int_0^T \left| \frac{H_t}{T-t} - \frac{\mathcal{H}_t^n}{\left(T-t+\frac{\eta^\star}{n}\right)} \right| dt = 0$$

and for any $\varrho > 0$

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \left| \frac{H_t}{T-t} - \frac{\mathcal{H}_t^n}{\left(T-t+\frac{\eta^\star}{n}\right)} \right|^\varrho dt\right] = 0.$$

Proof. For any $0 \leq t < T$ and any n, we have

$$\frac{H_t}{T-t} - \frac{\mathcal{H}_t^n}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)} = (T-t)^{p-1} Y_t - \left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1} Y_t^n$$

and Y and Y^n are non-negative processes. Using (3.5), we have

$$0 \leq (T-t)^{p-1} Y_t \leq \frac{1}{T-t}\mathbb{E}\left[\int_t^T \left(\frac{1}{\eta_s} + (T-s+1)^p \gamma_s\right) ds \middle| \mathcal{F}_t\right] \leq \frac{1}{\eta_\star} + (T+1)^p \gamma^\star.$$

Now from the a priori estimate (3.4):

$$\begin{aligned} Y_t^n &\leq \frac{1}{(T-t+\frac{1}{n^{q-1}})^p}\left\{\frac{1}{n^{q-1}} + \mathbb{E}\left[\int_t^T \left(\left(\frac{p-1}{\eta_s}\right)^{p-1} + (T-s+1)^p\gamma_s\right) ds \middle| \mathcal{F}_t\right]\right\} \\ &\leq \frac{1}{(T-t+\frac{1}{n^{q-1}})^p}\left\{\frac{1}{n^{q-1}} + (T-t)\left(\left(\frac{p-1}{\eta_\star}\right)^{p-1} + (T+1)^p\gamma^\star\right)\right\}. \end{aligned}$$

Moreover

$$\begin{aligned} \frac{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}}{n^{q-1}\left(T-t+\frac{1}{n^{q-1}}\right)^p} &= \left(\frac{T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}}{T-t+\frac{1}{n^{q-1}}}\right)^{p-1} \frac{1}{n^{q-1}\left(T-t+\frac{1}{n^{q-1}}\right)} \\ &= \left(1+\frac{(\eta^\star)^{q-1}-1}{n^{q-1}(T-t)+1}\right)^{p-1}\frac{1}{(n^{q-1}(T-t)+1)} \leq \eta^\star. \end{aligned}$$

And

$$\begin{aligned} \frac{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}}{\left(T-t+\frac{1}{n^{q-1}}\right)^p}(T-t) &= \left(\frac{T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}}{T-t+\frac{1}{n^{q-1}}}\right)^{p-1} \frac{T-t}{\left(T-t+\frac{1}{n^{q-1}}\right)} \\ &= \left(1+\frac{(\eta^\star)^{q-1}-1}{n^{q-1}(T-t)+1}\right)^{p-1}\frac{n^{q-1}(T-t)}{(n^{q-1}(T-t)+1)} \leq \eta^\star. \end{aligned}$$

Therefore

$$\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1} Y_t^n \leq \eta^\star\left[1+\left(\left(\frac{p-1}{\eta_\star}\right)^{p-1} + (T+1)^p\gamma^\star\right)\right].$$

Since Y^n converges to Y pointwise, the result is a consequence of the dominated convergence theorem. □

Now let us study the Malliavin derivatives. We already know that Y^n is Malliavin differentiable and the solution of the linear BSDE (3.6) is for $0 \leq \theta \leq t \leq T$:

$$D_\theta Y_t^n = \mathbb{E}\left[\int_t^T \left(\left(\frac{Y_s^n}{\eta_s}\right)^q D_\theta\eta_s + D_\theta\gamma_s\right)\exp\left(-p\int_t^s\left(\frac{Y_u^n}{\eta_u}\right)^{q-1}du\right)ds\middle|\mathcal{F}_t\right].$$

Proposition 3.3.3. *The processes* $(\mathcal{H}^n)_{n\in\mathbb{N}}$ *are Malliavin differentiable and their Malliavin derivatives satisfy, for any* $0 \le \theta \le t \le T$*:*

$$D_\theta Y_t^n = \frac{1}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}} D_\theta \eta_t + \frac{1}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p}} D_\theta \mathcal{H}_t^n.$$

Proof. It is due to the fact that the processes Y^n and η are Malliavin differentiable and (3.21). □

Evoke that $D_\theta H$ is the solution of the BSDE (3.16):

$$\begin{aligned} D_\theta H_t = \int_t^T \Bigg[&(T-s) D_\theta b_s^\eta + (T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) D_\theta \eta_s \\ &- \frac{\partial G}{\partial h}(s, H_s, \eta_s) D_\theta H_s \Bigg] ds - \int_t^T D_\theta Z_s^H dW_s. \end{aligned}$$

The Malliavin derivative $D_\theta \mathcal{H}^n$ is the solution of:

$$\begin{aligned} D_\theta \mathcal{H}_t^n = &- \left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta \eta_T \\ &+ \int_t^T \left[\left(T - s + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) D_\theta b_s^\eta + \left(T - s + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p D_\theta \gamma_s \right] ds \\ &- \int_t^T \left[\frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right) D_\theta \eta_s + \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right) D_\theta \mathcal{H}_s^n \right] ds \\ &- \int_t^T D_\theta \mathcal{Z}_s^n dW_s. \end{aligned}$$

3.3.1 Convergence of $D_\theta \mathcal{H}^n$

Our goal is to prove that $D_\theta \mathcal{H}^n$ converges to $D_\theta H$. The difference $D_\theta H - D_\theta \mathcal{H}^n$ satisfies the next equation:

$$\begin{aligned}
\Delta_t^n &= D_\theta H_t - D_\theta \mathcal{H}_t^n \\
&= \left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta \eta_T - \left(\frac{\eta^\star}{n}\right)^{q-1} \int_t^T \left[D_\theta b_s^\eta + p \left(\int_0^1 \left(T - s + a \left(\frac{\eta^\star}{n}\right)^{q-1} \right) da \right) D_\theta \gamma_s \right] ds \\
&\quad - \int_t^T \left[\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s \right) \right] D_\theta \eta_s ds \\
&\quad - \int_t^T (D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n) dW_s \\
&\quad - \int_t^T \left[\frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s \right) \right] D_\theta H_s ds \\
&\quad - \quad \int_t^T \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s \right) \left[D_\theta H_s - D_\theta \mathcal{H}_s^n \right] ds. \qquad (3.28)
\end{aligned}$$

Our first statement is:

Lemma 3.3.4. *There exists a constant $\kappa_1 > 0$ such that a.s. for any $n \geq n_0$ and $s \in [0, T]$*

$$\frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s \right) \geq -\kappa_1.$$

Proof. Evoke that from (3.13)

$$\frac{\partial G}{\partial h}(t, h, \eta) = \frac{qh}{\eta(T-t)^2} \int_0^1 \left| 1 + a \frac{1}{\eta(T-t)} h \right|^{q-2} \operatorname{sign}\left(1 + a \frac{1}{\eta(T-t)} h \right) da.$$

Lemma 3.3.1 implies that for any $T - \delta \leq s \leq T$ and $n \geq n_0$:

$$-\frac{C_1 \zeta^\star}{\eta_\star} \leq -\frac{C_1 \zeta^\star}{\eta_s} \leq \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s \right) \leq \frac{C_2 \zeta^\star}{\eta_s \left(T - s + \left(\frac{\eta^\star}{n}\right)^{q-1} \right)}. \qquad (3.29)$$

Now from (3.11) and since $\mathcal{H}^n$ is bounded by C, if $s \leq T-\delta$

$$\left|\frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right| = \frac{p}{(T-s+\left(\frac{\eta^\star}{n}\right)^{q-1})}\left|\left|1+\frac{1}{\eta_s(T-s+\left(\frac{\eta^\star}{n}\right)^{q-1})}\mathcal{H}^n_s\right|^{q-1}-1\right|$$
$$\leq \frac{p}{\delta}\left(\left(1+\frac{C}{\eta_\star(T-\delta)}\right)^{q-1}+1\right).$$

□

The second result concerns the control of the next difference terms.

Lemma 3.3.5. *There exist two processes Υ^n and $\widetilde{\Upsilon}^n$ such that*

$$\left|\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right| \leq \Upsilon^n_s\left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} \tag{3.30}$$

and

$$\left|\frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right|$$
$$\leq \frac{\widetilde{\Upsilon}^n_s}{(T-s)}\left[\left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} + \left(\frac{\eta^\star}{n}\right)^{q-1}\frac{p}{(T-s+(\eta^\star/n)^{q-1})}\right]. \tag{3.31}$$

And Υ^n and $\widetilde{\Upsilon}^n$ are bounded uniformly w.r.t $s \in [0,T]$ and $n \geq n_0$.

Proof. For (s,h) and $(u,\hat{h})$ and if

$$\delta = \frac{h}{\eta(T-s)} - \frac{\hat{h}}{\eta(T-u)}$$

then from (3.12) and (3.14)

$$\begin{aligned}
&\frac{\partial G}{\partial \eta}(s,h,\eta) - \frac{\partial G}{\partial \eta}\left(u,\hat{h},\eta\right) \\
&\quad = -q\delta \int_0^1 \left|1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right|^{q-1} da + p\left|1 + \frac{h}{\eta(T-s)}\right|^{q-1} - p\left|1 + \frac{\hat{h}}{\eta(T-u)}\right|^{q-1} \qquad (3.32) \\
&\quad = -q\delta \int_0^1 \left|1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right|^{q-1} da \\
&\qquad + p(q-1)\delta \int_0^1 \left|1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right|^{q-2} \operatorname{sign}\left(1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right) da \\
&\quad = -q\delta \int_0^1 \left|1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right|^{q-2} \left(\frac{\hat{h}}{\eta(T-u)} + a\delta\right) \operatorname{sign}\left(1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right) da \qquad (3.33)
\end{aligned}$$

provided that there is no division by zero when $q < 2$.

If (s,h) and $(u,\hat{h})$ are chosen such that

$$\frac{h}{\eta(T-s)} > -\frac{1}{2}, \quad \frac{\hat{h}}{\eta(T-u)} > -\frac{1}{2},$$

for any $a \in [0,1]$

$$1 + \frac{\hat{h}}{\eta(T-u)} + a\delta = 1 + (1-a)\frac{\hat{h}}{\eta(T-u)} + a\frac{h}{\eta(T-s)} > \frac{1}{2}.$$

Evoke that from our choice of δ and n_0,

$$\frac{H_s}{\eta_s(T-s)} > -\frac{1}{2}, \quad \frac{\mathcal{H}^n_s}{\eta_s(T-s+(\eta^\star/n)^{q-1})} > -\frac{1}{2},$$

and from Lemma 3.3.1

$$\frac{1}{2} \leq 1 + \frac{\mathcal{H}^n_s}{\eta_s(T-s+(\eta^\star/n)^{q-1})} + a\delta^n_s \leq 1 + \frac{C_2+R}{\eta_\star},$$

with

$$\delta^n_s = \frac{H_s}{\eta_s(T-s)} - \frac{\mathcal{H}^n_s}{\eta_s(T-s+(\eta^\star/n)^{q-1})}.$$

Hence we deduce that for any $T-\delta \leq s \leq T$ and $n \geq n_0$

$$\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right) = \Upsilon^n_s \delta^n_s$$

with the bounded process

$$\Upsilon^n_s = -q \int_0^1 \left(1 + \frac{\mathcal{H}^n_s}{\eta_s(T-s+(\eta^\star/n)^{q-1})} + a\delta^n_s\right)^{q-2} \left(\frac{\mathcal{H}^n_s}{\eta_s(T-s+(\eta^\star/n)^{q-1})} + a\delta^n_s\right) da.$$

On the rest of the time interval $[0, T-\delta]$, since $s \mapsto 1/(T-s)$, H and $\mathcal{H}^n$ are bounded, from (3.32), we easily deduce that if $q \geq 2$, the functions are Lipschitz continuous and thus

$$\left|\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right| \leq \Upsilon^n_s |\delta^n_s|.$$

But for $1 < q < 2$, since $x \mapsto |x|^{q-1}$ is $(q-1)$-Hölder continuous, we only obtain that

$$\left|\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right| \leq \Upsilon^n_s (|\delta^n_s|^{q-1} \vee |\delta^n_s|).$$

Similarly with (3.11) and the same previous notations as previously:

$$\begin{aligned}
&\frac{\partial G}{\partial h}(s, h, \eta) - \frac{\partial G}{\partial h}\left(u, \hat{h}, \eta\right) \\
&= \frac{p}{(T-s)}\left|1 + \frac{h}{\eta(T-s)}\right|^{q-1} - \frac{p}{(T-u)}\left|1 + \frac{\hat{h}}{\eta(T-u)}\right|^{q-1} \\
&= \frac{p}{(T-s)}\left|1 + \frac{h}{\eta(T-s)}\right|^{q-1} - \frac{p}{(T-s)}\left|1 + \frac{\hat{h}}{\eta(T-u)}\right|^{q-1} \\
&\quad + \left(\frac{p}{T-s} - \frac{p}{T-u}\right)\left|1 + \frac{\hat{h}}{\eta(T-u)}\right|^{q-1} \\
&= \frac{p(q-1)}{T-s}\delta \int_0^1 \left|1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right|^{q-2} \operatorname{sign}\left(1 + \frac{\hat{h}}{\eta(T-u)} + a\delta\right) da \\
&\quad + \frac{p(s-u)}{(T-s)(T-u)}\left|1 + \frac{\hat{h}}{\eta(T-u)}\right|^{q-1}.
\end{aligned}$$

Hence there exists a uniformly bounded process $\widetilde{\Upsilon}^n$ such that a.s. on $[T-\delta, T]$ and for

$n \geq n_0$

$$\frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)$$
$$= \frac{\widetilde{\Upsilon}^n_s}{(T-s)}\left[\left(\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right) + \left(\frac{\eta^\star}{n}\right)^{q-1}\frac{p}{(T-s+(\eta^\star/n)^{q-1})}\right].$$

And on $[0, T-\delta]$, we obtain the inequality by Lipschitz-continuity if $q \geq 2$ or Hölder-continuity if $q < 2$, and by boundedness of H and $\mathcal{H}^n$. □

Let us start with the next result:

Lemma 3.3.6. *Assume that for some $\varrho > 1$,*

$$\mathbb{E}\left[\int_0^T \left[|D_\theta \eta_T|^\varrho + \int_0^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho\right) ds\right] d\theta\right] < +\infty.$$

Then for any $0 \leq \theta \leq t \leq T$ $D_\theta \mathcal{H}^n_t$ converges a.s. to $D_\theta H_t$ and for $\ell < \varrho$, $D\mathcal{H}^n$ converges to DH in $L^\ell(\Omega \times [0,T]^2)$.

Proof. From Lemma 3.3.4

$$\Gamma^n_{t,s} = \exp\left[-\int_t^s \frac{\partial G}{\partial h}\left(u - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_u, \eta_u\right) du\right],$$

is bounded uniformly w.r.t. $n \geq n_0$ by $\exp(T\kappa_1)$. Since (3.28) is a linear BSDE, we have:

$$\begin{aligned}
&D_\theta H_t - D_\theta \mathcal{H}^n_t \\
&= \left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[D_\theta \eta_T \Gamma^n_{t,T} \middle| \mathcal{F}_t\right] \\
&\quad - \left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[\int_t^T \left[D_\theta b^\eta_s + p\left(\int_0^1 \left(T - s + a\left(\frac{\eta^\star}{n}\right)^{q-1}\right) da\right) D_\theta \gamma_s\right] \Gamma^n_{t,s} ds \middle| \mathcal{F}_t\right] \\
&\quad - \mathbb{E}\left[\int_t^T \left[\frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right] D_\theta \eta_s \Gamma^n_{t,s} ds \middle| \mathcal{F}_t\right] \\
&\quad - \mathbb{E}\left[\int_t^T \left[\frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right] D_\theta H_s \Gamma^n_{t,s} ds \middle| \mathcal{F}_t\right].
\end{aligned} \tag{3.34}$$

In the right-hand side of (3.34), the first two terms satisfy for any $0 \leq \theta \leq t \leq T$ and $n \geq n_0$

$$\left(\frac{\eta^\star}{n}\right)^{q-1} \left|\mathbb{E}\left[D_\theta \eta_T \Gamma^n_{t,T} \middle| \mathcal{F}_t\right]\right| \leq C\left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[|D_\theta \eta_T| \middle| \mathcal{F}_t\right] \tag{3.35}$$

and

$$\begin{aligned}
&\left(\frac{\eta^\star}{n}\right)^{q-1}\left|\mathbb{E}\left[\int_t^T\left[D_\theta b_s^\eta+p\left(\int_0^1\left(T-s+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)da\right)D_\theta\gamma_s\right]\Gamma_{t,s}^n ds\middle|\mathcal{F}_t\right]\right| \\
&\quad\le C\left(\frac{\eta^\star}{n}\right)^{q-1}\mathbb{E}\left[\int_t^T\left(|D_\theta b_s^\eta|+p\left(T+(\eta^\star)^{q-1}\right)|D_\theta\gamma_s|\right)ds\middle|\mathcal{F}_t\right] \\
&\quad\le C\left(\frac{\eta^\star}{n}\right)^{q-1}\mathbb{E}\left[\int_t^T\left(|D_\theta b_s^\eta|+|D_\theta\gamma_s|\right)ds\middle|\mathcal{F}_t\right]. \qquad (3.36)
\end{aligned}$$

From (3.30), we obtain that

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_t^T\left[\frac{\partial G}{\partial\eta}(s,H_s,\eta_s)-\frac{\partial G}{\partial\eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}_s^n,\eta_s\right)\right]D_\theta\eta_s\Gamma_{t,s}^n ds\middle|\mathcal{F}_t\right]\right| \\
&\quad\le\mathbb{E}\left[\int_t^T\Upsilon_s^n\left|\frac{H_s}{(T-s)}-\frac{\mathcal{H}_s^n}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1}|D_\theta\eta_s|\Gamma_{t,s}^n ds\middle|\mathcal{F}_t\right] \\
&\quad\le C\mathbb{E}\left[\int_t^T\left|\frac{H_s}{(T-s)}-\frac{\mathcal{H}_s^n}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1}|D_\theta\eta_s|ds\middle|\mathcal{F}_t\right]. \qquad (3.37)
\end{aligned}$$

From (3.17), we have

$$|D_\theta H_s|\le C(T-s)\zeta_s,$$

where

$$\zeta_s=\mathbb{E}\left[\int_s^T\left(|D_\theta b_u^\eta|+|D_\theta\gamma_u|+|D_\theta\eta_u|\right)du\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\int_s^T\varpi_u du\middle|\mathcal{F}_s\right].$$

Thus using (3.31)

$$\begin{aligned}
&\left|\frac{\partial G}{\partial h}(s,H_s,\eta_s)-\frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}_s^n,\eta_s\right)\right||D_\theta H_s| \\
&\quad\le C\zeta_s|\widetilde{\Upsilon}_s^n|\left|\frac{H_s}{(T-s)}-\frac{\mathcal{H}_s^n}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} \\
&\quad\quad+Cp|\widetilde{\Upsilon}_s^n|\left(\frac{\eta^\star}{n}\right)^{q-1}\frac{\zeta_s}{(T-s+(\eta^\star/n)^{q-1})}
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathbb{E}\left[\int_t^T \left[\frac{\partial G}{\partial h}(s,H_s,\eta_s) - \frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right] D_\theta H_s \Gamma^n_{t,s} ds \middle| \mathcal{F}_t\right]\right| \\
&\quad \leq C\mathbb{E}\left[\int_t^T \zeta_s \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} ds \middle| \mathcal{F}_t\right] \\
&\qquad + pC\left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[\int_t^T \frac{\zeta_s}{(T-s+(\eta^\star/n)^{q-1})} ds \middle| \mathcal{F}_t\right]. \qquad (3.38)
\end{aligned}$$

For the second one

$$\begin{aligned}
0 &\leq \mathbb{E}\left[\int_t^T \frac{\zeta_s}{T-s+(\eta^\star/n)^{q-1}} ds \middle| \mathcal{F}_t\right] = \mathbb{E}\left[\int_t^T \frac{1}{T-s+(\eta^\star/n)^{q-1}} \left(\int_s^T \varpi_u du\right) ds \middle| \mathcal{F}_t\right] \\
&= \ln(T-t+(\eta^\star/n)^{q-1})\zeta_t - \mathbb{E}\left[\int_t^T \ln(T-s+(\eta^\star/n)^{q-1}) \varpi_s ds \middle| \mathcal{F}_t\right] \\
&\leq \ln(T+(\eta^\star)^{q-1})\zeta_t + \mathbb{E}\left[\int_t^T -\ln(T-s)\varpi_s ds \middle| \mathcal{F}_t\right] \\
&\leq \ln(T+(\eta^\star)^{q-1})\zeta_t + \mathbb{E}\left[\int_t^T |\ln(T-s)|\varpi_s ds \middle| \mathcal{F}_t\right].
\end{aligned}$$

Coming back to (3.34), with (3.35), (3.36), (3.37), (3.38), we obtain that for $0 \leq \theta \leq t \leq T$

$$\begin{aligned}
|D_\theta H_t - D_\theta \mathcal{H}^n_t| &\leq C\left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[|D_\theta \eta_T| + 2\int_t^T \left(|D_\theta b^\eta_s| + |D_\theta \gamma_s| + |D_\theta \eta_s|\right) ds \middle| \mathcal{F}_t\right] \\
&\quad + C\left(\frac{\eta^\star}{n}\right)^{q-1} \mathbb{E}\left[\int_t^T |\ln(T-s)|\varpi_s ds \middle| \mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} |D_\theta \eta_s| ds \middle| \mathcal{F}_t\right] \\
&\quad + C\mathbb{E}\left[\int_t^T \zeta_s \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} ds \middle| \mathcal{F}_t\right]. \qquad (3.39)
\end{aligned}$$

With Hölder's inequality ($1/\varrho + 1/\varrho_* = 1$):

$$\begin{aligned}
\mathbb{E}\left[\int_t^T |\ln(T-s)|\varpi_s ds \middle| \mathcal{F}_t\right] &\leq \left[\int_t^T |\ln(T-s)|^{\varrho_*} ds\right]^{1/\varrho_*} \left[\mathbb{E}\left[\int_t^T (\varpi_s)^\varrho ds \middle| \mathcal{F}_t\right]\right]^{1/\varrho} \\
&\leq C\left[\mathbb{E}\left[\int_t^T (\varpi_s)^\varrho ds \middle| \mathcal{F}_t\right]\right]^{1/\varrho}
\end{aligned}$$

and

$$\mathbb{E}\left[\int_t^T \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} |D_\theta \eta_s| ds \middle| \mathcal{F}_t\right] \leq \nu^n_t \left[\mathbb{E}\left[\int_t^T |D_\theta \eta_s|^\varrho ds \middle| \mathcal{F}_t\right]\right]^{1/\varrho},$$

$$\mathbb{E}\left[\int_t^T \zeta_s \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{(q-1)\wedge 1} ds \middle| \mathcal{F}_t\right] \leq \nu^n_t \left[\mathbb{E}\left[\int_t^T |\zeta_s|^\varrho ds \middle| \mathcal{F}_t\right]\right]^{1/\varrho}$$

with

$$\nu^n_t = \left[\mathbb{E}\left[\int_t^T \left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{\varrho^*((q-1)\wedge 1)} ds \middle| \mathcal{F}_t\right]\right]^{1/\varrho^*}.$$

Note that

$$\begin{aligned}\mathbb{E}\left[\int_t^T |\zeta_s|^\varrho ds \middle| \mathcal{F}_t\right] &= \mathbb{E}\left[\int_t^T \mathbb{E}\left[\int_s^T \varpi_u du \middle| \mathcal{F}_s\right]^\varrho ds \middle| \mathcal{F}_t\right] \\ &\leq \mathbb{E}\left[\int_t^T (T-s)^{\varrho-1} \mathbb{E}\left[\int_s^T (\varpi_u)^\varrho du \middle| \mathcal{F}_s\right] ds \middle| \mathcal{F}_t\right] \\ &\leq \frac{1}{\varrho}(T-t)^\varrho \mathbb{E}\left[\int_t^T (\varpi_u)^\varrho du \middle| \mathcal{F}_t\right].\end{aligned}$$

From Lemma 3.3.2 and its proof, we can use the dominated convergence theorem to deduce that for a fixed t, ν^n converges to zero a.s. Thus coming back to (3.39), we obtain the a.s. convergence of $D_\theta \mathcal{H}^n_t$ to $D_\theta H_t$.

To obtain the convergence in mean, we raise to the power $1 < \ell < \varrho$ and use the expectation in (3.39):

$$
\begin{aligned}
&\mathbb{E}\left[|D_\theta H_t - D_\theta \mathcal{H}^n_t|^\ell\right] \\
&\leq C\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} \mathbb{E}\left[|D_\theta \eta_T|^\ell + \int_t^T \left(|D_\theta b^\eta_s|^\ell + |D_\theta \gamma_s|^\ell + |D_\theta \eta_s|^\ell\right) ds\right] \\
&\quad + C\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} \left[\mathbb{E}\left[\int_t^T (\varpi_s)^\varrho ds\right]\right]^{\ell/\varrho} \\
&\quad + C\mathbb{E}\left[(\nu^n_t)^\ell \left[\mathbb{E}\left[\int_t^T |D_\theta \eta_s|^\varrho ds \middle| \mathcal{F}_t\right]\right]^{\ell/\varrho}\right] + C\mathbb{E}\left[(\nu^n_t)^\ell \left[\mathbb{E}\left[\int_t^T (\varpi_u)^\varrho du \middle| \mathcal{F}_t\right]\right]^{\ell/\rho}\right] \\
&\leq C\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} \mathbb{E}\left[|D_\theta \eta_T|^\varrho + \int_t^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho\right) ds\right] \\
&\quad + C\left[\mathbb{E}\left[(\nu^n_t)^{\ell\varrho/(\varrho-\ell)}\right]\right]^{(\varrho-\ell)/\varrho} \left(\left[\mathbb{E}\left[\int_t^T |D_\theta \eta_s|^\varrho ds\right]\right]^{\ell/\varrho} + \left[\mathbb{E}\left[\int_t^T (\varpi_u)^\varrho du\right]\right]^{\ell/\rho}\right) \\
&\leq C\varepsilon_n \mathbb{E}\left[|D_\theta \eta_T|^\varrho + \int_t^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho\right) ds\right]
\end{aligned}
$$

with

$$\varepsilon_n = \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} + \left[\mathbb{E}\left[(\nu^n_0)^{\ell\varrho/(\varrho-\ell)}\right]\right]^{(\varrho-\ell)/\varrho}.$$

Therefore with Lemma 3.3.2

$$\lim_{n\to+\infty} \mathbb{E}\left[\int_0^T \int_0^T |D_\theta H_t - D_\theta \mathcal{H}^n_t|^\ell \, d\theta dt\right] = 0,$$

which achieves the proof of the lemma. □

Remark that if the condition is:

$$\sup_{\theta\in[0,T]} \mathbb{E}\left[|D_\theta \eta_T|^\varrho + \int_0^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho\right) ds\right] < +\infty,$$

then

$$\lim_{n\to+\infty} \sup_{\theta\in[0,T]} \mathbb{E}\left[\int_0^T |D_\theta H_t - D_\theta \mathcal{H}^n_t|^\ell \, dt\right] = 0.$$

Now we prove a stronger convergence result.

Lemma 3.3.7. *Assume that for some $\varrho > 1$,*

$$\sup_{\theta\in[0,T]} \mathbb{E}\left[|D_\theta \eta_T|^\varrho + \int_0^T \left(|D_\theta b^\eta_s|^\varrho + |D_\theta \gamma_s|^\varrho + |D_\theta \eta_s|^\varrho\right) ds\right] < +\infty.$$

Then for any $1 < \ell < \varrho$

$$\lim_{n\to+\infty} \sup_{\theta\in[0,T]} \mathbb{E}\left[\sup_{t\in[0,T]} |D_\theta H_t - D_\theta \mathcal{H}^n_t|^\ell\right] = 0.$$

Proof. We apply Itô's formula to $\Delta^n_t = D_\theta H_t - D_\theta \mathcal{H}^n_t$ with the function $x \mapsto |x|^\ell$ for $\ell > 1$, $0 \le \theta \le t \le T$ and $n \ge n_0$. Using the BSDE representation (3.28), we obtain

$$\begin{aligned}
&e^{\mu t}|\Delta^n_t|^\ell + \frac{\ell(1\wedge(\ell-1))}{2}\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}(D_\theta Z^H_s - D_\theta \mathcal{Z}^n_s)^2 ds \\
&\le \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} e^{\mu T}|D_\theta \eta_T|^\ell - \int_t^T \mu e^{\mu s}|\Delta^n_s|^\ell ds \\
&\quad - \ell\left(\frac{\eta^\star}{n}\right)^{q-1}\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\left[D_\theta b^\eta_s + p\left(\int_0^1\left(T-s+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)da\right)D_\theta\gamma_s\right]\Delta^n_s ds \\
&\quad + \ell\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\left[\frac{\partial G}{\partial \eta}(s,H_s,\eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right]D_\theta \eta_s \Delta^n_s ds \\
&\quad - \ell\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\Delta^n_s(D_\theta Z^H_s - D_\theta \mathcal{Z}^n_s)dW_s \\
&\quad - \ell\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\left[\frac{\partial G}{\partial h}(s,H_s,\eta_s) - \frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)\right]D_\theta H_s \Delta^n_s ds \\
&\quad - \ell\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}^n_s, \eta_s\right)(\Delta^n_s)^2 ds.
\end{aligned}$$

With Young's inequality we have

$$\begin{aligned}
&\left(\frac{\eta^\star}{n}\right)^{q-1}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\left|D_\theta b^\eta_s + p\left(\int_0^1\left(T-s+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)da\right)D_\theta\gamma_s\right||\Delta^n_s| \\
&\qquad \le \frac{1}{\ell}\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)}\left|D_\theta b^\eta_s + p\left(\int_0^1\left(T-s+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)da\right)D_\theta\gamma_s\right|^\ell + \frac{\ell-1}{\ell}|\Delta^n_s|^\ell \\
&\qquad \le \frac{1}{\ell}\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)}\left|D_\theta b^\eta_s + p\left(T+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)D_\theta\gamma_s\right|^\ell + \frac{\ell-1}{\ell}|\Delta^n_s|^\ell,
\end{aligned}$$

$$
\begin{aligned}
&|\Delta_s^n|^{\ell-2} 1_{\Delta_s^n \neq 0} \left| \frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right| |D_\theta H_s| |\Delta_s^n| \\
&\qquad \leq \frac{1}{\ell} \left| \frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right|^\ell |D_\theta H_s|^\ell + \frac{\ell - 1}{\ell} |\Delta_s^n|^\ell,
\end{aligned}
$$

and

$$
\begin{aligned}
&|\Delta_s^n|^{\ell-2} 1_{\Delta_s^n \neq 0} \left| \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right| |D_\theta \eta_s| |\Delta_s^n| \\
&\qquad \leq \frac{1}{\ell} \left| \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right|^\ell |D_\theta \eta_s|^\ell + \frac{\ell - 1}{\ell} |\Delta_s^n|^\ell.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&e^{\mu t} |\Delta_t^n|^\ell + \frac{\ell(1 \wedge (\ell - 1))}{2} \int_t^T e^{\mu s} |\Delta_s^n|^{\ell - 2} 1_{\Delta_s^n \neq 0} (D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n)^2 ds \\
&\quad \leq \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} e^{\mu T} |D_\theta \eta_T|^\ell - \ell \int_t^T e^{\mu s} |\Delta_s^n|^{\ell-2} 1_{\Delta_s^n \neq 0} \Delta_s^n (D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n) dW_s \\
&\qquad + \int_t^T \left[3(\ell - 1) - \mu - \ell \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right] e^{\mu s} |\Delta_s^n|^\ell ds \\
&\qquad + \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} \int_t^T e^{\mu s} \left| D_\theta b_s^\eta + p\left(T + \left(\frac{\eta^\star}{n}\right)^{q-1}\right) D_\theta \gamma_s\right|^\ell ds \\
&\qquad + \int_t^T e^{\mu s} \left| \frac{\partial G}{\partial \eta}(s, H_s, \eta_s) - \frac{\partial G}{\partial \eta}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right|^\ell |D_\theta \eta_s|^\ell ds \\
&\qquad + \int_t^T e^{\mu s} \left| \frac{\partial G}{\partial h}(s, H_s, \eta_s) - \frac{\partial G}{\partial h}\left(s - \left(\frac{\eta^\star}{n}\right)^{q-1}, \mathcal{H}_s^n, \eta_s\right)\right|^\ell |D_\theta H_s|^\ell ds.
\end{aligned}
$$

Using Lemma 3.3.4, we choose

$$
\mu = 1 + 3(\ell - 1) + \ell \kappa_1
$$

such that

$$
\begin{aligned}
&e^{\mu t}|\Delta^n_t|^\ell + \int_t^T e^{\mu s}|\Delta^n_s|^\ell ds + \frac{\ell(1\wedge(\ell-1))}{2}\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}(D_\theta Z^H_s - D_\theta \mathcal{Z}^n_s)^2 ds \\
&\leq \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)} e^{\mu T}|D_\theta \eta_T|^\ell - \ell\int_t^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}\Delta^n_s(D_\theta Z^H_s - D_\theta \mathcal{Z}^n_s)dW_s \\
&+ \left(\frac{\eta^\star}{n}\right)^{\ell(q-1)}\int_t^T e^{\mu s}\left|D_\theta b^\eta_s + p\left(T + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)D_\theta\gamma_s\right|^\ell ds \\
&+ \int_t^T e^{\mu s}\left|\frac{\partial G}{\partial \eta}(s,H_s,\eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell |D_\theta\eta_s|^\ell ds \\
&+ \int_t^T e^{\mu s}\left|\frac{\partial G}{\partial h}(s,H_s,\eta_s) - \frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell |D_\theta H_s|^\ell ds. \qquad (3.40)
\end{aligned}
$$

From Lemma 3.3.5, with Young's inequality with $\kappa > 1$

$$
\begin{aligned}
&\left|\frac{\partial G}{\partial \eta}(s,H_s,\eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell |D_\theta\eta_s|^\ell \\
&\quad\leq \frac{\kappa-1}{\kappa}\left|\frac{\partial G}{\partial \eta}(s,H_s,\eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^{\ell\frac{\kappa}{\kappa-1}} + \frac{1}{\kappa}|D_\theta\eta_s|^{\ell\kappa} \\
&\quad\leq \frac{\kappa-1}{\kappa}\left|\Upsilon^n_s\left(\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right)\right|^{\ell\frac{\kappa}{\kappa-1}} + \frac{1}{\kappa}|D_\theta\eta_s|^{\ell\kappa} \\
&\quad\leq C\frac{\kappa-1}{\kappa}\left|\frac{H_s}{(T-s)} - \frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^{\ell\frac{\kappa}{\kappa-1}} + \frac{1}{\kappa}|D_\theta\eta_s|^{\ell\kappa}.
\end{aligned}
$$

If $\ell < \varrho$, then there exists $\kappa > 1$ such that $\ell\kappa \leq \varrho$. With Lemma 3.3.2, our assumption on $D_\theta\eta$, and the dominated convergence theorem, we deduce that if $\ell < \varrho$

$$
\lim_{n\to+\infty}\mathbb{E}\left[\int_0^T e^{\mu s}\left|\frac{\partial G}{\partial \eta}(s,H_s,\eta_s) - \frac{\partial G}{\partial \eta}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell |D_\theta\eta_s|^\ell ds\right] = 0.
$$

Again with Lemma 3.3.5,

$$\begin{aligned}
&\mathbb{E}\left[\int_{T-\delta}^{T} e^{\mu s}\left|\frac{\partial G}{\partial h}(s,H_s,\eta_s)-\frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell |D_\theta H_s|^\ell ds\right]\\
&\quad\leq 2^{\ell-1}C\mathbb{E}\left[\int_{T-\delta}^{T}|\zeta_s|^\ell\left|\frac{H_s}{(T-s)}-\frac{\mathcal{H}^n_s}{(T-s+(\eta^\star/n)^{q-1})}\right|^\ell ds\right]\\
&\qquad+2^{\ell-1}C\left(\frac{\eta^\star}{n}\right)^{\ell(q-1)}\mathbb{E}\left[\int_{T-\delta}^{T}\frac{|\zeta_s|^\ell}{(T-s+(\eta^\star/n)^{(q-1)})^\ell}ds\right].
\end{aligned}$$

where again

$$\zeta_s=\mathbb{E}\left[\int_s^T\left(|D_\theta b^\eta_u|+|D_\theta\gamma_u|+|D_\theta\eta_u|\right)du\middle|\mathcal{F}_s\right]=\mathbb{E}\left[\int_s^T\varpi_u du\middle|\mathcal{F}_s\right].$$

Thus

$$|\zeta_s|^\ell\leq\mathbb{E}\left[\int_s^T|\varpi_u|^\ell du\middle|\mathcal{F}_s\right].$$

Next we can argue as in the proof of the previous lemma to deduce that

$$\lim_{n\to+\infty}\mathbb{E}\int_0^T e^{\mu s}\left|\frac{\partial G}{\partial h}(s,H_s,\eta_s)-\frac{\partial G}{\partial h}\left(s-\left(\frac{\eta^\star}{n}\right)^{q-1},\mathcal{H}^n_s,\eta_s\right)\right|^\ell|D_\theta H_s|^\ell ds=0$$

Taking the expectation in (3.40), we deduce that

$$\begin{aligned}
\lim_{n\to+\infty}\sup_{\theta\in[0,T]}\Bigg(\sup_{t\in[0,T]}\mathbb{E}\left[e^{\mu t}|\Delta^n_t|^\ell\right]+\mathbb{E}\left[\int_0^T e^{\mu s}|\Delta^n_s|^{\ell-2}\mathbf{1}_{\Delta^n_s\neq 0}(D_\theta Z^H_s-D_\theta\mathcal{Z}^n_s)^2ds\right]&\\
+\mathbb{E}\left[\int_0^T e^{\mu s}|\Delta^n_s|^\ell ds\right]\Bigg)&=0.
\end{aligned}$$

The Burkholder-Davis-Gundy inequality leads to:

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_t^T e^{\mu s}|\Delta_s^n|^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}\Delta_s^n(D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n)\Delta_s^n dW_s\right|\right]\\
&\leq C\mathbb{E}\left[\left(\int_0^T e^{2\mu s}|\Delta_s^n|^{2\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}(D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n)^2 ds\right)^{\frac{1}{2}}\right]\\
&\leq C\mathbb{E}\left[\left(\sup_{t\in[0,T]} e^{\mu t}|\Delta_t^n|^{\ell}\int_0^T e^{\mu s}|\Delta_s^n|^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}(D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n)^2 ds\right)^{\frac{1}{2}}\right]\\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\mu t}|\Delta_t^n|^{\ell}\right] + \frac{C^2}{2}\mathbb{E}\left[\int_0^T e^{\mu s}|\Delta_s^n|^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}(D_\theta Z_s^H - D_\theta \mathcal{Z}_s^n)^2 ds\right].
\end{aligned}$$

With the same arguments as above, we obtain that:

$$\lim_{n\to+\infty}\sup_{\theta\in[0,T]}\mathbb{E}\left[\sup_{t\in[0,T]} e^{\mu t}|\Delta_t^n|^{\ell}\right] = 0.$$

To obtain the convergence of the sequence $\Delta Z_s^n = D_\theta Z^H - D_\theta \mathcal{Z}^n$ for $1 < \ell < 2$:

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^T |\Delta Z_s^n|^2 ds\right)^{\ell/2}\right] &= \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\Delta_s^n\neq 0}|\Delta Z_s^n|^2 ds\right)^{\ell/2}\right]\\
&= \mathbb{E}\left[\left(\int_0^T (\Delta_s^n)^{2-\ell}(\Delta_s^n)^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}|\Delta Z_s^n|^2 ds\right)^{\ell/2}\right]\\
&\leq \mathbb{E}\left[(\Delta_*^n)^{\ell(2-\ell)/2}\left(\int_0^T (\Delta_s^n)^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}|\Delta Z_s^n|^2 ds\right)^{\ell/2}\right]\\
&\leq \left\{\mathbb{E}\left[(\Delta_*^n)^{\ell}\right]\right\}^{(2-\ell)/2}\left\{\mathbb{E}\left[\int_0^T (\Delta_s^n)^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}|\Delta Z_s^n|^2 ds\right]\right\}^{\ell/2}\\
&\leq \frac{2-\ell}{2}\mathbb{E}\left[(\Delta_*^n)^{\ell}\right] + \frac{\ell}{2}\mathbb{E}\left[\int_0^T (\Delta_s^n)^{\ell-2}\mathbf{1}_{\Delta_s^n\neq 0}|\Delta Z_s^n|^2 ds\right]
\end{aligned}$$

where we have used Hölder's and Young's inequality with $\frac{2-\ell}{2} + \frac{\ell}{2} = 1$ and $\Delta_*^n = \sup_{s\in[0,T]}|\Delta_s^n|$. This achieves the proof of the lemma. □

3.3.2 Convergence of $D_\theta Y^n$

Let us start with the convergence for fixed parameters θ and t.

Proposition 3.3.8. *If we set $D_\theta Y_T = 0$ for $0 \leq \theta \leq T$, then for any $0 \leq \theta \leq t \leq T$, we* *have the almost surely convergence:*

$$\lim_{n\to+\infty} D_\theta Y_t^n = D_\theta Y_t.$$

Proof. For any $0 \leq \theta \leq t < T$, according to Theorem 3.2.3, Proposition 3.3.3 and Lemma 3.3.6,

$$D_\theta Y_t^n = \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}} D_\theta \eta_t + \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p}} D_\theta \mathcal{H}_t^n$$

converges a.s. to

$$\frac{D_\theta \eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} D_\theta H_t = D_\theta Y_t.$$

For $t = T$, since $Y_T^n = n$, $D_\theta Y_T^n = 0$ for any n. Another way to obtain this fact consists of using Proposition 3.3.3 and the terminal condition of the process $\mathcal{H}^n$:

$$D_\theta Y_T^n = \frac{n}{\eta^\star} D_\theta \eta_T + \left(\frac{n}{\eta^\star}\right)^q D_\theta \mathcal{H}_T^n = \frac{n}{\eta^\star} D_\theta \eta_T - \left(\frac{n}{\eta^\star}\right)^q (\eta^\star)^q \frac{1}{n^{q-1}} \frac{D_\theta \eta_T}{\eta^\star} = 0.$$

Hence the sequence $D_\theta Y_T^n$ converges a.s. to $0 = D_\theta Y_T$. □

Note that

$$\begin{aligned}
&D_\theta H_t - D_\theta \mathcal{H}_t^n \\
&= (T-t)^p D_\theta Y_t - \left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p D_\theta Y_t^n + \left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta \eta_t \\
&= (T-t)^p (D_\theta Y_t - D_\theta Y_t^n) - \left[\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p - (T-t)^p\right] D_\theta Y_t^n + \frac{\eta^\star}{n} D_\theta \eta_t.
\end{aligned}$$

This equality allows us to prove the following result.

Theorem 3.3.9. *Under the conditions of Lemma 3.3.7,*

$$\lim_{n\to+\infty} \sup_{\theta\in[0,T]} \mathbb{E}\left[\sup_{t\in[0,T]} (T-t)^{\ell p} |D_\theta Y_t - D_\theta Y_t^n|^\ell\right] = 0.$$

In particular for any $0 \leq \tau < T$

$$\lim_{n\to+\infty} \sup_{\theta\in[0,T]} \mathbb{E}\left[\sup_{t\in[0,\tau]} |D_\theta Y_t - D_\theta Y_t^n|^\ell\right] = 0.$$

Proof. From the previous remark,

$$
\begin{aligned}
&(T-t)^p(D_\theta Y_t - D_\theta Y_t^n)\\
&= D_\theta H_t - D_\theta \mathcal{H}_t^n + \left[\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p - (T-t)^p\right] D_\theta Y_t^n - \frac{\eta^\star}{n} D_\theta \eta_t.
\end{aligned}
$$

Furthermore with Proposition 3.3.3

$$
\begin{aligned}
&\left[\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p - (T-t)^p\right] D_\theta Y_t^n\\
&= p\left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta Y_t^n \int_0^1 \left(T-t+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1} da\\
&= p\left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta \eta_t A_t^n + p\left(\frac{\eta^\star}{n}\right)^{q-1} \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)} D_\theta \mathcal{H}_t^n A_t^n\\
&= p\left(\frac{\eta^\star}{n}\right)^{q-1} D_\theta \eta_t A_t^n + p\left(\frac{\eta^\star}{n}\right)^{q-1} \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)} (D_\theta \mathcal{H}_t^n - D_\theta H_t) A_t^n\\
&\quad + p\left(\frac{\eta^\star}{n}\right)^{q-1} \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)} D_\theta H_t A_t^n
\end{aligned}
$$

with

$$
0 \le A_t^n = \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}} \int_0^1 \left(T-t+a\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1} da \le 1.
$$

Note that for any n and t

$$
0 \le \left(\frac{\eta^\star}{n}\right)^{q-1} \frac{1}{\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)} \le 1
$$

and from (3.17) for $t \in [0,T]$,

$$
|D_\theta H_t| \le C(T-t)\mathbb{E}\left[\int_t^T (|D_\theta b_s^\eta| + |D_\theta \gamma_s| + |D_\theta \eta_s|)\, ds \middle| \mathcal{F}_t\right].
$$

Therefore

$$\begin{aligned}
&\left[\left(T-t+\left(\frac{\eta^\star}{n}\right)^{q-1}\right)^p-(T-t)^p\right]|D_\theta Y^n_t| \\
&\leq p\left(\frac{\eta^\star}{n}\right)^{q-1}|D_\theta \eta_t|+p|D_\theta \mathcal{H}^n_t-D_\theta H_t| \\
&\quad +p\left(\frac{\eta^\star}{n}\right)^{q-1}\mathbb{E}\left[\int_t^T (|D_\theta b^\eta_s|+|D_\theta \gamma_s|+|D_\theta\eta_s|)\,ds\middle|\mathcal{F}_t\right].
\end{aligned}$$

In other words

$$\begin{aligned}
(T-t)^p|D_\theta Y_t-D_\theta Y^n_t| \leq (1+p)|D_\theta H_t-D_\theta\mathcal{H}^n_t|+\left(\frac{\eta^\star}{n}+p\left(\frac{\eta^\star}{n}\right)^{q-1}\right)|D_\theta\eta_t| \\
+p\left(\frac{\eta^\star}{n}\right)^{q-1}\mathbb{E}\left[\int_t^T (|D_\theta b^\eta_s|+|D_\theta \gamma_s|+|D_\theta\eta_s|)\,ds\middle|\mathcal{F}_t\right].
\end{aligned}$$

The conclusion directly comes from Lemma 3.3.7. □

3.4 Applications and Examples

3.4.1 Gradient of the related PDE

Here we consider that η and γ are smooth functions, $\eta_t=\eta(t,X_t)$ and $\gamma_t=\gamma(t,X_t)$, of the solution $X=X^x$ of the SDE: for $x\in\mathbb{R}^d$

$$X_t = x+\int_0^t b(s,X_s)ds+\int_0^t \sigma(s,X_s)dW_s,\quad 0\leq t\leq T. \tag{3.41}$$

We suppose that b and σ are continuous on $[0,T]\times\mathbb{R}^d$ and of class C^1 with respect to x with bounded first derivatives. According to [69, Theorem 2.2.1], we have :

Lemma 3.4.1. *The SDE admits a unique solution X in $S^\infty((0,T),\mathbb{R}^d)$ such that:*

1. *For any $i\in\{1,\cdots,d\}$, for all $t\in[0,T]$, $X^i_t\in\mathbb{D}^{1,\infty}$ and for all $p\in[1,+\infty)$,*

$$\sup_{0\leq\theta\leq t}\mathbb{E}\left[\sup_{\theta\leq s\leq T}|D_\theta X^i_s|^p\right]<+\infty. \tag{3.42}$$

2. *The process DX^i satisfies the linear SDE*

$$D_\theta X^i_t = \sigma_i(\theta, X_\theta) + \sum_{k=1}^d \int_\theta^t \frac{\partial b_i}{\partial x_k}(s, X_s) D_\theta X^k_s ds + \sum_{j=1}^d \sum_{k=1}^d \int_\theta^t \frac{\partial \sigma^j_i}{\partial x_k}(s, X_s) D_\theta X^k_s dW^j_s.$$

Under this setting,

$$\eta_t = \eta(t, X_t) = \eta(0, x) + \int_0^t (\mathcal{L}\eta)(s, X_s) ds + \int_0^t \partial_x \eta(s, X_s)\sigma(s, X_s) dW_s,$$

where $\mathcal{L}$ is the infinitesimal generator of the SDE (3.41):

$$\mathcal{L}\phi = \langle b, \partial_x \phi \rangle + \frac{1}{2}\mathrm{tr}\left(\sigma\sigma^* \partial^2_x \phi\right). \tag{3.43}$$

Hence $b^\eta_s = (\mathcal{L}\eta)(s, X_s)$ and $\sigma^\eta_s = \partial_x \eta(s, X_s)\sigma(s, X_s)$.

