

THÈSE PRÉSENTÉE
POUR OBTENIR LE GRADE DE
DOCTEUR
DE L'UNIVERSITÉ DE BORDEAUX
ECOLE DOCTORALE MATHÉMATIQUES ET
INFORMATIQUE

MATHÉMATIQUES PURES

Par **Éloan RAPION**

Hyperbolicité intermédiaire des variétés admettant une
variation de structure de Hodge et applications en
caractéristique positive

Sous la direction de : **Yohan BRUNEBARBE**

Soutenue le 25 juin 2026

Rapporteurs et membres du jury :

M. Yohan BRUNEBARBE	Chargé de recherche	CNRS, Université de Bordeaux	Directeur de thèse
M. Benoît CADOREL	Maître de conférences	Université de Lorraine	Rapporteur
M. Ariyan JAVANPEYKAR	Assistant Professor	Radboud University Nijmegen	Rapporteur hors jury
Mme Ekaterina AMERIK	Professeure	Université de Paris-Sud	Présidente
M. Benjamin BAKKER	Associate Professor	University of Illinois	Examineur
M. Sébastien BOUCKSOM	Directeur de recherche	CNRS, Sorbonne Université	Examineur
M. Simone DIVERIO	Professeur	Sapienza Università di Roma	Examineur
M. Henri GUENANCIA	Directeur de recherche	CNRS, Université de Bordeaux	Examineur

Hyperbolicité intermédiaire des variétés admettant une variation de structure de Hodge et applications en caractéristique positive

Résumé : Soit V une variété quasi-projective complexe lisse. Supposons qu'elle admette une variation de structure de Hodge dont l'application des périodes est génériquement immersive. Alors V présente diverses propriétés d'hyperbolicité. En particulier, soit (W, D) une compactification logarithmique lisse de V . Alors d'après les travaux de Zuo, Brunebarbe, Klingler, Totaro et Cadorel, les fibrés vectoriels cotangent logarithmique et canonique logarithmique sont big.

Dans cette thèse, on étudie la positivité des fibrés vectoriels $\Omega_W^p(\log D)$ pour tout p . On montre qu'ils sont tous big. On montre également que pour p assez grand, ils sont big au sens (plus fort) de Viehweg. On développe l'exemple de l'espace des modules des solides quintiques polarisés.

On s'intéresse ensuite au cas particulier des variétés localement symétriques. On calcule alors explicitement les lieux de base augmentés associés aux $\Omega_W^p(\log D)$. Ces ensembles décrivent le "défaut d'amplitude" d'un fibré vectoriel. On calcule également le lieu de base stable associé au fibré cotangent logarithmique (cas $p = 1$). Ce lieu de base stable décrit l'ensemble des vecteurs tangents sur lesquels toute forme différentielle symétrique logarithmique de degré 1 s'annule. Ces deux résultats généralisent un calcul fait par Mok sous les hypothèses $p = 1$ et V compact.

On donne deux applications en caractéristique positive. Premièrement, on prouve un théorème d'isotrivialité des familles de courbes paramétrées par l'espace des modules des variétés abéliennes principalement polarisées. La preuve nécessite l'étude des formes différentielles absolument logarithmiques définies par Abramovich (pour pallier le manque de compactifications logarithmiques lisses en caractéristique positive).

Deuxièmement, on considère le "théorème d'Arakelov" pour les schémas abéliens prouvé par Faltings. Celui-ci énonce que sur une variété algébrique complexe lisse, il n'existe qu'un nombre fini de schémas abéliens vérifiant une certaine propriété. On démontre qu'une version de ce théorème est vérifiée en caractéristique positive. On prouve aussi un énoncé similaire pour les familles de surfaces K3.

Mots-clés : Hyperbolicité, Variation de structure de Hodge, Variété localement symétrique

Intermediate hyperbolicity of varieties supporting a variation of Hodge structure and applications in positive characteristic

Abstract: Let V be a smooth complex quasiprojective variety. Suppose it supports a variation of Hodge structure whose period map is generically immersive. Then V exhibits various hyperbolicity properties. In particular, let (W, D) be a smooth logarithmic compactification of V . Then, according to the work of Zuo, Brunebarbe, Klingler, Totaro, and Cadorel, the logarithmic cotangent and logarithmic canonical vector bundles are big.

In this thesis, we study the positivity of the vector bundles $\Omega_W^p(\log D)$ for all p . We show that they are all big. We also show that for sufficiently large p , they are big in the (stronger) Viehweg sense. We develop the example of the moduli space of polarized quintic threefolds.

We then turn our attention to the special case of locally symmetric varieties. We then explicitly compute the augmented base loci associated with the $\Omega_W^p(\log D)$. These sets describe the "amplitude defect" of a vector bundle. We also compute the stable base locus associated with the logarithmic cotangent bundle (case $p = 1$). This stable base locus describes the set of tangent vectors on which every logarithmic symmetric differential form of degree 1 vanishes. These two results generalize a calculation made by Mok under the assumptions $p = 1$ and V compact.

We present two applications in positive characteristic. First, we prove an isotriviality theorem for families of curves parameterized by the moduli space of principally polarized Abelian varieties. The proof requires the study of the absolutely logarithmic differential forms defined by Abramovich (to compensate for the lack of smooth logarithmic compactifications in positive characteristic).

Second, we consider the "Arakelov theorem" for Abelian schemes proven by Faltings. This states that on a smooth complex algebraic variety, there are only a finite number of Abelian schemes satisfying a certain property. We demonstrate that a version of this theorem holds in positive characteristic. We also prove a similar statement for families of K3 surfaces.

Keywords: Hyperbolicity, Variation of Hodge structure, Locally symmetric variety

Contents

Remerciements	6
1 Introduction (français)	10
1.1 Hyperbolicité des variétés localement symétriques	10
1.1.1 Courbure de la métrique canonique	11
1.1.2 Hyperbolicité au sens de Brody, Kobayashi, Demailly et Borel	12
1.1.3 Positivité des fibrés vectoriels de formes différentielles logarithmiques	14
1.1.4 Finitude d'ensembles de morphismes	15
1.2 Hyperbolicité des variétés admettant une variation de struc- ture de Hodge	17
1.2.1 Courbure de la métrique	18
1.2.2 Hyperbolicité au sens de Brody, Kobayashi, Demailly et Borel	18
1.2.3 Positivité des fibrés vectoriels de formes différentielles logarithmiques	19
1.2.4 Finitude d'ensembles de morphismes	19
1.3 Résultats de la thèse	20
1.3.1 Lieux de base des formes différentielles logarithmiques de caractéristique quelconque	20
1.3.2 Hyperbolicité intermédiaire des variétés admettant une variation de structure de Hodge	22
1.3.3 Hyperbolicité des variétés localement symétriques . . .	23
1.3.4 Applications en caractéristique positive	25
1.4 Conventions	27
2 Introduction (English)	29
2.1 Hyperbolicity of locally symmetric varieties	29
2.1.1 Curvature of the canonical metric	30

2.1.2	Hyperbolicity in the sense of Brody, Kobayashi, Demailly, and Borel	31
2.1.3	Positivity of vector bundles of logarithmic differential forms	33
2.1.4	Finiteness of sets of morphisms	34
2.2	Hyperbolicity of varieties supporting a variation of Hodge structure	36
2.2.1	Curvature of the metric	36
2.2.2	Hyperbolicity in the sense of Brody, Kobayashi, Demailly, and Borel	37
2.2.3	Positivity of vector bundles of logarithmic differential forms	37
2.2.4	Finiteness of sets of morphisms	38
2.3	Results of the thesis	38
2.3.1	Base loci of logarithmic differential forms in arbitrary characteristic	39
2.3.2	Intermediate hyperbolicity of varieties supporting a variation of Hodge structure	40
2.3.3	Hyperbolicity of locally symmetric varieties	42
2.3.4	Applications in positive characteristic	43
2.4	Conventions	45
3	Absolutely logarithmic forms and base loci	47
3.1	Introduction	47
3.2	Base loci of a line bundle	48
3.2.1	Basic properties	48
3.2.2	Augmented base locus and Kodaira morphism	48
3.2.3	Nakamaye's theorem	49
3.2.4	Base change and spreading-out	49
3.2.5	Positivity of vector bundles	50
3.2.6	Non-Kähler locus	51
3.3	Absolutely logarithmic forms	53
3.3.1	Definition and basic properties	53
3.3.2	Compatibility with finite étale covering maps	55
3.3.3	Finiteness of the dimension	56
3.4	Base loci of the absolutely logarithmic forms	56
3.4.1	Partial functoriality	57
3.4.2	Descent of finite étale covering maps	58
3.4.3	Base change and spreading-out	59
3.4.4	Metric tools	61
3.5	Finiteness of sets of complete rational maps	62

3.5.1	Corollaries	64
3.5.2	Outline of the proof	65
3.5.3	Families of rational maps	66
3.5.4	Pulling back absolutely logarithmic forms	67
3.5.5	Notations	69
3.5.6	Finding again the initial morphism	70
3.5.7	End of the proof	73
4	Varieties supporting a variation of Hodge structure	75
4.1	Introduction	75
4.1.1	Bigness	75
4.1.2	Augmented base locus	76
4.1.3	Outline of the proofs	77
4.2	Definitions and notations	78
4.3	L-bigness	79
4.4	Curvature of the Hodge metric	80
4.5	Augmented base locus	81
4.6	Decomposition of polarized infinitesimal variations of Hodge structure	84
4.7	Rank criterion	86
4.7.1	Decomposed PIVHS and rank	87
4.7.2	Augmented base locus and rank	88
4.7.3	Positivity and rank	89
4.8	Example: the moduli space of quintic threefolds	89
5	Singularities of the Hodge metric	94
5.1	Introduction	94
5.2	Growth moderation near singularities	94
5.2.1	Functions with moderate growth	95
5.2.2	Differential forms with moderate growth on the punc- tured polydisk	96
5.2.3	Differential forms with moderate growth on a complex manifold	99
5.3	Real polarized variations of Hodge structure and singularities .	101
5.3.1	Singularities of the Hodge metric	102
6	Locally symmetric varieties	104
6.1	Introduction	104
6.1.1	Bigness	104
6.1.2	Augmented base locus	105
6.1.3	Stable base locus in degree one	105

6.1.4	Outline of the proofs	106
6.2	PVHS on locally symmetric varieties	106
6.2.1	The construction of Zucker	107
6.2.2	Higgs field and conjugate Higgs field	109
6.2.3	A useful PVHS	111
6.3	Characteristic subvarieties	112
6.3.1	Characteristic subvarieties	113
6.3.2	Higher degree characteristic subvarieties	114
6.4	Augmented base locus	119
6.5	Stable base locus	120
6.6	Application: isotriviality of families of curves	126
7	Applications in positive characteristic	127
7.1	Introduction	127
7.2	Isotriviality of families of curves parametrized by the moduli space of Abelian varieties	130
7.3	Infinitesimal Arakelov's theorems	131
7.3.1	Families of Abelian varieties	131
7.3.2	Families of K3 surfaces	135
7.3.3	Families of polarized varieties in mixed characteristic	138

Remerciements

Tout d’abord, je remercie sincèrement mon directeur de thèse Yohan Brunebarbe pour son encadrement d’une grande qualité. Merci pour ta disponibilité, tout le savoir que tu as partagé, ton soutien et ta bienveillance tout au long de ces trois ans et demi.

Je remercie mes deux rapporteurs, Benoît Cadorel et Ariyan Javanpeykar, pour leur minutieux travail de lecture du manuscrit et leurs nombreuses suggestions pour améliorer celui-ci ou en étendre les résultats. Merci à Ekaterina Amerik, Benjamin Bakker, Sébastien Boucksom, Simone Diverio et Henri Guenancia d’avoir accepté de faire partie de mon jury.

Un chaleureux merci à Lucas, Shourya, Kirsten, Florian et Elena, qui m’ont accueilli dans le laboratoire de l’IMB et dans la ville de Bordeaux.

Je remercie les chercheurs et chercheuses permanents ainsi que les membres du personnel de l’IMB, qui a constitué un cadre idéal pour cette thèse. Merci en particulier à Cyril Mauvillain, Nadia Mechalakh et Thomas Ribeyron, qui m’ont beaucoup aidé.

Merci à l’ensemble des doctorantes et doctorants de l’IMB. Merci aux membres de Lambda pour les divers événements organisés et la super ambiance qu’ils ont pu installer : Kylian, Virginie et Loïs pour leur accueil et les soirées jeux, Antoine avec qui j’ai eu le plaisir de préparer de nombreux TD, Julien pour son soutien, Agathe, Marco et Ishak pour leurs conseils pour l’organisation des séminaires, Gabriel mon chaleureux co-organisateur de séminaires, Afonso, Alix et Alexis pour avoir pris la succession, la présidente Erell pour son organisation et son humour, Ali pour nos discussions musicales, Melvin pour les situations traquenardesques, Jonathan pour son sérieux et le nouveau président Tom pour les nombreux événements mis en place. Merci aux doctorantes et doctorants qui étaient déjà là quand je suis arrivé : Samuel, Jean, Pierre-Jean, Magalie, Gauthier, Issa, Adrien, Jean, Fabrice, Khaoula, Guillaume... Merci à celles et ceux qui ont été mes collègues tout

au long de ces trois ans : Marien et Xiaoqiu pour les passionnantes discussions historiques, Théo, mon plus ancien ami dans ce laboratoire, pour son aide sur l'administratif, Julien pour son humour singulier, mon collègue de bureau Omar pour ses explications de maths appliqués, et bien sûr Simon, Henry, Guilhem, Mathias, Nicolas, Zhuo et Zhubin.

Merci aux doctorantes et doctorants que j'ai pu voir arriver au cours de ma thèse : Aymeric pour nos échanges cinématographiques, Kaplan qui lui non plus ne cessera jamais de me surprendre, Laurent pour ses explications sur le tennis, Alexis pour m'avoir appris entre autres le mot hypercactus, ainsi que Emeric, Samy et Elise. Je remercie également celles et ceux qui ont commencé leur thèse cette année : Briec pour nos riches conversations plus ou moins brainrot à l'interface complexe / p -adique, mon co-bureau Fabien pour nos discussions (et parfois le désarroi mathématique que nous avons partagé), Anouk pour les pauses de 16h, Leo pour sa bonne humeur, ainsi que Guido, Rayane, Alix, Dylan, Héloïse, Antonin, Gabriele, Adam, Valentine et Martin. Merci aux posts-docs Julien (pour les conseils cinéma), Guillaume (pour ses cours sur les transformers), Beppe et Wenwen. Merci à Florian, Matthieu, Giacomo, Gurwane et Léon, les futurs doctorants et doctorante. Une pensée pour Lucas, camarade de Rennes décédé durant sa thèse à Bordeaux l'année dernière.

Je souhaite remercier les membres de l'ANR Cyclades pour les passionnantes rencontres mathématiques, ainsi que les nombreux mathématiciennes et mathématiciens que j'ai eu le plaisir de rencontrer en conférences. Merci tout particulièrement à André, Thomas, Bastien et Ariane.

Merci aux nombreux professeurs, directrices et directeurs de stages qui m'ont formés et soutenus tout au long de ma vie et de mes études, sans qui cette thèse n'auraient bien sûr pas pu exister. Je ne peux pas être exhaustif, mais je souhaite mentionner Béatrice Tual, Olivier Le Poittevin, Benoît Bregeon, Lionel Chaussade, François Sauvageot, Cécile Pierrot, Emiliano Lorini et Gerard Freixas à Montplet.

Merci à l'ensemble du "Fredoreso" nantais, à commencer par Antoine, pour ton immense soutien, ta disponibilité, ta sincérité, ta générosité omniprésentes tout au long de cette thèse. Merci également à Jules, Alice, Clément, Greg et Yami. Enfin merci à Marion de m'avoir fait confiance pour être témoin de ce magnifique mariage. Je réitère mes vœux de bonheur pour toi ainsi que pour Irénée.

Merci à l'équipe des crêpiers rennais : Téofil pour sa disponibilité, pour les longs appels téléphoniques et pour son hospitalité, Jérôme pour continuer

à partager mes mauvais goûts musicaux, Matthieu pour son organisation, Julius pour son imprévisibilité, Maxence pour sa bienveillance et Antoine pour son honnêteté.

À Bordeaux en dehors du laboratoire, merci à Adèle, Coline, Ronan et Auriane pour leur amitié et leur hospitalité.

Dans les inclassables, je remercie dans le désordre Armand, Sandrine et Sylvain pour les soirées jeux, Éric et Christophe pour leur soutien, Karine Beauchard pour ses cours d'excellente qualité et sa technique d'effacement de tableau, Inès pour les cours de yoga, Emmanuelle (de l'équipe géométrie) pour toutes les discussions du vendredi midi et Clovis pour l'hébergement.

Finalement, je tiens à remercier l'ensemble de ma (bruyante) famille, mes grand-mères Marie-Jo et Rose-Andrée, mes oncles, tantes, cousins et cousines. Une pensée pour mon grand-père Raymond décédé peu de temps avant le début de cette thèse. Merci pour tout à mes parents Hélène et Dodo, sans qui rien n'aurait été possible et grâce à qui j'ai pu travailler sereinement sur cette thèse. Merci enfin à ma sœur Maëlys, c'est toujours avec une grande joie que je te retrouve à Sainte-Luce.

Chapter 1

Introduction (français)

1.1 Hyperbolicité des variétés localement symétriques

Définissons le premier objet central de cette thèse : les variétés localement symétriques. Soit Σ un domaine symétrique borné dans $\mathbb{C}^d$ (pour un $d \in \mathbb{N}$). Son groupe de biholomorphismes $\text{Aut}(\Sigma)$ admet une unique structure de groupe de Lie réel. Le groupe $\text{Aut}(\Sigma)$ a un nombre fini de composantes connexes. Notons $\text{Aut}(\Sigma)^0$ sa composante neutre. Tout réseau sans torsion Γ de $\text{Aut}(\Sigma)^0$ agit librement et proprement discontinûment sur Σ . Ainsi, le quotient $X := \Gamma \backslash \Sigma$ admet une (unique) structure de variété analytique complexe pour laquelle le passage au quotient est holomorphe.

Définition 1.1.1. *Une variété localement symétrique X est une variété analytique complexe biholomorphe au quotient d'un domaine symétrique borné par l'action d'un réseau Γ sans torsion de la composante neutre de son groupe de biholomorphismes.*

La variété localement symétrique X est arithmétique si le réseau Γ est arithmétique.

Par le théorème d'arithmécité de Margulis (voir [54, Theorem 1]), si Σ est irréductible et de rang au moins 2, le réseau Γ est arithmétique. Dans ce cas, toute variété localement symétrique ayant Σ pour revêtement universel est arithmétique. Alors d'après Baily et Borel, X admet une structure de variété algébrique.

Théorème 1.1.2 (voir [4]). *Une variété arithmétique localement symétrique admet une structure de variété quasi-projective complexe (lisse) compatible avec sa structure analytique.*

Borel a montré dans ce cas l'unicité de la structure de variété algébrique complexe compatible avec la structure analytique (voir [12]). Une variété

arithmétique localement symétrique admet une compactification projective normale canonique $\bar{X}^{SBB}$ appelée *compactification de Satake-Baily-Borel*.

Un domaine symétrique borné admet une métrique de Kähler-Einstein canonique : la métrique de Bergman. Elle est invariante par l'action de $\text{Aut}(\Sigma)^0$ (dans le cas Σ irréductible, cela la caractérise à une constante multiplicative près parmi les métriques hermitiennes). En particulier, elle est invariante par l'action de Γ . Par conséquent, la métrique de Bergman induit une métrique de Kähler-Einstein sur X que nous appellerons la *métrique canonique* de X .

1.1.1 Courbure de la métrique canonique

La courbure de la métrique canonique d'une variété localement symétrique admet plusieurs propriétés de négativité.

Proposition 1.1.3 (voir [58, Proposition 2 p.46]). *Soit X une variété localement symétrique. Alors les courbures sectionnelle, bisectionnelle holomorphe et de Ricci de la métrique canonique de X sont semi-négatives. De plus les courbures sectionnelle holomorphe et scalaire de la métrique canonique de X sont strictement négatives.*

La propriété précédente est une propriété d'hyperbolicité des variétés localement symétriques. Dans l'autre sens, le théorème suivant est une propriété de “non-hyperbolicité” : elle interdit la construction d'une métrique à courbure “plus négative” que la métrique canonique. Par exemple, il n'existe pas de métrique hermitienne à courbure bisectionnelle holomorphe strictement négative. Ce théorème a été démontré par Mok dans le cas compact (voir [60] ou [58, Theorem 1 p.99]) puis par To dans le cas général.

Théorème 1.1.4 (voir [82]). *Soit X une variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2. Alors toute métrique hermitienne à courbure bisectionnelle holomorphe semi-négative est un multiple constant de la métrique canonique.*

Pour prouver le théorème précédent, Mok définit les “(higher) characteristic subvarieties”. Ces sous-variétés décrivent le défaut de négativité de la courbure de la métrique canonique (et donc d'après le Théorème 1.1.4 le défaut d'hyperbolicité de X). Nous ne définissons ici que les deux extrêmes $\mathcal{S}_1$ (la “characteristic subvariety” originale) et $\mathcal{S}_{r-1}$ (la plus grande des “higher characteristic subvarieties”, où r désigne le rang du revêtement universel de X en tant que domaine symétrique borné). Remarquons d'abord que si x est

un point d'une variété localement symétrique et v est un vecteur tangent en x , alors l'ensemble $\mathcal{N}(v)$ des vecteurs tangents w en x tels que la métrique canonique est à courbure bisectionnelle holomorphe nulle sur le plan engendré par v et w est un espace vectoriel.

Définition 1.1.5. *La sous-variété caractéristique $\mathcal{S}_1$ (resp. $\mathcal{S}_{r-1}$) de X est l'ensemble des $[v] \in \mathbb{P}\Omega_X^1$ tels que la dimension de $\mathcal{N}(v)$ soit maximale (resp. soit non nulle).*

Mok démontre lorsque X est compacte une variante Finsler du Théorème 1.1.4 (il montre aussi que l'on peut déduire le cas compact du Théorème 1.1.4 de cette variante Finsler). Elle est implicite dans [58, §3 p.113] et énoncée dans [59, Theorem 3].

Théorème 1.1.6. *Soit X une variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2. Alors toute métrique hermitienne à courbure semi-négative sur le fibré en droites $\mathcal{O}(-1)$ sur $\mathbb{P}\Omega_X^1$ coïncide en restriction à $\mathcal{S}_1$ avec un multiple constant de la métrique canonique.*

1.1.2 Hyperbolicité au sens de Brody, Kobayashi, Demailly et Borel

Plusieurs notions classiques d'hyperbolicité d'une variété algébrique complexe V se définissent en considérant les morphismes d'une variété donnée vers V .

Définition 1.1.7. • *Un espace analytique complexe V est hyperbolique au sens de Brody si toute application holomorphe $\mathbb{C} \rightarrow V$ est constante. Une variété algébrique complexe est hyperbolique au sens de Brody si l'espace analytique associé est hyperbolique au sens de Brody.*

- *Un espace analytique complexe V est hyperbolique au sens de Kobayashi si l'ensemble de ses points admet une distance (au sens de la topologie générale) pour laquelle toute application holomorphe du disque de Poincaré vers V est lipschitzienne, le disque étant muni de la métrique de Poincaré. Une variété algébrique complexe est hyperbolique au sens de Kobayashi si l'espace analytique associé est hyperbolique au sens de Kobayashi.*
- *Une variété projective complexe V est algébriquement hyperbolique (ou hyperbolique au sens de Demailly) si pour tout fibré en droites ample A sur V , il existe $\varepsilon \in \mathbb{R}_{>0}$ tel que pour tout $g \in \mathbb{N}$, toute courbe projective*

complexe lisse connexe C de genre g et tout morphisme non constant $f : C \rightarrow V$, on a $2g - 2 \geq \varepsilon \deg f^* A$ (cela ne dépend pas du choix de A).

- Une variété algébrique complexe V est hyperbolique au sens de Borel si pour toute variété algébrique complexe réduite V' , toute application holomorphe $(V')^{an} \rightarrow V^{an}$ est induite par un morphisme de variétés algébriques $V' \rightarrow V$.

Ces différentes notions sont liées de la manière suivante.

Proposition 1.1.8 (voir [49, 46]). *Un espace analytique complexe hyperbolique au sens de Kobayashi est hyperbolique au sens de Brody. Une variété algébrique complexe hyperbolique au sens de Kobayashi ou de Borel est hyperbolique au sens de Brody. Pour une variété projective complexe, les hyperbolicités au sens de Kobayashi, Brody et Borel sont équivalentes, et elles impliquent l'hyperbolicité algébrique.*

Dans le cas des variétés localement symétriques, on a les énoncés suivants.

Proposition 1.1.9. *Une variété localement symétrique est hyperbolique au sens de Brody et de Kobayashi. Si elle est arithmétique et compacte, elle est en plus algébriquement hyperbolique.*

Preuve. La courbure sectionnelle holomorphe de la métrique de Bergman d'un domaine symétrique borné est négative. Par homogénéité, elle est bornée supérieurement par une constante $A \in \mathbb{R}_{<0}$. La courbure sectionnelle holomorphe de la métrique canonique d'une variété localement symétrique X est donc aussi bornée supérieurement par une constante $A \in \mathbb{R}_{<0}$. On peut alors lui appliquer le lemme d'Ahlfors-Schwarz (voir [49, Theorem 2.1.4]) : il existe une constante $C \in \mathbb{R}_{>0}$ telle que les applications holomorphes du disque de Poincaré vers X sont C -lipschitziennes. Ainsi X est hyperbolique au sens de Kobayashi. Le reste découle immédiatement de la proposition précédente. $\square$

C'est en référence au théorème de Borel suivant que Javanpeykar et Kucharczyk ont donné son nom à l'hyperbolicité au sens de Borel dans [46]. Il implique en particulier que la structure algébrique de Baily-Borel sur X est la seule compatible avec la structure analytique.

Théorème 1.1.10 (voir [12, Theorem 3.10]). *Les variétés arithmétiques localement symétriques sont hyperboliques au sens de Borel.*

On peut prouver des énoncés d’hyperbolicité de la compactification de Satake-Baily-Borel. Ceux-ci renforcent les théorèmes qui précèdent, à ceci près qu’il faut éventuellement remplacer X par un revêtement étale fini.

Théorème 1.1.11. *Soit X une variété arithmétique localement symétrique. Alors X admet un revêtement étale fini X' tel que la compactification de Satake-Baily-Borel $\bar{X}'^{SBB}$ de X' est hyperbolique au sens de Brody, Kobayashi, Demailly et Borel.*

Nadel a démontré une version intermédiaire entre l’hyperbolicité au sens de Brody de X' et celle de $\bar{X}'^{SBB}$: l’hyperbolicité au sens de Brody “logarithmique” de X' (voir [63, Theorem 0.2]). Rousseau a démontré dans [74, Theorem 1.1] l’hyperbolicité au sens de Kobayashi “logarithmique” de X' . Enfin, Brunenbarbe a remarqué que par un argument sur la structure du bord de la compactification de Satake-Baily-Borel, le théorème de Nadel implique le Théorème 1.1.11, et donc en particulier le théorème de Rousseau (voir [20, Sous-section 1.2]).

1.1.3 Positivité des fibrés vectoriels de formes différentielles logarithmiques

Une variété algébrique intègre est dite *de type général* si elle est birationnelle à une variété algébrique propre et lisse de type général (c’est-à-dire dont le fibré en droites canonique est big). Cette propriété ne dépend pas du choix du modèle birationnel propre et lisse. Tai a montré que toute variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2 admet un revêtement étale fini de type général (voir [3, Chapter IV Theorem 1.4]). Brunenbarbe a démontré l’énoncé plus fort qui suit.

Théorème 1.1.12 (voir [20, Corollary 1.7]). *Soit X une variété arithmétique localement symétrique. Alors X admet un revêtement étale fini X' tel que toute sous-variété algébrique de la compactification de Satake-Baily-Borel $\bar{X}'^{SBB}$ de X' est de type général.*

Cet énoncé devient faux si on ne permet pas de prendre un revêtement étale fini : une variété arithmétique localement symétrique n’est pas nécessairement de type général (voir [20, Remark 1.2(ii)]). Cadorel a déterminé explicitement des espaces de modules de variétés abéliennes principalement polarisées avec structure de niveau qui vérifient l’énoncé précédent, ou qui le vérifient pour des sous-variétés de codimension assez petite (voir [25]).

Pour les formes différentielles de degré 1, le résultat suivant apparaît comme corollaire du Théorème 1.2.4 qui sera énoncé dans la section suivante

(voir aussi [26]). Un fibré vectoriel E sur une variété algébrique complexe propre W est dit *L-big* s'il existe $\varepsilon \in \mathbb{R}_{>0}$ tel que pour tout $k \gg 0$, on a $\dim H^0(W, \text{Sym}^k E) \geq \varepsilon k^{\dim W + \text{rk } E - 1}$. Le fibré vectoriel E est dit *faiblement positif* s'il existe un ouvert Zariski-dense $U \subset W$ tel que pour tout fibré en droites ample A sur W et tout entier strictement positif α , il existe un entier strictement positif β tel que $\text{Sym}^{\alpha\beta} \Omega_V^1(\log D) \otimes A^\beta$ est engendré sur U par ses sections globales.

Théorème 1.1.13. *Soit X une variété arithmétique localement symétrique. Alors pour toute log-compactification $(\bar{X}, D)$ de X , le fibré vectoriel $\Omega_{\bar{X}}^1(\log D)$ est L-big et faiblement positif.*

Mok a montré que dans le cas compact, le fibré vectoriel Ω_X^1 n'est pas ample (plus précisément, il n'est même pas big au sens de Viehweg). L'obstruction est décrite par le lieu de base augmenté $\mathbb{B}_+(\mathcal{O}(1))$ du fibré en droites tautologique $\mathcal{O}(1)$ sur $\mathbb{P}\Omega_X^1$.

Théorème 1.1.14 (voir [58, Proposition 4 p.262]). *Soit X une variété localement symétrique compacte dont le revêtement universel est irréductible et de rang au moins 2. Alors la sous-variété caractéristique $\mathcal{S}_{r-1} \subset \mathbb{P}\Omega_X^1$ est le lieu de base stable et le lieu de base augmenté du fibré en droites tautologique $\mathcal{O}(1)$ de $\mathbb{P}\Omega_X^1$.*

L'inclusion $\mathbb{B}_+(\mathcal{O}(1)) \subset \mathcal{S}_{r-1}$ est un résultat d'hyperbolicité. L'inclusion $\mathcal{S}_{r-1} \subset \mathbb{B}_+(\mathcal{O}(1))$ est un résultat de non-hyperbolicité (c'est une obstruction à la positivité de Ω_X^1). L'inclusion de $\mathcal{S}_{r-1}$ dans le lieu de base stable de $\mathcal{O}(1)$ est un résultat plus fort (il signifie que pour tout $n \in \mathbb{N}_{>0}$, toute section globale de $\text{Sym}^n \Omega_X^1$ s'annule sur $\mathcal{S}_{r-1}$). La preuve de celle-ci utilise le théorème de rigidité Finsler (Théorème 1.1.6).

1.1.4 Finitude d'ensembles de morphismes

Borel et Narasimhan ont montré la restriction suivante pour définir des morphismes vers une variété localement symétrique.

Théorème 1.1.15 (voir [14, Proposition 2.1, Theorem 4.6]). *Soit X une variété localement symétrique. Soit M le complémentaire d'un diviseur à croisements normaux dans une variété analytique complexe compacte connexe. Pour tout $x \in M$, $y \in X$ et morphisme de groupes $\rho : \pi_1(M, x) \rightarrow \pi_1(X, y)$, il existe au plus une application holomorphe $f : M \rightarrow X^{an}$ telle que $f(x) = y$ qui induise ρ sur les groupes fondamentaux.*

Pour une variété arithmétique localement symétrique X de revêtement universel noté Σ et de groupe fondamental noté Γ , notons $l(\Sigma)$ la dimension maximale de $\mathcal{N}(v)$ pour v parmi les vecteurs tangents non nuls de X ($l(\Sigma)$ est également la dimension maximale d’une “composante de bord propre” de Σ , voir [88, Section 5]) et $l(\Gamma)$ la dimension maximale pour une composante de bord de $\bar{X}^{SBB}$. On a $l(\Sigma) \geq l(\Gamma)$. Noguchi a démontré le théorème suivant.

Théorème 1.1.16 (voir [67]). *Soit X une variété arithmétique localement symétrique de revêtement universel noté Σ et de groupe fondamental noté Γ . Soit M le complémentaire d’un diviseur à croisements normaux dans une variété analytique complexe compacte connexe.*

- *Il existe un “espace des modules des morphismes $M \rightarrow X$ ”, c’est-à-dire un espace analytique complexe $\text{Hol}(M, X)$ et une application holomorphe $f : \text{Hol}(M, X) \times M \rightarrow X$ tels que pour tout espace analytique M' et toute application holomorphe $g : M' \times M \rightarrow X$, il existe une unique application holomorphe $h : M' \rightarrow \text{Hol}(M, X)$ telle que $g = f \circ (h \times \text{id}_M)$.*
- *L’espace des modules $\text{Hol}(M, X)$ est un ouvert de Zariski d’un espace analytique complexe compact.*
- *Pour tout $k \in \mathbb{N}$, les applications holomorphes $M \rightarrow X$ d’image de dimension k sont paramétrées par un ouvert-fermé $\text{Hol}(M, X, k) \subset \text{Hol}(M, X)$.*
- *Si $k > l(\Gamma)$, alors $\text{Hol}(M, X, k)$ est compact. Si $k > l(\Sigma)$, alors $\text{Hol}(M, X, k)$ est fini.*
- *Soit C une composante connexe de $\text{Hol}(M, X)$, soit $x \in M$. Alors l’application :*

$$\begin{aligned} C &\rightarrow X \\ c &\mapsto f(c, x) \end{aligned}$$

est un plongement fermé sur une sous-variété complexe totalement géodésique.

