Motivic Homotopy Theory and Infinite Loop Spaces

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Introduction

This text is a report written for my master's second year internship from April to June 2024 with Joseph Ayoub, whom I warmly thank.

The purpose is to summarize most of what I studied in motivic homotopy theory, and possible directions for further work. The main (short-term) goal was to study [EHK+21] and the recognition principle for motivic infinite loop spaces. This theorem is a motivic analogue for the classical recognition principle due to Boardman and Vogt [BV68], stating that connective spectra are equivalent to grouplike $\mathcal{E}_\infty$-spaces. Here, connective spectra will become *very effective motivic spectra*, and grouplike $\mathcal{E}_\infty$-spaces will become a notion of "grouplike motivic spaces with transfers".

In this report, the term *space* will as usual denote ∞-groupoids (as in [Lur09b]). We are interested in *motivic spaces*, which are modeled as presheaves of spaces on some category of schemes. The base underlying scheme will firstly be a qcqs noetherian scheme S of finite Krull dimension to give the definitions in a general context, however the main theorems will have us choose $S = \operatorname{Spec} k$ for a perfect field k. For the sake of conciseness, the proofs of the main theorems will also assume that k is infinite.

Contents

Notations

The following notations are used throughout the text and to avoid any confusion, they mean the following :

- X_+ denotes X with an added disjoint base-point, whenever X is a scheme or a finite set ;
- C_* denotes the pointed version under the terminal object $C_{/*}$, whenever C is an ∞-category ;
- Sch_S denotes the category of S-schemes and Sm_S the category of finitely presented smooth S-schemes, whenever S is a base scheme ;
- $\mathrm{PSh}(C)$ denotes the ∞-category of presheaves of spaces on C, whenever C is an ∞-category ;
- $\mathrm{sPre}(C)$ denotes the (model) category of simplicial presheaves on C, whenever C is a category.

Also, $\mathrm{K}(C)$ denotes the K-theory space of a stable ∞-category C, satisfying the following defining properties :

1. $\mathrm{K}(0) \simeq *$;
2. If $C' \to C \to C''$ is a split exact sequence of stable ∞-categories, then the induced map of spaces
$$\mathrm{K}(C) \to \mathrm{K}(C') \times \mathrm{K}(C'')$$
is an equivalence ;
3. 1 and 2 induce an $\mathcal{E}_\infty$-space structure (cf. definition 1.14) on K, which is grouplike (cf. definition 1.15).

For a scheme X, the space $\mathrm{K}(X)$ denotes $\mathrm{K}(\mathbf{Perf}(X))$, where $\mathbf{Perf}(X)$ is the stable ∞-category of perfect complexes on X.

1 Motivic spaces and spectra

1.1 The ∞-category of motivic spaces

The construction of this ∞-category is due to Morel and Voevodsky. See [AE16] for further details.

Let S be a quasi-compact, quasi-separated noetherian base scheme of finite Krull dimension. Denote by Sm_S the category of finitely presented smooth S-schemes.

Definition 1.1. A finite family $\{p_i : V_i \to U\}_{i\in I}$ in Sm_S is a *Nisnevich cover* if each p_i is étale and for every $x \in U$, there is a $i \in I$ and a $y \in V_i$ such that $p_i(y) = x$ and $k(x) \to k(y)$ is an isomorphism. The associated Grothendieck topology on Sm_S is called the *Nisnevich topology.*

One can show that this topology lies between the étale and the Zariski topology on Sm_S, enjoying good properties of both topologies. In particular, Zariski covers are clearly Nisnevich covers. Another important property is that the Nisnevich topology is subcanonical (since it is coarser than the étale topology, which already is) : the representable presheaves are already Nisnevich sheaves. Lurie defines the Nisnevich topology in another equivalent way in [Lur18] (Definition 3.7.1.1, page 341) :

Definition 1.2. The Nisnevich topology on Sm_S is the topology generated by the finite families of étale morphisms $\{p_i : V_i \to U\}_{i\in I}$ satisfying the condition that there is a finite sequence

$$\emptyset \subseteq Z_n \subseteq Z_{n-1} \subseteq \ldots \subseteq Z_1 \subseteq Z_0 = U$$

of finitely presented closed subschemes of U such that for every $0 \leq m \leq n-1$, the morphism

$$\coprod_{i\in I} p_i^{-1}(Z_m - Z_{m+1}) \to Z_m - Z_{m+1}$$

admits a section.

Definition 1.3. A cartesian square in Sm_S

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ i\ } & X \end{array}$$

is an *elementary distinguished Nisnevich square* if :

- i is an open immersion ;
- p is étale ;
- $p^{-1}(X - U)_{\mathrm{red}} \to (X - U)_{\mathrm{red}}$ is an isomorphism of schemes.

With this notation, the family $\{i, p\}$ is a Nisnevich cover of X.

Example 1.4. If k is a field with characteristic different than 2 and $a \in k$ is a non-zero square, then one can Nisnevich-cover $\mathbb{A}^1$ with the open immersion $\mathbb{A}^1 - \{a\} \to \mathbb{A}^1$ and the étale $\mathbb{A}^1 - 0 \to \mathbb{A}^1$ sending x to x^2. This cover does not come from a distinguished Nisnevich square, since a has two square roots.

Definition 1.5. A presheaf of simplicial sets F on Sm_S *satisfies Nisnevich descent* (or *is Nisnevich-fibrant*) if it is a presheaf of spaces, $F(\emptyset)$ is contractible, and for every distinguished Nisnevich square as in definition 1.3, the canonical map

$$F(X) \to F(U) \times_{F(U \times_X V)} F(V)$$

is a weak equivalence. In other words, the distinguished square becomes homotopy cartesian after applying F.

The term Nisnevich-fibrant comes from the fact that the presheaves satisfying Nisnevich descent are precisely the fibrant objects in $\mathrm{L}_{\mathrm{Nis}}\,\mathrm{sPre}(\mathrm{Sm}_S)$, the Bousfield localization of the projective model structure on simplicial presheaves with respect to Nisnevich hypercovers.

In particular, one can *localize* a simplicial presheaf F via Nisnevich sheafification, yielding a universal morphism $F \to \mathrm{L}_{\mathrm{Nis}}F$ from F to a Nisnevich-fibrant simplicial sheaf.

Definition 1.6. A *motivic space over* S is a simplicial presheaf F on Sm_S that satisfies Nisnevich descent and is $\mathbb{A}^1$-homotopy invariant : for every $X \in \mathrm{Sm}_S$, the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence $F(X) \simeq F(X \times \mathbb{A}^1)$. Denote by $\mathbf{H}(S)$ the ∞-category of motivic spaces over S.

The notion of $\mathbb{A}^1$-homotopy invariance can also be defined by means of Bousfield localizations. Indeed, the motivic spaces are precisely the fibrant objects in the model category $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\,\mathrm{sPre}(\mathrm{Sm}_S)$ obtained as the Bousfield localization of $\mathrm{L}_{\mathrm{Nis}}\,\mathrm{sPre}(\mathrm{Sm}_S)$ with respect to the maps $\mathbb{A}^1 \times_S X \to X$.

Example 1.7. The multiplicative group $\mathbb{G}_m$ (say, over a field) is already a motivic space, without even having to $\mathbb{A}^1$-localize it. Indeed, one already has $\mathrm{Hom}(X, \mathbb{G}_m) \simeq \mathrm{Hom}(X \times \mathbb{A}^1, \mathbb{G}_m)$ since any morphism $\mathbb{A}^1 \to \mathbb{G}_m$ has to be constant. This property, known as $\mathbb{A}^1$-rigidity, will not be used here. One can consult subsection 4.4 in [AE16] for a few details.

In particular, one can (and usually has to) $\mathbb{A}^1$-localize a Nisnevich-fibrant simplicial presheaf to obtain a motivic space. In practice, this is hard to compute. However, one has the following useful interpretation. Denote by $\Delta^\bullet$ the cosimplicial scheme with

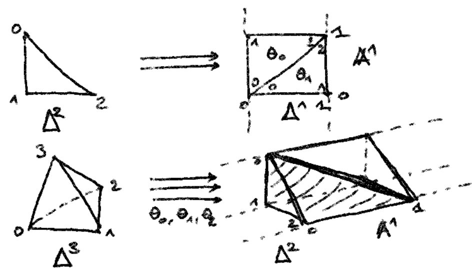
$$\Delta^n = \mathrm{Spec}\, k[x_0, \dots, x_n]/\left(x_0 + \cdots + x_n = 1\right)$$

and the usual faces and degeneracies. Define a functor $\mathrm{Sing}^{\mathbb{A}^1} : \mathrm{sPre}(\mathrm{Sm}_S) \to \mathrm{sPre}(\mathrm{Sm}_S)$ by letting

$$\mathrm{Sing}^{\mathbb{A}^1}(F) : X \mapsto |F(X \times \Delta^\bullet)|.$$

Theorem 1.8. *Let* $F \in \mathrm{sPre}(\mathrm{Sm}_S)$. *Then* $\mathrm{Sing}^{\mathbb{A}^1}(F)$ *is* $\mathbb{A}^1$*-invariant. Moreover, the functor* $\mathrm{L}_{\mathrm{mot}} = \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}$ *is naturally equivalent to the countable composition power* $\left(\mathrm{L}_{\mathrm{Nis}}\,\mathrm{Sing}^{\mathbb{A}^1}\right)^{\circ\mathbb{N}}$.