Example 3.4.2. *If we apply Itô's formula to $\eta = \varphi(X)$, with $d = 1$ and $\varphi(x) = \frac{\eta^\star - \eta_\star}{\pi}\arctan(x) + \frac{\eta^\star + \eta_\star}{2}$, then η satisfies all required conditions (Assumptions 13 and 14) of the previous sections.*

In this section, the superscript t, x indicates the dependence of the solution on the initial data (t, x), and it will be omitted when the context is clear. In this Markovian setting, it is known that the coupled system of Equations (3.41)-(3.2) is related to the solution of the PDE:

$$\frac{\partial u}{\partial t} + \mathcal{L}u - (p-1)\frac{|u|^{q-1}}{\eta(t,x)^{q-1}} u + \gamma(t, x) = 0, \quad u(T, \cdot) = +\infty \tag{3.44}$$

If $Y^{t,x}$ solves the BSDE (3.2) when η_s and γ_s are replaced by $\eta(s, X^{t,x}_s)$ and $\gamma(s, X^{t,x}_s)$, then for any $0 \leq t \leq s < T$, $Y^{t,x}_s = u(s, X^{t,x}_s)$ and u is the viscosity solution of the previous PDE. See [74, 41, 75, 21]. Furthermore the expansion (3.7) of Y corresponds to:

$$\forall (t, x) \in [0, T) \times \mathbb{R}^d, \qquad u(t, x) = \frac{\eta(t, x)}{(T-t)^{p-1}} + \frac{h(t, x)}{(T-t)^p}, \tag{3.45}$$

with $|h(t, x)| = O(T-t)^2$ at time T. This property is also given in [41, Lemma 4.1]. Here h is defined thanks to the relation: $h(s, X^{t,x}_s) = H^{t,x}_s$.

Proposition 3.4.3. *We assume that:*

- *The functions* $(t,x) \mapsto \eta(t,x)$, $(t,x) \mapsto \mathcal{L}\eta(t,x)$ *and* $(t,x) \mapsto \gamma(t,x)$ *are continuous w.r.t.* (t,x) *and of class* C^1 *w.r.t.* x*, with bounded derivatives.*
- *The matrix* σ *is uniformly elliptic, that is if there exists* $\lambda > 0$ *such that*

$$\forall s \in [0,T],\ \forall (x,y) \in \mathbb{R}^d \times \mathbb{R}^d, \quad \langle \sigma(s,x)\sigma^*(s,x)y, y\rangle \geq \lambda |y|^2.$$

The solution u *is of class* C^1 *w.r.t.* x *and for* $0 \leq t < T$ *and* $x \in \mathbb{R}^d$

$$\partial_x u(t,x) = \frac{\partial_x \eta(t,x)}{(T-t)^{p-1}} + \frac{\partial_x h(t,x)}{(T-t)^p}.$$

Finally the approximating sequences u^n *and* $\partial_x u^n$ *(resp.* h^n *and* $\partial_x h^n$*) pointwise converge to* u *and* $\partial_x u$ *on* $[0,T) \times \mathbb{R}^d$ *(resp. to* h *and* $\partial_x h$ *on* $[0,T] \times \mathbb{R}^d$*), where* u^n *is the continuous viscosity solution of the PDE* (3.44) *with terminal condition* $u^n(T,\cdot) = n$ *and* h^n *is defined by*

$$u^n(t,x) = \frac{\eta(t,x)}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p-1}} + \frac{h^n(t,x)}{\left(T - t + \left(\frac{\eta^\star}{n}\right)^{q-1}\right)^{p}}, \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Proof. The function h^n is linked to $\mathcal{H}^n$ by the relation: $h^n(s, X_s^{t,x}) = \mathcal{H}_s^{n,t,x}$. We know that u^n and h^n pointwise converge to u and h (from the convergence of Y^n and $\mathcal{H}^n$ to Y and H), which gives the asymptotic expansion of u.

Under our setting, we use [64, Lemma 2.4 and Theorem 3.1] to deduce that $D_\theta Y_s^n$ and $D_\theta \mathcal{H}_s^n$ are of the form

$$D_\theta Y_s^n = \nabla Y_s^n (\nabla X_\theta)^{-1} \sigma(\theta, X_\theta) 1_{\theta \leq s}, \quad D_\theta \mathcal{H}_s^n = \nabla \mathcal{H}_s^n (\nabla X_\theta)^{-1} \sigma(\theta, X_\theta) 1_{\theta \leq s},$$

where the processes ∇X, ∇Y^n and $\nabla \mathcal{H}^n$ are solution of the variational equations related to (3.41), (3.3) and (3.22). Furthermore h^n is of class C^1 w.r.t. x with $\partial_x h^n = \nabla \mathcal{H}^n$ and $\partial_x h^n$ is continuous on $[0,T] \times \mathbb{R}^d$. Let us emphasize that these results cannot be directly used for Y (singularity in the terminal condition) or H (singularity in the generator).

However we can define the solution $(\nabla H, \nabla Z^H)$ of the variational equation related to

(3.8)

$$\nabla H_s = \int_s^T \left[(T-u)\partial_x b^\eta(u,X_u)\nabla X_u + (T-u)^p \partial_x \gamma(u,X_u)\nabla X_u\right] du$$
$$- \int_s^T \frac{\partial G}{\partial \eta}(u,H_u,\eta_u)\partial_x \eta(u,X_u)\nabla X_u du - \int_s^T \frac{\partial G}{\partial h}(u,H_u,\eta_u)\nabla H_u du - \int_s^T \nabla Z^H_u dW_u$$

using again that a.s. $|H_u| \leq C(T-u)^2$ close to T. And from Lemma 3.2.1, we also have

$$D_\theta H_s = \nabla H_s (\nabla X_\theta)^{-1} \sigma(\theta, X_\theta) 1_{\theta \leq s}.$$

Then we check the proof of [64, Theorem 3.1] to deduce that $\partial_x h$ exists. We define

$$\nabla X^\varepsilon_s = \frac{1}{\varepsilon}\left(X^{t,x+\varepsilon}_s - X^{t,x}_s\right), \quad \nabla H^\varepsilon_s = \frac{1}{\varepsilon}\left(H^{t,x+\varepsilon}_s - H^{t,x}_s\right), \quad \nabla Z^\varepsilon_s = \frac{1}{\varepsilon}\left(Z^{H,t,x+\varepsilon}_s - Z^{H,t,x}_s\right)$$

and prove that ∇H^ε converges to ∇H when ε goes to zero. First note that

$$\nabla H^\varepsilon_s = \int_s^T \left[(T-u)\partial_x \widetilde{b}^\eta(u) + (T-u)^p \partial_x \widetilde{\gamma}(u) - \partial_\eta \widetilde{G}(u)\right] \nabla X^\varepsilon_u du$$
$$- \int_s^T \partial_h \widetilde{G}(u) \nabla H^\varepsilon_u du - \int_s^T \nabla Z^\varepsilon_u dW_u$$

where

$$\partial_x \widetilde{b}^\eta(u) = \int_0^1 \partial_x b^\eta(u, X^{t,x}_u + a(X^{t,x+\varepsilon}_u - X^{t,x}_u))da$$
$$\partial_x \widetilde{\gamma}(u) = \int_0^1 \partial_x \gamma(u, X^{t,x}_u + a(X^{t,x+\varepsilon}_u - X^{t,x}_u))da$$
$$\partial_\eta \widetilde{G}(u) = \int_0^1 \partial_\eta G(u, \eta^{t,x}_u + a(\eta^{t,x+\varepsilon}_u - \eta^{t,x}_u), H^{t,x+\varepsilon}_u)da$$
$$\times \int_0^1 \partial_x \eta(u, X^{t,x}_u + \alpha(X^{t,x+\varepsilon}_u - X^{t,x}_u))d\alpha$$
$$\partial_h \widetilde{G}(u) = \int_0^1 \partial_h G(u, \eta^{t,x}_u, H^{t,x}_u + a(H^{t,x+\varepsilon}_u - H^{t,x}_u))da$$

The key point now is that $|H^{t,x}_u| \leq C(T-u)^2$ on the interval $[T-\delta, T]$ and both constants C and δ depend only on the bounds on η and γ, and not on x. Therefore the processes $\partial_\eta \widetilde{G}$ and $\partial_h \widetilde{G}$ are bounded, uniformly w.r.t. ε. The rest of the proof can be copied from [64], to conclude that $\partial_x h$ exists and is equal to ∇H and that $\partial_x h$ is continuous on $[0,T] \times \mathbb{R}^d$.

From (3.45), we deduce that $\partial_x u$ exists on $[0,T) \times \mathbb{R}^d$ and is given by the statement of the proposition. From our convergence result (Lemma 3.3.7), we deduce that the sequence

$\nabla\mathcal{H}^n$ converges to ∇H. Hence the sequence $\partial_x h^n$ converges to $\partial_x h$ on $[0,T]\times\mathbb{R}^d$. The result follows immediately for $\partial_x u^n$. □

Estimate (3.17) becomes for $0\leq\theta\leq t\leq T$

$$\begin{aligned}|D_\theta H_t| &= |\nabla H_t(\nabla X_\theta)^{-1}\sigma(\theta,X_\theta)|\\ &\leq C(T-t)\mathbb{E}\left[\int_t^T\left(|\partial_x(\mathcal{L}u)(s,X_s)|+|\partial_x\eta(s,X_s)|+|\partial_x\gamma(s,X_s)|\right)|D_\theta X_s|ds\middle|\mathcal{F}_t\right]\\ &\leq C(T-t)\mathbb{E}\left[\int_t^T|D_\theta X_s|ds\middle|\mathcal{F}_t\right].\end{aligned}$$

From Lemma 3.4.1 and Hölder's inequality, we deduce that

$$|\partial_x h(t,x)| = |\nabla H_t| \leq C(T-t).$$

Remark 3.4.4. *In [41, Theorem 2.9], it is already proved that u is of class C^1 w.r.t. t.*

3.4.2 Sensitivity in liquidation problem

Malliavin calculus is a useful tool to analyze the sensitivity in finance, see among many others [35, 39]. In the liquidation problem mentioned in the introduction, the optimal state process is given by

$$\Xi_s = x\exp\left(-\int_t^s\left(\frac{Y_u}{\eta_u}\right)^{q-1}du\right)$$

or with the previous notations:

$$\begin{aligned}\Xi_s &= x\exp\left(-\int_t^s\frac{1}{(T-u)}\left(1+\frac{H_u}{\eta_u(T-u)}\right)^{q-1}du\right)\\ &= x\frac{T-s}{T-t}\exp\left(-\int_t^s\frac{1}{(T-u)}\left[\left(1+\frac{H_u}{\eta_u(T-u)}\right)^{q-1}-1\right]du\right).\end{aligned}$$

In particular for $0\leq\theta\leq s<T$

$$D_\theta\Xi_s = -(q-1)\Xi_s\int_t^s\left|\frac{Y_u}{\eta_u}\right|^{q-2}\text{sign }(Y_u)D_\theta\left(\frac{Y_u}{\eta_u}\right)du. \tag{3.46}$$

A key argument in the greeks computations is the positivity of the Malliavin covariance matrix. This property is ensured if there is a diffusion part in Ξ with an elliptic diffusion matrix (see again [35, 39]). In our case, there is no diffusion part for Ξ. Worse than this, we know some degenerate examples.

Indeed in [5, Section 5], the authors consider the case where $\gamma = 0$ and η has uncorrelated multiplicative increments. In our setting, it means that:

Lemma 3.4.5. *When η is an Itô process, η has uncorrelated multiplicative increments if and only if the drift b^η is of the form: $b^\eta_t = g(t)\eta_t$ where g is a deterministic function.*

Proof. If the drift b^η is of the form: $b^\eta_t = g(t)\eta_t$, from [5, Example 5.1], η has uncorrelated multiplicative increments. Conversely we have

$$\eta_t = \eta_0 + \int_0^t b^\eta_s ds + \int_0^t \sigma_s dW_s \Longrightarrow \mathbb{E}[\eta_t] = \eta_0 + \int_0^t \mathbb{E}[b^\eta_s] ds.$$

If we consider $M_t = \eta_t / \mathbb{E}[\eta_t]$, from [5, Lemma 5.1], η has uncorrelated multiplicative increments if and only if M is a martingale. But

$$dM_t = \frac{1}{(\mathbb{E}[\eta_t])^2} \left[\mathbb{E}[\eta_t] b^\eta_t - \mathbb{E}[b^\eta_t]\eta_t \right] dt + \frac{\sigma_t}{\mathbb{E}[\eta_t]} dW_t.$$

Hence $b^\eta_t = \left(\dfrac{\mathbb{E}[b^\eta_t]}{\mathbb{E}[\eta_t]} \right) \eta_t$. □

From [5, Propositions 5.2 and 5.3], we know that η has uncorrelated multiplicative increments if and only if Ξ is deterministic, that is $D_\theta \Xi_s = 0$ for any $0 \leq \theta \leq s \leq T$. We also have another result.

Lemma 3.4.6. *When $\gamma = 0$, η has uncorrelated multiplicative increments if and only if for any $0 \leq \theta \leq u < T$, $D_\theta \left(\dfrac{Y_u}{\eta_u} \right) = 0$.*

Proof. From the proof of [5, Proposition 5.3], we also know that if η has uncorrelated multiplicative increments, then $\dfrac{Y}{\eta}$ is deterministic, hence its Malliavin derivative is zero. Conversely if the Malliavin derivaitive of Y/η is null, using (3.46), $D_\theta \Xi_s = 0$, thus Ξ is deterministic and η has uncorrelated multiplicative increments. □

Moreover we can explicitly compute H, either from [5, Proposition 5.3] where Y is given, or directly.

Lemma 3.4.7. *Assume that η has uncorrelated multiplicative increments, with $b_t^\eta = g(t)\eta_t$. Then $H_t = \eta_t(T-t)h(t)$ with*

$$h(t) = -1 + \left(\frac{1}{T-t}\int_t^T \exp\left(-(q-1)\int_t^s g(u)du\right) ds\right)^{1-p}.$$

Proof. Since $\gamma = 0$ and $b_t^\eta = g(t)\eta_t$, H is the solution of the BSDE with generator

$$F(t,h) = (T-t)\eta_t g(t) - (p-1)\eta_t \left[\left(1 + \frac{1}{\eta_t(T-t)}h\right)\left|1+\frac{1}{\eta_t(T-t)}h\right|^{q-1} - 1 \right.$$
$$\left. - q\frac{1}{\eta_t(T-t)}h\right]$$

and terminal condition 0. Make the ansatz that $H_t = \eta_t(T-t)h(t)$. Then

$$dH_t = \left[(T-t)h(t)\eta_t g(t) + \eta_t(T-t)h'(t) - \eta_t h(t)\right]dt + \sigma_t^\eta dW_t,$$
$$F(t,H_t) = (T-t)\eta_t g(t) - (p-1)\eta_t\left[(1+h(t))\left|1+h(t)\right|^{q-1} - 1 - qh(t)\right].$$

Therefore $dH_t + F(t,H_t)dt$ is a martingale if

$$(T-t)i(t)g(t) + (T-t)i'(t) - (p-1)\left[i(t)\left|i(t)\right|^{q-1} - i(t)\right] = 0$$

with $i(t) = h(t)+1$. Define G as the solution of $G'(t) = g(t)G(t)$ with $G(0) = 1$. Then

$$(T-t)(i(t)G(t))' = (p-1)G(t)\left[i(t)\left|i(t)\right|^{q-1} - i(t)\right].$$

We can verify that

$$i(t)G(t) = \left(\frac{1}{T-t}\int_t^T \frac{1}{G(s)^{q-1}}ds\right)^{1-p}.$$

Thus

$$h(t) = -1 + \left(\frac{1}{T-t}\int_t^T \left(\frac{G(t)}{G(s)}\right)^{q-1} ds\right)^{1-p}.$$

Note that $h(t) \sim (T-t)$ as t goes to T. By uniqueness of H, we obtain the result. □

Remark that we obtain an explicit expression for Y:

$$Y_t = \frac{\eta_t}{(T-t)^{p-1}} + \frac{H_t}{(T-t)^p} = \frac{\eta_t}{(T-t)^{p-1}} \left(\frac{1}{T-t} \int_t^T \exp\left(-(q-1) \int_t^s g(u)du \right) ds \right)^{1-p}$$
$$= \eta_t \left(\int_t^T \exp\left(-(q-1) \int_t^s g(u)du \right) ds \right)^{1-p}.$$

Let us consider the case where η is deterministic. Then for $\theta \leq s < T$

$$D_\theta \Xi_s = -(q-1)\Xi_s \int_t^s \frac{1}{(T-u)^2 \eta_u} \left| 1 + \frac{H_u}{\eta_u(T-u)} \right|^{q-2} \operatorname{sign} \left(1 + \frac{H_u}{\eta_u(T-u)} \right) D_\theta H_u du.$$

and $D_\theta H$ is the solution of the BSDE (3.16), which becomes:

$$D_\theta H_t = \int_t^T \left[(T-s)^p D_\theta \gamma_s - \frac{\partial G}{\partial h}(s, H_s, \eta_s) D_\theta H_s \right] ds - \int_t^T D_\theta Z_s^H dW_s$$
$$= \mathbb{E}\left[\int_t^T (T-s)^p D_\theta \gamma_s \Gamma_{t,s} ds \middle| \mathcal{F}_t \right]$$

where $\Gamma_{t,s}$ is given by (3.19). If $\gamma_s = \gamma(s, X_s)$, then $D_\theta \gamma_s = (\partial_x \gamma)(s, X_s) D_\theta X_s$. In this case, we can find easy conditions on $\partial_x \gamma$ and the coefficients b and σ of the SDE of X such that the Malliavin covariance matrix of $D_\theta H$ is definite positive. For example if $\partial_x \gamma$ is bounded away from zero and if the parameters b and σ satisfy the conditions of Lemma 3.4.1 and if σ is uniformly elliptic (see Proposition 3.4.3 for the definition), we can apply [69, Theorems 2.3.1 and 2.3.3].

Now in general we have for $0 \leq \theta \leq s < T$

$$D_\theta \Xi_s = -(q-1)\Xi_s \int_t^s \left| \frac{Y_u}{\eta_u} \right|^{q-2} \operatorname{sign}(Y_u) D_\theta \left(\frac{Y_u}{\eta_u} \right) du$$
$$= -(q-1)\Xi_s \int_t^s \frac{1}{(T-u)^2 \eta_u^2} \left| 1 + \frac{H_u}{\eta_u(T-u)} \right|^{q-2}$$
$$\times \operatorname{sign} \left(1 + \frac{H_u}{\eta_u(T-u)} \right) \left[\eta_u D_\theta H_u - H_u D_\theta \eta_u \right] du$$

with $D_\theta H$ given by (3.16). Existence of tractable conditions such that the Malliavin covariance matrix is definite positive is left for further research.

3.5 Appendix

Let us evoke the arguments of [42, Theorem 23]. We want to solve the BSDE (3.8)

$$H_t = \int_t^T F(s, H_s)ds - \int_t^T Z_s^H dW_s = \mathbb{E}\left[\int_t^T F(s, H_s)ds \middle| \mathcal{F}_t\right]$$

where F is gievn by (3.9). We define the operator

$$\Gamma(H)_t = \mathbb{E}\left[\int_t^T F(s, H_s)ds \middle| \mathcal{F}_t\right]$$

and a solution is a fixed point of this operator Γ.

Proposition 3.5.1. *If η is bounded away from zero by $\eta_\star$ and if the drift b^η of η is bounded, there exists a process (H, Z^H) solution of the previous BSDE and there exists three constants $\delta > 0$, $R > 0$ and C such that a.s. on $[T-\delta, T]$, $|H_t| \leq R(T-t)^2$ and on $[0, T]$, $|H_t| \leq C$.*

Proof. Remark that

$$\begin{aligned} G(t, h, \eta) &= (p-1)\eta \left[\left(1 + \frac{1}{\eta(T-t)}h\right)\left|1 + \frac{1}{\eta(T-t)}h\right|^{q-1} - 1 - q\frac{1}{\eta(T-t)}h\right] \\ &= \frac{qh^2}{\eta(T-t)^2}\int_0^1 \left|1 + a\frac{1}{\eta(T-t)}h\right|^{q-2} \operatorname{sign}\left(1 + a\frac{1}{\eta(T-t)}h\right)(1-a)da. \end{aligned}$$

Now suppose for a while that $|H_t| \leq R(T-t)^2$ on $[T-\delta, T]$. Then for any $a \in [0, 1]$ and $T - \delta \leq t \leq T$

$$a\frac{|H_t|}{\eta_t(T-t)} \leq a\frac{R(T-t)}{\eta_t} \leq \frac{R\delta}{\eta_\star} \leq \frac{1}{2},$$

if we choose $\delta \leq \dfrac{\eta_\star}{2R}$. Moreover

$$\begin{aligned} \frac{\partial G}{\partial h}(t, h, \eta) &= \frac{p}{(T-t)}\left(\left|1 + \frac{1}{\eta(T-t)}h\right|^{q-1} - 1\right) \\ &= \frac{qh}{\eta(T-t)^2}\int_0^1 \left|1 + a\frac{1}{\eta(T-t)}h\right|^{q-2} \operatorname{sign}\left(1 + a\frac{1}{\eta(T-t)}h\right)da. \end{aligned}$$

Thus if again $|H_t| \leq R(T-t)^2$ on $[T-\delta, T]$, and under our previous condition on δ, we have

$$\left|\frac{\partial G}{\partial h}(t, H_t, \eta_t)\right| \leq \frac{qR}{\eta_\star} 2^{|q-2|} = L.$$

Therefore if both H and $\widetilde{H}$ are bounded from above by $t \mapsto R(T-t)^2$ on $[T-\delta, T]$, then

$$\begin{aligned}
&|\Gamma(H)_t - \Gamma(\widetilde{H})_t| \\
&\leq \mathbb{E}\left[\int_t^T |F(s, H_s) - F(s, \widetilde{H}_s)| ds \middle| \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\int_t^T L|H_s - \widetilde{H}_s| ds \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[\int_t^T L\frac{|H_s - \widetilde{H}_s|}{(T-s)^2}(T-s)^2 ds \middle| \mathcal{F}_t\right] \\
&\leq L(T-t)^3\|H - \widetilde{H}\|_{\mathcal{H}^\delta} \leq \delta L\|H - \widetilde{H}\|_{\mathcal{H}^\delta}(T-t)^2 \leq \frac{1}{2}\|H - \widetilde{H}\|_{\mathcal{H}^\delta}(T-t)^2
\end{aligned}$$

if $\delta \leq 1/(2L)$. Hence

$$\|\Gamma(H) - \Gamma(\widetilde{H})\|_{\mathcal{H}^\delta} \leq \frac{1}{2}\|H - \widetilde{H}\|_{\mathcal{H}^\delta}$$

that is Γ is a contraction on the ball of $\mathcal{H}^\delta$ with radius R. Finally

$$\begin{aligned}
|\Gamma(H)_t| &\leq |\Gamma(H)_t - \Gamma(0)_t| + |\Gamma(0)_t| \\
&\leq \frac{1}{2}\|H\|_{\mathcal{H}^\delta}(T-t)^2 + \mathbb{E}\left[\int_t^T |(T-s)b^\eta_s + (T-s)^p\gamma_s|\, ds \middle| \mathcal{F}_t\right] \\
&\leq \frac{R}{2}(T-t)^2 + (T-t)^2\left[\frac{1}{2}\|b^\eta\|_\infty + \frac{1}{p+1}(T-t)^{p-1}\gamma^\star\right] \\
&\leq \left[\frac{1}{2}\|b^\eta\|_\infty + \frac{1}{p+1}\gamma^\star + \frac{R}{2}\right](T-t)^2 \leq R(T-t)^2
\end{aligned}$$

if $\delta \leq 1$ and $R = \|b^\eta\|_\infty + \dfrac{2}{p+1}\gamma^\star$. To summarize, if

$$R = \|b^\eta\|_\infty + \frac{2}{p+1}\gamma^\star, \quad L = \frac{qR}{\eta_\star}2^{|q-2|}, \quad \delta = \min\left(1, T, \frac{1}{2L}, \frac{\eta_\star}{2R}\right)$$

then Γ is a contraction from the ball of $\mathcal{H}^\delta$ with radius R into itself, thus has a unique fixed point H, which is the solution of the wanted BSDE. Moreover the solution (H, Z^H) is the limit in $\mathcal{H}^\delta$ of the sequence $(H^k, Z^{H,k})$ unique solution in $\mathcal{H}^\delta$ of

$$H^k_t = \int_t^T F(s, H^{k-1}_s) ds - \int_t^T Z^{H,k}_s dW_s$$

with $(H^0, Z^{H,0}) = (0,0)$ and for any k and $t \in [T-\delta, T]$, $|H^k_t| \leq R(T-t)^2$.

Note that the generator F is continuous and monotone in h on $[0, T-\delta]$:

$$\begin{aligned}
&F(t,h) - F(t,\widetilde{h}) \\
&= -(p-1)\eta_t \left[\left(1 + \frac{1}{\eta_t(T-t)} h\right) \left|1 + \frac{1}{\eta_t(T-t)} h\right|^{q-1} \right. \\
&\quad \left. - \left(1 + \frac{1}{\eta_t(T-t)} \widetilde{h}\right) \left|1 + \frac{1}{\eta_t(T-t)} \widetilde{h}\right|^{q-1} \right] + \frac{p}{T-t}(h - \widetilde{h}) \\
&= -\frac{p}{(T-t)}(h-\widetilde{h}) \int_0^1 \left|1 + \frac{1}{\eta_t(T-t)}\widetilde{h} + a\frac{1}{\eta_t(T-t)}(h-\widetilde{h})\right|^{q-1} da + \frac{p}{T-t}(h-\widetilde{h}),
\end{aligned}$$

thus for $t \leq T - \delta$

$$(F(t,h) - F(t,\widetilde{h}))(h-\widetilde{h}) \leq \frac{p}{\delta}(h-\widetilde{h})^2.$$

And

$$\begin{aligned}
|F(t,h)| &\leq (T-t)\|b^\eta\|_\infty + (T-t)^p \gamma^\star + (p-1)\eta_t \left[\left|1 + \frac{1}{\eta_t \delta} h\right|^q + 1 + \frac{q}{\eta_t \delta}|h| \right] \\
&\leq T\|b^\eta\|_\infty + T^p\gamma^\star + (p-1)\eta_t 2^{q-1}\left(1 + \frac{1}{\eta_t^q \delta^q}|h|^q\right) + \frac{p}{\delta}|h| \\
&\leq T\|b^\eta\|_\infty + T^p\gamma^\star + \frac{p}{\delta}|h| + (p-1)2^{q-1}\frac{1}{\eta_\star^{q-1}\delta^q}|h|^q + (p-1)(2^{q-1}+1)\eta_t.
\end{aligned}$$

In particular

$$\mathbb{E}\left[\sup_{|h|\leq M} |F(t,h)| \leq C(1 + \mathbb{E}[\eta_t])\right] < +\infty.$$

Since $H_{T-\delta}$ is a bounded random variable, the BSDE

$$H_t = H_{T-\delta} + \int_t^{T-\delta} F(s, H_s) ds - \int_t^{T-\delta} Z^H_s dW_s$$

has a unique solution on $[0, T-\delta]$ and there exists a constant C such that $|H_t| \leq C$ on $[0, T-\delta]$. See [72, Proposition 5.24] and [13, Proposition 3.3]. Notice that the martingale $\int Z^H dW$ is a BMO-martingale on $[0, T-\delta]$. Since H is bounded (by C), we can modify the generator F outside the interval $[-C, C]$, such that F is Lipschitz continuous and with linear growth w.r.t. h. Then we can define the sequence H^k on $[0, T-\delta]$, converging to H and such that a.s. $|H^k_t| \leq C$. □

We define on $[0,T)$ the process

$$\widehat{Y}_t = \frac{\eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} H_t.$$

We can easily verify that for any $0 \leq t \leq s < T$

$$\widehat{Y}_t = \widehat{Y}_s + \int_t^s \left(-(p-1)\frac{|\widehat{Y}_u|^{q-1}}{\eta_u^{q-1}} \widehat{Y}_u + \gamma_u \right) du - \int_t^s \widehat{Z}_u dW_u,$$

and a.s.

$$\lim_{t \to T} \widehat{Y}_t = +\infty.$$

Moreover on $[T-\delta, T]$,

$$\widehat{Y}_t \geq \frac{\eta_t}{(T-t)^{p-1}} - R\frac{1}{(T-t)^{p-2}} \geq \frac{1}{(T-t)^{p-1}} \left(\eta_\star - R(T-t)\right) \geq \frac{\eta_\star}{2(T-t)^{p-1}}.$$

Thus $\widehat{Y}$ is non-negative on $[T-\delta, T]$. By standard comparison principle on $[0, T-\delta]$ (see [72, Section 5.3.6]), $\widehat{Y}$ is also non-negative on $[0,T]$. Since Y is the minimal non-negative solution, a.s. for any $t \in [0,T]$, $Y_t \leq \widehat{Y}_t$.

From the uniqueness result of [42, Theorem 10], $\widehat{Y} = Y$ and thus the minimal solution of (3.2) is given by (3.7). Let us evoke the main arguments.

Proposition 3.5.2. *If η and the process $(\sigma_u^\eta)^2 \eta_u^{-q-1}$ are bounded, $\widehat{Y} = Y$.*

Proof. We split Y as follows

$$Y_t = \frac{\eta_t}{(T-t)^{p-1}} + \frac{1}{(T-t)^p} \mathcal{H}_t.$$

Since $\widehat{Y} \geq Y$, we deduce that a.s. for any t, $\mathcal{H}_t \leq H_t$. Our goal is to prove that $\mathcal{H} = H$, thus $\widehat{Y} = Y$.

Since $Y_t \geq 0$, for any t, $-(T-t)\eta_t \leq \mathcal{H}_t$, and on $[T-\delta, T]$, $\mathcal{H}_t \leq H_t \leq R(T-t)^2$. Thus a.s. $\lim_{t \to T} \mathcal{H}_t = 0$. From the dynamics of Y, $\mathcal{H}$ solves the BSDE (3.8) on $[0,\tau]$ for any $\tau < T$. Finally

$$1 + \frac{\mathcal{H}_t}{\eta_t (T-t)} = (T-t)^{p-1} \frac{Y_t}{\eta_t} \geq 0.$$

Hence

$$
\begin{aligned}
F(t,\mathcal{H}_t) &= [(T-t)b_t^\eta + (T-t)^p\gamma_t] \\
&\quad - (p-1)\eta_t \left[\left(1+\frac{1}{\eta_t(T-t)}\mathcal{H}_t\right)\left|1+\frac{1}{\eta_t(T-t)}\mathcal{H}_t\right|^{q-1} - 1 - q\frac{1}{\eta_t(T-t)}\mathcal{H}_t \right]
\end{aligned}
$$

is controlled on $[0,T]$:

$$
\begin{aligned}
-\eta_t - (p-1)\eta_t\left(1+\frac{R}{\eta_t}(T-t)\right)^q &\leq F(t,\mathcal{H}_t) - [(T-t)b_t^\eta + (T-t)^p\gamma_t] \\
&\leq (p-1)\eta_t + pR(T-t).
\end{aligned}
$$

Hence $\mathcal{H}$ is also a solution of the BSDE (3.8).

Now let us consider $\Delta H = H - \mathcal{H}$, $\Delta Z = Z^H - \mathcal{Z}$ and we proceed as in the proof of [42, Proposition 20]. We denote

$$
g(y) = |y|^{q-1}y, \quad g'(y) = q|y|^{q-1}, \quad g''(y) = q(q-1)|y|^{q-2}\text{sgn}(y).
$$

Then for $0 \leq t \leq T$

$$
\begin{aligned}
\Delta H_t &= \int_t^T F(s,H_s) - F(s,\mathcal{H}_s)ds - \int_t^T \Delta Z_s dW_s \\
&= -(p-1)\int_t^T \eta_s \left[g\left(1+\frac{H_s}{\eta_s(T-s)}\right) - g\left(1+\frac{\mathcal{H}_s}{\eta_s(T-s)}\right) \right. \\
&\quad \left. -g'\left(1+\frac{\mathcal{H}_s}{\eta_s(T-s)}\right)\frac{\Delta H_s}{\eta_s(T-s)} \right] ds \\
&\quad - (p-1)\int_t^T \eta_s \left[g'\left(1+\frac{\mathcal{H}_s}{\eta_s(T-s)}\right) - q\right]\frac{\Delta H_s}{\eta_s(T-s)}ds - \int_t^T \Delta Z_s dW_s.
\end{aligned}
$$

Define for $T-\delta \leq t \leq s < T$

$$
\begin{aligned}
\Xi_s &= \eta_s \left[g\left(1+\frac{H_s}{\eta_s(T-s)}\right) - g\left(1+\frac{\mathcal{H}_s}{\eta_s(T-s)}\right) - g'\left(1+\frac{\mathcal{H}_s}{\eta_s(T-s)}\right)\frac{\Delta H_s}{\eta_s(T-s)} \right], \\
\Upsilon_{t,s} &= \int_t^s \eta_u \left[g'\left(1+\frac{\mathcal{H}_u}{\eta_u(T-u)}\right) - q\right]\frac{1}{\eta_u(T-u)}du \\
&= \int_t^s q\left[\left|1+\frac{\mathcal{H}_u}{\eta_u(T-u)}\right|^{q-1} - 1\right]\frac{1}{(T-u)}du.
\end{aligned}
$$

Then for any $\varepsilon > 0$

$$\Delta H_t = \mathbb{E}\left[\Delta H_{T-\varepsilon}\exp(-(p-1)\Upsilon_{t,T-\varepsilon})\Big|\mathcal{F}_t\right] - (p-1)\mathbb{E}\left[\int_t^{T-\varepsilon}\Xi_s\exp(-(p-1)\Upsilon_{t,s})ds\Big|\mathcal{F}_t\right]$$
$$\leq \mathbb{E}\left[\Delta H_{T-\varepsilon}\exp(-(p-1)\Upsilon_{t,T-\varepsilon})\Big|\mathcal{F}_t\right]$$

since $\Xi_s \geq 0$ because g is a convex function on $[0,\infty)$.

Let us explain how to control the negative part of $\Upsilon_{t,s}$. From [42, Lemma 6], we also know that a.s. for any $t \in [0,T]$

$$Y_t \geq \left[\mathbb{E}\left[\int_t^T \eta_s^{1-q}ds\Big|\mathcal{F}_t\right]\right]^{1-p}.$$

Thus

$$1 + \frac{\mathcal{H}_t}{\eta_t(T-t)} = (T-t)^{p-1}\frac{Y_t}{\eta_t} \geq \left[\frac{\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T \eta_s^{1-q}ds\Big|\mathcal{F}_t\right]\right]^{1-p} \geq \frac{\eta_\star}{\eta_t}.$$

Evoke that η is an Itô process, bounded from below by $\eta_\star > 0$. By Itô's formula for $t \leq s \leq T$

$$\eta_s^{1-q} = \eta_t^{1-q} + \int_t^s (1-q)\eta_u^{-q}b_u^\eta du + \int_t^s (1-q)\eta_u^{-q}\sigma_u^\eta dW_u + \frac{q(q-1)}{2}\int_t^s \eta_u^{-q-1}|\sigma_u^\eta|^2 du.$$

Hence

$$\mathbb{E}\left[\int_t^T \eta_s^{1-q}ds\Big|\mathcal{F}_t\right] = (T-t)\eta_t^{1-q} + (q-1)\mathbb{E}\left[\int_t^T\left(\int_t^s(-\eta_u^{-q}b_u^\eta + \frac{q}{2}\eta_u^{-q-1}|\sigma_u^\eta|^2)du\right)ds\Big|\mathcal{F}_t\right]$$
$$= (T-t)\eta_t^{1-q} + (q-1)\mathbb{E}\left[\int_t^T (T-u)\theta_u du\Big|\mathcal{F}_t\right]$$

with

$$\theta_u = -\eta_u^{-q}b_u^\eta + \frac{q}{2}\eta_u^{-q-1}|\sigma_u^\eta|^2 = \frac{1}{\eta_u^{q-1}}\left(\frac{q}{2}\frac{|\sigma_u^\eta|^2}{\eta_u^2} - \frac{b_u^\eta}{\eta_u}\right).$$

Thus

$$Y_t \geq \left[(T-t)\eta_t^{1-q} + (q-1)\mathbb{E}\left[\int_t^T (T-u)\theta_u du\Big|\mathcal{F}_t\right]\right]^{1-p}$$
$$\geq \frac{\eta_t}{(T-t)^{p-1}}\left[1 + \frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du\Big|\mathcal{F}_t\right]\right]^{1-p},$$

and

$$(T-t)^{p-1}\frac{Y_t}{\eta_t} = 1 + \frac{\mathcal{H}_t}{\eta_t(T-t)} \geq \left[1 + \frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du \middle| \mathcal{F}_t\right]\right]^{1-p}.$$

We deduce that

$$\begin{aligned}
&\left|1 + \frac{\mathcal{H}_t}{\eta_t(T-t)}\right|^{q-1} - 1 \\
&\geq \left[1 + \frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du \middle| \mathcal{F}_t\right]\right]^{-1} - 1 \\
&= -\left[1 + \frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du \middle| \mathcal{F}_t\right]\right]^{-1} \frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du \middle| \mathcal{F}_t\right] \\
&\geq -\frac{(q-1)\eta_t^{q-1}}{T-t}\mathbb{E}\left[\int_t^T (T-u)\theta_u du \middle| \mathcal{F}_t\right].
\end{aligned}$$

Hence

$$\begin{aligned}
(p-1)\Upsilon_{t,s} &= (p-1)\int_t^s q\left[\left|1 + \frac{\mathcal{H}_u}{\eta_u(T-u)}\right|^{q-1} - 1\right]\frac{1}{(T-u)}du \\
&\geq -q\int_t^s \frac{\eta_u^{q-1}}{(T-u)^2}\mathbb{E}\left[\int_u^T (T-r)\theta_r dr \middle| \mathcal{F}_u\right] du.
\end{aligned}$$

If η and θ are bounded by $\eta^\star$ and $\|\theta\|_\infty$, we obtain

$$(p-1)\Upsilon_{t,s} \geq -\frac{q}{2}(\eta^\star)^{q-1}\|\theta\|_\infty.$$

Since for any $\varepsilon > 0$,

$$\Delta H_t \leq \mathbb{E}\left[\Delta H_{T-\varepsilon}\exp(-(p-1)\Upsilon_{t,T-\varepsilon}) \middle| \mathcal{F}_t\right] \leq C\mathbb{E}\left[\Delta H_{T-\varepsilon} \middle| \mathcal{F}_t\right]$$

we can pass to the limit to deduce that $\Delta H \leq 0$, that is $H \leq \mathcal{H}$. □

Remark that another sufficient condition would be: if $\eta \in L^\varrho(\Omega)$ and

$$\exp\left(q\int_t^T \frac{\eta_u^{q-1}}{(T-u)^2}\mathbb{E}\left[\int_u^T (T-r)\theta_r dr \middle| \mathcal{F}_u\right] du\right) \in L^{\varrho_*}(\Omega).$$

For example, η geometric Brownian motion: $b^\eta = \mu\eta$, $\sigma^\eta = \sigma\eta$ and

$$\eta_u^{q-1}\mathbb{E}\left[\theta_r \middle| \mathcal{F}_u\right] = f(u, r).$$

Here $\varrho_* = \infty$.

Chapter 4

Continuity problem for BSDE and IPDE with singular terminal condition

---

The notion of backward stochastic differential equations (BSDEs) was first introduced by Bismut in [15] in the linear setting and by Pardoux & Peng in [70] for non linear equation. One particular interest for the study of BSDE is the application to partial differential equations (PDEs). Indeed as proved by Pardoux & Peng in [71], BSDEs can be seen as generalization of the Feynman-Kac formula for non linear PDEs. Roughly speaking, if we can solve a system of two SDEs with one forward in time and one backward in time, then the solution is a deterministic function and is a (weak) solution of the related PDE. This is a method of characteristics to solve parabolic PDE. The converse assertion can be proved provided the solution of the PDE is enough regular to apply Itô's formula (see [32, Chapter 6]). Since then a large literature has been developped on this topic (see in particular the books [26], [32], [72] and the references therein). The extension to quasi-linear PDEs or to fully non linear PDEs has been already developed (see among other [63], [83] or [87]).

Among all semi-linear PDEs, a particular form has been widely studied:

$$\frac{\partial u}{\partial t}(t,x) + \Delta u(t,x) - u(t,x)|u(t,x)|^{q-1} = 0. \tag{4.1}$$

Baras & Pierre [11], Marcus & Veron [65] (and many other papers) have given existence and uniqueness results for this PDE. In [65] it is shown that every positive solution of (4.1) possesses a uniquely determined final trace g which can be represented by a couple $(\mathcal{S},\mu)$ where $\mathcal{S}$ is a closed subset of $\mathbb{R}^d$ and μ a non-negative Radon measure on $\mathcal{R} = \mathbb{R}^d \setminus \mathcal{S}$:

$$\lim_{t\to T} \int_{\mathcal{R}} u(t,x)\varphi(x)dx = \int_{\mathcal{R}} \varphi(x)d\mu(x), \quad \forall \varphi \in C_c(\mathcal{R}).$$

The final trace can also be represented by a positive, outer regular Borel measure ν, and ν is not necessary locally bounded. The two representations are related by:

$$\forall A \subset \mathbb{R}^d, A \text{ Borel}, \begin{cases} \nu(A) = +\infty & \text{if } A \cap \mathcal{S} \neq \emptyset \\ \nu(A) = \mu(A) & \text{if } A \subset \mathcal{R}. \end{cases}$$

The set $\mathcal{S}$ is the set of singular final points of u and it corresponds to a "blow-up" set of u. From the probabilistic point of view, Dynkin & Kuznetsov [30] and Le Gall [61] have proved similar results for the PDE (4.1) in the case $1 < q \leq 2$ using the theory of superprocesses. Now if we want to represent the solution u of (4.1) using an FBSDE (F means forward), we have to deal with a *singular* terminal condition ξ in the BSDE, which means that $\mathbb{P}(\xi = +\infty) > 0$. This singular case and the link between the solution of the BSDE with singular terminal condition and the viscosity solution of the PDE (4.1) have been studied first in [74] and developed in [75].

Besides PDEs motivation, BSDEs are a powerful tool to solve stochastic optimal control problems (see e.g. the survey article [33] or the books [73, 86]). In [5] and [56], it is proved that BSDEs with singular terminal condition provide a purely probabilistic solution of a stochastic control problem with a terminal constraint on the controlled process, motivated by models of optimal portfolio liquidation under stochastic price impact. On liquidation models see, e.g. [2, 3, 34, 43, 37, 47, 54], among many others. The related BSDEs are of the following form

$$-dY_t = -\frac{Y_t|Y_t|^{q-1}}{\eta_t} dt + \lambda_t dt - Z_t dW_t \tag{4.2}$$

with $\lim_{t \to T} Y_t = +\infty$ on $\mathcal{S}$. Parameter η is a measure of the illiquidity of the market, whereas λ penalizes the size of the remaining position of the portfolio. Here the singular set $\mathcal{S}$ corresponds to the scenarios with mandatory liquidation. The important feature is that the previous BSDE is directly related to a PDE similar to (4.1) and this link and PDEs technics have been used in [40, 41, 48, 80] to solve the same optimal liquidation problem.

In the standard L^p setting (see [26, 72]), the solution of the BSDE, with terminal condition ξ, is càdlàg [1] on $[0, T]$ and verifies

$$\lim_{t \to T} Y_t = \xi. \tag{4.3}$$

1. French acronym for right-continuous with left-limits

When ξ is not integrable, in particular if $\mathbb{P}(\xi = +\infty) > 0$, the classical notion of solution has to be adapted. As proved in [56], the minimal solution only satisfies: a.s.

$$\liminf_{t \to T} Y_t \geq \xi = Y_T.$$

Therefore it is called a super-solution in [56].

We refer to the problem of establishing that a candidate solution satisfies (4.3) as the "continuity problem". In the PDE's context, if it is quite immediately that under weak conditions, there exists a minimal (viscosity) solution u such that

$$\liminf_{(t,x) \to (T,x_0)} u(t,x) \geq u(T,x_0).$$

The continuity at time T is not obvious.