On peut donner un énoncé de finitude plus précis dans le cas particulier où M est une variété algébrique et X est l’espace des modules $\mathcal{A}_g(n)$ de variétés abéliennes principalement polarisées de dimension $g \in \mathbb{N}_{>0}$ avec structure de niveau $n \geq 3$. Si V est une variété algébrique complexe, un morphisme $f : V \rightarrow \mathcal{A}_g(n)$ correspond à la donnée d’une classe d’isomorphismes de schémas abéliens principalement polarisés avec structure de niveau n sur V . Comme il n’y a qu’un nombre fini de structures de niveau n sur une famille donnée, un énoncé de finitude d’ensemble de morphismes $V \rightarrow \mathcal{A}_g(n)$ équivaut à un énoncé de finitude d’ensemble de classes d’isomorphismes de

schémas abéliens principalement polarisés. Si V est de dimension 1, on a le théorème de Faltings suivant.

Théorème 1.1.17 (voir [38]). *Soit C une courbe algébrique complexe.*

- *L'ensemble des classes d'isomorphisme de schémas abéliens principalement polarisés sur C est paramétré par un schéma de type fini sur $\mathbb{C}$.*
- *En particulier, il n'y a qu'un nombre fini de schémas abéliens principalement polarisés rigides (c'est-à-dire non-déformables).*
- *Une famille A de schémas abéliens principalement polarisés sur C est déformable si et seulement s'il existe un endomorphisme antisymétrique du système local $R^1\pi_*\mathbb{Z}_A$ qui ne provient pas d'un endomorphisme du C -schéma abélien principalement polarisé A .*

Deligne a remarqué dans [32, Section 3] que l'énoncé de finitude se réduit au cas de la dimension 1, ce qui donne le théorème suivant.

Théorème 1.1.18. *Soit V une variété algébrique complexe. Alors pour tous les schémas abéliens A principalement polarisés sur V sauf un nombre fini d'entre eux, il existe un endomorphisme antisymétrique du système local $R^1\pi_*\mathbb{Z}_A$ qui ne provient pas d'un endomorphisme de la famille A .*

En caractéristique positive, la variété $\mathcal{A}_g(n)$ perd une partie des propriétés d'hyperbolicité qu'elle présente en caractéristique 0. En caractéristique au moins 3, Moret-Bailly a prouvé que $\mathcal{A}_g(n)$ contient des courbes rationnelles (voir [61]). Cela contredit l'énoncé analogue à Théorème 1.1.12 car $\mathbb{P}^1$ n'est pas de type général. Cela contredit aussi l'énoncé analogue au premier point de Théorème 1.1.17 car en composant un morphisme non constant $\mathbb{P}^1 \rightarrow \mathcal{A}_g(n)$ par les endomorphismes de $\mathbb{P}^1$, on obtient un ensemble de morphismes $\mathbb{P}^1 \rightarrow \mathcal{A}_g(n)$ de degré non borné.

1.2 Hyperbolicité des variétés admettant une variation de structure de Hodge

Les variétés arithmétiques localement symétriques sont un cas particulier de variété algébrique admettant une variation de structure de Hodge entière polarisable, ou $\mathbb{Z}$ -VSHP. Une partie des énoncés d'hyperbolicité de la section précédente se généralise au cas des variétés admettant une VSHP. Lorsque l'on parlera de variété admettant une VSHP, on supposera toujours l'immersivité de l'application des périodes, au moins génériquement.

1.2.1 Courbure de la métrique

Considérons une variété analytique complexe M munie d'une variation de structure de Hodge complexe polarisable ($\mathbb{C}$ -VSHP) $\mathbb{V}$. Le choix d'une polarisation de $\mathbb{V}$ donne une métrique hermitienne dite *de Hodge* sur le fibré vectoriel holomorphe sous-jacent à $\mathbb{V}$. Comme la polarisation de $\mathbb{V}$ induit une polarisation de la $\mathbb{C}$ -VSHP $\text{End}_{\mathbb{C}}(\mathbb{V})$, elle définit aussi une métrique de Hodge sur $\text{End}_{\mathbb{C}}(\mathbb{V})$. Supposons que l'application des périodes soit immersive. Cela signifie que le morphisme de $\mathcal{O}_M$ -modules défini par le champ de Higgs $\theta : T_M \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}) \otimes_{\mathbb{C}} \mathcal{O}_M$ est une injection localement scindée. Ainsi, on peut tirer en arrière la métrique de Hodge sur le fibré sous-jacent à $\text{End}_{\mathbb{C}}(\mathbb{V})$ pour obtenir une métrique hermitienne sur T_M . Lu a démontré que cette métrique est Kähler ([53, Theorem 1.2]), et que sa courbure a les propriétés de négativité suivantes.

Théorème 1.2.1 (voir [53, Theorem 1.2]). *Soit M une variété analytique complexe munie d'une variation de structure de Hodge complexe polarisée $\mathbb{V}$. Supposons que l'application des périodes soit immersive. Soit g la métrique de Kähler définie sur M par $\mathbb{V}$.*

Alors la courbure bisectionnelle holomorphe de g est semi-négative. Ses courbures sectionnelle holomorphe, de Ricci et scalaire sont strictement négatives.

1.2.2 Hyperbolicité au sens de Brody, Kobayashi, Demailly et Borel

En utilisant la métrique précédemment définie, Griffiths et Schmid ont démontré l'hyperbolicité au sens de Kobayashi et donc aussi de Brody des variétés admettant une $\mathbb{C}$ -VSHP.

Théorème 1.2.2 (voir [40, Corollary 9.3]). *Soit M une variété analytique complexe munie d'une variation de structure de Hodge complexe polarisable. Supposons que l'application des périodes soit immersive.*

Alors M est hyperbolique au sens de Brody et de Kobayashi.

Bakker, Brunenbarbe et Tsimerman ont montré l'hyperbolicité au sens de Borel pour les variétés admettant une $\mathbb{Z}$ -VSHP dans [5, Corollary 7.1]. En utilisant cet article, Javanpeykar et Litt ont montré l'hyperbolicité algébrique des variétés admettant une $\mathbb{Z}$ -VSHP à monodromie locale unipotente (voir [47, Theorem 4.2]). Deng a généralisé ces deux résultats aux variétés admettant une $\mathbb{C}$ -VSHP (voir [34]).

Théorème 1.2.3. *Soit V une variété algébrique complexe lisse munie d'une $\mathbb{C}$ -VSHP. Supposons que l'application des périodes soit quasi-finie.*

Alors V est hyperbolique au sens de Demailly et Borel.

Cadorel et Deng ont prouvé la “pseudo-hyperbolicité algébrique” dans le cadre plus général des variétés admettant un fibré harmonique nilpotent dont le champ de Higgs est injectif en un point (voir [27, Theorem A]).

1.2.3 Positivité des fibrés vectoriels de formes différentielles logarithmiques

Le fibré en droites canonique logarithmique et le fibré vectoriel cotangent logarithmique d’une variété quasi-projective complexe lisse admettant une $\mathbb{C}$ -VSHP ont les propriétés de positivité suivantes.

Théorème 1.2.4. *Soit $\bar{V}$ une variété projective complexe lisse. Soit $D \subset \bar{V}$ un diviseur à croisements normaux. Notons $V := \bar{V} \setminus D$. Soit $\mathbb{V}$ une $\mathbb{C}$ -VSHP sur V . Supposons que l’application des périodes soit immersive en un point de V . Alors :*

- *V est de type log-général, c’est-à-dire que le fibré en droites canonique logarithmique $\omega_{\bar{V}}(D)$ est big,*
- *le fibré vectoriel cotangent logarithmique $\Omega_{\bar{V}}^1(\log D)$ est L -big et faiblement positif.*

Dans le théorème précédent, le fait que $\omega_{\bar{V}}(D)$ soit big et $\Omega_{\bar{V}}^1(\log D)$ soit faiblement positif a été démontré par Zuo (voir [91, Theorem 0.1]). Brunebarbe, Klingler et Totaro ont prouvé que $\Omega_{\bar{V}}^1$ est L -big quand $V = \bar{V}$ (voir [24, Theorem 3.1]). Brunebarbe a ensuite montré que $\Omega_{\bar{V}}^1(\log D)$ est L -big si la monodromie de $\mathbb{V}$ est entière (voir [22]). Finalement, le cas général a été démontré par Brunebarbe et Cadorel (voir [23, Theorem 1.1]). Remarquons que dans la preuve, ils supposent simplement que $\bar{V}$ est une variété kählérienne compacte. Alors la conclusion implique que $\bar{V}$ est une variété projective (voir [23, Corollary 1.3]).

Cadorel et Deng ont généralisé le premier point du théorème précédent aux variétés admettant un fibré harmonique nilpotent dont le champ de Higgs est injectif en un point (voir [27, Theorem A]).

1.2.4 Finitude d’ensembles de morphismes

Les points du Théorème 1.1.17 de Faltings admettent des généralisations dans le contexte des variations de structure de Hodge. La preuve du premier point du Théorème 1.1.17 repose sur une “inégalité d’Arakelov” pour les

schémas abéliens. Cette inégalité d'Arakelov admet des généralisations pour les variétés admettant une $\mathbb{C}$ -VSHP (voir par exemple le survey [86]). Une condition nécessaire de déformabilité des variations de structure de Hodge est prouvée par Peters dans [70].

Deligne a prouvé que sur une variété algébrique complexe lisse fixée, il n'y a qu'un nombre fini de systèmes locaux entiers sous-jacents à une $\mathbb{Z}$ -VSHP (voir [32]). La théorie de Hodge variationnelle permet aussi de montrer des analogues du Théorème 1.1.18 pour d'autres espaces de modules (voir par exemple [37]).

1.3 Résultats de la thèse

On démontre dans cette thèse de nouveaux résultats d'hyperbolicité des variétés localement symétriques et des variétés admettant une variation de structure de Hodge. Plus précisément, on s'intéresse à la positivité des fibrés vectoriels de formes différentielles logarithmiques et à la finitude d'ensembles de morphismes.

1.3.1 Lieux de base des formes différentielles logarithmiques de caractéristique quelconque

Soit k un corps. Considérons une k -variété algébrique lisse log-compactifiable V . Si $(\bar{V}, D)$ est une compactification logarithmique de V et m et p sont deux entiers positifs avec $1 \leq p \leq \dim V$, on note $L_{V,m}^p$ l'ensemble des sections globales du fibré vectoriel $\mathrm{Sym}^m \Omega_{\bar{V}}^p(\log D)$ sur $\bar{V}$. Considéré en tant que sous-espace vectoriel de $H^0(V, \mathrm{Sym}^m \Omega_V^p)$, $L_{V,m}^p$ ne dépend pas du choix de $(\bar{V}, D)$. Si on note $\mathcal{O}_p(m) := \mathcal{O}_p(1)^{\otimes m}$ avec $\mathcal{O}_p(1)$ le fibré en droites tautologique de $\mathbb{P} \Omega_{\bar{V}}^p(\log D)$, les sections globales de $\mathrm{Sym}^m \Omega_{\bar{V}}^p(\log D)$ sur $\bar{V}$ (resp. V) s'identifient à celles de $\mathcal{O}_p(m)$ sur $\mathbb{P} \Omega_{\bar{V}}^p(\log D)$ (resp. $\mathbb{P} \Omega_V^p$). On s'intéresse à deux fermés de (l'espace total de) $\Lambda^p T_V$ définis par $L_{V,m}^p$.

Notation 1.3.1. Soit V une variété algébrique lisse admettant une log-compactification $(\bar{V}, D)$. On note :

- $\mathbb{B}^p(V)$ le fermé de $\mathbb{P} \Omega_V^p$ défini comme le lieu d'annulation commun de tous les éléments de $L_{V,m}^p$ vus comme des sections de $\mathcal{O}_p(m)$, pour tout $m \in \mathbb{N}_{>0}$,
- ${}^{\mathrm{T}}\mathbb{B}^p(V)$ le fermé de $\Lambda^p T_V$ défini comme l'ensemble des tenseurs sur lesquels tous les éléments de $L_{V,m}^p$ s'annulent pour tout $m \in \mathbb{N}_{>0}$,

- $\mathbb{B}_+^p(V)$ le fermé de $\mathbb{P}\Omega_V^p$ défini comme l'intersection de $\mathbb{P}\Omega_V^p$ avec le lieu de base augmenté de $\mathcal{O}_p(1)$ sur $\mathbb{P}\Omega_{\bar{V}}^p(\log D)$ (cette intersection ne dépend pas du choix de $(\bar{V}, D)$),
- ${}^{\mathbb{T}}\mathbb{B}_+^p(V)$ le fermé de $\Lambda^p T_V$ défini comme l'ensemble d'une part des tenseurs nuls et d'autre part des tenseurs non nuls dont la classe dans $\mathbb{P}\Omega_V^p$ est dans $\mathbb{B}_+^p(V)$.

Le fermé ${}^{\mathbb{T}}\mathbb{B}^p(V)$ est égal à l'ensemble d'une part des tenseurs nuls de $\Lambda^p T_V$ et d'autre part des tenseurs non nuls dont la classe dans $\mathbb{P}\Omega_V^p$ est dans $\mathbb{B}^p(V)$. Ainsi, l'ensemble ${}^{\mathbb{T}}\mathbb{B}^p(V)$ (resp. ${}^{\mathbb{T}}\mathbb{B}_+^p(V)$) contient la même information que $\mathbb{B}^p(V)$ (resp. $\mathbb{B}_+^p(V)$). Remarquons de plus que $\mathbb{B}^p(V) \subset \mathbb{B}_+^p(V)$ (de manière équivalente ${}^{\mathbb{T}}\mathbb{B}^p(V) \subset {}^{\mathbb{T}}\mathbb{B}_+^p(V)$). Informellement, ${}^{\mathbb{T}}\mathbb{B}_+^p(V)$ est l'ensemble des tenseurs de $\Lambda^p T_V$ “mal distingués” par les éléments de $L_{V,m}^p$ pour tout m .

Ces ensembles nous intéressent pour deux raisons. Premièrement, $\mathbb{B}_+^p(V)$ nous renseigne sur la positivité de $\Omega_{\bar{V}}^p(\log D)$. En effet, ce fibré vectoriel est L-big si et seulement si $\mathbb{B}_+^p(V) \neq \mathbb{P}\Omega_V^p$, et il est ample relativement à V si et seulement si la projection de $\mathbb{B}_+^p(V)$ sur $\bar{V}$ est incluse dans D .

Deuxièmement, $\mathbb{B}^p$ et $\mathbb{B}_+^p$ nous renseignent sur les morphismes entre variétés. Pour V et V' deux k -variétés algébriques lisses, un morphisme $f : V \rightarrow V'$ induit un morphisme $\Lambda^p df : \Lambda^p T_V \rightarrow \Lambda^p T_{V'}$. Alors ${}^{\mathbb{T}}\mathbb{B}^p$ est fonctoriel. Ce n'est pas le cas de ${}^{\mathbb{T}}\mathbb{B}_+^p$ (sauf si f est non-ramifiée). On montre cependant le théorème suivant.

Théorème 1.3.2 ($\simeq$ Theorem 3.5.3). *Soit k un corps, soit V et V' deux k -variétés algébriques lisses avec V' log-compactifiable. Alors pour tous les morphismes $f : V \rightarrow V'$ sauf un nombre fini d'entre eux, pour tout entier p avec $1 \leq p \leq \dim V'$, l'image de $\Lambda^p df$ est incluse dans ${}^{\mathbb{T}}\mathbb{B}_+^p(V')$.*

On montrera en fait un énoncé un peu plus général, en autorisant f à être une application rationnelle, à condition qu'elle soit “complète” (ce qui est le cas si c'est un morphisme, ou si V' est propre). Si $\Omega_{V'}^p(\log D')$ est ample relativement à V' (ce qui équivaut à $\mathbb{B}_+^p(V')$ vide), le théorème précédent dit que le rang de df est majoré par $p - 1$ pour presque tout f . Cela généralise des théorèmes de Noguchi et Sunada (cas $V' = \bar{V}'$, voir [68]) et de Noguchi (cas $k = \mathbb{C}$, voir [66]). Si V' est de type log-général (ce qui équivaut à $\mathbb{B}_+^p(V') \neq \mathbb{P}\Omega_{V'}^p \cong V'$ pour $p = \dim V'$), le théorème précédent dit que le rang de df est majoré par $\dim V' - 1$ pour presque tout f . Cela généralise des théorèmes de Kobayashi et Ochiai (cas $k = \mathbb{C}$ et $V' = \bar{V}'$, voir [50]), Martin-Deschamps et Lewin-Ménégaux (cas $V' = \bar{V}'$, voir [55]) et Tsushima (cas $k = \mathbb{C}$, voir [84]).

Enfin, notons que pour pouvoir définir $L_{V,m}^p$ (et donc $\mathbb{B}_+^p$ et $\mathbb{B}^p$) comme précédemment, il faut que V admette une compactification logarithmique. Cependant, la preuve du Theorem 7.1.1 nécessitera l'utilisation de ces objets pour une variété qui n'est a priori pas log-compactifiable. On donne donc une définition inconditionnelle de $L_{V,m}^p$ en utilisant la notion de forme absolument logarithmique définie par Abramovich dans [1, Definition 1]. Dans le chapitre 3, on définit $L_{V,m}^p$, $\mathbb{B}_+^p$ et $\mathbb{B}^p$ en toute généralité. On démontre les propriétés dont nous aurons besoin. On finit par la preuve du Théorème 1.3.2.

1.3.2 Hyperbolicité intermédiaire des variétés admettant une variation de structure de Hodge

Dans le chapitre 4, on commence par généraliser la L-bigness dans le Théorème 1.2.4 à tous les degrés.

Théorème 1.3.3 (= Theorem 4.1.1). *Soit V une variété quasi-projective complexe lisse. Soit $\mathbb{V}$ une $\mathbb{C}$ -VSHP à monodromie locale quasi-unipotente sur une log-compactification projective $(\bar{V}, D)$. Supposons qu'il existe $x \in V$ un point où l'application des périodes de $\mathbb{V}$ est immersive. Soit p un entier tel que $1 \leq p \leq \dim V$. Alors $\Omega_V^p(\log D)$ est L-big.*

Notre preuve est similaire à celle de Brunebarbe et Cadorel. Ensuite, on étudie le $\mathbb{B}_+^p$ des variétés admettant une variation de structure de Hodge. Commençons par introduire une notation.

Notation 1.3.4. *Soit M une variété analytique complexe, $x \in M$, $\mathbb{V}$ une $\mathbb{C}$ -VSHP sur M . Soit $\theta_x : T_{M,x} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}_x)$ le champ de Higgs et $\theta_x^* : T_{M,x}^{0,1} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}_x)$ le champ de Higgs conjugué en x . Pour $v \in T_{M,x}$, on note $C_{\mathbb{V}}(v) := \{w \in T_{M,x}, [\theta_x^*(\bar{v}), \theta_x(w)] = 0\}$.*

On démontre le théorème suivant.

Théorème 1.3.5 (= Theorem 4.1.5). *Soit V une variété quasi-projective complexe lisse. Soit $\mathbb{V}$ une $\mathbb{C}$ -VSHP à monodromie locale quasi-unipotente sur une log-compactification projective. Soit $x \in V$ un point où l'application des périodes de $\mathbb{V}$ est immersive. Soit p un entier tel que $1 \leq p \leq \dim V$. Soit $\alpha \in \Lambda^p T_{V,x}$ tel que $[\alpha] \in \mathbb{B}_+^p(V)$. Alors il existe $v \in T_{V,x} \setminus \{0\}$ tel que $\alpha \in \Lambda^p C_{\mathbb{V}}(v)$.*

Une variété algébrique complexe lisse est d'autant plus hyperbolique que son $\mathbb{B}_+^p$ est petit. Le théorème précédent est donc un résultat d'hyperbolicité.

On donne aussi un énoncé moins précis mais qui ne fait pas intervenir le champ de Higgs conjugué (voir la Proposition 4.7.4). On en déduit un

critère pour que $\Omega_V^p(\log D)$ soit Viehweg-big (c'est-à-dire ample relativement à un ouvert non vide) lorsque V admet une variation de structure de Hodge. Pour simplifier l'énoncé de ce critère pour cette introduction, on ajoute une hypothèse (*) de "type Torelli" et une hypothèse (**) de "type Calabi-Yau" à $\mathbb{V}$ (voir le Theorem 4.7.6 pour l'énoncé général sans ces deux hypothèses). Pour une $\mathbb{C}$ -VSHP $\mathbb{V}$ sur une variété lisse V et un point $x \in V$, on a une décomposition $\mathbb{V}_x = \bigoplus_{i \in \mathbb{Z}} \mathbb{V}_x^i$. Pour tout $i, j \in \mathbb{Z}$ avec $j \leq i$, on peut restreindre la $(i - j)$ -ième itérée θ^{i-j} du champ de Higgs θ associé à $\mathbb{V}$ pour définir une application $\theta_x^{i,j} : T_{V,x} \rightarrow \text{Hom}_{\mathbb{C}}(\mathbb{V}_x^i, \mathbb{V}_x^j)$.

Théorème 1.3.6 (= Theorem 4.1.2). *Soit V une variété quasi-projective complexe lisse. Soit $\mathbb{V}$ une $\mathbb{C}$ -VSHP à monodromie locale quasi-unipotente sur une log-compactification projective $(\bar{V}, D)$. Soit $x \in V$ un point où l'application des périodes de $\mathbb{V}$ est immersive. Soit $i \in \mathbb{Z}$ le plus grand entier tel que $\theta_x^{i,i-1}$ ne soit pas nul. Supposons que :*

(*) $\theta_x^{i,i-1}$ est injective,

(**) $\dim \mathbb{V}_x^i = 1$.

Soit j le plus petit entier avec $j < i$ tel que $\theta_x^{i,j}$ n'est pas nul. Soit m la dimension maximale pour un sous-espace vectoriel $S \subset T_{V,x}$ tel que pour tout $v \in S$, $\theta_x^{i,j}(v) = 0$. Soit $p \in \mathbb{Z}$ tel que $m < p \leq \dim V$. Alors $\Omega_V^p(\log D)$ est Viehweg-big.

Ce théorème peut s'utiliser en pratique. À l'aide d'un système de calcul formel, on en déduit le théorème suivant.

Théorème 1.3.7 (= Theorem 4.1.3). *Soit $\mathcal{M}$ l'espace des modules fin des solides quintiques complexes polarisés, qui est un champ de Deligne-Mumford lisse sur $\mathbb{C}$ admettant un revêtement étale fini $M \rightarrow \mathcal{M}$ avec M une variété quasi-projective complexe lisse. Soit $(\bar{M}, D)$ une log-compactification projective de M .*

Alors $\Omega_{\bar{M}}^p(\log D)$ est Viehweg-big si $91 \leq p \leq 101 = \dim M$.

Dans le théorème précédent, le "91" n'est probablement pas optimal (nous avons été limités par notre puissance de calcul). Remarquons que M n'est pas une variété localement symétrique, donc ce résultat n'est pas redondant avec ceux du chapitre 6.

1.3.3 Hyperbolicité des variétés localement symétriques

Dans le court chapitre 5, on développe des outils qui serviront dans le chapitre 6. Dans ce dernier, on précise les résultats du chapitre 4 dans le cas des variétés localement symétriques. On commence par généraliser la sous-variété caractéristique $\mathcal{S}_{r-1}$ de Mok en degré supérieur.

Définition 1.3.8. Soit X une variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2, soit R le tenseur de courbure de Riemann de la métrique canonique, soit $1 \leq p \leq \dim X$. La sous-variété caractéristique de degré supérieur $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ est l'ensemble des $[\alpha] \in \mathbb{P}\Omega_X^p$ tels qu'il existe $v \in T_X$ avec $v \neq 0$ tel que $\alpha \in \Lambda^p \ker R(v, \bar{v})$.

On a $\mathcal{S}^1 = \mathcal{S}_{r-1}$. On montre alors le théorème suivant.

Théorème 1.3.9 (= Theorem 6.1.4). Soit X une variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2. Soit $1 \leq p \leq \dim X$. Alors $\mathbb{B}_+^p(X) = \mathcal{S}^p$.

L'inclusion $\mathbb{B}_+^p(X) \subset \mathcal{S}^p$ peut être montrée avec le chapitre précédent, mais l'on peut aussi presque éviter l'usage des variations de structures de Hodge en utilisant [49, Chapter 6.8] (voir Remarks 6.4.2.4). L'inclusion $\mathcal{S}^p \subset \mathbb{B}_+^p(X)$ est un résultat de non-hyperbolicité. On en déduit le théorème suivant.

Théorème 1.3.10 (= Theorem 6.1.1). Soit $(\bar{X}, D)$ une log-compactification projective d'une variété localement symétrique X dont le revêtement universel est irréductible et de rang au moins 2. Soit $1 \leq p \leq \dim X$.

Alors $\Omega_{\bar{X}}^p(\log D)$ est L -big. De plus, les assertions suivantes sont équivalentes :

- $\Omega_{\bar{X}}^p(\log D)$ est Viehweg-big,
- $\Omega_{\bar{X}}^p(\log D)$ est ample relativement à X ,
- $p > l$ avec l la dimension maximale de $\ker R(v, \bar{v})$ pour $v \in T_X$ avec $v \neq 0$.

Le nombre l est égal au $l(\Sigma)$ du Théorème 1.1.16. Dans le cas où X est compact, l'amplitude de Ω_X^p quand $p > l$ a été prouvée par Noguchi et Sunada (voir [68]). En degré 1, on calcule aussi le lieu de base stable.

Théorème 1.3.11 (= Theorem 6.1.5). Soit X une variété localement symétrique dont le revêtement universel est irréductible et de rang au moins 2. Alors $\mathbb{B}^1(X) = \mathbb{B}_+^1(X) = \mathcal{S}^1$.

Le cas X compact du théorème précédent a été montré par Mok (voir [58, Appendix IV], rappelons-nous que notre $\mathcal{S}^1$ est le $\mathcal{S}_{r-1}$ de Mok). Sa preuve repose sur le Théorème 1.1.6. La difficulté de la généralisation au cas non compact réside dans la nécessité d'avoir une théorie de Chern-Weil pour des métriques singulières. Le principal ingrédient est le travail de Kollár dans [51,

Section 5] sur les singularités des métriques de Hodge (basé sur les travaux de Cattani, Kaplan et Schmid [29]). Nous faisons quelques compléments à l'article de Kollár dans le chapitre 5. Nous utilisons cela pour montrer une version non compacte du Théorème 1.1.6 pour un type (très spécifique à notre problème) de métrique singulière.

1.3.4 Applications en caractéristique positive

Si on applique le Théorème 1.3.2 avec les $\mathbb{B}_+^p$ calculés dans le chapitre 6, le résultat obtenu est un énoncé plus faible que les énoncés rappelés dans la Sous-section 1.1.4. De plus, nous ne connaissons pas de conséquence intéressante de la fonctorialité de ${}^{\mathbb{T}}\mathbb{B}^p$ appliquée avec le Théorème 1.3.11 qui ne puisse pas être directement démontrée avec le théorème de rigidité métrique de Mok et To (Théorème 1.1.4).

Cependant, les formes différentielles logarithmiques sont des objets algébriques, contrairement aux variations de structure de Hodge et aux métriques. Des arguments de spreading-out nous permettent de transférer des informations sur les formes différentielles logarithmiques en caractéristique positive. On peut alors s'en servir pour montrer en caractéristique positive des énoncés qui avaient une preuve transcendante en géométrie complexe. On explique cela dans le chapitre 7.

D'abord, le Théorème 1.3.11 associé à un spreading-out et à la fonctorialité de ${}^{\mathbb{T}}\mathbb{B}^p$ nous permet de montrer le Théorème 1.3.12 ci-dessous. On note $\mathcal{M}_g$ l'espace des modules des courbes projectives lisses connexes de genre $g \geq 2$. C'est un champ de Deligne-Mumford. Une famille $f : V \rightarrow \mathcal{M}_g$ de courbes projectives lisses connexes de genre g sur une k -variété intègre V (k un corps) est *séparable* si pour toute k -variété V' (resp. M), tout morphisme étale dominant $p : V' \rightarrow V$ (resp. $q : M \rightarrow \mathcal{M}_g \otimes k$) et tout relèvement $f' : V' \rightarrow M$ de f , la corestriction $f' : V' \rightarrow \overline{\text{Im } f'}$ est séparable. La famille est *isotriviale* si l'application de V vers l'espace des modules grossiers des courbes projectives lisses connexes de genre g induite par f est constante.

Théorème 1.3.12 (= Theorem 7.1.1). *Pour tous $g, h \geq 2, n \geq 3$, pour tout nombre premier p sauf un nombre fini d'entre eux, pour tout corps k de caractéristique 0 ou p avec une racine primitive n -ième de l'unité, toute famille séparable de courbes projectives lisses connexes de genre h sur $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} k$ est isotriviale.*

Dans le cas $k = \mathbb{C}$, une preuve métrique du théorème précédent a été donnée par Liu, Sun, Yang et Yau pour une variété localement symétrique

compacte dont le revêtement universel est irréductible et de rang au moins 2 à la place de $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} k$ dans [52, Corollary 1.6]. Leur preuve fonctionne aussi dans le cas non-compact, elle s'applique donc en particulier à $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} \mathbb{C}$.

Ensuite, le Théorème 1.3.9 associé à un spreading-out et au Théorème 1.3.2 permet de montrer le Théorème 1.3.13 ci-dessous. Pour un corps k de caractéristique différente de 2 et une k -variété lisse V , considérons un V -schéma abélien principalement polarisé $\pi : A \rightarrow V$ de dimension relative $g \in \mathbb{N}$ sur V . Dans ce cas, il y a un isomorphisme canonique $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{A/V}^{\otimes 2}$ avec $\underline{\mathrm{Lie}}_{A/V}$ le fibré vectoriel de rang g défini par l'algèbre de Lie des fibres de π . De plus, si on note KS l'application de Kodaira-Spencer $T_V \rightarrow R^1\pi_*T_{A/V}$, alors l'image de $\beta \circ \mathrm{KS}$ est dans le facteur direct $\mathrm{Sym}^2 \underline{\mathrm{Lie}}_{A/V}$ de $\underline{\mathrm{Lie}}_{A/V}^{\otimes 2}$ (voir [2, Section 1]).

Théorème 1.3.13 (= Theorem 7.1.2). *Soit $g \in \mathbb{N}$. Dans toutes les caractéristiques positives sauf un nombre fini d'entre elles et en caractéristique 0, pour tout corps k algébriquement clos de cette caractéristique, on a le résultat suivant.*

*Soit V une k -variété lisse. Alors pour tous les V -schémas abéliens principalement polarisés $\pi : A \rightarrow V$ de dimension relative g sauf un nombre fini d'entre eux, pour tout point fermé $x \in V$, il existe un sous-espace vectoriel $S \subset \mathrm{Lie} A_x$ de dimension $g - 1$ tel que $\mathrm{Im}(\beta_x \circ \mathrm{KS}_x) \subset \mathrm{Sym}^2 S$, avec $\mathrm{KS} : T_V \rightarrow R^1\pi_*T_{A/V}$ l'application de Kodaira-Spencer et $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{A/V}^{\otimes 2}$ l'isomorphisme canonique.*

En particulier, le rang de KS est presque sûrement majoré par $\frac{g(g-1)}{2}$. Si la famille est séparable (dans le même sens que pour les familles de courbes), cela implique que l'adhérence de l'image d'un morphisme $V \rightarrow \mathcal{A}_g$ est presque sûrement de dimension au plus $\frac{g(g-1)}{2}$. Pour $k = \mathbb{C}$, le Théorème 1.3.13 est une conséquence du Théorème 1.1.18.

On montre aussi un énoncé similaire pour les familles de surfaces K3. Soit $\pi : S \rightarrow V$ une famille polarisée de surfaces K3 sur une k -variété lisse V . La polarisation permet de définir la “partie primitive” de $R^1\pi_*T_{S/V}$, qui en est un facteur direct. L'image de l'application de Kodaira-Spencer $\mathrm{KS}_{S/V}$ est dans cette partie primitive. Pour $x \in V$, le cup-produit et une identification $T_{S/V,x} \cong \Omega_{S/V,x}^1$ munissent la fibre $R^1\pi_*T_{S/V,x}$ de $R^1\pi_*T_{S/V}$ d'une forme quadratique non-dégénérée (déterminée à un scalaire près). On peut alors définir un “sous- $\mathcal{O}_V$ -module isotrope” de $R^1\pi_*T_{S/V}$ comme un sous-module dont toutes les sections définissent un élément isotrope sur chaque fibre.