Proof. For $n \in \mathbb{N}$ and $0 \leq i \leq n$, define $\theta_i : \Delta^{n+1} \to \Delta^n \times_S \mathbb{A}^1$ sending the vertex v_j to $v_j \times \{0\}$ for $j \leq i$ and to $v_{j-1} \times \{1\}$ for $j > i$. We obtain a simplicial decomposition of $\Delta^n \times_S \mathbb{A}^1$:



For $X \in \mathrm{Sm}_S$, one obtains a diagram

$$\begin{array}{ccccc}
\cdots \Lleftarrow \Delta^2 \times_S X & \Lleftarrow & \Delta^1 \times_S X & \Lleftarrow & X \\
{\scriptstyle 0}\downdownarrows{\scriptstyle 1} & \searrow^{\theta_i \times_S X} & {\scriptstyle 0}\downdownarrows{\scriptstyle 1} & \searrow^{\theta_i \times_S X} & \\
\cdots \Lleftarrow \Delta^2 \times_S \mathbb{A}^1 \times_S X & \Lleftarrow & \Delta^1 \times_S \mathbb{A}^1 \times_S X & \Lleftarrow & \mathbb{A}^1 \times_S X
\end{array}$$

Apply the presheaf F to get a simplicial homotopy between the maps

$$F(0), F(1) : \mathrm{Sing}^{\mathbb{A}^1}(F)(X \times_S \mathbb{A}^1) \to \mathrm{Sing}^{\mathbb{A}^1}(F)(X).$$

Denoting $\mu : \mathbb{A}^1 \times_S \mathbb{A}^1 \to \mathbb{A}^1$ the multiplication $(x, y) \mapsto xy$, the following diagram then commutes :

$$\begin{array}{ccccc}
 & & F(X \times_S \mathbb{A}^1) & \xrightarrow{F(0)} & F(X) \\
 & \swarrow^{\mathrm{id}} & \downarrow{\scriptstyle F(\mathrm{id}_X \times \mu)} & & \downarrow{\scriptstyle F(\mathrm{pr}_X)} \\
F(X \times_S \mathbb{A}^1) & \xleftarrow{F(1 \times \mathrm{id}_{\mathbb{A}^1})} & F(X \times_S \mathbb{A}^1 \times_S \mathbb{A}^1) & \xrightarrow{F(0 \times \mathrm{id}_{\mathbb{A}^1})} & F(X \times_S \mathbb{A}^1)
\end{array}$$

Since the two bottom maps are homotopic, the composite $F(\mathrm{pr}_X) \circ F(0)$ is homotopic to the identity, and since pr_X has a section, $F(\mathrm{pr}_X)$ is an equivalence.

The last statement is rather technical. Writing $\Phi = \mathrm{L}_{\mathrm{Nis}} \mathrm{Sing}^{\mathbb{A}^1}$, for any simplicial presheaf F, the countable composition

$$\Phi^{\circ \mathbb{N}}(F) \simeq \mathrm{hocolim}_n \left(\mathrm{L}_{\mathrm{Nis}} \mathrm{Sing}^{\mathbb{A}^1} \right)^{\circ n} (F) \simeq \mathrm{hocolim}_n \left(\mathrm{Sing}^{\mathbb{A}^1} \mathrm{L}_{\mathrm{Nis}} \right)^{\circ n} (F)$$

is both a filtered homotopy colimit of Nisnevich local presheaves of spaces, and a homotopy colimit of $\mathbb{A}^1$-invariant presheaves. Hence the functor $\Phi^{\circ \mathbb{N}}$ takes values in motivic spaces. To conclude, it suffices to show that Φ preserves $\mathbb{A}^1$-local weak equivalences, from which will follow the right-hand equivalence in :

$$\Phi^{\circ\mathbb{N}}(F) \simeq \Phi^{\circ\mathbb{N}}(\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}F) \simeq \mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}F.$$

See [AE16], theorem 4.27 for the remaining details. □

Example 1.9. Once $\mathbb{A}^1$-localized, the affine line $\mathbb{A}^1$ evidently becomes contractible - this is the main purpose of $\mathbb{A}^1$-localization here. The projection $\mathbb{A}^1 \simeq \mathbb{A}^1 \times_S * \to *$ is among the morphisms which became weak equivalences after the Bousfield localization $\mathrm{L}_{\mathbb{A}^1}$.

Example 1.10. As in classical homotopy theory, one can construct spaces as homotopy cofibers of pointed spaces, that is, "topological quotients". The most important in our study is the *Tate sphere*, which is the homotopy cofiber $\mathbb{T} = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ of the open inclusion $\mathbb{A}^1 - 0 \subseteq \mathbb{A}^1$, seen as a morphism of pointed motivic spaces (at 1). Recall that by definition, this is the homotopy pushout of the diagram $* \leftarrow \mathbb{A}^1 - 0 \to \mathbb{A}^1$.

Proposition 1.11. *If the base scheme S is a noetherian scheme of finite Krull dimension, then any elementary distinguished Nisnevich square (as in definition 1.3) in* Sm_S*, seen as a diagram of simplicial presheaves, is a homotopy pushout in* $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\,\mathrm{sPre}(\mathrm{Sm}_S)$.

Proof. Recall from earlier that the Nisnevich topology is coarser than the étale topology, so it is subcanonical. Hence, an elementary distinguished Nisnevich square can indeed be seen without modification as a diagram in $\mathrm{L}_{\mathbb{A}^1}\mathrm{L}_{\mathrm{Nis}}\,\mathrm{sPre}(\mathrm{Sm}_S)$. For any motivic space X, its representable presheaf is a Nisnevich sheaf, so it satisfies Nisnevich descent : after applying it, the square becomes homotopy cartesian. Since it is true for all spaces X, the original square is indeed a homotopy pushout. □

Corollary 1.12. *The open (Zariski, hence Nisnevich) covering of the projective line by two copies of the affine line (say, pointed at* 1*) determines the following distinguished square :*

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1 \end{array}$$

Since $\mathbb{A}^1$ is contractible, this exhibits $\mathbb{P}^1$ as the homotopy cofiber $\mathbb{A}^1/\mathbb{G}_m$. This in turns is equivalent to the Tate sphere, as $\mathbb{G}_m \simeq \mathbb{A}^1 - 0$.

Remark 1.13. *More generally, one can define a collection of bigraded* motivic spheres $S^{a,b} = \mathbb{G}_m^{\wedge b} \wedge (S^1)^{\wedge(a-b)}$ *for any pair of integers $a \geq b \geq 0$ and prove that $S^{2n-1,n}$ is weakly equivalent to $\mathbb{A}^n - 0$. The case of the multiplicative group is $n = 1$, and $S^{2,1}$ is the Tate sphere.*

The equivalence $\mathbb{T} \simeq S^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$ will be used implicitely from now on. Motivic spectra will be introduced to formally invert the operation $- \wedge \mathbb{P}^1$, which will automatically invert smashing with S^1 or $\mathbb{G}_m$.

1.2 Unframed recognition

To finish this section, we study the *unframed* version of the main recognition theorem. Let $\mathbf{SH}^{S^1}(S) = \mathrm{Stab}(\mathbf{H}(S)_*) = \mathbf{H}(S)_* \otimes \mathrm{Spt}$ be the usual ∞-categorical stabilization of $\mathbf{H}(S)_*$ (see [Lur17] for more details). An object of this ∞-category is an S^1-motivic spectrum : a sequence $(X_n)_{n\geq 0}$ of pointed motivic spaces equipped with equivalences $X_n \simeq \Omega_{S^1} X_{n+1}$, where the functor $\Omega_{S^1} = \mathrm{Hom}_*(S^1, -)$ is the usual loop space functor, right adjoint to the suspension functor $\Sigma_{S^1} = S^1 \wedge -$.

Definition 1.14. Let C be an ∞-category with finite products. A *commutative monoid* or $\mathcal{E}_\infty$*-object* in C is a functor $X : \mathrm{Fin}_* \to C$ from the category of finite pointed sets such that for every $n \geq 0$, the maps $\{1, \dots, n\}_+ \to \{i\}_+$ induce together an equivalence :

$$X(\{1, \dots, n\}_+) \simeq \prod_{i=1}^{n} X(\{i\}_+).$$

A *symmetric monoidal* ∞*-category* is an $\mathcal{E}_\infty$-object in the ∞-category of ∞-categories.

In [Lur17] the *Bar construction* is defined, providing n-fold deloopings, over any (presentable) ∞-category C, of $\mathcal{E}_n$-objects[1]. The details of the construction are of little interest here, the important result being that it provides left adjoints $\mathbb{B}^n_C \dashv \Omega^n$ and $\mathbb{B}^\infty_C \dashv \Omega^\infty$ when one considers the target ∞-category of Ω^n (resp. Ω^∞) to be that of $\mathcal{E}_n$-objects (resp. $\mathcal{E}_\infty$-objects).

In this subsection and thereafter, we will write $\mathbb{B}$ (resp. $\mathbb{B}_{\mathrm{Nis}}$ and $\mathbb{B}_{\mathrm{mot}}$) for the Bar construction over $\mathrm{PSh}(\mathrm{Sm}_S)$ (resp. $\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ and $\mathbf{H}(S)_*$). When the compositions make sense, the Bar constructions commute with the localizations $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\mathrm{mot}}$.

Definition 1.15. A monoid $X \in \mathrm{Mon}(C)$ is *grouplike* if the two shearing maps $X \times X \to X \times X$ are equivalences. In the context of Nisnevich sheaves, this is equivalent to $\pi_0^{\mathrm{Nis}}(X)$ being a sheaf of groups, as the intuition would suggest.

