O-minimal geometry and applications

Séminaire Lambda 2024

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1. What is an o-minimal structure ?
2. What are the examples ?
3. What happens in an o-minimal structure ?
4. What are the applications ?

What is an o-minimal structure ?

If $A, B \subset \mathbb{R}^n$ are definable sets, we want $A \cup B$, $A \cap B$, $A \setminus B$ to be definable. . .

What is an o-minimal structure ?

If $A, B \subset \mathbb{R}^n$ are definable sets, we want $A \cup B$, $A \cap B$, $A \setminus B$ to be definable...

But also the closure, the interior, the border of A, or the set of points at distance at least 1 to A...

An example of "reasonable" definition

$$\begin{aligned}\bar{A} &= \{x \in \mathbb{R}^2 | \forall \varepsilon \in \mathbb{R}_+^*, \exists a \in A, d(x,a) \leq \varepsilon\} \\ &= \{x \in \mathbb{R}^2 | \forall \varepsilon \in \mathbb{R}, (\varepsilon > 0 \Rightarrow \exists a \in A, (x_1 - a_1)^2 + (x_2 - a_2)^2 \leq \varepsilon^2)\}.\end{aligned}$$

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For $x \in \mathbb{R}^2$:

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Ingredients : $\wedge, \vee, \wedge, \forall, \exists$ in $\mathbb{R}$ and A, coordinates, $+, -, \times, =, \leq$.

Logical operators correspond to geometrical constructions

Let ϕ_1, ϕ_2 be two formulas with n free variables (parameters).

$$\begin{aligned}
\{x \in \mathbb{R}^n | \phi_1(x) \wedge \phi_2(x)\} &= \{x \in \mathbb{R}^n | \phi_1(x)\} \cap \{x \in \mathbb{R}^n | \phi_2(x)\} \\
\{x \in \mathbb{R}^n | \phi_1(x) \vee \phi_2(x)\} &= \{x \in \mathbb{R}^n | \phi_1(x)\} \cup \{x \in \mathbb{R}^n | \phi_2(x)\} \\
\{x \in \mathbb{R}^n | \neg\phi_1(x)\} &= \{x \in \mathbb{R}^n | \phi_1(x)\}^c
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Let ϕ be a formula with $n + m$ free variables.

$$\{x \in \mathbb{R}^n | \exists y \in \mathbb{R}^m, \phi(x, y)\} = \pi\{(x, y) \in \mathbb{R}^{n+m} | \phi(x, y)\}$$

with $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$.

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with $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$.

$$\forall x \in \mathbb{R}^n, \phi(x) \leftrightarrow \neg\exists x \in \mathbb{R}^n, \neg\phi(x)$$

Logical operators correspond to geometrical constructions

Let ϕ be a formula with 1 free variable.

$$\{x \in \mathbb{R}^n, \phi(x_1)\} = \{x' \in \mathbb{R}, \phi(x')\} \times \mathbb{R}^{n-1}$$

Finally, we need sets of the form $\{x \in \mathbb{R}^n, P(x) = 0\}$ and $\{x \in \mathbb{R}^n, P(x) \geq 0\}$ for P a polynomial.

Definition of a structure

Definition

A structure is the data for each $n \in \mathbb{N}$ of a set $\mathcal{S}_n$ of subsets of $\mathbb{R}^n$ such that :

1. $\emptyset \in \mathcal{S}_n$, $\forall A, B \in \mathcal{S}_n, A \cup B, A \cap B, A^c \in \mathcal{S}_n$;
2. $\forall A \in \mathcal{S}_n, \mathbb{R} \times A, A \times \mathbb{R} \in \mathcal{S}_{n+1}$;
3. $\forall m \in \mathbb{N}$, *for each projection* $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ *on some coordinates*, $\forall A \in \mathcal{S}_{n+m}$, $\pi(A) \in \mathcal{S}_n$;
4. *for each* $P \in \mathbb{R}[X_1, \ldots, X_n]$, $\{x \in \mathbb{R}^n, P(x) = 0\}, \{x \in \mathbb{R}^n, P(x) \geq 0\} \in \mathcal{S}_n$.

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2. $\forall A \in \mathcal{S}_n, \mathbb{R} \times A, A \times \mathbb{R} \in \mathcal{S}_{n+1}$;
3. *For each projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ on the n first coordinates,* $\forall A \in \mathcal{S}_{n+1}, \pi(A) \in \mathcal{S}_n$;
4.
   - $\forall a \in \mathbb{R}, \{a\} \in \mathcal{S}_1$;
   - $\{(x, y) \in \mathbb{R}^2, x = y\}, \{(x, y) \in \mathbb{R}^2, x \leq y\} \in \mathcal{S}_2$;
   - $\{(x, y, z) \in \mathbb{R}^2, x + y = z\}, \{(x, y, z) \in \mathbb{R}^2, xy = z\} \in \mathcal{S}_3$.