Théorème 1.3.14 (= Theorem 7.1.5). *Soit $d \in \mathbb{N}_{>0}$. Dans toutes les caractéristiques positives sauf un nombre fini d'entre elles et en caractéristique 0, pour tout corps k algébriquement clos de cette caractéristique, on a le résultat suivant.*

*Soit V une k -variété lisse. Alors pour toutes les familles primitivement polarisées de degré $2d$ de surfaces K3 $\pi : S \rightarrow V$ sauf un nombre fini d'entre elles, soit $\mathrm{KS}_{S/V}$ est nulle, soit il existe un sous- $\mathcal{O}_V$ -module inversible isotrope L de la partie primitive de $R^1\pi_*T_{S/V}$ tel que $\mathrm{Im}(\mathrm{KS}_{S/V}) \subset L$.*

En particulier, le rang de $\mathrm{KS}_{S/V}$ est presque sûrement 0 ou 1. Si la famille est séparable (dans le même sens que pour les familles de courbes), cela implique que la dimension de l'adhérence de l'image d'un morphisme séparable de V vers le champ (de Deligne-Mumford) des modules des surfaces K3 primitivement polarisées de degré $2d$ est presque sûrement 0 ou 1.

Deux propriétés des espaces des modules des variétés abéliennes principalement polarisées et des surfaces K3 sont utilisées pour prouver les deux théorèmes précédents. Premièrement, leur existence en caractéristique mixte. Deuxièmement, un théorème de Torelli infinitésimal (sur $\mathbb{C}$). On démontre un théorème général pour les familles de variétés dont l'espace des modules partage ces deux propriétés (voir Theorem 7.3.14). Il généralise les Théorèmes 1.3.13 et 1.3.14.

1.4 Conventions

Soit k un corps. Une *variété* V sur k est un schéma séparé et de type fini sur k . Une *log-compactification* de V est la donnée $(\bar{V}, D)$ d'une k -variété propre et lisse $\bar{V}$ et d'un diviseur à croisements normaux $D \subset \bar{V}$ tel que V s'identifie à la sous-variété ouverte $\bar{V} \setminus D$. Une k -variété est *log-compactifiable* si elle admet une log-compactification. En particulier, une k -variété log-compactifiable est lisse. Par le théorème de compactification de Nagata (voir [64]) et le théorème de résolution des singularités de Hironaka, si k est de caractéristique 0, une k -variété est log-compactifiable si et seulement si elle est lisse. Une *log-compactification projective* de V est une log-compactification $(\bar{V}, D)$ de V telle que $\bar{V}$ soit une variété projective. Une k -variété est *projectivement log-compactifiable* si elle admet une log-compactification projective. En particulier, une k -variété projectivement log-compactifiable est quasi-projective et lisse. Par le théorème de résolution des singularités de Hironaka, si k est de caractéristique 0, une k -variété est projectivement log-compactifiable si et seulement si elle est quasi-projective et lisse.

Soit E un fibré vectoriel sur un schéma S . On notera aussi E l'espace total associé. La notation $\mathbb{P} E$ désigne le schéma qui paramètre les hyperplans de E (en particulier, on utilise la convention de Grothendieck). Il est muni d'un morphisme projectif $\mathbb{P} E \rightarrow S$ de dimension relative le rang de E moins un. Lorsque V est une k -variété lisse admettant une log-compactification $(\bar{V}, D)$ et p est un entier naturel, on note $\mathcal{O}_p(1)$ le fibré en droites tautologique de $\mathbb{P} \Omega_V^p(\log D)$. Pour $m \in \mathbb{N}$, on note $\mathcal{O}_p(m)$ la m -ième puissance tensorielle de $\mathcal{O}_p(1)$.

Soit V une k -variété lisse. Soit $p, m \in \mathbb{N}$. La notation $L_{V,m}^p$ désigne le sous-espace vectoriel de $H^0(V, \text{Sym}^m \Omega_V^p)$ composé des formes "absolument logarithmiques". Sa définition précise est donnée dans la Sous-section 3.3.1. Si V admet une log-compactification $(\bar{V}, D)$, alors $L_{V,m}^p$ est l'ensemble des éléments de $H^0(V, \text{Sym}^m \Omega_V^p)$ qui s'étendent en une section globale de $\text{Sym}^m \Omega_{\bar{V}}^p(\log D)$ sur $\bar{V}$.

Les notations $\text{Bs}_m^p(V)$, $\mathbb{B}^p(V)$ et $\mathbb{B}_+^p(V)$ sont définies dans la Section 3.4. Toutes désignent un fermé de $\mathbb{P} \Omega_V^p$. Si V admet une log-compactification projective $(\bar{V}, D)$, alors $\text{Bs}_m^p(V)$ (resp. $\mathbb{B}^p(V)$, $\mathbb{B}_+^p(V)$) est l'intersection de $\mathbb{P} \Omega_V^p$ avec le lieu de base de $\mathcal{O}_p(m)$ (resp. le lieu de base stable de $\mathcal{O}_p(1)$, le lieu de base augmenté de $\mathcal{O}_p(1)$) sur $\mathbb{P} \Omega_{\bar{V}}^p(\log D)$.

Les notations ${}^t\text{Bs}_m^p(V)$, ${}^t\mathbb{B}^p(V)$ et ${}^t\mathbb{B}_+^p(V)$ sont définies dans la Section 3.4. Toutes désignent un fermé de (l'espace total de) $\Lambda^p T_V$. Le fermé ${}^t\text{Bs}_m^p(V)$ (resp. ${}^t\mathbb{B}^p(V)$, ${}^t\mathbb{B}_+^p(V)$) est l'union de la section nulle de $\Lambda^p T_V$ et de l'ensemble des tenseurs non nuls dont l'image dans $\mathbb{P} \Omega_V^p$ est dans $\text{Bs}_m^p(V)$ (resp. $\mathbb{B}^p(V)$, $\mathbb{B}_+^p(V)$).

Une *variété localement symétrique* X est une variété quasi-projective complexe lisse biholomorphe au quotient d'un domaine symétrique borné Σ de $\mathbb{C}^d$ (pour un $d \in \mathbb{N}$) par l'action d'un réseau sans torsion de la composante neutre du groupe des biholomorphismes de Σ . Nous dirons que le revêtement universel de X est *irréductible* s'il n'est pas biholomorphe à un produit non trivial de deux domaines symétriques bornés.

Chapter 2

Introduction (English)

2.1 Hyperbolicity of locally symmetric varieties

Let us define the first central object of this thesis: locally symmetric varieties. Let Σ be a bounded symmetric domain in $\mathbb{C}^d$ (for some $d \in \mathbb{N}$). Its biholomorphism group $\text{Aut}(\Sigma)$ admits a unique real Lie group structure. The group $\text{Aut}(\Sigma)$ has a finite number of connected components. Let $\text{Aut}(\Sigma)^0$ denote its identity component. Any torsion-free lattice Γ of $\text{Aut}(\Sigma)^0$ acts freely and properly discontinuously on Σ . Thus, the quotient $X := \Gamma \backslash \Sigma$ has a (unique) complex analytic variety structure for which the passage to the quotient is holomorphic.

Definition 2.1.1. *A locally symmetric variety X is a complex analytic variety that is biholomorphic to the quotient of a bounded symmetric domain under the action of a torsion-free lattice Γ of the identity component of its biholomorphism group.*

The locally symmetric variety X is arithmetic if the lattice Γ is arithmetic.

By Margulis's arithmeticity theorem (see [54, Theorem 1]), if Σ is irreducible and of rank at least 2, the lattice Γ is arithmetic. In this case, any locally symmetric variety with Σ as its universal covering is arithmetic. Therefore, according to Baily and Borel, X has the structure of an algebraic variety.

Theorem 2.1.2 (see [4]). *An arithmetic locally symmetric variety has a (smooth) complex quasiprojective variety structure compatible with its analytic structure.*

Borel showed in this case the uniqueness of the complex algebraic variety structure compatible with the analytic structure (see [12]). An arithmetic locally symmetric variety admits a canonical normal projective compactification $\bar{X}^{SBB}$ called the *Satake-Baily-Borel compactification*.

A bounded symmetric domain admits a canonical Kähler-Einstein metric: the Bergman metric. It is invariant under the action of $\text{Aut}(\Sigma)^0$ (in the case where Σ is irreducible, this characterizes it up to a multiplicative constant among Hermitian metrics). In particular, it is invariant under the action of Γ . Consequently, the Bergman metric induces a Kähler-Einstein metric on X that we will call the *canonical metric* of X .

2.1.1 Curvature of the canonical metric

The curvature of the canonical metric of a locally symmetric variety has several properties of negativity.

Proposition 2.1.3 (see [58, Proposition 2 p.46]). *Let X be a locally symmetric variety. Then the sectional, holomorphic bisectional, and Ricci curvatures of the canonical metric of X are semi-negative. Furthermore, the holomorphic sectional and scalar curvatures of the canonical metric of X are strictly negative.*

The previous property is a property of hyperbolicity of locally symmetric varieties. Conversely, the following theorem is a property of “non-hyperbolicity”: it prohibits the construction of a metric with curvature “more negative” than the canonical metric. For example, there exists no Hermitian metric with strictly negative holomorphic bisectional curvature. This theorem was proven by Mok in the compact case (see [60] or [58, Theorem 1 p.99]) and then by To in the general case.

Theorem 2.1.4 (see [82]). *Let X be a locally symmetric variety whose universal covering is irreducible and of rank at least 2. Then every Hermitian metric with holomorphic bisectional curvature that is semi-negative is a constant multiple of the canonical metric.*

To prove the previous theorem, Mok defines the “(higher) characteristic subvarieties”. These subvarieties describe the lack of negativity of the curvature of the canonical metric (and thus, according to Theorem 2.1.4, the lack of hyperbolicity of X). Here we define only the two extremes $\mathcal{S}_1$ (the original “characteristic subvariety”) and $\mathcal{S}_{r-1}$ (the largest of the “higher characteristic subvarieties”, where r denotes the rank of the universal covering of X as a bounded symmetric domain). Note that if x is a point on a

locally symmetric variety and v is a tangent vector at x , then the set $\mathcal{N}(v)$ of tangent vectors w at x such that the canonical metric has zero holomorphic bisectional curvature on the plane spanned by v and w is a vector space.

Definition 2.1.5. *The characteristic subvariety $\mathcal{S}_1$ (resp. $\mathcal{S}_{r-1}$) of X is the set of $[v] \in \mathbb{P}\Omega_X^1$ such that the dimension of $\mathcal{N}(v)$ is maximal (resp. is nonzero).*

Mok proves that when X is compact, a Finsler variant of Theorem 2.1.4 holds (he also shows that the compact case of Theorem 2.1.4 can be deduced from this Finsler variant). It is implicit in [58, §3 p.113] and stated in [59, Theorem 3].

Theorem 2.1.6. *Let X be a locally symmetric variety whose universal covering is irreducible and of rank at least 2. Then every Hermitian metric with semi-negative curvature on the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}\Omega_X^1$ coincides, when restricted to $\mathcal{S}_1$, with a constant multiple of the canonical metric.*

2.1.2 Hyperbolicity in the sense of Brody, Kobayashi, Demailly, and Borel

Several classical notions of hyperbolicity of a complex algebraic variety V are defined by considering morphisms from a given variety to V .

Definition 2.1.7. • *A complex analytic space V is hyperbolic in the sense of Brody if every holomorphic map $\mathbb{C} \rightarrow V$ is constant. A complex algebraic variety is hyperbolic in the sense of Brody if the associated analytic space is hyperbolic in the sense of Brody.*

- *A complex analytic space V is hyperbolic in the sense of Kobayashi if the set of its points admits a distance (in the sense of general topology) such that every holomorphic map from the Poincaré disk to V is Lipschitz, with the Poincaré disk equipped with the Poincaré metric. A complex algebraic variety is hyperbolic in the sense of Kobayashi if the associated analytic space is hyperbolic in the sense of Kobayashi.*
- *A complex projective variety V is algebraically hyperbolic (or hyperbolic in the sense of Demailly) if for every ample line bundle A on V , there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for every $g \in \mathbb{N}$, every smooth, connected complex projective curve C of genus g , and every non-constant morphism $f : C \rightarrow V$, we have $2g - 2 \geq \varepsilon \deg f^*A$ (this does not depend on the choice of A).*

- A complex algebraic variety V is hyperbolic in the sense of Borel if for every reduced complex algebraic variety V' , every holomorphic map $(V')^{an} \rightarrow V^{an}$ is induced by a morphism of algebraic varieties $V' \rightarrow V$.

These different concepts are related as follows.

Proposition 2.1.8 (see [49, 46]). *A complex analytic space that is hyperbolic in the sense of Kobayashi is hyperbolic in the sense of Brody. A complex algebraic variety that is hyperbolic in the sense of Kobayashi or Borel is hyperbolic in the sense of Brody. For a complex projective variety, the hyperbolicities in the senses of Kobayashi, Brody, and Borel are equivalent, and they imply algebraic hyperbolicity.*

In the case of locally symmetric varieties, we have the following statements.

Proposition 2.1.9. *A locally symmetric variety is hyperbolic in the sense of Brody and Kobayashi. If it is arithmetic and compact, it is also algebraically hyperbolic.*

Proof. The holomorphic sectional curvature of the Bergman metric on a bounded symmetric domain is negative. By homogeneity, it is bounded above by a constant $A \in \mathbb{R}_{<0}$. The holomorphic sectional curvature of the canonical metric of a locally symmetric variety X is therefore also bounded above by a constant $A \in \mathbb{R}_{<0}$. We can then apply the Ahlfors-Schwarz lemma (see [49, Theorem 2.1.4]) to it: there exists a constant $C \in \mathbb{R}_{>0}$ such that the holomorphic maps from the Poincaré disk to X are C -Lipschitz. Thus X is hyperbolic in the Kobayashi sense. The rest follows immediately from the previous proposition. $\square$

It is in reference to the following Borel theorem that Javanpeykar and Kucharczyk named Borel hyperbolicity in [46]. In particular, it implies that the Baily-Borel algebraic structure on X is the only one compatible with the analytic structure.

Theorem 2.1.10 (see [12, Theorem 3.10]). *Arithmetic locally symmetric varieties are hyperbolic in the sense of Borel.*

We can prove statements regarding the hyperbolicity of the Satake-Baily-Borel compactification. These reinforce the preceding theorems, except that X may need to be replaced by a finite étale covering.

Theorem 2.1.11. *Let X be an arithmetic locally symmetric variety. Then X has a finite étale covering X' such that the Satake-Baily-Borel compactification $\bar{X}'^{SBB}$ of X' is hyperbolic in the sense of Brody, Kobayashi, Demailly, and Borel.*

Nadel proved an intermediate version between the Brody hyperbolicity of X' and that of $\bar{X}'^{SBB}$: the “logarithmic” Brody hyperbolicity of X' (see [63, Theorem 0.2]). Rousseau proved in [74, Theorem 1.1] the “logarithmic” Kobayashi hyperbolicity of X' . Finally, Brunebarbe noted that by an argument concerning the boundary structure of the Satake-Baily-Borel compactification, Nadel’s theorem implies Theorem 2.1.11, and thus in particular Rousseau’s theorem (see [20, Subsection 1.2]).

2.1.3 Positivity of vector bundles of logarithmic differential forms

An integral algebraic variety is said to be of *general type* if it is birational to a proper and smooth algebraic variety of general type (i.e., whose canonical line bundle is big). This property does not depend on the choice of the proper and smooth birational model. Tai showed that every locally symmetric variety whose universal covering is irreducible and of rank at least 2 admits a finite étale covering of general type (see [3, Chapter IV Theorem 1.4]). Brunebarbe proved the following stronger statement.

Theorem 2.1.12 (see [20, Corollary 1.7]). *Let X be an arithmetic locally symmetric variety. Then X has a finite étale covering X' such that every algebraic subvariety of the Satake-Baily-Borel compactification $\bar{X}'^{SBB}$ of X' is of general type.*

This statement becomes false if we do not allow a finite étale covering: an arithmetic locally symmetric variety is not necessarily of general type (see [20, Remark 1.2(ii)]). Cadorel has explicitly determined moduli spaces of principally polarized Abelian varieties with level structure that satisfy the previous statement, or that satisfy it for subvarieties of sufficiently small codimension (see [25]).

For differential forms of degree 1, the following result appears as a corollary of Theorem 2.2.4, which will be stated in the next section (see also [26]). A vector bundle E over a proper complex algebraic variety W is said to be *L-big* if there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for all $k \gg 0$, we have $\dim H^0(W, \text{Sym}^k E) \geq \varepsilon k^{\dim W + \text{rk } E - 1}$. The vector bundle E is said to be *weakly positive* if there exists a Zariski-dense open set $U \subset W$ such that for every ample line bundle A on W and every strictly positive integer α , there exists a strictly positive integer β such that $\text{Sym}^{\alpha\beta} \Omega_V^1(\log D) \otimes A^\beta$ is generated over U by its global sections.

Theorem 2.1.13. *Let X be an arithmetic locally symmetric variety. Then for any log-compactification $(\bar{X}, D)$ of X , the vector bundle $\Omega_{\bar{X}}^1(\log D)$ is L -big and weakly positive.*

Mok showed that in the compact case, the vector bundle Ω_X^1 is not ample (more precisely, it is not even big in the Viehweg sense). The obstruction is described by the augmented base locus $\mathbb{B}_+(\mathcal{O}(1))$ of the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P}\Omega_X^1$.

Theorem 2.1.14 (see [58, Proposition 4 p.262]). *Let X be a compact locally symmetric variety whose universal covering is irreducible and of rank at least 2. Then the characteristic subvariety $\mathcal{S}_{r-1} \subset \mathbb{P}\Omega_X^1$ is the stable base locus and the augmented base locus of the tautological line bundle $\mathcal{O}(1)$ of $\mathbb{P}\Omega_X^1$.*

The inclusion $\mathbb{B}_+(\mathcal{O}(1)) \subset \mathcal{S}_{r-1}$ is a hyperbolicity result. The inclusion $\mathcal{S}_{r-1} \subset \mathbb{B}_+(\mathcal{O}(1))$ is a non-hyperbolicity result (this is an obstruction to the positivity of Ω_X^1). The inclusion of $\mathcal{S}_{r-1}$ in the stable base locus of $\mathcal{O}(1)$ is a stronger result (it means that for every $n \in \mathbb{N}_{>0}$, every global section of $\text{Sym}^n \Omega_X^1$ vanishes on $\mathcal{S}_{r-1}$). The proof of this uses the Finsler rigidity theorem (Theorem 2.1.6).

2.1.4 Finiteness of sets of morphisms

Borel and Narasimhan proved the following restriction for defining morphisms to a locally symmetric variety.

Theorem 2.1.15 (see [14, Proposition 2.1, Theorem 4.6]). *Let X be a locally symmetric variety. Let M be the complement of a normal crossing divisor in a connected compact complex analytic variety. For every $x \in M$, $y \in X$, and group homomorphism $\rho : \pi_1(M, x) \rightarrow \pi_1(X, y)$, there exists at most one holomorphic map $f : M \rightarrow X^{an}$ such that $f(x) = y$ which induces ρ on the fundamental groups.*

For an arithmetic locally symmetric variety X with a universal covering denoted by Σ and a fundamental group denoted by Γ , let $l(\Sigma)$ denote the maximum dimension of $\mathcal{N}(v)$ for v among the nonzero tangent vectors of X ($l(\Sigma)$ is also the maximum dimension of a “proper boundary component” of Σ , see [88, Section 5]) and $l(\Gamma)$ denote the maximum dimension of a boundary component of $\bar{X}^{SBB}$. We have $l(\Sigma) \geq l(\Gamma)$. Noguchi proved the following theorem.

Theorem 2.1.16 (see [67]). *Let X be an arithmetic locally symmetric variety with a universal covering denoted by Σ and a fundamental group denoted by*

Γ . Let M be the complement of a normal crossing divisor in a compact, connected, complex analytic manifold.

- There exists a “moduli space of morphisms $M \rightarrow X$ ”, that is, a complex analytic space $\text{Hol}(M, X)$ and a holomorphic map $f : \text{Hol}(M, X) \times M \rightarrow X$ such that for every analytic space M' and every holomorphic map $g : M' \times M \rightarrow X$, there exists a unique holomorphic map $h : M' \rightarrow \text{Hol}(M, X)$ such that $g = f \circ (h \times \text{id}_M)$.
- The moduli space $\text{Hol}(M, X)$ is a Zariski open subset of a compact complex analytic space.
- For every $k \in \mathbb{N}$, the holomorphic maps $M \rightarrow X$ with image of dimension k are parameterized by a clopen set $\text{Hol}(M, X, k) \subset \text{Hol}(M, X)$.
- If $k > l(\Gamma)$, then $\text{Hol}(M, X, k)$ is compact. If $k > l(\Sigma)$, then $\text{Hol}(M, X, k)$ is finite.
- Let C be a connected component of $\text{Hol}(M, X)$, and let $x \in M$. Then the map:

$$\begin{aligned} C &\rightarrow X \\ c &\mapsto f(c, x) \end{aligned}$$

is a closed embedding onto a totally geodesic complex submanifold.

A more precise finiteness statement can be given in the special case where M is an algebraic variety and X is the moduli space $\mathcal{A}_g(n)$ of principally polarized Abelian varieties of dimension $g \in \mathbb{N}_{>0}$ with level structure $n \geq 3$. If V is a complex algebraic variety, a morphism $f : V \rightarrow \mathcal{A}_g(n)$ corresponds to the data of an isomorphism class of principally polarized Abelian schemes with level- n structure n on V . Since there are only a finite number of level- n structures on a given family, a statement of finiteness for the set of morphisms $V \rightarrow \mathcal{A}_g(n)$ is equivalent to a statement of finiteness for the set of isomorphism classes of principally polarized Abelian schemes. If V is of dimension 1, we have the following theorem of Faltings.

Theorem 2.1.17 (see [38]). *Let C be a complex algebraic curve.*

- *The set of isomorphism classes of principally polarized Abelian schemes over C is parameterized by a scheme of finite type over $\mathbb{C}$.*
- *In particular, there are only a finite number of rigid (i.e., non-deformable) principally polarized Abelian schemes.*

- A family A of principally polarized Abelian schemes over C is deformable if and only if there exists an antisymmetric endomorphism of the local system $R^1\pi_*\underline{\mathbb{Z}}_A$ that does not arise from an endomorphism of the principally polarized Abelian C -scheme A .

Deligne noted in [32, Section 3] that the finiteness statement reduces to the case of dimension 1, which yields the following theorem.

Theorem 2.1.18. *Let V be a complex algebraic variety. Then, for all principally polarized Abelian schemes A over V except finitely many, there exists an antisymmetric endomorphism of the local system $R^1\pi_*\underline{\mathbb{Z}}_A$ that does not arise from an endomorphism of the family A .*

In positive characteristic, the variety $\mathcal{A}_g(n)$ loses some of the hyperbolicity properties it possesses in characteristic 0. In characteristic at least 3, Moret-Bailly proved that $\mathcal{A}_g(n)$ contains rational curves (see [61]). This contradicts the statement analogous to Theorem 2.1.12 since $\mathbb{P}^1$ is not of general type. This also contradicts the statement analogous to the first point of Theorem 2.1.17 because by composing a non-constant morphism $\mathbb{P}^1 \rightarrow \mathcal{A}_g(n)$ with the endomorphisms of $\mathbb{P}^1$, one obtains a set of morphisms $\mathbb{P}^1 \rightarrow \mathcal{A}_g(n)$ of unbounded degree.

2.2 Hyperbolicity of varieties supporting a variation of Hodge structure

Arithmetic locally symmetric varieties are a special case of algebraic varieties supporting an integral polarizable variation of Hodge structure, or $\mathbb{Z}$ -PVHS. Some of the hyperbolicity statements from the previous section generalize to the case of varieties supporting a PVHS. When we speak of a variety supporting a PVHS, we will always assume the immersivity of the period map, at least generically.

2.2.1 Curvature of the metric

Consider a complex analytic manifold M equipped with a polarizable complex variation of Hodge structure ($\mathbb{C}$ -PVHS) $\mathbb{V}$. The choice of a polarization of $\mathbb{V}$ yields a Hermitian metric known as the *Hodge metric* on the underlying holomorphic vector bundle of $\mathbb{V}$. Since the polarization of $\mathbb{V}$ induces a polarization of the $\mathbb{C}$ -PVHS $\text{End}_{\mathbb{C}}(\mathbb{V})$, it also defines a Hodge metric on $\text{End}_{\mathbb{C}}(\mathbb{V})$. Suppose the period map is immersive. This means that the $\mathcal{O}_M$ -module morphism defined by the Higgs field $\theta : T_M \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}) \otimes_{\mathbb{C}} \mathcal{O}_M$ is

a locally split injection. Thus, we can pull back the Hodge metric on the underlying vector bundle of $\text{End}_{\mathbb{C}}(\mathbb{V})$ to obtain a Hermitian metric on T_M . Lu has shown that this metric is Kähler ([53, Theorem 1.2]), and that its curvature has the following properties of negativity.

Theorem 2.2.1 (see [53, Theorem 1.2]). *Let M be a complex analytic manifold equipped with a complex polarized variation of Hodge structure $\mathbb{V}$. Suppose that the period map is immersive. Let g be the Kähler metric defined on M by $\mathbb{V}$. Then the holomorphic bisectional curvature of g is semi-negative. Its holomorphic sectional, Ricci, and scalar curvatures are strictly negative.*

2.2.2 Hyperbolicity in the sense of Brody, Kobayashi, Demailly, and Borel

Using the previously defined metric, Griffiths and Schmid proved the hyperbolicity in the sense of Kobayashi, and thus also of Brody, of varieties supporting a $\mathbb{C}$ -PVHS.

Theorem 2.2.2 (see [40, Corollary 9.3]). *Let M be a complex analytic manifold equipped with a polarizable complex variation of Hodge structure. Suppose that the period map is immersive.*

Then M is hyperbolic in the sense of Brody and Kobayashi.

Bakker, Brunebarbe, and Tsimerman proved hyperbolicity in the sense of Borel for varieties supporting a $\mathbb{Z}$ -PVHS in [5, Corollary 7.1]. Using this paper, Javanpeykar and Litt proved the algebraic hyperbolicity of varieties supporting a $\mathbb{Z}$ -PVHS with unipotent local monodromy (see [47, Theorem 4.2]). Deng generalized these two results to varieties with a $\mathbb{C}$ -PVHS (see [34]).

Theorem 2.2.3. *Let V be a smooth complex algebraic variety equipped with a $\mathbb{C}$ -PVHS. Suppose that the period map is quasi-finite. Then V is hyperbolic in the sense of Demailly and Borel.*

Cadorel and Deng have proven a “pseudo-algebraic hyperbolicity” in the more general case of varieties supporting a nilpotent harmonic bundle whose Higgs field is injective at a point (see [27, Theorem A]).

2.2.3 Positivity of vector bundles of logarithmic differential forms

The logarithmic canonical line bundle and the logarithmic cotangent vector bundle of a smooth complex quasiprojective variety supporting a $\mathbb{C}$ -PVHS have the following positivity properties.

Theorem 2.2.4. *Let $\bar{V}$ be a smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $V := \bar{V} \setminus D$. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on V . Suppose that the period map is immersive at a point of V . Then:*

- *V is of log-general type, that is, the logarithmic canonical line bundle $\omega_{\bar{V}}(D)$ is big,*
- *the logarithmic cotangent vector bundle $\Omega_{\bar{V}}^1(\log D)$ is L-big and weakly positive.*

In the previous theorem, the fact that $\omega_{\bar{V}}(D)$ is big and $\Omega_{\bar{V}}^1(\log D)$ is weakly positive was proven by Zuo (see [91, Theorem 0.1]). Brunebarbe, Klingler, and Totaro proved that $\Omega_{\bar{V}}^1$ is L-big when $V = \bar{V}$ (see [24, Theorem 3.1]). Brunebarbe then showed that $\Omega_{\bar{V}}^1(\log D)$ is L-big if the monodromy of $\mathbb{V}$ is integral (see [22]). Finally, the general case was proven by Brunebarbe and Cadorel (see [23, Theorem 1.1]). Note that in the proof, they simply assume that $\bar{V}$ is a compact Kähler manifold. Thus, the conclusion implies that $\bar{V}$ is a projective variety (see [23, Corollary 1.3]).

Cadorel and Deng generalized the first point of the previous theorem to varieties supporting a nilpotent harmonic bundle whose Higgs field is injective at a point (see [27, Theorem A]).

2.2.4 Finiteness of sets of morphisms

The points of Faltings’s Theorem 2.1.17 admit generalizations in the context of variations of Hodge structure. The proof of the first point of Theorem 2.1.17 relies on an “Arakelov inequality” for Abelian schemes. This Arakelov inequality admits generalizations for varieties supporting a $\mathbb{C}$ -PVHS (see, for example, the survey [86]). A necessary condition for the deformability of variations of Hodge structure is proven by Peters in [70].

Deligne proved that on a fixed smooth complex algebraic variety, there are only a finite number of integral local systems underlying a $\mathbb{Z}$ -PVHS (see [32]). Variational Hodge theory also allows one to show analogues of Theorem 2.1.18 for other moduli spaces (see, for example, [37]).

2.3 Results of the thesis

In this thesis, we prove new results on the hyperbolicity of locally symmetric varieties and varieties supporting a variation of Hodge structure. More precisely, we focus on the positivity of vector bundles of logarithmic differential forms and the finiteness of sets of morphisms.

2.3.1 Base loci of logarithmic differential forms in arbitrary characteristic

Let k be a field. Consider a smooth, log-compactifiable k -algebraic variety V . If $(\bar{V}, D)$ is a log-compactification of V and m and p are two positive integers with $1 \leq p \leq \dim V$, we denote by $L_{V,m}^p$ the set of global sections of the vector bundle $\mathrm{Sym}^m \Omega_{\bar{V}}^p(\log D)$ over $\bar{V}$. Considered as a vector subspace of $H^0(V, \mathrm{Sym}^m \Omega_V^p)$, $L_{V,m}^p$ does not depend on the choice of $(\bar{V}, D)$. If we denote $\mathcal{O}_p(m) := \mathcal{O}_p(1)^{\otimes m}$ with $\mathcal{O}_p(1)$ the tautological line bundle of $\mathbb{P} \Omega_V^p(\log D)$, the global sections of $\mathrm{Sym}^m \Omega_{\bar{V}}^p(\log D)$ on $\bar{V}$ (resp. V) coincide with those of $\mathcal{O}_p(m)$ on $\mathbb{P} \Omega_{\bar{V}}^p(\log D)$ (resp. $\mathbb{P} \Omega_V^p$). We are interested in two closed subsets of (the total space of) $\Lambda^p T_V$ defined by $L_{V,m}^p$.

Notation 2.3.1. *Let V be a smooth algebraic variety that admits a log-compactification $(\bar{V}, D)$. We denote by:*

- $\mathbb{B}^p(V)$ the closed subset of $\mathbb{P} \Omega_V^p$ defined as the common vanishing locus of all elements of $L_{V,m}^p$ viewed as sections of $\mathcal{O}_p(m)$, for all $m \in \mathbb{N}_{>0}$,
- ${}^{\mathrm{T}}\mathbb{B}^p(V)$ the closed subset of $\Lambda^p T_V$ defined as the set of tensors on which all elements of $L_{V,m}^p$ vanish for all $m \in \mathbb{N}_{>0}$,
- $\mathbb{B}_+^p(V)$ the closed subset of $\mathbb{P} \Omega_V^p$ defined as the intersection of $\mathbb{P} \Omega_V^p$ with the augmented base locus of $\mathcal{O}_p(1)$ on $\mathbb{P} \Omega_V^p(\log D)$ (this intersection does not depend on the choice of $(\bar{V}, D)$),
- ${}^{\mathrm{T}}\mathbb{B}_+^p(V)$ the closed subset of $\Lambda^p T_V$ defined as the set consisting, on the one hand, of the zero tensors and, on the other hand, of the non-zero tensors whose class in $\mathbb{P} \Omega_V^p$ lies in $\mathbb{B}_+^p(V)$.

The closed set ${}^{\mathrm{T}}\mathbb{B}^p(V)$ is equal to the set consisting, on the one hand, of the zero tensors of $\Lambda^p T_V$ and, on the other hand, of the non-zero tensors whose class in $\mathbb{P} \Omega_V^p$ is in $\mathbb{B}^p(V)$. Thus, the set ${}^{\mathrm{T}}\mathbb{B}^p(V)$ (resp. ${}^{\mathrm{T}}\mathbb{B}_+^p(V)$) contains the same information as $\mathbb{B}^p(V)$ (resp. $\mathbb{B}_+^p(V)$). Note further that $\mathbb{B}^p(V) \subset \mathbb{B}_+^p(V)$ (equivalently, ${}^{\mathrm{T}}\mathbb{B}^p(V) \subset {}^{\mathrm{T}}\mathbb{B}_+^p(V)$). Informally, ${}^{\mathrm{T}}\mathbb{B}_+^p(V)$ is the set of tensors in $\Lambda^p T_V$ that are “poorly distinguished” by the elements of $L_{V,m}^p$ for all m .

These sets are of interest to us for two reasons. First, $\mathbb{B}_+^p(V)$ tells us about the positivity of $\Omega_V^p(\log D)$. Indeed, this vector bundle is L-big if and only if $\mathbb{B}_+^p(V) \neq \mathbb{P} \Omega_V^p$, and it is ample relative to V if and only if the projection of $\mathbb{B}_+^p(V)$ onto $\bar{V}$ is contained in D .