A monoid $X \in \mathrm{Mon}(\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S))$ is *strongly* $\mathbb{A}^1$*-invariant* if both X and $\mathbb{B}_{\mathrm{Nis}} X$ are $\mathbb{A}^1$-invariant. Denote by $\mathrm{Mon}_{\mathrm{mot}}(\mathbf{H}(S))$ the corresponding full sub-∞-category of $\mathrm{Mon}(\mathbf{H}(S))$.

A commutative monoid $X \in \mathrm{CMon}(\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S))$ is *strictly* $\mathbb{A}^1$*-invariant* if all the deloopings $\mathbb{B}^n_{\mathrm{Nis}} X$ are $\mathbb{A}^1$-invariant. Denote by $\mathrm{CMon}_{\mathrm{mot}}(\mathbf{H}(S))$ the corresponding full sub-∞-category of $\mathrm{CMon}(\mathbf{H}(S))$.

Proposition 1.16. *The adjunction*

$$\mathbb{B}^\infty_{\mathrm{mot}} : \mathrm{CMon}(\mathbf{H}(S)) \rightleftarrows \mathbf{SH}^{S^1}(S) : \Omega^\infty_{S^1}$$

restricts to an equivalence between grouplike and strictly $\mathbb{A}^1$*-invariant motivic spaces and connective motivic* S^1*-spectra :*

$$\mathrm{CMon}^{\mathrm{gp}}_{\mathrm{mot}}(\mathbf{H}(S)) \simeq \mathbf{SH}^{S^1}(S)_{\geq 0}.$$

[1] The definition of an $\mathcal{E}_n$-object will not be given here since it will not be used.

Proof. In the following commutative diagram, the vertical inclusions are those of $\mathbb{A}^1$-invariant objects. Both rows are limit diagrams of ∞-categories.

$$\begin{array}{ccccccc}
\mathbf{SH}^{S^1}(S) & \longrightarrow & \dots & \xrightarrow{\Omega} & \mathbf{H}(S)_* & \xrightarrow{\Omega} & \mathbf{H}(S)_* \\
\big\downarrow & & & & \big\downarrow & & \big\downarrow \\
\mathrm{Stab}(\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)) & \longrightarrow & \dots & \xrightarrow{\Omega} & \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)_* & \xrightarrow{\Omega} & \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*
\end{array}$$

The spectra in $\mathbf{SH}^{S^1}(S)$ that are connective are then exactly the connective spectra $X \in \mathrm{Stab}(\mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S))_{\geq 0}$ such that every component $\Omega^{\infty-n}X \in \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$ is $\mathbb{A}^1$-invariant. These components being the deloopings of the $\mathcal{E}_\infty$-object $\Omega^\infty X \in \mathrm{PSh}_{\mathrm{Nis}}(\mathrm{Sm}_S)_*$, this is the definition of $\Omega^\infty X$ being in $\mathrm{CMon}^{\mathrm{gp}}_{\mathrm{mot}}(\mathbf{H}(S))$. □

When S is the spectrum of a perfect field, there is an explicit description of both sides of this equivalence, whose proof (mostly due to Morel) will not be discussed here :

Corollary 1.17. *Let k be a perfect field. Then the above adjunction*

$$\mathbb{B}^\infty_{\mathrm{mot}} : \mathrm{CMon}(\mathbf{H}(k)) \rightleftarrows \mathbf{SH}^{S^1}(k) : \Omega^\infty_{S^1}$$

restricts to an equivalence between :

- $\mathrm{CMon}^{\mathrm{gp}}_{\mathrm{mot}}(\mathbf{H}(k))$ *which consists of the commutative monoids X in $\mathbf{H}(k)$ such that $\pi_0^{\mathrm{Nis}}(X)$ is a strongly $\mathbb{A}^1$-invariant sheaf of groups, and*
- $\mathbf{SH}^{S^1}(k)_{\geq 0}$ *is the smallest full sub-∞-category of $\mathbf{SH}^{S^1}(k)$ which is stable by taking homotopy colimits and which contains the suspension spectra $\Sigma^\infty_{S^1} X_+$ for any smooth k-scheme X.*

Remark 1.18. *The usual recognition theorem from classical homotopy theory states that* over ∞-toposes, *in particular over the ∞-category of spaces, the Bar construction and the loop space functors restrict to equivalences between grouplike $\mathcal{E}_n$-algebras (resp. grouplike commutative monoids) and pointed n-connective objects (resp. pointed connective spectra). This theorem could not apply to $\mathbf{H}(S)$ which is not an ∞-topos, and indeed one has to restrict further to $\mathbb{A}^1$-invariant objects to obtain the desired equivalence.*

Remark 1.19. *The assumption that $S = \operatorname{Spec} k$ with k a perfect field will come up again in later theorems. The reason why k has to be perfect comes from the fact (see [MVW06], 13.8) that in this case, for all homotopy invariant presheaves with transfers F, the cohomology presheaves $H^n(-, F_{\mathrm{Nis}})$ are also homotopy invariant for $n \geq 0$. This in turn is a consequence of the fact that if k is a perfect field, then every regular local k-algebra is formally smooth over k.*

2 Framed correspondences

Following Voevodsky and [EHK+21], we introduce the notion of framed correspondences : they form an ∞-categorical version of the classical notion of correspondences. As presheaves with transfers are a central object of study in Voevodsky's theory of motives, motivic spaces with framed transfers will play a central role in the study of motivic spectra.

2.1 Three types of framing

Definition 2.1. Let X and Y be S-schemes and $n \geq 0$. An *equationally framed correspondence of level* n from X to Y is the data (Z, U, φ, g) of :

1. A closed subscheme $Z \subseteq \mathbb{A}^n_X$ which is finite over X ;
2. An étale neighborhood $U \supset Z$ in $\mathbb{A}^n_X$;
3. A morphism $\varphi : U \to \mathbb{A}^n$ such that $\varphi^{-1}(0) = Z$;
4. A morphism of S-algebraic spaces $g : U \to Y$.

Two such correspondences (Z, U, φ, g) and (Z', U', φ', g') are *equivalent* if $Z = Z'$, and if there is an étale neighborhood of it refining both U and U', on which φ and φ' agree, as well as g and g'.

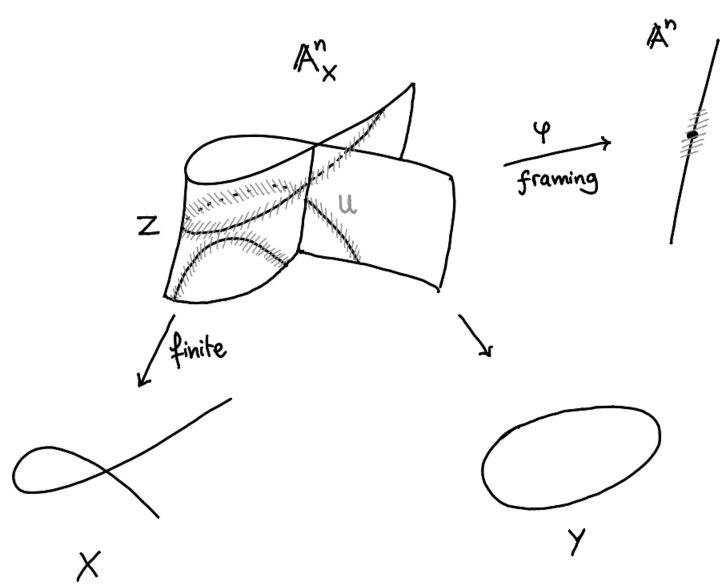


Figure 1: An equationally framed correspondence of level 1

Voevodsky proved that there is a natural bijection :

$$\mathrm{Corr}_S^{\mathrm{efr},n}(X,Y) \simeq \mathrm{Maps}(X_+ \wedge (\mathbb{P}^1)^{\wedge n}, \mathrm{L}_{\mathrm{Nis}}(Y_+ \wedge \mathbb{T}^{\wedge n}))$$

realizing the set of equivalence classes of equationally framed correspondences of level n from X to Y as a mapping space between pointed presheaves of spaces.

Definition 2.2. For each $n \geq 0$, define the *suspension map* as a natural transformation :

$$\sigma_{X,Y} : \mathrm{Corr}_S^{\mathrm{efr},n}(X,Y) \to \mathrm{Corr}_S^{\mathrm{efr},n+1}(X,Y)$$

by sending (Z, U, φ, g) to $(Z \times \{0\}, U \times \mathbb{A}^1, \varphi \times \mathrm{id}_{\mathbb{A}^1}, g \circ \mathrm{pr}_U)$. In other words, one can add trivial additional dimensions. The set of *equationally framed correspondences* from X to Y is the colimit

$$\mathrm{Corr}_S^{\mathrm{efr}}(X,Y) = \mathrm{colim}(\mathrm{Corr}_S^{\mathrm{efr},0}(X,Y) \xrightarrow{\sigma} \mathrm{Corr}_S^{\mathrm{efr},1}(X,Y) \xrightarrow{\sigma} \dots)$$

Denote by $h_S^{\mathrm{efr},n}(Y)$ and $h_S^{\mathrm{efr}}(Y)$ the presheaves $\mathrm{Corr}_S^{\mathrm{efr},n}(-,Y)$ and $\mathrm{Corr}_S^{\mathrm{efr}}(-,Y)$ respectively.

Definition 2.3. A presheaf F on Sch_S *satisfies closed gluing* if it sends pushout squares of closed immersions to pullback squares, and if $F(\emptyset) \simeq *$.

One can geometrically realize nonsingular simplicial sets K as S-schemes $|K|_S$ (instead of CW-complexes), using the cosimplicial scheme $\Delta^\bullet$ from the first section. This is done by gluing the affine cells along closed immersions of the faces.