Definition of a structure

An element of some $\mathcal{S}_n$ is said to be *definable* in $\mathcal{S}$.

Proposition

Let $\mathcal{S}$ be a structure. Let $A_1, \ldots, A_k$ be definable in $\mathcal{S}$. If ϕ is a formula with $n \in \mathbb{N}$ free variables which quantifies in $A_1, \ldots, A_k$, then $\{x \in \mathbb{R}^n | \phi(x)\}$ is definable in $\mathcal{S}$.

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Caution

We can quantify only in definable sets.

Definable functions

Definition

Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ be definable in $\mathcal{S}$. A map $f : A \to B$ is definable if its graph is definable in $\mathbb{R}^{n+m}$.

- If $f : A \to B$ and $g : B \to C$ are definable, then $g \circ f$ is definable.
- If f is injective, f^{-1} is definable.
- Images and inverse images of definable sets by definable maps are definable.
- If $f : A \times B \to C$ is definable, then $a \mapsto \lim_{b \to 0} f(a, b)$ is definable.
- If $f, g : A \to \mathbb{R}$ are definable, $f + g, f - g, fg, \frac{f}{g}, f'$ are definable.

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Caution

The antiderivative of a definable function is not necessarily definable.

Definition of an o-minimal structure

Definition

A structure $\mathcal{S}$ is o-minimal if $\mathcal{S}_1$ is the set of finite unions of points and intervals.

What are the examples ?

Definition

The structure $\mathbb{R}_{alg}$ *is the smallest structure.*

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Definition

The structure $\mathbb{R}_{alg}$ is the smallest structure.

Definition

A basic semialgebraic subset of $\mathbb{R}^n$ is a subset of the form :

$$\{x \in \mathbb{R}^n | P(x) = 0 \wedge Q_1(x) > 0 \wedge \cdots \wedge Q_k(x) > 0\}$$

for $P, Q_1, \ldots, Q_k \in R[X_1, \ldots, X_n]$.

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A semialgebraic subset of $\mathbb{R}^n$ is a finite union of basic semialgebraic subsets of $\mathbb{R}^n$.

Semialgebraic subset are the subset of points which satisfies a formula without quantifiers.

Theorem (Tarski–Seidenberg)

A projection of a semialgebraic subset is semialgebraic.

Example

A projection of $\{(a,b,c,x) \in \mathbb{R}^4 | ax^2 + bx + c = 0\}$ *is* $\{(a,b,c) \in \mathbb{R}^3 | b^2 - 4ac > 0\}$.

Equivalently, every formula which quantifies over $\mathbb{R}$ is equivalent to a formula without quantifiers :

$$(\exists x \in \mathbb{R}, ax^2 + bx + c = 0) \equiv (b^2 - 4ac > 0).$$

Corollary

Definable sets of $\mathbb{R}_{alg}$ *are semialgebraic sets.*
In particular, $\mathbb{R}_{alg}$ *is o-minimal.*

Other examples

$\mathbb{R}_{\text{an}}$: structure generated by functions on $[0,1]^n$ which are restrictions of (real) analytic functions defined on an open set. If A is bounded and definable, Ω is open, $\bar{A} \subset \Omega$, and f is analytic on Ω, then its restriction to A is definable.

Caution

An analytic function on a (bounded) open subset is not necessarily definable in $\mathbb{R}_{an}$.

Sets of $\mathbb{R}_{\text{an}}$ are *globally subanalytic sets*.

Theorem (Gabrielov 1969)

$\mathbb{R}_{an}$ is o-minimal.

Other examples

$\mathbb{R}_{\text{exp}}$: structure generated by the (global) exponential.
$\mathbb{R}_{\text{an, exp}}$: structure generated by $\mathbb{R}_{\text{an}}$ and $\mathbb{R}_{\text{exp}}$.

Theorem (van den Dries, Miller 1994)

$\mathbb{R}_{exp}$ and $\mathbb{R}_{an,\ exp}$ are o-minimal.

What happens in an o-minimal structure ?

Choose some o-minimal structure $\mathcal{S}$.

Theorem (Monotonicity)

Let $I \subset \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$ be a definable function. Then, there are $x_1 < \cdots < x_k \in I$ such that on each $]x_i, x_{i+1}[$, f is constant, or continuous and strictly monotone.

Cell decomposition

A cell decomposition is a partition of $\mathbb{R}^n$ in definable sets. We define it inductively.

A cell decomposition of $\mathbb{R}$ is a partition of $\mathbb{R}$ in points and open intervals.

