Families of curves parametrized by $\mathcal{A}_g(n)$

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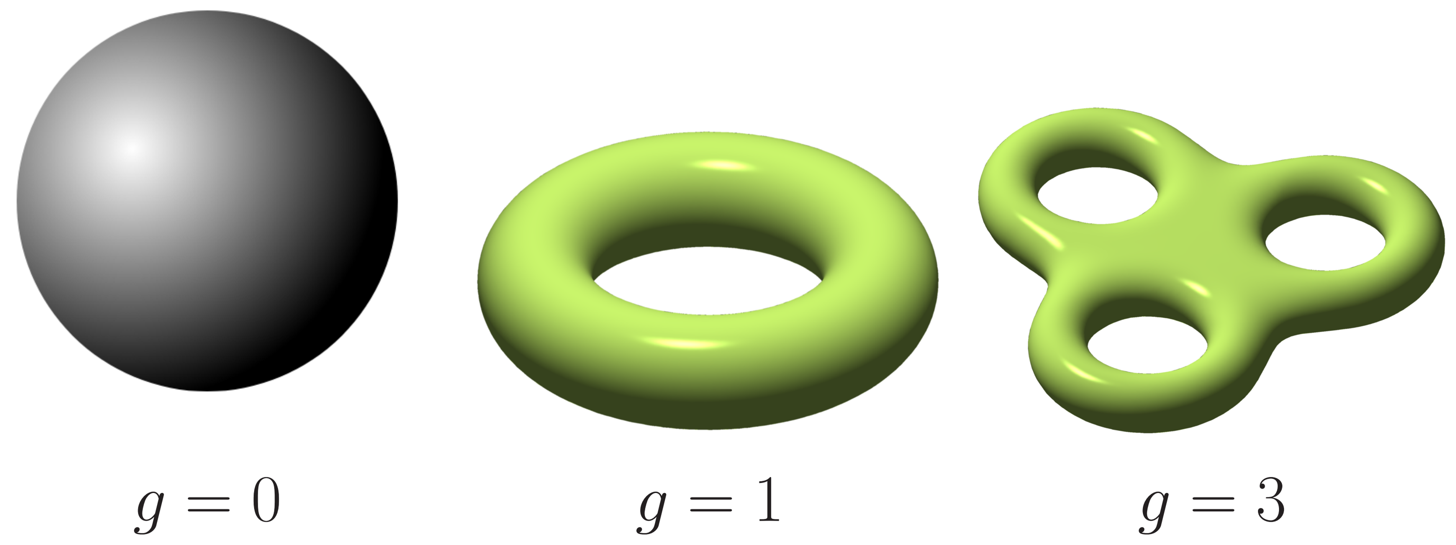
Algebraic varieties

Let k be a field.
Affine k-variety: zeros in $\bar{k}^n$ of polynomials with coefficients in k.
Projective k-variety: affine variety $\sqcup$ points at infinity.
Morphism of k-varieties: application whose coordinates are rational functions with coefficients in k.
Smooth: without singularities (complex case: manifold).

Curves

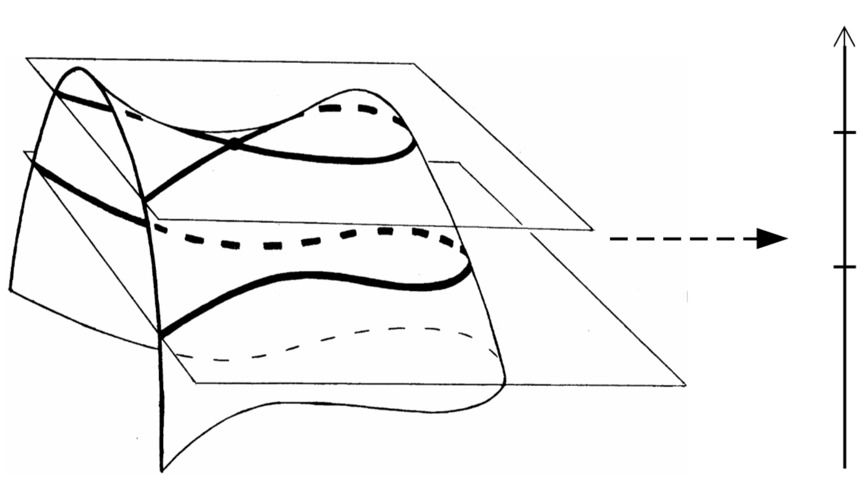
Curve: Algebraic variety of dimension 1.
A connected curve has a **genus** $g \in \mathbb{N}$.
Complex case: smooth projective curves are compact Riemann surfaces.



Connected smooth complex projective curves

Families

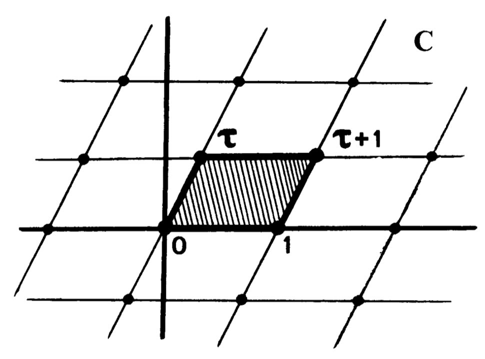
Let B be an algebraic variety.
Family of [curves] over B: Data of an algebraic k-variety F and a morphism $F \to B$ whose fibers are [curves].
Trivial family: Projection $C \times B \to B$ with C a [curve].
Isotrivial family: Families with isomorphic geometric fibers.



Family of curves parametrized by $\mathbb{A}^1_{\mathbb{R}}$

Abelian varieties

Abelian variety: Connected projective variety with a compatible group structure.
Abelian varieties are smooth and are Abelian groups.
Complex case: quotients of $\mathbb{C}^n$ by lattices satisfying Riemann condition. Isomorphic to $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ as real Lie groups.



Lattice of $\mathbb{C}$

Moduli spaces

Moduli space of [curves]: "Object" $\mathcal{M}$ such that we have the following correspondence:

$$\begin{matrix} F \\ \downarrow \\ B \end{matrix} \quad \Leftrightarrow \quad B \to \mathcal{M}$$

Families of [curves] over B — Morphisms

Allows to define "separable families" (a technical condition for positive characteristic).
$\mathcal{M}_g$: Moduli space of connected smooth projective curves of genus g.
$\mathcal{A}_g$: Moduli space of principally polarized Abelian varieties of dimension g.
$\mathcal{A}_g(n)$: Moduli space of principally polarized Abelian varieties of dimension g with n-level structure.
$\mathcal{A}_g(n)$ is an algebraic variety.

Theorem

For every integers $g, h \geq 2, n \geq 3$, for all but finitely many prime numbers p, for every field k of characteristic 0 or p, every separable family of curves of genus h over $\mathcal{A}_g(n)$ is isotrivial.

References

[1] Pierre Deligne and David Mumford. "The irreducibility of the space of curves of given genus". In: *Publications Mathématiques de l'IHES* 36 (1969), pp. 75–109.

[2] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.