Lemma 2.4. *Let F be a presheaf on Sch_S satisfying closed gluing and let K be a finite nonsingular simplicial set. Then the natural map*

$$F(- \times_S |K|_S) \to \mathrm{Maps}(K, F(- \times \Delta^\bullet))$$

is an equivalence.

Proof. The presheaf F satisfies closed gluing and $|K|_S$ is inductively constructed by a sequence of pushouts of closed immersions. □

Denote by $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)$ the full sub-∞-category of $\mathrm{PSh}(\mathrm{Sch}_S)$ spanned by the presheaves which transform finite coproducts into finite products.

Proposition 2.5. *Let Y be an S-scheme, and h be either $h_S^{\mathrm{efr}}(Y)$ or $h_S^{\mathrm{efr},n}(Y)$ for some $n \geq 0$.*

1. *$h \in \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)$;*

2. *h satisfies closed gluing ;*

3. *For τ the Nisnevich or the étale topology, if $R \hookrightarrow Y$ is a τ-covering sieve generated by a single map then $h(R) \to h(Y)$ is a τ-local equivalence.*

Remark 2.6. *Even though $h_S^{\mathrm{efr}}(Y)$ transforms finite disjoint unions into finite products, this is not the case of h_S^{efr} itself on-the-nose, and there are no reasons to believe it could be. In subsection 2.2, it will be proved that it is the case once $\mathbb{A}^1$-localized.*

One can easily extend the definition of $\mathrm{Corr}_S^{\mathrm{efr},n}(-,-)$ and $\mathrm{Corr}_S^{\mathrm{efr}}(-,-)$ to make sense (and still be functorial) on Sch_{S+}, the full subcategory of $(\mathrm{Sch}_S)_*$ spanned by the pointed S-schemes of the form X_+. The resulting functors $h_S^{\mathrm{efr},n}(Y)$ and $h_S^{\mathrm{efr}}(Y)$ can then be extended from Sch_{S+} to functors $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_* \to \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$ by asking that they preserve sifted colimits.

Definition 2.7 (Linear structure)**.** Let X and Y be S-schemes and $n \geq 0$. Then there is a natural pointed map

$$X_+ \wedge h_S^{\mathrm{efr},n}(Y) \to h_S^{\mathrm{efr},n}(X \times_S Y)$$

sending a map $a : Z \to X$ and an equationally framed correspondence (W, U, φ, g) to the correspondence $(W, U, \varphi, (a \times g) \circ \gamma)$ where $\gamma : U \to Z \times_S U$ is the graph of $U \to Z$.

This construction clearly commutes with suspension (drawing the situation suffices to convince oneself), so at the colimit $n \to +\infty$ this induces a natural pointed map

$$X_+ \wedge h_S^{\mathrm{efr}}(Y) \to h_S^{\mathrm{efr}}(X \times_S Y).$$

These Sch_{S+}-linear structures on the original $h_S^{\mathrm{efr},n}$ and h_S^{efr} can be extended as before to $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$-linear structures on the extensions of these presheaves discussed above.

This linearity agrees with the natural maps $F \wedge G \to h_S^{\mathrm{efr}}(F \wedge G)$ coming from the fact that $h_S^{\mathrm{efr},0}(Y)$ is the presheaf represented by Y_+.

Proposition 2.8. *Let X and Y be S-schemes, and (Z, U, φ, g) be an equationally framed correspondence of level n from X to Y. Denote by $i : Z \to \mathbb{A}^n_X$ the inclusion.*

1. *The conormal sheaf $\mathcal{N}_i^\vee$ is free of rank n ;*
2. *i is (Koszul-)regular ;*
3. *$Z \to X$ is syntomic ;*
4. *If X is affine, then $Z \to X$ is a relative global complete intersection.*

Proof. Let $j : Z \to U$ be the inclusion, which becomes i after composing with $U \to \mathbb{A}^n_X$. Since the latter is étale, i and j have the same conormal sheaves. Since $Z = \varphi^{-1}(0)$, we have a cartesian square :

$$\begin{array}{ccc} Z & \overset{j}{\hookrightarrow} & U \\ \downarrow & & \downarrow{\scriptstyle \varphi} \\ \{0\} & \hookrightarrow & \mathbb{A}^n_X \end{array}$$

which by lemmas 29.31.2 and 29.31.3 in Stacks, 01R1, induces a surjective map at the level of conormal sheaves, $\mathcal{O}_Z^n \to \mathcal{N}_j^\vee$. By [Vak17], proposition 21.2.16, the regularity of i will ensure that $\mathcal{N}_i^\vee$ is locally free of rank n, which together with the above surjection will prove 1.

It remains to show statements 2 - 4. Since 2 and 3 are local on X, assume that it is affine, so that U may also be assumed affine. One deduces with Nakayama's lemma that $Z \to X$ is a relative global complete intersection, which proves statements 2 - 4 at once (see Stacks, 00SW and 069G). □

As seen in the proof, in an equationally framed correspondence the components of φ induce an explicit isomorphism of the conormal sheaf $\mathcal{N}_i^\vee$ with $\mathcal{O}_Z^n$. Also, it has been seen that the finite morphism $Z \to X$ is automatically syntomic. One can then forget the heavy datum of U and φ to obtain the lighter notion of *normally* framed correspondences.

Definition 2.9. Let X and Y be S-schemes and $n \geq 0$. A *normally framed correspondence of level* n from X to Y is the data (Z, f, h, i, τ) of :

1. A span $X \xleftarrow{f} Z \xrightarrow{h}$ with f finite syntomic ;
2. A closed immersion $i : Z \to \mathbb{A}^n_X$ over X ;
3. A trivialization $\tau : \mathcal{O}_Z^n \to \mathcal{N}_i^\vee$ of the conormal sheaf of i.

Even though this notion is much lighter to handle than equationally framed correspondences, they will turn out to be motivically equivalent. As before, one can define the functors $\mathrm{Corr}_S^{\mathrm{nfr},n}$ and $h_S^{\mathrm{nfr},n}$, and the suspension morphism is even easier to define, since it amounts to composing the closed immersion i with $\mathbb{A}^n_X \to \mathbb{A}^{n+1}_X, x \mapsto (x, 0)$. Define $\mathrm{Corr}_S^{\mathrm{nfr}}$ and h_S^{nfr} as before, by taking the colimit with respect to these suspension morphisms.

This proposition is the exact same as 2.5, with the equational framing replaced by a normal framing.

Proposition 2.10. *Let Y be an S-scheme, and h be either $h_S^{\mathrm{nfr}}(Y)$ or $h_S^{\mathrm{nfr},n}(Y)$ for some $n \geq 0$.*

1. *$h \in \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)$;*
2. *h satisfies closed gluing ;*
3. *For τ the Nisnevich or the étale topology, if $R \hookrightarrow Y$ is a τ-covering sieve generated by a single map then $h(R) \to h(Y)$ is a τ-local equivalence.*

The discussion inside and after proposition 2.8 show that equationally framings are special cases of normal framings. We thus obtain natural maps $\mathrm{Corr}_S^{\mathrm{efr},n} \to \mathrm{Corr}_S^{\mathrm{nfr},n}$ commuting with suspension (again, this is obvious) and inducing a natural map

$$\mathrm{Corr}_S^{\mathrm{efr}} \to \mathrm{Corr}_S^{\mathrm{nfr}} .$$

Similarly, one easily defines an Sch_{S+}-linear structure on h_S^{nfr} and a way to extend this functor to $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$ by asking it to preserve sifted colimits. As before, this extension inherits a $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$-linear structure.

One can now make the data of a normal framing even lighter by dropping the closed immersion i. Indeed, f being finite syntomic in definition 2.9 ensures that its cotangent complex $\mathcal{L}_f$ is canonically equivalent to the complex $\mathcal{N}_i^\vee \to \mathcal{O}_Z^n$. The additional datum of the trivialization τ of $\mathcal{N}_i^\vee$ gives a specific path $\mathcal{L}_f \simeq 0$ in the K-theory space of Z. One can now forget about i and τ and only remember this path. Notice that n does not appear anymore.

Definition 2.11. Let X and Y be S-schemes. A *framed correspondence* from X to Y is the data (Z, f, h, τ) of :

1. A span $X \xleftarrow{f} Z \xrightarrow{h}$ with f finite syntomic ;
2. A trivialization $\tau : 0 \simeq \mathcal{L}_f$ in $\mathrm{K}(Z)$.

In addition to being a somewhat lighter notion to handle than normally framed correspondences, the framed correspondences have the nice property of involving the space $\mathrm{K}(Z)$, making $\mathrm{Corr}_S^{\mathrm{fr}}(X, Y)$ itself form an ∞-groupoid :

$$\mathrm{Corr}_S^{\mathrm{fr}}(X, Y) = \mathrm{colim}_{(Z,f,h)} \mathrm{Maps}_{\mathrm{K}(Z)}(0, \mathcal{L}_f).$$

The same results as before apply to framed correspondences :

Proposition 2.12. *Let Y be an S-scheme.*

1. $h_S^{\mathrm{fr}}(Y) \in \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)$ *;*
2. *For τ the Nisnevich or the étale topology, if $R \hookrightarrow Y$ is a τ-covering sieve generated by a single map then $h_S^{\mathrm{fr}}(R) \to h_S^{\mathrm{fr}}(Y)$ is a τ-local equivalence.*

As before, h_S^{fr} can be extended to a sifted colimit preserving functor $\mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_* \to \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$.