We define a cell decomposition of $\mathbb{R}^n$ in the following way : consider a cell decomposition of $\mathbb{R}^n$. Choose a cell C of $\mathbb{R}^n$. Choose continuous definable functions $f_1, \ldots, f_k : C \to \mathbb{R}$ such that $f_1 < \cdots < f_n$. Cells of $\mathbb{R}^{n+1}$ above $\mathbb{R}^n$ are graphs of the functions f_i, sets of points between the graphs of f_i and f_{i+1}, the set of points below the graph of f_1 and the set of points above the graph of f_k.

Proposition

Let C be a cell of a cell decomposition of $\mathbb{R}^n$. Then there exists $0 \leq m \leq n$ and a definable homeomorphism $g : C \to \mathbb{R}^m$.

Cell decomposition theorem

Theorem (Cell decomposition)

Let $A_1, \ldots, A_k \subset \mathbb{R}^n$ be definable. Then there exists a cell decomposition of $\mathbb{R}^n$ such that each A_i is a union of cells.

Theorem (Piecewise continuity)

Let $A \subset \mathbb{R}^n$ be definable, let $f : A \to \mathbb{R}$ be definable. Then there exists a cell decomposition of $\mathbb{R}^n$ such that A is a union of cells on which f is continuous.

Uniform finiteness

Theorem (Uniform finiteness)

Let $A \subset \mathbb{R}^{n+1}$ *be definable such that for each* $x \in \mathbb{R}^n, \{y \in \mathbb{R} | (x, y) \in A\}$ *is finite. Then there exists* $N \in \mathbb{N}$ *such that for each* $x \in \mathbb{R}^n, \operatorname{card}\{y \in \mathbb{R} | (x, y) \in A\} \leq N$.

Smooth version

Let $k \in \mathbb{N}$.
In the cell decomposition theorem, we can suppose the functions to be C^k.
In the piecewise continuity theorem, we can suppose the cell decomposition to be C^k and f to be C^k on each cell.

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Caution

It is not true for $k = \infty$.

But it is actually the case in $\mathbb{R}_{\text{an, exp}}$.

Topology of definable sets

Lemma (Curve selection)

Let A be definable, let $x \in \bar{A}$. Then, there exists a continuous definable map $\gamma : [0,1] \to \bar{A}$ such that $\gamma(0) = x$ and $\gamma(]0,1]) \subset A$.

Proposition

A definable set is connected iff it is (definably) path-connected.

Dimension

Definition

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Proposition

- *If* $B \subset A$ *is nonempty definable :* $\dim(B) \leq \dim(A)$ *;*
- *if* $g : A \to B$ *is a definable bijection,* $\dim(A) = \dim(B)$ *;*
- $\dim(\mathbb{R}^n) = n$ *;*
- $\dim(A) \in \mathbb{N}$ *;*
- $\dim(A \cup B) = \max(\dim(A), \dim(B))$ *;*
- *if a cell decomposition of* A *is given,* $\dim(A)$ *is the max of the dimensions of these cells.*

Euler characteristic

Definition

Let A be definable. Suppose we have a cell decomposition of A. For all $k \in \mathbb{N}$, let n_k be the number of cells of dimension k. The Euler characteristic of A is :

$$\chi(A) := \sum_k (-1)^k n_k.$$

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Proposition

- $\chi(A)$ *does not depend on the choice of the cell decomposition ;*
- *if $g : A \to B$ is a definable bijection, $\chi(A) = \chi(B)$.*

Definable equivalence

Theorem

Let A and B be definable. There exists a definable bijection $g : A \to B$ iff $\dim(A) = \dim(B)$ and $\chi(A) = \chi(B)$.

Kurdyka-Łojasiewicz functions

Definition

Let $\Omega \subset \mathbb{R}^n$ *be open,* $f : \Omega \to \mathbb{R}$ *be smooth. Then* f *has the Kurdyka-Łojasiewicz property if for every* $x \in \Omega$*, there is an open neighborhood* U *of* a *in* Ω*,* $\eta \in \mathbb{R}_+^*$ *and a continuous function* $\phi : [0, \eta[\to \mathbb{R}_+$ *such that :*

- $\phi(0) = 0$ *;*
- ϕ *is* C^1 *on* $]0, \eta[$ *;*
- $\phi' > 0$ *on* $]0, \eta[$

such that on $U \cap \{f(a) < f < f(a) + \eta\}$, $\|\nabla(\phi \circ (f - f(a)))\| \geq 1$.