Second, $\mathbb{B}^p$ and $\mathbb{B}_+^p$ provide information about morphisms between varieties. For V and V' two smooth k -algebraic varieties, a morphism $f : V \rightarrow V'$ induces a morphism $\Lambda^p df : \Lambda^p T_V \rightarrow \Lambda^p T_{V'}$. Then ${}^T\mathbb{B}^p$ is functorial. This is not the case for ${}^T\mathbb{B}_+^p$ (unless f is unramified). However, we prove the following theorem.

Theorem 2.3.2 ($\simeq$ Theorem 3.5.3). *Let k be a field, and let V and V' be two smooth k -algebraic varieties with V' log-compactifiable. Then for all morphisms $f : V \rightarrow V'$ except finitely many of them, for every integer p such that $1 \leq p \leq \dim V'$, the image of $\Lambda^p df$ is contained in ${}^T\mathbb{B}_+^p(V')$.*

We will in fact prove a slightly more general statement, allowing f to be a rational map, provided that it is “complete” (which is the case if it is a morphism, or if V' is proper). If $\Omega_{V'}^p(\log D')$ is ample relative to V' (which is equivalent to $\mathbb{B}_+^p(V')$ being empty), the previous theorem states that the rank of df is bounded above by $p - 1$ for almost all f . This generalizes theorems of Noguchi and Sunada (case $V' = \bar{V}'$, see [68]) and Noguchi (case $k = \mathbb{C}$, see [66]). If V' is of log-general type (which is equivalent to $\mathbb{B}_+^p(V') \neq \mathbb{P}\Omega_{V'}^p \cong V'$ for $p = \dim V'$), the previous theorem states that the rank of df is bounded above by $\dim V' - 1$ for almost every f . This generalizes theorems of Kobayashi and Ochiai (case $k = \mathbb{C}$ and $V' = \bar{V}'$, see [50]), Martin-Deschamps and Lewin-Ménégaux (case $V' = \bar{V}'$, see [55]) and Tsushima (case $k = \mathbb{C}$, see [84]).

Finally, note that in order to define $L_{V,m}^p$ (and thus $\mathbb{B}_+^p$ and $\mathbb{B}^p$) as before, V must admit a log-compactification. However, the proof of Theorem 7.1.1 will require the use of these objects for a variety that is not a priori log-compactifiable. We therefore give an unconditional definition of $L_{V,m}^p$ using the notion of absolutely logarithmic form defined by Abramovich in [1, Definition 1]. In Chapter 3, we define $L_{V,m}^p$, $\mathbb{B}_+^p$, and $\mathbb{B}^p$ in a completely general way. We prove the properties we will need. We conclude with the proof of Theorem 2.3.2.

2.3.2 Intermediate hyperbolicity of varieties supporting a variation of Hodge structure

In Chapter 4, we begin by generalizing the L-bigness in Theorem 2.2.4 to all degrees.

Theorem 2.3.3 (= Theorem 4.1.1). *Let V be a smooth complex quasiprojective variety. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS with quasi-unipotent local monodromy on a projective log-compactification $(\bar{V}, D)$. Suppose there exists $x \in V$ such*

that the period map of $\mathbb{V}$ is immersive at x . Let p be an integer such that $1 \leq p \leq \dim V$. Then $\Omega_V^p(\log D)$ is L -big.

Our proof is similar to that of Brunebarbe and Cadorel. Next, we study the $\mathbb{B}_+^p$ of varieties supporting a variation of Hodge structure. Let us begin by introducing some notation.

Notation 2.3.4. Let M be a complex analytic manifold, $x \in M$, and $\mathbb{V}$ a $\mathbb{C}$ -PVHS on M . Let $\theta_x : T_{M,x} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}_x)$ be the Higgs field and $\theta_x^* : T_{M,x}^{0,1} \rightarrow \text{End}_{\mathbb{C}}(\mathbb{V}_x)$ the conjugate Higgs field at x . For $v \in T_{M,x}$, we define $C_{\mathbb{V}}(v) := \{w \in T_{M,x}, [\theta_x^*(\bar{v}), \theta_x(w)] = 0\}$.

We prove the following theorem.

Theorem 2.3.5 (= Theorem 4.1.5). Let V be a smooth complex quasiprojective variety. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS with quasi-unipotent local monodromy on a projective log-compactification. Let $x \in V$ be a point where the period map of $\mathbb{V}$ is immersive. Let p be an integer such that $1 \leq p \leq \dim V$. Let $\alpha \in \Lambda^p T_{V,x}$ such that $[\alpha] \in \mathbb{B}_+^p(V)$. Then there exists $v \in T_{V,x} \setminus \{0\}$ such that $\alpha \in \Lambda^p C_{\mathbb{V}}(v)$.

A smooth complex algebraic variety is all the more hyperbolic the smaller its $\mathbb{B}_+^p$ is. The previous theorem is therefore a hyperbolicity result.

We also give a less precise statement that does not involve the conjugate Higgs field (see Proposition 4.7.4). From this, we deduce a criterion for $\Omega_V^p(\log D)$ to be Viehweg-big (i.e., ample relative to a non-empty open set) when V admits a variation of Hodge structure. To simplify the statement of this criterion for this introduction, we add a ‘‘Torelli-type’’ assumption (*) and a ‘‘Calabi-Yau-type’’ assumption (**) to $\mathbb{V}$ (see Theorem 4.7.6 for the general statement without these two assumptions). For a $\mathbb{C}$ -PVHS $\mathbb{V}$ on a smooth variety V and a point $x \in V$, we have a decomposition $\mathbb{V}_x = \bigoplus_{i \in \mathbb{Z}} \mathbb{V}_x^i$. For any $i, j \in \mathbb{Z}$ with $j \leq i$, one can restrict the $(i - j)$ -th iterate θ^{i-j} of the Higgs field θ associated with $\mathbb{V}$ to define a map $\theta_x^{i,j} : T_{V,x} \rightarrow \text{Hom}_{\mathbb{C}}(\mathbb{V}_x^i, \mathbb{V}_x^j)$.

Theorem 2.3.6 (= Theorem 4.1.2). Let V be a smooth complex quasiprojective variety. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS with quasi-unipotent local monodromy on a projective log-compactification $(\bar{V}, D)$. Let $x \in V$ be a point where the period map of $\mathbb{V}$ is immersive. Let $i \in \mathbb{Z}$ be the largest integer such that $\theta_x^{i,i-1}$ is nonzero. Suppose that:

- (*) $\theta_x^{i,i-1}$ is injective,
- (**) $\dim \mathbb{V}_x^i = 1$.

Let j be the smallest integer with $j < i$ such that $\theta_x^{i,j}$ is nonzero. Let m be the maximum dimension of a vector subspace $S \subset T_{V,x}$ such that for every

$v \in S$, $\theta_x^{i,j}(v) = 0$. Let $p \in \mathbb{Z}$ such that $m < p \leq \dim V$. Then $\Omega_V^p(\log D)$ is Viehweg-big.

This theorem can be applied in practice. Using a computer algebra system, we deduce the following theorem.

Theorem 2.3.7 (= Theorem 4.1.3). *Let $\mathcal{M}$ be the fine moduli space of polarized complex quintic threefolds, which is a smooth Deligne-Mumford stack over $\mathbb{C}$ with a finite étale covering $M \rightarrow \mathcal{M}$ where M is a smooth complex quasiprojective variety. Let $(\bar{M}, D)$ be a projective log-compactification of M . Then $\Omega_{\bar{M}}^p(\log D)$ is Viehweg-big if $91 \leq p \leq 101 = \dim M$.*

In the previous theorem, the “91” is probably not optimal (we were limited by our computational power). Note that M is not a locally symmetric variety, so this result is not redundant with those in Chapter 6.

2.3.3 Hyperbolicity of locally symmetric varieties

In the short Chapter 5, we develop tools that will be used in Chapter 6. In the latter, we refine the results of Chapter 4 for locally symmetric varieties. We begin by generalizing Mok’s characteristic subvariety $\mathcal{S}_{r-1}$ to higher degrees.

Definition 2.3.8. *Let X be a locally symmetric variety whose universal covering is irreducible and of rank at least 2, let R be the Riemann curvature tensor of the canonical metric, and let $1 \leq p \leq \dim X$. The characteristic subvariety of higher degree $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is the set of $[\alpha] \in \mathbb{P}\Omega_X^p$ such that there exists $v \in T_X$ with $v \neq 0$ such that $\alpha \in \Lambda^p \ker R(v, \bar{v})$.*

We have $\mathcal{S}^1 = \mathcal{S}_{r-1}$. We then prove the following theorem.

Theorem 2.3.9 (= Theorem 6.1.4). *Let X be a locally symmetric variety whose universal covering is irreducible and of rank at least 2. Let $1 \leq p \leq \dim X$. Then $\mathbb{B}_+^p(X) = \mathcal{S}^p$.*

The inclusion $\mathbb{B}_+^p(X) \subset \mathcal{S}^p$ can be shown using the previous chapter, but one can also almost avoid the use of variations of Hodge structure by using [49, Chapter 6.8] (see Remarks 6.4.2.4). The inclusion $\mathcal{S}^p \subset \mathbb{B}_+^p(X)$ is a statement of non-hyperbolicity. From this, we deduce the following theorem.

Theorem 2.3.10 (= Theorem 6.1.1). *Let $(\bar{X}, D)$ be a projective log-compactification of a locally symmetric variety X whose universal covering is irreducible and of rank at least 2. Let $1 \leq p \leq \dim X$.*

Then $\Omega_{\bar{X}}^p(\log D)$ is L -big. Furthermore, the following statements are equivalent:

- $\Omega_X^p(\log D)$ is Viehweg-big,
- $\Omega_X^p(\log D)$ is ample relative to X ,
- $p > l$ where l is the maximum dimension of $\ker R(v, \bar{v})$ for $v \in T_X$ with $v \neq 0$.

The number l is equal to $l(\Sigma)$ from Theorem 2.1.16. When X is compact, the ampleness of Ω_X^p when $p > l$ was proven by Noguchi and Sunada (see [68]). In degree 1, we also compute the stable base locus.

Theorem 2.3.11 (= Theorem 6.1.5). *Let X be a locally symmetric variety whose universal covering is irreducible and of rank at least 2. Then $\mathbb{B}^1(X) = \mathbb{B}_+^1(X) = \mathcal{S}^1$.*

The case where X is compact in the previous theorem was proved by Mok (see [58, Appendix IV], recall that our $\mathcal{S}^1$ is Mok's $\mathcal{S}_{r-1}$). His proof relies on Theorem 2.1.6. The difficulty of generalizing to the non-compact case lies in the need for a Chern-Weil theory for singular metrics. The main ingredient is Kollár's work in [51, Section 5] on the singularities of Hodge metrics (based on the work of Cattani, Kaplan, and Schmid [29]). We provide some complements to Kollár's article in Chapter 5. We use this to prove a non-compact version of Theorem 2.1.6 for a type of singular metric (very specific to our problem).

2.3.4 Applications in positive characteristic

If we apply Theorem 2.3.2 with the $\mathbb{B}_+^p$ computed in Chapter 6, the result obtained is a statement weaker than the statements recalled in Subsection 2.1.4. Furthermore, we know of no interesting consequence of the functoriality of ${}^{\text{T}}\mathbb{B}^p$ applied with Theorem 2.3.11 that cannot be directly proven using Mok and To's metric rigidity theorem (Theorem 2.1.4).

However, logarithmic differential forms are algebraic objects, unlike variations of Hodge structure and metrics. Spreading-out arguments allow us to transfer information about logarithmic differential forms to positive characteristic. We can then use this to prove in positive characteristic statements that had a transcendental proof in complex geometry. This is explained in Chapter 7.

First, Theorem 2.3.11 combined with a spreading-out and the functoriality of ${}^{\text{T}}\mathbb{B}^p$ allows us to prove the following Theorem 2.3.12. Let $\mathcal{M}_g$ denote the moduli space of smooth, connected projective curves of genus $g \geq 2$.

This is a Deligne-Mumford stack. A family $f : V \rightarrow \mathcal{M}_g$ of smooth connected projective curves of genus g over an integral k -variety V (k a field) is *separable* if for every k -variety V' (resp. M), every dominant étale morphism $p : V' \rightarrow V$ (resp. $q : M \rightarrow \mathcal{M}_g \otimes k$) and every lift $f' : V' \rightarrow M$ of f , the corestriction $f' : V' \rightarrow \overline{\text{Im } f'}$ is separable. The family is *isotrivial* if the map from V to the coarse moduli space of smooth connected projective curves of genus g induced by f is constant.

Theorem 2.3.12 (= Theorem 7.1.1). *For all $g, h \geq 2, n \geq 3$, for all but finitely many prime numbers p , for every field k of characteristic 0 or p with a primitive n -th root of unity, every separable family of smooth connected projective curves of genus h over $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} k$ is isotrivial.*

In the case $k = \mathbb{C}$, a metric proof of the previous theorem was given by Liu, Sun, Yang, and Yau for a compact locally symmetric variety whose universal covering is irreducible and of rank at least 2, in place of $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} k$ in [52, Corollary 1.6]. Their proof also works in the non-compact case, so it applies in particular to $\mathcal{A}_g(n) \otimes_{\mathbb{Z}[e^{2i\pi/n}]} \mathbb{C}$.

Next, Theorem 2.3.9 combined with a spreading-out and Theorem 2.3.2 allows us to prove Theorem 2.3.13 below. For a field k of characteristic other than 2 and a smooth k -variety V , consider a principally polarized Abelian V -scheme $\pi : A \rightarrow V$ of relative dimension $g \in \mathbb{N}$ over V . In this case, there is a canonical isomorphism $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\text{Lie}}_{A/V}^{\otimes 2}$ where $\underline{\text{Lie}}_{A/V}$ is the vector bundle of rank g defined by the Lie algebra of the fibers of π . Furthermore, if we denote by KS the Kodaira-Spencer map $T_V \rightarrow R^1\pi_*T_{A/V}$, then the image of $\beta \circ \text{KS}$ lies in the direct factor $\text{Sym}^2 \underline{\text{Lie}}_{A/V}$ of $\underline{\text{Lie}}_{A/V}^{\otimes 2}$ (see [2, Section 1]).

Theorem 2.3.13 (= Theorem 7.1.2). *Let $g \in \mathbb{N}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of that characteristic, the following result holds.*

*Let V be a smooth k -variety. Then for all but finitely many principally polarized Abelian V -schemes $\pi : A \rightarrow V$ of relative dimension g , for every closed point $x \in V$, there exists a vector subspace $S \subset \text{Lie } A_x$ of dimension $g - 1$ such that $\text{Im}(\beta_x \circ \text{KS}_x) \subset \text{Sym}^2 S$, where $\text{KS} : T_V \rightarrow R^1\pi_*T_{A/V}$ is the Kodaira-Spencer map and $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\text{Lie}}_{A/V}^{\otimes 2}$ the canonical isomorphism.*

In particular, the rank of KS is almost surely bounded above by $\frac{g(g-1)}{2}$. If the family is separable (in the same sense as for families of curves), this implies that the closure of the image of a morphism $V \rightarrow \mathcal{A}_g$ is almost

surely of dimension at most $\frac{g(g-1)}{2}$. For $k = \mathbb{C}$, Theorem 2.3.13 follows from Theorem 2.1.18.

We also prove a similar statement for families of K3 surfaces. Let $\pi : S \rightarrow V$ be a polarized family of K3 surfaces on a smooth k -variety V . The polarization allows us to define the “primitive part” of $R^1\pi_*T_{S/V}$, which is a direct factor of it. The image of the Kodaira-Spencer map $\mathrm{KS}_{S/V}$ lies in this primitive part. For $x \in V$, the cup product and an identification $T_{S/V,x} \cong \Omega_{S/V,x}^1$ endow the fiber $R^1\pi_*T_{S/V,x}$ of $R^1\pi_*T_{S/V}$ with a non-degenerate quadratic form (defined up to a scalar). We can then define an “isotropic $\mathcal{O}_V$ -submodule” of $R^1\pi_*T_{S/V}$ as a submodule all of whose sections define an isotropic element on each fiber.

Theorem 2.3.14 (= Theorem 7.1.5). *Let $d \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of that characteristic, the following result holds.*

*Let V be a smooth k -variety. Then for all but finitely many primitively polarized families of K3 surfaces $\pi : S \rightarrow V$ of degree $2d$, either $\mathrm{KS}_{S/V}$ is zero, or there exists an isotropic invertible $\mathcal{O}_V$ -submodule L of the primitive part of $R^1\pi_*T_{S/V}$ such that $\mathrm{Im}(\mathrm{KS}_{S/V}) \subset L$.*

In particular, the rank of $\mathrm{KS}_{S/V}$ is almost surely 0 or 1. If the family is separable (in the same sense as for families of curves), this implies that the dimension of the closure of the image of a separable morphism from V to the moduli (Deligne-Mumford) stack of primitively polarized K3 surfaces of degree $2d$ is almost surely 0 or 1.

Two properties of the moduli spaces of principally polarized Abelian varieties and K3 surfaces are used to prove the two previous theorems. First, their existence in mixed characteristic. Second, an infinitesimal Torelli theorem (over $\mathbb{C}$). We prove a general theorem for families of varieties whose moduli space shares these two properties (see Theorem 7.3.14). It generalizes Theorems 2.3.13 and 2.3.14.

2.4 Conventions

Let k be a field. A *variety* V over k is a separated scheme of finite type over k . A *log-compactification* of V is the data $(\bar{V}, D)$ of a proper and smooth k -variety $\bar{V}$ and a divisor with normal crossings $D \subset \bar{V}$ such that V identifies with the open subvariety $\bar{V} \setminus D$. A k -variety is *log-compactifiable* if it admits a log-compactification. In particular, a log-compactifiable k -variety

is smooth. By Nagata's compactification theorem (see [64]) and Hironaka's resolution of singularities theorem, if k has characteristic 0, a k -variety is log-compactifiable if and only if it is smooth. A *projective log-compactification* of V is a log-compactification $(\bar{V}, D)$ of V such that $\bar{V}$ is a projective variety. A k -variety is *projectively log-compactifiable* if it admits a projective log-compactification. In particular, a projectively log-compactifiable k -variety is quasiprojective and smooth. By Hironaka's resolution of singularities theorem, if k is of characteristic 0, a k -variety is projectively log-compactifiable if and only if it is quasiprojective and smooth.

Let E be a vector bundle over a scheme S . We also denote by E the associated total space. The notation $\mathbb{P}E$ denotes the scheme that parametrizes the hyperplanes of E (in particular, we use Grothendieck's convention). It is equipped with a projective morphism $\mathbb{P}E \rightarrow S$ of relative dimension equal to the rank of E minus one. When V is a smooth k -variety with a log-compactification $(\bar{V}, D)$ and p is a natural number, we denote by $\mathcal{O}_p(1)$ the tautological line bundle of $\mathbb{P}\Omega_V^p(\log D)$. For $m \in \mathbb{N}$, we denote by $\mathcal{O}_p(m)$ the m -th tensor power of $\mathcal{O}_p(1)$.

Let V be a smooth k -variety. Let $p, m \in \mathbb{N}$. The notation $L_{V,m}^p$ denotes the vector subspace of $H^0(V, \text{Sym}^m \Omega_V^p)$ consisting of "absolutely logarithmic" forms. Its precise definition is given in Subsection 3.3.1. If V admits a log-compactification $(\bar{V}, D)$, then $L_{V,m}^p$ is the set of elements of $H^0(V, \text{Sym}^m \Omega_V^p)$ that extend to a global section of $\text{Sym}^m \Omega_{\bar{V}}^p(\log D)$ over $\bar{V}$.

The notations $\text{Bs}_m^p(V)$, $\mathbb{B}^p(V)$, and $\mathbb{B}_+^p(V)$ are defined in Section 3.4. All denote a closed set in $\mathbb{P}\Omega_V^p$. If V admits a projective log-compactification $(\bar{V}, D)$, then $\text{Bs}_m^p(V)$ (resp. $\mathbb{B}^p(V)$, $\mathbb{B}_+^p(V)$) is the intersection of $\mathbb{P}\Omega_V^p$ with the base locus of $\mathcal{O}_p(m)$ (resp. the stable base locus of $\mathcal{O}_p(1)$, the augmented base locus of $\mathcal{O}_p(1)$) on $\mathbb{P}\Omega_{\bar{V}}^p(\log D)$.

The notations ${}^{\text{T}}\text{Bs}_m^p(V)$, ${}^{\text{T}}\mathbb{B}^p(V)$, and ${}^{\text{T}}\mathbb{B}_+^p(V)$ are defined in Section 3.4. All denote a closed set in (the total space of) $\Lambda^p T_V$. The closed set ${}^{\text{T}}\text{Bs}_m^p(V)$ (resp. ${}^{\text{T}}\mathbb{B}^p(V)$, ${}^{\text{T}}\mathbb{B}_+^p(V)$) is the union of the zero section of $\Lambda^p T_V$ and the set of non-zero tensors whose image in $\mathbb{P}\Omega_V^p$ lies in $\text{Bs}_m^p(V)$ (resp. $\mathbb{B}^p(V)$, $\mathbb{B}_+^p(V)$).

A *locally symmetric variety* X is a smooth complex quasiprojective variety that is biholomorphic to the quotient of a bounded symmetric domain Σ of $\mathbb{C}^d$ (for $d \in \mathbb{N}$) by the action of a torsion-free lattice of the identity component of the group of biholomorphisms of Σ . We say that the universal covering of X is *irreducible* if it is not biholomorphic to a non-trivial product of two bounded symmetric domains.

Chapter 3

Absolutely logarithmic forms and base loci

3.1 Introduction

In this chapter k is any field. We prove some properties of the absolutely logarithmic forms of a smooth algebraic variety V (defined by Abramovich) and of their associated base loci. In Section 3.2, we recall the main properties of the “basic”, stable and augmented base locus of a line bundle. In Section 3.3, we recall the definition of absolutely logarithmic forms and we prove their main properties (partial functoriality and descent of finite étale covers). In Section 3.4, we define the “basic”, stable and augmented base loci $\text{Bs}_m^p(V)$, $\mathbb{B}^p(V)$ and $\mathbb{B}_+^p(V)$ associated with the absolutely logarithmic forms. In the case of a log-compactifiable algebraic variety, they correspond to the base locus of a line bundle. Hence we can often apply the results of the first section to them. We prove the partial functoriality, the descent of finite étale covers and the compatibility with base change and spreading-out. In Section 3.5, we state and prove the main theorem of this chapter (Theorem 3.5.3): a finiteness statement for a set of rational maps between two algebraic varieties $V \rightarrow V'$ whose image is “not included in $\mathbb{B}_+^p(V')$ ” in some sense.

A log-compactification allows one to define logarithmic forms, but it is not enough for us because in the proof of Theorem 7.1.1, we do not know if the k -variety C admits a log-compactification. That is why we use absolutely logarithmic forms.

3.2 Base loci of a line bundle

Let L be a line bundle on a smooth proper k -variety W . Its *base locus* is the closed subset $\text{Bs}(L) \subset W$ defined as the intersection of the vanishing loci of every global section of L on W . Its *stable base locus* is the closed subset $\mathbb{B}(L) \subset W$ defined as the intersection of the base loci $\text{Bs}(L^m)$ for every $m \in \mathbb{N}_{>0}$ (the exponent denotes the tensor power).

When W is projective, we can also define the augmented base locus, which was first defined in [36], in continuation of [65]. Let A be an ample line bundle on W . Then, one can easily check that $\bigcap_{m \in \mathbb{N}_{>0}} \mathbb{B}(L^m \otimes A^{-1})$ does not depend on the choice of A . So the following definition makes sense.

Definition 3.2.1. *Let W be a smooth projective k -variety and L a line bundle on it. The augmented base locus of L is:*

$$\mathbb{B}_+(L) := \bigcap_{m \in \mathbb{N}_{>0}} \mathbb{B}(L^m \otimes A^{-1})$$

where A is any ample line bundle.

3.2.1 Basic properties

The following lemma is easy to deduce from the definition of $\mathbb{B}_+$.

Lemma 3.2.2. *Let L and L' be two line bundles on a smooth projective k -variety W . Let $f : L \rightarrow L'$ be a morphism of $\mathcal{O}_W$ -modules. Let $U \subset W$ be an open subset in restriction to which f is an isomorphism. Then $\mathbb{B}_+(L') \cap U \subset \mathbb{B}_+(L)$.*

We will also need the following proposition.

Proposition 3.2.3 (see [18, Proposition 2.3]). *Let W and W' be two smooth complex projective varieties. Let $f : W \rightarrow W'$ be a birational morphism, let $C \subset W$ be its exceptional locus. Let L be a line bundle on W' . Then $\mathbb{B}_+(f^*L) = f^{-1}(\mathbb{B}_+(L)) \cup C$.*

3.2.2 Augmented base locus and Kodaira morphism

Thanks to [19], we have the following useful characterizations of the augmented base locus.

Theorem 3.2.4. *Let W be a smooth projective k -variety and L a line bundle on it.*

1. The complement of $\mathbb{B}_+(L)$ in W is the maximum for inclusion among Zariski open subsets U of $W \setminus \mathbb{B}(L)$ such that for $m \in \mathbb{N}_{>0}$ divisible enough, $\text{Bs}(L^m) = \mathbb{B}(L)$ and the m -th Kodaira morphism $W \setminus \mathbb{B}(L) \rightarrow \mathbb{P} H^0(W, L^m)$ defined by L is unramified on U .
2. In the previous statement, we can replace “unramified” by “an isomorphism on its image”.
3. For $m \in \mathbb{N}_{>0}$ divisible enough, for every $x \in W \setminus \mathbb{B}(L)$, we have $x \notin \mathbb{B}_+(L)$ if and only if for every $v \in T_{W,x}$, there exists a global section of L^m which vanishes at x but not on v .

Proof. The second point is [19, Theorem A]. The first point is an immediate consequence of [19, Theorem A, Theorem B, Corollary 2.5]. The third point is a reformulation of the first one. $\square$

3.2.3 Nakamaye’s theorem

The augmented base locus of a nef line bundle can be determined by the computation of its exceptional locus.

Definition 3.2.5. *Let W be a smooth projective k -variety and L a line bundle on it. The exceptional locus of L , denoted by $\text{Exc}(L)$, is the union of the irreducible closed subsets $Z \subset W$ of positive dimension such that the intersection number $L^{\dim Z} \cdot Z$ is 0.*

Then, the augmented base locus can be computed thanks to the following theorem (first proved by Nakamaye in [65, Theorem 1.1] when k is algebraically closed of characteristic 0).

Theorem 3.2.6 (see [10, Theorem 1.4]). *Let W be a smooth projective k -variety and L a line bundle on it. If L is nef, then $\mathbb{B}_+(L) = \text{Exc}(L)$.*

3.2.4 Base change and spreading-out

The base loci are compatible with base change and partially compatible with spreading-out.

Proposition 3.2.7. *Let k be a field, k'/k a field extension, W a k -variety, L a line bundle on W . Denote by f the projection $W \times_k k' \rightarrow W$. Then $\text{Bs}(f^*L) = f^{-1}(\text{Bs}(L))$ and $\mathbb{B}(f^*L) = f^{-1}(\mathbb{B}(L))$. If moreover W is smooth projective, then $\mathbb{B}_+(f^*L) = f^{-1}(\mathbb{B}_+(L))$.*

Proof. It is a direct consequence of the canonical isomorphisms $H^0(W \times_k k', f^* L^m) \cong H^0(W, L^m) \otimes_k k'$ for every $m \in \mathbb{N}_{>0}$ (see [80, Lemma 02KH]) and of Theorem 3.2.4.1. $\square$

Proposition 3.2.8. *Let S be an irreducible locally Noetherian scheme, η the generic point of S , W a scheme, $p : W \rightarrow S$ a proper morphism, L a line bundle on W . For $y \in S$, denote by $\kappa(y)$ the residue field of S at y , $\{y\} := \text{Spec}(\kappa(y))$, W_y the proper $\kappa(y)$ -variety $W \times_S \{y\}$, and L_y the restriction of L to W_y .*

Denote by $B \subset W$ the union of the base loci of L_y for every $y \in S$. Then there exists a non-empty open subset $U \subset S$ such that $B \cap p^{-1}(U)$ is the closure of $B \cap p^{-1}(\{\eta\})$.

If moreover p is smooth projective, denote by $B_+ \subset W$ the union of the augmented base loci of L_y for every $y \in S$. Then there exists a non-empty open subset $U \subset S$ such that $B_+ \cap p^{-1}(U)$ is included in the closure of $B_+ \cap p^{-1}(\{\eta\})$.

Proof. By the coherence of proper direct images (see [80, Lemma 02O5]), there exists a non-empty open subset $U \subset S$ such that $p_* L$ is free on U . If $\sigma_1, \dots, \sigma_n \in H^0(U, p_* L) = H^0(p^{-1}(U), L)$ are a family of global sections that generate $p_* L$ on U , then $B \times_S U$ is the common vanishing locus of the $\sigma_1, \dots, \sigma_n$, so it is closed. As η is generic in U , then up to shrinking U , $B \cap p^{-1}(U)$ is the closure of $B \cap p^{-1}(\{\eta\})$.

For the second part, let $m \in \mathbb{N}_{>0}$ be such that the Kodaira map given by L_η^m gives a locally closed embedding $\phi_m : W_\eta \setminus (B_+ \cap p^{-1}(\{\eta\})) \hookrightarrow \mathbb{P} H^0(W_\eta, L_\eta^m)$. As above, let U be a non-empty open subset such that $p_* L^m$ is free on U . Denote by $B_m \subset W$ the union of the base loci of L_y^m for every $y \in S$. Then we have a Kodaira U -morphism $\bar{\phi}_m : p^{-1}(U) \setminus B_m \rightarrow \mathbb{P}(p_* L^m)|_U$. Its ramification locus R is closed, thus, as η is generic in U , up to shrinking U , R is the closure of $(B_+ \setminus B_m) \cap p^{-1}(\{\eta\})$ in $p^{-1}(U) \setminus B_m$. Finally, because of Theorem 3.2.4.1, for every $y \in U$, the augmented base locus of L_y is included in $(B_m \cup R) \cap p^{-1}(\{y\})$. $\square$

3.2.5 Positivity of vector bundles

Let E be a vector bundle on a connected smooth complex projective variety W . Denote by π the projection $\mathbb{P} E \rightarrow W$. The augmented base locus $\mathbb{B}_+(\mathcal{O}(1))$ of the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P} E$ is linked with positivity properties of E . See [8] for the proof of the following facts.

We have $\mathbb{B}_+(\mathcal{O}(1)) \neq \mathbb{P} E$ if and only if E is L-big, which means that $\mathcal{O}(1)$ is big, or equivalently that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that for every $m \gg 0$, we have $\dim H^0(W, \text{Sym}^m E) \geq \varepsilon m^{\dim(W) + \text{rk}(E) - 1}$.

Let $U \subset W$ be an open subset. We have $\pi(\mathbb{B}_+(\mathcal{O}(1))) \cap U = \emptyset$ if and only if E is ample with respect to U . That means that there exists an ample line bundle A on W , two integers $a, b \in \mathbb{N}_{>0}$ and a morphism $A^{\oplus a} \rightarrow \text{Sym}^b E$ that is surjective over U . The vector bundle E is said *Viehweg-big* if $\pi(\mathbb{B}_+(\mathcal{O}(1))) \neq W$, that is equivalent to the existence of a non-empty open subset with respect to which E is ample. Viehweg-bigness is stronger than L-bigness, and both notions coincide with usual bigness for line bundles. Finally, E is ample if and only if $\mathbb{B}_+(\mathcal{O}(1)) = \emptyset$.

3.2.6 Non-Kähler locus

The non-Kähler locus is an analytic analog of the augmented base locus. Let M be a compact complex manifold. Consider the real Bott-Chern cohomology group $H_{\partial\bar{\partial}}^{1,1}(M, \mathbb{R})$ (if M is Kähler, this group is $H^{1,1}(M) \cap H^2(M, \mathbb{R})$).

Definition 3.2.9 (see [16, Definition 3.16]). *The non-Kähler locus of a big class $c \in H_{\partial\bar{\partial}}^{1,1}(M, \mathbb{R})$ is the (closed) set of points of M at which the Lelong number of every Kähler current in the class c is positive.*

In case of a non big class $c \in H_{\partial\bar{\partial}}^{1,1}(M, \mathbb{R})$, we will call non-Kähler locus of c the whole manifold M . The non-Kähler locus is interesting for us as it allows one to compute augmented base loci using singular metrics of semipositive curvature thanks to the following theorem.

Theorem 3.2.10 (see [15, Corollaire 2.2.8] or [83, Theorem 2.3]). *Let W be a smooth complex projective variety. Let L be a line bundle on it. Then the non-Kähler locus of $c_1(L)$ is $\mathbb{B}_+(L)$.*

If a class is big and nef, then we can compute its non-Kähler locus using positive currents instead of Kähler currents thanks to the following theorem. It is an analytic analog of Theorem 3.2.6.