2.2 These three types are motivically equivalent

A key result in the study of framed correspondences is that the three notions described above are equivalent, after applying motivic localization $\mathrm{L}_{\mathrm{mot}}$. The precise details of this theorem will not all be discussed for the sake of conciseness, only the main ideas.

One of the important steps is that after $\mathbb{A}^1$-localization, the presheaves $h_S^{\mathrm{efr}}(Y)$, $h_S^{\mathrm{nfr}}(Y)$ and $h_S^{\mathrm{fr}}(Y)$ are $\mathcal{E}_\infty$-objects. Let us detail why this is the case for $h_S^{\mathrm{nfr}}(Y)$, the other cases are analogous (see [GP21] for equationally framed correspondences). Following definition 1.14, put :

$$h_S^{\mathrm{nfr}}(Y)(I_+) = h_S^{\mathrm{nfr}}(Y^{\sqcup I})$$

The desired $\mathbb{A}^1$-equivalences induced by the maps $\{1, \ldots, n\}_+ \to \{i\}_+$ are part of a more general result, as follows.

Proposition 2.13. *Let $Y_1, \dots, Y_k$ be S-schemes. Then the canonical map*

$$h_S^{\mathrm{nfr}}(Y_1 \sqcup \cdots \sqcup Y_k) \to h_S^{\mathrm{nfr}}(Y_1) \times \cdots \times h_S^{\mathrm{nfr}}(Y_k)$$

is an $\mathrm{L}_{\mathbb{A}^1}$-equivalence. In particular, $\mathrm{L}_{\mathbb{A}^1} h_S^{\mathrm{nfr}}(Y)$ is always an $\mathcal{E}_\infty$-object.

Proof. It is clear that proving the case $k = 2$ suffices. The canonical map α is the colimit of the maps

$$\alpha_n : h_S^{\mathrm{nfr},n}(Y_1 \sqcup Y_2) \to h_S^{\mathrm{nfr},n}(Y_1) \times h_S^{\mathrm{nfr},n}(Y_2).$$

One can define a map

$$\beta_n : h_S^{\mathrm{nfr},n}(Y_1) \times h_S^{\mathrm{nfr},n}(Y_2) \to h_S^{\mathrm{nfr},n+1}(Y_1 \sqcup Y_2)$$

in the following manner. Let $c_1 = (Z_1, f_1, h_1, i_1, \tau_1)$ and $c_2 = (Z_2, f_2, h_2, i_2, \tau_2)$ be normally framed correspondences of level n from some X to Y_1 and Y_2 respectively. Then let $\beta_n(c_1, c_2)$ be the normally framed correspondence $(Z_1 \sqcup Z_2, f_1 \sqcup f_2, h_1 \sqcup h_2, i, \tau)$ from X to $Y_1 \sqcup Y_2$ of level $n+1$, with closed immersion $i = (0, i_1) + (1, i_2) : Z_1 \sqcup Z_2 \to \mathbb{A}^1 \times \mathbb{A}^n_X$ and trivialization τ induced by τ_1 and τ_2.

Taking into account the suspension morphisms from before, one has the following non-necessarily commutative diagram :

$$\begin{array}{ccc}
h_S^{\mathrm{nfr},n}(Y_1 \sqcup Y_2) & \xrightarrow{\quad\sigma\quad} & h_S^{\mathrm{nfr},n+1}(Y_1 \sqcup Y_2) \\
\Big\downarrow{\scriptstyle \alpha_n} & \overset{\beta_n}{\nearrow} & \Big\downarrow{\scriptstyle \alpha_{n+1}} \\
h_S^{\mathrm{nfr},n}(Y_1) \times h_S^{\mathrm{nfr},n}(Y_2) & \xrightarrow{\ \sigma\times\sigma\ } & h_S^{\mathrm{nfr},n+1}(Y_1) \times h_S^{\mathrm{nfr},n+1}(Y_2)
\end{array}$$

When n is even, one can show that there are $\mathbb{A}^1$-homotopies $\sigma \simeq \beta_n \alpha_n$ and $\sigma \times \sigma \simeq \alpha_{n+1} \beta_n$, and that composing these two homotopies result in the identity homotopy (the outer square commutes). At the colimit, the maps $\mathrm{L}_{\mathbb{A}^1} \beta_n$ provide an inverse to $\mathrm{L}_{\mathbb{A}^1} \alpha$. $\square$

Proposition 2.14. *Let X and Y be S-schemes with X affine and Y étale over some affine bundle over S. Then for every closed subscheme $X_0 \subseteq X$ and $n \geq 0$, the restriction-forgetful map*

$$\mathrm{Corr}_S^{\mathrm{efr},n}(X, Y) \to \mathrm{Corr}_S^{\mathrm{efr},n}(X_0, Y) \times_{\mathrm{Corr}_S^{\mathrm{nfr},n}(X_0, Y)} \mathrm{Corr}_S^{\mathrm{nfr},n}(X, Y)$$

is surjective.

The proof of this proposition is not profound, it amounts to finding an extension of the equational framing of the correspondence from X_0 to X, matching the trivialization of the conormal sheaf. The interesting part here is that together with lemma 2.4, it is enough to prove the motivic equivalence of equational and normal framings.

Corollary 2.15. *Let Y be a smooth S-scheme which is a finite coproduct of schemes étale over affine bundles over S. Then the natural map of $\mathcal{E}_\infty$-objects*

$$\mathrm{L}_{\mathbb{A}^1} h_S^{\mathrm{efr}}(Y) \to \mathrm{L}_{\mathbb{A}^1} h_S^{\mathrm{nfr}}(Y)$$

is an equivalence on affine S-schemes.

Proof. Since the two presheaves are $\mathcal{E}_\infty$, it suffices to assume that Y itself is étale over an affine bundle. Apply the previous proposition 2.14 to $X = - \times \mathbb{A}^n$ and $X_0 = - \times \partial\mathbb{A}^n$ to obtain the surjectivity of the map

$$h_S^{\mathrm{efr}}(Y)^{\mathbb{A}^n} \to h_S^{\mathrm{efr}}(Y)^{\partial\mathbb{A}^n} \times_{h_S^{\mathrm{nfr}}(Y)^{\partial\mathbb{A}^n}} h_S^{\mathrm{nfr}}(Y)^{\mathbb{A}^n}$$

on affine S-schemes. Now, apply lemma 2.4, which is possible since both $h_S^{\mathrm{efr}}(Y)$ and $h_S^{\mathrm{nfr}}(Y)$ satisfy closed gluing (propositions 2.5, 2.10). We obtain the equivalences

$$h_S^{\mathrm{efr}}(Y)^{\Delta^\bullet} \to h_S^{\mathrm{nfr}}(Y)^{\Delta^\bullet}$$

on affine S-schemes. □

Corollary 2.16. *Let $F \in \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$. Then the natural map*

$$\mathrm{L}_{\mathrm{mot}} h_S^{\mathrm{efr}}(Y) \to \mathrm{L}_{\mathrm{mot}} h_S^{\mathrm{nfr}}(Y)$$

is an equivalence of presheaves of $\mathcal{E}_\infty$-spaces.

It now remains to compare $\mathrm{L}_{\mathrm{mot}} h_S^{\mathrm{efr}} \simeq \mathrm{L}_{\mathrm{mot}} h_S^{\mathrm{nfr}}$ to $\mathrm{L}_{\mathrm{mot}} h_S^{\mathrm{fr}}$. To this end, one first has to describe a map :

$$\mathrm{Corr}_S^{\mathrm{nfr}}(X, Y) \to \mathrm{Corr}_S^{\mathrm{fr}}(X, Y).$$

To this end, one can describe the functor $\mathrm{Corr}_S^{\mathrm{nfr}}$ in a way much more similar to $\mathrm{Corr}_S^{\mathrm{fr}}$:

Definition 2.17. For $Z \to X$ finite syntomic, let $\mathrm{Emb}_X(Z, \mathbb{A}_X^\infty)$ denote the colimit of the sets $\mathrm{Emb}_X(Z, \mathbb{A}_X^n)$ of closed X-immersions $Z \to \mathbb{A}_X^n$.

To take the normal framing into account, let similarly $\mathrm{Emb}_X^{\mathrm{fr}}(Z, \mathbb{A}_X^\infty)$ denote the colimit of the sets $\mathrm{Emb}_X^{\mathrm{fr}}(Z, \mathbb{A}_X^n)$ of closed X-immersions together with trivializations of the conormal sheaf.

Also denote by $\mathrm{sVect}_0(Z)$ the groupoid of stable vector bundles of rank 0 over Z, that is, the colimit

$$\mathrm{Vect}_0(Z) \xrightarrow{\oplus \mathcal{O}_Z} \mathrm{Vect}_1(Z) \xrightarrow{\oplus \mathcal{O}_Z} \cdots \to \mathrm{sVect}_0(Z).$$

There are natural maps $\mathrm{Emb}_X(Z, \mathbb{A}_X^n) \to \mathrm{Vect}_n(Z), i \mapsto \mathcal{N}_i^\vee$, inducing a map

$$\mathrm{Emb}_X(Z, \mathbb{A}_X^\infty) \to \mathrm{sVect}_0(Z)$$

at the colimit. Together with the map forgetting the framing, we obtain cartesian squares :

$$\begin{array}{ccc} \mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^n_X) & \longrightarrow & \mathrm{Emb}_X(Z,\mathbb{A}^n_X) \\ \downarrow & & \downarrow \\ * & \xrightarrow{\mathcal{O}^n_Z} & \mathrm{Vect}_n(Z) \end{array} \qquad \begin{array}{ccc} \mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^\infty_X) & \longrightarrow & \mathrm{Emb}_X(Z,\mathbb{A}^\infty_X) \\ \downarrow & & \downarrow \\ * & \xrightarrow{0} & \mathrm{sVect}_0(Z) \end{array}$$

natural in the finite syntomic S-morphism $Z \to X$ (with respect to cartesian squares).