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Definition

Let $\Omega \subset \mathbb{R}^n$ be open, $f : \Omega \to \mathbb{R}$ be smooth. Then f has the Kurdyka-Łojasiewicz property if for every $x \in \Omega$, there is an open neighborhood U of a in Ω, $\eta \in \mathbb{R}_+^*$ and a continuous function $\phi : [0, \eta[\to \mathbb{R}_+$ such that :

- $\phi(0) = 0$;
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such that on $U \cap \{f(a) < f < f(a) + \eta\}$, $\|\nabla(\phi \circ (f - f(a)))\| \geq 1$.

Theorem (Kurdyka, 1998)

If f is definable in an o-minimal structure, then f has the Kurdyka-Łojasiewicz property. Moreover, we can suppose ϕ definable and concave.

Non Kurdyka-Łojasiewicz functions

Counter-examples

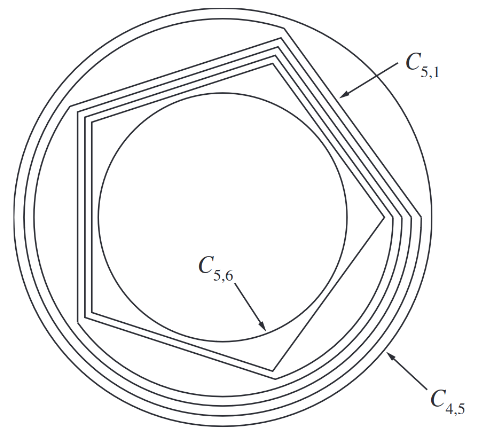
- $x \mapsto \sin(\frac{1}{x})e^{-x^{-2}}$
- 

Figure – *Bolte, Daniilidis, Ley, Mazet*

Definable spaces

Definition

Let S be a topological space. A definable atlas on S is the data of a finite open cover $(U_i)_{i\in I}$ of S and of homeomorphisms $g_i : U_i \to V_i \subset \mathbb{R}^{n_i}$, such that :

- $\forall i \in I, V_i$ *is definable ;*
- $\forall i, j \in I, g_i(U_i \cap U_j)$ *is definable ;*
- $\forall i, j \in I, g_j \circ g_i^{-1} : g_i(U_i \cap U_j) \to g_j(U_i \cap U_j)$ *is definable.*

Two definable atlases are equivalent if their union is a definable atlas.
A definable space is the data of a topological space and an equivalence class of definable atlases.

There is a notion of definable subset of a definable space. There is also a notion of morphism between definable spaces.

Proposition

A complex algebraic variety has a canonical structure of $\mathbb{R}_{alg}$-definable space.

Then it has also a structure of definable space for every o-minimal structure.

O-minimal Chow lemma

Theorem (Peterzil, Starchenko 2009)

A closed analytic subset of $\mathbb{C}^n$ is algebraic iff it is definable in an o-minimal structure.

GAGA

Theorem (Serre 1956)

Let V be a complex projective variety. Let $\mathrm{Coh}(V)$ (resp. $\mathrm{Coh}(V^{an})$) be the abelian category of coherent algebraic (resp. analytic) modules over V. Then the analytification functor $\mathrm{Coh}(V) \to \mathrm{Coh}(V^{an})$ is an equivalence of abelian categories.

Theorem (Bakker, Brunebarbe, Tsimerman 2023)

Let V be a complex variety. Let $\mathrm{Coh}(V)$ (resp. $\mathrm{Coh}(V^{def})$) be the abelian category of coherent algebraic (resp. definable) modules over V. Then the definabilisation functor $\mathrm{Coh}(V) \to \mathrm{Coh}(V^{def})$ is fully faithful, exact, and its essential image is stable under subobjects and quotients.

Pila-Wilkie theorem

Definition

Let $x = \left(\frac{p_1}{q_1}, \ldots, \frac{p_n}{q_n}\right) \in \mathbb{Q}^n$ *such that* $\forall i$, p_i *and* q_i *are coprime integers.*

The height of x *is* $H(x) := \max(|p_1|, \ldots, |p_n|, |q_1|, \ldots, |q_n|)$.

For $A \subset \mathbb{R}^n$, $T \in \mathbb{N}$, we note
$N(A, T) := \text{card}(A \cap \{x \in \mathbb{Q}^n, H(x) \leq T\})$.

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For $A \subset \mathbb{R}^n$, $T \in \mathbb{N}$, we note
$N(A, T) := \mathrm{card}(A \cap \{x \in \mathbb{Q}^n, H(x) \leq T\})$.

Theorem (Pila, Wilkie 2006)

If A *is definable and does not contain an infinite semialgebraic set, then for every* $\varepsilon \in \mathbb{R}_+^*$, $N(A, T) = o(T^\varepsilon)$.

Thanks to Téofil Adamski, Jérôme Milot, Yohan Brunebarbe and Julien Hermant for their advice.