As explained in details in [1, Section 1.1], solving this problem is crucial to ensure:

- Uniqueness of the solution of the BSDE,
- Tight control for the liquidation problem (no extra liquidation cost or no strict super-hedging),
- Condition in optimal targeting problem [10].

From [76], it is known that the existence of the limit at time T essentially depends on the generator of the BSDE. The solution is càdlàg on $[0,T]$ provided we can control the growth of the generator w.r.t. y. But replacing $\geq$ by $=$ is more delicate and has been studied in [76, 81, 66, 1].

If some partial results are available for general condition ξ, the more accurate results are given in the Markovian case, that is when $\xi = g(X_T)$, where X is a diffusion process and g is a function defined on $\mathbb{R}^d$ with values in $[0,+\infty]$ [2]. In this case, the corresponding trace is $\mathcal{S} = \{g = +\infty\}$ and the measure μ has a density w.r.t. the Lebesgue measure given by g. In the rest of this chapter, we only consider this Markovian framework.

Let us now distinguish two different cases. In the first one, the forward diffusion process X is continuous. Then the related PDE is a semi-linear parabolic PDE with only local differential operator, as for example Equation (4.1). Then in [76], it is proved that if the generator is sufficiently non linear ($q > 3$ for PDE (4.1) and BSDE (4.2)), Condition (4.3) holds. Otherwise Malliavin's calculus is a useful tool to prove that (4.3) holds under some uniform ellipticity condition on the matrix diffusion of X. It has been done in [74] for the specific generator related to PDE (4.1) ; the general case is studied in the Section

2. The non-negativity of g is not necessary but it simplifies the presentation of the results.

2, which is in the final stages of writing for upcoming submission. To summarize, with non degenerate diffusion matrix, for continuous diffusion process X or equivalently for parabolic PDE, the continuity property holds and this property is coherent with the results obtained in [65, 31, 61] for the PDE (4.1).

In the second case, X is also driven by a Poisson process (or more generally by a Poisson random measure). Then the corresponding integro-partial differential equation (IPDE) (4.11) has a non-local integral operator. This kind of IPDE with terminal singularity has not been studied with analytical methods and is only considered (to our best knowledge) in [75]. From [76, 75], if the generator is sufficiently non linear, continuity property again holds. In other words, the behaviors with or without jumps (or with local or non-local operators) are the same.

The goal of this chapter is to provide an explicit example for which continuity property fails. We study the BSDE (4.2)

$$Y_t = g(N_T) - \int_t^T Y_s|Y_s|^{q-1}ds - \int_t^T U_s d\widetilde{N}_s, \quad 0 \leq t \leq T, \tag{4.4}$$

where N is a Poisson process with intensity λ and $\widetilde{N}$ is the compensated Poisson process: $\widetilde{N}_t = N_t - \lambda t$. We show that:

- The value $q = 2$ is critical. We construct an example for which $\mathcal{S} = \{g = +\infty\} = [x_0, \infty)$ and the minimal solution of the BSDE (4.4) or of the related PDE is the function $t \mapsto 1/(T-t)$. Hence the continuity problem (4.3) does not hold whatever $g\mathbf{1}_{g<+\infty}$ is. We also prove that the lack of continuity is due to the jump part of X ; adding a Brownian part does not change this fact.
- For $q < 2$, the solution of the BSDE (4.4) (or of the PDE) explodes at time T. The solution is compared with the solution for $q = 2$ and the first one is greater than the second. Again this behavior does not depend on the terminal value.
- For $2 < q$, continuity property holds. Note that for $q > 3$, the result is already proved in [76].

The main novelty of the chapter is the lack of continuity at time T for BSDEs (resp. for PDEs), when there are jumps (resp. when the operator is non-local). Here the regularity of the terminal condition (or of the trace in the context of [65]) does not influence the behavior of the solution. Somehow the solution forgets the terminal constraint. This property has never been observed in the literature.

This chapter is organized as follows. In Section 4.1, we present the setting for BSDEs and PDEs and the known results. Since our goal is to provide an example of discontinuity, the setting is not the most general (see [26, 55, 72, 85] for the wider framework of BSDEs with or without jumps).

Section 4.2 studies in details the quadratic case (the generator is $y \mapsto -y|y|$) when the forward process is the Poisson process. The terminal condition is equal to $+\infty$ on an interval $[x_0, +\infty)$ for a fixed threshold x_0 and is finite on the complement of this interval. Our main result is in Theorem 4.2.7 and Corollary 4.2.10: any approximating sequence of the solution (of the BSDE or of the PDE) converges to $u : t \mapsto 1/(T-t)$ on $[0, T)$, whatever the value of the terminal condition on $(-\infty, x_0)$ is. In other words, for this singular terminal condition, there is only one solution equal to u on $[0, T)$. As a consequence, the solution of the BSDE or of the PDE does not depend on the terminal condition. Theorem 4.2.16 shows that adding a diffusion part does not change the result. The discontinuity comes from the jump part and cannot be overcome by the smoothing effect of the diffusion part.

In Section 4.3, we deduce that the quadratic case is critical. With less non linear generators ($q < 2$), the discontinuity holds (Theorem 4.3.2), whereas for more non linear generators ($q > 2$), continuity property holds (Proposition 4.3.3).

The non-decreasing Poisson process X has a tendency to go into the singular set $\mathcal{S} = [x_0, \infty)$, which intuitively explains the observed discontinuity. Indeed in Section 4.4, we show that if $\mathcal{S} = (-\infty, x_0]$, then continuity again holds. Here the Poisson process tends to exit from $\mathcal{S}$.

To illustrate this final discontinuity, we also study the related numerical scheme in Section 4.5. In our setting, we solve an ordinary Ricatti differential equation. The implicit Euler scheme is well posed and approximates the solution for bounded terminal condition with standard convergence rate. We prove that the same scheme can be used in the singular case and that it behaves according to the theoretical analysis, that is it forgets the terminal value and explodes at time T, when the discretization step tends to zero.

Let us emphasize that most of the results of this chapter are true if we work with a compound Poisson process with positive jumps that are bounded away from zero. Nonetheless the extension to more general Poisson random measures or to multi-dimensional processes are left for further research.

4.1 Framework and definitions

We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$. We assume that this set supports a one-dimensional Brownian motion W and a Poisson process N with intensity λ. The filtration $\mathbb{F}$ is generated by W and N. The compensated process $\widetilde{N} = (N_t - \lambda t, t \geq 0)$ is a martingale with respect to $\mathbb{F}$.

For a given $T \geq 0$, we denote by $\mathcal{P}$ the predictable σ-field on $\Omega \times [0,T]$. On $\Omega \times [0,T]$, a function that is $\mathcal{P}$-measurable, is called predictable. $\mathcal{D}$ (resp. $\mathcal{D}(0,T)$) is the set of all predictable processes on $[0,+\infty)$ (resp. on $[0,T]$).

Now to define the solution of our BSDE, let us remember the following spaces for $p \geq 1$.

- $S^p_c(0,T)$ is the space of all adapted processes X with right-continuous with left limits paths, such that $\mathbb{E}\left[\sup_{t\in[0,T]} |X_t|^p\right] < \infty$.
- $H^p(0,T)$ denotes the subspace of all processes $X \in \mathcal{D}(0,T)$ such that the expectation $\mathbb{E}\left[\left(\int_0^T |X_t|^2 dt\right)^{p/2}\right]$ is finite.

Finally we also define

$$M^p(0,T) = S^p_c(0,T) \times H^p(0,T) \times H^p(0,T).$$

We consider the BSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, U_s)ds - \int_t^T Z_s dW_s - \int_t^T U_s d\widetilde{N}_s, \quad 0 \leq t \leq T. \tag{4.5}$$

Here, the random variable ξ is $\mathcal{F}_T$-measurable with values in $\mathbb{R}$ and the generator f : $\Omega \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is a random function, measurable with respect to $Prog \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$ where $Prog$ denotes the sigma-field of progressive subsets of $\Omega \times [0,T]$. The unknowns are (Y, Z, U) such that Y is progressively measurable and càdlàg with values in $\mathbb{R}$, $Z \in \mathcal{D}(0,T)$ such that a.s. $\int_0^T |Z_s|^2 ds < +\infty$ and $U \in \mathcal{D}(0,T)$.

4.1.1 Existence of a solution for the BSDE

The next conditions on f are very standard in the BSDE theory (see for example [26, 72]). For notational convenience we will denote $f^0_t = f(t, 0, 0, 0)$.

- The function $y \mapsto f(t, y, z, u)$ is continuous and monotone: there exists $\mu \in \mathbb{R}$ such that a.s. and for any $t \in [0,T]$, $(z,u) \in \mathbb{R}^2$

$$(f(t,y,z,u) - f(t,y',z,u))(y-y') \leq \mu(y-y')^2. \tag{A1}$$

- For every $n > 0$ the function

$$\sup_{|y|\leq n} |f(t,y,0,0) - f_t^0| \in L^1((0,T)\times\Omega). \tag{A2}$$

- f is Lispchitz continuous in z, uniformly w.r.t. all parameters: there exists $L > 0$ such that for any (t,y,u), z and z': a.s.

$$|f(t,y,z,u) - f(t,y,z',u)| \leq L|z-z'|. \tag{A3}$$

- There exists a progressively measurable process $\kappa = \kappa^{y,z,u,v} : \Omega \times [0,T] \to \mathbb{R}$ such that

$$f(t,y,z,u) - f(t,y,z,v) \leq (u-v)\kappa_t^{y,z,u,v} \tag{A4}$$

  with $\mathbb{P} \otimes Leb$-a.e. for any (y,z,u,v), $-1 \leq \kappa_t^{y,z,u,v}$ and $|\kappa_t^{y,z,u,v}| \leq \vartheta$, where ϑ is a constant.
- There exists $\varrho > 1$ such that

$$\mathbb{E}\left[\int_0^T |f_s^0|^\varrho ds\right] < +\infty. \tag{A5}$$

In [55, 85], it is proved that under Conditions (A1)-(A5) and if $\xi \in L^\varrho(\Omega)$, then the BSDE (4.5) has a unique solution (Y,Z,U) in $M^\varrho(0,T)$. Moreover the comparison principle holds: roughly speaking, if $\xi' \geq \xi$ and $f' \geq f$, then $Y' \geq Y$.

Now if ξ is not integrable or if $\mathbb{P}(\xi = +\infty) > 0$, to ensure the existence of a solution which is finite before time T, we suppose that there exists a constant $q > 1$ and a positive constant η such that for any $y \geq 0$

$$f(t,y,z,u) \leq -\eta y|y|^{q-1} + f(t,0,z,u). \tag{A6}$$

Definition 4.1.1. *The generator f satisfies Condition* **(A)** *if all assumptions* (A1)–(A6) *hold.*

Example 4.1.2. *The function $f(t,y,z,u) = -y|y|^{q-1}$ satisfies Condition* **(A)***.*

In [56], the following result is proved.

Theorem 4.1.3 (Theorem 1 in [56]). *Under Condition* **(A)** *and if* ξ *and* f^0 *are non-negative, then there exists a process* (Y, Z, U) *such that*

- (Y, Z, U) *belongs to* $M^{\varrho}(0, t)$ *for any* $t < T$.
- Y *is non-negative;*
- *For all* $0 \leq s \leq t < T$:

$$Y_s = Y_t + \int_s^t f(t, Y_r, Z_r, U_r)dr - \int_s^t Z_r dW_r - \int_s^t U_s d\widetilde{N}_s.$$

- (Y, Z, U) *is a super-solution in the sense that a.s.*

$$\liminf_{t \to T} Y_t \geq \xi. \tag{4.6}$$

Any process $(\widetilde{Y}, \widetilde{Z}, \widetilde{U})$ *satisfying the previous four items is called* ***super-solution*** *of the BSDE* (4.5) *with singular terminal condition* ξ. *Finally the process* (Y, Z, U) *is the minimal super-solution, in the sense that for any other supersolution, a.s. for any* t, $\widetilde{Y}_t \geq Y_t$.

Note that this result holds in the more general framework with Poisson random measure and general filtration.

As explained in the introduction, Condition (4.6) is too weak to ensure uniqueness of the solution and is interpreted as an extra cost for liquidation in finance. Instead of (4.6), we want to have (4.3):

$$\lim_{t \to T} Y_t = \xi.$$

It is proved in [76, Section 3] that the existence of a left-limit at time T for Y only depends on f. A sufficient condition is the existence of a non-increasing, of class C^1 and concave function h and of a positive constant $\widetilde{\eta}$ such that for any $y \geq 0$

$$\widetilde{\eta} h(y) \leq f(t, y, z, u) - f(t, 0, z, u).$$

In this chapter, we only discuss if a.s.

$$\liminf_{t \to T} Y_t = \xi. \tag{4.7}$$

If some partial results have been obtained for the non-Markovian setting ([81, 66, 1]), more complete results have been obtained in the Markovian setting.

4.1.2 Markovian setting

For $x \in \mathbb{R}$, we consider the forward SDE: for any $0 \leq t \leq T$

$$X_t = x + \int_0^t b(r, X_r)dr + \int_0^t \sigma(r, X_r)dW_r + \int_0^t \beta(r, X_{r-})d\widetilde{N}_r. \tag{4.8}$$

The coefficients $b : [0, T] \times \mathbb{R} \to \mathbb{R}$, $\sigma : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\beta : [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy:

1. b, σ and β are jointly continuous w.r.t. (t, x) and Lipschitz continuous w.r.t. x uniformly in t, i.e. there exists a constant K such that for any $t \in [0, T]$, for any x and y in $\mathbb{R}$: a.s.

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| + |\beta(t, x) - \beta(t, y)| \leq K|x - y|$$

2. b and σ growth at most linearly:

$$|b(t, x)| + |\sigma(t, x)| \leq C_{b,\sigma}(1 + |x|).$$

3. β is bounded w.r.t. t and x: there exists a constant C_β such that

$$|\beta(t, x)| \leq C_\beta.$$

Under these assumptions, the forward SDE (4.8) has a unique strong solution X (see [78]).

We assume that

$$\xi = g(X_T)$$

where the function g is defined on $\mathbb{R}$ with values in $[0, +\infty] = [0, +\infty) \cup \{+\infty\}$. We denote

$$\mathcal{S} = \{x \in \mathbb{R} \quad s.t. \quad g(x) = +\infty\}$$

the set of singularity points for the terminal condition induced by g. We suppose that $\mathcal{S}$ is closed and that for all closed set $\mathcal{K} \subset \mathbb{R} \setminus \mathcal{S}$

$$g(X_T)1_{\mathcal{K}}(X_T) \in \; L^1(\Omega, \mathcal{F}_T, \mathbb{P}).$$

In [76, Theorem 4.5], under Condition **(A)** on f and under this setting for $\xi = g(X_T)$, it is proved that (4.7) holds provided that in (A6) $q > 3$. There are also some technical conditions between the jumps of X and the singular set $\mathcal{S}$; these conditions are discussed in Sections 4.2 and 4.4. If $\mathbb{F}$ is only generated by W (in particular $\beta = 0$ in (4.8)), then (4.7) holds for any $q > 1$, provided that the diffusion coefficient σ is uniformly elliptic. Indeed in this case, we can use the representation of the process Z as the Malliavin derivative of Y.

The rest of this chapter shows that the presence of jumps can completely destroy (4.7), that is for $q \leq 2$, it is possible to have a.s.

$$\liminf_{t \to T} Y_t = +\infty,$$

even if $\mathbb{P}(\xi = +\infty) < 1$.

4.1.3 Related PDEs

A key feature of BSDE is the link with parabolic PDE. Let us now define for $(t, x) \in [0, T] \times \mathbb{R}$, the forward SDE: for any $0 \leq t \leq s \leq T$

$$X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x})dr + \int_t^s \sigma(r, X_r^{t,x})dW_r + \int_t^s \beta(r, X_{r-}^{t,x})d\widetilde{N}_r. \tag{4.9}$$

The coefficients b, σ and β still satisfy the previously mentioned conditions: Lipschitz continuity w.r.t. x and at most linear growth. Then (4.9) has a unique strong solution $X^{t,x}$ belonging in $S_c^p(0, T)$ for any $p > 1$. Together with the SDE (4.9), we solve the BSDE: for any $0 \leq t \leq s \leq T$

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, U_r^{t,x})dr - \int_s^T Z_r^{t,x}dW_r - \int_t^T U_r^{t,x}d\widetilde{N}_r. \tag{4.10}$$

Now the generator $f : [0, T] \times \mathbb{R}^4 \to \mathbb{R}$ is a deterministic function, such that the random function $f^X(s, y, z, u) = f(s, X_s^{t,x}, y, z, u)$ satisfies Condition **(A)** uniformly w.r.t. x. The function $g : \mathbb{R} \to \mathbb{R}$ is measurable and non-negative and $\xi = g(X_T^{t,x})$. Hence we can apply the previous results to ensure the existence of a minimal super-solution $(Y^{t,x}, Z^{t,x}, U^{t,x})$.

To make the link with IPDE, we also suppose that the generator f verifies some extra regularity assumptions (see [12, 75]):

- The function f is locally Lipschitz continuous w.r.t. y: for all $R > 0$, there exists L_R such that for any y and y' and any (t, x, z, u)

$$|y| \leq R, |y'| \leq R \Longrightarrow |f(t, x, y, z, u) - f(t, x, y', z, u)| \leq L_R|y - y'|.$$

- The function $u \in \mathbb{R} \mapsto f(t,x,y,z,u)$ is non-decreasing for all $t \in [0,T]$ and $(x,y,z) \in \mathbb{R}^3$:

$$\forall u \leq u', \quad 0 \leq f(t,x,y,z,u') - f(t,x,y,z,u) \leq \vartheta(u'-u).$$

  ϑ is the constant of Condition (A4).
- $(t,x) \mapsto f(t,x,y,z,u)$ is continuous and for all $R > 0$, $t \in [0,T]$, $|x| \leq R$, $|x'| \leq R$, $|y| \leq R$, $(z,u) \in \mathbb{R}^2$,

$$|f(t,x,y,z,u) - f(t,x',y,z,u)| \leq \omega_R(|x-x'|(1+|z|)),$$

  where $\omega_R(s)$ tends to 0 when $s \searrow 0$.
- $x \mapsto f(t,x,0,0,0)$ is of at most polynomial growth.

The generator of Example 4.1.2 satisfies these conditions. Now from [12, Proposition 2.5 and Theorem 3.4]:

Proposition 4.1.4. *If $g : \mathbb{R} \to \mathbb{R}$ is continuous and with polynomial growth, the function $u(t,x) = Y_t^{t,x}$ is the unique continuous viscosity solution of the IPDE:*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + \mathcal{I}(t,x,u) + f(t,x,u,u'\sigma,\mathcal{B}(t,x,u)) = 0 \\ \\ u(T,x) = g(x) \end{cases} \tag{4.11}$$

(among the functions with polynomial growth). Moreover if g is bounded, u is also bounded.

In the previous IPDE, we have:

- $\mathcal{L}$ is the local second-order differential operator, due to the continuous part of the forward SDE:
$$\mathcal{L}(t,x,\phi) = \frac{1}{2}\sigma^2(t,x)\phi''(x) + b(t,x)\phi'(x) \ ;$$
- $\mathcal{I}$ is a non local differential operator and comes from the jump part of the forward SDE:
$$\mathcal{I}(t,x,\phi) = \phi(t,x+\beta(t,x)) - \phi(t,x) - \phi'(t,x)\beta(t,x) \ ;$$
- $\mathcal{B}$ is also a non local operator coming from the generator of the BSDE:

$$\mathcal{B}(t,x,\phi) = \phi(t,x+\beta(t,x)) - \phi(t,x).$$

From [75], we obtain the next statement [3]

Proposition 4.1.5. *If $g : \mathbb{R} \to [0, +\infty]$ is a continuous function such that for any compact set $\mathfrak{K}$ in $\mathbb{R} \setminus \mathcal{S}$, $g(X_T)1_{\mathfrak{K}}(X_T)$ is integrable, where $\mathcal{S} = \{x \in \mathbb{R},\ g(x) = \infty\}$, then u is the minimal non-negative viscosity solution of* (4.11)*, such that:*

$$\liminf_{(t,x)\to(T,x_0)} u(t,x) \geq g(x_0) \tag{4.12}$$

holds.

The continuity problem for BSDE can be written here: does the minimal viscosity solution u satisfy

$$\lim_{(t,x)\to(T,x_0)} u(t,x) = g(x_0) \ ?$$

A natural question concerns the regularity of the solution u. In [75, Section 4.3], it is proved that if b and σ are bounded functions and σ is uniformly elliptic, if f is Hölder continuous w.r.t. (t,x), then $u \in C^{1,2}([0,T) \times \mathbb{R})$ (see [75, Lemmas 5 and 6, Proposition 5]).

4.2 Quadratic case with right barrier

In this section, we still assume that $X = N$ is the Poisson process ($\sigma = 0$ and $\beta = 1$), denoted $X^{t,x}$ if we want to emphasize that it starts at time t from point x:

$$X_s^{t,x} = x + \int_t^s \lambda dr + \int_t^s d\widetilde{N}_r = x + N_s - N_t.$$

We study the quadratic case: $f(s,y,z,u) = -y|y|$ (Example 4.1.2 with $q = 2$), so the BSDEs (4.2) and (4.5) become

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\widetilde{N}_s, \quad 0 \leq t \leq T. \tag{4.13}$$

We note $Y^{t,x}$ the solution of the BSDE whose dynamics is that of the BSDE (4.13) on $[t,T]$ when $X = X^{t,x}$. Moreover we consider the following function g : for $x_0 \in \mathbb{R}$ and $\varphi : \mathbb{R} \to [0, +\infty)$

$$g(x) = (+\infty)1_{\{x \geq x_0\}} + \varphi(x)1_{\{x < x_0\}}. \tag{4.14}$$

3. Note that continuity of the minimal solution is not guaranteed in this proposition.

For this case, it is obvious that the singularity set $\mathcal{S} = [x_0, \infty)$ has a compact and regular boundary $\{x_0\}$ and obviously if $x \geq x_0$, $x+1 > x_0$. In other words it satisfies the technical conditions (called **(E)** in [76]) mentioned in Section 4.1.2. But $q = 2$ is too small to apply some known result about the continuity at time T. Let us remark that φ plays a role only if X starts below x_0.

Let us evoke some properties for this BSDE and the truncated BSDE: for any $K > 0$ and for $0 \leq t \leq T$

$$Y_t^K = g(X_T) \wedge K - \int_t^T Y_s^K |Y_s^K| ds - \int_t^T U_s^K d\widetilde{N}_s. \tag{4.15}$$

From Section 4.1.1 and [56], there exists a unique solution (Y^K, U^K) for (4.15) and a minimal solution (Y, U) for (4.13) such that Y is the increasing limit of Y^K and since g is non-negative: a.s.

$$\forall t \in [0, T], \quad 0 \leq Y_t^K \leq Y_t \leq \frac{1}{T-t}.$$

Note that these estimates do not depend on g. Moreover a.s.

$$\lim_{t \to T} Y_t \geq \xi = g(X_T).$$

The existence of the limit follows from [76, Theorem 3.1].

Finally the related IPDE (4.11) is: for any $(t, x) \in [0, T] \times \mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t, x+1) \\ \\ u(T, x) = g(x), \end{cases} \tag{4.16}$$

and the truncated PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda u^K(t, x+1) \\ \\ u^K(T, x) = g(x) \wedge K. \end{cases} \tag{4.17}$$

If $g : \mathbb{R} \to [0, +\infty]$ is continuous, then there exists a unique continuous viscosity solution u^K for (4.17) and a minimal viscosity solution u for (4.16) such that u is the increasing limit of u^K and for any (t,x)

$$0 \le u^K(t,x) \le u(t,x) \le \frac{1}{T-t}.$$

Recall that $u^K(t,x) = Y_t^{K,t,x}$ and $u(t,x) = Y_t^{t,x}$.

We are going to show that we have

$$\lim_{t\to T} Y_t > \xi = Y_T, \quad Y_t = \frac{1}{T-t} \quad \forall t \in [0,T).$$

In other words we show that there exist cases for which the solution is non continuous at time T. Contrary to [76, 75], the main changements are the quadratic driver (corresponding to $q = 2$ in (A6)) and simple jumps associated to a Poisson process (without jumps, for uniformly elliptic diffusions, we have continuity whatever the power q).

4.2.1 Solving the PDE and behavior at time T

Here we are going to resolve the PDEs (4.16) and (4.17), without the help of BSDE's theory. Let us state some results concerning the ODE:

$$\begin{cases} y'(t) - \lambda y(t) - y(t)|y(t)| + \lambda\psi(t) = 0, \quad 0 \le t \le T, \\ \\ y(T) = \chi \in \mathbb{R}. \end{cases} \tag{4.18}$$

Lemma 4.2.1. *If $\psi \in C^0([0,T])$, then there exists a unique bounded solution y. Moreover if $\chi \ge 0$ and if for any t $\psi(t) \ge 0$, then $y(t) \ge 0$ for any t. The solution satisfies:*

$$\forall t < T, \quad y(t) \le \frac{1}{(T-t)^2} \int_t^T \left[\lambda\psi(s)(T-s)^2 + 1\right] e^{-\lambda(s-t)} ds.$$

As a consequence, if $\psi(t) \le 1/(T-t)$, then the same estimate holds for y.

Proof. The function $(t,y) \mapsto \lambda y + y|y| - \lambda\psi(t)$ is continuous w.r.t. t and is locally Lipschitz continuous w.r.t. y. Hence there exists a unique solution of the ODE, defined on an interval $(\tau, T]$. We also have

$$y'(t) = (\lambda + |y(t)|)y(t) - \lambda\psi(t).$$

So for $\tau < t \leq T$

$$y(t) = \chi \exp\left(-\int_t^T (\lambda + |y(s)|)ds\right) + \lambda \int_t^T \psi(s) \exp\left(-\int_t^s (\lambda + |y(u)|)du\right) ds.$$

Thus:

$$|y(t)| \leq |\chi| + \lambda \int_t^T |\psi(s)|ds \leq |y(T)| + \lambda \int_0^T |\psi(s)|ds.$$

As $\psi \in C^0([0,T])$, the function y is bounded on $(\tau, T]$, independently of τ. Hence there exists a global solution defined on $[0,T]$. And if χ and ψ are non-negative, the function y is also non-negative.

Now we prove the a priori estimate on y (adaptation of [56]). Let us solve the following linear ODE on the interval $[0, T-\varepsilon]$ for $0 < \varepsilon < T$,

$$(y^\varepsilon)' - \lambda y^\varepsilon - 2\frac{1}{T-t}y^\varepsilon + \frac{1}{(T-t)^2} + \lambda\psi(t), \quad y^\varepsilon(T-\varepsilon) = y(T-\varepsilon).$$

The solution y^ε is given by

$$\begin{aligned}
y^\varepsilon(t) &= y(T-\varepsilon)\exp\left(-\int_t^{T-\varepsilon}\left(\lambda + 2\frac{1}{T-s}\right)ds\right) \\
&\quad + \lambda \int_t^{T-\varepsilon} \psi(s) \exp\left(-\int_t^s \left(\lambda + 2\frac{1}{T-u}\right)du\right) ds \\
&\quad + \int_t^{T-\varepsilon} \frac{1}{(T-s)^2} \exp\left(-\int_t^s \left(\lambda + 2\frac{1}{T-u}\right)du\right) ds \\
&= y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)} \exp\left(-2\int_t^{T-\varepsilon} \frac{1}{T-s}ds\right) \\
&\quad + \lambda \int_t^{T-\varepsilon} \psi(s) e^{-\lambda(s-t)} \exp\left(-2\int_t^s \frac{1}{T-u}du\right) ds \\
&\quad + \int_t^{T-\varepsilon} \frac{1}{(T-s)^2} e^{-\lambda(s-t)} \exp\left(-2\int_t^s \frac{1}{T-u}du\right) ds
\end{aligned}$$

But

$$\exp\left(-2\int_t^s \frac{1}{T-u}du\right) = \left(\frac{T-s}{T-t}\right)^2.$$

Thus

$$\begin{aligned} y^\varepsilon(t) =& \frac{1}{(T-t)^2}\varepsilon^2 y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)} \\ &+ \frac{1}{(T-t)^2}\int_t^{T-\varepsilon}\left[\lambda\psi(s)+\frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds. \end{aligned}$$

Using the inequality $y^2 \geq 2cy - c^2$ with $c = 1/(T-t)$, we have the inequality between the two generators

$$\lambda y + 2\frac{1}{T-t}y - \frac{1}{(T-t)^2} - \lambda\psi(t) \leq \lambda y + y^2 - \lambda\psi(t).$$

Thus, with the comparison result for backward ODE, we deduce that for any $t \in [0, T-\varepsilon]$

$$\begin{aligned} y(t) &\leq y^\varepsilon(t) \\ &= \frac{1}{(T-t)^2}\varepsilon^2 y(T-\varepsilon)e^{-\lambda(T-\varepsilon-t)} \\ &+ \frac{1}{(T-t)^2}\int_t^{T-\varepsilon}\left[\lambda\psi(s)+\frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds. \end{aligned}$$

Letting ε go to zero, since the function y is bounded, we deduce that for any $t < T$

$$\begin{aligned} y(t) &\leq \frac{1}{(T-t)^2}\int_t^{T}\left[\lambda\psi(s)+\frac{1}{(T-s)^2}\right]e^{-\lambda(s-t)}(T-s)^2 ds \\ &= \frac{1}{(T-t)^2}\int_t^{T}\left[\lambda\psi(s)(T-s)^2+1\right]e^{-\lambda(s-t)} ds. \end{aligned}$$

If $\psi(t) \leq \dfrac{1}{T-t}$, a computation shows that the same estimate holds for y:

$$\begin{aligned} y(t) &\leq \frac{1}{(T-t)^2}\int_t^{T}\left[\lambda(T-s)+1\right]e^{-\lambda(s-t)} ds \\ &= \frac{1}{(T-t)^2}\left(-\frac{e^{-\lambda(T-t)}}{\lambda} + \frac{\lambda(T-t)+1}{\lambda} + \frac{e^{-\lambda(T-t)}}{\lambda} - \frac{1}{\lambda}\right) = \frac{1}{T-t}. \end{aligned}$$

This achieves the proof of the lemma. □

Remark 4.2.2. *Let us emphasize that the mapping* $(y(T), \psi) \mapsto y$ *is non-decreasing: if* $\widehat{\chi} \geq \chi \geq 0$ *and* $\widehat{\psi}(t) \geq \psi(t) \geq 0$ *for any* t*, then* $\widehat{y}(t) \geq y(t) \geq 0$*.*

Indeed we have

$$
\begin{aligned}
&\widehat{y}(t) - y(t) \\
&= \widehat{\chi} - \chi - \int_t^T (\lambda \widehat{y}(s) + \widehat{y}(s)|\widehat{y}(s)| - \lambda \widehat{\psi}(s) - \lambda y(s) - y(s)|y(s)| + \lambda \psi(s)) ds \\
&= \widehat{\chi} - \chi - \int_t^T (\lambda \widehat{y}(s) - \lambda y(s) + \underbrace{\widehat{y}(s)|\widehat{y}(s)|}_{=\widehat{y}(s)^2} - \underbrace{y(s)|y(s)|}_{=y(s)^2}) ds + \lambda \int_t^T \widehat{\psi}(s) - \psi(s) ds \\
&= \widehat{\chi} - \chi - \int_t^T (\lambda + a(s)) (\widehat{y}(s) - y(s)) ds + \lambda \int_t^T \widehat{\psi}(s) - \psi(s) ds,
\end{aligned}
$$

with

$$
a(s) = \frac{\widehat{y}(s)|\widehat{y}(s)| - y(s)|y(s)|}{\widehat{y}(s) - y(s)} 1_{\widehat{y}(s) \leq y(s)} \geq 0.
$$

Thus

$$
\begin{aligned}
\widehat{y}(t) - y(t) = (\widehat{\chi} - \chi) \exp\left(- \int_t^T (\lambda + a(s)) ds \right) \\
+ \lambda \int_t^T (\widehat{\psi}(s) - \psi(s)) \exp\left(- \int_t^s (\lambda + a(u)) du \right) ds \geq 0.
\end{aligned}
$$

We begin with the case $x \geq x_0$. We rewrite the PDEs

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1), \\
\\
u(T,x) = +\infty,
\end{cases}
\tag{4.19}
$$

and

$$
\begin{cases}
\dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda u^K(t,x+1), \\
\\
u^K(T,x) = K.
\end{cases}
\tag{4.20}
$$

Lemma 4.2.3. *On* $[0,T] \times [x_0, \infty)$, *for any* $K > 0$, *the solutions of* (4.19) *and* (4.20) *are:*

$$
u(t,x) = \frac{1}{T-t}, \quad u^K(t,x) = \frac{1}{T - t + \frac{1}{K}}.
$$

Proof. For the equation (4.20), we notice that the function $t \longmapsto \dfrac{1}{T - t + \frac{1}{K}}$ satisfies the PDE and is continuous and bounded. By uniqueness of the viscosity solution ([12, Theorem 3.5]), we have

$$u^K(t,x) = \frac{1}{T-t+\dfrac{1}{K}}.$$

Then the minimal solution is the increasing limit of u^K:

$$u(t,x) = \lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}.$$

Clearly, it satisfies PDE (4.19) on $[0,T)\times[x_0,+\infty)$. □

We consider now the case $x \in [x_0-1,x_0)$. So $x+1 \geq x_0$ and we rewrite the PDE (4.17) for u^K

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda \dfrac{1}{T-t+\dfrac{1}{K}}, \\ \\ u^K(T,x) = \varphi(x) \wedge K. \end{cases}$$

Note that it is an ODE with parameter x.

Lemma 4.2.4. *On $[0,T]\times[x_0-1,x_0)$, for any $K>0$, the solution of* (4.21) *is*

$$u^K(t,x) = \frac{1}{T-t+\dfrac{1}{K}} - \frac{1_{\varphi(x)<K}}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2\left(\dfrac{1}{K-\varphi(x)} + \displaystyle\int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2}ds\right)}, \tag{4.21}$$

and

$$u(t,x) = \lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}1_{\{t<T\}} + \varphi(x)1_{\{t=T\}}.$$

Proof. Here x is a fixed parameter in $[x_0-1,x_0)$. We begin with the equation in u^K. We recognize a Riccati equation whose a particular solution of the dynamic is $t \longmapsto \dfrac{1}{T-t+\dfrac{1}{K}}$. We make the variable changement

$$u^K(t,x) = \frac{1}{T-t+\dfrac{1}{K}} - w^K(t,x)$$

where w^K is a non-negative function. The sign of w^K comes from the a priori estimate on

u^k given by [56, Lemma 1]. So the function $w^K(\cdot, x)$ satisfies the ODE

$$\frac{\partial w^K}{\partial t}(t,x) - \left(\lambda + \frac{2}{T - t + \frac{1}{K}}\right) w^K(t,x) + w^K(t,x)^2 = 0.$$

We recognize a Bernoulli equation. We make the variable changement, under reserve of non cancellation,

$$y^K(t,x) = \frac{1}{w^K(t,x)}.$$

So the function $y^K(\cdot, x)$ satisfies a first order linear differential equation. Solving this ODE and going back to u^K, we obtain that if $\varphi(x) < K$

$$u^K(t,x) = \frac{1}{T - t + \frac{1}{K}} - \frac{1}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2 \left(\frac{1}{K - \varphi(x)} + \int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds\right)}$$

and for $\varphi(x) = K$

$$u^K(t,x) = \frac{1}{T - t + \frac{1}{K}}.$$

Let us pass to the limit on K for $t < T$. Since $\varphi(x) < +\infty$, we have

$$\lim_{K \to +\infty} \left(K(T-t)+1\right)^2 \frac{1}{K - \varphi(x) \wedge K} = +\infty$$

whereas

$$0 \leq \int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds \leq \int_t^T \frac{1}{(K(T-s)+1)^2} ds = \frac{T-t}{K(T-t)+1}.$$

Therefore we obtain for any $t < T$.

$$u(t,x) = \lim_{K \to +\infty} u^K(t,x) = \frac{1}{T-t}.$$

□

This function u satisfies the PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda \dfrac{1}{T-t}, \\ \\ u(T,x) = \varphi(x). \end{cases} \tag{4.22}$$

Again it is an ODE with parameter x. Since $\psi = \dfrac{1}{T-\cdot} \notin C^0([0,T])$, we cannot apply Lemma 4.2.1. Nonetheless we have:

Lemma 4.2.5. *This function* $u(\cdot,x) = \dfrac{1}{T-\cdot}$ *is the unique non-negative solution of* (4.22) *defined on* $[0,T)$.

Proof. Again x is fixed and we assume there exists a non-negative solution $u(\cdot,x)$ defined on $[0,T)$. So the function $u(\cdot,x)$ satisfies the forward PDE

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)^2 = -\lambda \dfrac{1}{T-t} \\ \\ u(0,x) = a, \end{cases} \tag{4.23}$$

with $a \geq 0$ defined by $a = u(0,x)$. According to the Cauchy-Lipschitz theorem, this ODE has a unique solution $u(\cdot,x)$ defined on $[0,\tau^a)$ (τ^a also depends on x, but we do not write this dependence). If $a = \dfrac{1}{T}$ then the function $t \longmapsto \dfrac{1}{T-t}$ is solution and well defined on $[0,T)$. In that case

$$u(t,x) = \frac{1}{T-t}.$$

We are going to show it is the only possibility. We suppose by absurd $a \neq \dfrac{1}{T}$. We also have a Riccati equation whose a particular solution is $t \longmapsto \dfrac{1}{T-t}$. So, by applying the previous method, we obtain an explicit solution

$$u(t,x) = \frac{1}{T-t} + \frac{1}{\dfrac{(T-t)^2}{T^2} e^{-\lambda t} \left(c - T^2 \displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds \right)}, \tag{4.24}$$

with $c = \left(a - \dfrac{1}{T}\right)^{-1} = \left(u(0,x) - \dfrac{1}{T}\right)^{-1}$. Now if $a > \dfrac{1}{T}$ then by divergence of the integral

$\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds$ in T, there exists $\tau^a \in [0, T)$ such that

$$\int_0^{\tau^a} \frac{e^{\lambda s}}{(T-s)^2} ds = \frac{c}{T^2} = \frac{1}{T^2 a - T} > 0.$$

So the function $u(\cdot, x)$ is defined only on $[0, \tau^a)$ with $\tau^a < T$, what contradicts our assumption on u. Now if $a < \frac{1}{T}$ then the function $u(\cdot, x)$ is defined on $[0, T)$ and

$$\begin{aligned} u(t,x) &= \frac{1}{T-t} + \frac{1}{\frac{(T-t)^2}{T^2} e^{-\lambda t} \left(\frac{1}{a - \frac{1}{T}} - T^2 \int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds \right)} \\ &= \frac{1}{T-t} + \frac{1}{(T-t)^2 e^{-\lambda t} \frac{1}{aT^2 - T} - e^{-\lambda t} (T-t)^2 \int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds}. \end{aligned}$$

Integrating by parts leads to

$$\begin{aligned} &u(t,x) \\ &= \frac{1}{T-t} \Bigg[1 \\ &\quad + \frac{1}{(T-t)e^{-\lambda t} \left(\frac{a}{aT-1} + \lambda \ln(T) + \lambda^2 \int_0^t e^{\lambda s} \ln(T-s) ds \right) - 1 - \lambda (T-t) \ln(T-t)} \Bigg] \\ &= \frac{e^{-\lambda t} \left(\frac{a}{1-aT} - \lambda \ln(T) - \lambda^2 \int_0^t e^{\lambda s} \ln(T-s) ds \right) + \lambda \ln(T-t)}{(T-t)e^{-\lambda t} \left(\frac{a}{1-aT} - \lambda \ln(T) - \lambda^2 \int_0^t e^{\lambda s} \ln(T-s) ds \right) + 1 + \lambda (T-t) \ln(T-t)}. \end{aligned}$$

Therefore we obtain for any x

$$u(t,x) \underset{t \to T}{\sim} \lambda \ln(T-t). \tag{4.25}$$

In particular, $u(\cdot, x)$ becomes negative when t tends to T, which contradicts our assumption on u. Thus $a = \frac{1}{T}$ and

$$u(t,x) = \frac{1}{T-t}$$

is the only non-negative solution defined on $[0, T)$. □

We suppose now $x \in [x_0 - 2, x_0 - 1)$. Then $x + 1 \in [x_0 - 1, x_0)$ and we rewrite the PDE (4.17)

$$\begin{cases} \frac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)^2 = -\lambda u^K(t,x+1), \\ \\ u^K(T,x) = \varphi(x) \wedge K, \end{cases} \tag{4.26}$$

where $u^K(t, x + 1)$ is given by (4.21) with $\varphi(x + 1)$ instead of $\varphi(x)$:

$$u^K(t,x+1) \\ = \frac{1}{T - t + \frac{1}{K}} - \frac{\mathbf{1}_{\varphi(x+1)<K}}{e^{\lambda(T-t)}\left(K(T-t)+1\right)^2 \left(\int_t^T \frac{e^{-\lambda(T-s)}}{(K(T-s)+1)^2} ds + \frac{1}{K - \varphi(x+1)}\right)}.$$

Existence of u^K, solution of (4.26), is given by Lemma 4.2.1 (x is a parameter) since $u^K(\cdot, x+1)$ is a bounded function. Moreover since $u^K(\cdot, x+1)$ is bounded from above by $1/(T - \cdot)$ and is non-decreasing w.r.t. K, and with Remark 4.2.2, we have the estimate : for $K \leq \widehat{K}$

$$0 \leq u^K(t,x) \leq u^{\widehat{K}}(t,x) \leq \frac{1}{T-t}.$$

Nonetheless we cannot derive the explicit expression of u^K, but we prove that u^K still converges to $t \mapsto 1/(T - t)$. And from our previous result (Lemma 4.2.5), this function is the unique non-negative solution of (4.22) on $[x_0 - 2, x_0 - 1)$.

Lemma 4.2.6. *For any $x \in [x_0 - 2, x_0 - 1)$, the solution $u^K(\cdot, x)$ of (4.26) converges:*

$$u^K(t,x) \underset{K\to+\infty}{\longrightarrow} \frac{1}{T-t}.$$

Proof. Since $u^K(\cdot, x)$ is a non-decreasing sequence of functions, it converges to some limit function $u(\cdot, x)$ such that for any $t < T$:

$$0 \leq u^K(t,x) \underset{K\to+\infty}{\nearrow} u(t,x) \leq \frac{1}{T-t}.$$

For any $t < T$

$$u^K(t,x) = u^K(0,x) + \int_0^t \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)| - \lambda u^K(s,x+1)\right] ds.$$

Using dominated convergence theorem and Lemma 4.2.4, we can pass to the limit:

$$u(t,x) = u(0,x) + \int_0^t \left[\lambda u(s,x) + u(s,x)|u(s,x)| - \lambda \frac{1}{T-s}\right] ds.$$

Thus u solves the PDE (4.22): on $[0,T)$

$$\frac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda \frac{1}{T-t}.$$

If we assume

$$u(0,x) = a < \frac{1}{T} - \varepsilon$$

for some $\varepsilon > 0$, then, according to the performed reasoning in the proof of Lemma 4.2.5, the solution u is equal to zero in a time $\tau \in [0,T)$ what it cannot be. Thus $u(0,x) = \frac{1}{T}$ and

$$u^K(0,x) \underset{K\to+\infty}{\longrightarrow} \frac{1}{T}.$$

Now we consider the difference function

$$e^K(t,x) = u^K(t,x) - \frac{1}{T-t} = u^K(t,x) - u^\infty(t), \quad 0 \le t < T.$$

By performing the difference between the two PDEs

$$\frac{\partial e^K}{\partial t}(t,x) - \lambda e^K(t,x) - (u^K(t,x)^2 - u^\infty(t)^2) = -\lambda(u^K(t,x+1) - u^\infty(t)),$$

i.e., according to the identity $a^2 - b^2 = (a-b)(a+b)$,

$$\frac{\partial e^K}{\partial t}(t,x) - \left(\lambda + u^K(t,x) + u^\infty(t)\right) e^K(t,x) = -\lambda(u^K(t,x+1) - u^\infty(t)).$$

If we denote

$$a^K(t,x) = \lambda + u^K(t,x) + u^\infty(t), \quad c_K = e^K(0,x) = u^K(0,x) - \frac{1}{T}$$

the difference function e^K is given by

$$\begin{aligned} & e^K(t,x) \\ &= \left(c_K - \lambda \int_0^t (u^K(s,x+1) - u^\infty(s)) \exp\left(-\int_0^s a^K(r,x)dr\right) ds\right) \exp\left(\int_0^t a^K(s,x)ds\right). \end{aligned}$$

We are going to study the behavior of each term when $K \to +\infty$. We already know that

$$c_K = e^K(0,x) = u^K(0,x) - \frac{1}{T} \underset{K\to+\infty}{\longrightarrow} 0.$$

The term

$$a^K(t,x) = \lambda + u^K(t,x) + u^\infty(t) \in [0, \lambda + 2u^\infty(t)]$$

is bounded w.r.t. K, so $\exp\left(\int_0^t a^K(s,x)ds\right)$ also. Finally for the last, apply the dominated convergence theorem:

$$\int_0^t (u^K(s,x+1) - u^\infty(s)) \exp\left(-\int_0^s a^K(r,x)dr\right) ds \underset{K\to+\infty}{\longrightarrow} 0.$$

Therefore we obtain

$$e^K(t,x) \underset{K\to+\infty}{\longrightarrow} 0,$$

i.e.

$$u^K(t,x) \underset{K\to+\infty}{\longrightarrow} u^\infty(t) = \frac{1}{T-t}.$$

□

Theorem 4.2.7. *The solution u^K of* (4.17) *converges to the solution u of* (4.16)*, which is given by*

$$u(t,x) = \frac{1}{T-t} 1_{\{t<T\}} + g(x) 1_{\{t=T\}}, \quad 0 \le t \le T, \quad x \in \mathbb{R}.$$

This solution u is the unique non-negative solution defined on $[0,T]$*.*

Proof. For $x \ge x_0$, it results from Lemma 4.2.3. Then we argue by recursion on the intervals $[x_0 - k - 1, x_0 - k)$, with $k \in \mathbb{N}$. The initialization step comes from Lemma 4.2.4. We suppose the result for the $x \in [x_0 - k - 1, x_0 - k)$, then, by applying the proof of Lemma 4.2.6 we obtain the result for the $x \in [x_0 - k - 2, x_0 - k - 1)$. The recurrence principle allows to conclude. Uniqueness comes from Lemma 4.2.5. □

Remark 4.2.8. *Of course the same study can be done for $f(t,x,y,u) = -\eta y|y|$ with some constant $\eta > 0$. Then the solution becomes $u(t,x) = \dfrac{1}{\eta(T-t)}$.*

The next result is used below to control the martingale part of the BSDE.