The key insight to the equivalence $\mathrm{L}_{\mathrm{mot}} h^{\mathrm{nfr}}_S \simeq \mathrm{L}_{\mathrm{mot}} h^{\mathrm{fr}}_S$ is that

$$\mathrm{Corr}^{\mathrm{nfr}}_S(X,Y) = \mathrm{colim}_{(Z,f,h)} \mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^\infty_X),$$

which looks a lot like the description of $\mathrm{Corr}^{\mathrm{fr}}_S(X,Y)$ given after definition 2.11.

There is a natural map $\mathrm{sVect}_0(Z) \to \mathrm{K}(Z)$ sending a rank n vector bundle $\mathcal{V}$ to $\mathcal{O}^n_Z - \mathcal{V}$. The discussion before definition 2.11 about $\mathcal{L}_f$ being canonically equivalent to $\mathcal{N}^\vee_i \to \mathcal{O}^n_Z$ shows that we can extend the cartesian square above to :

$$\begin{array}{ccccc} \mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^\infty_X) & \longrightarrow & \mathrm{Emb}_X(Z,\mathbb{A}^\infty_X) & \longrightarrow & * \\ \downarrow & \lrcorner & \downarrow & & \downarrow{\scriptstyle \mathcal{L}_f} \\ * & \xrightarrow{0} & \mathrm{sVect}_0(Z) & \longrightarrow & \mathrm{K}(Z) \end{array}$$

The outer square being commutative exhibits an equivalence $0 \simeq \mathcal{L}_f$ for every element of $\mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^\infty_X)$. Hence there is a natural map

$$\mathrm{Emb}^{\mathrm{fr}}_X(Z,\mathbb{A}^\infty_X) \to \mathrm{Maps}_{\mathrm{K}(Z)}(0,\mathcal{L}_f).$$

At the colimit over the groupoid of spans $X \xleftarrow{f} Z \to Y$ with f finite syntomic, one obtains the desired comparison map $h^{\mathrm{nfr}}_S(Y) \to h^{\mathrm{fr}}_S(Y)$.

See [EHK+21] for the proof of the following comparison theorem.

Proposition 2.18. *The natural* $\mathrm{L}_{\mathbb{A}^1}$*-localized map*

$$\mathrm{L}_{\mathbb{A}^1} \mathrm{Emb}^{\mathrm{fr}}(-,\mathbb{A}^\infty) \to \mathrm{L}_{\mathbb{A}^1} \mathrm{Maps}_{\mathrm{K}}(0,\mathcal{L})$$

between presheaves of spaces on the category FSyn *(whose objects are finite syntomic morphisms and whose morphisms are the cartesian squares) is an equivalence on affine* S*-schemes.*

We can now conclude that the three types of framed correspondences described in this section are all motivically equivalent :

Corollary 2.19. *Let* $F \in \mathrm{PSh}_\Sigma(\mathrm{Sch}_S)_*$*. Then the map*

$$\mathrm{L}_{\mathbb{A}^1} h^{\mathrm{nfr}}_S(F) \to \mathrm{L}_{\mathbb{A}^1} h^{\mathrm{fr}}_S(F)$$

is an equivalence on affine S*-schemes. In particular, it induces an equivalence of presheaves of* $\mathcal{E}_\infty$*-spaces*

$$\mathrm{L}_{\mathrm{mot}} h^{\mathrm{nfr}}_S(F) \simeq \mathrm{L}_{\mathrm{mot}} h^{\mathrm{fr}}_S(F).$$

3 Framed motivic spaces and spectra

3.1 The ∞-category of framed motivic spaces

The construction of the ∞-category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$ of framed correspondences is extremely technical and is the subject of the section 4 in [EHK+21]. To avoid unnecessary complications, only the relevant properties will be presented. Intuitively, it is a symmetric monoidal ∞-category whose mapping spaces are the ∞-groupoids $\mathrm{Corr}^{\mathrm{fr}}_S(X,Y)$.

Theorem 3.1. *There is a symmetric monoidal ∞-category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$ and a symmetric monoidal functor $\gamma : \mathrm{Sm}_{S+} \to \mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$, inducing an adjunction*

$$\gamma^* : \mathrm{PSh}(\mathrm{Sm}_{S+}) \rightleftarrows \mathrm{PSh}(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)) : \gamma_*.$$

This data satisfies the following intuitive properties :

1. *γ is essentially surjective*
2. *There is a natural equivalence $\gamma_*\gamma^* \simeq h^{\mathrm{fr}}_S$, hence a natural equivalence of spaces*

$$\mathrm{Corr}^{\mathrm{fr}}_S(X,Y) \simeq \mathrm{Maps}_{\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)}(\gamma(X_+),\gamma(Y_+))$$

3. *The natural framing map $X_+ \to h^{\mathrm{fr}}_S(X)$ agrees with the unit $X_+ \to \gamma_*\gamma^*(X_+)$*
4. *The linear structure maps $X_+ \wedge h^{\mathrm{fr}}_S(Y) \to h^{\mathrm{fr}}_S(X \times_S Y)$ agrees with the composition*

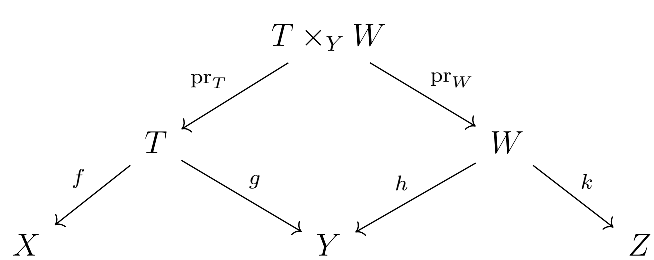
$$X_+ \wedge \gamma_*\gamma^*(Y_+) \xrightarrow{unit} \gamma_*\gamma^*(X_+) \wedge \gamma_*\gamma^*(Y_+) \xrightarrow{monoidal} \gamma_*\gamma^*((X \times_S Y)_+)$$

Intuitively, the functor γ embeds Sm_{S+} in the ∞-category of framed correspondences. The functor γ_* forgets the framed transfers of a presheaf, and the functor γ^* adds framed transfers to a classical presheaf, in a universal way.

Remark 3.2. *The composition map*

$$\pi_0\,\mathrm{Corr}^{\mathrm{fr}}_S(X,Y) \times \pi_0\,\mathrm{Corr}^{\mathrm{fr}}_S(Y,Z) \to \pi_0\,\mathrm{Corr}^{\mathrm{fr}}_S(X,Z)$$

in the homotopy category $\mathrm{h}\,\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$ sends framed correspondences $(T,f,g,\alpha : 0 \simeq \mathcal{L}_f)$ and $(W,h,k,\beta : 0 \simeq \mathcal{L}_h)$ to the following framed correspondence. First, form the pullback :



to get a span $(T \times_Y W, f \circ \mathrm{pr}_T, k \circ \mathrm{pr}_W)$*. Now the canonical cofiber sequence of cotangent complexes*

$$\mathrm{pr}_T^* \mathcal{L}_f \to \mathcal{L}_{f \circ \mathrm{pr}_T} \to \mathcal{L}_{\mathrm{pr}_T}$$

induces an isomorphism $\varphi : \mathrm{pr}_T^* \mathcal{L}_f \oplus \mathcal{L}_{\mathrm{pr}_T} \to \mathcal{L}_{f \circ \mathrm{pr}_T}$ *in* $\tau_{\leq 1} \mathrm{K}(T \times_Y W)$*. Letting* ψ *be the canonical isomorphism* $g^* \mathcal{L}_h \to \mathcal{L}_{\mathrm{pr}_T}$*, we obtain a trivialization of the cotangent complex of* $f \circ \mathrm{pr}_T$ *as required :*

$$\varphi \circ (\mathrm{pr}_T^* \alpha \oplus (\psi \circ g^* \beta)) \in \pi_0 \operatorname{Maps}_{\mathrm{K}(T \times_Y W)}(0, \mathcal{L}_{f \circ \mathrm{pr}_T}).$$

This ∞-category lets us define *framed* motivic spaces in the same fashion as motivic spaces, with additional framed transfers.

Definition 3.3. A *framed motivic space* over S is a presheaf $F \in \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S))$ whose restriction to Sm_S is both $\mathbb{A}^1$-invariant and Nisnevich-local. Denote by $\mathbf{H}^{\mathrm{fr}}(S)$ the full sub-∞-category of $\mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S))$ spanned by framed motivic spaces.

Again, there are localization functors $\mathrm{L}_{\mathbb{A}^1}$, $\mathrm{L}_{\mathrm{Nis}}$ and $\mathrm{L}_{\mathrm{mot}}$ from $\mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S))$ to the three corresponding full sub-∞-categories.

The next three propositions about framed motivic spaces and the functor γ will be admitted, since their proofs are technical and not enlightening.