Theorem 3.2.11 (see [83, Theorem 4.5]). *Let M be a compact complex manifold. Let c be a nef and big real $(1,1)$ -class on M . Let T be a positive closed current in the class c . Let $U \subset M$ be an open subset on which the restriction of T is a smooth Kähler form. Then the intersection of U and of the non-Kähler locus of c is empty.*

We consider now the case of a vector bundle. Let E be a holomorphic vector bundle of rank $r \in \mathbb{N}_{>0}$ on a complex manifold M of dimension $d \in \mathbb{N}$. Let h be a Hermitian metric on E . Let R be the curvature tensor of the Chern connection of h . Let $x \in M$ and $e \in E_x$ be such that $h(e, e) = 1$. Then there exists an open neighborhood U of x in M and holomorphic sections

$e_1, \dots, e_r$ of E on U such that $e_r(x) = e$, $h_x(e_k, e_l) = \delta_{kl}$ and $dh_x(e_k, e_l) = 0$ for every $1 \leq k, l \leq r$, with δ the Kronecker symbol (see [58, Lemma 1 p.26]). Denote by $(e_1^*, \dots, e_r^*)$ the dual basis of $(e_1, \dots, e_r)$ (each e_k^* is a holomorphic section of the dual $E^\vee$ of E). For $1 \leq k < r$, denote by u_k the quotient $\frac{e_k^*}{e_r^*}$. Up to shrinking U , it admits holomorphic coordinates $z_1, \dots, z_d$. Denote by $v_1, \dots, v_d$ the associated basis of holomorphic vector fields.

Consider $\mathbb{P} E^\vee$ and the projection $\pi : \mathbb{P} E^\vee \rightarrow M$. Then $(\pi^* z_1, \dots, \pi^* z_d, u_1, \dots, u_{r-1})$ defines holomorphic coordinates in a neighborhood of $[e]$ in $\mathbb{P} E^\vee$. Let $\hat{h}$ be the Hermitian metric induced by h on the tautological line bundle $\mathcal{O}(1)$ on $\mathbb{P} E^\vee$. Denote by $C_1(\mathcal{O}(1), \hat{h})$ its first Chern form. Then at $[e] \in \mathbb{P} E^\vee$ we have the following equality (see [58, Proposition 1 p.38]):

$$C_1(\mathcal{O}(1), \hat{h})_{[e]} = -\frac{\mathbf{i}}{2\pi} \sum_{k=1}^{r-1} du_k \wedge d\bar{u}_k + \frac{\mathbf{i}}{2\pi} \sum_{1 \leq i, j \leq d} h_{[e]}(R(v_i, \bar{v}_j).e, e) dz_i \wedge d\bar{z}_j. \quad (3.1)$$

Lemma 3.2.12. *Let M be a complex manifold. Let E be a holomorphic vector bundle on M . Let h be a Hermitian metric on E . Let R be the curvature tensor of the Chern connection of h , suppose it is seminegative in the sense of Griffiths. Let $1 \leq p \leq \text{rk } E$. Let $\hat{h}$ be the Hermitian metric induced by h on the tautological line bundle $\mathcal{O}_p(1)$ on $\mathbb{P} \Lambda^p E^\vee$. Let $C_1(\mathcal{O}_p(1), \hat{h})$ be the first Chern form of $\hat{h}$. Let $x \in M$, let $\alpha \in \Lambda^p E_x \setminus \{0\}$.*

Then the real $(1, 1)$ -form $C_1(\mathcal{O}_p(1), \hat{h})$ is semipositive. Moreover, it is not positive definite at $[\alpha]$ if and only if there exists $v \in T_{M, x} \setminus \{0\}$ such that $\alpha \in \Lambda^p K(v)$ with $K(v)$ the kernel of the Hermitian endomorphism $R(v, \bar{v})$ of E_x .

Proof. Denote by R_p the curvature tensor of the Chern connection of $\Lambda^p E$ with the Hermitian metric induced by h . For $v \in T_{M, x}$ and $e_1, \dots, e_p \in E_x$, we have:

$$R_p(v, \bar{v}).(e_1 \wedge \dots \wedge e_p) = \sum_{i=1}^p e_1 \wedge \dots \wedge e_{i-1} \wedge R(v, \bar{v}).e_i \wedge e_{i+1} \wedge \dots \wedge e_p.$$

As $R(v, \bar{v})$ is seminegative Hermitian, $R_p(v, \bar{v})$ is seminegative Hermitian too. Hence formula (3.1) gives that $C_1(\mathcal{O}_p(1), \hat{h})$ is semipositive, and that it is not positive definite at $[\alpha]$ if and only if there exists $v \in T_{M, x}$ such that $\alpha \in \ker R_p(v, \bar{v})$. Finally, we have $\ker R_p(v, \bar{v}) = \Lambda^p \ker R(v, \bar{v})$ as it holds for every semipositive or seminegative Hermitian endomorphism by the spectral theorem. $\square$

3.3 Absolutely logarithmic forms

3.3.1 Definition and basic properties

In the next section, we will have to consider the graded k -algebra of logarithmic symmetric forms on a log-compactification of a log-compactifiable smooth k -variety. This graded k -algebra is actually independent of the choice of the log-compactification, and Abramovich generalized its definition to every k -variety in [1, Definition 1]. We give here a reformulation of Abramovich's definition in the smooth case.

Let K be a separable function field over k . Let d be its transcendence degree. A divisorial valuation of K/k is a valuation which vanishes on k and whose residue field has transcendence degree $d - 1$ over k . A valuation is divisorial if and only if it is the valuation defined by a prime divisor of a normal k -variety with K as function field.

Let ν be a divisorial valuation of K/k . By multiplying by a constant, we can suppose $\text{Im}(\nu) = \mathbb{Z}$. Let A be the valuation ring $\{x \in K, \nu(x) \geq 0\}$, $\mathfrak{m}$ its maximal ideal $\{x \in K, \nu(x) \geq 1\}$, $\mathfrak{m}^{-1}$ the A -submodule $\{x \in K, \nu(x) \geq -1\}$ of K , $\kappa := A/\mathfrak{m}$ the residue field of ν . Let $p \in \mathbb{N}_{>0}$ with $1 \leq p \leq d$. The canonical maps $A \rightarrow \kappa$ and $A \hookrightarrow K$ induce morphisms of A -modules $\varpi^p : \Omega_{A/k}^p \rightarrow \Omega_{\kappa/k}^p$ and $f^p : \Omega_{A/k}^p \rightarrow \Omega_{K/k}^p$. We denote by $F_{\nu,1}^p$ (resp. $L_{\nu,1}^p$) the A -submodule $\text{Im } f^p$ (resp. $\mathfrak{m}^{-1} f^p(\ker \varpi^p)$) of $\Omega_{K/k}^p$. As suggested by the following proposition, an element of $L_{\nu,1}^p$ can be thought of as “logarithmic with respect to ν ”.

Proposition 3.3.1. *The A -module $L_{\nu,1}^p$ is the A -submodule of $\Omega_{K/k}^p$ generated by $F_{\nu,1}^p$ and by the elements $u^{-1} du \wedge \alpha$ for every $u \in \mathfrak{m}$ and $\alpha \in F_{\nu,1}^{p-1}$.*

Proof. Choose $z \in \mathfrak{m}$ such that $\nu(z) = 1$. On the one hand, $dz \in \ker \varpi^1$ and for every $a \in A$, $z da \in \ker \varpi^1$. On the other hand, by [80, Lemma 00RR], the A -module $\ker \varpi^1$ is generated by elements of the form dm for $m \in \mathfrak{m}$. As $\mathfrak{m}$ is the ideal of A generated by z , there exists $a \in A$ such that $m = za$. Hence $dm = d(za) = a dz + z da$. Thus the A -module $\ker \varpi^1$ is generated by dz and the elements of the form $z da$.

Hence, because of [80, Lemma 00DO], the A -module $\ker \varpi^p$ is generated by the elements $dz \wedge \beta$ and $z da \wedge \beta$ for every $a \in A$ and $\beta \in \Omega_{A/k}^{p-1}$. As the A -module $\mathfrak{m}^{-1}$ is generated by z^{-1} , $L_{\nu,1}^p$ is generated by the elements $z^{-1} dz \wedge \alpha$ and $da \wedge \alpha$ for every $a \in A$ and $\alpha \in F_{\nu,1}^{p-1}$. But the A -module $F_{\nu,1}^p$ is generated by the elements of the form $da \wedge \alpha$. Finally, for every $u \in \mathfrak{m}$, it is immediate from the definition that $u^{-1} du \wedge \alpha \in L_{\nu,1}^p$. $\square$

Lemma 3.3.2. 1. The A -modules $F_{\nu,1}^p$ and $L_{\nu,1}^p$ are free and finite.

2. Let $z \in \mathfrak{m}$ be such that $\nu(z) = 1$. We have $F_{\nu,1}^p[z^{-1}] = L_{\nu,1}^p[z^{-1}] = \Omega_{K/k}^p$.

Proof. As A is essentially of finite type over k , the A -module $\Omega_{A/k}^p$ is finite, hence $\ker \varpi^p$ is finite, so $F_{\nu,1}^p$ and $L_{\nu,1}^p$ are finite. As they are submodules of $\Omega_{K/k}^p$, they have no torsion. As A is a discrete valuation ring, they are free: we get (1). Let $a \in A$ and $n \in \mathbb{N}$. We have $d(z^{-n}a) = -nz^{-n-1}a \, dz + z^{-n}da \in F_{\nu,1}^1[z^{-1}] \subset L_{\nu,1}^1[z^{-1}]$. This proves (2). $\square$

We denote by $F_{\nu,\bullet}^p = \bigoplus_{m \in \mathbb{N}} F_{\nu,m}^p$ (resp. $L_{\nu,\bullet}^p = \bigoplus_{m \in \mathbb{N}} L_{\nu,m}^p$) the free symmetric A -algebra generated by $F_{\nu,1}^p$ (resp. $L_{\nu,1}^p$) with its canonical grading. The inclusions of $F_{\nu,1}^p$ (resp. $L_{\nu,1}^p$) in $\Omega_{K/k}^p$ induce a morphism ι_F^p (resp. ι_L^p) of graded A -algebras from $F_{\nu,\bullet}^p$ (resp. $L_{\nu,\bullet}^p$) to $D_{K,\bullet}^p := \bigoplus_{m \in \mathbb{N}} \text{Sym}_K^m \Omega_{K/k}^p$. This last graded A -algebra is a graded algebra of polynomials over K . The following lemma is easy to prove.

Lemma 3.3.3. 1. The graded A -algebras $F_{\nu,\bullet}^p$ and $L_{\nu,\bullet}^p$ are graded algebras of polynomials over A .

2. The morphisms ι_F^p and ι_L^p are injective and induce isomorphisms $\text{Frac}(F_{\nu,\bullet}^p) = \text{Frac}(L_{\nu,\bullet}^p) = \text{Frac}(D_{K,\bullet}^p)$.

3. The domains $\iota_F^p(F_{\nu,\bullet}^p)$, $\iota_L^p(L_{\nu,\bullet}^p)$ and $D_{K,\bullet}^p$ are integrally closed in $\text{Frac}(D_{K,\bullet}^p)$.

Proof. The point (1) is an immediate consequence of Lemma 3.3.2.1. The injectivity follows, and the equality of fraction fields (2) follows from Lemma 3.3.2.2. The domains $F_{\nu,\bullet}^p$, $L_{\nu,\bullet}^p$ and $D_{K,\bullet}^p$ are normal as by (1) they are rings of polynomials over a normal domain. Hence (3) follows from (2). $\square$

In the following, we will omit ι_F^p and ι_L^p and consider $F_{\nu,\bullet}^p$ and $L_{\nu,\bullet}^p$ as subsets of $D_{K,\bullet}^p$.

Let V be a smooth k -variety. We denote by $D_{V,\bullet}^p$ the graded k -algebra $\bigoplus_{m \in \mathbb{N}} H^0(V, \text{Sym}^m \Omega_V^p)$. Suppose V connected. Let K be its function field. Then $D_{V,\bullet}^p$ is a graded k -subalgebra of $D_{K,\bullet}^p$. As every symmetric differential form defined in codimension 1 on V extends as a global symmetric differential form on V , $D_{V,\bullet}^p$ identifies with the intersection of the $F_{\nu,\bullet}^p$'s over all divisorial valuations ν of K/k centered in V .

If V is connected, we define the graded k -algebra $L_{V,\bullet}^p$ as the intersection of $D_{V,\bullet}^p$ with the $L_{\nu,\bullet}^p$'s for every divisorial valuation ν of K/k (with K the function field of V). In the general case (without connectedness assumption),

we define the graded k -algebra $L_{V,\bullet}^p$ as the direct sum of the $L_{C,\bullet}^p$'s for every connected component C of V . Proposition 3.3.1 shows that $L_{V,\bullet}^p$ is the algebra of *absolutely logarithmic* symmetric differential forms defined by Abramovich in [1, Definition 1]. Then we get the following properties.

Proposition 3.3.4 (see [1, Remark p.46]). *If $(\bar{V}, D)$ is a log-compactification of V , then $L_{V,\bullet}^p$ is the graded k -algebra $\bigoplus_{m \in \mathbb{N}} H^0(\bar{V}, \text{Sym}^m \Omega_{\bar{V}}^p(\log D))$.*

Let V, V' be smooth k -varieties, let $f : V \rightarrow V'$ be a morphism. Then the pull-back of differential forms induces a morphism of graded k -algebras $f^* : D_{V',\bullet}^p \rightarrow D_{V,\bullet}^p$.

Proposition 3.3.5 (see [1, Lemma 5]). *Suppose V' is log-compactifiable. Then $f^* L_{V',\bullet}^p \subset L_{V,\bullet}^p$.*

3.3.2 Compatibility with finite étale covering maps

Proposition 3.3.6. *Let V be a connected smooth k -variety. Let K be the function field of V . Then the domains $D_{V,\bullet}^p$ and $L_{V,\bullet}^p$ are integrally closed in $\text{Frac}(D_{K,\bullet}^p)$.*

Proof. The domains $D_{V,\bullet}^p$ and $L_{V,\bullet}^p$ are intersections of some subrings $F_{\nu,\bullet}^p$'s and $L_{\nu,\bullet}^p$'s of $\text{Frac}(D_{K,\bullet}^p)$. They are integrally closed in it by Lemma 3.3.3.3. $\square$

Thanks to the previous proposition, we can prove the following property of compatibility with finite étale covering maps.

Proposition 3.3.7. *Let V, S be two connected smooth k -varieties. Let $f : V \rightarrow S$ be a finite étale covering map. Then $L_{V,\bullet}^p$ is the integral closure of $f^* L_{S,\bullet}^p$ in $\text{Frac}(D_{V,\bullet}^p)$.*

Proof. As $f^* : D_{S,\bullet}^p \rightarrow D_{V,\bullet}^p$ is injective, we consider $D_{S,\bullet}^p$ as a graded k -subalgebra of $D_{V,\bullet}^p$. Let K (resp. K') be the function field of S (resp. V). Then f induces a finite separable extension of fields K'/K . Hence, every divisorial valuation of K/k is the restriction of a divisorial valuation of K'/k . Moreover, if ν' is a divisorial valuation of K'/k and ν is its restriction to K , then ν is divisorial and $L_{\nu,\bullet}^p = L_{\nu',\bullet}^p \cap D_{K,\bullet}^p$. Therefore $L_{S,\bullet}^p = L_{V,\bullet}^p \cap D_{S,\bullet}^p$ and, with Proposition 3.3.6, the integral closure of $L_{S,\bullet}^p$ in $\text{Frac}(D_{V,\bullet}^p)$ is included in $L_{V,\bullet}^p$.

To prove that $L_{V,\bullet}^p$ is integral over $L_{S,\bullet}^p$, suppose first that f is Galois. The field extension $\text{Frac}(D_{V,\bullet}^p)/\text{Frac}(D_{S,\bullet}^p)$ is finite and Galois with the same degree and Galois group as K'/K . As $L_{V,\bullet}^p$ is stable under the action of the

Galois group, the minimal polynomial over $\text{Frac}(D_{S,\bullet}^p)$ of an element of $L_{V,\bullet}^p$ has coefficients in $L_{V,\bullet}^p \cap D_{S,\bullet}^p = L_{S,\bullet}^p$.

In the general case, there exists a Galois finite étale covering map $V' \rightarrow S$ which dominates V . We have $L_{V,\bullet}^p \subset L_{V',\bullet}^p$ and the latter is integral over $L_{S,\bullet}^p$. $\square$

3.3.3 Finiteness of the dimension

As when there exists a log-compactification, $L_{V,m}^p$ is finite dimensional.

Proposition 3.3.8. *Let V be a smooth k -variety. For every $m \in \mathbb{N}_{>0}$, the dimension of $L_{V,m}^p$ is finite.*

Proof. By Nagata's theorem [64], there exists a proper k -variety $\bar{V}$ with an open embedding $V \hookrightarrow \bar{V}$. As we can replace it by its normalization, we can suppose $\bar{V}$ normal. Let $D \subset \bar{V}$ be the union of the irreducible closed subsets of codimension 1 that do not intersect V . Then by definition, every element of $L_{V,m}^p$ defines a section of $\text{Sym}^m \Omega_{\bar{V}}^p$ on V and of $\text{Sym}^m \Omega_{\bar{V}}^p(D')$ in the neighborhood of the generic point of every irreducible component D' of D . Hence it defines a section of $\text{Sym}^m \Omega_{\bar{V}}^p(D)$ on an open subset of $\bar{V}$ whose complement has codimension at least 2. Thus it defines a global section of the reflexive hull $\text{Sym}^{[m]} \Omega_{\bar{V}}^p(D)$ of $\text{Sym}^m \Omega_{\bar{V}}^p(D)$ (see [43, Proposition 1.6]). We have defined an injection $L_{V,m}^p \hookrightarrow H^0(\bar{V}, \text{Sym}^{[m]} \Omega_{\bar{V}}^p(D))$, and the latter space has finite dimension. $\square$

3.4 Base loci of the absolutely logarithmic forms

Consider a smooth k -variety V and the projective bundle $\mathbb{P} \Omega_V^p \rightarrow V$ (with Grothendieck's convention: the fiber at a point x is the set of hyperplanes of $\Omega_{V,x}^p$, or equivalently the set of lines of $\Lambda^p T_{V,x}$). Then the vanishing locus of a homogeneous element of $L_{V,\bullet}^p$ defines a closed subset of $\mathbb{P} \Omega_V^p$ and a closed subset of $\Lambda^p T_V$. Let $m \in \mathbb{N}_{>0}$. We denote by $\text{Bs}_m^p(V) \subset \mathbb{P} \Omega_V^p$ and ${}^T\text{Bs}_m^p(V) \subset \Lambda^p T_V$ the intersection of the vanishing loci of the elements of $L_{V,m}^p$. Remark that Bs_m^p and ${}^T\text{Bs}_m^p$ carry the same information. We denote by $\mathbb{B}^p(V)$ the closed subset $\bigcap_{m \in \mathbb{N}_{>0}} \text{Bs}_m^p(V) \subset \mathbb{P} \Omega_V^p$ and by ${}^T\mathbb{B}^p(V)$ the closed subset $\bigcap_{m \in \mathbb{N}_{>0}} {}^T\text{Bs}_m^p(V) \subset \Lambda^p T_V$. Remark that there exists $m \in \mathbb{N}_{>0}$ such that $\mathbb{B}^p(V) = \text{Bs}_m^p(V)$ and ${}^T\mathbb{B}^p(V) = {}^T\text{Bs}_m^p(V)$, and that $\mathbb{B}^p$ and ${}^T\mathbb{B}^p$ carry the same information.

Let $\mathcal{O}_p(1)$ be the tautological line bundle over $\mathbb{P}\Omega_V^p$ and $\mathcal{O}_p(m)$ its tensor power $\mathcal{O}_p(1)^{\otimes m}$. If π^p denotes the projection $\mathbb{P}\Omega_V^p \rightarrow V$, then we have $\pi_*^p \mathcal{O}_p(m) = \text{Sym}^m \Omega_V^p$. In particular $H^0(\mathbb{P}\Omega_V^p, \mathcal{O}_p(m)) = H^0(V, \text{Sym}^m \Omega_V^p)$. Hence the finite dimensional (see Proposition 3.3.8) linear series $L_{V,m}^p \subset H^0(V, \text{Sym}^m \Omega_V^p) = H^0(\mathbb{P}\Omega_V^p, \mathcal{O}_p(m))$ defines a Kodaira morphism $\phi_m^p : \mathbb{P}\Omega_V^p \setminus \text{Bs}_m^p(V) \rightarrow \mathbb{P}L_{V,m}^p$. Consider $R_m^p \subset \mathbb{P}\Omega_V^p$ the union of $\text{Bs}_m^p(V)$ and of the ramification locus of ϕ_m^p . We denote by $\mathbb{B}_+^p(V)$ the closed subset $\bigcap_{m \in \mathbb{N}_{>0}} R_m^p \subset \mathbb{P}\Omega_V^p$. In concrete terms, we have $\mathbb{B}^p(V) \subset \mathbb{B}_+^p(V)$, and a point $x \in \mathbb{P}\Omega_V^p \setminus \mathbb{B}^p(V)$ is in $\mathbb{B}_+^p(V)$ if and only if there exists a tangent vector $v \in T_{\mathbb{P}\Omega_{V,x}^p} \setminus \{0\}$ such that every homogeneous element of $L_{V,\bullet}^p$ that vanishes at x vanishes on v .

We denote by ${}^T\mathbb{B}_+^p(V)$ the closed subset of $\Lambda^p T_V$ defined as the union of the zero section Z and of the inverse image of $\mathbb{B}_+^p(V)$ by the quotient morphism $\Lambda^p T_V \setminus Z \rightarrow \mathbb{P}\Omega_V^p$. Again, it carries the same information as $\mathbb{B}_+^p(V)$.

Suppose V is log-compactifiable. Let $(\bar{V}, D)$ be a log-compactification. Let $\mathcal{O}_p(1)$ be the tautological line bundle on $\mathbb{P}\Omega_{\bar{V}}^p(\log D)$, let $\mathcal{O}_p(m)$ be its m -th tensor power. Then for every $m \in \mathbb{N}_{>0}$ we have $\text{Bs}_m^p(V) = \text{Bs}(\mathcal{O}_p(m)) \cap \mathbb{P}\Omega_V^p$, and $\mathbb{B}^p(V) = \mathbb{B}(\mathcal{O}_p(1)) \cap \mathbb{P}\Omega_V^p$. If moreover V is projectively log-compactifiable, we can choose $\bar{V}$ projective. In this case we have $\mathbb{B}_+^p(V) = \mathbb{B}_+(\mathcal{O}_p(1)) \cap \mathbb{P}\Omega_V^p$ as a consequence of Theorem 3.2.4.

3.4.1 Partial functoriality

The partial functoriality of $L_\bullet^p$ (Proposition 3.3.5) gives the partial functoriality of ${}^T\text{Bs}_m^p$ and ${}^T\mathbb{B}^p$.

Proposition 3.4.1. *Let $f : V \rightarrow V'$ be a morphism between smooth k -varieties with V' log-compactifiable. Then for every $m \in \mathbb{N}_{>0}$, $\Lambda^p df({}^T\text{Bs}_m^p(V)) \subset {}^T\text{Bs}_m^p(V')$, and $\Lambda^p df({}^T\mathbb{B}^p(V)) \subset {}^T\mathbb{B}^p(V')$.*

The following definition and proposition give the form under which we will apply the previous proposition.

Definition 3.4.2. *A closed subset C of $\mathbb{P}\Omega_V^p$ is linearly non-degenerate if for each closed point $x \in V$, the fiber C_x is not included in a hyperplane of $\mathbb{P}\Omega_{V,x}^p$.*

Proposition 3.4.3. *Let $f : V \rightarrow V'$ be a morphism between smooth k -varieties with V connected and V' log-compactifiable. Suppose the corestriction $f : V \rightarrow \overline{\text{Im } f}$ is separable. Suppose there exists $m \in \mathbb{N}$ such that $\text{Bs}_m^1(V)$ is linearly non-degenerate and $\text{Bs}_m^1(V')$ is empty. Then f is constant. The*

same statement holds with $\mathbb{B}^1(V)$ instead of $\text{Bs}_m^1(V)$ and $\mathbb{B}^1(V')$ instead of $\text{Bs}_m^1(V')$.

Proof. Thanks to Proposition 3.4.1 and by the assumption on m , df is zero. Hence, thanks to the separability assumption, f is constant. $\square$

Remark 3.4.4. In the statement of the previous proposition, if we replace the exponent “1” in the hypothesis by a positive integer p , then a similar proof gives $\dim \overline{\text{Im } f} < p$.

There is also a functoriality for $\mathbb{B}_+^p$, but only where the morphism is not ramified. It is an immediate consequence of Propositions 3.4.1 and 3.3.5.

Proposition 3.4.5. *Let $f : V \rightarrow V'$ be a morphism between smooth k -varieties with V' log-compactifiable. Let $R \subset V$ be the ramification locus of f . Let $f^p : \mathbb{P}\Omega_{V \setminus R}^p \rightarrow \mathbb{P}\Omega_{V'}^p$ be the morphism induced by f . Then $f^p(\mathbb{B}_+^p(V) \cap \mathbb{P}\Omega_{V \setminus R}^p) \subset \mathbb{B}_+^p(V')$.*

3.4.2 Descent of finite étale covering maps

When we have a finite étale covering map $V \rightarrow S$ between smooth k -varieties, we consider $D_{S,\bullet}^p$ as a graded k -subalgebra of $D_{V,\bullet}^p$.

Proposition 3.4.6. *Let $f : V \rightarrow S$ be a finite étale covering map of degree $n \in \mathbb{N}_{>0}$ between smooth k -varieties. Let f^p be the map $\mathbb{P}\Omega_V^p \rightarrow \mathbb{P}\Omega_S^p$ induced by f . Let $m \in \mathbb{N}_{>0}$. Then we have:*

1. $\text{Bs}_m^p(V) \subset (f^p)^{-1}(\text{Bs}_m^p(S))$,
2. $(f^p)^{-1} \left(\bigcap_{r=1}^n \text{Bs}_{mr}^p(S) \right) \subset \text{Bs}_m^p(V)$,
3. $\mathbb{B}^p(V) = (f^p)^{-1}(\mathbb{B}^p(S))$,
4. $\mathbb{B}_+^p(V) \subset (f^p)^{-1}(\mathbb{B}_+^p(S))$,
5. if k has characteristic 0 and V is quasiprojective: $\mathbb{B}_+^p(V) = (f^p)^{-1}(\mathbb{B}_+^p(S))$.

Proof. We can freely suppose V and S connected. The inclusion (1) results from the inclusion $L_{S,\bullet}^p \subset L_{V,\bullet}^p$ given by Proposition 3.3.7.

For the second inclusion, let $x \in (f^p)^{-1}(\bigcap_{r=1}^n \text{Bs}_{mr}^p(S))$. Let $\alpha \in L_{V,m}^p$. The field extension $\text{Frac}(D_{V,\bullet}^p)/\text{Frac}(D_{S,\bullet}^p)$ is finite of degree n , hence the minimal polynomial χ of α on $\text{Frac}(D_{S,\bullet}^p)$ is a monic polynomial of degree at most n . By Proposition 3.3.7, α is integral over $L_{S,\bullet}^p$. Its conjugates and

then the coefficients of χ are integral over $L_{S,\bullet}^p$. By Proposition 3.3.6, the coefficients of χ are in $L_{S,\bullet}^p$. As α is homogeneous of degree m , each non-leading coefficients of χ is homogeneous of degree mr for some $1 \leq r \leq n$. Thus, by definition of x , we have $\chi_{f(x)} = X^n$. Hence $\alpha_x^n = 0$, so $\alpha_x = 0$: this gives (2).

The equality (3) is a direct consequence of (1) and (2). The inclusion (4) results from (3) and the inclusion $L_{S,\bullet}^p \subset L_{V,\bullet}^p$.

It remains to prove the inclusion $(f^p)^{-1}(\mathbb{B}_+^p(S)) \subset \mathbb{B}_+^p(V)$, under the hypothesis that k has characteristic 0 and that V is quasiprojective. We first suppose that the finite étale covering map f is Galois. As $\mathbb{B}_+^p(V)$ is stable under the action of the Galois group, the inclusion $(f^p)^{-1}(\mathbb{B}_+^p(S)) \subset \mathbb{B}_+^p(V)$ is equivalent to $\mathbb{B}_+^p(S) \subset f^p(\mathbb{B}_+^p(V))$. Let $x \in \mathbb{P}\Omega_V^p \setminus \mathbb{B}_+^p(V)$, let us prove that $f^p(x) \notin \mathbb{B}_+^p(S)$. First we have $x \notin \mathbb{B}^p(V)$, so by (3) we have $x \notin \mathbb{B}^p(S)$. Now, let $v \in T_{\mathbb{P}\Omega_V^p, x} \cong T_{\mathbb{P}\Omega_S^p, f^p(x)}$. The additional hypothesis implies that V is projectively log-compactifiable. Hence we can apply Theorem 3.2.4.2: there exists $m \in \mathbb{N}_{>0}$ such that $\text{Bs}_m^p(V) = \mathbb{B}^p(V)$ and the Kodaira morphism $\mathbb{P}\Omega_S^p \setminus \mathbb{B}^p(V) \rightarrow \mathbb{P}\Omega_{V,m}^p$ is an isomorphism from $\mathbb{P}\Omega_S^p \setminus \mathbb{B}_+^p(V)$ to its image. In particular, there exists $\alpha \in L_{V,m}^p$ that vanishes at x but not on v and not at the conjugates of x (for the action of the Galois group). Hence the norm $N(\alpha) \in L_{S,mn}^p$ of α (which is the product of the conjugates of α under the action of the Galois group) vanishes at $f^p(x)$ but not on $df_x^p.v$. Thus $f^p(x) \notin \mathbb{B}_+^p(S)$.

In the general case, there exists a smooth quasiprojective k -variety V' and a finite étale covering map $g : V' \rightarrow V$ such that the finite étale covering map $f' := f \circ g : V' \rightarrow S$ is Galois. Using the Galois case and (4), we get:

$$\begin{aligned} (f^p)^{-1}(\mathbb{B}_+^p(S)) &= g^p((f'^p)^{-1}(\mathbb{B}_+^p(S))) \\ &= g^p(\mathbb{B}_+^p(V')) \\ &\subset \mathbb{B}_+^p(V). \end{aligned}$$

Thus we obtain (5). □

3.4.3 Base change and spreading-out

The base loci are compatible with base change.

Proposition 3.4.7. *Let k be a field, k'/k a field extension, V a projectively log-compactifiable smooth k -variety. Let $p, m \in \mathbb{N}_{>0}$. Denote by f^p the projection $\mathbb{P}\Omega_{V \times_k k'}^p \rightarrow \mathbb{P}\Omega_V^p$. Then we have $\text{Bs}_m^p(V \times_k k') = (f^p)^{-1}(\text{Bs}_m^p(V))$, $\mathbb{B}^p(V \times_k k') = (f^p)^{-1}(\mathbb{B}^p(V))$ and $\mathbb{B}_+^p(V \times_k k') = (f^p)^{-1}(\mathbb{B}_+^p(V))$.*

Proof. Apply Proposition 3.2.7 to $\mathcal{O}_p(1)$. □

Let S be an irreducible Noetherian scheme, W a scheme and $q : W \rightarrow S$ a smooth morphism. A closed subset $D \subset W$ of codimension 1 with irreducible components $(D_i)_{i \in I}$ is a *horizontal strict normal crossing* divisor over S if for every $I' \subset I$, the schematic intersection $\bigcap_{i \in I'} D'_i$ is smooth over S and its codimension is the cardinality of I' . A closed subset $D \subset W$ of codimension 1 is a *horizontal normal crossing* divisor over S if there exists a scheme W' and a surjective étale morphism $\pi : W' \rightarrow W$ such that $\pi^{-1}(D)$ is a horizontal strict normal crossing divisor over S . In this case, for every $p \in \mathbb{N}$ that is at most the relative dimension of W over S , there is a well-defined vector bundle $\Omega_{W/S}^p(\log D)$, compatible with base change. If k is a field and V is a k -variety, then the horizontal (strict) normal crossing divisors of V over k are the (strict) normal crossing divisors of V .

To prove that the base loci are compatible with spreading-out, we need the following lemma.

Lemma 3.4.8. *Let S be an irreducible Noetherian scheme, η the generic point of S , V a scheme, $q : V \rightarrow S$ a smooth morphism. For $s \in S$, denote by $\kappa(s)$ the residue field of S at s , $\{s\} := \text{Spec}(\kappa(s))$, and V_s the $\kappa(s)$ -variety $V \times_S \{s\}$. Suppose V_η is (projectively) log-compactifiable.*

Then there exists a non-empty open subset $U \subset S$, a smooth proper (resp. projective) U -scheme $\bar{V}$, an open embedding $j : q^{-1}(U) \hookrightarrow \bar{V}$ over U such that $D := \bar{V} \setminus \text{Im}(j)$ is a horizontal normal crossing divisor of $\bar{V}$ over S .

In particular, for every $s \in U$, V_s is (projectively) log-compactifiable, $(\bar{V} \times_U \{s\}, D \times_U \{s\})$ is a (projective) log-compactification of V_s , and for every $p \in \mathbb{N}_{>0}$, there is a well-defined vector bundle $\Omega_{\bar{V}/U}^p(\log D)$ on $\bar{V}$.