Proposition 4.2.9. *For any $K > 0$, the difference $u^K(t,x+1) - u^K(t,x)$ is the sum of* *a non-negative term and a bounded term controlled for $x < x_0$ by*

$$|x_0 - x| \sup_{y \in [x, x_0)} |\varphi(y+1) - \varphi(y)|.$$

Proof. For $x \geq x_0 - 1$, we have: $u^K(t, x+1) - u^K(t, x) \geq 0$; it is an immediate consequence of Lemmata 4.2.3 and 4.2.4 and the formulas therein. Now for $x < x_0 - 1$, the difference $\Delta(t, x) = u^K(t, x+1) - u^K(t, x)$ satisfies: for any $t \in [0, T]$

$$\Delta'(t, x) - (\lambda + u^K(t, x+1) + u^K(t, x))\Delta(t, x) + \lambda\Delta(t, x+1) = 0.$$

Hence for any $x < x_0 - 1$:

$$\begin{aligned} \Delta(t, x) = {} & (\varphi(x+1) - \varphi(x)) \exp\left(-\int_t^T (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) \\ & + \int_t^T \lambda\Delta(r, x+1) \exp\left(-\int_t^r (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) dr. \end{aligned}$$

Since u^K is non-negative, the first term is bounded by $|\varphi(x+1) - \varphi(x)|$.

Now for $x \in [x_0 - 2, x_0 - 1)$, $x + 1 \in [x_0 - 1, x_0)$, thus $\Delta(r, x+1) \geq 0$. Thus the claim is true on $[x_0 - 2, x_0 - 1)$:

$$\Delta(t, x) = \Delta^+(t, x) + \Gamma(t, x)$$

where $\Delta^+(t, x) \geq 0$ and $|\Gamma(t, x)| \leq |\varphi(x+1) - \varphi(x)|$.

For $x \in [x_0 - 3, x_0 - 2)$,

$$\begin{aligned} & \int_t^T \lambda\Delta(r, x+1) \exp\left(-\int_t^r (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) dr \\ & = \int_t^T \lambda\Delta^+(r, x+1) \exp\left(-\int_t^r (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) dr \\ & + \int_t^T \lambda\Gamma(r, x+1) \exp\left(-\int_t^r (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) dr. \end{aligned}$$

And

$$\begin{aligned} & \int_t^T \lambda|\Gamma(r, x+1)| \exp\left(-\int_t^r (\lambda + u^K(s, x+1) + u^K(s, x)) ds\right) dr \\ & \leq \int_t^T \lambda|\varphi(x+2) - \varphi(x+1)| \exp\left(-\lambda(r - t)\right) dr \leq |\varphi(x+2) - \varphi(x+1)|. \end{aligned}$$

Thus again for $2 < x_0 - x \leq 3$

$$\Delta(t,x) = \Delta^+(t,x) + \Gamma(t,x)$$

where $\Delta^+(t,x) \geq 0$ and $|\Gamma(t,x)| \leq 2\sup_{y\in[x,x_0)} |\varphi(y+1) - \varphi(y)|$.

We conclude by recursion. □

4.2.2 Consequence for the BSDE (4.13)

Here we still consider the terminal condition $\xi = g(X_T)$, with g given by (4.14) and the BSDE (4.13) is:

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T U_s d\widetilde{N}_s, \quad 0 \leq t < T.$$

We denote by (Y^K, U^K) the solution of the same BSDE (4.13) with terminal condition $g(X_T) \wedge K$. The first immediate consequence is the next result:

Corollary 4.2.10. *A.s. for any* $t \in [0,T)$

$$\lim_{K\to+\infty} Y_t^K = \frac{1}{T-t}, \quad \lim_{K\to+\infty} U_t^K = 0.$$

The solution Y *of the BSDE* (4.13) *is given by*

$$Y_t = \frac{1}{T-t}1_{\{t<T\}} + g(X_T)1_{\{t=T\}}, \quad 0 \leq t \leq T.$$

Moreover the process U^K *is the sum of a non-negative term and of a term controlled by* $\Phi(X_{s-})$ *where* $\Phi(x) = |x_0 - x| \sup_{y\in[x,x_0)} |\varphi(y+1) - \varphi(y)|$.

Proof. We can apply Itô's formula to $u^K(t,X_t)$ (only regularity w.r.t. t is required) to obtain the solution Y^K:

$$\begin{aligned}
Y_t^K = u^K(t,X_t) &= g(X_T)\wedge K - \int_t^T \frac{\partial u^K}{\partial t}(s,X_s)ds - \sum_{t<s\leq T}\left[u^K(s,X_s) - u^K(s,X_{s-})\right] \\
&= g(X_T)\wedge K - \int_t^T [\lambda(u^K(s,X_{s-}+1) - u^K(s,X_{s-})) + u^K(s,X_{s-})^2]ds \\
&\quad - \int_t^T \left[u^K(s,X_{s-}+1) - u^K(s,X_{s-})\right] dN_s \\
&= g(X_T)\wedge K - \int_t^T (Y_s^K)^2 ds - \int_t^T \left[u^K(s,X_{s-}+1) - u^K(s,X_{s-})\right](dN_s - \lambda ds).
\end{aligned}$$

Hence $U_s^K = u^K(s,X_{s-}+1) - u^K(s,X_{s-})$. The conclusion follows from Proposition 4.2.9 and when we pass to the limit when K tends to ∞. □

Therefore we do not have the continuity of the process Y at the terminal time T: a.s.

$$\lim_{t\to T} Y_t = +\infty > \xi = Y_T.$$

This property does not depend on a particular choice of φ on $(-\infty, x_0)$. The singularity is propagated by the jumps of the forward process.

Note that Remark 4.2.8 still holds for the BSDE.

4.2.3 When we add a diffusion term

As mentioned in the introduction, if there is no jump and if the diffusion is uniformly elliptic, continuity at time T holds. Here we study the BSDE (4.13)

$$Y_t = g(X_T) - \int_t^T Y_s|Y_s|ds - \int_t^T Z_s dW_s - \int_t^T U_s \widetilde{N}(ds),$$

with g given by (4.14):

$$g(x) = (+\infty)1_{\{x_0\leq x\}} + \varphi(x)1_{\{x<x_0\}},$$

and now we add a diffusion part in X, namely:

$$X_t = x + N_t + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s,$$

So the associated PDE (4.16) becomes

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)| = -\lambda u(t,x+1) \\ \\ u(T,x) = g(x), \end{cases} \tag{4.27}$$

with

$$\mathcal{L}u(t,x) = b(x)\frac{\partial u}{\partial x}(t,x) + \frac{1}{2}\sigma(x)^2\frac{\partial^2 u}{\partial x^2}(t,x).$$

If $\lambda = 0$, we have a standard parabolic PDE which is studied in [65]. We want to prove that the minimal solution is again $u(t,x) = \dfrac{1}{T-t}$. Compared to Theorem 4.2.7, the differential operator $\mathcal{L}$ does not change the behavior of the solution.

First note that $t \mapsto \dfrac{1}{T-t}$ solves the PDE (4.27) on $[x_0,\infty)$ where $g(x) = +\infty$. We also consider the truncated version of the PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) + \mathcal{L}u^K(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda u^K(t,x+1) \\ \\ u^K(T,x) = g(x) \wedge K. \end{cases} \tag{4.28}$$

An auxiliary function. Let us consider the following PDE on $[0,T] \times \mathbb{R}$

$$\frac{\partial \overline{u}^K}{\partial t}(t,x) + \mathcal{L}\overline{u}^K(t,x) - \lambda \overline{u}^K(t,x) - \overline{u}^K(t,x)|\overline{u}^K(t,x)| = -\lambda \frac{1}{T-t+\dfrac{1}{K}}, \tag{4.29}$$

with $\overline{u}^K(T,x) = g(x) \wedge K$. This PDE is related with the BSDE without jumps

$$\overline{Y}_t^K = g(\mathfrak{X}_t) \wedge K - \int_t^T \overline{Y}_s^K |\overline{Y}_s^K| ds - \lambda \int_t^T \overline{Y}_s^K ds + \lambda \int_t^T \frac{ds}{T-s+\dfrac{1}{K}} - \int_t^T \overline{Z}_s^K dW_s, \tag{4.30}$$

where

$$\mathfrak{X}_t = x + \int_0^t b(\mathfrak{X}_s)ds + \int_0^t \sigma(\mathfrak{X}_s)dW_s.$$

Lemma 4.2.11. *There exists a unique solution* $(\overline{Y}^K, \overline{Z}^K)$ *to the BSDE* (4.30) *such that a.s. for any* t,

$$0 \leq \overline{Y}_t^K \leq \frac{1}{T-t+\dfrac{1}{K}}.$$

This sequence converges to $(\overline{Y}, \overline{Z})$ *in* $M^p(0, T-\varepsilon)$ *for any* $p > 1$ *and* $\varepsilon > 0$ *and for any* $0 \leq t \leq s < T$

$$\overline{Y}_t = \overline{Y}_s - \int_t^s \overline{Y}_r^2 dr - \lambda \int_t^s \overline{Y}_r dr + \lambda \int_t^s \frac{dr}{T-r} - \int_t^s \overline{Z}_r dW_r.$$

Proof. The driver is given by

$$f_K(s, y) = -y|y| - \lambda y + \frac{\lambda}{T - s + \dfrac{1}{K}}.$$

It is continuous and monotone w.r.t. y and bounded w.r.t. s. The terminal condition is bounded. Thus the solution $(\overline{Y}^K, \overline{Z}^K)$ exists and is unique in $M^\varrho(0, T)$ for any $\varrho > 1$. Comparison principle implies that a.s. for any t,

$$0 \leq \overline{Y}_t^K \leq \frac{1}{T - t + \dfrac{1}{K}},$$

since $f_K(s, y) \geq -y|y| - \lambda y$ and $\left(\dfrac{1}{T - t + \dfrac{1}{K}}, 0 \right)$ is the solution with terminal value K. Moreover for $K \leq K'$, a.s. $\overline{Y}_t^K \leq \overline{Y}_t^{K'}$. Hence

$$\overline{Y}_t = \lim_{K \to \infty} \overline{Y}_t^K$$

exists and satisfies $0 \leq \overline{Y}_t \leq 1/(T-t)$. The rest of the Lemma can be deduced with the same arguments as in [56, Proposition 3]. □

Proposition 4.2.12. *For any* $t < T$, $\overline{Y}_t = \dfrac{1}{T-t}$. *In particular*

$$\lim_{t \to T} \overline{Y}_t = +\infty.$$

Proof. The BSDE (4.30) can be considered as a linear BSDE and thus

$$\begin{aligned}
\overline{Y}^K_t &= \mathbb{E}\Bigg[(g(\mathfrak{X}_t)\wedge K)e^{-\lambda(T-t)}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right) \\
&\quad + \lambda\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^s |\overline{Y}^K_r|dr\right)ds\bigg|\mathcal{F}_t\Bigg] \\
&\geq \lambda\mathbb{E}\left[\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^s |\overline{Y}^K_r|dr\right)ds\bigg|\mathcal{F}_t\right] \\
&\geq \lambda\mathbb{E}\left[\int_t^T \frac{e^{-\lambda(s-t)}}{T-s+\frac{1}{K}}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)ds\bigg|\mathcal{F}_t\right] \\
&\geq \lambda e^{-\lambda T}\int_t^T \frac{1}{T-s+\frac{1}{K}}ds\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\bigg|\mathcal{F}_t\right] \\
&= \lambda e^{-\lambda T}\ln(K(T-t)+1)\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\bigg|\mathcal{F}_t\right].
\end{aligned}$$

Since $\overline{Y}^K$ converges to $\overline{Y}$ and $\overline{Y}$ is finite on $[0,T)$, we deduce that for any $t<T$

$$\lim_{K\to\infty}\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\bigg|\mathcal{F}_t\right] = 0,$$

in particular by Fatou's lemma,

$$\mathbb{E}\left[\exp\left(-\int_t^T |\overline{Y}_r|dr\right)\bigg|\mathcal{F}_t\right] = 0. \tag{4.31}$$

Now consider the difference

$$\begin{aligned}
\frac{1}{T-t+\frac{1}{K}} - \overline{Y}^K_t &= (K-g(\mathfrak{X}_T)\wedge K) \qquad (4.32)\\
&\quad - \int_t^T\left(\lambda+\frac{1}{T-s+\frac{1}{K}}+\overline{Y}^K_s\right)\left(\frac{1}{T-s+\frac{1}{K}}-\overline{Y}^K_s\right)ds + \int_t^T \overline{Z}^K_s dW_s \\
&= \mathbb{E}\left[\frac{K-g(\mathfrak{X}_T)\wedge K}{K(T-t)+1}e^{-\lambda(T-t)}\exp\left(-\int_t^T |\overline{Y}^K_r|dr\right)\bigg|\mathcal{F}_t\right]. \qquad (4.33)
\end{aligned}$$

Since for $t < T$,

$$0 \leq \frac{K - (g(\mathfrak{X}_T) \wedge K)}{K(T-t)+1} \leq \frac{1}{T-t}, \text{ and } \lim_{K\to+\infty} \frac{K - (g(\mathfrak{X}_T) \wedge K)}{K(T-t)+1} = \frac{1}{T-t} \mathbf{1}_{g(\mathfrak{X}_T)<+\infty},$$

we deduce

$$\lim_{K\to+\infty} \left(\frac{1}{T-t+\frac{1}{K}} - \overline{Y}_t^K \right) = 0$$

and that $\overline{Y}_t = 1/(T-t)$, which achieves the proof. □

According to [12, 75], we have the following link between the BSDE and the PDE:

Lemma 4.2.13. *If the function $g \wedge K$ is continuous, then there exists a unique bounded and continuous viscosity solution $\overline{u}^K$ to (4.29). Moreover for any t, x and K*

$$0 \leq \overline{u}^K(t,x) \leq \frac{1}{T-t+\frac{1}{K}}.$$

Finally if b is bounded and σ is uniformly elliptic, that is, there exists $\nu > 0$ such that for any (t,x)

$$\nu \leq \sigma(t,x)^2 \leq \frac{1}{\nu},$$

then for any ε, $\overline{u}^K$ belongs to $C^{1,2}([0,T-\varepsilon]\times\mathbb{R})$.

Proof. The regularity of $\overline{u}^K$ is proved in [75, Section 4.3] and follows from classical results for PDEs, see among others [36, 57]. □

Now evoke that for any (t,x)

$$\overline{u}^K(t,x) = \overline{Y}_t^{K,t,x}.$$

From this proposition and this lemma, we deduce that for $x \in \mathbb{R}$ and $t < T$

$$\lim_{K\to+\infty} \overline{u}^K(t,x) = \lim_{K\to+\infty} \overline{Y}_t^{K,t,x} = \frac{1}{T-t}.$$

Back to the PDEs (4.28) **and** (4.27)**.** On $[x_0,+\infty)$, $g(x)\wedge K = K$ and the solution is $u^K(t,x) = \dfrac{1}{T-t+\dfrac{1}{K}}$. Therefore we can deduce that for $x \in [x_0-1, x_0)$ the PDE (4.28)

is
$$\frac{\partial u^K}{\partial t}(t,x) + \mathcal{L}u^K(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)| = -\lambda \frac{1}{T-t+\dfrac{1}{K}},$$
with $u^K(T,x) = g(x) \wedge K$ and thus on $[0,T] \times [x_0-1, x_0)$, $u^K(t,x) = \overline{u}^K(t,x)$. Hence the minimal solution of (4.27) satisfies for $x \in [x_0-1,x_0)$ and $t<T$: $u(t,x) = \dfrac{1}{T-t}$. In particular
$$\lim_{t\to T} u(t,x) = +\infty > g(x) = u(T,x).$$

Now to handle the case $x \in [x_0-2, x_0-1)$, let us introduce a second auxiliary PDE: on $[0,T]\times\mathbb{R}$
$$\begin{cases} \dfrac{\partial \widetilde{u}^K}{\partial t}(t,x) + \mathcal{L}\widetilde{u}^K(t,x) - \lambda \widetilde{u}^K(t,x) - \widetilde{u}^K(t,x)|\widetilde{u}^K(t,x)| = -\lambda \overline{u}^K(t,x+1) \\ \\ \widetilde{u}^K(T,x) = g(x) \wedge K. \end{cases}$$

The associated BSDE is
$$\widetilde{Y}_t^K = g(\mathfrak{X}_T) \wedge K - \int_t^T \widetilde{Y}_s^K |\widetilde{Y}_s^K| ds - \lambda \int_t^T \widetilde{Y}_s^K ds + \lambda \int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) ds - \int_t^T \widetilde{Z}_s^K dW_s. \quad (4.34)$$

Lemma 4.2.14. *There exists a unique solution $(\widetilde{Y}^K, \widetilde{Z}^K)$ to the BSDE* (4.34) *such that a.s. for any t,*
$$0 \leq \widetilde{Y}_t^K \leq \frac{1}{T-t+\dfrac{1}{K}}.$$
This sequence converges to $(\widetilde{Y}, \widetilde{Z})$ in $M^p(0,T-\varepsilon)$ for any $p>1$ and $\varepsilon>0$ and for any $0 \leq t \leq s < T$
$$\widetilde{Y}_t = \widetilde{Y}_s - \int_t^s \widetilde{Y}_r^2 dr - \lambda \int_t^s \widetilde{Y}_r dr + \lambda \int_t^s \frac{dr}{T-r} - \int_t^s \widetilde{Z}_r dW_r.$$

Proof. This proof can be deduced with the same arguments as in Lemma 4.2.11. □

Now we state the same result as in Proposition 4.2.12:

Proposition 4.2.15. *For any $t<T$, a.s. $\widetilde{Y}_t = \dfrac{1}{T-t}$. In particular*
$$\lim_{t\to T} \widetilde{Y}_t = +\infty.$$

Proof. The proof is rather similar as the proof of Proposition 4.2.12. However since the BSDE (4.34) contains a stochastic term $\overline{u}^K(s, \mathfrak{X}_s + 1)$ (instead of the deterministic and explicit $1/(T - s + 1/K)$), new arguments have to be used.

We notice that $(\widetilde{Y}^K, \widetilde{Z}^K)$ is solution of a linear BSDE:

$$\begin{aligned}
\widetilde{Y}_t^K &= \mathbb{E}\left[(\exp\left(-\int_t^T |\widetilde{Y}_s^K| ds \right) e^{-\lambda(T-t)} g(\mathfrak{X}_T) \wedge K) \middle| \mathcal{F}_t \right] \\
&\quad + \lambda \mathbb{E}\left[\int_t^T \exp\left(-\int_t^s |\widetilde{Y}_r^K| dr \right) e^{-\lambda(s-t)} \overline{u}^K(s, \mathfrak{X}_s + 1) ds \middle| \mathcal{F}_t \right] \\
&\geq \lambda e^{-\lambda T} \mathbb{E}\left[\exp\left(-\int_t^T |\widetilde{Y}_r^K| dr \right) \int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) ds \middle| \mathcal{F}_t \right],
\end{aligned}$$

with for $t < t$, $\widetilde{Y}_t^K \leq 1/(T - t) < +\infty$,

$$\int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) ds \xrightarrow[K \to +\infty]{a.s.} \int_t^T \frac{1}{T - s} ds = +\infty,$$

and

$$\exp\left(-\int_t^T |\widetilde{Y}_r^K| dr \right) \underset{K \to +\infty}{\overset{a.s.}{\searrow}} \exp\left(-\int_t^T |\widetilde{Y}_r| dr \right).$$

Thus:

$$\exp\left(-\int_t^T |\widetilde{Y}_r^K| dr \right) \underset{K \to +\infty}{\overset{a.s.}{\searrow}} 0. \tag{4.35}$$

Then with $\widetilde{Y}_s^K \geq 0$, for any $t < T$

$$\begin{aligned}
0 &\leq \frac{1}{T - t + \frac{1}{K}} - \widetilde{Y}_t^K \\
&= K - \int_t^T \frac{1}{(T - s + \frac{1}{K})^2} ds \\
&\quad - g(\mathfrak{X}_T) \wedge K + \int_t^T \widetilde{Y}_s^K |\widetilde{Y}_s^K| ds + \lambda \int_t^T \widetilde{Y}_s^K ds - \lambda \int_t^T \overline{u}^K(s, \mathfrak{X}_s + 1) + \int_t^T \widetilde{Z}_s dW_s \\
&= K - g(\mathfrak{X}_T) \wedge K - \int_t^T \left(\lambda + \frac{1}{T - s + \frac{1}{K}} + \widetilde{Y}_s^K \right) \left(\frac{1}{T - s + \frac{1}{K}} - \widetilde{Y}_s^K \right) ds \\
&\quad + \lambda \int_t^T \left(\frac{1}{T - s + \frac{1}{K}} - \overline{u}^K(s, \mathfrak{X}_s + 1) ds \right) + \int_t^T \widetilde{Z}_s dW_s.
\end{aligned}$$

Using the explicit formula for the solution of a linear BSDE leads to:

$$
\begin{aligned}
0 &\leq \frac{1}{T-t+\frac{1}{K}} - \widetilde{Y}^K_t \\
&= \mathbb{E}\left[e^{-\lambda(T-t)} \frac{\frac{1}{K}}{T-t+\frac{1}{K}} \exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) (K - g(\mathfrak{X}_T)\wedge K) \middle| \mathcal{F}_t \right] \\
&\quad + \lambda \mathbb{E}\left[\int_t^T e^{-\lambda(s-t)} \frac{T-s+\frac{1}{K}}{T-t+\frac{1}{K}} \exp\left(-\int_t^s \widetilde{Y}^K_r dr\right) \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \middle| \mathcal{F}_t \right] \\
&= e^{-\lambda(T-t)} \frac{1}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) \left(1 - \frac{g(\mathfrak{X}_T)\wedge K}{K}\right) \middle| \mathcal{F}_t \right] + \lambda \frac{1}{T-t+\frac{1}{K}} \\
&\quad \times \mathbb{E}\left[\int_t^T \left(T-s+\frac{1}{K}\right) e^{-\lambda(s-t)} \exp\left(-\int_t^s \widetilde{Y}^K_r dr\right) \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \middle| \mathcal{F}_t \right] \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) + \int_t^T \left(T-s+\frac{1}{K}\right) \exp\left(-\int_t^s \widetilde{Y}^K_r dr\right) \right. \\
&\qquad \left. \left(\frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \right) ds \middle| \mathcal{F}_t \right],
\end{aligned}
$$

with, according to (4.33),

$$
0 \leq \frac{1}{T-s+\frac{1}{K}} - \overline{u}^K(s,\mathfrak{X}_s+1) \leq \frac{e^{-\lambda T}}{T-s+\frac{1}{K}} \mathbb{E}\left[\exp\left(-\int_s^T \overline{Y}^K_r dr\right) \middle| \mathcal{F}_s \right].
$$

Thus:

$$
\begin{aligned}
0 &\leq \frac{1}{T-t+\frac{1}{K}} - \widetilde{Y}^K_t \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\Bigg[\exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) + \int_t^T \exp\left(-\int_t^s \widetilde{Y}^K_r dr\right) \\
&\qquad \times \mathbb{E}\left[\exp\left(-\int_s^T \overline{Y}^K_r dr\right) \middle| \mathcal{F}_s \right] ds \Bigg| \mathcal{F}_t \Bigg] \\
&= \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) + \int_t^T \exp\left(-\int_t^s \widetilde{Y}^K_r dr\right) \exp\left(-\int_s^T \overline{Y}^K_r dr\right) ds \middle| \mathcal{F}_t \right] \\
&\leq \frac{C_{\lambda,T}}{T-t+\frac{1}{K}} \mathbb{E}\left[\exp\left(-\int_t^T \widetilde{Y}^K_s ds\right) + \int_t^T \exp\left(-\int_s^T \overline{Y}^K_r dr\right) ds \middle| \mathcal{F}_t \right].
\end{aligned}
$$

From (4.31) and (4.35), we deduce the statement of the proposition. □

Therefore for any $t < T$ and any $x \in \mathbb{R}$,

$$\lim_{K\to+\infty} \widetilde{u}^K(t,x) = \lim_{K\to+\infty} \widetilde{Y}_t^{K,t,x} = \frac{1}{T-t}.$$

And for such $x \in [x_0 - 2, x_0 - 1)$ and for all $t \in [0,T]$,

$$u^K(t,x) = \widetilde{u}^K(t,x).$$

We deduce for $x \in [x_0 - 2, x_0 - 1)$

$$\lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}.$$

Then, by recurrence, for all $x \in \mathbb{R}$:

$$u(t,x) = \lim_{K\to+\infty} u^K(t,x) = \frac{1}{T-t}.$$

Theorem 4.2.16. *The minimal super-solution (Y,Z,U) verifies: a.s.*

$$\lim_{t\to T} Y_t = +\infty > \xi = g(X_T).$$

Proof. From Lemma 4.2.13, the solutions u^K are smooth. Thus $Y_t^K = u^K(t,X_t)$. Passing through the limit on K, we deduce that $Y_t = 1/(T-t)$ a.s. The conclusion follows immediately. □

4.3 Other generators

We now use the results of the previous section, to show that the quadratic case is pivotal. We consider the BSDE (4.5) and we assume that the generator f satisfies Condition **(A)**, such that existence of a solution is guaranteed.

We suppose that X is the Poisson process and that the terminal condition is still given by (4.14). We denote by $(Y^{(2),K}, Z^{(2),K}, U^{(2),K}) = (Y^{(2),K}, 0, U^{(2),K})$ the solution of BSDE (4.13) with terminal condition $g(X_T) \wedge K$.

4.3.1 For $q < 2$

Let us start with the particular case: $f(t,x,y,u) = -y|y|^{q-1}$ for $1 < q < 2$.

Proposition 4.3.1. *The minimal super-solution (Y, U) of the BSDE (4.5) with generator $y \mapsto -y|y|^{q-1}$ for $1 < q < 2$ satisfies: a.s. for any $t \in [(T-1) \vee 0, T)$*

$$Y_t \geq \frac{1}{T-t}.$$

In particular a.s. $\lim_{t\to T} Y_t = +\infty$.

Proof. $(Y^{(q),K}, U^{(q),K})$ denotes the solution of the BSDE with terminal condition $g(X_T) \wedge K$. Then

$$\begin{aligned} Y_t^{(q),K} - Y_t^{(2),K} &= \int_t^T (-Y_s^{(q),K}|Y_s^{(q),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|)ds \\ &\quad - \int_t^T \left(U_s^{(q),K} - U_s^{(2),K}\right) d\widetilde{N}_s \\ &= \int_t^T a_s^K (Y_s^{(q),K} - Y_s^{(2),K}) + \int_t^T (-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|)ds \\ &\quad - \int_t^T \left(U_s^{(q),K} - U_s^{(2),K}\right) d\widetilde{N}_s \end{aligned}$$

with, by decrease of the function $y \longmapsto -y|y|^{q-1}$,

$$a_s^K = \frac{-Y_s^{(q),K}|Y_s^{(q),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|^{q-1}}{Y_s^{(q),K} - Y_s^{(2),K}} 1_{\{Y_s^{(q),K} \neq Y_s^{(2),K}\}} \leq 0.$$

The formula for linear BSDE implies that

$$Y_t^{(q),K} - Y_t^{(2),K} = \mathbb{E}\left[\int_t^T \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|\right) \Gamma_{t,s}^K ds \middle| \mathcal{F}_t\right]$$

with $\Gamma_{t,s}^K = \exp\left(\int_t^s a_u^K du\right) \in [0,1]$. In other words

$$Y_t^{(q),K} - Y_t^{(2),K} \geq \mathbb{E}\left[\int_t^T \Gamma_{t,s}^K \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}|\right) 1_{[0,1]}(Y_s^{(2),K}) ds \middle| \mathcal{F}_t\right]$$

since for $y \geq 1$, $y|y|^{q-1} = y^q \leq y^2 = y|y|$. By the dominated convergence theorem, we deduce that for any $T-1 < t < T$

$$Y_t^{(q)} - Y_t^{(2)} = \lim_{K\to+\infty} Y_t^{(q),K} - Y_t^{(2),K} \geq 0,$$

that is : a.s. for $(T-1) \vee 0 \leq t < T$

$$Y_t^{(q)} \geq \frac{1}{T-t}.$$

This achieves the proof. □

Our proof shows that more general generators can be considered.

Theorem 4.3.2. *If Condition* **(A)** *holds, if* f^0 *is non-negative, if* $u \mapsto f(t,y,z,u)$ *is non-decreasing and if for* $q < 2$ *and some* $R > 0$

$$\forall y > R, \quad f(t,y,z,u) - f(t,0,z,u) \geq -y^q,$$

then the minimal super-solution of the BSDE (4.5) *verifies: a.s.*

$$\lim_{t \to T} Y_t = +\infty.$$

Proof. If (Y^K, Z^K, U^K) denotes the solution of the BSDE (4.5) with terminal condition $g(X_T) \wedge K$, then using a standard linearization method :

$$\begin{aligned}
Y_t^K - Y_t^{(2),K} &= \int_t^T \left(f(s, Y_s^K, Z_s^K, U_s^K) + Y_s^{(2),K} |Y_s^{(2),K}| \right) ds - \int_t^T Z_s^K dW_s \\
&- \int_t^T \left(U_s^K - U_s^{(2),K} \right) d\widetilde{N}_s \\
&= \int_t^T a_s^K (Y_s^{(q),K} - Y_s^{(2),K}) ds + \int_t^T b_s^K Z_s^K ds - \int_t^T Z_s^K dW_s \\
&+ \int_t^T \left(f(s, Y_s^{(2),K}, 0, U_s^K) - f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) \right) ds \\
&- \int_t^T \left(U_s^K - U_s^{(2),K} \right) d\widetilde{N}_s \\
&+ \int_t^T f(s,0,0,0) ds \\
&+ \int_t^T \left(f(s, Y_s^{(2),K}, 0, 0) - f(s,0,0,0) + Y_s^{(2),K} |Y_s^{(2),K}| \right) ds \\
&+ \int_t^T \left(f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, 0) \right) ds
\end{aligned}$$

with

$$a_s^K = \frac{f(s,Y_s^K,Z_s^K,U_s^K) - f(s,Y_s^{(2),K},Z_s^K,U_s^K)}{Y_s^K - Y_s^{(2),K}} \mathbf{1}_{\{Y_s^K \neq Y_s^{(2),K}\}},$$
$$b_s^K = \frac{f(s,Y_s^{(2),K},Z_s^K,U_s^K) - f(s,Y_s^{(2),K},0,U_s^K)}{Z_s^K} \mathbf{1}_{\{Z_s^K \neq 0\}}.$$

The process b^K is bounded and the process a^K is bounded from above. Solving this linear BSDE leads to

$$Y_t^K - Y_t^{(2),K} = \mathbb{E}\left[\int_t^T \left(f(s,0,0,0) + A_s^K + B_s^K \right) \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t \right]$$

where $\mathcal{E}^K$ is non-negative and belongs to any $L^\varrho([0,T]\times\Omega)$,

$$B_s^K = f(s,Y_s^{(2),K},0,U_s^{(2),K}) - f(s,Y_s^{(2),K},0,0),$$

and

$$\begin{aligned} A_s^K &= f(s,Y_s^{(2),K},0,0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \\ &\geq \left(f(s,Y_s^{(2),K},0,0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \leq R\right\}} \\ &\quad + \left(-Y_s^{(2),K}|Y_s^{(2),K}|^{q-1} + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \geq R\right\}} \\ &\geq \left(f(s,Y_s^{(2),K},0,0) - f(s,0,0,0) + Y_s^{(2),K}|Y_s^{(2),K}| \right) \mathbf{1}_{\left\{Y_s^{(2),K} \leq R\right\}} \end{aligned}$$

if $R > 1$. From our assumptions, with the dominated convergence theorem, we obtain that for every $t \in \left[T - \frac{1}{R}, T\right]$

$$\liminf_{K\to\infty} \mathbb{E}\left[\int_t^T A_s^K \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t \right] \geq 0.$$

Moreover

$$|B_s^K| \leq C|U_s^{(2),K}| = C|u^K(s,X_{s-}+1) - u^K(s,X_{s-})| \leq C\frac{2}{T-s}.$$

Thus for any $\varepsilon > 0$

$$\lim_{K\to\infty} \mathbb{E}\left[\int_t^{T-\varepsilon} B_s^K \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t\right] = 0$$

Let us decompose $U_s^{(2),K} = \widehat{U}_s^{(2),K} + \widetilde{U}_s^{(2),K}$, with $\widehat{U}_s^{(2),K} \geq 0$. Then

$$\begin{aligned} B_s^K &= f(s, Y_s^{(2),K}, 0, U_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, \widetilde{U}_s^{(2),K}) \\ &\quad + f(s, Y_s^{(2),K}, 0, \widetilde{U}_s^{(2),K}) - f(s, Y_s^{(2),K}, 0, 0). \end{aligned}$$

Since $u \mapsto f(t, y, z, u)$ is non-decreasing, the first term is non-negative, whereas the second is bounded by $C|\widetilde{U}_s^{(2),K}| \leq C\Phi(X_{s-})$ (see Corollary 4.2.10). Hence for any $\varepsilon > 0$

$$\mathbb{E}\left[\int_{T-\varepsilon}^T B_s^K \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t\right] \geq -C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \middle| \mathcal{F}_t\right].$$

Evoke that $f(s, 0, 0, 0)$ is also non-negative. Hence

$$Y_t^K \geq Y_t^{(2),K} + \mathbb{E}\left[\int_t^T A_s^K \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t\right] + \mathbb{E}\left[\int_t^{T-\varepsilon} B_s^K \mathcal{E}_{t,s}^K ds \middle| \mathcal{F}_t\right] - C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \middle| \mathcal{F}_t\right].$$

Then passing to the limit on K gives : a.s. $Y_t \geq Y_t^{(2)} - C\mathbb{E}\left[\int_{T-\varepsilon}^T |\Phi(X_{s-})|^{\varrho^*} ds \middle| \mathcal{F}_t\right]$ on $[T - \frac{1}{R}, T]$. Note that

$$\Phi(X_{s-}) \leq |X_{s-} - x_0| \sup_{y\in[X_0,x_0)} |\varphi(y+1) - \varphi(y)|.$$

Letting ε go to zero, we obtain that a.s. $Y_t \geq Y_t^{(2)}$ on $[T - \frac{1}{R}, T]$, which achieves the proof of the proposition. □

4.3.2 The case $q > 2$

From [76, Section 4] we already know that for $q > 3$, continuity holds: a.s. $\lim_{t\to T} Y_t = g(X_T)$. From the previous section, we also know that continuity fails for $q \leq 2$. In this part, we prove that continuity remains true for $2 < q \leq 3$.

We still consider the terminal condition $g(X_T)$ with g given by (4.14) and the truncated BSDE is : for $0 \leq t \leq T$

$$Y_t^K = g(X_T) \wedge K - \int_t^T Y_s^K |Y_s^K|^{q-1} ds - \int_t^T U_s^K d\widetilde{N}_s. \tag{4.36}$$

Again from [56], there exists a unique solution (Y^K, U^K) for (4.36) and a minimal solution (Y, U) for (4.13) such that Y is the increasing limit of Y and since g is non-negative: a.s.

$$\forall t \in [0,T], \quad 0 \leq Y_t^K \leq Y_t \leq \left(\frac{p-1}{T-t}\right)^{p-1}.$$

Here p is the Hölder conjugate of q. Note that these estimates do not depend on g. Moreover a.s.

$$\lim_{t\to T} Y_t \geq \xi = g(X_T).$$

Finally the related IPDE (4.11) is: for any $(t,x) \in [0,T] \times \mathbb{R}$

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \lambda u(t,x) - u(t,x)|u(t,x)|^{q-1} = -\lambda u(t,x+1) \\ \\ u(T,x) = g(x), \end{cases} \tag{4.37}$$

and the truncated PDE:

$$\begin{cases} \dfrac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)|^{q-1} = -\lambda u^K(t,x+1) \\ \\ u^K(T,x) = g(x) \wedge K. \end{cases} \tag{4.38}$$

With the same arguments as for Lemma 4.2.3, on $[0,T] \times [x_0, \infty)$, the solutions are

$$u(t,x) = \left(\frac{p-1}{T-t}\right)^{p-1}, \quad u^K(t,x) = \left(\frac{p-1}{T-t+K^{1-q}}\right)^{p-1}.$$

Thus on $[0,T] \times [x_0 - 1, x_0)$, the PDE (4.38) becomes

$$\frac{\partial u^K}{\partial t}(t,x) - \lambda u^K(t,x) - u^K(t,x)|u^K(t,x)|^{q-1} = -\lambda u^K(t,x+1) = -\lambda \left(\frac{p-1}{T-t+K^{1-q}}\right)^{p-1}.$$

The statement of Lemma 4.2.1 remains valid, that is u^K is well-defined, is non-decreasing w.r.t. K and converges to u as K tends to ∞. Moreover for any $t < T$

$$\begin{aligned}
u^K(t,x) &= u^K(T,x) - \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds \\
&\quad + \lambda \int_t^T \left(\frac{p-1}{T-s+K^{1-q}}\right)^{p-1} ds \\
&= g(x) \wedge K - \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds \\
&\quad + \lambda \frac{(p-1)^{p-1}}{2-p} \left((T+K^{1-q})^{2-p} - (T-t+K^{1-q})^{2-p}\right).
\end{aligned}$$

It is equivalent to

$$\begin{aligned}
u^K(t,x) + \int_t^T \left[\lambda u^K(s,x) + u^K(s,x)|u^K(s,x)|^{q-1}\right] ds &= g(x) \wedge K \\
+ \lambda \frac{(p-1)^{p-1}}{2-p} \left((T+K^{1-q})^{2-p} - (T-t+K^{1-q})^{2-p}\right).
\end{aligned}$$

Here is the key point : $q > 2$ implies that $p < 2$ or $2 - p > 0$. Hence we can pass to the limit w.r.t. K and the right-hand side is finite and equal to

$$g(x) + \lambda \frac{(p-1)^{p-1}}{2-p} \left(T^{2-p} - (T-t)^{2-p}\right).$$

By the monotone convergence theorem, the left-hand side converges to

$$u(t,x) + \int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds$$

and is larger than $u(t,x)$. We deduce that

$$u(t,x) + \int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds = g(x) + \lambda \frac{(p-1)^{p-1}}{2-p} \left(T^{2-p} - (T-t)^{2-p}\right)$$

and that $t \longmapsto u(t,x)$ is bounded by $g(x) + \lambda \frac{(p-1)^{p-1}}{2-p} T^{2-p}$. Therefore

$$\lim_{t \to T} \int_t^T \left[\lambda u(s,x) + u(s,x)|u(s,x)|^{q-1}\right] ds = 0$$

and for $x \in [x_0 - 1, x_0)$

$$\lim_{t \to T} u(t,x) = g(x).$$

We can iterate these arguments on $[x_0 - 2, x_0 - 1)$ since $u^K(t, x+1) \leq u(t, x+1)$ and $t \mapsto u(t, x+1)$ is a bounded function. Then by recursion we prove that:

Proposition 4.3.3. *If $q > 2$, the PDE (4.37) has a unique solution u, which is equal to $t \mapsto \left(\frac{p-1}{T-t}\right)^{p-1}$ for $x \geq x_0$, such that $t \mapsto u(t,x)$ is bounded for any $x < x_0$ and that $\lim_{t\to T} u(t,x) = g(x)$.*

In other words continuity holds for $q > 2$. For the BSDE, from the representation $Y_t^K = u^K(t, X_t)$, we immediately deduce that a.s.

$$\lim_{t\to T} Y_t = g(X_T).$$

Remark 4.3.4. *Note that the same result holds if the generator is of the form*

$$(s, y) \mapsto -y|y|^{q-1} + f_s^0$$

where f^0 is a deterministic and integrable function.

We replace the explicit expression of the solution on $[x_0, +\infty)$, by the solution of the ODE

$$y' = y|y|^{q-1} - f_s^0, \quad y(T) = +\infty$$

which is bounded by

$$\frac{1}{(T-t)^p} \int_0^T \left[(p-1)^{p-1} + (T-s)^p f_s^0\right] ds$$

(see [56]) and is still integrable on $(0, T)$ if $q > 2$.

As a consequence, under condition **(A)** with $q > 2$, and if f^0 is deterministic and integrable (or bounded from above by a deterministic and integrable function), then the solution of the BSDE (4.5) satisfies a.s.

$$\lim_{t\to T} Y_t = g(X_T).$$

The proof is based on a comparison principle between the solution of (4.5) and the solution of the BSDE with generator $(s, y) \mapsto -y|y|^{q-1} + f_s^0$.

4.4 Poisson case with left barrier

As mentioned in Section 4.1.2, continuity property is proved in [75] under a sufficient condition, which link the set $\mathcal{S}$ to the jumps of X. This assumption is verified for the terminal value given by (4.14).

Let us show here that this condition is unnecessary. We again consider that X is a Poisson process. Now the function g is defined with $x_0 \in \mathbb{R}$ and $\varphi : \mathbb{R} \longrightarrow [0, \infty)$ a continuous function with polynomial growth:

$$g(x) = (+\infty)\mathbf{1}_{\{x \leq x_0\}} + \varphi(x)\mathbf{1}_{\{x > x_0\}}. \tag{4.39}$$

Note that $\mathcal{S} = (-\infty, x_0]$ has a compact boundary, but $x \in \mathcal{S}$ does not imply that $x+1 \in \mathcal{S}$. Moreover in general, the truncated function $g \wedge K$ is not continuous at the point x_0.

Nonetheless since the forward process is a Poisson process, if the process is greater than x_0 at some time τ, it remains greater than x_0 after. Let us consider the unique solution of the BSDE:

$$\mathcal{Y}^K_t = \varphi(X_T) \wedge K + \int_t^T f(s, \mathcal{Y}^K_s, \mathcal{U}^K_s) ds - \int_t^T \mathcal{U}^K_s d\widetilde{N}_s,$$

where f still verifies Condition **(A)**.

Let us fix $\tau < T$ and consider the $\mathcal{F}_\tau$-measurable set $A_\tau = \{X_\tau > x_0\}$. Then on A_τ, for any $\tau \leq t \leq T$, $X_t \geq X_\tau > x_0$ and $g(X_T) = \varphi(X_T)$. Multiplying the two BSDEs by $\mathbf{1}_{A_\tau}$, we deduce that for any $\tau < t \leq T$, $\mathcal{Y}^K_t = Y^K_t$ on A_τ. Letting K go to $+\infty$ leads to: $Y_t = \mathcal{Y}_t$ on the set A_τ, where $(\mathcal{Y}, \mathcal{U})$ solves the BSDE:

$$\mathcal{Y}_t = \varphi(X_T) + \int_t^T f(s, \mathcal{Y}_s, \mathcal{U}_s) ds - \int_t^T \mathcal{U}_s d\widetilde{N}_s.$$

Note that the existence and uniqueness of $(\mathcal{Y}, \mathcal{U})$ is ensured by the growth assumption on φ. In particular a.s. on the set A_τ

$$\lim_{t \to T} Y_t = \varphi(X_T).$$

Now we take a increasing sequence of τ_n converging to T. Since the family A_τ is a non-decreasing family of sets, a.s.

$$\bigcup_{n\in\mathbb{N}} \{X_{\tau_n} > x_0\} = \{X_{T-} > x_0\} = \{X_T > x_0\}$$

since T cannot be a jump time of X. According to the Theorem 4.1.3, we deduce that a.s.

$$\lim_{t\to T} Y_t = g(X_T).$$

We proved that

Proposition 4.4.1. *The minimal super-solution* (Y, U) *of the BSDE* (4.5) *with terminal condition* $\xi = g(X_T)$ *for* g *given by* (4.39)*, satisfies* (4.3)*.*

With our choice of X, the PDE (4.11) becomes:

$$\frac{\partial u}{\partial t}(t,x) + \lambda u(t,x+1) - \lambda u(t,x) + f(t,x,u(t,x),\lambda(u(t,x+1)-u(t,x))) = 0. \quad (4.40)$$

Here the function f satisfies Condition **(A)** (uniformly in x) and the regularity conditions mentioned just before Proposition 4.1.4.

Proposition 4.4.2. *The minimal viscosity solution* u *of the PDE* (4.40)*, with terminal condition* g *given by* (4.39) *is continuous on* $[0,T]\times\mathbb{R}\setminus\{x_0\}$*.*

Proof. For a starting point $x > x_0$ at time t, we have : for any $t \le s \le T$, $X_s^{t,x} > x_0$. Hence a.s. $g(X_T^{t,x}) = \varphi(X_T^{t,x})$. We can apply Proposition 4.1.4 to deduce that the quantity $\widehat{u}(t,x) = \mathcal{Y}_t^{t,x}$ solves the PDE (4.40) with terminal condition φ, and that on $\{x > x_0\}$, $\widehat{u} = u$. In other words, the solution u^K of the PDE (4.40), with terminal condition $g(x)\wedge K$, converges to $\widehat{u}$ on $(x_0, +\infty)$. And we know that $u = \widehat{u}$ is continuous on $[0,T]\times(x_0,\infty)$.

Now for $x \in (x_0-1, x_0]$, to solve the PDE (4.40), let us consider the ordinary differential equation with parameter x:

$$(y^{K,x})'(t) + \lambda\widehat{u}(t,x+1) - \lambda y^{K,x}(t) + f(t,x,y^{K,x}(t),\lambda(\widehat{u}(t,x+1)-y^{K,x}(t))) = 0$$

with terminal condition $y^K(T) = K$. It is equivalent to

$$y^{K,x}(t) = K + \int_t^T F(s,x,y^{K,x}(s))ds.$$

The generator F satisfies Conditions (A1) to (A5), hence the solution $y^{K,x}$ exists and is unique. Since $\widehat{u} = u$ for $x > x_0$, it is immediate that $\check{u}^K(t,x) = y^{K,x}(t)$ is the solution of (4.40) with terminal condition $\check{u}^K(T,x) = K$.

Moreover F is continuous w.r.t. x. Hence standard stability estimate on BSDE (see [71, Theorem 2.9] or [49] for the Lipschitz case or [72, Theorem 5.10]) implies that $x \mapsto y^{K,x}$ is also continuous w.r.t. x, uniformly in K. Roughly speaking for (x,x')

$$
\begin{aligned}
(y^{K,x}(t) - y^{K,x'}(t))^2 &= 2\int_t^T (y^{K,x}(s) - y^{K,x'}(s))(F(s,x,y^{K,x}(s)) - F(s,x',y^{K,x'}(s)))ds \\
&\leq C_{\mu,\vartheta} \int_t^T (y^{K,x}(s) - y^{K,x'}(s))^2 ds + \lambda^2 \int_t^T (\widehat{u}(s,x+1) - \widehat{u}(s,x'+1))^2 ds \\
&\quad + \int_t^T (F(s,x,y^{K,x}(s)) - F(s,x',y^{K,x}(s)))^2 ds.
\end{aligned}
$$

Using regularity condition on f w.r.t. x and Gronwall's lemma, we deduce the regularity of $x \mapsto y^{K,x}$ uniformly w.r.t. K. Hence passing to the limit on K for $\check{u}^K$, $(t,x) \mapsto u(t,x)$ is also continuous on $[0,T] \times (x_0 - 1, x_0]$. Iterating this procedure, we deduce that u is continuous on $[0,T] \times (-\infty, x_0]$. □

At the point x_0, g is not continuous. But if $\lim_{x\to x_0} \varphi(x) = +\infty$, then u is also continuous at x_0.