Proposition 3.4. *The* ∞*-category* $\mathbf{H}^{\mathrm{fr}}(S)$ *of framed motivic spaces is generated under sifted colimits by the objects* $\mathrm{L}_{\mathrm{mot}} \gamma(X_+)$ *for affine smooth* S*-schemes* X*. Moreover, this object is compact for every* $X \in \mathrm{Sm}_S$*.*

Proposition 3.5. *The functor forgetting the framing*

$$\gamma_* : \mathrm{PSh}_\Sigma(\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)) \to \mathrm{PSh}_\Sigma(\mathrm{Sm}_S)_*$$

commutes with the three localizations $\mathrm{L}_{\mathbb{A}^1}$*,* $\mathrm{L}_{\mathrm{Nis}}$ *and* $\mathrm{L}_{\mathrm{mot}}$*. The functor* $\gamma_* \gamma^*$ *preserves motivic equivalences.*

Proposition 3.6. *The functor* $\gamma_* : \mathbf{H}^{\mathrm{fr}}(S) \to \mathbf{H}(S)_*$ *is conservative and preserves sifted colimits.*

3.2 Effective and framed motivic spectra

The Tate sphere $\mathbb{T} = (\mathbb{A}^1 / \mathbb{A}^1 - 0, 1)$ and the multiplicative group $\mathbb{G} = (\mathbb{G}_m, 1)$ in $\mathrm{PSh}_\Sigma(\mathrm{Sm}_S)_*$ may be promoted to presheaves with framed transfers using the functor γ^* (see theorem 3.1). From now on, we let $\mathbb{T}^{\mathrm{fr}} = \gamma^* \mathbb{T}$ and $\mathbb{G}^{\mathrm{fr}} = \gamma^* \mathbb{G}$. Notice that much like the unframed version, $\mathbb{T}^{\mathrm{fr}} \simeq S^1 \wedge \mathbb{G}^{\mathrm{fr}}$ in $\mathbf{H}^{\mathrm{fr}}(S)$.

Definition 3.7. The symmetric monoidal ∞-category of *framed motivic spectra* is obtained from $\mathbf{H}^{\mathrm{fr}}(S)$ by formally inverting the suspension operation $- \otimes \mathbb{T}^{\mathrm{fr}}$. Denote $\mathbf{SH}^{\mathrm{fr}}(S) = \mathbf{H}^{\mathrm{fr}}(S) \left[(\mathbb{T}^{\mathrm{fr}})^{-1} \right]$ the corresponding ∞-category. One has the usual suspension-loop adjunction :

$$\Sigma^\infty_{\mathbb{T},\mathrm{fr}} : \mathbf{H}^{\mathrm{fr}}(S) \rightleftarrows \mathbf{SH}^{\mathrm{fr}}(S) : \Omega^\infty_{\mathbb{T},\mathrm{fr}}.$$

Proposition 3.8. *The ∞-category $\mathbf{SH}^{\mathrm{fr}}(S)$ of framed motivic spectra is stable, and generated under sifted colimits by the objects $(\mathbb{T}^{\mathrm{fr}})^{\otimes n} \otimes \Sigma^\infty_{\mathbb{T},\mathrm{fr}}\gamma^*(X_+)$ for X an affine smooth S-scheme and $n \leq 0$.*

The framed suspension $\mathbb{T}$-spectrum $\Sigma^\infty_{\mathbb{T},\mathrm{fr}}\gamma^*(X_+)$ is compact for every smooth S-scheme X.

Proof. The stability is obvious. The second and third statements come directly from proposition 3.4 □

Definition 3.9. The ∞-category $\mathbf{SH}^{\mathrm{eff}}(S)$ of *effective motivic spectra* is the smallest full sub-∞-category of $\mathbf{SH}(S)$ closed under taking homotopy colimits containing $\Sigma^{-n}\Sigma^\infty_{\mathbb{T}}X_+$ for $X \in \mathrm{Sm}_S$ and $n \geq 0$. The ∞-category $\mathbf{SH}^{\mathrm{veff}}(S)$ of *very effective motivic spectra* is the smallest sub-∞-category of $\mathbf{SH}(S)$ closed under taking extensions and homotopy colimits containing the suspension spectra $\Sigma^\infty_{\mathbb{T}}X_+$ for $X \in \mathrm{Sm}_S$.

Lemma 3.10. *The ∞-category $\mathbf{SH}^{\mathrm{veff}}(S)$ of very effective motivic spectra is closed under taking smash products.*

Proof. We begin by proving it when one argument is a suspension spectrum. Let $E \in \mathbf{SH}^{\mathrm{veff}}(S)$ and $X \in \mathrm{Sm}_S$. Then the smash product $\Sigma^\infty_{\mathbb{T}}X \wedge E$ is clearly very effective in the case where E itself is a suspension spectrum. In the general induction case :

- either E is a homotopy colimit $\operatorname{hocolim} E_i$ of spectra whose smash product with $\Sigma^\infty_{\mathbb{T}}X_+$ is very effective, and in this case

  $$\Sigma^\infty_{\mathbb{T}}X_+ \wedge E \simeq \operatorname{hocolim} \Sigma^\infty_{\mathbb{T}}X_+ \wedge E_i$$

  is very effective since $\mathbf{SH}^{\mathrm{veff}}(S)$ is stable under taking homotopy colimits ;

- or E is an extension $A \to E \to B \xrightarrow{+1}$ of spectra whose smash product with $\Sigma^\infty_{\mathbb{T}}X_+$ is very effective, and in this case $\Sigma^\infty_{\mathbb{T}}X_+$ is an extension of very effective spectra, hence it is also very effective.

Now that we know that $\Sigma^\infty_{\mathbb{T}}X_+ \wedge E$ is very effective for $X \in \mathrm{Sm}_S$ and $E \in \mathbf{SH}^{\mathrm{veff}}(S)$, the exact same argument shows that $\mathbf{SH}^{\mathrm{veff}}(S)$ is stable under taking smah products in the general case. □

In particular, $\mathbf{SH}^{\mathrm{veff}}(S)$ is a symmetric monoidal ∞-category. The same argument applies to $\mathbf{SH}^{\mathrm{eff}}(S)$.

4 Motivic recognition

Let us first recall the several (∞-)categories in play. From Sm_{S+}, one can construct the ∞-category of framed correspondences (theorem 3.1) $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$. Considering Nisnevich-local and $\mathbb{A}^1$-invariant presheaves of spaces on these lead to the definition of unframed pointed and framed motivic spaces, respectively $\mathbf{H}(S)_*$ and $\mathbf{H}^{\mathrm{fr}}(S)$. As seen in subsection 1.2, one can then stabilize $\mathbf{H}(S)_*$ to obtain $\mathbf{SH}^{S^1}(S)$ and the corresponding suspension and loop space functors $\Sigma^\infty_{S^1} \dashv \Omega^\infty_{S^1}$. The same construction applied to the ∞-category $\mathbf{H}^{\mathrm{fr}}(S)$ of framed motivic spaces yields the ∞-category of framed motivic S^1-spectra $\mathbf{SH}^{S^1,\mathrm{fr}}(S)$. By further inverting the multiplicative group $\mathbb{G}$ or $\mathbb{G}^{\mathrm{fr}}$ as in subsection 3.2, one obtains the ∞-categories $\mathbf{SH}(S)$ of motivic ($\mathbb{T}$-)spectra and $\mathbf{SH}^{\mathrm{fr}}(S)$ of framed motivic ($\mathbb{T}$-)spectra.

Notice that since $\mathbb{T} \simeq S^{2,1} \simeq \mathbb{G} \wedge S^1$, inverting S^1 and then $\mathbb{G}$ amounts to inverting $\mathbb{T}$. In this sense, the corresponding suspension and loop space functors factorize as $\Sigma^\infty_{\mathbb{T}} \simeq \Sigma^\infty_{\mathbb{G}} \circ \Sigma^\infty_{S^1}$ (and the same with Ω^∞ and their framed versions). All in all, there is a commutative diagram of adjunctions :

$$\begin{array}{ccccccc}
\mathrm{Sm}_{S+} & \longrightarrow & \mathbf{H}(S)_* & \underset{\Omega^\infty_{S^1}}{\overset{\Sigma^\infty_{S^1}}{\rightleftarrows}} & \mathbf{SH}^{S^1}(S) & \underset{\Omega^\infty_{\mathbb{G}}}{\overset{\Sigma^\infty_{\mathbb{G}}}{\rightleftarrows}} & \mathbf{SH}(S) \\
\gamma\downarrow & & \gamma^*\downarrow\uparrow\gamma_* & & \gamma^*\downarrow\uparrow\gamma_* & & \gamma^*\downarrow\uparrow\gamma_* \\
\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S) & \longrightarrow & \mathbf{H}^{\mathrm{fr}}(S) & \underset{\Omega^\infty_{S^1,\mathrm{fr}}}{\overset{\Sigma^\infty_{S^1,\mathrm{fr}}}{\rightleftarrows}} & \mathbf{SH}^{S^1,\mathrm{fr}}(S) & \underset{\Omega^\infty_{\mathbb{G},\mathrm{fr}}}{\overset{\Sigma^\infty_{\mathbb{G},\mathrm{fr}}}{\rightleftarrows}} & \mathbf{SH}^{\mathrm{fr}}(S)
\end{array}$$

Proposition 4.1. *The two functors γ_* on the right of this diagram, forgetting the framing of motivic S^1- and $\mathbb{T}$-spectra, are conservative and preserve colimits.*

Proof. By proposition 3.6, they are conservative and preserve filtered colimits. Being right adjoints and the ∞-categories being stable, they are right exact, so they preserve all finite colimits. □

Proposition 4.2. *The restriction of the S^1-suspension spectrum functor*

$$\Sigma^\infty_{S^1,\mathrm{fr}} : \mathbf{H}^{\mathrm{fr}}(S) \to \mathbf{SH}^{S^1,\mathrm{fr}}(S)$$

to the full sub-∞-category spanned by grouplike and strictly $\mathbb{A}^1$-invariant framed motivic spaces is fully faithful.