Proof. It is an easy consequence of the definitions and of classical spreading-out arguments from EGA (see [73, Theorem 3.2.1] and [80, Lemma 05F9]). $\square$

The “basic” base loci are compatible with spreading-out, and the augmented base loci are partially compatible.

Proposition 3.4.9. *Let S be an irreducible Noetherian scheme, η the generic point of S , V a scheme, $q : V \rightarrow S$ a smooth morphism. For $s \in S$, denote by $\kappa(s)$ the residue field of S at s , $\{s\} := \text{Spec}(\kappa(s))$, and V_s the $\kappa(s)$ -variety $V \times_S \{s\}$. Suppose V_η is log-compactifiable. Let $m \in \mathbb{N}_{>0}$. For every $p \in \mathbb{N}$, denote by π^p the smooth morphism $\mathbb{P} \Omega_{V/S}^p \rightarrow S$.*

Denote by $B^p \subset \mathbb{P} \Omega_{V/S}^p$ the union of the $\text{Bs}_m^p(V_s)$ ’s for every $s \in S$. Then there exists a non-empty open subset $U \subset S$ such that for every $p \in \mathbb{N}$, $B^p \cap (\pi^p)^{-1}(U)$ is the closure of $B^p \cap (\pi^p)^{-1}(\{\eta\})$.

If moreover V_η is projectively log-compactifiable, denote by $B_+^p \subset \mathbb{P}\Omega_{V/S}^p$ the union of $\mathbb{B}_+^p(V_s)$'s for every $s \in S$. Then there exists a non-empty open subset $U \subset S$ such that for every $p \in \mathbb{N}$, $B_+^p \cap (\pi^p)^{-1}(U)$ is included in the closure of $B_+^p \cap (\pi^p)^{-1}(\{\eta\})$.

Proof. Consider $(U, \bar{V}, D)$ given by Lemma 3.4.8, replace S by U , then apply Proposition 3.2.8 to the tautological line bundle on $\mathbb{P}\Omega_{\bar{V}/S}^p(\log D)$. Take the intersection of the open subsets obtained for the different values of p with $1 \leq p \leq \dim V_\eta$. $\square$

The previous proposition allows one to spread-out emptiness and linear-non-degeneracy.

Corollary 3.4.10. *Let S be an irreducible Noetherian scheme, η the generic point of S , V a scheme, $q : V \rightarrow S$ a smooth morphism. For $s \in S$, denote by $\kappa(s)$ the residue field of S at s , $\{s\} := \text{Spec}(\kappa(s))$, and V_s the $\kappa(s)$ -variety $V \times_S \{s\}$. Suppose V_η is log-compactifiable. Let $m, p \in \mathbb{N}$.*

Suppose $\text{Bs}_m^p(V_\eta)$ is empty (resp. linearly non-degenerate). Then there exists a non-empty open subset $U \subset S$ such that for every $s \in U$, $\text{Bs}_m^p(V_s)$ is empty (resp. linearly non-degenerate).

Suppose V_η is projectively log-compactifiable and $\mathbb{B}_+^p(V_\eta)$ is empty. Then there exists a non-empty open subset $U \subset S$ such that for every $s \in U$, $\mathbb{B}_+^p(V_s)$ is empty.

Proof. The statements of emptiness are immediate consequences of Proposition 3.4.9. To prove the statement of linear-non-degeneracy, replace S by a non-empty open subset given by Proposition 3.4.9 so that the union of the $\text{Bs}_m^p(V_s)$'s for every $s \in S$ is a closed subset B of $\mathbb{P}\Omega_{V/S}^p$. Let d be the relative dimension of V over S . Consider P the fiber product over V : $P := \prod_{i=1}^d \mathbb{P}\Omega_{V/S}^p$. Let $C \subset P$ be the open subset of maximal rank d -uples of lines. Let $B^d := \prod_{i=1}^d B \subset P$: then $B^d \cap C$ is a locally closed subset of P , let Q be its projection on V . Chevalley's theorem gives that Q is constructible, hence that $q(V \setminus Q)$ is constructible. By assumption, it does not contain the generic point of S . Hence the complement of $q(V \setminus Q)$ in S contains a non-empty open subset. $\square$

3.4.4 Metric tools

The following proposition shows how to use Lemma 3.2.12 to compute non-Kähler loci. We will not directly use this proposition, as the hypotheses of compactness and nefness are too restrictive for our cases. However its proof gives the outline of the proof of Theorem 4.1.5.

Proposition 3.4.11. *Let M be a compact Kähler manifold. Suppose the curvature R is seminegative in the sense of Griffiths. Let $1 \leq p \leq \dim M$. Suppose the tautological line bundle $\mathcal{O}_p(1)$ on $\mathbb{P}\Omega_M^p$ is nef. Let $\alpha \in \Lambda^p T_{M,x} \setminus \{0\}$, suppose $[\alpha]$ is in the non-Kähler locus of the first Chern class $c_1(\mathcal{O}_p(1))$. Then there exists $v \in T_M$ with $v \neq 0$ such that $\alpha \in \Lambda^p K(v)$ with $K(v)$ the kernel of the Hermitian endomorphism $R(v, \bar{v})$ of $T_{M,x}$.*

Proof. Denote by U the complement of the set of $[\alpha] \in \mathbb{P}\Omega_M^p$ such that there exists $v \in T_M$ with $v \neq 0$ such that $\alpha \in \Lambda^p K(v)$. By Lemma 3.2.12, U is open and the semipositive real $(1, 1)$ -form $C_1(\mathcal{O}_p(1), \hat{h})$ is Kähler on U (with h the Kähler metric and $\hat{h}$ the metric induced on $\mathcal{O}_p(1)$).

Suppose U is not empty (otherwise there is nothing to prove). Then we have $\int_{\mathbb{P}\Omega_M^p} C_1(\mathcal{O}_p(1), \hat{h})^{\dim \mathbb{P}\Omega_M^p} > 0$. Hence Theorem 3.4.13 below gives that $\mathcal{O}_p(1)$ is a big line bundle.

As the real $(1, 1)$ -class $c_1(\mathcal{O}_p(1))$ is big and, by hypothesis, nef, Theorem 3.2.11 applied with the closed positive real $(1, 1)$ -form $C_1(\mathcal{O}_p(1), \hat{h})$ gives that the non-Kähler locus of $c_1(\mathcal{O}_p(1))$ does not intersect U . $\square$

Remark 3.4.12. Let M be a Kähler manifold, suppose the curvature R is seminegative in the sense of Griffiths. Let $x \in M$ and $v, w \in T_{M,x}$. Then $w \in K(v)$ if and only if $v \in K(w)$. It is obvious if v and w are linearly dependent. Otherwise, as $R(v, \bar{v})$ is seminegative, it vanishes on w if and only if $\langle R(v, \bar{v})(w), w \rangle = 0$, which means that the holomorphic bisectional curvature in the direction of the plane generated by v and w is zero.

The bigness of a line bundle is deduced from metric information thanks to the following theorem.

Theorem 3.4.13 (see [17, Theorem 1.2]). *Let M be a connected compact Kähler manifold. Let L be a line bundle on M . Suppose that there exists a nonempty open subset $U \subset M$ and a singular Hermitian metric of semipositive curvature on L such that the restriction of its curvature to U is a Kähler form. Then L is big.*

3.5 Finiteness of sets of complete rational maps

Let k be a field, V and V' two k -varieties and $f : V \dashrightarrow V'$ a rational map. We denote by $\bar{\Gamma}(f)$ the closure in $V \times V'$ of the graph of f and by $\overline{\text{Im}}(f)$ the closure of the projection of $\bar{\Gamma}(f)$ on V' . Remark that $\bar{\Gamma}(f)$ and $\overline{\text{Im}}(f)$ do not depend on the choice of a dense open subset U of definition of f and that $\overline{\text{Im}}(f)$ is the closure of $f(U)$ in V' .

By [80, Lemma 056V], the variety V is generically smooth if and only if it is geometrically reduced. In this case, denote by V^{reg} its smooth locus.

Definition 3.5.1. *Let k be a field, V and V' two k -varieties with V irreducible and geometrically reduced. A rational map $f : V \dashrightarrow V'$ is complete if there exists an open subset $U \subset V^{\text{reg}}$ such that $V^{\text{reg}} \setminus U$ has codimension at least 2 in V^{reg} and f is defined on U .*

The hypothesis of completeness is useful because it allows one to define the pull-back of a logarithmic form (Lemma 3.5.12 below).

Examples 3.5.2. 1. *A morphism $V \rightarrow V'$ is a complete rational map.*

2. *If V' is proper, then every rational map $V \dashrightarrow V'$ is complete.*

3. *Noguchi defines in [66] a proper rational map as a rational map $f : V \dashrightarrow V'$ such that $\bar{\Gamma}(f)$ is proper over V^1 . A proper rational map is complete. Indeed, by the valuative criterion of properness, there exists an open subset $U \subset V^{\text{reg}}$ such that $V^{\text{reg}} \setminus U$ has codimension at least 2 and the birational map $V \dashrightarrow \bar{\Gamma}(f)$ is defined on U . Composing with the projection $\bar{\Gamma}(f) \rightarrow V'$, we obtain that f is defined on U . This example is more general than the two previous ones.*

If V is irreducible and geometrically reduced and V' is smooth, then a rational map $f : V \dashrightarrow V'$ induces a rational map from the total space of $\Lambda^p T_{V^{\text{reg}}}$ to the total space of $\Lambda^p T_{V'}$, denoted by $\Lambda^p df$.

In this section, we prove the following theorem.

Theorem 3.5.3. *Let k be a field, V an irreducible and geometrically reduced k -variety and V' a smooth and projectively log-compactifiable k -variety. Then for all but finitely many complete rational maps $f : V \dashrightarrow V'$, for every $p \in \mathbb{N}$ with $1 \leq p \leq \dim V'$, we have $\bar{\text{Im}}(\Lambda^p df) \subset \mathbb{T}\mathbb{B}_+^p(V')$.*

Remark 3.5.4. By definition, $\mathbb{T}\mathbb{B}_+^p(V')$ contains the zero section of $\Lambda^p T_{V'}$. Thus the inclusion $\bar{\text{Im}}(\Lambda^p df) \subset \mathbb{T}\mathbb{B}_+^p(V')$ is verified if the rank of df is everywhere strictly less than p .

¹The graph is supposed proper over the source, not over the target. In particular, it has no link with the usual notion of properness for morphisms.

3.5.1 Corollaries

Let us firstly prove some corollaries.

Corollary 3.5.5. *Let k be a field, V an irreducible and geometrically reduced k -variety and V' a smooth irreducible k -variety with a log-compactification $(\bar{V}', D')$. Suppose the log-canonical line bundle $\omega_{\bar{V}'}(D')$ is big. Then for all but finitely many complete rational maps $f : V \dashrightarrow V'$, the rank of df is bounded above by $\dim V' - 1$.*

In particular, if f is separable on $\overline{\text{Im}}(f)$, then f is not dominant.

This corollary was proven by Kobayashi and Ochiai in the case $V' = \bar{V}'$ and $k = \mathbb{C}$ (with a generalization to an analytic setting, see [50]), by Martin-Deschamps and Lewin-Ménégaux in the case $V' = \bar{V}'$ (see [55]) and by Tsushima in the case $k = \mathbb{C}$ with f proper in the sense of Noguchi (see [84]).

Proof. Denote $\dim V'$ by p and identify $\mathbb{P}\Omega_{V'}^p$ with V' . For a complete rational map $f : V \dashrightarrow V'$, if there is a point at which the rank of df is p , then $\dim \overline{\text{Im}}(f) = p$. Thus $\overline{\text{Im}}(f) = V'$. As $\overline{\text{Im}}(\Lambda^p df)$ is not included in the zero section of $\Lambda^p T_{V'}$, we have $\overline{\text{Im}}(\Lambda^p df) = \Lambda^p T_{V'}$. By Subsection 3.2.5, the hypothesis on $\omega_{\bar{V}'}(D')$ is equivalent to $\mathbb{B}_+^p(V') \neq V'$. Hence $\overline{\text{Im}}(\Lambda^p df) \not\subset {}^t\mathbb{B}_+^p(V')$ so the corollary follows from Theorem 3.5.3. $\square$

For $p < \dim V'$, the bigness hypothesis is not sufficient.

Corollary 3.5.6. *Let k be a field, V an irreducible and geometrically reduced k -variety and V' a smooth k -variety with a log-compactification $(\bar{V}', D')$. Let $p \in \mathbb{N}$ with $1 \leq p \leq \dim V'$. Suppose $\Omega_{\bar{V}'}^p(\log D')$ is ample with respect to V' . Then for all but finitely many complete rational maps $f : V \dashrightarrow V'$, the rank of df is bounded above by $p - 1$.*

In particular, if f is separable on $\overline{\text{Im}}(f)$, then $\dim \overline{\text{Im}}(f) < p$.

This corollary was proven by Noguchi and Sunada in the case $V' = \bar{V}'$ (see [68]) and by Noguchi in the case $k = \mathbb{C}$ with f proper (see [66]).

Proof. By Subsection 3.2.5, the hypothesis is equivalent to $\mathbb{B}_+^p(V') = \emptyset$, thus the corollary is immediate with Theorem 3.5.3. $\square$

In some cases, we can partially extend the theorem with a Deligne-Mumford stack as codomain.

Corollary 3.5.7. *Let k be an algebraically closed field, V a smooth irreducible k -variety, $\mathcal{M}$ a Deligne-Mumford k -stack, V' a smooth and projectively log-compactifiable k -variety and $q : V' \rightarrow \mathcal{M}$ a finite étale covering map of degree $n \in \mathbb{N}_{>0}$. Suppose the characteristic of k does not divide n . Then for all but finitely many morphisms $f : V \rightarrow \mathcal{M}$, for every $p \in \mathbb{N}$ with $1 \leq p \leq \dim \mathcal{M}$, we have $\text{Im}(\Lambda^p \text{d}f) \subset \mathbb{T}\mathbb{B}_+^p(V')$ with $\tilde{f} : V \times_{\mathcal{M}} V' \rightarrow V'$ the pull-back of f by q .*

Proof. For every $f : V \rightarrow \mathcal{M}$, $V \times_{\mathcal{M}} V'$ is a finite étale cover of degree n of V . As the characteristic of k does not divide n and k is algebraically closed, there exist only finitely many isomorphism classes of finite étale covers of V of degree n (see [69, Théorème 4.11]). Thus the statement results from Theorem 3.5.3, applied to every finite étale cover of degree n of V . $\square$

3.5.2 Outline of the proof

The outline of the proof is based on Martin-Deschamps and Lewin-Ménégaux's article [55] mentioned above. We denote by $\mathcal{C}$ the category of reduced k -varieties. The Yoneda functor associates to every object A (resp. morphism $f : A \rightarrow B$) of $\mathcal{C}$ a presheaf (resp. morphism of presheaves) on $\mathcal{C}$ denoted by $\underline{A}$ (resp. $\underline{f}$) such that for every object C of $\mathcal{C}$, $\underline{A}(C)$ is the set of morphisms from C to A (resp. $\underline{f} : \underline{A}(C) \rightarrow \underline{B}(C)$ is the composition with f).

We will define a presheaf $\mathcal{D}$ on $\mathcal{C}$ such that $\mathcal{D}(k) := \mathcal{D}(\text{Spec } k)$ is the set of complete rational maps such that $\overline{\text{Im}}(\Lambda^p \text{d}f) \not\subset \mathbb{T}\mathbb{B}_+^p(V')$. Our goal is to prove that $\mathcal{D}(k)$ is finite. Recall that $L_{V,n}^p$ and $L_{V',n}^p$ are finite dimensional by Proposition 3.3.8. We choose $n \in \mathbb{N}_{>0}$ such that the Kodaira map $\phi_n : \mathbb{P}\Omega_{V'}^p \dashrightarrow \mathbb{P}L_{V',n}^p$ defined by the line bundle $\mathcal{O}_p(n)$ on $\mathbb{P}\Omega_{V'}^p$, gives a locally closed embedding $\mathbb{P}\Omega_{V'}^p \setminus \mathbb{B}_+^p(V') \hookrightarrow \mathbb{P}L_{V',n}^p$ (such a n exists by Theorem 3.2.4.2). We denote by E the vector space $\text{Hom}_k(L_{V',n}^p, L_{V,n}^p)^\vee$. Denote by $\mathbb{V}E$ the vector space $\text{Hom}_k(L_{V',n}^p, L_{V,n}^p)$ considered as an affine k -space (in particular the set of rational points of $\mathbb{V}E$ is $\text{Hom}_k(L_{V',n}^p, L_{V,n}^p)$). It is an object of $\mathcal{C}$. For every object S of $\mathcal{C}$, we have $\underline{\mathbb{V}E}(S) = H^0(S, \mathcal{O}_S \otimes_k \text{Hom}_k(L_{V',n}^p, L_{V,n}^p))$. Denote $\mathbb{V}^*E := \mathbb{V}E \setminus \{0\}$.

We will see that we can pull back absolutely logarithmic forms by complete rational maps. Hence to every complete rational map $V \dashrightarrow V'$, we can associate a k -linear map $L_{V',n}^p \rightarrow L_{V,n}^p$. This procedure gives a map $\mathcal{D}(k) \rightarrow \mathbb{V}^*E(k)$. We will see that this map extends as a morphism of presheaves $u : \mathcal{D} \rightarrow \underline{\mathbb{V}^*E}$.

Thanks to the condition $\overline{\text{Im}}(\Lambda^p \text{d}f) \not\subset \mathbb{T}\mathbb{B}_+^p(V')$, we can find again $f \in \mathcal{D}(k)$ from the datum of $u(f) \in \underline{\mathbb{V}^*E}(k)$. Actually, it is even sufficient to

know $q(u(f)) \in \mathbb{P}E$ with $q : \mathbb{V}^*E \rightarrow \mathbb{P}E$ the canonical projection. We would like to define a projective variety $A \subset \mathbb{P}E$ such that $\text{Im}(\underline{q} \circ u) \subset \underline{A}$ and a morphism $r : \underline{A} \rightarrow \mathcal{D}$ such that $r \circ \underline{q} \circ u = \text{id}_{\mathcal{D}}$. In this case, by Yoneda lemma, we would have $u \circ r = \underline{e}$ for some morphism $e : A \rightarrow \mathbb{V}E$. The crucial point is that as A is projective and $\mathbb{V}E$ is affine, the image of e would be finite. Hence the image of $A(k)$ by $u \circ r$ would be a finite set of rational points of $\mathbb{V}E$. As the composition $\mathcal{D} \xrightarrow{q \circ u} \underline{A} \xrightarrow{r} \mathcal{D} \xrightarrow{u} \underline{q}^{-1}(\underline{A}) \xrightarrow{q} \underline{A} \xrightarrow{r} \mathcal{D}$ is the identity and contains $u \circ r$, $\mathcal{D}(k)$ would be finite.

We will not exactly do this. Instead of a projective variety $A \subset \mathbb{P}E$, we will define a finite union of smooth locally closed subvarieties $A_i \subset \mathbb{P}E$ for i in a finite set I . Then we will define a morphism $r_i : \underline{A}_i \rightarrow \mathcal{D}$ for every i . We will prove that every $u \circ r_i$ extends on a compactification of A_i . As above, this implies that the image of every $A_i(k)$ by $u \circ r_i$ is finite, allowing us to finish the proof in the same way.

In Subsection 3.5.3, we define (complete) rational maps in families (that will allow us to define $\mathcal{D}$). In Subsection 3.5.4, we explain how to pull back absolutely logarithmic forms by families of complete rational maps (that will allow us to define u). In Subsection 3.5.5, we give the definition of $\mathcal{D}$ and u . In Subsection 3.5.6, we explain the construction of the A_i 's and the r_i 's. We conclude the proof in Subsection 3.5.7.

3.5.3 Families of rational maps

To define the presheaf $\mathcal{D}$, we need to define a convenient notion of family of rational maps. In this subsection, k is any field.

If S, V and V' are k -varieties and $f : V \rightarrow V'$ is a morphism, we denote $V_S := V \times_k S$ and $f_S : V_S \rightarrow V'_S$ the pulled back morphism. If s is a point of S , W and W' are S -varieties and $f : W \rightarrow W'$ is a S -morphism, we denote by W_s the fiber of W above s and by $f_s : W_s \rightarrow W'_s$ the induced morphism.

Definition 3.5.8 (see [55]). *Let S be a k -variety. Let W and W' be S -varieties such that the fibers of W are geometrically irreducible. A S -rational map $W \dashrightarrow W'$ is a rational map $f : W \dashrightarrow W'$ such that there exists an open subset $U \subset W$ such that the structural morphism $U \rightarrow S$ is surjective and f defines a S -morphism $U \rightarrow W'$.*

We denote by $\text{Rat}_S(W, W')$ the set of S -rational maps $W \dashrightarrow W'$. The condition of surjectivity of $U \rightarrow S$ makes this notion functorial: if T is a k -variety, then every morphism $g : T \rightarrow S$ induces a pull back map $g^* : \text{Rat}_S(W, W') \rightarrow \text{Rat}_T(W \times_S T, W' \times_S T)$.

Definition 3.5.9. *Let S be a k -variety. Let W and W' be S -varieties such that the fibers of W are geometrically integral. A S -rational map $f : W \dashrightarrow W'$ is S -complete if for every $s \in S$, $f_s : W_s \dashrightarrow W'_s$ is complete.*

Remark that this property is preserved by pull backs.

For $f \in \text{Rat}_S(W, W')$, denote by $\overline{\Gamma}_S(f)$ the closure in $W \times_S W'$ of the graph of f . Again, $\overline{\Gamma}_S(f)$ does not depend on the choice of a dense open subset U of definition of f . Remark that $\overline{\Gamma}_T(g^*f) \times_S U = \overline{\Gamma}_S(f) \times_S T \times_S U$ and $\overline{\text{Im}}(g^*f) \times_S U = \overline{\text{Im}}(f) \times_S T \times_S U$. Thus we have $\overline{\Gamma}_T(g^*f) \subset \overline{\Gamma}_S(f) \times_S T$ and $\overline{\text{Im}}(g^*f) \subset \overline{\text{Im}}(f) \times_S T$, but there is not equality in general.² For this reason, we define strict S -rational maps, which are more compatible with base change.

Definition 3.5.10. *Let S be a k -variety. Let W and W' be S -varieties such that the fibers of W are geometrically irreducible. A S -rational map $f : W \dashrightarrow W'$ is strict if for every $s \in S$, $\overline{\Gamma}_S(f)_s$ is geometrically irreducible.*

Strictness is preserved by pull backs. Moreover, for every S -variety $g : T \rightarrow S$ and every strict S -rational map $f : W \dashrightarrow W'$, we have $\overline{\Gamma}_T(g^*f) = \overline{\Gamma}_S(f) \times_S T$ and $\overline{\text{Im}}(g^*f) = \overline{\text{Im}}(f) \times_S T$.

Lemma 3.5.11. *Let S be a k -variety. Let W and W' be S -varieties such that the fibers of W are geometrically irreducible. Let f be a S -rational map $W \dashrightarrow W'$. Then there exists a finite partition $S = \bigsqcup_{i \in I} S_i$ such that for every $i \in I$, S_i is a locally closed subset of S , and if we denote by $\iota_i : S_i \hookrightarrow S$ the inclusion, the S_i -rational map ι_i^*f is strict.*

Proof. The fiber of $\overline{\Gamma}_S(f)$ above the generic point of an irreducible component of S is geometrically irreducible. Hence by [80, Lemma 0559], there exists a dense open subset $U \subset S$ such that the fibers of $\overline{\Gamma}_S(f)$ above U are geometrically irreducible. Then the pull-back of f on U is strict. Now we restrict to $S \setminus U$ and conclude by induction on the dimension. $\square$

3.5.4 Pulling back absolutely logarithmic forms

First, we prove that one can pull-back absolutely logarithmic forms by complete rational maps.

²A counterexample is given by S a variety, $\pi : W \rightarrow S$ a S -variety, W' the blow-up of W centered at a closed point $x \in W$ such that $\dim_x W > \dim_{\pi(x)} S > 0$, $f : W \dashrightarrow W'$ the rational inverse of the blow-up, T the point $\pi(x)$ and g the inclusion of T in S . Then the exceptional divisor is included in $\overline{\text{Im}}(f) \times_S T$ but not in $\overline{\text{Im}}(g^*f)$.

Lemma 3.5.12. *Let k be a field, let V and V' be two smooth k -varieties with V irreducible and V' log-compactifiable. Let $f : V \dashrightarrow V'$ be a complete rational map. Let $U \subset V$ be a dense open subset on which f is defined. Let $p \in \mathbb{N}$ with $1 \leq p \leq \dim V'$, let $\alpha \in L_{V',\bullet}^p$. Then the form $f^*\alpha$ on U is the restriction of an element of $L_{V,\bullet}^p$.*

Proof. By Proposition 3.3.5, the pull-back by $f : U \rightarrow V'$ gives a morphism of graded k -algebras $L_{V',\bullet}^p \rightarrow L_{U,\bullet}^p$. As f is complete, we can suppose that $V \setminus U$ has codimension at least 2. Thus, as V is normal, a symmetric differential form of U extends to V . Finally, by definition, a global form β on V is absolutely logarithmic if and only if its restriction to a dense open subset U is absolutely logarithmic (indeed, it is equivalent to $\beta \in L_{\nu,\bullet}^p$ for every divisorial valuation ν , and the set of divisorial valuations only depends on the function field). $\square$

Now, we prove that we can pull-back elements of $L_{V',\bullet}^p$ “in families” by S -complete S -rational maps, with S a reduced k -variety.

Lemma 3.5.13. *Let k be an algebraically closed field, let S be a reduced k -variety, let V and V' be two smooth k -varieties with V irreducible and V' log-compactifiable. Let $n \in \mathbb{N}$. Let $f \in \text{Rat}_S(V_S, V'_S)$ be S -complete. Let $U \subset V_S$ be a dense open subset on which f is defined. Denote by π (resp. π') the projection $V_S \rightarrow S$ (resp. $V'_S \rightarrow S$). Denote by f^* the morphism of $\mathcal{O}_S$ -modules $\pi'_* \text{Sym}^n \Omega_{V'_S/S}^p \rightarrow \pi_* \text{Sym}^n \Omega_{V_S/S}^p$ induced by f . Then there exists a unique morphism of $\mathcal{O}_S$ -modules $f_L^* : \mathcal{O}_S \otimes_k L_{V',n}^p \rightarrow \mathcal{O}_S \otimes_k L_{V,n}^p$ such that the following diagram commutes.*

$$\begin{array}{ccccccc}
 \mathcal{O}_S \otimes_k L_{V',n}^p & \longrightarrow & \mathcal{O}_S \otimes_k H^0(V', \text{Sym}^n \Omega_{V'}^p) & \longrightarrow & \pi'_* \text{Sym}^n \Omega_{V'_S/S}^p & & \\
 \downarrow f_L^* & & & & \searrow f^* & & \\
 \mathcal{O}_S \otimes_k L_{V,n}^p & \longrightarrow & \mathcal{O}_S \otimes_k H^0(V, \text{Sym}^n \Omega_V^p) & \longrightarrow & \pi_* \text{Sym}^n \Omega_{V_S/S}^p & \longrightarrow & \pi_* \text{Sym}^n \Omega_{U/S}^p
 \end{array}$$

Proof. If f_L^* exists, it is uniquely determined because of the injectivity of the morphisms of the bottom line in the diagram. Now we prove the existence. As f is S -rational, we can freely enlarge U to suppose that $\pi(U) = S$.

We have to check that for every open subset $S' \subset S$ and every $\alpha \in H^0(S', \mathcal{O}_{S'}) \otimes_k L_{V',n}^p$, there exists $\rho \in H^0(S', \mathcal{O}_{S'}) \otimes_k L_{V,n}^p$ whose image in $H^0(V_{S'}, \text{Sym}^n \Omega_{V_{S'}/S'}^p)$ restricts as $f^*\alpha$ on $U \times_S S'$. It is sufficient to prove it for every S' in a base of open subsets of S . Hence, as $L_{V,n}^p$ has finite dimension (see Proposition 3.3.8), we can suppose that there exists a finite

set of closed points $P \subset V$ such that $P_{S'} \subset U$ and the linear composition $\iota : L_{V,n}^p \hookrightarrow H^0(V, \text{Sym}^n \Omega_V^p) \rightarrow H^0(P, (\text{Sym}^n \Omega_V^p)|_P)$ is injective.

The restriction of $f^* \alpha$ to $P_{S'}$ defines an element ρ' of $H^0(P_{S'}, (\text{Sym}^n \Omega_{V_{S'}/S'}^p)|_{P_{S'}}) \cong H^0(S', \mathcal{O}_{S'}) \otimes_k H^0(P, (\text{Sym}^n \Omega_V^p)|_P)$. By Lemma 3.5.12, for every closed point $s \in S'$, $(f^* \alpha)_s$ is the restriction to U_s of an element of $L_{V,n}^p$. Hence we have $\rho' \in H^0(S', \mathcal{O}_{S'}) \otimes_k \iota(L_{V,n}^p)$. As $\iota(L_{V,n}^p) \cong L_{V,n}^p$, we have defined an element ρ of $H^0(S', \mathcal{O}_{S'}) \otimes_k L_{V,n}^p$, whose image in $H^0(U \times_S S', \text{Sym}^n \Omega_{V_{S'}/S'}^p)$ is $f^* \alpha$. $\square$

3.5.5 Notations

Let k be a field, V an irreducible and geometrically reduced k -variety, V' a smooth and projectively log-compactifiable k -variety and $f : V \dashrightarrow V'$ a rational map. Let k' be a field extension of k . Let $f' : V \otimes_k k' \dashrightarrow V' \otimes_k k'$ be the pull-back of f . By Proposition 3.4.7, for every $p \in \mathbb{N}$ with $1 \leq p \leq \dim V'$, we have $\overline{\text{Im}}(\Lambda^p df) \subset {}^{\text{T}}\mathbb{B}_+^p(V')$ if and only if $\overline{\text{Im}}(\Lambda^p df') \subset {}^{\text{T}}\mathbb{B}_+^p(V' \otimes_k k')$. Moreover, this last inclusion is realized in restriction to an irreducible component of $V \otimes_k k'$ if and only if it is realized in restriction to every irreducible component. Hence, we can suppose k algebraically closed to prove Theorem 3.5.3.

Moreover, one can freely replace V with a nonempty open subset to prove the theorem. Thus, as k is perfect, we can suppose V smooth. Finally, we can consider each irreducible component of V' independently, so we can suppose V' (geometrically) integral. From now on, we fix V a smooth integral variety and V' a projectively log-compactifiable smooth integral variety over an algebraically closed field k , and $p \in \mathbb{N}$ with $1 \leq p \leq \dim V'$.

Notation 3.5.14. *We denote by $\mathcal{D}$ the presheaf on the category $\mathcal{C}$ of reduced k -varieties such that:*

- *for every object S of $\mathcal{C}$: $\mathcal{D}(S)$ is the set of S -complete S -rational maps $f : V_S \dashrightarrow V'_S$ such that for every $s \in S$, $\overline{\text{Im}}(\Lambda^p df_s) \not\subset {}^{\text{T}}\mathbb{B}_+^p(V') \otimes_k \kappa(s)$ (with $\kappa(s)$ the residue field of S at s),*
- *for every morphism f of $\mathcal{C}$: $\mathcal{D}(f) := f^*$.*

To define u , we first need the following notations.

Notations 3.5.15. • *We denote by E the vector space $\text{Hom}_k(L_{V',n}^p, L_{V,n}^p)^\vee$.*

- *We denote by $\mathbb{V}E$ the vector space $\text{Hom}_k(L_{V',n}^p, L_{V,n}^p)$ viewed as an affine k -variety (which means that for every object S of $\mathcal{C}$, we have $\underline{\mathbb{V}}E(S) = H^0(S, \mathcal{O}_S \otimes_k \text{Hom}_k(L_{V',n}^p, L_{V,n}^p))$).*

- We denote $\mathbb{V}^*E := \mathbb{V}E \setminus \{0\}$.

In Lemma 3.5.13, we have defined a morphism of $\mathcal{O}_S$ -modules $f_L^* : \mathcal{O}_S \otimes_k L_{V',n}^p \rightarrow \mathcal{O}_S \otimes_k L_{V,n}^p$ for every object S of $\mathcal{C}$ and every S -complete element f of $\text{Rat}_S(V_S, V'_S)$. It corresponds to an element of $\mathbb{V}E(S) = H^0(S, \mathcal{O}_S \otimes_k \text{Hom}_k(L_{V',n}^p, L_{V,n}^p))$. As f^* is compatible with base changes $T \rightarrow S$, f_L^* is compatible with base changes too, so that procedure is functorial in S . Moreover, by definition of $\mathbb{B}_+^p$, the image of an element of $\mathcal{D}(S)$ is in $\mathbb{V}^*E(S)$. Hence we get a morphism of presheaves $\mathcal{D} \rightarrow \mathbb{V}^*E$.

Notation 3.5.16. We denote by u the morphism $\mathcal{D} \rightarrow \mathbb{V}^*E$ of presheaves on the category $\mathcal{C}$ such that for every object S of $\mathcal{C}$ and every element f of $\mathcal{D}(S)$, $u(f)$ is the element of $\mathbb{V}^*E(S) \subset H^0(S, \mathcal{O}_S \otimes_k \text{Hom}_k(L_{V',n}^p, L_{V,n}^p))$ given by f_L^* (from Lemma 3.5.13).