Remark 4.4.3. *Note that the arguments can be generalized to any non-decreasing forward process X. For example X can be a Lévy subordinator and the BSDE is driven by the Poisson random measure associated to X.*

4.5 Associated Euler scheme

We are interesting here in the convergence of the numerical scheme for the ODE: for $t \in [0,T)$

$$
\begin{cases}
u'(t) - \lambda u(t) - u(t)|u(t)| = -\lambda \dfrac{1}{T-t}, \\[2ex]
u(T) = \chi \in [0,+\infty).
\end{cases}
$$

This ODE is the same as the ODE (4.23), but with a terminal condition, and has been used to solve the PDE (4.22) and in Lemma 4.2.5, we prove that the unique non-negative solution u is given by:

$$
u(t) = \frac{1}{T-t}, \quad \forall t < T,
$$

whatever χ is. Our aim is to illustrate this behavior for the numerical scheme and to show that the approximating sequence generated to the scheme converges to u for any χ.

We consider a regular subdivision $0 = t_0 < ... < t_N = T$ of the interval $[0, T]$ with a step $h_N = \frac{T}{N}$. We use here the implicit Euler method to define the scheme $u_N(t_k)$ by

$$u_N(t_N) = \chi$$

and by the implicit descending recurrence relation

$$u_N(t_{k+1}) = u_N(t_k) - h_N f(t_k, u_N(t_k)),$$

with

$$f(t, u) = \lambda u + u^2 - \lambda \frac{1}{T - t}.$$

We have the convergence on all closed intervals of $[0, T)$:

Theorem 4.5.1. *For all* $0 < \alpha < 1$*, we have*

$$\max_{0 \le k \le \lfloor \alpha N \rfloor} \left| u_N(t_k) - \frac{1}{T - t_k} \right| \underset{N \to +\infty}{\longrightarrow} 0.$$

We implemented the scheme. On Figure 4.1, on the left graph, the terminal value $\chi = 10$ is fixed and N increases. On the interval $[0, 0.8]$, the curves are overlaid on each other. On the right, N is equal to 1000 and χ increases; again on $[0, 0.93]$, the curves are overlaid.

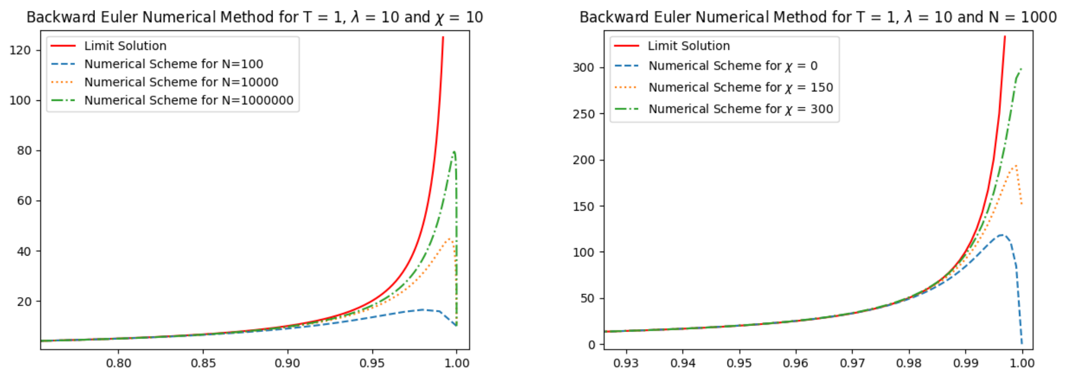

Figure 4.1 – Backward Euler Numerical Method for $T = 1$ and $\lambda = 10$. On the left, $\chi = 10$; on the right $N = 1000$.

To proof this theorem, firstly we are going to study the behavior of the scheme at the time $t_0 = 0$ thanks to the inferior and superior limits. Secondly we are using the results of convergence of forward schemes. To study the behavior of the scheme, we can explicit its expression.

Lemma 4.5.2. *The implicit backward Euler scheme can be written explicit : for all* $k \in [\![0, N-1]\!]$*, we have*

$$u_N(t_k) = \frac{\sqrt{(1 + h_N\lambda)^2 + 4h_N\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T - t_k}\right)} - (1 + h_N\lambda)}{2h_N} \geq 0.$$

Proof. We prove by recurrence the nonnegativity of $u_N(t_k)$. For $k = N$, we have $u_N(t_N) = \chi \geq 0$. Then if we assume $u_N(t_{k+1}) \geq 0$ for $k \in [\![0, N-1]\!]$, thus

$$0 \leq u_N(t_{k+1}) + \lambda h_N\frac{1}{T - t_k} = u_N(t_k) + h_N\left(\lambda u_N(t_k) + u_N(t_k)|u_N(t_k)|\right) = F(u_N(t_k)),$$

with $F(x) = h_N(\lambda x + x|x|)$. F is non-decreasing with $F(0) = 0$. Thus $u_N(t_k) \geq 0$.

Then we have that $u_N(t_k)$ is a non-negative root of the polynome

$$P = h_N X^2 + (1 + \lambda h_N)X - \left(u_N(t_{k+1}) + \lambda h_N\frac{1}{T - t_k}\right),$$

of discriminant

$$\Delta = (1 + \lambda h_N)^2 + 4h_N\left(u_N(t_{k+1}) + \lambda h_N\frac{1}{T - t_k}\right) > 0,$$

so the root are

$$x_1 = \frac{-(1 + \lambda h_N) + \sqrt{\Delta}}{2h_N} > 0, \quad x_2 = \frac{-(1 + h_N) - \sqrt{\Delta}}{2h_N} < 0.$$

Therefore we obtain the expression of the lemma. □

With this lemma, we have the following inequalities which will be useful to obtain contradictions if we assume the inferior and superior limits are different from the wished limit.

Lemma 4.5.3. *For all $k \in [\![0, N]\!]$,*

$$0 \leq u_N(t_k) \leq \chi + \lambda \sum_{i=1}^{N} \frac{1}{i}.$$

Proof. We show by descending recurrence on $k \in [\![0, N]\!]$ the property

$$u_N(t_k) \leq \chi + \lambda \sum_{i=1}^{N-k} \frac{1}{i}.$$

For $k = N$, we directly have

$$u_N(t_k) = u_N(t_N) = \chi.$$

Then we suppose the result at rank $k+1$ for $k \in [\![0, N-1]\!]$. Then, we have the expression of $u_N(t_k)$ in function of $u_N(t_{k+1})$ according to the lemma 4.5.2,

$$\begin{aligned}
u_N(t_k) &= \frac{\sqrt{(1+h_N\lambda)^2 + 4h_N\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)} - (1+h_N\lambda)}{2h_N} \\
&= \frac{2\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)}{\sqrt{(1+h_N\lambda)^2 + 4h_N\left(u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k}\right)} + (1+h_N\lambda)} \\
&\leq u_N(t_{k+1}) + h_N\lambda\frac{1}{T-t_k} \\
&\leq \chi + \lambda \sum_{i=1}^{N-k-1} \frac{1}{i} + h_N\lambda\frac{1}{Nh_N - kh_N} = \chi + \lambda\sum_{i=1}^{N-k}\frac{1}{i}.
\end{aligned}$$

The recurrence principle allows to conclude and to obtain the inequality of this lemma. □

From this lemma, we deduce the rough estimate that for any $0 \leq k \leq N$:

$$u_N(t_k) \leq \chi + \lambda(1+\gamma) + \lambda\ln(N),$$

with the Euler's constant γ.

Lemma 4.5.4. *The inferior limit in $t_0 = 0$ satisfies*

$$\liminf_{N\to+\infty} u_N(t_0) \geq \frac{1}{T}.$$

Proof. We assume by contradiction that $\liminf_{N\to+\infty} u_N(t_0) < \dfrac{1}{T}$. Then, for all $\varepsilon > 0$, there exists a subsequence of $(u_N(t_0))_{N\in\mathbb{N}}$ which we note $(U_N)_{N\in\mathbb{N}}$, and $N_0 \in \mathbb{N}$ such that

$$\lim_{N\to+\infty} U_N = \ell < \frac{1}{T} - 2\varepsilon, \quad \forall N \geq N_0,\ 0 \leq U_N \leq \frac{1}{T} - \varepsilon.$$

For all $a \in \left[0, \dfrac{1}{T} - \varepsilon\right]$ the solution v^a of the ODE (4.23) with initial condition a is given by (4.24):

$$v^a(t) = \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{T(1-aT)}e^{-\lambda t} + (T-t)e^{-\lambda t}\displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2}ds}\right), \quad t \in [0,T).$$

So, for $\varepsilon < \dfrac{e^{-\lambda T}}{T}$,

$$\begin{aligned}
v^a(t) \leq v^{\frac{1}{T}-\varepsilon}(t) &\leq \frac{1}{T-t}\left(1 - \frac{1}{\dfrac{T-t}{\varepsilon T^2}e^{-\lambda t} + (T-t)\displaystyle\int_0^t \frac{1}{(T-s)^2}ds}\right) \\
&= \frac{e^{-\lambda t} - \varepsilon T}{(T-t)e^{-\lambda t} + \varepsilon t T} \leq \frac{1}{(T-t)e^{-\lambda T} + \varepsilon t T} \\
&= \frac{1}{Te^{-\lambda T} + (\varepsilon T - e^{-\lambda T})t} \leq \frac{1}{Te^{-\lambda T} + (\varepsilon T - e^{-\lambda T})T} = \frac{1}{\varepsilon T^2},
\end{aligned}$$

According to (4.25), we have

$$v^{\frac{1}{T}-\varepsilon}(t) \underset{t\to T}{\sim} \lambda \ln(T-t) < 0.$$

So, for $\eta > 0$, there exists $\tau^\varepsilon \in (0, T)$ such that: $v^{\frac{1}{T}-\varepsilon}(\tau^\varepsilon) < -\eta$. Moreover

$$v^a(t) \geq v^0(t) = \frac{1}{T-t}\left(1 - \frac{1}{\frac{T-t}{T}e^{-\lambda t} + (T-t)e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{(T-s)^2}ds}\right)$$
$$= -\frac{\lambda e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{T-s}ds}{1 - \lambda(T-t)e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{T-s}ds}.$$

By continuity on the interval $[0, \tau^\varepsilon]$, the function v^0 is bounded from below by a constant $K_\varepsilon < 0$. So, on this interval, each solution v^a is bounded between K_ε and $\frac{1}{T^2\varepsilon}$. Then we can consider the ODE (4.23) starting at time 0 from U_N, with a driver $\widetilde{f}$ which is bounded, of class C^1 with bounded derivative, such that the bounds for $\widetilde{f}$ do not depend on a or N. Thus the associated Euler scheme

$$\begin{cases} w_N(t_0) = U_N \\ \\ w_N(t_{k+1}) = w_N(t_k) + h_N\left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda\frac{1}{T-t_k}\right) \end{cases}$$

satisfies the standard conistency and stability results for Euler's scheme (see [8, Theorem 2.4] or [27, Chapter VIII]): there exists a constant $C > 0$ which depends on the driver $\widetilde{f}$, such that

$$\max_{0 \leq k \leq \left\lfloor \frac{\tau^\varepsilon N}{T} \right\rfloor} |w_N(t_k) - v^\ell(t_k)| \leq C|w_N(t_0) - \ell| + \frac{C}{N}, \quad C > 0.$$

where we have chosen $a = \ell < \frac{1}{T} - 2\varepsilon < \frac{1}{T} - \varepsilon$. But we have

$$w_N(t_0) = U_N \underset{N \to +\infty}{\longrightarrow} \ell, \quad \frac{1}{N} \underset{N \to +\infty}{\longrightarrow} 0.$$

So, for N big enough,

$$\max_{0 \leq k \leq \left\lfloor \frac{\tau^\varepsilon N}{T} \right\rfloor} |w_N(t_k) - v^\ell(t_k)| \leq \frac{\eta}{2}.$$

Furthermore

$$v^\ell(\tau^\varepsilon) \leq v^{\frac{1}{T}-\varepsilon}(\tau_\varepsilon) < -\eta,$$

the function v^ℓ is continuous on $[0,\tau^\varepsilon]$ and $t^N = \left\lfloor \frac{\tau^\varepsilon N}{T} \right\rfloor \frac{T}{N} \underset{N\to+\infty}{\longrightarrow} \tau^\varepsilon$. So, for N big enough,

$$v^\ell(t^N) < -\frac{3\eta}{4}.$$

Thus

$$\begin{aligned} w_N(t^N) &= w_N(t^N) - v^\ell(t^N) + v^\ell(t^N) \\ &\leq \max_{0\leq k\leq \left\lfloor \frac{\tau^\varepsilon}{T} \right\rfloor} |w_N(t_k) - v^\ell(t_k)| + v^\ell(t^N) \\ &< \frac{\eta}{2} - \frac{3\eta}{4} \\ &= -\frac{\eta}{4} \\ &< 0. \end{aligned}$$

But, after extraction, the sequence $w_N(t_k)$ satisfies the same Euler scheme than $u_N(t_k)$, so, after extraction,

$$u_N\left(t^N\right) = w_N\left(t^N\right) < 0.$$

However the scheme $u_N(t_k)$ cannot be negative according to the lemma 4.5.2. We obtain a contradiction, therefore

$$\liminf_{N\to+\infty} u_N(t_0) \geq \frac{1}{T}.$$

□

Lemma 4.5.5. *The superior limit in $t_0 = 0$ satisfies*

$$\limsup_{N\to+\infty} u_N(t_0) \leq \frac{1}{T}.$$

Proof. We again prove this result by contradiction. We assume

$$\limsup_{N\to+\infty} u_N(t_0) > \frac{1}{T}.$$

Then, for all $\varepsilon > 0$, there exists a subsequence of $(u_N(t_0))_{N\in\mathbb{N}}$, which we note $(U_N)_{N\in\mathbb{N}}$, and $N_0 \in \mathbb{N}$ such that

$$\lim_{N\to+\infty} U_N = \ell > \frac{1}{T} + 2\varepsilon, \quad \forall N \geq N_0,\ U_N \geq \frac{1}{T} + \varepsilon.$$

But, for $a > \frac{1}{T}$, the differential equation (4.23) does not admit solution on $[0, T)$ (Lemma 4.2.5). More precisely, the solution is defined on $[0, \tau)$ with τ defined like the first time in $[0, T)$ such that

$$\frac{1}{Ta - 1} = T \int_0^\tau \frac{e^{\lambda s}}{(T-s)^2} ds.$$

If $a \geq \frac{1}{T} + \varepsilon$, then we have

$$\frac{T}{T-\tau} - 1 = T \int_0^\tau \frac{1}{(T-s)^2} ds \leq T \int_0^\tau \frac{e^{\lambda s}}{(T-s)^2} ds = \frac{1}{Ta-1} \leq \frac{1}{\varepsilon T}.$$

So

$$\tau \leq \frac{T}{1 + \varepsilon T}.$$

In other words, each solution v^a of the differential equation (4.23) which satisfies $v^a(0) = a \geq \frac{1}{T} + \varepsilon$, explodes before the time $\frac{T}{1+\varepsilon T}$. Now we consider the Euler scheme

$$\begin{cases} w_N(t_0) = a \geq \dfrac{1}{T} + \varepsilon \\ \\ w_N(t_{k+1}) = w_N(t_k) + h_N \left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda \dfrac{1}{T - t_k} \right), \end{cases}$$

for $0 \leq t_k \leq \frac{T}{1+\varepsilon T}$ i.e. $k \in \left[\!\left[0, \left\lfloor \frac{N}{1+\varepsilon T} \right\rfloor \right]\!\right]$. On this interval we have

$$0 \leq \frac{1}{T - t_k} \leq \frac{1+\varepsilon}{\varepsilon T^2} < +\infty.$$

Let us now prove by recursion that for N larger than some constant depending on T and ε, and for any k

$$w_N(t_k) \geq \frac{1}{T - t_k} + \varepsilon.$$

This property holds for $k = 0$. If the property is satisfied for k then

$$\begin{aligned} w_N(t_{k+1}) &= w_N(t_k) + h_N \left(\lambda w_N(t_k) + w_N(t_k)^2 - \lambda \frac{1}{T - t_k} \right) \\ &\geq w_N(t_k) + h_N w_N(t_k)^2 + h_N \lambda \varepsilon \geq w_N(t_k) + h_N w_N(t_k)^2. \end{aligned}$$

But
$$\frac{1}{T-t_{k+1}} = \frac{1}{T-t_k-h_N} = \frac{1}{T-t_k} + \frac{h_N}{(T-t_k)(T-t_k-h_N)}.$$
So
$$\begin{aligned} w_N(t_{k+1}) - \frac{1}{T-t_{k+1}} &\geq w_N(t_k) - \frac{1}{T-t_k} + h_N\left(w_N(t_k)^2 - \frac{1}{(T-t_k)(T-t_k-h_N)}\right) \\ &\geq \varepsilon + h_N\left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} + \frac{1}{(T-t_k)^2} - \frac{1}{(T-t_k)(T-t_k-h_N)}\right) \\ &= \varepsilon + h_N\left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} - \frac{h_N}{(T-t_k)^2(T-t_k-h_N)}\right), \end{aligned}$$

with
$$\frac{1}{(T-t_k)^2(T-t_k-h_N)} = \frac{1}{(T-t_k)^2}\frac{1}{T-t_{k+1}} \leq \frac{(1+\varepsilon)^2}{\varepsilon^2T^4}\frac{1+\varepsilon}{\varepsilon T^2} = \frac{(1+\varepsilon)^3}{\varepsilon^3T^6}.$$
So, for N larger than $\varepsilon^5T^5/(1+\varepsilon)^3$
$$w_N(t_{k+1}) - \frac{1}{T-t_{k+1}} \geq \varepsilon + h_N\left(\varepsilon^2 + \frac{2\varepsilon}{T-t_k} - h_N\frac{(1+\varepsilon)^3}{\varepsilon^3T^6}\right) \geq \varepsilon.$$
Hence the property is proved for any k and the recurrence formula can be rewritten
$$w_n(t_{k+1}) = w_N(t_k) + h_Nw_N(t_k)^2 + h_N\lambda\left(w_N(t_k) - \frac{1}{T-t_k}\right) \geq w_N(t_k) + h_Nw_N(t_k)^2.$$

If we define $\overline{w}_N(t_k)$ the sequence defined by
$$\begin{cases} \overline{w}_N(t_0) = \dfrac{1}{T} + \varepsilon \\ \\ \overline{w}_N(t_{k+1}) = \overline{w}_N(t_k) + h_N\overline{w}_N(t_k)|\overline{w}_N(t_k)| \end{cases}$$
this sequence is well-defined, non-negative and non-decreasing. From the previous property of $w_N(t_k)$, a direct comparison for the schemes leads to:
$$\overline{w}_N(t_k) \leq w_N(t_k).$$

We consider the sequence

$$y_N(t_k) = \frac{1}{\overline{w}_N(t_k)}.$$

So

$$y_N(t_{k+1}) = \frac{1}{\overline{w}_N(t_{k+1})} = \frac{1}{\overline{w}_N(t_k) + h_N \overline{w}_N(t_k)^2} = \frac{1}{\dfrac{1}{y_N(t_k)} + \dfrac{h_N}{y_N(t_k)^2}} = \frac{y_N(t_k)}{1 + \dfrac{h_N}{y_N(t_k)}},$$

and

$$y_N(t_0) = \frac{1}{\overline{w}_N(t_0)} = \frac{1}{\dfrac{1}{T} + \varepsilon} = \frac{T}{1 + \varepsilon T}.$$

Assume that for the biggest $k : \hat{k} = \left\lfloor \dfrac{N}{1+\varepsilon T} \right\rfloor$,

$$y_N(t_{\hat{k}}) > 2\sqrt{T}\sqrt{h_N}.$$

So, since the sequence $y_N(t_k)$ is non-increasing in k, we have

$$y_N(t_0) \geq ... \geq y_N(t_k) \geq y_N(t_{k+1}) \geq ... \geq y_N(t_{\hat{k}}) > 2\sqrt{T}\sqrt{h_N}.$$

Thus, from the inequality $\dfrac{1}{1+u} \leq 1 - u + u^2$,

$$\begin{aligned} y_N(t_{k+1}) &= y_N(t_k) \frac{1}{1 + \dfrac{h_N}{y_N(t_k)}} \leq y_N(t_k) \left(1 - \frac{h_N}{y_N(t_k)} + \frac{h_N^2}{y_N(t_k)^2}\right) \\ &= y_N(t_k) - h_N + \frac{h_N^2}{y_N(t_k)} < y_N(t_k) - h_N + \frac{1}{2\sqrt{T}} h_N^{\frac{3}{2}}. \end{aligned}$$

So, by successive iterations,

$$\begin{aligned} y_N(t_k) &< y_N(t_0) - k h_N + k \frac{1}{2\sqrt{T}} h_N^{\frac{3}{2}} \\ &= \frac{T}{1+\varepsilon T} - k\frac{T}{N} + k\frac{T}{N}\frac{1}{2\sqrt{T}}\sqrt{h_N}. \end{aligned}$$

But, for $\hat{k} \geq \dfrac{N}{1+\varepsilon T} - 1$, we have

$$\frac{T}{1+\varepsilon T} - \hat{k}\frac{T}{N} \leq \frac{T}{1+\varepsilon T} - \frac{T}{1+\varepsilon T} + \frac{T}{N} = \frac{T}{N}.$$

Hence with $\hat{k} \leq N$

$$y_N(t_{\hat{k}}) < \frac{T}{N} + \hat{k}\frac{T}{N}\frac{1}{2\sqrt{T}}\sqrt{h_N} \leq \frac{T}{N} + \frac{T}{2\sqrt{T}}\sqrt{h_N} = \sqrt{h_N}\left(\frac{\sqrt{T}}{\sqrt{N}} + \frac{\sqrt{T}}{2}\right) \leq 2\sqrt{T}\sqrt{h_N}$$

what contradicts the assumption. Therefore

$$0 \leq y_N(t_{\hat{k}}) \leq 2\sqrt{T}\sqrt{h_N} = 2T\frac{1}{\sqrt{N}}.$$

Thus

$$w_N(t_{\hat{k}}) \geq \overline{w}_N(t_{\hat{k}}) = \frac{1}{y_N(t_{\hat{k}})} \geq \frac{1}{2T}\sqrt{N}.$$

Now if we consider $a = \ell > \dfrac{1}{T} + 2\varepsilon > \dfrac{1}{T} + \varepsilon$, we can deduce, after extraction,

$$u_N(t_{\hat{k}}) \geq \frac{1}{2T}\sqrt{N},$$

what cannot be according to the lemma 4.5.2. Therefore we have

$$\limsup_{N\to+\infty} u_N(t_0) \leq \frac{1}{T}.$$

□

As a consequence of the two previous lemmata, we state

Proposition 4.5.6. *We have the limit in* $t_0 = 0$ *:*

$$\lim_{N\to+\infty} u_N(t_0) = \frac{1}{T}.$$

Finally we obtain Theorem 4.5.1 by using the convergence results about the forward Euler scheme.

Proof of Theorem 4.5.1. We consider the forward numerical scheme

$$\begin{cases} w_N(t_{k+1}) = w_N(t_k) + h_N \left(\lambda w_N(t_k) + w_N(t_k)|w_N(t_k)| - \lambda \dfrac{1}{T - t_k} \right) \\ \\ w_N(t_0) = u_N(t_0) \end{cases},$$

associated to the differential equation

$$\begin{cases} w'(t) = \lambda w(t) + w(t)|w(t)| - \lambda \dfrac{1}{T-t} = f(t, w(t)), \quad 0 \le t < T \\ \\ w(0) = u_N(t_0). \end{cases}$$

To obtain the exact solution, we have to distinguish if $u_N(t_0) < \frac{1}{T}, u_N(t_0) > \frac{1}{T}$ or $u_N(t_0) = \frac{1}{T}$. The last case is direct because, in that case, the exact solution is $w^N(t) = \frac{1}{T-t}$, $0 \le t < T$. In a first time, if

$$u_N(t_0) < \frac{1}{T},$$

then the solution is given by (4.24):

$$w^N(t) = \frac{1}{T-t} \left(1 - \frac{1}{\dfrac{T-t}{T(1-u_N(t_0)T)} e^{-\lambda t} + (T-t)e^{-\lambda t} \displaystyle\int_0^t \frac{e^{\lambda s}}{(T-s)^2} ds} \right).$$

Let us prove that for N large enough (depending on the convergence proved in the previous proposition), w^N is non-negative and bounded from above on $[0, \alpha T]$. Indeed with an integration by part, we obtain

$$w^N(t) = \frac{1}{T-t} \left(1 - \frac{1}{\dfrac{T-t}{T(1-u_N(t_0)T)} e^{-\lambda t} + 1 - \dfrac{(T-t)e^{-\lambda t}}{T} - (T-t)\lambda e^{-\lambda t} \displaystyle\int_0^t \frac{e^{\lambda s}}{T-s} ds} \right).$$

Therefore the function w^N is positive until the first time $t = t(N)$ such that

$$\frac{T-t}{T(1-u_N(t_0)T)}e^{-\lambda t} - \frac{(T-t)e^{-\lambda t}}{T} - (T-t)\lambda e^{-\lambda t}\int_0^t \frac{e^{\lambda s}}{T-s}ds = 0,$$

which is equivalent to

$$\frac{1}{\frac{1}{u_N(t_0)} - T} = \int_0^t \frac{e^{\lambda s}}{T-s}ds.$$

Thus

$$\frac{1}{\frac{1}{u_N(t_0)} - T} = \int_0^t \frac{e^{\lambda s}}{T-s}ds < e^{\lambda T}\int_0^t \frac{ds}{T-s} = e^{\lambda T}\ln\left(\frac{T}{T-t}\right),$$

and we deduce that

$$t(N) > T\left(1 - \exp\left(-\frac{e^{-\lambda T}}{\frac{1}{u_N(t_0)} - T}\right)\right) \underset{N\to+\infty}{\longrightarrow} T,$$

because $u_N(t_0) \underset{N\to+\infty}{\longrightarrow} \frac{1}{T}$. Therefore for N large enough, the time $t(N)$ is greater than αT, that w^N is positive on $[0, \alpha T]$. Furthermore the function w^N is bounded from above on $[0, \alpha T]$ by

$$C_\alpha = \frac{1}{T - \alpha T},$$

because

$$w^N(t) \leq \frac{1}{T-t}, \quad 0 \leq t \leq \alpha T.$$

Now if $u_N(t_0) > \frac{1}{T}$, then the exact solution w^N is still given by (4.24), is non-decreasing, but is only defined on $[0, \tau)$ with τ defined by

$$\frac{1}{u_N(t_0) - \frac{1}{T}} = \int_0^\tau \frac{e^{\lambda s}}{(T-s)^2}ds.$$

But $u_N(t_0) \underset{N\to+\infty}{\longrightarrow} \frac{1}{T}$, so $\frac{1}{u_N(t_0) - \frac{1}{T}} \underset{N\to+\infty}{\longrightarrow} +\infty$, thus, for N big enough, we have $\tau > \alpha T$. So the function w^N is defined and continuous on $[0, \alpha T]$ and bounded from above by, for $\varepsilon > 0$ and N large enough such that $u_N(t_0) < \frac{1}{T} + 1$

$$\begin{aligned}
C_{\alpha,N} &= w^N(\alpha T) \\
&= \frac{1}{T-\alpha T}\left(1 - \frac{1}{\dfrac{T-\alpha T}{T(1-u_N(t_0)T)}e^{-\lambda\alpha T} + (T-\alpha T)e^{-\lambda\alpha T}\displaystyle\int_0^{\alpha T}\frac{e^{\lambda s}}{(T-s)^2}ds}\right) \\
&\leq \frac{1}{T-\alpha T}\left(1 - \frac{1}{-(1-\alpha)e^{-\lambda\alpha T} + (T-\alpha T)e^{-\lambda\alpha T}\displaystyle\int_0^{\alpha T}\frac{e^{\lambda s}}{(T-s)^2}ds}\right) =: C_\alpha.
\end{aligned}$$

Therefore, in each case, the solution is non-negative and bounded from above by some constant C_α on $[0,\alpha T]$.

We can consider the function ψ defined by

$$\psi(w) = \begin{cases} \lambda w & \text{si} \quad w < 0 \\ \lambda w + w^2 & \text{si} \quad 0 \leq w \leq C_\alpha \\ (\lambda + 2C_\alpha)w - C_\alpha^2 & \text{si} \quad w > C_\alpha. \end{cases}$$

The function ψ is of class C^1 on $\mathbb{R}$ and Lipschitz continuous with a Lipschitz constant equal to $\lambda + 2C_\alpha$. We consider the function $\widetilde{f}$ defined by

$$\widetilde{f}(t,w) = \psi(w) - \lambda\frac{1}{T-t}, \quad w \in \mathbb{R},\ 0 \leq t < T.$$

This function is equal to f on $[0,T) \times [0, C-\alpha]$ and inherits the regularity property of ψ w.r.t. w. Therefore, with the previous inequalities, the function w^N satisfies the differential equation

$$\begin{cases} w'(t) = \widetilde{f}(t, w(t)), \quad 0 \leq t < T \\ \\ w(0) = u_N(t_0) \end{cases}$$

with driver $\widetilde{f}$. So, according to [8] or [27], there exists a constant C such that

$$\max_{0\leq k\leq\lfloor\alpha N\rfloor}|w_N(t_k) - w^N(t_k)| \leq C(|w_N(t_0) - \underbrace{w^N(t_0)}_{=u_N(t_0)}| + Th_N) = CTh_N.$$

This constant C depends on T, λ and α, but not on N. A direct computation shows that

C can be chosen equal to

$$e^{\lambda+2C_\alpha}\left[\lambda\frac{1}{T^2(1-\alpha)^2}+\lambda^2\frac{1}{T(1-\alpha)}+\left(\lambda^2+2\lambda\frac{1}{T(1-\alpha)}\right)C_\alpha+3\lambda C_\alpha^2+2\lambda C_\alpha^3\right],$$

which shows the dependence w.r.t. α. Therefore, since $u_N(t_k)=w_N(t_k)$ by definition of the numerical schemes satisfied by these two sequences,

$$\max_{0\leq k\leq\lfloor\alpha N\rfloor}|u_N(t_k)-w^N(t_k)|\leq CTh_N\underset{N\to+\infty}{\longrightarrow}0.$$

Therefore all that remains is to study the second term in the inequality

$$\max_{0\leq k\leq\lfloor\alpha N\rfloor}|u_N(t_k)-u(t_k)|\leq\max_{0\leq k\leq\lfloor\alpha N\rfloor}|u_N(t_k)-w^N(t_k)|+\max_{0\leq k\leq\lfloor\alpha N\rfloor}|w^N(t_k)-u(t_k)|,$$

with

$$u(t)=\frac{1}{T-t},\quad 0\leq t<T.$$

We have

$$\begin{aligned}|w^N(t_k)-u(t_k)|&=\frac{1}{\frac{T-t_k}{T(1-u_N(t_0)T)}e^{-\lambda t_k}+(T-t_k)e^{-\lambda t_k}\int_0^{t_k}\frac{e^{\lambda s}}{(T-s)^2}ds}\\&\leq\frac{1}{\frac{T-t_k}{T(1-u_N(t_0)T)}e^{-\lambda t_k}}=\frac{T(1-u_N(t_0)T)e^{\lambda t_k}}{T-t_k}\\&\leq\frac{T(1-u_N(t_0)T)e^{-\lambda t_{\lfloor\alpha N\rfloor}}}{T-t_{\lfloor\alpha N\rfloor}}.\end{aligned}$$

So

$$\max_{0\leq k\leq\lfloor\alpha N\rfloor}|w^N(t_k)-u(t_k)|\leq\frac{T(1-u_N(t_0)T)e^{-\lambda t_{\lfloor\alpha N\rfloor}}}{T-t_{\lfloor\alpha N\rfloor}}\underset{N\to+\infty}{\longrightarrow}0,$$

because

$$u_N(t_0)\underset{N\to+\infty}{\longrightarrow}\frac{1}{T}\quad\text{and}\quad t_{\lfloor\alpha N\rfloor}=\lfloor\alpha N\rfloor\frac{T}{N}\underset{N\to+\infty}{\longrightarrow}\alpha T<T.$$

Finally we have shown

$$\max_{0\leq k\leq\lfloor\alpha N\rfloor}|u_N(t_k)-u(t_k)|\underset{N\to+\infty}{\longrightarrow}0$$

which achieves the proof of Theorem 4.5.1. □

Chapter 5

Malliavin calculus with respect to a Hawkes process

In this chapter we aim to develop a local Malliavin calculus with respect to a Hawkes process. Malliavin calculus is a mathematical framework used to study the smoothness of random variables and functionals of stochastic processes, especially those driven by Brownian motion. The central concept is the Malliavin derivative, a type of derivative that extends the classical notion of differentiation to the space of random variables. This approach allows one to analyze the regularity and differentiability of random processes, providing powerful tools for studying stochastic differential equations (SDEs) and probabilistic systems. The key ideas of Malliavin calculus were introduced by Paul Malliavin in the 1970s. There is a huge literature on this subject, encompassing Malliavin calculus for Lévy processes (see among others [69, 14, 29, 16] and the references therein).

Hawkes processes have been introduced by Alan Hawkes in the 1970s as a class of self-exciting point processes. They are widely used to model events or occurrences where the occurrence of one event increases the likelihood of subsequent events in the near future. These processes have been used to model earthquakes, and for some time now, they have been experiencing a renewed interest due to their applications in finance and actuarial science [9, 25, 58].

A Hawkes process is characterized by an intensity function that depends on both a baseline intensity and a history of past events. More specifically, the intensity at time t is given by:

$$\lambda^*(t) = \lambda + \int_{(0,t]} \mu(t-s) dN_s$$

where λ is the baseline intensity, μ is a function that describes the impact of past events, and N is the counting process that represents the occurrences of events. The key feature of Hawkes processes is that the function μ is typically a non-negative function, which means that past events increase the probability of future events — hence the term "self-exciting."

Recently a Malliavin calculus with respect to the Hawkes process N is developed in [45] and [46]. Roughly speaking, the main ingredient consists to perturb the system by adding a particle (or a jump) (see [45, Lemma 3.5]) leading to an expansion formula for functionals of the Hawkes process [45, Theorem 3.13]). Let us mention that the derivative operator is not local. With these results, they are able to develop a Stein method (see [46]) and to compute some prices of financial or insurance derivatives (see [45, Section 4]).

Our method is different and follows the approach of Carlen-Pardoux [22]. We perturb the jump times and formally differentiate with respect to these jump times. This allows us to define a local derivative, satisfying the chain rule. We apply it to the study of absolute continuity for the law of Hawkes functionals and to the computation of Greeks.

Breakdown of the chapter. The first step of our construction of a Malliavin derivative with respect to a Hawkes process is to define a directional derivative with respect to a function $m \in \mathcal{H}$, where $\mathcal{H}$ is the ad hoc Cameron-Martin space. An integration by parts formula is obtained thanks to the absolute continuity property of the law of the perturbed jump times w.r.t. the initial probability measure (see Proposition 5.1.7, Theorem 5.1.13, Proposition 5.1.15 and Corollary 5.1.16).

The second step is to define the Malliavin derivative D in all directions by considering a Hilbert basis of $\mathcal{H}$. We get a local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ which admits a carré du champ Γ and a gradient D (see Proposition 5.2.1). Therefore we get similar properties to the directional derivative as the chain rule. Moreover we are interested in the associated divergence operator δ, for which we get an explicit expression of $\delta(u)$ when u is predictable (see Proposition 5.2.4, Remark 5.2.5 and Corollary 5.2.7).

We then establish an absolute continuity criterion: conditionally to

$$\Gamma[F] = (\Gamma[F_i, F_j])_{1\leq i,j\leq d} \in GL_d(\mathbb{R}),$$

the random vector $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$ admits a absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^d$ (see Theorem 5.3.5 and Corollary 5.3.6).

This criterion is firstly applied to the solution of stochastic differential equation driven by the Hawkes process (see Theorem 5.4.9, Corollary 5.4.11 and Proposition 5.4.13). As a second application, we compute Greeks for a financial payoff when the underlying process is driven by the Hawkes process (see Proposition 5.4.17).

5.1 Framework and directional derivation

5.1.1 Setting and first notations

We consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the space of càdlàg trajectories

$$\omega(t) = \sum_i i 1_{[t_i, t_{i+1})},$$

with $0 < t_1 < \cdots < t_i < \cdots$.

We define

$$N_t(\omega) = \sum_{s \leq t} \Delta\omega_s, \quad t \geq 0,$$

the process which counts the jumps between 0 and t, where $\Delta w_s = w_s - w_{s-}$ and $w_{s-} = \lim_{u \to s^-} w_u$.

We assume that, under $\mathbb{P}$, $(N_t)_{t \in \mathbb{R}_+}$ is a Hawkes process with conditional intensity

$$\lambda^*(t) = \lambda + \int_{(0,t)} \mu(t-s) dN_s = \lambda + \sum_{i=1}^{+\infty} \mu(t - T_i) 1_{\{t > T_i\}} = \lambda + \sum_{i=1}^{N_t} \mu(t - T_i).$$

Throughout this chapter, we suppose that

Assumption 15.

- $\lambda \in (0, +\infty)$
- $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *differentiable with bounded derivative and such that*

$$\|\mu\|_1 = \int_0^{+\infty} \mu(t) dt < 1.$$

We introduce $(T_i)_{i \in \mathbb{N}^*}$ the jump instants of the Hawkes process N, and, for any $n \in \mathbb{N}^*$, $0 < t_1 < \cdots < t_n$ and $s \in \mathbb{R}_+^*$:

$$\lambda^*(s; t_1, \cdots, t_n) = \lambda + \sum_{i=1}^{n-1} \mu(s - t_i) 1_{\{s > t_i\}}.$$

Thus, for any $n \in \mathbb{N}^*$, on the event $\{N_t = n\}$,

$$\lambda^*(t) = \lambda^*(t; T_1, \cdots, T_n).$$

We consider the $\mathbb{P}$-complete right continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ generated by the

Hawkes process N where $T \in \mathbb{R}_+^*$ is a fixed time horizon.

We apply the same approach as in [22] to define the directional derivative using the reparametrization of time with respect to a function in a Cameron-Martin space.

Let $L^2([0,T])$ be the usual space of square integrable function on $[0,T]$ with respect to the Lebesgue measure and $\mathcal{H}$ be the closed subspace of $L^2([0,T])$ orthogonal to the constant functions, i.e.,

$$\mathcal{H} = \left\{ m \in L^2([0,T]) \quad \int_0^T m(s)ds = 0 \right\}. \tag{5.1}$$

We denote $\widehat{m} = \int_0^\cdot m(s)ds$ for every $m \in \mathcal{H}$, then $\widehat{m}(0) = \widehat{m}(T) = 0$. In a natural way, $\mathcal{H}$ inherits the Hilbert structure of $L^2([0,T])$ and we denote by $\| \, \|_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the norm and the scalar product on it. From now on in this section, we fix a function $m \in \mathcal{H}$. The condition $\int_0^T m(s)ds = 0$ ensures that the change of intensity that we are about to define simply shifts the jump times without affecting the total number of jumps. Let us define

$$\widetilde{m}_\varepsilon(s) = \begin{cases} -\frac{1}{3\varepsilon} & \text{if } m(s) \leq -\frac{1}{3\varepsilon}; \\ m(s) & \text{if } \frac{-1}{3\varepsilon} \leq m(s) \leq \frac{1}{3\varepsilon}; \\ \frac{1}{3\varepsilon} & \text{if } m(s) \geq \frac{1}{3\varepsilon}. \end{cases}$$

and $m_\varepsilon \in \mathcal{H}$ such that

$$m_\varepsilon(s) = \widetilde{m}_\varepsilon(s) - \frac{1}{T}\int_0^T \widetilde{m}_\varepsilon(s)ds. \tag{5.2}$$

We remark that $\frac{1}{3} \leq 1 + \varepsilon m_\varepsilon(s) \leq \frac{5}{3}$ (since $-\frac{1}{3\varepsilon} \leq \widetilde{m}_\varepsilon(s) \leq \frac{1}{3\varepsilon}$).

Lemma 5.1.1. *We have the following convergence*

$$||m - m_\varepsilon||_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

Proof. We have $||m - m_\varepsilon||_{\mathcal{H}} \leq ||m - \widetilde{m}_\varepsilon||_{\mathcal{H}} + ||\widetilde{m}_\varepsilon - m_\varepsilon||_{\mathcal{H}}$ with for the first term, for almost every $s \in [0,T]$,

$$m(s) - \widetilde{m}_\varepsilon(s) = \begin{cases} m(s) + \dfrac{1}{3\varepsilon} & \text{if } \; m(s) \leq -\dfrac{1}{3\varepsilon} \\ 0 & \text{if } \; -\dfrac{1}{3\varepsilon} \leq m(s) \leq \dfrac{1}{3\varepsilon} \\ m(s) - \dfrac{1}{3\varepsilon} & \text{if } \; \dfrac{1}{3\varepsilon} \leq m(s). \end{cases} \tag{5.3}$$

Thus

$$|m(s) - \widetilde{m}_\varepsilon(s)| \leq \left(|m(s)| + \frac{1}{3\varepsilon}\right) 1_{\{|m(s)|\geq \frac{1}{3\varepsilon}\}} \leq 2|m(s)| 1_{\{|m(s)|\geq \frac{1}{3\varepsilon}\}}.$$

Therefore, by dominated convergence theorem,

$$\|m - \widetilde{m}_\varepsilon\|_{\mathcal{H}} \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Indeed $|m(s)|^2 1_{\{|m(s)|\geq \frac{1}{3\varepsilon}\}} \underset{\varepsilon\to 0}{\longrightarrow} 0$ and $|m(s)|^2 1_{\{|m(s)|\geq \frac{1}{3\varepsilon}\}} \leq |m(s)|^2 \in L^1([0,T])$. Now for the second term, as $\int_0^T m(s)ds = 0$ and by Cauchy-Schwarz inequality,

$$\begin{aligned} ||\widetilde{m}_\varepsilon - m_\varepsilon||_{\mathcal{H}} &= \frac{1}{T}\left|\int_0^T \widetilde{m}_\varepsilon(s)ds\right| = \frac{1}{T}\left|\int_0^T (\widetilde{m}_\varepsilon(s) - m(s))ds\right| \\ &\leq \frac{1}{\sqrt{T}}||\widetilde{m}_\varepsilon - m||_{\mathcal{H}} \underset{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned}$$

□

We define the reparametrization of time with respect to m_ε as follow

$$\tau_\varepsilon(s) = s + \varepsilon\widehat{m}_\varepsilon(s) = \int_0^s (1 + \varepsilon m_\varepsilon(u))du, \quad s \in \mathbb{R}_+.$$

Notice that $\tau_\varepsilon(0) = 0, \tau_\varepsilon(T) = T$, and since $1 + \varepsilon m_\varepsilon(s) \in \left[\frac{1}{3}, \frac{5}{3}\right] \subset \mathbb{R}_+^*$, τ_ε is an increasing function hence invertible so the number and the order of jump times between 0 and T remain unchanged. Moreover, a direct calculation gives

$$\forall s \in [0,T], \quad \tau_\varepsilon^{-1}(s) = \int_0^s \frac{1}{1 + \varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))}du.$$

Let $\mathcal{T}_\varepsilon : \Omega \to \Omega$ be the map defined by, for any $\omega \in \Omega$,

$$(\mathcal{T}_\varepsilon(\omega))(s) = \omega(\tau_\varepsilon(s)),$$

$$\mathcal{T}_\varepsilon F = F \circ \mathcal{T}_\varepsilon \text{ for all } F \in L^2(\Omega),$$

and $\mathbb{P}^\varepsilon$ be the probability measure $\mathbb{P}\mathcal{T}_\varepsilon^{-1}$ defined on $\mathcal{F}_T$.

5.1.2 Directional derivation

Definition 5.1.2. *We denote*

$$\mathbb{D}_m^0 = \left\{ F \in L^2(\Omega) : \frac{\partial \mathcal{T}_\varepsilon F}{\partial \varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon F - F) \text{ in } L^2(\Omega) \text{ exists} \right\}.$$

For $F \in \mathbb{D}_m^0, D_m F$ is defined as the limit

$$D_m F = \frac{\partial \mathcal{T}_\varepsilon F}{\partial \varepsilon}|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}(\mathcal{T}_\varepsilon F - F). \tag{5.4}$$

Definition 5.1.3. *Let define the set $\mathcal{S}$ of "smooth" functions. We say that a map $F : \Omega \to \mathbb{R}$ belongs to $\mathcal{S}$ if there exists $a \in \mathbb{R}, d \in \mathbb{N}^*$ and for any $n \in \{1, \cdots, d\}$, a function $f_n : \mathbb{R}^n \to \mathbb{R}$ such that:*

1. *The random variable F can be written*

$$F = a 1_{\{N_T = 0\}} + \sum_{n=1}^{d} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T = n\}}. \tag{5.5}$$

2. *For any $n \in \{1, \cdots, d\}$, the function f_n is smooth with bounded derivatives of any order.*

Remark 5.1.4. *The space $\mathcal{S}$ is dense in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.*

Here are some basic properties of directional derivatives on $\mathcal{S}$.