Proof. From the adjunction $\Sigma^\infty_{S^1,\mathrm{fr}} \dashv \Omega^\infty_{S^1,\mathrm{fr}}$, it suffices to prove that the unit map $X \to \Omega^\infty_{S^1,\mathrm{fr}}\Sigma^\infty_{S^1,\mathrm{fr}}X$ is an equivalence for any grouplike and strictly $\mathbb{A}^1$-invariant framed motivic space X.

By conservativity of γ_* (proposition 4.1), it suffices to prove that the unframed map

$$\gamma_*X \to \Omega^\infty_{S^1}\mathbb{B}^\infty_{\mathrm{mot}}\gamma_*X$$

is an equivalence. This is the unit map on grouplike and strictly $\mathbb{A}^1$-invariant objects from proposition 1.16. □

Theorem 4.3 (Cancellation theorem)**.** *Let k be a perfect field and $M \in \mathbf{SH}^{S^1,\mathrm{fr}}(k)$. Then the unit map of the suspension-loop adjunction $M \to \Omega_{\mathbb{G}}(\mathbb{G}^{\mathrm{fr}} \otimes M)$ is an equivalence.*

Proof. We only treat the case where k is infinite. See appendix B in [EHK+21] for the general case. By 4.1, it suffices to treat the case where $M = \gamma_*\gamma^*\Sigma^\infty_{S^1} X_+$ for some smooth k-scheme X. In this case, using the equivalence $\gamma_*\gamma^* \simeq h^{\mathrm{fr}}$ from theorem 3.1, the unit map becomes

$$\mathbb{B}^\infty_{\mathrm{mot}}\mathrm{L}_{\mathrm{mot}}h^{\mathrm{fr}}(X) \to \Omega_{\mathbb{G}}\mathbb{B}^\infty_{\mathrm{mot}}\mathrm{L}_{\mathrm{mot}}h^{\mathrm{fr}}(\mathbb{G} \wedge X_+)$$

induced by linearity of h^{fr}.

Since framed correspondences are motivically equivalent to equationally framed correspondences, it now suffices to prove that

$$\mathbb{B}^\infty_{\mathrm{mot}}\mathrm{L}_{\mathrm{mot}}h^{\mathrm{efr}}(X) \to \Omega_{\mathbb{G}}\mathbb{B}^\infty_{\mathrm{mot}}\mathrm{L}_{\mathrm{mot}}h^{\mathrm{efr}}(\mathbb{G} \wedge X_+)$$

is an equivalence. This comes directly from theorem B in [AGP21], after unwrapping the definition of the framed motive of X from definition 5.2 in [GP21] : the map of interest here is one of the structure maps of the spectrum in theorem B. □

Corollary 4.4. *Let k be a perfect field. Then the framed $\mathbb{G}$-suspension spectrum functor*

$$\Sigma^\infty_{\mathbb{G},\mathrm{fr}} : \mathbf{SH}^{S^1,\mathrm{fr}}(k) \to \mathbf{SH}^{\mathrm{fr}}(k)$$

is fully faithful.

Proof. From the adjunction $\Sigma^\infty_{\mathbb{G},\mathrm{fr}} \dashv \Omega^\infty_{\mathbb{G},\mathrm{fr}}$, it suffices to show that the unit map $M \to \Omega^\infty_{\mathbb{G},\mathrm{fr}}\Sigma^\infty_{\mathbb{G},\mathrm{fr}}M$ is an equivalence for every $M \in \mathbf{SH}^{S^1,\mathrm{fr}}(k)$. This map is the colimit of the unit maps appearing in the cancellation theorem :

$$M \to \Omega_{\mathbb{G}}(\mathbb{G}^{\mathrm{fr}} \otimes M) \to \Omega^2_{\mathbb{G}}((\mathbb{G}^{\mathrm{fr}})^{\otimes 2} \otimes M) \to \dots$$

each of which is an equivalence. □

The same result is true when we replace $\mathbb{G}$-suspension spectra by $\mathbb{T}$-suspension spectra :

Corollary 4.5. *Let k be a perfect field. Then the framed $\mathbb{T}$-suspension spectrum functor*

$$\Sigma^\infty_{\mathbb{T},\mathrm{fr}} : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathbf{SH}^{\mathrm{fr}}(k)$$

is fully faithful.

Proof. We have the factorization $\Sigma^\infty_{\mathbb{T},\mathrm{fr}} \simeq \Sigma^\infty_{\mathbb{G},\mathrm{fr}} \circ \Sigma^\infty_{S^1,\mathrm{fr}}$. The $\mathbb{G}$-suspension functor is fully faithful by the previous corollary, and the S^1-suspension functor is fully faithful by 4.2. □

Theorem 4.6 (Reconstruction theorem)**.** *Let k be a perfect field. Then the adjunction*

$$\gamma^* : \mathbf{SH}(k) \rightleftarrows \mathbf{SH}^{\mathrm{fr}}(k) : \gamma_*$$

is an equivalence of symmetric monoidal ∞-categories.

Proof. Let $\eta : \mathrm{id} \to \gamma_*\gamma^*$ and $\varepsilon : \gamma^*\gamma_* \to \mathrm{id}$ be the unit and counit of the adjunction respectively. It suffices to show that η is an equivalence. Indeed, the triangle identity $\gamma_*\varepsilon \circ \eta_{\gamma^*} \simeq \mathrm{id}$ will ensure that $\gamma_*\varepsilon$ is an equivalence, which in turn shows that ε is an equivalence, by the conservativity of γ_* (proposition 4.1).

By proposition 4.1, it suffices to show that $\Sigma^{\infty-n}_{\mathbb{T}} X_+ \to \gamma_*\gamma^*\Sigma^{\infty-n}_{\mathbb{T}} X_+$ is an equivalence for all $X \in \mathrm{Sm}_S$ and $n \geq 0$. Since $\gamma_*\gamma^*$ commutes with $\Sigma_{\mathbb{T}}$, it suffices to show it for $n = 0$.

By proposition 3.5, this motivic spectrum $\gamma_*\gamma^*\Sigma^{\infty}_{\mathbb{T}} X_+$ is the $\mathbb{G}$-spectrification of the $\mathbb{G}$-prespectrum $(\mathbb{B}^{\infty}_{\mathrm{mot}}\mathrm{L}_{\mathrm{mot}}\gamma_*\gamma^*(\mathbb{G}^{\wedge k} \wedge X_+))_{k\geq 0}$. By theorem 3.1, one can replace $\gamma_*\gamma^*$ by h^{fr}_k. The corresponding unit map from $\Sigma^{\infty}_{\mathbb{T}} X_+$ is induced by the sequence of maps

$$\mathbb{G}^{\wedge k} \wedge X_+ \to h^{\mathrm{efr}}(\mathbb{G}^{\wedge k} \wedge X_+) \to h^{\mathrm{fr}}(\mathbb{G}^{\wedge k} \wedge X_+)$$

whose second map is a motivic equivalence by corollaries 2.16 and 2.19, and whose first map is also a motivic equivalence by theorem 11 in [GP21]. □

In other words, over a perfect field, the framed stable motivic homotopy theory is equivalent to the classical stable motivic homotopy theory.

All the hard work to prove the motivic recognition principle is now done.

Theorem 4.7 (Motivic recognition principle)**.** *Let k be a perfect field. Then the functor*

$$\gamma_*\Sigma^{\infty}_{\mathbb{T},\mathrm{fr}} : \mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \to \mathbf{SH}(k)$$

is fully faithful and induces an equivalence of symmetric monoidal ∞-categories

$$\mathbf{H}^{\mathrm{fr}}(k)^{\mathrm{gp}} \simeq \mathbf{SH}^{\mathrm{veff}}(k).$$

Proof. The fully faithfulness is immediate from the fully faithfulness of $\Sigma^{\infty}_{\mathbb{T},\mathrm{fr}}$ (corollary 4.5) and that of γ_* (reconstruction theorem 4.6). To obtain an equivalence of ∞-categories, it now suffices to compute the essential image of this functor $\gamma_*\Sigma^{\infty}_{\mathbb{T},\mathrm{fr}}$.

This image is the full sub-∞-category of $\mathbf{SH}(k)$ generated under taking homotopy colimits by the $\mathbb{T}$-suspension spectra $\Sigma^{\infty}_{\mathbb{T},\mathrm{fr}} X_+$ for $X \in \mathrm{Sm}_k$. One can show (see [Bac17], proposition 4) that the construction of $\mathbf{SH}^{\mathrm{veff}}(S)$ doesn't necessitate stability under extensions, hence this essential image is exactly the very effective motivic spectra. □

In other words, the underlying pointed motivic space $X = \Omega^{\infty}_{\mathbb{T}} E \in \mathbf{H}(S)_*$ of a very effective (intuitively, connective) motivic $\mathbb{P}^1$-spectrum $E \in \mathbf{SH}^{\mathrm{veff}}(S)$, together with its framing $\gamma^* X$ and its grouplike $\mathcal{E}_\infty$ structure, determines the whole delooping spectrum E up to motivic homotopy equivalence.

Directions for further study

- Understand the details of the construction of the ∞-category $\mathbf{Corr}^{\mathrm{fr}}(\mathrm{Sm}_S)$ of framed correspondences

- Read more about the consequences of theorem 4.7, especially the representability of the motivic sphere spectrum. (But first, learn about Hilbert schemes.)
- Is there a way to weaken the assumption $S = \operatorname{Spec} k$ for k perfect ? The case $\dim(S) \geq 2$ over an algebraically closed field breaks down, but maybe when S is a curve, or even just the spectrum of a non-perfect field ?

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