3.5.6 Finding again the initial morphism

Let q be the canonical projection $\mathbb{V}^*E \rightarrow \mathbb{P}E$. In this subsection, we prove the following lemma.

Lemma 3.5.17. *There exists:*

- a finite set I ,
- a disjoint family (A_i) of smooth locally closed subvarieties of $\mathbb{P}E$ indexed by I ,
- for every $i \in I$, a morphism $r_i : \underline{A}_i \rightarrow \mathcal{D}$ of presheaves on $\mathcal{C}$,

such that for every $f \in \mathcal{D}(k)$, there exists $i \in I$ such that $q \circ u(f) \in A_i(k)$ and $r_i(q \circ u(f)) = f$.

For every $n \in \mathbb{N}_{>0}$, an element s of $\mathbb{P}E(k)$ gives a nonzero k -linear map $L_{V',n}^p \rightarrow L_{V,n}^p$, up to a scalar. As the elements of $L_{V,n}^p$ give global sections of the line bundle $\mathcal{O}_p(n)$ on $\mathbb{P}\Omega_V^p$, there is an associated Kodaira rational map $f_s^1 : \mathbb{P}\Omega_V^p \dashrightarrow \mathbb{P}L_{V',n}^p$. On the other hand, we have a Kodaira map $\phi_n : \mathbb{P}\Omega_{V'}^p \dashrightarrow \mathbb{P}L_{V',n}^p$. Thanks to Theorem 3.2.4.2, we can choose $n \in \mathbb{N}_{>0}$ such that ϕ_n defines a locally closed embedding $\mathbb{P}\Omega_{V'}^p \setminus \mathbb{B}_+^p(V') \hookrightarrow \mathbb{P}L_{V',n}^p$. We suppose $\mathbb{B}_+^p(V') \neq \mathbb{P}\Omega_{V'}^p$, so that ϕ_n is a rational map (otherwise, the statement of the theorem is obvious).

Our goal is to define a complete rational map $f_s : V \dashrightarrow V'$ in $\mathcal{D}(k)$ such that the following diagram commutes.

$$\begin{array}{ccc}
& & \mathbb{P} L_{V',v}^p \\
& \nearrow f_s^1 & \uparrow \phi_n \\
\mathbb{P} \Omega_V^p & & \mathbb{P} \Omega_{V'}^p \\
\downarrow & & \downarrow \\
V & \xrightarrow{\quad f_s \quad} & V'
\end{array}$$

The family (A_i) of locally closed subsets of $\mathbb{P} E$ will be defined such that s is an element of one of the $A_i(k)$'s if and only if it is possible to define f_s as above, and the morphism $r_i : \underline{A}_i \rightarrow \mathcal{D}$ will be defined such that $r_i(s) = f_s$.

But to define a morphism of presheaves $\underline{A}_i \rightarrow \mathcal{D}$, we have to do this in family. Using the universal property of $\mathbb{P} E$, it is easy to check that we can assemble the f_s^1 's to form a $\mathbb{P} E$ -rational map $f^1 : (\mathbb{P} \Omega_V^p)_{\mathbb{P} E} \dashrightarrow (\mathbb{P} L_{V',n}^p)_{\mathbb{P} E}$. Apply Lemma 3.5.11: there exists a finite set I^1 and a family of locally closed embeddings $\iota_i : A_i^1 \hookrightarrow \mathbb{P} E$ indexed by $i \in I^1$ such that $\mathbb{P} E = \bigsqcup_{i \in I^1} A_i^1$ and for every $i \in I^1$, $f_i^1 := \iota_i^* f^1$ is a strict A_i^1 -rational map. Denote $\overline{\text{Im}}_{A_i^1}(f_i^1)$ by $\overline{\text{Im}}_i$.

For $s \in \mathbb{P} E(k)$, we have to define a lift $\mathbb{P} \Omega_V^p \dashrightarrow \mathbb{P} \Omega_{V'}^p$ of f_s^1 when it is possible. Denote by W the locally closed subset of $\mathbb{P} L_{V',n}^p$ defined as the image of the Kodaira map $\phi_n : \mathbb{P} \Omega_{V'}^p \setminus \mathbb{B}_+^p(V') \hookrightarrow \mathbb{P} L_{V',n}^p$. Consider the set B_i of points $s \in A_i^1$ such that $(\overline{\text{Im}}_i)_s$ is included in $\bar{W} \otimes_k \kappa(s)$ and intersects $W \otimes_k \kappa(s)$, with $\kappa(s)$ the residue field of A_i^1 at s . The set B_i is constructible by Chevalley's theorem. Indeed, if we denote by q_i the projection $(\mathbb{P} L_{V',n}^p)_{A_i^1} \rightarrow A_i^1$, we have $B_i = q_i(\overline{\text{Im}}_i \cap W_{A_i^1}) \setminus q_i(\overline{\text{Im}}_i \setminus W_{A_i^1})$. The union of the B_i 's for every $i \in I^1$ is then constructible, so it is a disjoint finite union of locally closed subsets A_i^2 indexed by $i \in I^2$ with I^2 a finite set. We can suppose that for every $i \in I^2$, there exists $i' \in I^1$ such that $A_i^2 \subset A_{i'}^1$. Moreover, by definition, for every $i \in I^2$, we have a A_i^2 -rational map $(\mathbb{P} \Omega_V^p)_{A_i^2} \dashrightarrow (\mathbb{P} \Omega_{V'}^p)_{A_i^2}$. Composing with the projection $\mathbb{P} \Omega_{V'}^p \rightarrow V'$, we obtain a strict A_i^2 -rational map $f_i^2 : (\mathbb{P} \Omega_V^p)_{A_i^2} \dashrightarrow (V')_{A_i^2}$. We denote $\overline{\Gamma}_{A_i^2}(f_i^2)$ by $\overline{\Gamma}_i^2$.

For $s \in A_i^2(k)$, we need to factorize $(f_i^2)_s : \mathbb{P} \Omega_V^p \dashrightarrow V'$ by a rational map $(f_i^3)_s : V \dashrightarrow V'$ when it is possible. To do this, we need the following lemma.

Lemma 3.5.18. *Let k be an algebraically closed field. Let S be a geometrically irreducible k -variety. Let W , W' and P be three S -varieties with W and P geometrically irreducible. Let $\pi : P \rightarrow W$ a faithfully flat S -morphism*

and $f : P \dashrightarrow W'$ be a S -rational map. Then there exists a (unique) S -rational map $h : W \dashrightarrow W'$ such that $f = h \circ \pi$ (i.e. the diagram below commutes) if and only if there exists a closed subset $G \subset W \times_S W'$ such that $\bar{\Gamma}_S(f) = \pi'^{-1}(G)$ with $\pi' : P \times_S W' \rightarrow W \times_S W'$ the pull-back of π .

$$\begin{array}{ccc} P & & \\ \downarrow \pi & \searrow f & \\ W & \xrightarrow{\quad h \quad} & W' \end{array}$$

Moreover, in this case, $G = \bar{\Gamma}_S(h)$. In particular h is strict if and only if f is strict.

Proof. The direct implication is obvious with $G = \bar{\Gamma}_S(h)$. For the converse, let $U \subset P$ be the maximal open subset on which f is defined. Denote by O the open subset $\pi(U)$ of W . As f is S -rational, O intersect every fiber above S . The restriction $U \rightarrow O$ of π is faithfully flat. By the definition of U and by the faithfully flat descent [41, Exposé VIII Corollaire 5.4], $G \times_W O \rightarrow O$ is an isomorphism. Hence we can define h as the composition $W \leftarrow O \cong G \times_W O \rightarrow W'$. Moreover, we have $G = \pi'(\bar{\Gamma}_S(f))$, so G is geometrically irreducible, hence $G = \bar{\Gamma}_S(h)$. $\square$

Denote by G_i the set of points of $(V \times V')_{A_i^2}$ above which the fiber of $\bar{\Gamma}_i^2$ is equal to the whole fiber of $(\mathbb{P}\Omega_V^p \times V')_{A_i^2}$. Let F_i be the projection of G_i on $V_{A_i^2}$. Denote by A_i^3 the set of points of A_i^2 above which the fiber of F_i is equal to the whole fiber of $V_{A_i^2}$. By Chevalley's theorem, A_i^3 is a constructible subset of A_i^2 . We can freely refine the partition $A^2 := \bigsqcup_i A_i^2$ to suppose that either A_i^3 is empty or $A_i^3 = A_i^2$. In the first case, Lemma 3.5.18 implies that for every $s \in A_i^2(k)$, there is no factorization of $(f_i^2)_s : \mathbb{P}\Omega_V^p \dashrightarrow V'$ by a rational map $V \dashrightarrow V'$. In the second case, denote $H_i := G_i \times_V \mathbb{P}\Omega_V^p \subset (\mathbb{P}\Omega_V^p \times V')_{A_i^2}$. Then H_i is included in $\bar{\Gamma}_i^2$. Moreover, the fibers of H_i and $\bar{\Gamma}_i^2$ above A_i^2 have the same dimension and the fibers of $\bar{\Gamma}_i^2$ are irreducible. Hence $H_i = \bar{\Gamma}_i^2$. Now we apply Lemma 3.5.18: we obtain that $\bar{\Gamma}_i^3 := G_i$ is the graph of a strict A_i^3 -rational map $f_i^3 : V_{A_i^3} \dashrightarrow V'_{A_i^3}$.

We have to restrict to the set of points $s \in A_i^3$ such that $(f_i^3)_s$ is in $\mathcal{D}(\kappa(s))$. Let A_i^4 be the set of points $s \in A_i^3$ such that $(f_i^3)_s$ is complete. Let us check that A_i^4 is constructible. Let $(\bar{V}', D')$ be a log-compactification of V' . Denote by $\bar{f}_i^3 : V_{A_i^3} \dashrightarrow \bar{V}'_{A_i^3}$ the composition of f_i^3 with the inclusion of $V'_{A_i^3}$ in $\bar{V}'_{A_i^3}$. Then $\bar{f}_i^3$ is a A_i^3 -complete A_i^3 -rational map (see Examples 3.5.2.2). By Lemma 3.5.11, we can freely refine the partition $A^3 := \bigsqcup_i A_i^3$ to suppose

that every $\bar{f}_i^3$ is strict. Let K_i be the projection of $\bar{\Gamma}_i^3 \cap D'_{A_i^3}$ on $V_{A_i^3}$. Then f_i^3 is not defined on K_i , and outside of K_i , $\bar{f}_i^3$ and f_i^3 are defined on the same open subset. Thus A_i^4 is the set of points of A_i^3 above which the codimension of the fiber of K_i in $V_{A_i^3}$ is at least 2. Hence A_i^4 is constructible (see [80, Lemma 05F9]).

Finally, we define A_i as the set of points $s \in A_i^4$ such that $\overline{\text{Im}}(\Lambda^p d(f_i^4)_s) \not\subset \mathbb{T}\mathbb{B}_+^p(V')_{A_i^4}$. It is constructible by Chevalley's theorem. We can freely refine the partition (A_i) so that every A_i is locally closed and smooth.

For each A_i , the pull-back of f_i^4 gives a (strict) A_i -complete A_i -rational map $f_i : V_{A_i} \dashrightarrow V'_{A_i}$ such that for every $s \in A_i$, $\overline{\text{Im}}(\Lambda^p d(f_i)_s) \not\subset \mathbb{T}\mathbb{B}_+^p(V')_{\{s\}}$. By definition of $\mathcal{D}$, f_i is an element of $\mathcal{D}(A_i)$. By Yoneda lemma, it corresponds to a morphism $r_i : \underline{A}_i \rightarrow \mathcal{D}$. Following the construction of A_i and r_i , one can check that for every $f \in \mathcal{D}(k)$, there exists $i \in I$ such that $\underline{q}(u(f)) \in \underline{A}_i(k)$, and $r_i(\underline{q}(u(f))) = f$. That concludes the proof of Lemma 3.5.17.

3.5.7 End of the proof

Denote by $\bar{A}_i$ a normal compactification of A_i such that $\bar{A}_i \setminus A_i$ is a divisor (take for instance the normalization of a blow-up of the closure of A_i in $\mathbb{P}(E)$). As $\bar{A}_i$ is normal and $\bar{V}'$ is proper, there exists an open subset $O \subset \bar{A}_i$ such that $A_i \subset O$, $\bar{A}_i \setminus O$ has codimension at least 2 and f_i extends as a O -rational map $\bar{f}_i : V_O \dashrightarrow \bar{V}'_O$. Let B be the divisor $O \setminus A_i$ of O . Then the pull-back of an element $\alpha \in L_{V',n}^p$ by $\bar{f}_i$ defines an element $\bar{f}_i^* \alpha$ of $H^0(V_O, \text{Sym}^n \Omega_{V_O}^p(\log V_B))$. As the divisor V_B of V_O is vertical above O , there is a canonical surjection $\text{Sym}^n \Omega_{V_O}^p(\log V_B) \rightarrow \text{Sym}^n \Omega_{V_O/O}^p$. Hence $\bar{f}_i^* \alpha$ defines an element $\beta \in H^0(V_O, \text{Sym}^n \Omega_{V_O/O}^p)$. Let us prove that β comes from an element ρ of $H^0(V_O, \mathcal{O}_{V_O}) \otimes L_{V,n}^p$.

As $L_{V,n}^p$ has finite dimension, there exists a finite set of closed points $P \subset V$ such that the linear composition $\iota : L_{V,n}^p \hookrightarrow H^0(V, \text{Sym}^n \Omega_V^p) \rightarrow H^0(P, (\text{Sym}^n \Omega_V^p)|_P)$ is injective. Then β defines an element ρ' of $H^0(P_O, (\text{Sym}^n \Omega_{V_O/O}^p)|_{P_O})$. As P is finite, we have:

$$H^0(P_O, (\text{Sym}^n \Omega_{V_O/O}^p)|_{P_O}) \cong H^0(O, \mathcal{O}_O) \otimes_k H^0(P, (\text{Sym}^n \Omega_V^p)|_P).$$

By Lemma 3.5.12, the restriction of ρ' to P_{A_i} is in $H^0(A_i, \mathcal{O}_{A_i}) \otimes_k \text{Im}(\iota)$. By density of A_i in O , ρ' is in $H^0(O, \mathcal{O}_O) \otimes_k \text{Im}(\iota)$. Thus it defines an element ρ of $H^0(V_O, \mathcal{O}_{V_O}) \otimes L_{V,n}^p$.

We got a k -linear map $L_{V',n}^p \rightarrow H^0(V_O, \mathcal{O}_{V_O}) \otimes L_{V,n}^p$. It defines a morphism $O \rightarrow \mathbb{V}E$. As $\overline{A}_i$ is normal, $\overline{A}_i \setminus O$ has codimension at least two and $\mathbb{V}E$ is affine, this morphism extends as a morphism $e_i : \overline{A}_i \rightarrow \mathbb{V}E$. By construction of e_i , for every $s \in A_i(k)$, we have $e_i(s) = u(r_i(s))$. Moreover, as $\overline{A}_i$ is proper and $\mathbb{V}E$ is affine, e_i has finite image. Denote by $F_i \subset \mathbb{V}E(k)$ the finite set of closed points that are both in the image of e_i and in $q^{-1}(A_i(k))$.

Finally, let $f \in \mathcal{D}(k)$. Let $i \in I$ be such that $q(u(f)) \in A_i(k)$. We have $u(f) = u(r_i(q(u(f)))) \in F_i$, thus $f = r_i(q(u(f))) \in r_i(q(F_i))$. Therefore $\mathcal{D}(k) = \bigsqcup_{i \in I} r_i(q(F_i))$, so it is finite. This concludes the proof of Theorem 3.5.3.

Remark 3.5.19. Absolutely logarithmic forms are not essential in the previous proof: one can replace $L_{V,n}^p$ by $H^0(\bar{V}, \text{Sym}^{[n]} \Omega_{\bar{V}}^p(D))$ as in the proof of Proposition 3.3.8.

Chapter 4

Varieties supporting a variation of Hodge structure

4.1 Introduction

If a smooth complex quasiprojective variety admits a polarizable variation of Hodge structure, then the logarithmic cotangent and the logarithmic canonical bundles on a log-compactification are known to have some positivity properties. In this chapter, we investigate the positivity of the vector bundles of logarithmic differential forms of every degree. See Subsection 3.2.5 for the definition of the different notions of positivity.

4.1.1 Bigness

We begin by a result of L-bigness.

Theorem 4.1.1. *Let $\bar{V}$ be a connected smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $\mathbb{V}$ be a complex polarizable variation of Hodge structure ($\mathbb{C}$ -PVHS) on $V := \bar{V} \setminus D$. Let p be an integer such that $1 \leq p \leq \dim V$. Suppose there exists a point x in V at which the period map of $\mathbb{V}$ is immersive. Then the vector bundle $\Omega_{\bar{V}}^p(\log D)$ is L-big.*

The cases of $p = 1$ and $p = \dim V$ of the previous theorem were proven by Brunebarbe and Cadorel [23] (generalizing results of Zuo [91], Brunebarbe-Klingler-Totaro [24] and Brunebarbe [22]). A similar proof gives the general case.

The notion of Viehweg-bigness is stronger than L-bigness, and is equivalent to it if E is a line bundle. In particular, in the setting of Theorem 4.1.1,

$\Omega_{\bar{V}}^{\dim \bar{V}}(\log D)$ is Viehweg-big. We prove that it is actually the case for every degree high enough.

To give a degree from which $\Omega_{\bar{V}}^p(\log D)$ is Viehweg-big, we consider some restrictions of iterations of the Higgs field. For a $\mathbb{C}$ -PVHS $\mathbb{V}$ on a smooth variety V and a point $x \in V$, we have a grading $\mathbb{V}_x = \bigoplus_{i \in \mathbb{Z}} \mathbb{V}_x^i$. For every $i, j \in \mathbb{Z}$ with $j \leq i$, we can restrict the $(i - j)$ -th iterate θ^{i-j} of the Higgs field θ associated with $\mathbb{V}$ to define a map $\theta_x^{i,j} : T_{V,x} \rightarrow \text{Hom}_{\mathbb{C}}(\mathbb{V}_x^i, \mathbb{V}_x^j)$.

In order to simplify the statement, we add here a ‘‘Torelli type’’ hypothesis (*) and a ‘‘Calabi-Yau type’’ hypothesis (**) (for the general statement without these hypotheses, see Theorem 4.7.6).

Theorem 4.1.2. *Let $\bar{V}$ be a connected smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on $V := \bar{V} \setminus D$. Suppose $\mathbb{V}$ has quasi-unipotent local monodromy (around every irreducible component of D). Let $x \in V$ be a point where the period map of $\mathbb{V}$ is immersive. Let $i \in \mathbb{Z}$ be the biggest integer such that $\theta_x^{i,i-1}$ is nonzero. Suppose that:*

- (*) $\theta_x^{i,i-1}$ is injective,
- (**) $\dim \mathbb{V}_x^i = 1$.

Let j be the smallest integer with $j < i$ such that $\theta_x^{i,j}$ is nonzero. Let m be the maximum dimension for a vector subspace $S \subset T_{V,x}$ such that for every $v \in S$, $\theta_x^{i,j}(v) = 0$. Let $p \in \mathbb{N}$ be such that $m < p \leq \dim V$. Then $\Omega_{\bar{V}}^p(\log D)$ is Viehweg-big.

The previous theorem applies to the moduli space of quintic threefolds. In the following theorem, the ‘‘91’’ stems from the limits of our computing power and could probably be improved.

Theorem 4.1.3. *Consider $\mathcal{M}$ the fine moduli space of smooth complex quintic threefolds, that is a smooth Deligne-Mumford stack that admits a finite étale cover $M \rightarrow \mathcal{M}$ with M a connected smooth complex quasiprojective variety. Let $(\bar{M}, D)$ be a projective log-compactification of M .*

Then $\Omega_{\bar{M}}^p(\log D)$ is Viehweg-big if $91 \leq p \leq 101 = \dim M$.

4.1.2 Augmented base locus

Consider $(\bar{V}, D)$ a projective log-compactification of a connected smooth quasiprojective variety V . Recall that for $p \in \mathbb{N}$ with $1 \leq p \leq \dim V$, the augmented base locus $\mathbb{B}_+^p(V)$ controls the positivity of $\Omega_{\bar{V}}^p(\log D)$:

- $\Omega_{\bar{V}}^p(\log D)$ is L-big if and only if $\mathbb{B}_+^p(V) \neq \mathbb{P} \Omega_{\bar{V}}^p$,

- $\Omega_V^p(\log D)$ is ample with respect to a Zariski open subset $U \subset V$ if and only if the projection of $\mathbb{B}_+^p(V)$ on V does not intersect U ,
- $\Omega_V^p(\log D)$ is Viehweg-big if and only if the projection of $\mathbb{B}_+^p(V)$ on V is not V .

We prove that when V supports a $\mathbb{C}$ -PVHS $\mathbb{V}$ with quasi-unipotent local monodromy, $\mathbb{B}_+^p(V)$ is controlled by the Higgs field and the conjugate Higgs field.

Notation 4.1.4. *Let M be a complex manifold, $x \in M$, $\mathbb{V}$ a $\mathbb{C}$ -PVHS on M . Let $\theta_x : T_{M,x} \rightarrow \text{End}(\mathbb{V}_x)$ be the Higgs field and $\theta_x^* : T_{M,x}^{0,1} \rightarrow \text{End}(\mathbb{V}_x)$ the conjugate Higgs field at x . For $v \in T_{M,x}$, we denote $C_{\mathbb{V}}(v) := \{w \in T_{M,x}, [\theta_x^*(\bar{v}), \theta_x(w)] = 0\}$.*

Theorem 4.1.5. *Let $\bar{V}$ be a smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS with quasi-unipotent local monodromy on $V := \bar{V} \setminus D$. Let $x \in V$ be a point where the period map of $\mathbb{V}$ is immersive. Let p be an integer such that $1 \leq p \leq \dim V$. Let $\alpha \in \Lambda^p T_{V,x}$ be such that $[\alpha] \in \mathbb{B}_+^p(V)$. Then there exists $v \in T_{V,x} \setminus \{0\}$ such that $\alpha \in \Lambda^p C_{\mathbb{V}}(v)$ (Notation 4.1.4).*

4.1.3 Outline of the proofs

We begin by the proof of Theorem 4.1.5, which is based on the study of the positivity of the curvature of the Hodge metric which comes from a polarization of the $\mathbb{C}$ -PVHS. Indeed, a polarization of $\mathbb{V}$ induces a polarization of $\text{End}(\mathbb{V})$, that gives a Hodge metric. Outside of the Zariski closed subset $\text{Deg}(\mathbb{V}) \subset V$ of points where the period map is not immersive, we can pull back this Hodge metric by the Higgs field θ to obtain a Kähler metric h on $V \setminus \text{Deg}(\mathbb{V})$. It induces a Hermitian metric on $\Lambda^p T_{V \setminus \text{Deg}(\mathbb{V})}$, and then a Hermitian metric $\hat{h}$ on $\mathcal{O}_p(1)$. The negativity properties of the Hodge metric imply positivity properties of $\hat{h}$. We obtain the following proposition, that is proved in Section 4.4.

Proposition 4.1.6. *Let M be a complex manifold. Let $\mathbb{V}$ be a complex polarized variation of Hodge structure on M with an immersive period map. Let p be an integer such that $1 \leq p \leq \dim M$. Let $C_1(\mathcal{O}_p(1), \hat{h})$ be the first Chern form of $\hat{h}$ (notation given above). Let $x \in M$, let $\alpha \in \Lambda^p T_{M,x} \setminus \{0\}$.*

Then the real $(1,1)$ -form $C_1(\mathcal{O}_p(1), \hat{h})$ is semipositive. Moreover, if it is not positive definite at $[\alpha]$ then there exists $v \in T_{M,x} \setminus \{0\}$ such that $\alpha \in \Lambda^p C_{\mathbb{V}}(v)$ (Notation 4.1.4).

We can deduce information on the augmented base locus from it thanks to Tosatti's analytic version of Nakamaye's theorem ([83, Theorem 4.5], see Theorem 3.2.11). The latter theorem has a nefness hypothesis. We do not prove that $\mathcal{O}_p(1)$ is nef, but we compare it with the dual of a subquotient of a log-Higgs bundle, that is nef (see Theorem 4.5.5). Then we obtain the proof of Theorem 4.1.5 (section 4.5).

The key result to deduce Theorem 4.1.2 from Theorem 4.1.5 is that a tangent vector $v \in T_{V,x}$ induces a decomposition of the complex Hodge structure $\mathbb{V}_x$ compatible with the endomorphisms $\theta_x(w)$ for every $w \in C_{\mathbb{V}}(v)$ with θ the Higgs field (Proposition 4.6.5). The proof of this decomposition is based on Schur's lemma, applied in a semisimple category of "complex polarizable infinitesimal variations of Hodge structure" ($\mathbb{C}$ -PIVHS, see Definition 4.6.1). We deduce Theorem 4.1.2 from Proposition 4.6.5 in Section 4.7.

4.2 Definitions and notations

As in this chapter, we will consider only *complex* Hodge structures, it is not a loss of generality to consider that the weight is always 0, and then to consider simple (and not double) gradings. Thus, a *complex Hodge structure* ($\mathbb{C}$ -HS) is a finite dimensional complex vector space H with a grading $H = \bigoplus_{i \in \mathbb{Z}} H^i$. A *polarization* Q of this $\mathbb{C}$ -HS is a conjugate-symmetric sesquilinear form on H with respect to which the decomposition $\bigoplus_{i \in \mathbb{Z}} H^i$ is orthogonal and such that for every $i \in \mathbb{Z}$, the restriction of $(-1)^i Q$ to H^i is positive definite. The *Hodge metric* associated with Q is the unique Hermitian product of H with respect to which the decomposition $\bigoplus_{i \in \mathbb{Z}} H^i$ is orthogonal such that its restriction to H^i is equal to $(-1)^i Q$ for every $i \in \mathbb{Z}$.

Let M be a complex manifold. Denote by C^∞ the sheaf of rings of smooth complex functions on M , by A^1 the C^∞ -module of smooth differential 1-forms and by $A^{0,1}$ (resp. $A^{1,0}$) the submodule of 1-forms of bidegree $(0,1)$ (resp. $(1,0)$). A *complex variation of Hodge structure* ($\mathbb{C}$ -VHS) on M is the data of a complex local system $\mathbb{V}$ on M and for every $x \in M$, of a $\mathbb{C}$ -HS $\bigoplus_{i \in \mathbb{Z}} \mathbb{V}_x^i$ on $\mathbb{V}_x$ such that for every $i \in \mathbb{Z}$:

- the $\mathbb{V}_x^i$'s define a C^∞ -submodule $\mathbb{V}^i$ of $\mathbb{V}_{C^\infty} := \mathbb{V} \otimes_{\mathbb{C}} C^\infty$,
- if ∇ is the flat connection induced by $\mathbb{V}$ on $\mathbb{V}_{C^\infty}$ and if σ is a local section of $\mathbb{V}^i$, then the local section $\nabla \sigma$ of $\mathbb{V} \otimes_{\mathbb{C}} A^1$ is a section of $(\mathbb{V}^{i-1} \otimes_{C^\infty} A^{1,0}) \oplus (\mathbb{V}^i \otimes_{C^\infty} A^1) \oplus (\mathbb{V}^{i+1} \otimes_{C^\infty} A^{0,1})$.

A *polarization* Q of $\mathbb{V}$ is a conjugate-symmetric sesquilinear form $\mathbb{V} \otimes_{\mathbb{C}} \bar{\mathbb{V}} \rightarrow \mathbb{C}$ such that for every $x \in M$, Q_x is a polarization of the $\mathbb{C}$ -HS $\mathbb{V}_x$. The *Hodge metric* associated with Q is the Hermitian metric defined on $\mathbb{V}_{C^\infty}$ by the Hodge metrics associated with the Q_x 's. A *complex polarizable variation of Hodge structure* ($\mathbb{C}$ -PVHS) is a $\mathbb{C}$ -VHS that admits a polarization.

For $\mathbb{V}$ a $\mathbb{C}$ -VHS, $i \in \mathbb{Z}$ and σ a local section of $\mathbb{V}^i$, we have a decomposition $\nabla\sigma = \theta\sigma + \delta'\sigma + \delta''\sigma + \theta^*\sigma$ where $\theta\sigma$ (resp. $\delta'\sigma$, $\delta''\sigma$, $\theta^*\sigma$) is a section of $\mathbb{V}^{i-1} \otimes_{C^\infty} A^{1,0}$ (resp. $\mathbb{V}^i \otimes_{C^\infty} A^{1,0}$, $\mathbb{V}^i \otimes_{C^\infty} A^{0,1}$, $\mathbb{V}^{i+1} \otimes_{C^\infty} A^{0,1}$). It is easy to see that θ (resp. θ^*) is defined by a section of $\text{End}(\mathbb{V}) \otimes_{\mathbb{C}} A^{1,0}$ (resp. $\text{End}(\mathbb{V}) \otimes_{\mathbb{C}} A^{0,1}$). We call it the *Higgs field* (resp. *conjugate Higgs field*). If $\mathbb{V}$ is polarizable, then θ and θ^* are adjoint with respect of every polarization.

The *Higgs bundle* associated with a $\mathbb{C}$ -VHS $\mathbb{V}$ is the data of the unique holomorphic vector bundle E whose sheaf of smooth sections is $\mathbb{V}_{C^\infty}$ and whose Dolbeault operator is δ'' defined above, and of the holomorphic global section of $\text{End}(E) \otimes \Omega_M^1$ defined by the Higgs field θ . If $\mathbb{V}$ is polarizable, then the Hodge metric associated with a polarization is a smooth Hermitian metric on E .

4.3 L-bigness

With Lemma 3.2.12, we can prove Theorem 4.1.1. The proof is close to the proof of [23] for the cases $p = 1$ and $p = \dim V$. Recall that a $\mathbb{C}$ -PVHS on a complex manifold M induces a Hermitian metric h of seminegative curvature on the open subset U of points of M at which the period map is immersive. The proof is based on the following fact due to Brunebarbe, Klingler and Totaro.

Lemma 4.3.1 (see [24, Lemma 1.4]). *For every $x \in U$, there exists $v \in T_{V,x} \setminus \{0\}$ such that $\ker R(v, \bar{v}) = \{0\}$ with R the Riemann curvature tensor of h .*

Proof of Theorem 4.1.1. By [23, Proposition 2.4], h extends as a singular Hermitian metric of seminegative curvature on $\Omega_V^1(\log D)$. Hence it induces a singular Hermitian metric of seminegative curvature on $\Omega_V^p(\log D)$, and then on $\mathcal{O}_p(1)$ (it is sufficient to check it over U thanks to [23, Lemma 2.3]).

Let x be any point of U . Let v be a vector as given by Lemma 4.3.1 just above. Let $v_2, \dots, v_p$ be tangent vectors at x such that $v, v_2, \dots, v_p$ are linearly independent. By Remark 3.4.12, for every $w \in T_{U,x} \setminus \{0\}$, we have $v \notin \ker R(w, \bar{w})$. Thus, $\alpha := v \wedge v_2 \wedge \dots \wedge v_p \notin \Lambda^p \ker R(w, \bar{w})$.

Thus by Lemma 3.2.12, the curvature of the singular Hermitian metric of $\mathcal{O}_p(1)$ is Kähler on a Euclidean open neighborhood of $[\alpha]$ in $\mathbb{P}\Omega_V^p(\log D)$. We conclude by applying Theorem 3.4.13. $\square$

4.4 Curvature of the Hodge metric

In this section, we prove Proposition 4.1.6. In the following, M is a complex manifold, $\mathbb{V}$ a complex polarized variation of Hodge structure on M , (E, θ) the associated Higgs bundle and θ^* the conjugate Higgs field.

The curvature of the Chern connection of the Hodge metric on E is $-\theta \wedge \theta^* - \theta^* \wedge \theta$ (see [75, Definition 4.2.8]). The local system $\text{End}(\mathbb{V})$ has an induced structure of $\mathbb{C}$ -PVHS and an induced polarization. It is easy to see that its Higgs field is given by:

$$\Theta(v).f = [\theta(v), f] \quad (4.1)$$

for $x \in M$, $v \in T_{M,x}$ and $f \in \text{End}(E_x)$. Hence the curvature tensor R_E of $\text{End}(E)$ over M is given for $x \in M$, $v, w \in T_{M,x}$ and $f \in \text{End}(E_x)$ by:

$$R_E(v, \bar{w}).f = -[[\theta(v), \theta^*(\bar{w})], f]. \quad (4.2)$$

Lemma 4.4.1. *Let $x \in M$, $v \in T_{M,x}$ and $f \in \text{End}(E_x)^{-1}$. Suppose $[\theta(v), f] = 0$. Then $[\theta^*(\bar{v}), f] = 0$ if and only if $R_E(v, \bar{v}).f = 0$.*

Proof. The direct implication follows from the formula of the curvature (4.2) and the Jacobi formula.