Lemma 5.1.5. *Let $j \in \mathbb{N}^*$ and $\overline{T}_j = T_j \wedge T$. Then $\overline{T}_j \in \mathbb{D}_m^0$ and*

$$D_m \overline{T}_j = -\widehat{m}(T_j).$$

Proof. We first remark that, for any $\omega \in \Omega$,

$$T_j(\omega \circ \tau_\varepsilon) = \tau_\varepsilon^{-1}(T_j(\omega)).$$

Then

$$
\begin{aligned}
|\mathcal{T}_\varepsilon \overline{T}_j(\omega) - \overline{T_j}(\omega) + \varepsilon\widehat{m}(\overline{T_j})(\omega)| &= \left|(\overline{T}_j \circ \mathcal{T}_\varepsilon)(\omega) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt\right| \\
&= \left|\overline{T}_j(\omega \circ \tau_\varepsilon) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt\right| = \left|\tau_\varepsilon^{-1}(\overline{T}_j(\omega)) - \overline{T}_j(\omega) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt\right| \\
&= \left|\tau_\varepsilon^{-1}(\overline{T}_j(\omega)) - \tau_\varepsilon(\tau_\varepsilon^{-1}(\overline{T}_j(\omega))) + \varepsilon \int_0^{\overline{T_j}(\omega)} m(t)dt\right| \\
&= \left|\tau_\varepsilon(s_\varepsilon) - s_\varepsilon - \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} m(t)dt\right|, \quad \text{with } s_\varepsilon = \tau_\varepsilon^{-1}(\overline{T}_j(\omega)) \\
&\le \left|\tau_\varepsilon(s_\varepsilon) - s_\varepsilon - \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} m_\varepsilon(t)dt\right| + \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} |m_\varepsilon(t) - m(t)|dt \\
&\le \varepsilon \left|\int_{s_\varepsilon}^{\tau_\varepsilon(s_\varepsilon)} m_\varepsilon(t)dt\right| + \varepsilon \int_0^{\tau_\varepsilon(s_\varepsilon)} |m_\varepsilon(t) - m(t)|dt \\
&\le \varepsilon\sqrt{\tau_\varepsilon(s_\varepsilon) - s_\varepsilon}\sqrt{\int_0^T |m_\varepsilon(t)|^2 dt} + \varepsilon\int_0^T |m_\varepsilon(t) - m(t)|dt.
\end{aligned}
$$

We have

$$
\begin{aligned}
|\tau_\varepsilon(s_\varepsilon) - s_\varepsilon| &= \overline{T}_j(\omega) - s_\varepsilon = \int_0^{\overline{T}_j(\omega)} \left(1 - \frac{1}{1 + \varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))}\right) du \\
&\le \int_0^T \left(1 - \frac{1}{1 + \varepsilon m_\varepsilon(\tau_\varepsilon^{-1}(u))}\right) du \underset{\varepsilon \to 0}{\longrightarrow} 0.
\end{aligned}
$$

Moreover $\lim_{\varepsilon\to 0} \int_0^T |m_\varepsilon(t) - m(t)|dt = 0$ and $\int_0^T |m_\varepsilon(t)|^2 dt$ is bounded so that we get by a dominated convergence argument that $\overline{T}_j$ belongs to $\mathbb{D}_m^0$ and $D_m\overline{T}_j = -\widehat{m}(\overline{T}_j)$. □

Proposition 5.1.6. *Let $n \in \mathbb{N}^*$ and $f : \mathbb{R}^n \to \mathbb{R}$ a function of class C^1. Then*

$$
f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) \in \mathbb{D}_m^0
$$

and

$$
D_m f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) = -\sum_{j=1}^n \frac{\partial f}{\partial t_j}(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)\,\widehat{m}(\overline{T}_j).
$$

Thus $\mathcal{S} \subset \mathbb{D}_m^0$ and for any $F \in \mathcal{S}$ of the form (5.5)*,*

$$
D_m F = -\sum_{n=1}^d \sum_{j=1}^n \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)\widehat{m}(T_j)1_{\{N_T = n\}}.
$$

Proof. By the definition of D_m given by (5.4) we have

$$\begin{aligned} D_m f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) &= \frac{\partial \mathcal{T}_\varepsilon f(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n)}{\partial \varepsilon}|_{\varepsilon=0} \\ &= \frac{\partial}{\partial \varepsilon} f(\mathcal{T}_\varepsilon \overline{T}_1, \mathcal{T}_\varepsilon \overline{T}_2, \cdots, \mathcal{T}_\varepsilon \overline{T}_n)|_{\varepsilon=0} = \sum_{j=1}^n \frac{\partial f}{\partial t_j} \frac{\partial}{\partial \varepsilon} \mathcal{T}_\varepsilon \overline{T}_j|_{\varepsilon=0} \\ &= \sum_{j=1}^n \frac{\partial f}{\partial t_j} D_m \overline{T}_j = -\sum_{j=1}^n \frac{\partial f}{\partial t_j}(\overline{T}_1, \overline{T}_2, \cdots, \overline{T}_n) \widehat{m}(\overline{T}_j) \end{aligned}$$

where the last equality is due to Lemma 5.1.5. We deduce the last assertion by linearity using the fact that $N_T \circ \mathcal{T}_\varepsilon = N_T$. □

Proposition 5.1.7.

1. *If $F, G \in \mathcal{S}$ then $FG \in \mathcal{S}$ and $D_m(FG) = (D_m F)G + F(D_m G)$.*

2. *We have the chain rule: If $F_1, F_2, \cdots, F_n \in \mathcal{S}$ and $\Phi : \mathbb{R}^n \to \mathbb{R}$ is a smooth function then*

$$\Phi(F_1, F_2, \cdots, F_n) \in \mathcal{S}$$

and

$$D_m \Phi(F_1, F_2, \cdots, F_n) = \sum_{j=1}^n \frac{\partial \Phi}{\partial x_j}(F_1, F_2, \cdots, F_n) D_m F_j.$$

Proof.

1. We assume that $F, G \in \mathcal{S}$. Then

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^d f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}$$

and

$$G = b1_{\{N_T=0\}} + \sum_{n=1}^d g_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}.$$

So

$$FG = ab1_{\{N_T=0\}} + \sum_{n=1}^d (f_n \times g_n)(T_1, \cdots, T_n) 1_{\{N_T=n\}}.$$

Thus $FG \in \mathcal{S}$ and

$$\begin{aligned}
& D_m(FG) \\
&= abD_m 1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m[(f_n \times g_n)(T_1,\cdots,T_n)1_{\{T=n\}}] \\
&= 0 - \sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial(f_n \times g_n)}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&= -\sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1,\cdots,T_n)g_n(T_1,\cdots,T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&\quad -\sum_{n=1}^{d}\sum_{j=1}^{n} f_n(T_1,\cdots,T_n)\frac{\partial g_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)1_{\{N_T=n\}} \\
&= \sum_{n=1}^{d}\left(-\sum_{j=1}^{n}\frac{\partial f_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right)g_n(T_1,\cdots,T_n)1_{\{N_T=n\}} \\
&\quad +\sum_{n=1}^{d}\left(-\sum_{j=1}^{n}\frac{\partial g_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right)f_n(T_1,\cdots,T_n)1_{\{N_T=n\}} \\
&= \sum_{n=1}^{d} D_m f_n(T_1,\cdots,T_n)g_n(T_1,\cdots,T_n)1_{\{N_T=n\}} \\
&\quad +\sum_{n=1}^{d} D_m g_n(T_1,\cdots,T_n)f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.
\end{aligned}$$

Moreover

$$\begin{aligned}
(D_mF)G &= \left(aD_m 1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m[f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}]\right) \\
&\quad \times \left(b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\right) \\
&= 0 + \sum_{n=1}^{d} D_m f_n(T_1,\cdots,T_n)g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}
\end{aligned}$$

and

$$F(D_mG) = \sum_{n=1}^{d} D_m g_n(T_1,\cdots,T_n)f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.$$

Thus we obtain

$$D_m(FG) = (D_mF)G + F(D_mG).$$

2. We assume that $F, \cdots, G \in \mathcal{S}$ and $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$ smooth. Then

$$\begin{aligned}
&\Phi(F,G)\\
&= \Phi\left(a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}, b1_{\{N_T=0\}} + \sum_{n=1}^{d} g_n(T_1,\cdots,T_n)\right)\\
&= \Phi(a,b)1_{\{N_T=0\}} + \sum_{n=1}^{d} \Phi(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n))1_{\{N_T=n\}}\\
&= \Phi(a,b)1_{\{N_T=0\}} + \sum_{n=1}^{d} (\Phi(f_n,g_n))(T_1,\cdots,T_n))1_{\{N_T=n\}}.
\end{aligned}$$

Thus $\Phi(F,G) \in \mathcal{S}$ and, according to the Proposition 5.1.6,

$$\begin{aligned}
&D_m(\Phi(f_n,g_n))(T_1,\cdots,T_n)\\
&= -\sum_{j=1}^{n} \frac{\partial \Phi(f_n,g_n)}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\\
&= -\sum_{j=1}^{n} \left(\frac{\partial \Phi}{\partial x}(f_n,g_n)\frac{\partial f_n}{\partial t_j} + \frac{\partial \Phi}{\partial y}(f_n,g_n)\frac{\partial g_n}{\partial t_j}\right)(T_1,\cdots,T_n)\widehat{m}(T_j)\\
&= \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)\left(-\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right)\\
&\quad + \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)\left(-\sum_{j=1}^{n} \frac{\partial g_n}{\partial t_j}(T_1,\cdots,T_n)\widehat{m}(T_j)\right)\\
&= \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)D_m f_n(T_1,\cdots,T_n)\\
&\quad + \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)D_m g_n(T_1,\cdots,T_n).
\end{aligned}$$

Then

$$\begin{aligned}
D_m(\Phi(F,G)) &= \Phi(a,b)D_m 1_{\{N_T=0\}}\\
&\quad + \sum_{n=1}^{d} \frac{\partial \Phi}{\partial x}(f_n,g_n)(T_1,\cdots,T_n)D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\\
&\quad + \sum_{n=1}^{d} \frac{\partial \Phi}{\partial y}(f_n,g_n)(T_1,\cdots,T_n)D_m g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\frac{\partial \Phi}{\partial x}(F,G)D_m F \\
&= \frac{\partial \Phi}{\partial x}\Bigg(a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}, b1_{\{N_T=0\}} \\
&\quad + \sum_{n=1}^{d} g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\Bigg) \\
&\quad \times \Bigg(aD_m 1_{\{N_T=0\}} + \sum_{n=1}^{d} D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\Bigg) \\
&= \sum_{n=1}^{d} \frac{\partial \Phi}{\partial x}(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n))D_m f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial \Phi}{\partial y}(F,G)D_m G \\
&= \sum_{n=1}^{d} \frac{\partial \Phi}{\partial y}(f_n(T_1,\cdots,T_n), g_n(T_1,\cdots,T_n)) \\
&\quad \times D_m g_n(T_1,\cdots,T_n)1_{\{N_T=n\}}.
\end{aligned}$$

Thus

$$D_m(\Phi(F,G)) = \frac{\partial \Phi}{\partial x}(F,G)D_m F + \frac{\partial \Phi}{\partial y}(F,G)D_m G.$$

The case with more than two random variables is left to the reader.

□

5.1.3 Absolute continuity of $\mathbb{P}^\varepsilon$ w.r.t. $\mathbb{P}$

Evoke that $\mathbb{P}^\varepsilon$ is defined at the end of Section 5.1.1. Let $\mathbb{P}^\varepsilon$ be the expectation under the probability $\mathbb{P}^\varepsilon$.

Let $k \in \mathbb{N}^*$ and $0 < t_1 < t_2 < \cdots < t_k$, knowing $T_1 = t_1, \cdots, T_k = t_k$, the conditional link for $t \geq t_k$ is given by:

$$\begin{aligned}
\mathbb{P}(T_{k+1} > t | T_1 = t_1, \cdots, T_k = t_k) &= \mathbb{P}(N_t - N_{t_k} = 0 | T_1 = t_1, \cdots, T_k = t_k) \\
&= e^{-\int_{t_k}^{t} \lambda^*(s;t_1,\cdots,t_k)ds}.
\end{aligned}$$

Thus the density of T_{k+1} knowing $T_1 = t_1, \cdots, T_k = t_k$ is

$$t \longmapsto \lambda^*(t; t_1, \cdots, t_k) e^{-\int_{t_k}^{t} \lambda^*(s; t_1, \cdots, t_k)\, ds} 1_{\{t > t_k\}}.$$

We deduce that $(T_1, \cdots, T_k, T_{k+1})$ admits for density

$$(t_1, \cdots, t_{k+1}) \longmapsto \left(\prod_{i=1}^{k+1} \lambda^*(t_i; t_1, \cdots, t_k) \right) e^{-\int_0^{t_{k+1}} \lambda^*(s; t_1, \cdots, t_k)\, ds} 1_{\{0 < t_1 < \cdots < t_{k+1}\}},$$

where we used the fact that $\lambda^*(s; t_1, \cdots, t_k) = \lambda^*(s; t_1, \cdots, t_{i-1})$ if $s \leq t_i$.

Let $n \in \mathbb{N}^*$ and a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\begin{aligned}
&\mathbb{E}^{\varepsilon}[f(T_1, \cdots, T_n) 1_{\{N_T = n\}}] \\
&\quad = \mathbb{E}[(f \circ \Phi_{\varepsilon}^{-1})(T_1, \cdots, T_n) 1_{\{T_n \leq T < T_{n+1}\}}] \\
&\quad = \iint_{0 < t_1 < \cdots < t_n \leq T < t_{n+1}} (f \circ \Phi_{\varepsilon}^{-1})(t_1, \cdots, t_n) \left(\prod_{i=1}^{n+1} \lambda^*(t_i; t_1, \cdots, t_n) \right) \\
&\qquad \times e^{-\int_0^{t_{n+1}} \lambda^*(s; t_1, \cdots, t_n)\, ds} dt_1 \cdots dt_{n+1} \\
&\quad = \iint_{0 < t_1 < \cdots < t_n \leq T} (f \circ \Phi_{\varepsilon}^{-1})(t_1, \cdots, t_n) \prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) dt_1 \cdots dt_n \\
&\qquad \times \int_T^{\infty} \lambda^*(t_{n+1}; t_1, \cdots, t_n) e^{-\int_0^{t_{n+1}} \lambda^*(s; t_1, \cdots, t_n)\, ds} dt_{n+1} \\
&\quad = \iint_{0 < t_1 < \cdots < t_n \leq T} (f \circ \Phi_{\varepsilon}^{-1})(t_1, \cdots, t_n) \varphi_n(t_1, \cdots, t_n) dt_1 \cdots dt_n \\
&\quad = \iint_{0 < u_1 < \cdots < u_n \leq T} f(u_1, \cdots, u_n)(\varphi_n \circ \Phi_{\varepsilon})(u_1, \cdots, u_n) |\det J_{\Phi_{\varepsilon}}| du_1 \cdots du_n \\
&\quad = \iint_{0 < u_1 < \cdots < u_n \leq T} f(u_1, \cdots, u_n)(\varphi_n \circ \Phi_{\varepsilon})(u_1, \cdots, u_n) \\
&\qquad \times \prod_{i=1}^{n} (1 + \varepsilon m_{\varepsilon}(u_i)) du_1 \cdots du_n \\
&\quad = \mathbb{E}\left[f(T_1, \cdots, T_n) 1_{\{N_T = n\}} Z_n^{\varepsilon} \right],
\end{aligned}$$

where

$$\Phi_{\varepsilon}(u_1, \cdots, u_n) = (u_1 + \varepsilon \widehat{m}_{\varepsilon}(u_1), \cdots, u_n + \varepsilon \widehat{m}_{\varepsilon}(u_n)), \tag{5.6}$$

$$\varphi_n(t_1, \cdots, t_n) = \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^T \lambda^*(s; t_1, \cdots, t_n) ds}, \tag{5.7}$$

$\det J_{\Phi_\varepsilon}$ denotes the determinant of the Jacobian matrix of Φ_ε and

$$Z_n^\varepsilon = \frac{(\varphi_n \circ \Phi_\varepsilon)(T_1, \cdots, T_n)}{\varphi_n(T_1, \cdots, T_n)} \prod_{i=1}^{n} (1 + \varepsilon m_\varepsilon(T_i)). \tag{5.8}$$

Let us emphasize that Assumption 15 is used to ensure that: $\lambda^*(s; t_1, \cdots, t_n) \geq \lambda > 0$. This yields:

Proposition 5.1.8. *$\mathbb{P}^\varepsilon$ is absolutely continuous with respect to $\mathbb{P}$ with density*

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} = \sum_{n=0}^{+\infty} Z_n^\varepsilon 1_{\{N_T = n\}} := G^\varepsilon.$$

Remark 5.1.9. *This series converges in L^1 uniformly in ε because for $f = 1$*

$$\mathbb{E}[G^\varepsilon] = \sum_{n=0}^{+\infty} \mathbb{E}[Z_n^\varepsilon 1_{\{N_T = n\}}] = \sum_{n=0}^{+\infty} \mathbb{E}^\varepsilon[1_{\{N_T = n\}}] = 1.$$

Remark 5.1.10. *In case of the Poisson process, φ_n is constant, equal to λ^n and we have again the result obtained in [22] for standard Poisson processes*

$$\frac{d\mathbb{P}^\varepsilon}{d\mathbb{P}} = \prod_{i=1}^{N_T} (1 + \varepsilon m_\varepsilon(T_i)).$$

5.1.4 Limit behavior of the density G^ε when $\varepsilon \to 0$

We begin with a first result about the limits when ε tends to 0.

Proposition 5.1.11. *For any $n \in \mathbb{N}^*$, a.s.*

$$\lim_{\varepsilon \to 0} Z_n^\varepsilon = \lim_{\varepsilon \to 0} G^\varepsilon = 1.$$

Proof. Firstly we have the almost surely convergences

$$\forall i \in \{1, \cdots, n\}, \quad \varepsilon m_\varepsilon(T_i) \xrightarrow[\varepsilon \to 0]{a.s.} 0.$$

Indeed we use (5.2) and (5.3) to get

$$m(s) - m_\varepsilon(s) \quad = \quad m(s) - \widetilde{m}_\varepsilon(s) + \frac{1}{T} \int_0^T (m(r) - \widetilde{m}_\varepsilon(r)) dr.$$

As $T_i < +\infty$ a.s., for $\varepsilon \in \mathbb{R}_+^*$ small enough,

$$\begin{aligned} |m(T_i) - m_\varepsilon(T_i)| &= \left| \frac{1}{T} \int_0^T (m(r) - \widetilde{m}_\varepsilon(r)) dr \right| \leq \frac{1}{T} \int_0^T |m(r) - \widetilde{m}_\varepsilon(r)| dr \\ &\leq \frac{1}{\sqrt{T}} \|m - \widetilde{m}_\varepsilon\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0. \end{aligned}$$

See Lemma 5.1.1. Thus

$$\varepsilon m_\varepsilon(T_i) = \varepsilon m(T_i) + \varepsilon(m_\varepsilon(T_i) - m(T_i)) \xrightarrow[\varepsilon \to 0]{a.s.} 0.$$

So we have the convergence of the product in (5.8)

$$\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) \xrightarrow[\varepsilon \to 0]{a.s.} 1.$$

Secondly for the convergence of (5.6), for a.e. $s \in [0, T]$,

$$|\widehat{m}(s) - \widehat{m}_\varepsilon(s)| \leq \int_0^T |m(r) - m_\varepsilon(r)| dr \leq \sqrt{T} \|m - m_\varepsilon\|_{\mathcal{H}} \underset{\varepsilon \to 0}{\longrightarrow} 0.$$

In particular we have the uniform convergence on $[0, T]$ of $\widehat{m}_\varepsilon$ to $\widehat{m}$. Then, for any $i \in \{1, \cdots, n\}$,

$$\varepsilon \widehat{m}_\varepsilon(T_i) = \varepsilon \widehat{m}(T_i) + \varepsilon(\widehat{m}_\varepsilon(T_i) - \widehat{m}(T_i)) \xrightarrow[\varepsilon \to 0]{a.s.} 0.$$

Moreover

$$\begin{aligned} \varphi_n(\Phi_\varepsilon(T_1, \cdots, T_n)) &= \varphi_n(T_1 + \varepsilon \widehat{m}_\varepsilon(T_1), \cdots, T_n + \varepsilon \widehat{m}_\varepsilon(T_n)) \\ &= \prod_{j=1}^n \lambda^*(T_j + \varepsilon \widehat{m}_\varepsilon(T_j); T_1 + \varepsilon \widehat{m}_\varepsilon(T_1), \cdots, T_n + \varepsilon \widehat{m}_\varepsilon(T_n)) \\ &\quad \times \exp\left(- \int_0^T \lambda^*(s; T_1 + \varepsilon \widehat{m}_\varepsilon(T_1), \cdots, T_n + \varepsilon \widehat{m}_\varepsilon(T_n)) ds \right) \\ &= \prod_{j=1}^n \left(\lambda + \sum_{i=1}^{j-1} \mu(T_j - T_i + \varepsilon(\widehat{m}_\varepsilon(T_j) - \widehat{m}_\varepsilon(T_i))) \right) \\ &\quad \times \exp\left(-\lambda T - \sum_{i=1}^{n-1} \int_0^{T - T_i - \varepsilon \widehat{m}_\varepsilon(T_i)} \mu(u) du \right). \end{aligned}$$

Thus, as the function μ is continuous,

$$\varphi_n(\Phi_\varepsilon(T_1,\cdots,T_n)) \xrightarrow[\varepsilon\to 0]{a.s.} \varphi_n(T_1,\cdots,T_n).$$

Therefore the a.s. convergence of Z_n^ε is proved. We have the same convergence for G^ε by dominated convergence theorem because

$$\sum_{n=1}^{+\infty}\mathbb{E}[Z_n^\varepsilon 1_{\{N_T=n\}}] = \sum_{n=1}^{+\infty}\mathbb{E}[1_{\{N_T=n\}}] = 1 < +\infty.$$

This achieves the proof. $\square$

5.1.5 Integration by parts in Bismut's way

Proposition 5.1.12. *Under our setting*

$$\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} \;=\; \int_{(0,T]}(\psi(m,s)+\widehat{m}(s)\mu(T-s)+m(s))dN_s$$

where

$$\begin{aligned}\psi(m,s) &= \frac{1}{\lambda^*(s)}\int_{(0,s)}(\widehat{m}(s)-\widehat{m}(t))\mu'(s-t)dN_t \\ &= \frac{\sum_{i=1}^{N_s-1}(\widehat{m}(s)-\widehat{m}(T_i))\mu'(s-T_i)}{\lambda+\sum_{i=1}^{N_s-1}\mu(T_i-s)}.\end{aligned} \tag{5.9}$$

Proof. Let $n\in\mathbb{N}^*$ and let's work on the event $\{N_T=n\}$. From (5.8), we have

$$Z_n^\varepsilon \;=\; \frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\ldots,T_n)}{\varphi_n(T_1,\ldots,T_n)}\prod_{i=1}^n(1+\varepsilon m_\varepsilon(T_i)).$$

Thus, for any $\varepsilon\in\mathbb{R}_+^*$, according to Proposition 5.1.11, Z_n^ε a.s. converges to $Z_n^0=1$ and

$$\begin{aligned}\frac{Z_n^\varepsilon-1}{\varepsilon} &= \frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\ldots,T_n)-\varphi_n(T_1,\ldots,T_n)}{\varepsilon\varphi_n(T_1,\ldots,T_n)}\prod_{i=1}^n(1+\varepsilon m_\varepsilon(T_i)) \\ &\quad+\frac{1}{\varepsilon}\left(\prod_{i=1}^n(1+\varepsilon m_\varepsilon(T_i))-1\right).\end{aligned} \tag{5.10}$$

For the first term

$$\begin{aligned}
& \frac{(\varphi_n \circ \Phi_\varepsilon)(T_1,\dots,T_n) - \varphi_n(T_1,\dots,T_n)}{\varepsilon \varphi_n(T_1,\dots,T_n)} \\
= & \frac{\varphi_n(T_1 + \varepsilon \widehat{m}_\varepsilon(T_1),\dots,T_n + \varepsilon \widehat{m}_\varepsilon(T_n)) - \varphi_n(T_1,\dots,T_n)}{\varepsilon \varphi_n(T_1,\dots,T_n)} \\
= & \sum_{i=1}^n \int_0^1 \frac{\partial \varphi_n}{\partial t_i}(T_1 + \alpha\varepsilon \widehat{m}_\varepsilon(T_1),\dots,T_n + \alpha\varepsilon \widehat{m}_\varepsilon(T_n)) \frac{\varepsilon \widehat{m}_\varepsilon(T_i)}{\varepsilon \varphi_n(T_1,\cdots,T_n)} d\alpha \\
= & \sum_{i=1}^n \frac{\widehat{m}_\varepsilon(T_i)}{\varphi_n(T_1,\cdots,T_n)} \int_0^1 \frac{\partial \varphi_n}{\partial t_i}(T_1 + \alpha\varepsilon \widehat{m}_\varepsilon(T_1),\dots,T_n + \alpha\varepsilon \widehat{m}_\varepsilon(T_n)) d\alpha \\
\xrightarrow[\varepsilon \to 0]{a.s.} & \sum_{i=1}^n \frac{\widehat{m}(T_i)}{\varphi_n(T_1,\dots,T_n)} \frac{\partial \varphi_n}{\partial t_i}(T_1,\dots,T_n)
\end{aligned}$$

where the almost surely convergence is justified by, for any $i \in \{1,...,n\}, \widehat{m}_\varepsilon(T_i) \xrightarrow[\varepsilon \to 0]{a.s.} \widehat{m}(T_i)$ and $\frac{\partial \varphi_n}{\partial t_i}$ bounded because μ admits a bounded derivative. And, as in the proof of Proposition 5.1.11,

$$\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) \xrightarrow[\varepsilon \to 0]{a.s.} 1.$$

For the second term

$$\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1 \quad = \quad \varepsilon \sum_{i=1}^n m_\varepsilon(T_i) + \varepsilon^2 \sum_{1 \le i < j \le n} m_\varepsilon(T_i) m_\varepsilon(T_j) + \cdots + \varepsilon^n \prod_{i=1}^n m_\varepsilon(T_i).$$

Thus

$$\frac{\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1}{\varepsilon} \quad = \quad \sum_{i=1}^n m_\varepsilon(T_i) + \varepsilon \sum_{1 \le i < j \le n} m_\varepsilon(T_i) m_\varepsilon(T_j) + \cdots + \varepsilon^{n-1} \prod_{i=1}^n m_\varepsilon(T_i).$$

with, for any $i \in \{1,\dots,n\}, m_\varepsilon(T_i) \xrightarrow[\varepsilon \to 0]{a.s.} m(T_i)$. Thus

$$\frac{\prod_{i=1}^n (1 + \varepsilon m_\varepsilon(T_i)) - 1}{\varepsilon} \xrightarrow[\varepsilon \to 0]{a.s.} \sum_{i=1}^n m(T_i).$$

Therefore
$$\frac{Z_n^\varepsilon - Z_n^0}{\varepsilon} \xrightarrow[\varepsilon \to 0]{a.s.} \sum_{i=1}^n \frac{\widehat{m}(T_i)}{\varphi_n(T_1,\dots,T_n)} \frac{\partial \varphi_n}{\partial t_i}(T_1,\dots,T_n) + \sum_{i=1}^n m(T_i)$$

with, for any $i_0 \in \{1, \dots, n\}$ and $0 < t_1 < \dots < t_n \le T$,

$$\begin{aligned}
& \frac{1}{\varphi_n(t_1, \dots, t_n)} \frac{\partial \varphi_n}{\partial t_{i_0}}(t_1, \dots, t_n) \\
= \; & \frac{\partial}{\partial t_{i_0}} \ln(\varphi_n(t_1, \dots, t_n)) \\
= \; & \sum_{i=1}^{n} \frac{\partial}{\partial t_{i_0}} \ln(\lambda^*(t_i; t_1, \dots, t_n)) - \frac{\partial}{\partial t_{i_0}} \left(\int_0^T \lambda^*(s; t_1, \dots, t_n) ds \right) \\
= \; & \frac{1}{\lambda^*(t_{i_0}; t_1, \dots t_{i_0-1})} \frac{\partial \lambda^*}{\partial t_{i_0}}(t_{i_0}; t_1, \dots, t_{i_0-1}) + \sum_{i=i_0+1}^{n} \frac{1}{\lambda^*(t_i; t_1, \dots t_{i-1})} \frac{\partial \lambda^*}{\partial t_{i_0}}(t_i; t_1, \dots, t_{i-1}) \\
& - \frac{\partial}{\partial t_{i_0}} \left(\int_0^T \lambda^*(s; t_1, \dots, t_n) ds \right).
\end{aligned}$$

Evoke that for any $i \in \{i_0 + 1, ..., n\}$ and $s \in [0, T]$,

$$\begin{aligned}
\lambda^*(t_{i_0}; t_1, \dots, t_{i_0-1}) &= \lambda + \sum_{j=1}^{i_0-1} \mu(t_{i_0} - t_j), \\
\lambda^*(t_i; t_1, \dots, t_{i-1}) &= \lambda + \sum_{j=1}^{i-1} \mu(t_i - t_j), \\
\lambda^*(s; t_1, \cdots, t_n) &= \lambda + \sum_{j=1}^{n} \mu(s - t_j) 1_{\{s > t_j\}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial \lambda^*}{\partial t_{i_0}}(t_{i_0}; t_1, \dots, t_{i_0-1}) &= \sum_{j=1}^{i_0-1} \mu'(t_{i_0} - t_j), \\
\frac{\partial \lambda^*}{\partial t_{i_0}}(t_i; t_1, \dots, t_{i-1}) &= -\mu'(t_i - t_{i_0}), \\
\int_0^T \lambda^*(s; t_1, \dots, t_n) ds &= \lambda T + \sum_{j=1}^{n} \int_{t_j}^{T} \mu(s - t_j) ds \\
&= \lambda T + \sum_{j=1}^{n} \int_0^{T - t_j} \mu(s) ds, \\
\frac{\partial}{\partial t_{i_0}} \int_0^T \lambda^*(s; t_1, \dots, t_n) ds &= -\mu(T - t_{i_0}).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{\varphi_n(t_1,\ldots,t_n)}\frac{\partial \varphi_n}{\partial t_{i_0}}(t_1,\ldots,t_n)\\
=\;& \frac{\sum_{j=1}^{i_0-1}\mu'(t_{i_0}-t_j)}{\lambda^*(t_{i_0};t_1,\ldots,t_n)} - \sum_{i=i_0+1}^{n}\frac{\mu'(t_i-t_{i_0})}{\lambda^*(t_i;t_1,\ldots,t_n)} + \mu(T-t_{i_0}).
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{i_0=1}^{n}\frac{\widehat{m}(T_{i_0})}{\varphi_n(T_1,\ldots,T_n)}\frac{\partial \varphi_n}{\partial t_{i_0}}(T_1,\ldots,T_n)\\
=\;& \sum_{i_0=1}^{n}\widehat{m}(T_{i_0})\frac{\sum_{j=1}^{i_0-1}\mu'(T_{i_0}-T_j)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} - \sum_{i_0=1}^{n}\widehat{m}(T_{i_0})\sum_{i=i_0+1}^{n}\frac{\mu'(T_i-T_{i_0})}{\lambda^*(T_i;T_1,\ldots,T_n)}\\
&+\sum_{i_0=1}^{n}\widehat{m}(T_{i_0})\mu(T-t_{i_0})\\
=\;& \sum_{i_0=1}^{n}\sum_{j=1}^{i_0-1}\widehat{m}(T_{i_0})\frac{\mu'(T_{i_0}-T_j)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)} - \sum_{i_0=1}^{n}\sum_{i=1}^{i_0-1}\widehat{m}(T_i)\frac{\mu'(T_{i_0}-T_i)}{\lambda^*(T_{i_0};T_1,\ldots,T_n)}\\
&+\sum_{i_0=1}^{n}\widehat{m}(T_{i_0})\mu(T-T_{i_0})\\
=\;& \sum_{i_0=1}^{n}\left(\frac{1}{\lambda^*(T_{i_0};T_1,\ldots,T_n)}\sum_{j=1}^{i_0-1}(\widehat{m}(T_{i_0})-\widehat{m}(T_i))\mu'(T_{i_0}-T_i)+\widehat{m}(T_{i_0})\mu(T-T_{i_0})\right).
\end{aligned}$$

Finally

$$\begin{aligned}
\frac{Z_n^{\varepsilon}-Z_n^0}{\varepsilon} \xrightarrow[\varepsilon\to 0]{a.s.} & \sum_{i_0=1}^{n}\left(\frac{1}{\lambda^*(T_{i_0};T_1,\ldots,T_n)}\sum_{j=1}^{i_0-1}(\widehat{m}(T_{i_0})-\widehat{m}(T_i))\mu'(T_{i_0}-T_i)\right.\\
&\left.+\widehat{m}(T_{i_0})\mu(T-T_{i_0})+m(T_{i_0})\right).
\end{aligned}$$

Then we would like the same result for $\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$. We have, according to Proposition 5.1.11,

$$\begin{aligned}
\frac{\partial G^\varepsilon}{\partial \varepsilon}_{|\varepsilon=0} 1_{\{N_T=n\}} &= \lim_{\varepsilon\to 0} \frac{G^\varepsilon - 1}{\varepsilon} 1_{\{N_T=n\}} \\
&= \lim_{\varepsilon\to 0} \frac{G^\varepsilon 1_{\{N_T=n\}} - 1_{\{N_T=n\}}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{Z_n^\varepsilon 1_{\{N_T=n\}} - 1_{\{N_T=n\}}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{Z_n^\varepsilon - 1}{\varepsilon} 1_{\{N_T=n\}} \\
&= \frac{\partial Z_n^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} &= \sum_{n=1}^{+\infty} \frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}} \\
&= \sum_{n=1}^{+\infty} \frac{\partial Z_n^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} 1_{\{N_T=n\}}. \\
&= \sum_{i_0=1}^{N_T} \left(\frac{1}{\lambda^*(T_{i_0}; T_1, \ldots, T_{N_T})} \sum_{j=1}^{i_0-1} (\widehat{m}(T_{i_0}) - \widehat{m}(T_j)) \mu'(T_{i_0} - T_j) \right. \\
&\quad \left. + \widehat{m}(T_{i_0}) \mu(T - T_{i_0}) + m(T_{i_0}) \right) \\
&= \int_{(0,T]} (\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s)) dN_s
\end{aligned}$$

where ψ is given by (5.9). □

Theorem 5.1.13. *For any $F \in \mathcal{S}$,*

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} F \right].$$

Proof. We consider $F = f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}} \in \mathcal{S}$. Then, as the vectors

$$(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n)) \quad \text{and} \quad (T_1, \cdots, T_n)$$

are in the compact set $[0,T]^n$ and the function f_n is smooth, there exists a constant

$C = C(f_n, n, T) \in \mathbb{R}_+^*$ such that

$$\begin{aligned}
\left|\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right| &= \frac{|f_n(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n)) - f_n(T_1, \cdots, T_n)|}{\varepsilon} 1_{\{N_T = n\}} \\
&\leq C \frac{\|(\tau_\varepsilon^{-1}(T_1), \cdots, \tau_\varepsilon^{-1}(T_n)) - (T_1, \cdots, T_n)\|}{\varepsilon} 1_{\{N_T=n\}} \\
&= C \frac{\|(\tau_\varepsilon^{-1}(T_1) - T_1, \cdots, \tau_\varepsilon^{-1}(T_n) - T_n)\|}{\varepsilon} 1_{\{N_T=n\}} \\
&= C \sup_{1 \leq i \leq n} \frac{|\tau_\varepsilon^{-1}(T_i) - T_i|}{\varepsilon} 1_{\{N_T=n\}} \\
&\overset{(*)}{\leq} C \int_0^T |m_\varepsilon(s)| ds 1_{\{N_T=n\}} \\
&\leq C \int_0^T |m_\varepsilon(s) - m(s)| ds 1_{\{N_T=n\}} + C \int_0^T |m(s)| ds 1_{\{N_T=n\}} \\
&\leq C^2 1_{\{N_T=n\}} + C \int_0^T |m(s)| ds 1_{\{N_T=n\}}
\end{aligned}$$

where the last inequality is true for ε small enough because $m_\varepsilon \xrightarrow[\varepsilon \to 0]{\mathcal{H}} m$ according to Lemma 5.1.1, and the inequality $(*)$ is due to the equality

$$T_i = \tau_\varepsilon(\tau_\varepsilon^{-1}(T_i)) = \tau_\varepsilon^{-1}(T_i) + \varepsilon \int_0^{\tau_\varepsilon^{-1}(T_i)} m_\varepsilon(s) ds.$$

Thus, by dominated convergence theorem and $\mathbb{E}[\mathcal{T}_\varepsilon F] = \mathbb{E}^\varepsilon[F] = \mathbb{E}[G^\varepsilon F]$,

$$\mathbb{E}[D_m F] = \mathbb{E}\left[\lim_{\varepsilon \to 0} \frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{\mathcal{T}_\varepsilon F - F}{\varepsilon}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} F\right].$$

Hence, as $F = f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}$,

$$\begin{aligned}
\mathbb{E}[D_m F] &= \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{G^\varepsilon - 1}{\varepsilon} f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}\right] \\
&= \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{G^\varepsilon 1_{\{N_T=n\}} - 1}{\varepsilon} f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}\right] \\
&= \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{Z_n^\varepsilon - 1}{\varepsilon} f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\frac{Z_n^\varepsilon - 1}{\varepsilon} F\right].
\end{aligned}$$

Let us come back to the definition (5.10) of the growth rate of Z_n^ε. For the last term in

(5.10), we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left[\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))-1\right]-\sum_{i=1}^{n}m(T_i)\right|\right]\\
\leq\ &\mathbb{E}\left[\left|\frac{1}{\varepsilon}\left[\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))-1\right]-\sum_{i=1}^{n}m_\varepsilon(T_i)\right|\right]+\sum_{i=1}^{n}\mathbb{E}\left[|m_\varepsilon(T_i)-m(T_i)|\right]\\
\leq\ &\varepsilon\sum_{1\leq i<j\leq n}\mathbb{E}[|m_\varepsilon(T_i)m_\varepsilon(T_j)|]+...+\varepsilon^{n-1}\mathbb{E}\left[\prod_{i=1}^{n}|m_\varepsilon(T_i)|\right]\\
&+\sum_{i=1}^{n}\mathbb{E}\left[|m_\varepsilon(T_i)-m(T_i)|\right] \qquad (5.11)
\end{aligned}$$

with, for any $i\in\{1,...,n\}$, $|m_\varepsilon(T_i)-m(T_i)|\leq\frac{1}{\sqrt{T}}\|m-\widetilde{m}_\varepsilon\|_{\mathcal{H}}\underset{\varepsilon\to 0}{\longrightarrow}0$ as in the proof of Proposition 5.1.11. Thus

$$\sum_{i=1}^{n}\mathbb{E}\left[|m_\varepsilon(T_i)-m(T_i)|\right]\underset{\varepsilon\to 0}{\longrightarrow}0.$$

Moreover we know that $(T_1,\cdots,T_{n+1})$ admits for density

$$(t_1,\cdots,t_{n+1})\longmapsto\left(\prod_{j=1}^{n+1}\lambda^*(t_j;t_1,\cdots,t_n)\right)e^{-\int_0^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)\,ds}\mathbb{1}_{\{0<t_1<\cdots<t_{n+1}\}}.$$

Thus, for any Borel function $f:\mathbb{R}^n\longrightarrow\mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}[f(T_1,\cdots,T_n)\mid N_T=n]=\frac{\mathbb{E}[f(T_1,\cdots,T_n)\mathbb{1}_{\{T_n\leq T<T_{n+1}\}}]}{\mathbb{P}(N_T=n)}\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\times\left(\int_{t_n}^{+\infty}\lambda^*(t_{n+1};t_1,\cdots,t_n)e^{-\int_{t_n}^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)ds}dt_{n+1}\right)\mathbb{1}_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\times\left[-e^{-\int_{t_n}^{t_{n+1}}\lambda^*(s;t_1,\cdots,t_n)ds}\right]_{t_n}^{+\infty}\mathbb{1}_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n\\
&=\frac{1}{\mathbb{P}(N_T=n)}\int_{\mathbb{R}^n}f(t_1,\cdots,t_n)\left(\prod_{j=1}^{n}\lambda^*(t_j;t_1,\cdots,t_{n-1})\right)e^{-\int_0^{t_n}\lambda^*(s;t_1,\cdots,t_{n-1})ds}\\
&\times\left(1-e^{-\int_{t_n}^{+\infty}\lambda^*(s;t_1,\cdots,t_n)ds}\right)\mathbb{1}_{\{0<t_1<\cdots<t_n\leq T\}}dt_1\cdots dt_n.
\end{aligned}$$

Therefore $(T_1, \cdots, T_n)$ knowing $\{N_T = n\}$ admits for density

$$\begin{aligned}(t_1, \cdots, t_n) \quad \longmapsto \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\prod_{j=1}^{n} \lambda^*(t_j; t_1, \cdots, t_{n-1}) \right) e^{-\int_0^{t_n} \lambda^*(s;t_1,\cdots,t_{n-1})\, ds} \\ & \times \left(1 - e^{-\int_{t_n}^{+\infty} \lambda^*(s;t_1,\cdots,t_n) ds} \right) 1_{\{0<t_1<\cdots<t_n<T\}}\end{aligned}$$

with, for any $0 < t_1 < \cdots < t_n < T$ and any $j \in \{1, \cdots, n\}$,

$$\lambda^*(t_j; t_1, \cdots, t_{n-1}) \quad = \quad \lambda + \sum_{i=1}^{j-1} \mu(t_j - t_i) \leq \lambda + n\|\mu\|_\infty.$$

Thus the density of $(T_1, \cdots, T_n)$ knowing $\{N_T = n\}$ is bounded by

$$(t_1, \cdots, t_n) \longmapsto \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n 1_{\{0<t_1<\cdots<t_n<T\}}.$$

Therefore, for any $I \subset \{1, ..., n\}$,

$$\begin{aligned}\mathbb{E}\left[\left| \prod_{i\in I} m_\varepsilon(T_i) \right| \right] \quad \leq \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n \\ & \times \int_{[0,T]^n} \left| \prod_{i\in I} m_\varepsilon(t_i) \right| 1_{\{0<t_1<\cdots<t_n<T\}} dt \\ \leq \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n \int_{[0,T]^n} \prod_{i\in I} |m_\varepsilon(t_i)| dt \\ = \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n T^{n-|I|} \prod_{i\in I} \int_0^T |m_\varepsilon(t_i)| dt_i \\ = \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n T^{n-|I|} \left(\int_0^T |m_\varepsilon(t)| dt \right)^{|I|} \\ \leq \quad & \frac{1}{\mathbb{P}(N_T = n)} \left(\lambda + n\|\mu\|_\infty\right)^n T^{n-|I|} \left(\int_0^T |m(t)| dt + 1 \right)^{|I|}.\end{aligned}$$

The last inequality is justified by the convergence proved in Lemma 5.1.1. Thus we deduced with (5.11) that

$$\frac{1}{\varepsilon} \left[\prod_{i=1}^{n} (1 + \varepsilon m_\varepsilon(T_i)) - 1 \right] \xrightarrow[\varepsilon\to 0]{L^1(\Omega)} \sum_{i=1}^{n} m(T_i).$$

For the first term in (5.10), we have, due to the construction of m_ε in Section 5.1.1,

$$\left|\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))\right| \leq \left(\frac{5}{3}\right)^n$$

and

$$\begin{aligned}&\frac{(\varphi_n \circ \Phi_\varepsilon)(T_1,\cdots,T_n)-\varphi_n(T_1,\cdots,T_n)}{\varepsilon}\\ =&\sum_{k=1}^{n}\int_0^1 \frac{\partial\varphi_n}{\partial t_k}\left(T_1,\ldots,T_{k-1},T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k),T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1}),\ldots\right)d\alpha\ \widehat{m}_\varepsilon(T_k).\end{aligned}$$

Let us define

$$\psi_n(t_1,\cdots,t_n)=\prod_{i=1}^{n}\lambda^*(t_i;t_1,\cdots,t_n).$$

Thus

$$\varphi_n(t_1,\cdots,t_n)=\psi_n(t_1,\cdots,t_n)e^{-\int_0^T\lambda^*(s;t_1,\cdots,t_n)ds}.$$

Therefore, as μ is differentiable,

$$\begin{aligned}&\frac{1}{\varphi_n(T_1,\cdots,T_n)}\frac{\partial\varphi_n}{\partial t_k}\left(T_1,\ldots,T_{k-1},T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k),T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1}),\ldots\right)\\ =&\Bigg[\frac{1}{\psi_n(T_1,\cdots,T_n)}\frac{\partial\psi_n}{\partial t_k}\left(T_1,\ldots,T_{k-1},T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k),T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1}),\ldots\right)\\ &\qquad -\frac{\partial}{\partial t_k}\int_0^T\lambda^*(s;T_1,\ldots,T_{k-1},T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k),T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1}),\ldots)ds\Bigg]\end{aligned}$$

with, for the first term

$$\psi_n(t_1,\cdots,t_n)=\prod_{i=1}^{n}\left(\lambda+\sum_{j=1}^{i-1}\mu(t_i-t_j)1_{\{t_i>t_j\}}\right).$$

Thus

$$|\psi_n(t_1,\cdots,t_n)|\geq\lambda^n$$

and for any $0<t_1<\cdots<t_n<T$,

$$\frac{\partial\psi_n}{\partial t_k}(t_1,\cdots,t_n)=\sum_{\ell=1}^{n}\left[\sum_{j=1}^{\ell-1}\frac{\partial}{\partial t_k}(\mu(t_\ell-t_j)1_{\{t_\ell>t_j\}})\prod_{i=1,i\neq\ell}^{n}\left(\lambda+\sum_{i=1}^{j-1}\mu(t_i-t_j)1_{\{t_i>t_j\}}\right)\right].$$

Therefore

$$\begin{aligned}\left|\frac{\partial\psi_n}{\partial t_k}(t_1,\cdots,t_n)\right| &\leq \sum_{\ell=1}^{n}\left[\sum_{j=1}^{\ell-1}\|\mu'\|_\infty\prod_{i=1,i\neq\ell}^{n}\left(\lambda+\sum_{i=1}^{j-1}\|\mu\|_\infty\right)\right]\\ &\leq n^2\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}.\end{aligned}$$

Now for the second term

$$\left|\frac{\partial\lambda^*}{\partial t_k}(t_1,\cdots,t_n)\right|\leq n\|\mu'\|_\infty.$$

Thus, as, for any $\alpha\in(0,1)$, $T_1<\cdots<T_{k-1}<T_k+\alpha\varepsilon\widehat{m}_\varepsilon(T_k)<T_{k+1}+\varepsilon\widehat{m}_\varepsilon(T_{k+1})<\cdots$,

$$\begin{aligned}&\left|\frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\cdots,T_n)-\varphi_n(T_1,\cdots,T_n)}{\varepsilon}\frac{1}{\varphi_n(T_1,\cdots,T_n)}\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))\right|\\ \leq\ & 2^n\left(\frac{n^2\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}}{\lambda^n}+Tn\|\mu'\|_\infty\right)\sum_{k=1}^{n}|\widehat{m}_\varepsilon(T_k)|\\ \leq\ & n^2 2^n\left(\frac{n\|\mu'\|_\infty\left(\lambda+n\|\mu\|_\infty\right)^{n-1}}{\lambda^n}+T\|\mu'\|_\infty\right)\end{aligned}$$

where the last inequality is due to the uniform convergence of $\widehat{m}_\varepsilon$ to 0. Thus, by dominated convergence theorem and the almost surely convergence of the first term in (5.10),

$$\frac{(\varphi_n\circ\Phi_\varepsilon)(T_1,\ldots,T_n)-\varphi_n(T_1,\ldots,T_n)}{\varepsilon\varphi_n(T_1,\ldots,T_n)}\prod_{i=1}^{n}(1+\varepsilon m_\varepsilon(T_i))$$

$$\xrightarrow[\varepsilon\to0]{L^1(\Omega)}\sum_{i=1}^{n}\frac{\widehat{m}(T_i)}{\varphi_n(T_1,\ldots,T_n)}\frac{\partial\varphi_n}{\partial t_i}(T_1,\ldots,T_n).$$

Thus the two previous convergence in $L^1(\Omega)$ and (5.10) give

$$\mathbb{E}[D_mF]\ =\ \mathbb{E}\left[\lim_{\varepsilon\to0}\frac{Z_n^\varepsilon-1}{\varepsilon}F\right]=\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}F\right].$$

Then, by linearity, we deduce the same equality for any $F\in\mathcal{S}$. □

Remark 5.1.14. *For $F=1$ in $\mathbb{E}[D_mF]=\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}F\right]$ we get:* $\mathbb{E}\left[\frac{\partial G^\varepsilon}{\partial\varepsilon}|_{\varepsilon=0}\right]=0.$

Note that we can also prove this property with the expression of the Proposition 5.1.12:

$$
\begin{aligned}
& \mathbb{E}\left[\frac{\partial G^{\varepsilon}}{\partial \varepsilon}|_{\varepsilon=0}\right] \\
= \; & \mathbb{E}\left[\int_0^T (\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))\lambda^*(s)ds\right] \\
= \; & \mathbb{E}\left[\int_0^T \left(\int_{(0,s)} (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)dN_t + (m(s) + \widehat{m}(s)\mu(T-s))\lambda^*(s)\right) ds\right] \\
= \; & \int_0^T \int_0^s (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)\mathbb{E}[\lambda^*(t)]dtds + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))\mathbb{E}[\lambda^*(s)]ds \\
= \; & \int_0^T \int_0^s (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)g(t)dtds + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \; & \int_0^T \left(\int_t^T (\widehat{m}(s) - \widehat{m}(t))\mu'(s-t)ds\right) g(t)dt + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \; & \int_0^T \left([(\widehat{m}(s) - \widehat{m}(t))\mu(s-t)]_t^T - \int_t^T m(s)\mu(s-t)ds\right) g(t)dt \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \; & \int_0^T \left((0 - \widehat{m}(t))\mu(T-t) - 0 - \int_t^T m(s)\mu(s-t)ds\right) g(t)dt \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \; & -\int_0^T \widehat{m}(t)\mu(T-t)g(t)dt - \int_0^T m(s) \underbrace{\left(\int_0^s \mu(s-t)g(t)dt\right)}_{=g(s)-\lambda} ds \\
& + \int_0^T (m(s) + \widehat{m}(s)\mu(T-s))g(s)ds \\
= \; & 0
\end{aligned}
$$

where we note $g(s) = \mathbb{E}[\lambda^*(s)]$ which satisfies, according to Proposition 1.3.8 or [59],

$$g(s) = \lambda + \int_0^s \mu(s-t)g(t)dt.$$

5.1.6 Directional Dirichlet space

Proposition 5.1.15 (and definition of $\mathbb{D}_m^{1,2}$)**.** *The quadratic bilinear form on $L^2(\Omega)$, $(\mathcal{S}, \mathcal{E}_m)$ defined by*

$$\forall X, Y \in \mathcal{S}, \quad \mathcal{E}_m(X,Y) = \mathbb{E}[D_m X D_m Y],$$

is closable. We denote by $(\mathbb{D}_m^{1,2}, \mathcal{E}_m)$ *its closed extension. As a consequence,* D_m *is also closable and we still denote by* D_m *its extension which is well-defined on the whole space* $\mathbb{D}_m^{1,2}$.

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}$ converging to 0 in $L^2(\Omega)$ and such that

$$\lim_{n,k\to+\infty} \mathcal{E}_m(X_n - X_k) = \lim_{n,k\to+\infty} \mathbb{E}[(D_m X_n - D_m X_k)^2] = 0.$$

Thus $(D_m X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega)$, so it converges to an element Z in $L^2(\Omega)$. Then, let Y be in $\mathcal{S}$, we have by integration by part formula:

$$\begin{aligned} \mathbb{E}[D_m X_n Y] &= \mathbb{E}[D_m(X_n Y)] - \mathbb{E}[X_n D_m Y] \\ &= \mathbb{E}\left[X_n Y \frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}\right] - \mathbb{E}[X_n D_m Y]. \end{aligned}$$

The last equality comes from Theorem 5.1.13. Since $Y \frac{\partial G^\varepsilon}{\partial \varepsilon}|_{\varepsilon=0}$ and $D_m Y$ belong to $L^2(\Omega)$, we get, by dominated convergence theorem,

$$\forall Y \in \mathcal{S}, \quad \mathbb{E}[ZY] = \lim_{n\to+\infty} \mathbb{E}[D_m X_n Y] = 0.$$

Hence $Z = 0$ by density. We deduce thanks to [18, Proposition 1.3.2] that $(\mathcal{S}, \mathcal{E}_m)$ is closable.

As a consequence, for any $X \in \mathbb{D}_m^{1,2}$, there exists a sequence $(X_n)_{n\in\mathbb{N}} \in \mathcal{S}$ converging to X in $\mathbb{D}_m^{1,2}$ since for all $n, k \in \mathbb{N}$

$$\mathcal{E}_m(X_n - X_k) = \mathbb{E}[|D_m X_n - D_m X_k|^2]$$

we deduce that $(D_m X_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in L^2 hence converges to an element that we still denote $D_m X$ and defines in a unique way the extension of D_m to $\mathbb{D}_m^{1,2}$. □

Corollary 5.1.16. *Proposition 5.1.7 and Theorem 5.1.13 remain valid for any* $F \in \mathbb{D}_m^{1,2}$.

5.2 The local Dirichlet form

5.2.1 Definition using a Hilbert basis

We would like to define an operator D with domain $\mathbb{D}^{1,2} \subset L^2(\Omega)$ and taking values in $L^2(\Omega; \mathcal{H})$ such that

$$\forall F \in \mathbb{D}^{1,2}, m \in \mathcal{H}, \quad D_m F = \langle DF, m \rangle_{\mathcal{H}} = \int_0^T D_t F m(t) dt.$$

Let $(m_i)_{i \in \mathbb{N}}$ be a Hilbert basis of the space $\mathcal{H}$. Then every function $m \in \mathcal{H}$ can be expressed as

$$m = \sum_{i=0}^{+\infty} \langle m, m_i \rangle_{\mathcal{H}} m_i.$$

We now set

$$\mathbb{D}^{1,2} = \left\{ X \in \bigcap_{i=0}^{+\infty} \mathbb{D}^{1,2}_{m_i}, \quad \sum_{i=0}^{+\infty} \| D_{m_i} X \|^2_{L^2(\Omega)} < +\infty \right\}$$

and

$$\forall X, Y \in \mathbb{D}^{1,2} \quad \mathcal{E}(X, Y) = \sum_{i=0}^{+\infty} \mathbb{E}[D_{m_i} X D_{m_i} Y].$$

We also note $\mathcal{E}(X) = \mathcal{E}(X, X)$. Then, according to [18, Proposition 4.2.1],

Proposition 5.2.1. *The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a local Dirichlet form admitting a carré du champ Γ and a gradient D given by, for all $X, Y \in \mathbb{D}^{1,2}$,*

$$\Gamma[X, Y] = \langle DX, DY \rangle_{\mathcal{H}}$$

and

$$DX = \sum_{i=0}^{+\infty} D_{m_i} X m_i \in L^2(\Omega, \mathcal{H}).$$

As a consequence $\mathbb{D}^{1,2}$ is a Hilbert space equipped with the norm

$$\| X \|^2_{\mathbb{D}^{1,2}} = \| X \|^2_{L^2(\Omega)} + \mathcal{E}(X).$$

Moreover, as $\mathcal{S}$ is dense in each $\mathbb{D}^{1,2}_{m_i}, i \in \mathbb{N}$, $\mathcal{S}$ is dense in $\mathbb{D}^{1,2}$.

Proof. We start by proving that $(\mathbb{D}^{1,2}, \mathcal{E})$ is a Dirichlet form on $L^2(\Omega)$ in sense of [16, Definition 2.14]:

- The bilinear form $(\mathbb{D}^{1,2}, \mathcal{E})$ is a closed form on $L^2(\Omega)$. Let $(X_n)_{n\in\mathbb{N}} \in (\mathbb{D}^{1,2})^{\mathbb{N}}$ such that $\mathbb{E}[(X_n)^2] \underset{n\to+\infty}{\longrightarrow} 0$ and

$$\lim_{n,k\to+\infty} \sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] = \lim_{n,k\to+\infty} \mathcal{E}(X_n - X_k) = 0.$$

Then, for any $i \in \mathbb{N}$,

$$\lim_{n,k\to+\infty} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] = 0.$$

However, according to Proposition 5.1.15, the operator D_{m_i} is closable. Thus

$$\lim_{n\to+\infty} \mathbb{E}[(D_{m_i}X_n)^2] = 0.$$

Let $\varepsilon \in \mathbb{R}_+^*$. There exists $n_0 \in \mathbb{N}$ such that, for any $n, k \geq n_0$,

$$\sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] \leq \frac{\varepsilon}{4}.$$

Thus for any $j \in \mathbb{N}$,

$$\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] \leq \sum_{i=0}^{+\infty} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] \leq \frac{\varepsilon}{4}.$$

Therefore

$$\begin{aligned}\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_n)^2] &\leq 2\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_n - D_{m_i}X_k)^2] + 2\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_k)^2] \\ &\leq \frac{\varepsilon}{2} + 2\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_k)^2]\end{aligned}$$

with $\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_k)^2] \underset{k\to+\infty}{\longrightarrow} 0$. Thus there exists $k = k_j \in \mathbb{N}$ such that

$$\sum_{i=0}^{j} \mathbb{E}[(D_{m_i}X_k)^2] \leq \frac{\varepsilon}{4}.$$

Therefore $\sum_{i=0}^{j}\mathbb{E}[(D_{m_i}X_n)^2] \leq \varepsilon$ and

$$\sum_{i=0}^{+\infty}\mathbb{E}[(D_{m_i}X_n)^2] \leq \varepsilon.$$

To conclude we get

$$\mathcal{E}(X_n) = \sum_{i=0}^{+\infty}\mathbb{E}[(D_{m_i}X_n)^2] \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore $(\mathbb{D}^{1,2},\mathcal{E})$ is a closed form on $L^2(\Omega)$.

- The closed form $(\mathbb{D}^{1,2},\mathcal{E})$ satisfies:

$$\forall F \in \mathbb{D}^{1,2}, \quad F\wedge 1 \in \mathbb{D}^{1,2}, \mathcal{E}(F\wedge 1) \leq \mathcal{E}(F).$$

Indeed let $F\in\mathbb{D}^{1,2}$ and $i\in\mathbb{N}$. Thus, by definition of $\mathbb{D}^{1,2}$, $F\in\mathbb{D}^{1,2}_{m_i}$. We have to prove that $F\wedge 1\in\mathbb{D}^{1,2}_{m_i}$ and that $\mathcal{E}_{m_i}(F\wedge 1)\leq\mathcal{E}_{m_i}(F)$. According to the definition of $\mathbb{D}^{1,2}_{m_i}$ in Proposition 5.1.15, there exists $(F_n)_{n\in\mathbb{N}}\in\mathcal{S}^{\mathbb{N}}$ such that $F_n \overset{L^2(\Omega)}{\underset{n\to+\infty}{\longrightarrow}} F$ and $D_{m_i}F = \lim_{n\to+\infty} D_{m_i}F_n$ exists in $L^2(\Omega)$. For any $n\in\mathbb{N}$, there exists $a_n\in\mathbb{R}, d_n\in\mathbb{N}^*$ and $f_1^n:\mathbb{R}\longrightarrow\mathbb{R},\ldots,f_{d_n}^n:\mathbb{R}^{d_n}\longrightarrow\mathbb{R}$ smooth functions with bounded derivatives of any order such

$$F_n = a_n 1\{N_T=0\} + \sum_{m=1}^{d_m} f_m^n(T_1,\cdots,T_m)1_{\{N_T=m\}}.$$

We consider a sequence of smooth functions $(\phi_k)_{k\in\mathbb{N}}$ with bounded derivatives of any order such that we have the uniform convergence of ϕ_k to the function $x\longmapsto x\wedge 1$ and $\|\phi_k'\|_\infty\leq 1$ for any $k\in\mathbb{N}$. Thus

$$\phi_k(F_n) = \phi_k(a_n)1_{\{N_T=0\}} + \sum_{m=1}^{d_m}\phi_k(f_m^n(T_1,\cdots,T_m))1_{\{N_T=m\}} \in \mathcal{S}.$$

Moreover the following convergence in $L^2(\Omega)$ holds:

$$F\wedge 1 = \lim_{n\to+\infty}\lim_{k\to+\infty}\phi_k(F_n).$$

Therefore $F\wedge 1\in\mathbb{D}^{1,2}_{m_i}$ and

$$D_{m_i}(F\wedge 1) = \lim_{n\to+\infty}\lim_{k\to+\infty} D_{m_i}(\phi_k(F_n))$$

with for any $n, k \in \mathbb{N}$, by the chain rule,

$$D_{m_i}(\phi_k(F_n)) = \phi_k'(F_n) D_{m_i} F_n.$$

Thus

$$\mathcal{E}_{m_i}(F \wedge 1) = \mathbb{E}[(D_{m_i}(F \wedge 1))^2] \leq \mathbb{E}[(D_{m_i} F)^2] = \mathcal{E}_{m_i}(F).$$

Therefore

$$\mathcal{E}(F \wedge 1) = \sum_{i=0}^{+\infty} \mathcal{E}_{m_i}(F \wedge 1) \leq \sum_{i=0}^{+\infty} \mathcal{E}_{m_i}(F) = \mathcal{E}(F).$$

Now we prove that Γ is a carré du champ of the local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ in sense of [16, Definition 2.19]. Indeed the application Γ is a positive symmetric continuous bilinear form from $\mathbb{D}^{1,2} \times \mathbb{D}^{1,2}$ into $L^1(\Omega)$ such that, for any $X, Y \in \mathbb{D}^{1,2}$,

$$\begin{aligned} \mathbb{E}[\Gamma[X,Y]] &= \mathbb{E}[\langle DX, DY \rangle_{\mathcal{H}}] \\ &= \mathbb{E}\left[\sum_{i=0}^{+\infty} \langle DX, m_i \rangle \langle DY, m_i \rangle\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{+\infty} D_{m_i} X D_{m_i} Y\right] \\ &= \mathcal{E}(X,Y) \end{aligned}$$

where we used the fact that the family $(m_i)_{i \in \mathbb{N}}$ is a Hilbert basis in $\mathcal{H}$.

To conclude we prove that the operator D is the gradient of the local Dirichlet form $(\mathbb{D}^{1,2}, \mathcal{E})$ in sense of [16, p. 16]. Indeed:

- For any $X \in \mathbb{D}^{1,2}$, as $(m_i)_{i \in \mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$,

$$\|DX\|_{\mathcal{H}}^2 = \left\| \sum_{i=0}^{+\infty} D_{m_i} X m_i \right\|_{\mathcal{H}}^2 = \sum_{i=0}^{+\infty} (D_{m_i} X)^2 = \Gamma[X].$$

- Let $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function and $X \in \mathbb{D}^{1,2}$. Then, according to Remark 5.2.3, $\Phi(X) \in \mathbb{D}^{1,2}$ and $D(\Phi(X)) = \Phi'(X) DX$ where $\Phi'(X)$ is the Lebesgue partial derivative of Φ almost everywhere defined.

□

Corollary 5.2.2. *For all $n \in \mathbb{N}^*$ and $j \in \{0, ..., n\}$, writing $\overline{T}_j = T_j \wedge T$, we have*

$$D\overline{T}_j = \frac{\overline{T}_j}{T} - 1_{[0, \overline{T}_j]}.$$

As consequence for all

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}} \in \mathcal{S},$$

we have $F \in \mathbb{D}^{1,2}$ *and*

$$DF = \sum_{n=1}^{d}\sum_{j=1}^{n} \frac{\partial f_n}{\partial t_j}(T_1, \cdots, T_n)\left(\frac{T_j}{T} - 1_{[0,T_j]}\right)1_{\{N_T=n\}}.$$

In particular this expression does not depend on the basis $(m_i)_{i\in\mathbb{N}}$.

Proof. We have

$$\begin{aligned} D\overline{T}_j &= \sum_{i=0}^{+\infty} D_{m_i}\overline{T}_j m_i = -\sum_{i=0}^{+\infty} \widehat{m}_i(\overline{T}_j)m_i \\ &= -\sum_{i=0}^{+\infty}\int_0^T m_i(s)1_{\{0\le s\le \overline{T}_j\}}ds m_i \\ &= \sum_{i=0}^{+\infty}\left\langle \frac{\overline{T}_j}{T} - 1_{[0,\overline{T}_j]}, m_i\right\rangle_{\mathcal{H}} m_i = \frac{\overline{T}_j}{T} - 1_{[0,\overline{T}_j]}. \end{aligned}$$

Notice that the term $\frac{\overline{T}_j}{T}$ is mandatory to belong to $\mathcal{H}$ defined by (5.1). □

We get the chain rule for the operator D on $\mathbb{D}^{1,2}$

Remark 5.2.3. *For any* $F_1, \ldots, F_n \in \mathbb{D}^{1,2}$ *and smooth function* $\Phi : \mathbb{R}^n \longrightarrow \mathbb{R}$*, the random variable* $\Phi(F_1, \ldots, F_n)$ *belongs to* $\mathbb{D}^{1,2}$ *and*

$$D\Phi(F_1, \ldots, F_n) = \sum_{j=1}^{n} \frac{\partial \Phi}{\partial x_j}(F_1, \ldots, F_n)DF_j.$$

Moreover we can extend this result with a Lipschitz function Φ *thanks to Proposition 5.3.3 where we replace* $\frac{\partial \Phi}{\partial x_j}$ *is the Lebesgue partial derivative of* Φ *almost everywhere defined (see [18, Proposition III.2.1.5] for more details).*

5.2.2 Divergence operator by duality

Let $\delta : L^2(\Omega, \mathcal{H}) \longrightarrow L^2(\Omega)$ be the adjoint operator of D. Its domain, $\mathrm{Dom}(\delta)$, is the set of $u \in L^2(\Omega, \mathcal{H})$ such that there exists $c \in \mathbb{R}_+^*$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \left| \mathbb{E}\left[\int_0^T D_t F u_t dt \right] \right| \leq c \|F\|_{\mathbb{D}^{1,2}}.$$

Hence, for all $u \in \mathrm{Dom}(\delta)$, $\delta(u)$ is the unique element in $L^2(\Omega)$ such that

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}[\delta(u)F] = \mathbb{E}[\langle u, DF \rangle_{\mathcal{H}}] = \mathbb{E}\left[\int_0^T u_t D_t F dt \right].$$

We now introduce the set $\widetilde{\mathcal{S}}$ of elementary processes u of the form

$$u = \sum_{i=1}^{n} A_i m_i, \quad n \in \mathbb{N}^*, \quad A_i \in \mathbb{D}^{1,2}.$$

Proposition 5.2.4. *For all u in $\widetilde{\mathcal{S}}$, we have $u \in Dom(\delta)$ and*

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s)) dN_s - \int_0^T D_t(u(t)) dt$$

where ψ is defined by (5.9).

Proof. Let $u = A m_{i_0}$ with $A \in \mathbb{D}^{1,2}$ and $i_0 \in \mathbb{N}$. For any $F \in \mathbb{D}^{1,2}$ we have

$$\begin{aligned} \left| \mathbb{E}\left[\int_0^T D_t F u_t dt \right] \right| &\leq \mathbb{E}\left[\int_0^T |D_t F| |A m_{i_0}(t)| dt \right] \\ &\leq \|F\|_{\mathbb{D}^{1,2}} \sqrt{\mathbb{E}\left[\int_0^T |A|^2 |m_{i_0}(t)|^2 dt \right]} \\ &= \|F\|_{\mathbb{D}^{1,2}} \|A\|_{L^2(\Omega)} \|m_{i_0}\|_{L^2(0,T)} = c \|F\|_{\mathbb{D}^{1,2}} \end{aligned}$$

with $c = \|A\|_{L^2(\Omega)} \|m_{i_0}\|_{L^2(0,T)}$. Thus $u \in \mathrm{Dom}(\delta)$ and for any $F \in \mathbb{D}^{1,2}$

$$\begin{aligned}
\mathbb{E}[\delta(u)F] &= \mathbb{E}\left[\int_0^T u_t D_t F dt\right] = \mathbb{E}\left[A \int_0^T m_{i_0}(t) D_t F dt\right] \\
&= \mathbb{E}\left[A \int_0^T m_{i_0}(t) \sum_{i=0}^{+\infty} D_{m_i} F m_i(t) dt\right] \\
&= \mathbb{E}\left[A \sum_{i=0}^{+\infty} D_{m_i} F \int_0^T m_{i_0}(t) m_i(t) dt\right] \\
&= \mathbb{E}\left[A \sum_{i=0}^{+\infty} D_{m_i} F \langle m_{i_0}, m_i \rangle_{\mathcal{H}}\right] = \mathbb{E}[A D_{m_{i_0}} F].
\end{aligned}$$

Thus, integrating by parts:

$$\begin{aligned}
\mathbb{E}[\delta(u)F] &= \mathbb{E}[D_{m_{i_0}}(AF)] - \mathbb{E}[F D_{m_{i_0}} A] \\
&= \mathbb{E}\left[\frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} AF\right] - \mathbb{E}[F D_{m_{i_0}} A] \\
&= \mathbb{E}\left[\left(\frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} A - D_{m_{i_0}} A\right) F\right].
\end{aligned}$$

Therefore, because $\langle DA, m_{i_0} \rangle_{\mathcal{H}} = \sum_{i=0}^{+\infty} D_{m_i} A \langle m_i, m_{i_0} \rangle_{\mathcal{H}} = D_{m_{i_0}} A$,

$$\begin{aligned}
\delta(u) &= \frac{\partial G^{\varepsilon}_{m_{i_0}}}{\partial \varepsilon}_{|\varepsilon=0} A - D_{m_{i_0}} A \\
&= \left(\int_{(0,T]} (\psi(m_{i_0}, s) + \widehat{m_{i_0}}(s)\mu(T-s) + m_{i_0}(s)) dN_s\right) A - \int_0^T m_{i_0}(t) D_t A dt \\
&= \int_{(0,T]} (\psi(A m_{i_0}, s) + \widehat{A m_{i_0}}(s)\mu(T-s) + A m_{i_0}(s)) dN_s - \int_0^T D_t(A m_{i_0}(t)) dt \\
&= \int_{(0,T]} (\psi(u, s) + \widehat{u}(s)\mu(T-s) + u(s)) dN_s - \int_0^T D_t(u(t)) dt.
\end{aligned}$$

We deduce the result for any $u \in \widetilde{\mathcal{S}}$ by linearity. □

Remark 5.2.5. *We can retain that:*

1. *For all $m \in \mathcal{H}$ and $A \in \mathbb{D}^{1,2}$,*

$$\delta(mA) = \delta(m)A - D_m A.$$

2. For all $m \in \mathcal{H}$ and $A, F \in \mathbb{D}^{1,2}$,

$$\mathbb{E}[AD_m F] = \mathbb{E}[F\delta(mA)].$$

3. For all $u \in \mathcal{H}$ we have $Du = 0$ and

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s.$$

Remark 5.2.6. *Contrary to the standard Malliavin calculus on the Wiener space (see [69]), we do not have a priori the inclusion of $\mathbb{D}^{1,2} \otimes \mathcal{H}$ in Dom(δ) (see [22, Example3.4] where $\mu = 0$).*

Corollary 5.2.7. *If $u \in L^2(\Omega, \mathcal{H})$ is a predictable process then*

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s$$

Proof. We establish this result for an elementary process of the form:

$$u(t) = f_0 1_{[0,t_1]}(t) + \sum_{j=1}^{n-1} f_j(\overline{T}_1, \cdots, \overline{T}_n) 1_{]t_j,t_{j+1}]}(t)$$

where $n \in \mathbb{N}^*$, $t_j = \frac{jT}{n}$, f_0 is a constant, for any $j \in \{1, \cdots, n\}$, f_j is an infinitely differentiable function from $\mathbb{R}^n$ into $\mathbb{R}$ vanishing outside the simplex

$$\Delta_n^j = \{(x_1, \cdots, x_n) \in \mathbb{R}^n,\ 0 < x_1 < \cdots < x_n \leq t_j\}$$

and $f_{n-1} = -f_0 - \sum_{j=1}^{n-2} f_j$. This last condition ensures that u belongs to $L^2(\Omega; \mathcal{H})$. As a consequence, we can rewrite u as

$$u(t) = f_0 \left(1_{[0,t_1]}(t) - \frac{T}{n}\right) + \sum_{j=1}^{n-1} f_j(\overline{T}_1, \cdots, \overline{T}_n) \left(1_{]t_j,t_{j+1}]}(t) - \frac{T}{n}\right),$$

this proves that u belongs to $\widetilde{\mathcal{S}}$. We have, by Proposition 5.2.4 and Corollary 5.2.2,

$$\begin{aligned} D_t u(t) &= \sum_{j=1}^{n-1} \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n) D_t \overline{T}_i 1_{]t_j,t_{j+1}]}(t) \\ &= \sum_{j=1}^{n-1} \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n) \left(\frac{\overline{T}_i}{T} - 1_{[0,\overline{T}_i]}(t)\right) 1_{]t_j,t_{j+1}]}(t). \end{aligned}$$

Now, since f_j vanishes outside Δ_n^j, we have for any $j \in \{1, \cdots, n-1\}$ and any $i \in \{1, \cdots, n\}$

$$\frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n) 1_{[0,\overline{T}_i]}(t) 1_{]t_j,t_{j+1}]}(t) = 0.$$

Moreover for any i, as u is valued in $\mathcal{H}$,

$$\int_0^T \sum_{j=1}^{q} \frac{\partial f_j}{\partial x_i}(\overline{T}_1, \cdots, \overline{T}_n) \frac{\overline{T}_i}{T} 1_{]t_j,t_{j+1}]}(t) dt = \frac{\overline{T}_i}{T} \frac{\partial}{\partial x_i} \int_0^T u(t) dt = 0.$$

Therefore

$$\int_0^T D_t u(t) dt = 0,$$

Hence from Proposition 5.2.4 we deduce that

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s)) dN_s.$$

We conclude by using a density argument. Indeed if $u \in L^2(\Omega, \mathcal{H})$ is a predictable process, there exists a sequence $(u_n)_{n\in\mathbb{N}}$ of elementary processes as above converging to u in $L^2(\Omega, \mathcal{H})$ and clearly $\delta(u_n)$ converges to $\int_{(0,T]}(\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s))dN_s$. Since δ is a closed operator, we conclude that u belongs to $\mathrm{Dom}(\delta)$ and that

$$\delta(u) = \int_{(0,T]} (\psi(u,s) + \widehat{u}(s)\mu(T-s) + u(s)) dN_s.$$

□

Proposition 5.2.8. *Let $F \in \mathbb{D}^{1,2}$ and $X \in Dom(\delta)$ such that*

$$F\delta(X) - \int_0^T D_t F X_t dt \in L^2(\Omega),$$

then $FX \in Dom(\delta)$ and

$$\delta(FX) = F\delta(X) - \int_0^T D_t F X_t dt.$$

Proof. For any $G \in \mathcal{S}$

$$\begin{aligned}
\mathbb{E}[\delta(FX)G] &= \mathbb{E}\left[\int_0^T FX_t D_t G dt\right] \\
&= \mathbb{E}\left[\int_0^T X_t(D_t(GF) - GD_tF)dt\right] \\
&= \mathbb{E}\left[\delta(X)GF - G\int_0^T X_t D_t F dt\right] \\
&= \mathbb{E}\left[G\left(F\delta(X) - \int_0^T D_t F X_t dt\right)\right].
\end{aligned}$$

□

In particular if $X = m \in \mathcal{H} \subset \mathrm{Dom}(\delta)$ then $\delta(m) = \int_{(0,T]}(\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s$ and $\int_0^T D_t F m(t)dt = D_m F$. Hence we have

$$\delta(mF) = F\int_{(0,T]}(\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s - D_m F.$$

Remark 5.2.9. *We do not have the Clark-Ocone formula because for $F = N_T \in \mathbb{D}^{1,2}$ we have $N_T \neq \mathbb{E}[N_T]$ and $D_t N_T = 0$. Indeed, for any $m \in \mathcal{H}$ and $\varepsilon \in \mathbb{R}_+^*, \mathcal{T}_\varepsilon N_T = N_T$.*

5.3 Absolute continuity criterion

5.3.1 Local criterion

Lemma 5.3.1. *The distribution of $(T_1, \cdots, T_n)$ conditionally to $\{N_T = n\}$ has a density*

$$\begin{aligned}
k_n : \quad & \mathbb{R}^n \longrightarrow \mathbb{R}_+ \\
& (t_1, \cdots, t_n) \longmapsto \frac{\kappa(t)}{\int_{\mathbb{R}^n} \kappa(s)ds}
\end{aligned}$$

with, for any $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$,

$$\kappa(t) = 1_{\{0<t_1<\cdots<t_n \leq T\}}\left(\prod_{i=1}^n \lambda^*(t_i; t_1, \cdots, t_n)\right)e^{-\int_0^T \lambda^*(s;t_1,\cdots,t_n)ds}.$$

Proof. Let f be a measurable function on $\mathbb{R}^n$. Then

$$\begin{aligned}
&\mathbb{E}[f(T_1, \cdots, T_n) \mid N_T = n] \\
&= \frac{\mathbb{E}[1_{\{N_T=n\}} f(T_1, \cdots, T_n)]}{\mathbb{P}(N_T = n)} \\
&= \frac{\mathbb{E}[1_{\{T_n \le T < T_{n+1}\}} f(T_1, \cdots, T_n)]}{\mathbb{P}(T_n \le T < T_{n+1})} \\
&= \frac{\int_{0<t_1<\cdots<t_n \le T < t_{n+1}} f(t_1, \cdots, t_n) \varphi(t_1, \cdots, t_{n+1}) dt_1 \cdots dt_{n+1}}{\int_{0<t_1<\cdots<t_n \le T < t_{n+1}} \varphi(t_1, \cdots, t_{n+1}) dt_1 \cdots dt_{n+1}}
\end{aligned}$$

with

$$\begin{aligned}
\varphi(t_1, \cdots, t_{n+1}) &= \left(\prod_{i=1}^{n+1} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^{t_{n+1}} \lambda^*(s; t_1, \cdots, t_n) ds} \\
&= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) \lambda^*(t_{n+1}; t_1, \cdots, t_n) e^{-\int_0^{t_{n+1}} \lambda^*(s; t_1, \cdots, t_n) ds} \\
&= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^{t_n} \lambda^*(s; t_1, \cdots, t_n) ds} \\
&\quad \times \left(\lambda + \sum_{i=1}^{n} \mu(t_{n+1} - t_i) \right) e^{-\int_{t_n}^{t_{n+1}} \left(\lambda + \sum_{i=1}^n \mu(s - t_i)\right) ds}.
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_{T<t_{n+1}} \varphi(t_1, \cdots, t_{n+1}) dt_{n+1} \\
&= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^{t_n} \lambda^*(s; t_1, \cdots, t_n) ds} \\
&\quad \times \int_T^{+\infty} \left(\lambda + \sum_{i=1}^{n} \mu(t_{n+1} - t_i) \right) e^{-\int_{t_n}^{t_{n+1}} \left(\lambda + \sum_{i=1}^n \mu(s - t_i)\right) ds} dt_{n+1} \\
&= \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n) \right) e^{-\int_0^{t_n} \lambda^*(s; t_1, \cdots, t_n) ds} e^{-\int_{t_n}^{T} \left(\lambda + \sum_{i=1}^n \mu(s - t_i)\right) ds}.
\end{aligned}$$

Therefore

$$\mathbb{E}[f(T_1, \cdots, T_n) \mid N_T = n] = \frac{\int_{\mathbb{R}^n} f(t) \kappa(t) dt}{\int_{\mathbb{R}^n} \kappa(t) dt} = \int_{\mathbb{R}^n} f(t) k_n(t) dt$$

with, for any $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$,

$$\begin{aligned}\kappa(t) &= 1_{\{0<t_1<\cdots<t_n\leq T\}} \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n)\right) e^{-\int_0^{t_n} \lambda^*(s;t_1,\cdots,t_n)ds} \\ &\quad \times e^{-\int_{t_n}^{T}\left(\lambda+\sum_{i=1}^{n}\mu(s-t_i)\right)ds} \\ &= 1_{\{0<t_1<\cdots<t_n\leq T\}} \left(\prod_{i=1}^{n} \lambda^*(t_i; t_1, \cdots, t_n)\right) e^{-\int_0^{T} \lambda^*(s;t_1,\cdots,t_n)ds},\end{aligned}$$

which implies the conclusion of the lemma. □

Now fix $n \in \mathbb{N}^*$, as usual $C^\infty(\mathbb{R}^n)$ denotes the set of infinitely differentiable functions on $\mathbb{R}^n$. We consider the following quadratic form on $C^\infty(\mathbb{R}^n)$:

$$e_n(u,v) = \frac{1}{2}\int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t)\frac{\partial v}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) k_n(t)dt$$

and

$$e_n(u) = e_n(u,u).$$

Proposition 5.3.2.

1. *$(C^\infty(\mathbb{R}^n), e_n)$ is closable, its closure (d_n, e_n) defines a local Dirichlet form on the space $L^2(k_n(t)dt)$ and each $u \in d_n$ is a $\mathcal{B}(\mathbb{R}^n)$-measurable function in $L^2(k_n(t)dt)$ such that for any $i \in \{1, \cdots, n\}$ and for almost all*

   $$\widetilde{t} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}^{n-1},$$

   the function

   $$s \longmapsto u_{\widetilde{t}}^{(i)}(s) = u(t_1, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)$$

   has an absolutely continuous version $\widetilde{u}_{\widetilde{t}}^{(i)}$ on $[t_{i-1}, t_{i+1}]$ such that

   $$\sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t)\frac{\partial u}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) \in L^1(k_n(t)dt)$$

   where $\dfrac{\partial u}{\partial t_i} = \dfrac{\partial \widetilde{u}_{\widetilde{t}}^{(i)}}{\partial s}$.
2. *The Dirichlet form (d_n, e_n) admits a carré du champ operator γ_n and a gradient*

operator $\widetilde{D}^n$ *given by*

$$\gamma_n[u,v](t) = \sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t)\frac{\partial v}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right)$$

and

$$\widetilde{D}^n_s u(t) = \sum_{i=1}^{n} \frac{\partial u}{\partial t_i}(t)\left(\frac{t_i}{T} - 1_{[0,t_i]}(s)\right)$$

for all $u, v \in d_n, t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ *and* $s \in [0,T]$.

3. *The structure* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n(t)dt, d_n, \gamma_n)$ *satisfies for every* $d \in \mathbb{N}^*, u \in (d_n)^d$,

$$u_*[\det(\gamma_n[u]) \cdot k_n\nu_n] \ll \nu_d$$

where $\gamma_n[u]$ *denotes the matrix* $(\gamma_n(u_i,u_j))_{1\le i,j\le d}$, ν_n *(resp.* ν_d*) the Lebesgue measure on* $\mathbb{R}^n$ *(resp.* $\mathbb{R}^d$*) and* $u_*[\det(\gamma_n[u]) \cdot k_n\nu_n]$ *the image measure defined by, for any* $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned}(u_*[\det(\gamma_n[u]) \cdot k_n\nu_n])(B) &= [\det(\gamma_n[u]) \cdot k_n\nu_n](u^{-1}(B)) \\ &= \int_{u^{-1}(B)} \det(\gamma_n[u,u](t))k_n(t)dt.\end{aligned}$$

Proof. We prove this result thanks to [16, Proposition 2.30 and Theorem 2.31] with

$$k = k_n, \quad d = \widetilde{d}_n, \quad \xi_{ij}(t) = t_i \wedge t_j - \frac{t_i t_j}{T},$$

where $\widetilde{d}_n$ is the set of $\mathcal{B}(\mathbb{R}^n)$-measurable functions $u \in L^2(k_n(t)dt)$ such that for any $i \in \{1, \cdots, n\}$ and for almost all

$$\widetilde{t} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}^{n-1},$$

the function

$$s \longmapsto u^{(i)}_{\widetilde{t}}(s) = u(t_1, \cdots, t_{i-1}, s, t_{i+1}, \cdots, t_n)$$

has an absolutely continuous version $\widetilde{u}^{(i)}_{\widetilde{t}}$ on $[t_{i-1}, t_{i+1}]$ such that

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t)\frac{\partial u}{\partial t_j}(t)\left(t_i \wedge t_j - \frac{t_i t_j}{T}\right) \in L^1(k_n(t)dt)$$

where $\dfrac{\partial u}{\partial t_i} = \dfrac{\partial \widetilde{u}^{(i)}_{\widetilde{t}}}{\partial s}$ and set for any $u, v \in \widetilde{d}_n$:

$$e_n(u,v) = \frac{1}{2} \int_{\mathbb{R}^n} \sum_{i,j=1}^{n} \frac{\partial u}{\partial t_i}(t) \frac{\partial v}{\partial t_j}(t) \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right) k_n(t) dt.$$

The function $k = k_n : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ is measurable and the functions $\xi_{i,j}$ are symmetric Borel function. We have to check if the two assumptions (HG) of [17] are satisfied.

Condition 1. For any $i \in \{1, ..., n\}$ and ν_{n-1}-a.e. $\bar{t} \in B_{n-1}$ with

$$B_{n-1} = \left\{ \widetilde{t} = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_n) \in \mathbb{R}^{n-1} \quad \int_{\mathbb{R}} k^{(i)}_{n,\widetilde{t}}(s) ds > 0 \right\},$$

we have $0 < t_1 < \cdots < t_{i-1} < t_{i+1} < \cdots < t_n < T$ because else

$$\kappa^{(i)}_{\widetilde{t}} = \kappa(t_1, \cdots, t_{i-1}, \cdot, t_{i+1}, \cdots, t_n) = 0$$

and $\int_{\mathbb{R}} k^{(i)}_{n,\bar{t}}(s) ds = 0$. Thus, for any $t_i \in]t_{i-1}, t_{i+1}[$,

$$k^{(i)}_{n,\bar{t}}(t_i) = \kappa(t_1, \cdots, t_n) > 0$$

because $\lambda > 0$ and $\mu \geq 0$. Therefore

$$k_n(t_1, \cdots, t_n) = \frac{\kappa(t_1, \cdots, t_n)}{\int_{\mathbb{R}^n} \kappa(t) dt} > 0.$$

In particular $k_n(t_1, \cdots, t_n)$ is invertible and

$$\begin{aligned} &(k_n(t_1, \cdots, t_n))^{-1} \\ &= \frac{\int_{\mathbb{R}^n} \kappa(t) dt}{\kappa(t_1, \cdots, t_n)} \\ &= 1_{\{0 < t_1 < \cdots < t_n \leq T\}} \left(\prod_{i=1}^{n} (\lambda^*(t_i; t_1, \cdots, t_n))^{-1} \right) e^{\int_0^T \lambda^*(s; t_1, \cdots, t_n) ds} \int_{\mathbb{R}^n} \kappa(t) dt. \end{aligned}$$

Therefore $t_i \longmapsto (k_n(t_1, \cdots, t_n))^{-1}$ is integrable on $\mathbb{R}$ because it is only the integral of a continuous function on $[t_{i-1}, t_{i+1}]$ and equal to 0 on $\mathbb{R} \backslash [t_{i-1}, t_{i+1}]$.

Condition 2. For any $t = (t_1, \cdots, t_n) \in \mathbb{R}^n$ and any $c \in \mathbb{R}^n$, such that $0 = t_0 < t_1 <$

$t_2 < \cdots < t_n < T = t_{n+1}$,

$$\begin{aligned}
\sum_{i,j=1}^{n} \xi_{ij}(t) c_i c_j &= \sum_{i,j=1}^{n} \left(t_i \wedge t_j - \frac{t_i t_j}{T} \right) c_i c_j \\
&= \frac{1}{T} \sum_{i=1}^{n} t_i \left(T - t_i \right) c_i^2 + \frac{2}{T} \sum_{1 \le i < j \le n} t_i \left(T - t_j \right) c_i c_j
\end{aligned}$$

This double sum can be split as follows:

$$\begin{aligned}
\sum_{i=1}^{n} t_i \left(T - t_i \right) c_i^2 &= \sum_{i=1}^{n} \sum_{k=1}^{i} (t_k - t_{k-1}) \sum_{\ell=i}^{n} (t_{\ell+1} - t_\ell) c_i^2 \\
&= \sum_{k=1}^{n} (t_k - t_{k-1}) \sum_{\ell=k}^{n} (t_{\ell+1} - t_\ell) \sum_{i=k}^{\ell} c_i^2 \\
&= \sum_{k=1}^{n} (t_k - t_{k-1})(t_{k+1} - t_k) c_k^2 + \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{\ell=k+1}^{n} (t_{\ell+1} - t_\ell) \sum_{i=k}^{\ell} c_i^2
\end{aligned}$$

and

$$\begin{aligned}
2 \sum_{1 \le i < j \le n} t_i \left(T - t_j \right) c_i c_j &= 2 \sum_{1 \le i < j \le n} \sum_{k=1}^{i} (t_k - t_{k-1}) \sum_{\ell=j}^{n} (t_{\ell+1} - t_\ell) c_i c_j \\
&= 2 \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{k \le i < j \le n} \sum_{\ell=j}^{n} (t_{\ell+1} - t_\ell) c_i c_j \\
&= 2 \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{\ell=k+1}^{n} (t_{\ell+1} - t_\ell) \sum_{k \le i < j \le \ell} c_i c_j
\end{aligned}$$

Coming back to the initial sum, we have

$$\begin{aligned}
\sum_{i,j=1}^{n} \xi_{ij}(t) c_i c_j &= \frac{1}{T} \sum_{k=1}^{n} (t_k - t_{k-1})(t_{k+1} - t_k) c_k^2 \\
&\quad + \frac{1}{T} \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{\ell=k+1}^{n} (t_{\ell+1} - t_\ell) \left[\sum_{i=k}^{\ell} c_i^2 + 2 \sum_{k \le i < j \le \ell} c_i c_j \right] \\
&= \frac{1}{T} \sum_{k=1}^{n} (t_k - t_{k-1})(t_{k+1} - t_k) c_k^2 \\
&\quad + \frac{1}{T} \sum_{k=1}^{n-1} (t_k - t_{k-1}) \sum_{\ell=k+1}^{n} (t_{\ell+1} - t_\ell) \left[\sum_{i,j=k}^{\ell} (c_i + c_j)^2 \right] \\
&\ge \frac{1}{T} \sum_{k=1}^{n} (t_k - t_{k-1})(t_{k+1} - t_k) c_k^2 .
\end{aligned}$$

Thus, for any compact $K \subset \{(t_1, \cdots, t_n), \quad 0 = t_0 < t_1 < \cdots < t_n < T = t_{n+1}\}$, there exists $c \in \mathbb{R}_+^*$ such that, for any $(t_1, \cdots, t_n) \in K$,

$$\sum_{i,j=1}^{n} \xi_{ij}(t) c_i c_j \geq \frac{c^2}{T} \sum_{k=1}^{n} c_k^2.$$

The hypotheses (HG) of [16, Proposition 2.30] being satisfied, we conclude that $(\widetilde{d}_n, e_n)$ is a local Dirichlet hence $(C^\infty(\mathbb{R}^n), e_n)$ is closable and its closure (d_n, e_n) is such that $d_n \subset \widetilde{d}_n$. The last assertion is a consequence of [16, Theorem 2.31]. □

Using this result, we consider $\|\cdot\|_{d_n}$ the norm on d_n defined by, for any $u \in d_n$,

$$\|u\|_{d_n}^2 = \|u\|_{L^2(k_n(t)dt)}^2 + 2e_n(u).$$

5.3.2 Global criterion

We remind that for any $F \in L^0(\Omega)$, there exists $a \in \mathbb{R}$ and $f_n : \mathbb{R}^n \longrightarrow \mathbb{R}, n \in \mathbb{N}^*$, measurable such that, $\mathbb{P}$-almost surely,

$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n(T_1, \cdots, T_n) 1_{\{N_T=n\}}. \tag{5.12}$$

Proposition 5.3.3. *Let $F \in L^0(\Omega)$ of the form* (5.12). *Then $F \in \mathbb{D}^{1,2}$ if and only if $f_n \in d_n$ for any $n \in \mathbb{N}^*$ and*

$$\sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n) < +\infty.$$

In this case

$$\|F\|_{\mathbb{D}^{1,2}}^2 = a^2 \mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Proof. Let $F = a1_{\{N_T=0\}} + \sum_{n=1}^{d} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$ be in $\mathcal{S}$. Then as a consequence of Corollary 5.2.2, F belongs to $\mathbb{D}^{1,2}$ and

$$\mathcal{E}(F, F) = \mathbb{E}\left[\int_0^T |D_s F|^2 ds\right] = 2 \sum_{n=1}^{d} P(N_T = n) e_n(f_n).$$

Hence

$$\|F\|_{\mathbb{D}^{1,2}}^2 = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{d} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

We conclude using a density argument. Indeed, if F belongs to $\mathbb{D}^{1,2}$, there exists a sequence $(F^k)_k$ in $\mathcal{S}$ converging to F in $\mathbb{D}^{1,2}$. Now if for any k

$$F^k = a_k 1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n^k(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$$

with $f_n^k \in C^\infty(\mathbb{R}^d)$ and $f_n^k = 0$ for n large, then clearly for all n, k, ℓ:

$$\|f_n^k - f_n^\ell\|_{d_n}^2 \mathbb{P}(N_T = n) \leq \|F^k - F^\ell\|_{\mathbb{D}^{1,2}}^2$$

hence $(f_n^k)_n$ converges to an element f_n in d_n, a^k tends to a real number a and we get that

$$F = a 1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}},$$

and

$$\|F\|_{\mathbb{D}^{1,2}}^2 = \lim_{m\to+\infty} \|F 1_{\{N_T\leq m\}}\|_{\mathbb{D}^{1,2}}^2 = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Conversely, if $F \in L^0(\Omega)$ of the form (5.12) is such that $f_n \in d_n$ for any $n \in \mathbb{N}^*$ and

$$\sum_{n=1}^{+\infty} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n) < +\infty,$$

then define for any m, $F^m = a 1_{\{N_T=0\}} + \sum_{n=1}^{m} f_n(T_1, T_2, \cdots, T_n) 1_{\{N_T=n\}}$ by approximating each f_n for $n \in \{1, \cdots, m\}$ by a sequence of functions in $C^\infty(\mathbb{R}^d)$ we easily get that F^m belong to $\mathbb{D}^{1,2}$ and

$$\|F^m\|_{\mathbb{D}^{1,2}}^2 = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{m} \|f_n\|_{d_n}^2 \mathbb{P}(N_T = n).$$

Then (F^m) is a Cauchy sequence in $\mathbb{D}^{1,2}$ converging to F in L^2, this ends the proof. □

Remark 5.3.4. *We can summarize the equalities between the Dirichlet structures*

$$(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{D}^{1,2}, \Gamma) \quad \textit{and} \quad (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), k_n(t)dt, d_n, \gamma_n), n \in \mathbb{N}^* :$$

for any $F \in \mathbb{D}^{1,2}$ *of the form* (5.12)

1.
$$\|F\|^2_{L^2(\Omega)} = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|^2_{L^2(k_n(t)dt)},$$

2. *For a.e.* $s \in [0, T]$,
$$D_s F = \sum_{n=1}^{+\infty} \widetilde{D^n_s} f_n(T_1, \cdots, T_n) 1_{\{N_T = n\}},$$
with
$$\widetilde{D^n_s} f_n(T_1, \cdots, T_n) = \sum_{i=1}^{n} \frac{\partial f_n}{\partial t_i}(T_1, \cdots, T_n)\left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right),$$

3.
$$\Gamma[F] = \sum_{n=1}^{+\infty} \gamma_n[f_n](T_1, \cdots, T_n) 1_{\{N_T = n\}},$$

4.
$$\mathcal{E}(F) = \sum_{n=1}^{+\infty} e_n(f_n)\mathbb{P}(N_T = n),$$

5.
$$\|F\|^2_{\mathbb{D}^{1,2}} = a^2\mathbb{P}(N_T = 0) + \sum_{n=1}^{+\infty} \|f_n\|^2_{d_n}\mathbb{P}(N_T = n).$$

Theorem 5.3.5. *Let* $d \in \mathbb{N}^*$ *and* $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$*. Then, noting*
$$\Gamma[F] = (\Gamma[F_i, F_j])_{1\le i,j\le d},$$
the image measure $F_*[\det(\Gamma[F]).\mathbb{P}]$ *is absolutely continuous with respect to the Lebesgue measure* ν_d *on* $\mathbb{R}^d$*.*

Proof. Let $B \subset \mathbb{R}^d$ such that $\nu_d(B) = 0$. We would like to get
$$\begin{aligned} 0 &= (F_*[\det(\Gamma[F]).\mathbb{P}])(B) = \int_{F^{-1}(B)} \det(\Gamma[F](\omega))d\mathbb{P}(\omega) \\ &= \int_\Omega \det(\Gamma[F](\omega))1_B(F(\omega))d\mathbb{P}(\omega) = \mathbb{E}\left[\det(\Gamma[F])1_B(F)\right]. \end{aligned}$$

But, according to Proposition 5.3.3, there exist $a \in \mathbb{R}$ and $f_n \in (d_n)^d, n \in \mathbb{N}^*$ such that
$$F = a1_{\{N_T=0\}} + \sum_{n=1}^{+\infty} f_n(T_1, \cdots, T_n)1_{\{N_T=n\}}.$$

Thus

$$\Gamma[F] = \Gamma[F,F] = \sum_{n=1}^{+\infty} \gamma_n[f_n](T_1,\cdots,T_n)1_{\{N_T=n\}}.$$

In particular

$$\det(\Gamma[F])1_{\{N_T=0\}} = 0,$$

for any $n \in \mathbb{N}^*$,

$$\det(\Gamma[F])1_{\{N_T=n\}} = \det(\gamma_n[f_n](T_1,\cdots,T_n))1_{\{N_T=n\}}$$

and also

$$1_B(F)1_{\{N_T=n\}} = 1_B(f_n(T_1,\cdots,T_n))1_{\{N_T=n\}}.$$

Therefore

$$\det(\Gamma[F])1_B(F) = \sum_{n=1}^{+\infty} \det(\gamma_n[f_n](T_1,\cdots,T_n))1_B(f_n(T_1,\cdots,T_n))1_{\{N_T=n\}}$$

and, according to Lemma 5.3.1,

$$\begin{aligned}
&\mathbb{E}\left[\det(\Gamma[F])1_B(F)\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left[\det(\gamma_n[f_n](T_1,\cdots,T_n)1_B(f_n(T_1,\cdots,T_n)1_{\{N_T=n\}}\right] \\
&= \sum_{n=1}^{+\infty} \mathbb{E}\left[\det(\gamma_n[f_n](T_1,\cdots,T_n)1_B(f_n(T_1,\cdots,T_n) \mid N_T=n\right]\mathbb{P}(N_T=n) \\
&= \sum_{n=1}^{+\infty} \left(\int_{\mathbb{R}^n} \det(\gamma_n[f_n](t))1_B(f_n(t))k_n(t)dt\right)\mathbb{P}(N_T=n) \\
&= \sum_{n=1}^{+\infty} ((f_n)_*[\det(\gamma_n[f_n]\cdot k_n\nu_n)])(B)\mathbb{P}(N_T=n).
\end{aligned}$$

However, according to Proposition 5.3.2 applied to $f_n \in (d_n)^d$, the measure $(f_n)_*[\det(\gamma_n[f_n]\cdot k_n\nu_n)]$ is absolutely continuous with respect to ν_d. Thus, for any $n \in \mathbb{N}^*$,

$$((f_n)_*[\det(\gamma_n[f_n]\cdot k_n\nu_n)])(B) = 0$$

and

$$\mathbb{E}\left[\det(\Gamma[F])1_B(F)\right] = 0.$$

This concludes the proof. □

Corollary 5.3.6. *Let $d \in \mathbb{N}^*$ and $F = (F_1, \cdots, F_d) \in (\mathbb{D}^{1,2})^d$. Then, conditionally to $\Gamma[F] \in GL_d(\mathbb{R})$, the law of the random variable F is absolutely continuous with respect to the Lebesgue measure ν_d.*

5.4 Applications

5.4.1 SDE and density of the solution

We consider the stochastic differential equation

$$X_t = x_0 + \int_0^t f(s, X_s)ds + \int_{(0,t]} g(s, X_{s-})dN_s, \quad 0 \leq t \leq T, \tag{5.13}$$

or in the differential form

$$dX_t = f(t, X_t)dt + g(t, X_{t-})dN_t, \quad X_0 = x_0.$$

We assume that:

Assumption 16. *The functions $f, g : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are measurable and satisfy*

1. *For any $t \in [0, T]$, the maps $f(t, \cdot), g(t, \cdot)$ are of class C^1.*
2. $\sup_{t,x}(|\nabla_x f(t, x)| + |\nabla_x g(t, x)|) < +\infty$.
3. *For any $x \in \mathbb{R}^d$, the map $g(\cdot, x)$ is differentiable.*

Remark 5.4.1. *Since N_T admits moments of any order (See [62]), according to [78, Chapter V.3 Theorem 7 and Chapter V.4 Theorem 10], there exists a unique solution X to* (5.13) *such that* $\sup_{0 \leq t \leq T} |X_t| \in L^2(\Omega)$.

We consider the deterministic flow Φ defined by the solution of the ordinary differential equation (ODE in short)

$$\Phi_{s,t}(x) = x + \int_s^t f(u, \Phi_{s,u}(x))du, \quad 0 \leq s \leq t \leq T, \quad x \in \mathbb{R}^d.$$

Proposition 5.4.2. *On the set $\{N_T = 0\}$, we have*

$$X_t = \Phi_{0,t}(x_0), \quad 0 \leq t \leq T.$$

And, for any $n \in \mathbb{N}^*$, on the set $\{N_T = n\}$, we have

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0), \quad T_n \le t \le T,$$

where

$$\Psi(t, x) = x + g(t, x), \quad 0 \le t \le T, \quad x \in \mathbb{R}^d.$$

Proof. We proceed by induction. On the set $\{N_T = 0\}$ we have

$$X_t = x_0 + \int_0^t f(s, X_s) ds, \quad 0 \le t \le T.$$

Thus for all $t \in [0, T]$, $X_t = \Phi_{0,t}(x_0)$.

On the $\{N_T = 1\}$ we have, for any $t \in [0, T_1)$,

$$X_t = x_0 + \int_0^t f(s, X_s) ds.$$

Thus $X_t = \Phi_{0,t}(x_0)$. Then for any $t \in [T_1, T]$ we have

$$\begin{aligned} X_t &= X_{T_1-} + \int_{T_1}^t f(s, X_s) ds + \int_{[T_1,t]} g(s, X_{s-}) dN_s \\ &= X_{T_1-} + \int_{T_1}^t f(s, X_s) ds + g(T_1, X_{T_1-}) \\ &= \Psi(T_1, X_{T_1-}) + \int_{T_1}^t f(s, X_s) ds. \end{aligned}$$

Hence

$$\begin{aligned} X_t &= \Phi_{T_1,t}(\Psi(T_1, X_{T_1-})) = \Phi_{T_1,t}(\Psi(T_1, \Phi_{0,T_1}(x_0))) \\ &= [\Phi_{T_1,t} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0). \end{aligned}$$

We assume that the equality is satisfied on $\{N_T = n\}$ for $n \in \mathbb{N}^*$:

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2}(\cdot) \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0), \quad T_n \le t \le T.$$

Let consider the set $\{N_T = n+1\}$. The process X satisfies, for any $t \in [T_{n+1}, T]$,

$$\begin{aligned} X_t &= X_{T_{n+1}-} + \int_{T_{n+1}}^t f(s, X_s)ds + \int_{[T_{n+1},t]} g(s, X_{s-})dN_s \\ &= \Psi(T_{n+1}, X_{T_{n+1}-}) + \int_{T_{n+1}}^t f(s, X_s)ds. \end{aligned}$$

Thus

$$X_T = \Phi_{T_{n+1},T}(\Psi(T_{n+1}, X_{T_{n+1}-})).$$

However, as we are on $\{N_T = n+1\}$, we are also on $\{N_t = n\}$ for any $t \in [T_n, T_{n+1})$. According to the recurrence hypothesis,

$$X_t = [\Phi_{T_n,t} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0).$$

Thus, by continuity of the map Φ with respect to t,

$$X_{T_{n+1}-} = [\Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0).$$

Therefore

$$\begin{aligned} X_T &= \Phi_{T_{n+1},T}(\Psi(T_{n+1}, X_{T_{n+1}-})) \\ &= \Phi_{T_{n+1},T}(\Psi(T_{n+1}, [\Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0))) \\ &= [\Phi_{T_{n+1},T} \circ \Psi(T_{n+1}, \cdot) \circ \Phi_{T_n,T_{n+1}} \circ \Psi(T_n, \cdot) \circ \cdots \circ \Phi_{T_1,T_2} \circ \Psi(T_1, \cdot) \circ \Phi_{0,T_1}](x_0). \end{aligned}$$

The statement is proved. □

We continue this section with the same ideas as in [28]. As the following results are formal computations, they are proved in the same way.

Remark 5.4.3. *The process $\nabla_x \Phi$ satisfies, for any $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}^d$,*

$$\frac{\partial}{\partial t} \nabla_x \Phi_{s,t}(x) = \nabla_x f(t, \Phi_{s,t}(x)) \nabla_x \Phi_{s,t}(x), \quad \nabla_x \Phi_{s,s}(x) = I_d.$$

Thus

$$\nabla_x \Phi_{s,t}(x) = \exp\left(\int_s^t \nabla_x f(u, \Phi_{s,u}(x)) du \right), \quad 0 \leq s \leq t \leq T.$$

Definition 5.4.4. *We define the process K as the derivative of the flow generated by X,* *solution of the SDE*

$$K_t = I_d + \int_0^t \nabla_x f(s, X_s) K_s ds + \int_{(0,t]} \nabla_x g(s, X_{s-}) K_{s-} dN_s, \quad 0 \leq t \leq T.$$

From now we assume that:

Assumption 17. *For any* $(t, x) \in [0, T] \times \mathbb{R}^d$,

$$\det(I_d + \nabla_x g(t, x)) \neq 0$$

and $(I_d + \nabla_x g)^{-1}$ *is uniformly bounded.*

We now define the process $\widetilde{K}$ as the solution of

$$\begin{aligned} \widetilde{K}_t &= I_d - \int_0^t \widetilde{K}_s \nabla_x f(s, X_s) ds \\ &\quad - \int_{(0,t]} \widetilde{K}_s \nabla_x g(s, X_{s-})(I_d - \nabla_x g(s, X_{s-})(I_d + \nabla_x g(s, X_{s-}))^{-1}) dN_s. \end{aligned}$$

Following [16, Proposition 8.7], we have:

Lemma 5.4.5. *Processes* K *and* $\widetilde{K}$ *satisfy*

$$K_t \widetilde{K}_t = I_d, \quad 0 \leq t \leq T.$$

Moreover:

$$K_{T_i} = (I_d + \nabla_x g(t, X_{T_i-})) K_{T_i-}, \quad \widetilde{K}_{T_i} = (I_d + \nabla_x g(T_i, X_{T_i-}))^{-1} \widetilde{K}_{T_i-}, \quad i \in \mathbb{N}^*.$$

Definition 5.4.6. *We define the process* $(K_t^s)_{0 \leq s \leq t \leq T}$ *by:*

$$K_t^s = K_t \widetilde{K}_s, \quad 0 \leq s \leq t \leq T.$$

Similarly to [28, Proposition 6.4], we get the following result.

Proposition 5.4.7. *Let* $\varphi : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ *defined by:*

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad \varphi(t, x) = f(t, x + g(t, x)) - (I_d + \nabla_x g(t, x)) f(t, x) - \frac{\partial g}{\partial t}(t, x).$$

Then $X_T \in \mathbb{D}^{1,2}$ and, for a.e. $s \in [0,T]$,

$$D_s X_T = -\int_{(0,T]} K_T^t \varphi(t, X_{t-}) \left(\frac{t}{T} - 1_{[0,t]}(s)\right) dN_t.$$

Moreover

$$\Gamma[X_T] = \int_{(0,T]} \int_{(0,T]} K_T^t \varphi(t, X_{t-})(\varphi(u, X_{u-}))^* (K_T^u)^* \left(u \wedge t - \frac{ut}{T}\right) dN_t dN_u.$$

Thanks to this expression of DX_T and $\Gamma[X_T]$, we can deduce results about the absolute continuity of the law of X_T using Corollary 5.3.6.

Corollary 5.4.8. *If we consider the event*

$$\mathcal{C} = \left\{\det\left(\int_{(0,T]} \int_{(0,T]} K_T^t \varphi(t, X_{t-})(\varphi(u, X_{u-}))^* (K_T^u)^* \left(u \wedge t - \frac{ut}{T}\right) dN_t dN_u\right) > 0\right\}$$

then if $\mathbb{P}(\mathcal{C}) > 0$, the law of X_T conditionally to $\mathcal{C}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$.

Proof. It is a direct consequence of Proposition 5.4.7 and Corollary 5.3.6. □

Theorem 5.4.9. *In dimension $d = 1$, if for any $t \in [0,T]$ and $x \in \mathbb{R}$,*

$$\varphi(t,x) = f(t, x + g(t,x)) - f(t,x) - \frac{\partial g}{\partial x}(t,x) f(t,x) - \frac{\partial g}{\partial t}(t,x) \neq 0,$$

then, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. As $d = 1$ we have, according to Proposition 5.4.7,

$$D_s X_T = -\sum_{i=1}^{N_T} K_T^{T_i} \varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right).$$

Let $\omega \in \Omega$ such that

$$\Gamma[X_T](\omega) = \int_0^T |D_s X_T(\omega)|^2 ds = 0,$$

then, for almost every $s \in [0,T]$, $D_s X_T(\omega) = 0$. Thus, for almost every $s \in [T_{N_T(\omega)}, T]$, we get, writing T_i instead of $T_i(\omega)$,

$$\sum_{i=1}^{N_T(\omega)} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} = 0.$$

Then for almost every $s \in [T_{N_T(\omega)-1}, T_{N_T(\omega)}]$

$$\sum_{i=1}^{N_T(\omega)-1} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} + K_T^{T_{N_T(\omega)}} \varphi(T_{N_T(\omega)}, X_{T_{N_T(\omega)}-}) \left(\frac{T_{N_T(\omega)}}{T} - 1 \right) = 0.$$

Therefore, by subtracting the two equations, we get

$$\varphi(T_{N_T(\omega)}, X_{T_{N_T(\omega)}-}) = 0$$

then

$$\sum_{i=1}^{N_T(\omega)-1} K_T^{T_i} \varphi(T_i, X_{T_i-}) \frac{T_i}{T} = 0.$$

Thus, by considering $s \in [T_{N_T(\omega)-2}, T_{N_T(\omega)-1}]$ then $s \in [T_{N_T(\omega)-3}, T_{N_T(\omega)-2}]$, ..., we get, by successive iterations, for any $i \in \{1, \cdots, N_T(\omega)\}$,

$$\varphi(T_i, X_{T_i-}) = 0.$$

Therefore, by contrapositive, if $\varphi(t, x) \neq 0$ for any $t \in [0, T]$ and $x \in \mathbb{R}$ then $\Gamma[X_T] > 0$ on the set $\{N_T \geq 1\}$. Thus, according to Corollary 5.3.6, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. □

Example 5.4.10 (Linear with constant coefficients in dimension $d = 1$)**.** *We consider X the solution of the linear SDE*

$$dX_t = (aX_t + b)dt + (\alpha X_{t-} + \beta)dN_t, \ 0 \leq t \leq T, \quad X_0 = x_0, \tag{5.14}$$

where $x_0, a, b, \alpha, \beta \in \mathbb{R}$ satisfy

$$a\beta - \alpha b \neq 0.$$

In this case we have, for any $t \in [0, T]$ and $x \in \mathbb{R}$,

$$\varphi(t, x) = a(x + \alpha x + \beta) + b - ax - b - \alpha(ax + b) = a\beta - \alpha b \neq 0.$$

Then, according to Theorem 5.4.9, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Corollary 5.4.11. *We assume that $d = 1$ and the parameters f, g do not depend on*

$t \in [0, T]$. *We consider the Wronskian of f and g:*

$$W(f, g) = g' \times f - f' \times g.$$

Thus if the function f is of class C^2 and

$$\forall x \in \mathbb{R}, \quad |W(f, g)(x)| > \frac{1}{2}\|f''\|_\infty \|g\|_\infty^2$$

then, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proof. As the parameters f and g do not depend on t, we have, for any $t \in [0, T]$ and $x \in \mathbb{R}$,

$$\varphi(t, x) = f(x + g(x)) - f(x) - g'(x)f(x).$$

Then, by Taylor expansion, there exists $y_x \in \mathbb{R}$ such that

$$\varphi(t, x) = g(x)f'(x) + \frac{1}{2}g(y_x)^2 f''(x) - g'(x)f(x) = \frac{1}{2}g(y_x)^2 f''(x) - W(f, g)(x).$$

Thus, if there exists $x \in \mathbb{R}$ such that $\varphi(t, x) = 0$ then, according to the assumption,

$$|W(f, g)(x)| = \frac{1}{2}g(y_x)^2 |f''(x)| \leq \frac{1}{2}\|g\|_\infty^2 \|f''\|_\infty < |W(f, g)(x)|$$

which is absurd. Therefore $\varphi(t, x) \neq 0$ for any $t \in [0, T]$ and $x \in \mathbb{R}$. We conclude by using Theorem 5.4.9. □

Example 5.4.12. *We consider X the solution of the SDE in dimension $d = 1$*

$$dX_t = \cos(X_t)dt + \sin(X_{t-})dN_t, \ 0 \leq t \leq T, \quad X_0 = x_0, \tag{5.15}$$

where $x_0 \in \mathbb{R}$. In particular

$$\forall x \in \mathbb{R}, \quad |W(f, g)(x)| = 1 > \frac{1}{2} = \frac{1}{2}\|\cos''\|_\infty \|\sin\|_\infty^2.$$

Thus, according to Corollary 5.4.11, conditionally to $\{N_T \geq 1\}$, the law of X_T is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

Proposition 5.4.13. *If there exists $\ell \in \mathbb{N}$ such that for any $n \geq \ell, 0 \leq t_1 < \cdots < t_n \leq T$*

and $x_1, \cdots, x_n \in \mathbb{R}^d$*, the family* $(K_T^{t_i}\varphi(t_i, x_i))_{1\le i\le n}$ *spans* $\mathbb{R}^d$ *then, conditionally to* $\{N_T \ge \ell\}$*, the law of* X_T *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$*.*

Proof. Let $\omega \in \{N_T \ge \ell\}$ such that $\Gamma[X_T](\omega)$ is non invertible. Then, as it is a nonnegative symetric matrix, there exists $u \in \mathbb{R}^d \backslash \{0\}$ such that

$$u^*\Gamma[X_T](\omega)u = \int_0^T (u^* D_s X_T(\omega))^2 ds = 0.$$

Then, according to Proposition 5.4.7,

$$0 = \int_0^T (u^* D_s X_T(\omega))^2 ds = \int_0^T \left(u^* \sum_{i=1}^{N_T(\omega)} K_T^{T_i}\varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right)\right)^2 ds.$$

Thus, for almost every $s \in [0, T]$,

$$u^* \sum_{i=1}^{N_T(\omega)} K_T^{T_i}\varphi(T_i, X_{T_i-}) \left(\frac{T_i}{T} - 1_{[0,T_i]}(s)\right) = 0.$$

We deduce, as in dimension $d = 1$, that, for any $i \in \{1, \ldots, N_T(\omega)\}$,

$$u^* K_T^{T_i}\varphi(T_i, X_{T_i-}) = 0$$

which is absurd because $(K_T^{T_i}\varphi(T_i, X_{T_i-}))_{1\le i\le N_T(\omega)}$ spans $\mathbb{R}^d$. □

Remark 5.4.14. *Necessarily we have* $\ell \ge d$*.*

Example 5.4.15 (Linear with constant coefficients in dimension $d \in \mathbb{N}^*$)**.** *We consider* X *the solution of the linear SDE*

$$dX_t = (AX_t + b)dt + (MX_{t-} + \beta)dN_t,\ 0 \le t \le T, \quad X_0 = x_0, \tag{5.16}$$

where $x_0, b, \beta \in \mathbb{R}^d$ *and* $A, M \in \mathbb{R}^{d\times d}$ *such that*

$$\det(M + I_d) \ne 0, \quad AM = MA$$

and if there exists $\ell \in \mathbb{N}$ *such that, for any* $n \ge \ell$ *and* $0 \le t_1 < \cdots < t_n \le T$*, the family*

$$(\exp(A(T - t_i))(I_d + M)^{n-i}(A\beta - Mb))_{1\le i\le n}$$

spans $\mathbb{R}^d$. *In this case we have:*

- *The process* K *defined by Definition 5.4.4 satisfies*

$$K_t = I_d + \int_0^t AK_s ds + \int_{(0,t]} MK_{s-} dN_s, \quad 0 \leq t \leq T,$$

i.e.

$$dK_t = AK_s ds + MK_{s-} dN_s, \quad 0 \leq t \leq T, \quad K_0 = I_d,$$

i.e., for any $t \in [0,T]$, *as* $AM = MA$,

$$K_t = \exp(At) \prod_{0<s\leq t} (I_d + M\Delta N_s) = \exp(At)(I_d + M)^{N_t}.$$

- *As* $\det(I_d + M) \neq 0$ *then we can consider the process* $\widetilde{K} = K^{-1}$ *defined just above Lemma 5.4.5:*

$$\widetilde{K}_t = I_d - \int_0^t A\widetilde{K}_s ds - M(I_d + M)^{-1} \int_{(0,t]} \widetilde{K}_{s-} dN_s, \quad 0 \leq t \leq T,$$

i.e.

$$d\widetilde{K}_t = -A\widetilde{K}_t dt - M(I_d + M)^{-1}\widetilde{K}_{t-} dN_t, \quad 0 \leq t \leq T, \quad \widetilde{K}_0 = 1,$$

i.e., for any $t \in [0,T]$, *as* $AM = MA$,

$$\begin{aligned} \widetilde{K}_t &= \exp(-At) \prod_{0<s\leq t} \left(I_d - M(I_d + M)^{-1}\Delta N_s\right) \\ &= \exp(-At)(I_d - M(I_d + M)^{-1})^{N_t} \\ &= \exp(-At)(I_d + M)^{-N_t}. \end{aligned}$$

- *The process* (K_t^s) *is equal to, for any* $0 \leq s \leq t \leq T$,

$$\begin{aligned} K_t^s &= K_t \widetilde{K}_s \\ &= \exp(A(t-s))(I_d + M)^{N_t - N_s}. \end{aligned}$$

- *According to Proposition 5.4.7, we consider the function* $\varphi : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$

defined by, for any $t \in [0,T]$ *and* $x \in \mathbb{R}$,

$$\begin{aligned}\varphi(t,x) &= A(x+Mx+\beta)+b-(I_d+M)(Ax+b)\\ &= A\beta - Mb.\end{aligned}$$

Therefore, according to the assumption about the spanning property, we get the following result by Proposition 5.4.13: conditionally to $\{N_T \geq \ell\}$, *the law of* X_T *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^d$.

This is the case for example when $A = I_d$, M *is a diagonalizable matrix with* d *distinct eigenvalues (different to* -1 *to have* $\det(M+I_d) \neq 0$*): there exists* $\lambda_1, \cdots, \lambda_d$ *in* $\mathbb{R}\backslash\{-1\}$ *(distinct) and* $P \in GL_d(\mathbb{R})$ *such that*

$$M = P \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_d \end{pmatrix} P^{-1},$$

and $v := [P^{-1}(\beta - Mb)]_j = [P^{-1}\beta - DP^{-1}b]_j \neq 0$ *for any* $j \in \{1, \cdots, d\}$. *Thus, for any* $n \geq \ell, 0 \leq t_1 < \cdots < t_n \leq T$ *and* $i \in \{1, \cdots, n\}$,

$$\begin{aligned}&\exp(A(T-t_i))(I_d+M)^{n-i}(A\beta - Mb)\\ &= e^{T-t_i} P \begin{pmatrix} (1+\lambda_1)^{n-i} & & (0) \\ & \ddots & \\ (0) & & (1+\lambda_d)^{n-i} \end{pmatrix} P^{-1}(\beta - Mb).\end{aligned}$$

Therefore the family $(\exp(A(T-t_i))(I_d+M)^{n-i}(A\beta - Mb))_{1\leq i\leq n}$ *spans* $\mathbb{R}^d$ *if and only if the family*

$$\left(\begin{pmatrix} (1+\lambda_1)^{n-1}v_1 \\ \vdots \\ (1+\lambda_d)^{n-1}v_d \end{pmatrix}, \cdots, \begin{pmatrix} (1+\lambda_1)v_1 \\ \vdots \\ (1+\lambda_d)v_d \end{pmatrix}, \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix}\right)$$

spans $\mathbb{R}^d$. *The determinant of the last* d *vectors of this family is related to a Vandermonde determinant:*

$$v_1 \cdots v_d \prod_{1\leq i<j\leq d} (\lambda_j - \lambda_i)$$

which is not null. Therefore, conditionally to $\{N_T \geq \ell\}$, *the law of* X_T *admits an absolutely continuous law with respect to the Lebesgue measure on* $\mathbb{R}^d$.

5.4.2 Application to Greek computation

We consider an asset price whose dynamics is given by

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{N}_t, \quad S_0 = x_0, \tag{5.17}$$

where N is a Hawkes process with conditional intensity λ^*, r the interest rate, σ the volatility and x_0 the initial wealth. In other words we have

$$dS_t = (r - \sigma\lambda^*(t))S_t dt + \sigma S_{t-} dN_t.$$

Thus, if we write $\alpha_t = r - \sigma\lambda^*(t)$ then the dynamic is equivalent to

$$dS_t = \alpha_t S_t dt + \sigma S_{t-} dN_t, \quad S_0 = x_0. \tag{5.18}$$

We also consider an European option

$$C = \mathbb{E}\left[f(S_T)\right]$$

with f a function which can be not continuous as $f = 1_{[K,+\infty)}$ for example for a binary European option, and we interest to compute Greeks as

$$\Delta = \frac{\partial C}{\partial x_0}, \quad \Gamma = \frac{\partial^2 C}{\partial x_0^2}, \quad \rho = \frac{\partial C}{\partial r}, \quad \nu = \frac{\partial C}{\partial \sigma}.$$

In the sequel x denotes a real number in an interval (a, b) which can be equal to x_0, r or σ.

We consider a class $\mathcal{L}$ of real functions f of the form

$$f(y) = \sum_{i=1}^{n} \Phi_i(y) 1_{A_i}(y), \quad y \in \mathbb{R}, \tag{5.19}$$

where $n \in \mathbb{N}$, Φ_i is continuous and bounded and A_i is an interval with endpoints in $\mathbb{T} = \mathbb{T}_0 \cup \{-\infty, +\infty\}$ with the set $\mathbb{T}_0 \subset \mathbb{R}$ defined by:

$$\mathbb{T}_0 = \left\{ y \in \mathbb{R} \quad \lim_{n\to+\infty} \sup_{a<x<b} \mathbb{P}\left(F^x \in \left(y - \frac{1}{n}, y + \frac{1}{n}\right)\right) = 0 \right\}.$$

Proposition 5.4.16. *Let (a, b) be an interval of $\mathbb{R}$. Let $(F^x)_{a<x<b}$ and $(G^x)_{a<x<b}$ be two*

families of random variables such that the maps $x \in (a,b) \longmapsto F^x \in \mathbb{D}^{1,2}$ *and* $x \in (a,b) \longmapsto \mathbb{D}^{1,2}$ *are continuously differentiable. Let* $m \in \mathcal{H}$ *such that for any* $x \in (a,b)$ *on* $\left\{\frac{\partial F^x}{\partial x} \neq 0\right\}$

$$D_m F^x \neq 0$$

and such that $mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x}$ *is continuous in* x *in Dom*(δ). *Thus for any* $f \in \mathcal{L}$ *the map* $x \longmapsto \mathbb{E}[f(F^x)]$ *is continuous differentiable and*

$$\frac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)] = \mathbb{E}\left[f(S_T^{x_0})\delta\left(G^x m \frac{\frac{\partial F^x}{\partial x}}{D_m F^x}\right)\right] + \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x)\right].$$

Proof. We follow the proof of [28, Proposition 7.2]. First if $f \in C_b^1(\mathbb{R})$ then, as the maps $x \longmapsto F^x$ and $x \longmapsto G^x$ are differentiable, the map $x \longmapsto G^x f(F^x)$ is differentiable and, for any $x \in (a,b)$,

$$\frac{\partial}{\partial x}(G^x f(F^x)) = \frac{\partial G^x}{\partial x} f(F^x) + \frac{\partial F^x}{\partial x} f'(F^x) G^x.$$

Then, as $f \in C_b^1(\mathbb{R})$, $D_m f(F^x) = f'(F_x) D_m F^x$, and, as $D_m F^x \neq 0$ on $\left\{\frac{\partial F^x}{\partial x} \neq 0\right\}$,

$$\frac{\partial}{\partial x}(G^x f(F^x)) = \frac{\partial G^x}{\partial x} f(F^x) + \frac{\partial F^x}{\partial x} \frac{D_m f(F^x)}{D_m F^x} G^x.$$

Thus, according to Remark 5.2.5 and $mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} \in \mathrm{Dom}(\delta)$,

$$\begin{aligned}
\mathbb{E}\left[\frac{\partial}{\partial x}(G^x f(F^x))\right] &= \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x)\right] + \mathbb{E}\left[G^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x} D_m f(F^x)\right] \\
&= \mathbb{E}\left[\frac{\partial G^x}{\partial x} f(F^x)\right] + \mathbb{E}\left[\delta\left(mG^x \frac{\frac{\partial F^x}{\partial x}}{D_m F^x}\right) f(F^x)\right].
\end{aligned}$$

Finally, as the function $x \longmapsto \frac{\partial}{\partial x}(G^x f(F^x))$ is continuous, we have

$$\frac{\partial}{\partial x}\mathbb{E}[G^x f(F^x)] = \mathbb{E}\left[\frac{\partial}{\partial x}(G^x f(F^x))\right]$$

and the result follows in that case. For the general case ($f \in \mathcal{L}$) we conclude as in [28, Proposition 7.2], using an approximation argument. □

As in [53], if we can apply the Proposition 5.4.16 with

$$F^x = S_T^x, \quad G^x = 1_{\{N_T>0\}}, \quad x \in (a,b),$$

then we can compute the Delta conditionally to $\{N_T \geq 1\}$

$$\frac{\partial}{\partial x_0}\mathbb{E}\left[1_{\{N_T>0\}} f\left(S_T^{x_0}\right)\right] = \mathbb{E}\left[f(S_T^{x_0})\delta\left(m 1_{\{N_T>0\}}\frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right].$$

The solution of (5.17)

$$dS_t = S_t\alpha_t dt + \sigma S_{t-} dN_t, \quad S_0 = x_0,$$

is given by

$$S_t^{x_0} = x_0 \exp\left(\int_0^t \alpha_s ds\right)(1+\sigma)^{N_t} = x_0 \exp\left(rt - \sigma\int_0^t \lambda^*(s)ds\right)(1+\sigma)^{N_t}, \quad 0 \leq t \leq T.$$

In particular

$$S_T^{x_0} = x_0 \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T}$$

with

$$\begin{aligned}
\Lambda_T &= \int_0^T \lambda^*(s)ds = \lambda T + \sum_{i=1}^{N_T}\int_0^T \mu(s-T_i)1_{\{T_i\leq s\}}ds \\
&= \lambda T + \sum_{i=1}^{N_T}\int_{T_i}^T \mu(s-T_i)ds = \lambda T + \sum_{i=1}^{N_T}\int_0^{T-T_i}\mu(s)ds \\
&= \lambda T + \sum_{i=1}^{N_T}\widehat{\mu}(T-T_i).
\end{aligned}$$

Thus the map $x_0 \longmapsto S_T^{x_0}$ is continuously differentiable and

$$\frac{\partial S_T^{x_0}}{\partial x_0} = \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T} = \frac{S_T^{x_0}}{x_0}.$$

To compute $DS_T^{x_0}$ we cannot use the results of Section 5.4.1 because the paramater of the SDE $f : (t,x) \longmapsto \alpha_t x$ is not deterministic. However we can directly compute $DS_T^{x_0}$.

Indeed we have

$$\begin{aligned}\mathcal{T}_\varepsilon \Lambda_T &= \Lambda_T \circ \mathcal{T}_\varepsilon \\ &= \lambda T + \sum_{i=1}^{N_T \circ \mathcal{T}_\varepsilon} \widehat{\mu}(T - T_i \circ \mathcal{T}_\varepsilon)\end{aligned}$$

where $N_T \circ \mathcal{T}_\varepsilon = N_T$ because $\tau_\varepsilon(T) = T$. Therefore

$$\begin{aligned}\frac{\mathcal{T}_\varepsilon \Lambda_T - \Lambda_T}{\varepsilon} &= \sum_{i=1}^{N_T} \frac{\widehat{\mu}(T - T_i \circ \mathcal{T}_\varepsilon) - \widehat{\mu}(T - T_i)}{\varepsilon} \\ &\underset{\varepsilon \to 0}{\overset{L^2(\Omega)}{\longrightarrow}} \sum_{i=1}^{N_T} (\widehat{\mu}(T - T_i))' D_m T_i = \sum_{i=1}^{N_T} \mu(T - T_i)\widehat{m}(T_i).\end{aligned}$$

Thus $\Lambda_T \in \mathbb{D}_m^0$ and

$$D_m \Lambda_T = \sum_{i=1}^{N_T} \mu(T - T_i)\widehat{m}(T_i) = \int_{(0,T]} \mu(T - t)\widehat{m}(t) dN_t.$$

Since $D_m N_T = 0$ (see Remark 5.2.9), we get, by chain rule, $S_T^{x_0} \in \mathbb{D}_m^{1,2}$ and

$$D_m S_T^{x_0} = -\sigma S_T^{x_0} D_m \Lambda_T = -\sigma S_T^{x_0} \sum_{i=1}^{N_T} \mu(T - T_i)\widehat{m}(T_i) = -\sigma S_T^{x_0} \int_{(0,T]} \mu(T - t)\widehat{m}(t) dN_t.$$

Thus we have to choose a function $m \in \mathcal{H}$ such that, for any $t \in [0,T], \widehat{m}(t) = 0$ if and only if $t \in \{0, T\}$. In this case we get

$$1_{\{N_T > 0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} = -\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T} \mu(T - T_i)\widehat{m}(T_i)} = -\frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T - t)\widehat{m}(t) dN_t}.$$

Finally, according to Remark 5.2.5, $m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} \in \mathrm{Dom}(\delta)$ because we have $m \in$

$\mathcal{H}$ and $1_{\{N_T>0\}}\frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} \in \mathbb{D}_m^{1,2}$, and

$$\begin{aligned}
&\delta\left(m1_{\{N_T>0\}}\frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\\
&= -\delta(m)\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)} + D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)}\right)\\
&= -\delta(m)\frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]}\mu(T-t)\widehat{m}(t)dN_t} + D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]}\mu(T-t)\widehat{m}(t)dN_t}\right)
\end{aligned}$$

with, on $\{N_T > 0\}$,

$$\begin{aligned}
&D_m\left(\frac{1_{\{N_T>0\}}}{\sigma x_0 \sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)}\right)\\
&= -\frac{\sigma x_0 \sum_{i=1}^{N_T} D_m(\mu(T-T_i)\widehat{m}(T_i))}{\sigma^2 x_0^2\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2}\\
&= -\frac{\sum_{i=1}^{N_T}[\widehat{m}(T_i)D_m(\mu(T-T_i)) + \mu(T-T_i)D_m\widehat{m}(T_i)]}{\sigma x_0\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2}\\
&= \frac{\sum_{i=1}^{N_T}\widehat{m}(T_i)\mu'(T-T_i)D_m T_i}{\sigma x_0\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2} - \frac{\sum_{i=1}^{N_T}\mu(T-T_i)m(T_i)D_m T_i}{\sigma x_0\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2}\\
&= -\frac{\sum_{i=1}^{N_T}\mu'(T-T_i)\widehat{m}(T_i)^2}{\sigma x_0\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2} + \frac{\sum_{i=1}^{N_T}\mu(T-T_i)m(T_i)\widehat{m}(T_i)}{\sigma x_0\left(\sum_{i=1}^{N_T}\mu(T-T_i)\widehat{m}(T_i)\right)^2}\\
&= -\frac{\int_{(0,T]}\mu'(T-s)\widehat{m}(s)^2 dN_s}{\sigma x_0\left(\int_{(0,T]}\mu(T-s)\widehat{m}(s)dN_s\right)^2} + \frac{\int_{(0,T]}\mu(T-s)m(s)\widehat{m}(s)dN_s}{\sigma x_0\left(\int_{(0,T]}\mu(T-s)\widehat{m}(s)dN_s\right)^2},
\end{aligned}$$

and, using Remark 5.2.5,

$$\begin{aligned}
\delta(m) &= \int_{(0,T]} (\psi(m,s) + \widehat{m}(s)\mu(T-s) + m(s))dN_s \\
&= \sum_{j=1}^{N_T} (\psi(m,T_j) + \widehat{m}(T-T_j) + m(T_j)) \\
&= \sum_{j=1}^{N_T} \left(\frac{1}{\lambda^*(T_j)} \int_{(0,T_j)} (\widehat{m}(T_j) - \widehat{m}(t))\mu'(T_j - t)dN_t + \widehat{m}(T-T_j) + m(T_j) \right) \\
&= \sum_{j=1}^{N_T} \left(\frac{\sum_{i=1}^{j-1} (\widehat{m}(T_j) - \widehat{m}(T_i))\mu'(T_j - T_i)}{\lambda + \sum_{i=1}^{j-1} \mu(T_j - T_i)} + \widehat{m}(T-T_j) + m(T_j) \right).
\end{aligned}$$

Thus we deduce the following expression of the Delta:

Proposition 5.4.17. *For any $f \in \mathcal{L}$, we have*

$$\begin{aligned}
&\frac{\partial}{\partial x_0} \mathbb{E}[1_{\{N_T \geq 1\}} f(S_T)] \\
&= \mathbb{E}\left[f(S_T^{x_0}) \delta \left(m 1_{\{N_T > 0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}} \right) \right] \\
&= -\mathbb{E}\left[\frac{f(S_T^{x_0})\delta(m) 1_{\{N_T>0\}}}{\sigma x_0 \int_{(0,T]} \mu(T-t)\widehat{m}(t)dN_t} \right] - \mathbb{E}\left[\frac{f(S_T^{x_0}) \int_{(0,T]} \mu'(T-s)\widehat{m}(s)^2 dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s \right)^2} 1_{\{N_T>0\}} \right] \\
&\quad + \mathbb{E}\left[\frac{f(S_T^{x_0}) \int_{(0,T]} \mu(T-s)m(s)\widehat{m}(s)dN_s}{\sigma x_0 \left(\int_{(0,T]} \mu(T-s)\widehat{m}(s)dN_s \right)^2} 1_{\{N_T>0\}} \right].
\end{aligned}$$

Remark 5.4.18. *Any term in the expression of Δ can be written from the Hawkes process*

N, the jump instants $(T_i)_{i\in\mathbb{N}^*}$ and the parameters $T, \lambda, \mu, \widehat{\mu}, \mu', m, \widehat{m}$ and f:

$$\begin{aligned}
S_T^{x_0} &= x_0 \exp\left(rT - \sigma\Lambda_T\right)(1+\sigma)^{N_T},\\
\Lambda_T &= \lambda T + \sum_{i=1}^{N_T} \widehat{\mu}(T-T_i),\\
\delta(m) &= \sum_{j=1}^{N_T}\left(\frac{\sum_{i=1}^{j-1}(\widehat{m}(T_j)-\widehat{m}(T_i))\mu'(T_j-T_i)}{\lambda+\sum_{i=1}^{j-1}\mu(T_j-T_i)}\right.\\
&\quad \left. +\widehat{m}(T-T_j)+m(T_j)\right),\\
\int_{(0,T]} \mu(T-t)\widehat{m}(t)dN_t &= \sum_{i=1}^{N_T} \mu(T-T_i)\widehat{m}(T_i)\\
\int_{(0,T]} \mu'(T-t)\widehat{m}(t)^2 dN_t &= \sum_{i=1}^{N_T} \mu'(T-T_i)\widehat{m}(T_i)^2\\
\int_{(0,T]} \mu(T-t)m(t)\widehat{m}(t)dN_t &= \sum_{i=1}^{N_T} \mu(T-T_i)m(T_i)\widehat{m}(T_i).
\end{aligned}$$

In other words, if we simulate a sample of Hawkes process then we can approach Δ conditionally to $\{N_T > 0\}$.

Remark 5.4.19. *On $\{N_T = 0\}$, the process S^{x_0} is deterministic and we have to know the derivative of the function f to compute Δ.*

Remark 5.4.20. *For the other Greeks we can notice that*

$$\begin{aligned}
\frac{\partial^2 S_T^{x_0}}{\partial x_0^2} &= 0,\\
\frac{\partial S_T^r}{\partial r} &= T S_T^{x_0}\\
\frac{\partial S_T^\sigma}{\partial \sigma} &= \left(-\Lambda_T + \frac{N_T}{1+\sigma}\right) S_T^{x_0}.
\end{aligned}$$

Then we can deduce similar expressions of the other Greeks conditionally to $\{N_T > 0\}$.

For $\Gamma = \dfrac{\partial^2 C}{\partial x_0^2}$ *we can start by writing*

$$\begin{aligned}
G^{x_0} &= m 1_{\{N_T>0\}} \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}, \\
\frac{\partial^2}{\partial x_0^2}\mathbb{E}[1_{\{N_T>0\}} f(S_T^{x_0})] &= \frac{\partial}{\partial x_0}\mathbb{E}\left[f(S_T^{x_0})\delta\left(G^{x_0}\right)\right] \\
&= \mathbb{E}\left[f(S_T^{x_0})\delta\left(G^x m \frac{\frac{\partial S_T^{x_0}}{\partial x_0}}{D_m S_T^{x_0}}\right)\right] + \mathbb{E}\left[f(S_T^{x_0})\frac{\partial}{\partial x_0}\delta\left(G^{x_0}\right)\right]
\end{aligned}$$

where we apply two times Proposition 5.4.16 with different processes G.

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COLLEGE DOCTORAL PAYS DE LA LOIRE | ED MATHEMATIQUES STIC

Le Mans Université

Titre : Calculs de Malliavin et applications aux équations différentielles stochastiques.

Mot clés : Equations différentielles stochastiques rétrogrades, calcul de Malliavin, équations intégro-différentielles, processus de Hawkes, calcul de Greeks, singularité terminale

Résumé : Cette thèse est consacrée à l'étude d'équations différentielles stochastiques grâce aux calculs de Malliavin. Dans le premier chapitre, nous étudions le comportement de la solution minimale d'une équation différentielle stochastique rétrograde (EDSR) à valeur terminale singulière. Le calcul de Malliavin appliqué aux EDSR permet de trouver une condition suffisante sur la croissance du générateur, pour obtenir la continuité de la solution à l'instant final. Puis dans le second chapitre est montré que la dérivée de Malliavin de cette solution possède aussi une singularité en l'instant terminal. Dans le troisième chapitre, nous reprenons la même équation que dans le premier chapitre en ajoutant des sauts et montrons que la présence de ces sauts peut empêcher la continuité de la solution. Le quatrième chapitre termine avec le développement d'un calcul de Malliavin local par rapport à un processus de Hawkes. Il est appliqué pour obtenir un critère de densité de la solution d'une équation différentielle stochastique et au calcul de Greeks.

Title: Malliavin calculus and applications in stochastic differential equations

Keywords: Backward stochastic differential equations, Malliavin calculus, integro-differential equation, Hawkes processes, Greeks computations, Final singularity

Abstract: This thesis is dedicated to the study of stochastic differential equations using Malliavin calculus. In the first chapter, the continuity at the terminal time of the minimal solution to a backward stochastic differential equation with a singular terminal condition is examined. Malliavin calculus is used to control the various terms that appear in the computations, specifically by using the relationship between the Malliavin derivative of the process of interest and the process adapted to the filtration of the motion. In the second chapter, it is shown that the Malliavin derivative of the solution exhibits a singularity at the terminal time. In the third chapter, we add a jump term to the same equation as in the first chapter and show that these jumps can prevent the continuity of the solution. The fourth chapter concludes with the development of a local Malliavin calculus with respect to a Hawkes process. This is applied to obtain a density criterion for the solution of a stochastic differential equation and for the computation of Greeks.