For the converse implication, let $\mathfrak{sl} \subset \text{End}(E_x)$ be the complex Lie subalgebra of endomorphisms of trace zero. Consider its real form $\mathfrak{su}$ of endomorphisms of trace zero which are skew-adjoint with respect to the polarization. Denote by f^* the conjugate of $f \in \mathfrak{sl}$ for this real form. This is consistent with the notation θ^* in the sense that $(\theta(v))^* = \theta^*(\bar{v})$. Denote by $\mathfrak{sl}_0$ (resp. $\mathfrak{su}_0$) the intersection of $\mathfrak{sl}$ (resp. $\mathfrak{su}$) with $\text{End}(E_x)^0$. Note that $\mathfrak{su}_0$ is a real form of $\mathfrak{sl}_0$ and is compactly embedded in $\mathfrak{su}$ (which means that $\mathfrak{su}_0$ is the Lie algebra associated with a compact Lie subgroup of the real Lie group of inner automorphisms of $\mathfrak{su}$). Let β be the Killing form of $\mathfrak{su}$. By [44, Proposition 6.8], β is negative definite on $\mathfrak{su}_0$.

Now suppose $R_E(v, \bar{v}).f = 0$. As $[\theta^*(\bar{v}), f] \in \mathfrak{sl}_0$, by the foregoing, it is

sufficient to prove $\beta([\theta^*(\bar{v}), f], [\theta(v), f^*]) = 0$. We have:

$$\begin{aligned}
\beta([\theta^*(\bar{v}), f], [\theta(v), f^*]) &= \beta([\theta^*(\bar{v}), f], \theta(v), f^*) \\
&= -\beta([f, \theta(v)], \theta^*(\bar{v}), f^*) - \beta([\theta(v), \theta^*(\bar{v})], f, f^*) \\
&= -\beta([\theta(v), \theta^*(\bar{v})], f, f^*) \\
&= \beta(R_E(v, \bar{v}).f, f^*) \\
&= 0.
\end{aligned}$$

□

Now, suppose that the period map is immersive: equivalently, for every point $x \in M$, the linear map $T_{M,x} \rightarrow \text{End}(E_x)$ given by θ is injective. Consider the Hermitian metric on T_M defined as the pull back by θ of the Hodge metric on $\text{End}(\mathbb{V})$. By [53, Theorem 1.2], it is a Kähler metric. Denote by R its Riemann curvature tensor.

Lemma 4.4.2. *Let $x \in M$, $v, w \in T_{M,x}$. If $R(v, \bar{v}).w = 0$, then $[\theta^*(\bar{v}), \theta(w)] = 0$*

Proof. The Hodge metric on $\text{End}(E)$ has Griffiths seminegative curvature. As the injection $\theta : T_M \hookrightarrow \text{End}(E)$ is locally split and the curvature decreases on direct factor subbundles, we have $\theta(\ker R(v, \bar{v})) \subset \ker R_E(v, \bar{v})$. Moreover, $\theta(v)$ and $\theta(w)$ commute. Hence we can apply Lemma 4.4.1. □

Proof of Proposition 4.1.6. Apply Lemma 3.2.12 to the tangent bundle and apply Lemma 4.4.2. □

4.5 Augmented base locus

In this section, we prove Theorem 4.1.5, using the same method as for Proposition 3.4.11. Let V be a smooth complex quasiprojective variety. Let $(\bar{V}, D)$ be a projective log-compactification of V . Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on V . Suppose its local monodromy is unipotent. This hypothesis is not restrictive to prove Theorem 4.1.5. Indeed, for every $\mathbb{C}$ -PVHS $\mathbb{V}$ with quasi-unipotent local monodromy on V , there exists a finite étale covering map $V' \rightarrow V$ such that the pull-back of $\mathbb{V}$ on V' has unipotent local monodromy (see [6, Proposition 8.2]), and the augmented base locus is compatible with finite étale covering maps (Proposition 3.4.6.5).

Let (E, θ) be the Higgs bundle associated with $\mathbb{V}$. Denote by θ^* the conjugate Higgs field. Denote by $\text{Deg}(\mathbb{V})$ the *degeneracy locus* of $\mathbb{V}$, that is

the set of points $x \in V$ such that the linear map $T_{V,x} \rightarrow \text{End}(E_x)$ defined by θ is not injective. Equivalently, it is the set of points at which the period map is not immersive. It is a Zariski closed subset of V .

The following lemma is essentially [42, Example 7.17.3].

Lemma 4.5.1. *Let k be a field. Let V be an algebraic k -variety. Let E be a locally free coherent $\mathcal{O}_V$ -module. Let L be an invertible coherent submodule of E . Let $U \subset V$ be the biggest open subset of V on which L is locally a direct factor of E . Then there exist a blow-up $b : V' \rightarrow V$ whose center does not intersect U and an invertible coherent submodule L' of b^*E that is locally a direct factor, includes b^*L and is equal to it above U .*

Proof. The injection of L in E induces a morphism $a : E^\vee \otimes L \rightarrow \mathcal{O}_V$. The image $\mathcal{I}$ of a is a coherent sheaf of ideals whose restriction to U is equal to the structural sheaf. Let $b : V' \rightarrow V$ be the blow-up with respect to $\mathcal{I}$. Let $\mathcal{I}' := b^{-1}\mathcal{I} \cdot \mathcal{O}_{V'}$ be the inverse image ideal sheaf of $\mathcal{I}$. Then $\mathcal{I}'$ is the image of the morphism $b^*a : b^*E^\vee \otimes b^*L \rightarrow \mathcal{O}_{V'}$, and it is an invertible coherent $\mathcal{O}_{V'}$ -module. Then define L' as $\mathcal{I}'^{-1} \cdot b^*L$. $\square$

Schmid's nilpotent orbit theorem (see [77]) gives a canonical extension of E as a holomorphic vector bundle ${}^\diamond E$ on $\bar{V}$. The Higgs field defines a morphism of holomorphic vector bundles ${}^\diamond\theta : T_{\bar{V}}(-\log D) \rightarrow \text{End}({}^\diamond E)$ called the *log-Higgs field*. It induces a morphism ${}^\diamond\theta_p : \Lambda^p T_{\bar{V}}(-\log D) \rightarrow \Lambda^p \text{End}({}^\diamond E)$.

We denote by π the projection $\mathbb{P}\Omega_V^p(\log D) \rightarrow \bar{V}$ and by $\mathcal{O}_p(-1)$ the dual of $\mathcal{O}_p(1)$. It is locally a direct factor of $\pi^*\Lambda^p T_{\bar{V}}(-\log D)$.

Lemma 4.5.2. *There exist a smooth projective variety P , a projective birational morphism $b : P \rightarrow \mathbb{P}\Omega_V^p(\log D)$ which is an isomorphism above $V \setminus \text{Deg}(\mathbb{V})$ and such that $b^{-1}(\pi^{-1}(D \cup \text{Deg}(\mathbb{V})))$ is a normal crossing divisor, and an invertible coherent submodule L of $b^*\pi^*\Lambda^p \text{End}({}^\diamond E)$ that is locally a direct factor, includes $b^*{}^\diamond\theta_p(\mathcal{O}_p(-1))$ and is equal to it above $V \setminus \text{Deg}(\mathbb{V})$.*

Proof. Apply Lemma 4.5.1 and define P as a desingularization of V' . $\square$

The following lemma shows that above $V \setminus \text{Deg}(\mathbb{V})$, the augmented base locus of $L^\vee$ restricts the one of $\mathcal{O}_p(1)$.

Lemma 4.5.3. *With the notations of the previous lemma, we have $\mathbb{B}_+(\mathcal{O}_p(1)) \cap \mathbb{P}\Omega_{V \setminus \text{Deg}(\mathbb{V})}^p \subset b(\mathbb{B}_+(L^\vee))$.*

Proof. Because of Proposition 3.2.3, we have:

$$\mathbb{B}_+(\mathcal{O}_p(1)) \cap \mathbb{P}\Omega_{V \setminus \text{Deg}(\mathbb{V})}^p \subset b(\mathbb{B}_+(b^*\mathcal{O}_p(1))).$$

Then Lemma 3.2.2 gives $\mathbb{B}_+(b^*\mathcal{O}_p(1)) \cap b^{-1}(\mathbb{P}\Omega_{V \setminus \text{Deg}(\mathbb{V})}^p) \subset \mathbb{B}_+(L^\vee)$. $\square$

To conclude the proof of Theorem 4.1.5, we need to prove that $L^\vee$ has properties close to the ones needed for $\mathcal{O}_p(1)$ in the hypotheses of Proposition 3.4.11. This is given by the two theorems below, that we can apply to L because of the following lemma.

Lemma 4.5.4. *The log-Higgs field of $b^*\pi^*\Lambda^p \text{End}({}^\diamond E)$ vanishes on the line bundle L defined in Lemma 4.5.2.*

Proof. Let $x \in V$, let $v \in T_{V,x}$. As $\theta(v)$ commutes with $\theta(w)$ for every $w \in T_{V,x}$, the Higgs field Θ of $\text{End}(E)$ vanishes at $\theta(v)$ because of formula (4.1). Hence the Higgs field Θ^p of $\Lambda^p \text{End}(E)$ vanishes on the image of $\Lambda^p T_V$ in $\Lambda^p \text{End}(E)$ (defined by θ and Leibniz rule). Therefore, the Higgs field of $\pi^*\Lambda^p \text{End}(E)$ vanishes on the image of $\mathcal{O}_p(-1)$. By density of $b^{-1}(\mathbb{P}\Omega_V^p)$ in P , it is equivalent to the vanishing of the log-Higgs field of $b^*\pi^*\Lambda^p \text{End}({}^\diamond E)$ on L . $\square$

The following two theorems allow us to use $L^\vee$ instead of $\mathcal{O}_p(1)$ to mimic the proof of Proposition 3.4.11.

Theorem 4.5.5 (see [91, Theorem 1.2] or [22, Theorem 1.8]). *Let L be an invertible coherent submodule of ${}^\diamond E$ which is locally a direct factor. Suppose the log-Higgs field vanishes on L . Then the dual $L^\vee$ of L is nef.*

Theorem 4.5.6 (see [21, Theorem 1.4]). *Let L be an invertible coherent submodule of ${}^\diamond E$. Suppose the log-Higgs field vanishes on L . Then the restriction of the Hodge metric of E (given by a polarization of E) to L is a singular Hermitian metric with seminegative curvature.*

Proof of Theorem 4.1.5. Denote by U the complement of the set of $[\alpha] \in \mathbb{P}\Omega_{V \setminus \text{Deg}(\mathbb{V})}^p$ such that there exists $v \in T_{V \setminus \text{Deg}(\mathbb{V})}$ with $v \neq 0$ such that $\alpha \in \Lambda^p C(v)$. We have to prove that U does not intersect $\mathbb{B}_+^p(V)$. By Lemma 4.5.3, it is sufficient to prove that $b^{-1}(U)$ does not intersect $\mathbb{B}_+(L^\vee)$ with b and L defined in Lemma 4.5.2.

By Theorem 4.5.6 applied to L thanks to Lemma 4.5.4, the Hodge metric induces a singular metric $\hat{h}$ of semipositive curvature on $L^\vee$. It means that $C_1(L^\vee, \hat{h})$ defines a positive closed real $(1,1)$ -current on P whose de Rham class is the first Chern form $c_1(L^\vee)$. By Proposition 4.1.6 applied on $V \setminus$

$\text{Deg}(\mathbb{V})$, and because $L^\vee$ and $\mathcal{O}_p(1)$ are equal above it, $b^{-1}(U)$ is included in a Euclidean open subset $U' \subset P$ on which $C_1(L^\vee, \hat{h})$ is Kähler.

Suppose U' is not empty (otherwise there is nothing to prove). Then we have $\int_P C_1(L^\vee, \hat{h})^{\dim P} > 0$. Hence Theorem 3.4.13 gives that $L^\vee$ is a big line bundle.

The real $(1, 1)$ -class $c_1(L^\vee)$ is big, and nef by Theorem 4.5.5 (applied to L thanks to Lemma 4.5.4). Hence we can apply Theorem 3.2.11 with the positive closed real $(1, 1)$ -form $C_1(L^\vee, \hat{h})$. It gives that the non-Kähler locus of $c_1(L^\vee)$ does not intersect U' . Thanks to Theorem 3.2.10, the non-Kähler locus of $c_1(L^\vee)$ is the augmented base locus of $L^\vee$. $\square$

4.6 Decomposition of polarized infinitesimal variations of Hodge structure

To deduce the positivity properties of the logarithmic cotangent bundles from Theorem 4.1.5, we will need Proposition 4.6.5 below. To state and prove it, we need to define a notion of complex polarizable infinitesimal variation of Hodge structure.

Definition 4.6.1. *Let T be a complex vector space. A complex polarizable infinitesimal variation of Hodge structure ($\mathbb{C}$ -PIVHS) on T is the data of $(H, H^i, \delta, \delta^*)$ with (H, H^i) a $\mathbb{C}$ -HS, $\delta : T \rightarrow \text{End}(H, H^i)^{-1}$ a linear map and $\delta^* : T \rightarrow \text{End}(H, H^i)^1$ an antilinear map such that:*

- *for every $v, w \in T$, $\delta(v)$ and $\delta(w)$ commute,*
- *there exists a polarization Q of (H, H^i) with respect to which for every $v \in T$, $\delta(v)$ and $\delta^*(v)$ are adjoint.*

A morphism of $\mathbb{C}$ -PIVHS $(H, H^i, \delta, \delta^*) \rightarrow (H', H'^i, \delta', \delta'^*)$ on T is a morphism $f : H \rightarrow H'$ of $\mathbb{C}$ -HS which is equivariant for δ and δ' , and for δ^* and δ'^* (i.e. for every $v \in T$ and $a \in H$, we have $f(\delta(v).a) = \delta'(v).f(a)$ and $f(\delta^*(v).a) = \delta'^*(v).f(a)$).

Examples 4.6.2. • *Consider M a complex manifold, $x \in M$, and $\mathbb{V}$ a $\mathbb{C}$ -PVHS on M . Denote by $\mathbb{V}_x$ the fiber of $\mathbb{V}$ above x , $\mathbb{V}_x^i$ its $\mathbb{C}$ -HS, θ_x the Higgs field and θ_x^* the conjugate Higgs field at x . Then $(\mathbb{V}_x, \mathbb{V}_x^i, \theta_x, \theta_x^*)$ is a $\mathbb{C}$ -PIVHS on $T_{M,x}$.*

- *If $(H_{\mathbb{Z}}, H^{i,j}, Q, T, \delta)$ is an infinitesimal variation of Hodge structure of weight $n \in \mathbb{Z}$ in the sense of Carlson-Green-Griffiths-Harris [28], then $(H_{\mathbb{Z}} \otimes \mathbb{C}, H^{i,n-i}, \delta, \bar{\delta})$ is a $\mathbb{C}$ -PIVHS on T , where $\bar{\delta}$ is the conjugate of δ for the natural real form of $H_{\mathbb{Z}} \otimes \mathbb{C}$.*

Remark 4.6.3. The definition is made so that we can apply Schur's lemma. In particular:

- We do not use the “integral” infinitesimal variations of Hodge structure defined by Carlson, Green, Griffiths and Harris. Indeed, even if we are only interested by integral, rational or real variations of Hodge structure, it is necessary to have a $\mathbb{C}$ -linear category to apply Schur's lemma.
- We do not use the “naive” complex analog of integral infinitesimal variations of Hodge structure, that would be the data of (H, H^i, δ) without δ^* . Indeed, the category of “naive” $\mathbb{C}$ -PIVHSs on a complex vector space T is not semi-simple. Consider for instance a line $T = \mathbb{C}v$, a $\mathbb{C}$ -HS $H = H^0 \oplus H^1$ with H^0 and H^1 two lines, and $\delta(v)$ an isomorphism $H^1 \xrightarrow{\sim} H^0$. Then H^0 is stable by $\delta(v)$ but has no complement in H stable by $\delta(v)$.

Proposition 4.6.4. *Let T be a complex vector space. The category $\mathcal{C}_T$ of $\mathbb{C}$ -PIVHSs on T is semi-simple Abelian.*

Proof. It is easy to see that $\mathcal{C}_T$ is additive and admits kernels. Let H be an object of $\mathcal{C}_T$ and $H' \subset H$ a sub-object, i.e. a sub- $\mathbb{C}$ -HS stable by $\delta(v)$ and $\delta^*(v)$ for every $v \in T$. Consider the orthogonal $H'^\perp$ for a polarization of H as in Definition 4.6.1. It is a sub- $\mathbb{C}$ -HS stable by $\delta(v)$ (resp. $\delta^*(v)$) because H' is stable by its adjoint $\delta^*(v)$ (resp. $\delta(v)$), for every v . Hence we have $H = H' \oplus H'^\perp$ in the category $\mathcal{C}_T$. This proves semi-simplicity and the existence of quotients. $\square$

Let H and H' be two objects of $\mathcal{C}_T$. Using Leibniz rule for δ and δ^* , we can define the tensor product $H \otimes H'$ as an object of $\mathcal{C}_T$. If T' is a complex vector space, then a linear map $f : T \rightarrow T'$ induces an exact functor $f^* : \mathcal{C}_{T'} \rightarrow \mathcal{C}_T$. Hence, using the pullbacks by the projections, for S a complex vector space, an object H of $\mathcal{C}_S$ and an object H' of $\mathcal{C}_T$ define an object $H \otimes H'$ of $\mathcal{C}_{S \oplus T}$.

Proposition 4.6.5. *Let S, T be two complex vector spaces. Let $(H, H^i, \delta, \delta^*)$ be a $\mathbb{C}$ -PIVHS on $S \oplus T$. Then the following are equivalent.*

- For every $v \in S$ and $w \in T$, $\delta^*(v)$ and $\delta(w)$ commute.
- There exists a finite set A and a family (V_a) (resp (W_a)) of $\mathbb{C}$ -PIVHS on S (resp. T) indexed by A such that H is isomorphic to $\bigoplus_{a \in A} V_a \otimes W_a$ (in the category $\mathcal{C}_{S \oplus T}$).

Proof. We prove the direct implication, the converse is obvious. Denote $H_S := i_S^* H$ with $i_S : S \hookrightarrow S \oplus T$ the inclusion and $H_T := i_T^* H$ with $i_T : T \hookrightarrow S \oplus T$ the inclusion. The group $\mathbb{Z}$ acts freely by shift of the grading on the set of isomorphism classes of simple objects of $\mathcal{C}_S$. Let A be the finite set of orbits under this action which are represented by a simple sub- $\mathbb{C}$ -PIVHS of H_S . For every $a \in A$, choose a $\mathbb{C}$ -PIVHS V_a on S which represents a .

Let W_a be the vector space of linear maps $V_a \rightarrow H$ which are equivariant for $\delta|_S$ and $\delta|_S^*$ (but that are not necessarily morphisms of $\mathbb{C}$ -HS). Then W_a is naturally a $\mathbb{C}$ -HS, so it can be considered as an object of $\mathcal{C}_S$ with zero δ and δ^* .

Consider the following morphism of $\mathbb{C}$ -PIVHS on S .

$$\begin{aligned} \alpha : \bigoplus_{a \in A} V_a \otimes W_a &\rightarrow H_S \\ v_a \otimes w_a &\mapsto w_a(v_a). \end{aligned}$$

Schur's lemma gives the injectivity of α , and Proposition 4.6.4 gives the surjectivity.

For every $w \in T$, $\delta(w)$ is an element of $\text{End}(H)^{-1}$. By definition of a $\mathbb{C}$ -PIVHS, $\delta(w)$ is equivariant for $\delta|_S$. By hypothesis, it is equivariant for $\delta|_S^*$. Hence, by Schur's lemma, using the isomorphism α , $\delta(w)$ can be considered as an element of $\bigoplus_{a \in A} \text{End}(W_a)^{-1}$. The same applies for $\delta^*(w)$, which can be considered as an element of $\bigoplus_{a \in A} \text{End}(W_a)^1$. Denote by δ_{W_a} and $\delta_{W_a}^*$ the morphisms $T \rightarrow \text{End}(W_a)$ defined in this way for every $a \in A$.

It only remains to check that for every $a \in A$, the structure $(W_a, W_a^i, \delta_{W_a}, \delta_{W_a}^*)$ that we just defined is a $\mathbb{C}$ -PIVHS on T , by proving that it admits a convenient polarization. A polarization Q of $\mathbb{C}$ -HS of H and a polarization Q' of $\mathbb{C}$ -HS of V_a induce a polarization Q'' of $\mathbb{C}$ -HS of W_a , as it is a sub- $\mathbb{C}$ -HS of $\text{Hom}_{\mathbb{C}}(V_a, H)$. By definition of $\mathbb{C}$ -PIVHS, we can choose Q such that for every $w \in T$, $\delta(w)$ and $\delta^*(w)$ are adjoint. Moreover, one can easily check that for every $w \in T$ and $f \in W_a$, we have $\delta_{W_a}(w).f = \delta(w) \circ f$ and $\delta_{W_a}^*(w).f = \delta^*(w) \circ f$, so $\delta_{W_a}(w)$ and $\delta_{W_a}^*(w)$ are adjoint with respect to Q'' . $\square$

4.7 Rank criterion

Let $(H, H^i, \delta, \delta^*)$ be a $\mathbb{C}$ -PIVHS on a complex vector space T . For $v \in T, i \geq j \in \mathbb{Z}$, we will denote by $v^{i,j}$ the restriction $H^i \rightarrow H^j$ of the iterate $\delta(v)^{i-j}$ and by $\delta^{i,j}$ the map $v \mapsto v^{i,j}$.

4.7.1 Decomposed PIVHS and rank

The existence of the decomposition given by Proposition 4.6.5 allows one to prove properties of the rank of $\theta_x^{i,j}(w)$ for $w \in C_V(v)$ and for some integers i and j , in the context of Theorem 4.1.5.

Lemma 4.7.1. *Let $(H, H^i, \delta, \delta^*)$ be a $\mathbb{C}$ -PIVHS on a complex vector space T . Let $S \subset T$ be a vector subspace. Let $v \in T$ be such that $\delta(v) \neq 0$ and for every $w \in S$, $[\delta^*(v), \delta(w)] = 0$. Let $i \in \mathbb{Z}$ be maximal such that $v^{i,i-1} \neq 0$. Then there exists $j \in \mathbb{Z}$, $\lambda \in \mathbb{C} \setminus \{0\}$ and $u \in S$ such that $j < i$ and for every $w \in S$, $\text{rk}((\lambda v + u)^{i,j}) > \text{rk}(w^{i,j})$.*

Proof. Applying Proposition 4.6.5 to the pull-back of H on $\mathbb{C}v \oplus S$, we get a decomposition $H = \bigoplus_{a \in A} X_a$ (with A a finite set) and for every $a \in A$, $X_a = V_a \otimes W_a$ with V_a a $\mathbb{C}$ -PIVHS on $\mathbb{C}v$ and W_a a $\mathbb{C}$ -PIVHS on S . We can refine the decomposition to suppose that every V_a is simple. For every $a \in A$ and $j \in \mathbb{Z}$ such that $j < i$, the set of $(\lambda, u) \in \mathbb{C} \times S$ for which $\text{rk}((\lambda v + u)^{i,j}_{|X_a})$ is maximal is a non-empty Zariski open subset of $\mathbb{C} \times S$. Hence, the intersection of these subsets is a non-empty Zariski open subset of $\mathbb{C} \times S$. In particular, there exists $(\lambda, u) \in (\mathbb{C} \setminus \{0\}) \times S$ such that for every $a \in A$, every $j \in \mathbb{Z}$ such that $j < i$ and every $w \in S$, we have $\text{rk}((\lambda v + u)^{i,j}_{|X_a}) \geq \text{rk}(w^{i,j}_{|X_a})$. Now, it is sufficient to prove that there exists a and j such that $(\lambda v + u)^{i,j}_{|X_a} \neq 0$ and for every $w \in S$, we have $w^{i,j}_{|X_a} = 0$.

By definition of i , there exists $a \in A$ such that the restriction of $v^{i,i-1}$ to $X := X_a$ is non-zero. From now on, we denote $V := V_a$, $W := W_a$ and the endomorphisms of H defined by δ are considered in restriction to X . We denote by v_V the action of v on V and by w_W the action of an element $w \in S$ on W .

Let $s \in \mathbb{Z}$ be maximal such that $W^s \neq 0$, let $t \in \mathbb{Z}$ with $t \leq s$ be minimal such that there exists $w \in S$ such that $w_W^{s,t} \neq 0$. Let $k \in \mathbb{Z}$ be maximal such that $V^k \neq 0$. By simplicity of V , k is also maximal such that $v_V^{k,k-1} \neq 0$. Let $l \in \mathbb{Z}$ with $l < k$ be minimal such that $v_V^{k,l} \neq 0$. Then we have $i = k + s$. Let $j := l + t$: because $l < k$ and by definition of t , we have $w^{i,j} = 0$ for every $w \in S$. It remains to check that $(\lambda v + u)^{i,j} \neq 0$:

$$\begin{aligned} (\lambda v + u)^{i,j} &= (\lambda v_V \otimes \text{id}_W + \text{id}_V \otimes u_W)^{i,j} \\ &= \sum_{l'+t'=j} \binom{i-j}{k-l'} \lambda^{k-l'} v_V^{k,l'} \otimes u_W^{s,t'} \\ &= \binom{i-j}{k-l} \lambda^{k-l} v_V^{k,l} \otimes u_W^{s,t} \\ &\neq 0. \end{aligned}$$

□

The following two propositions are simplified forms of the previous lemma.

Proposition 4.7.2. *Let $(H, H^i, \delta, \delta^*)$ be a $\mathbb{C}$ -PIVHS on a complex vector space T . Let $S \subset T$ be a vector subspace. Suppose there exists $v \in T$ such that $\delta(v) \neq 0$ and for every $w \in S$, $[\delta^*(v), \delta(w)] = 0$.*

Then there exists $i, j \in \mathbb{Z}$ and $z \in T$ such that $j < i$ and for every $w \in S$, $\text{rk}(z^{i,j}) > \text{rk}(w^{i,j})$.

Proof. Take $z := \lambda v + u$ as given by the previous lemma. □

Proposition 4.7.3. *In the setting of the previous proposition, let $i \in \mathbb{Z}$ be maximal such that $\delta^{i,i-1} \neq 0$.*

If $\delta^{i,i-1}$ is injective, then the conclusion of the proposition can be realized with this i .

If moreover $\dim H^i = 1$, then the conclusion of the proposition can be realized with this i and for j the minimal integer such that $\delta^{i,j} \neq 0$. In this case this conclusion is equivalent to: for every $w \in S$, $w^{i,j} = 0$.

Proof. For the first part, the injectivity of $\delta^{i,i-1}$ guarantees that $v^{i,i-1} \neq 0$, so i is the one in the hypothesis of Lemma 4.7.1.

For the second part, we have by the first part $j' \in \mathbb{Z}$ and $z' \in T$ such that $j' < i$ and for every $w \in S$, $\text{rk}(z'^{i,j'}) > \text{rk}(w^{i,j'})$. As $\dim H^i = 1$, this implies $w^{i,j'} = 0$. Composing with $w^{j',j}$, we have $w^{i,j} = 0$. By definition of j , there exists $z \in T$ such that $z^{i,j} \neq 0$. Hence $\text{rk}(z^{i,j}) = 1 > 0 = \text{rk}(w^{i,j})$. □

4.7.2 Augmented base locus and rank

Combining the study of infinitesimal variations of Hodge structures of the previous subsections with Theorem 4.1.5, we obtain the following descriptions of the augmented base loci associated with V .

Proposition 4.7.4. *Let $\bar{V}$ be a smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on $V := \bar{V} \setminus D$. Suppose $\mathbb{V}$ has quasi-unipotent local monodromy. Let θ be the associated Higgs field. Let $x \in V \setminus \text{Deg}(\mathbb{V})$. Let $p \in \mathbb{N}$ be such that $1 \leq p \leq \dim V$. Let $[\alpha] \in \mathbb{B}_+^p(V)$ be above x .*

Then there exists a vector subspace $S \subset T_{V,x}$, two integers i and j with $i > j$ and $z \in T_{V,x}$ such that $\alpha \in \Lambda^p S$ and for every $w \in S$, we have $\text{rk } \theta^{i,j}(w) < \text{rk } \theta^{i,j}(z)$.

Proof. Apply Theorem 4.1.5: then the conclusion results from Proposition 4.7.2 applied to the $\mathbb{C}$ -PIVHS defined by $\mathbb{V}$ on $T_{V,x}$, with $C_{\mathbb{V}}(v)$ as S . □

Proposition 4.7.5. *In the setting of the previous proposition, let $i \in \mathbb{Z}$ be the biggest integer such that $\theta_x^{i,i-1}$ is nonzero. If:*

() $\theta_x^{i,i-1}$ is injective,*

then the conclusion of the proposition can be realized with this i .

If moreover:

*(**) $\dim \mathbb{V}_x^i = 1$,*

then the conclusion of the proposition can be realized with i and for j the minimal integer such that $\theta_x^{i,j} \neq 0$. In this case this conclusion is equivalent to: for every $w \in S$, $\theta_x^{i,j}(w) = 0$.

Proof. Apply Proposition 4.7.3. □

4.7.3 Positivity and rank

Finally, we can deduce Theorem 4.1.2 and its generalization Theorem 4.7.6 from the properties of the augmented base locus proven in the previous subsection.

Theorem 4.7.6. *Let $\bar{V}$ be a connected smooth complex projective variety. Let $D \subset \bar{V}$ be a normal crossing divisor. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on $V := \bar{V} \setminus D$. Suppose $\mathbb{V}$ has quasi-unipotent local monodromy. Let $x \in V \setminus \text{Deg}(\mathbb{V})$. Let m be the maximum dimension for a vector subspace $S \subset T_{V,x}$ such that there exist two integers $j < i$ and a vector $z \in T_{V,x}$ such that for every $w \in S$, $\text{rk } \theta_x^{i,j}(w) < \text{rk } \theta_x^{i,j}(z)$. Let $p \in \mathbb{N}$ be such that $m < p \leq \dim V$. Then $\Omega_{\bar{V}}^p(\log D)$ is Viehweg-big.*

Proof. Combine Proposition 4.7.4 with Subsection 3.2.5. □

Remark 4.7.7. In the statement of the previous theorem, m depends on x . If the inequality $p > m$ is realized for every x in an open subset $U \subset V \setminus \text{Deg}(\mathbb{V})$, then $\Omega_{\bar{V}}^p(\log D)$ is ample with respect to U .

Proof of Theorem 4.1.2. Combine Proposition 4.7.5 with Subsection 3.2.5. □

4.8 Example: the moduli space of quintic three-folds

Let n and d be two positive integers. Denote by $\mathcal{M}$ the fine moduli space of smooth hypersurfaces of degree d of $\mathbb{P}_{\mathbb{C}}^{n+1}$ polarized by the tautological line bundle $\mathcal{O}(1)$. It is a smooth Deligne-Mumford stack of finite type over

$\mathbb{C}$. If $U \subset \mathbb{P} H^0(\mathbb{P}_{\mathbb{C}}^{n+1}, \mathcal{O}(d))^\vee$ is the open subset of points $[f]$ such that $\{f = 0\}$ is smooth, then $\mathcal{M}$ is the stack quotient $[\mathrm{PGL}_{n+2} \backslash U]$. Let M be a connected smooth complex quasiprojective variety with a finite étale covering map $M \rightarrow \mathcal{M}$. If $d \geq 3$ and $(n, d) \neq (1, 3)$, then there exists $a \in \mathbb{N}_{>0}$ such that for every $N \geq 3$ coprime to a , the moduli space of hypersurfaces with a N -level structure gives an example of such a M (see [48, Theorem 3.2]). We will apply Theorem 4.1.2 to M .

Let f be a non-zero homogeneous polynomial of $\mathbb{C}[X_0, \dots, X_{n+1}]$ of degree d . Consider the hypersurface $Y := \{f = 0\}$ polarized by $\mathcal{O}(1)|_Y$. For $p, q \in \mathbb{Z}$, denote by $H_{\mathrm{prim}}^{p,q}(Y)$ the primitive cohomology of Y of bidegree (p, q) . Denote by $I := \left(\frac{\partial f}{\partial X_0}, \dots, \frac{\partial f}{\partial X_{n+1}} \right)$ the Jacobian ideal of f and by $R := \mathbb{C}[X_0, \dots, X_{n+1}]/I$ its Jacobian ring. As I is graded, R has a structure of graded ring. By a theorem of Griffiths, there are canonical isomorphisms $H_{\mathrm{prim}}^{n-q,q}(Y) \cong R_{d(q+1)-n-2}$ for every $0 \leq q \leq n$ and $T_{M,x} \cong R_d$ (see [28, Example p.186]).

Let W be the pull-back on M of the universal family of $\mathcal{M}$. The primitive cohomology of degree n of fibers of W define an integral polarized variation of Hodge structure $\mathbb{V}_{\mathbb{Z}}$ on M . Let $x \in M$ be a point. Let $f \in H^0(\mathbb{P}_{\mathbb{C}}^{n+1}, \mathcal{O}(d))$ be such that W_x and $\{f = 0\}$ are isomorphic as polarized varieties. Griffiths proved that the following square commutes:

$$\begin{array}{ccc} T_{M,x} \otimes H_{\mathrm{prim}}^{n-q,q}(W_x) & \xrightarrow{\sim} & R_d \otimes R_{d(q+1)-n-2} \\ \downarrow & & \downarrow \\ H_{\mathrm{prim}}^{n-q-1,q+1}(W_x) & \xrightarrow{\sim} & R_{d(q+2)-n-2} \end{array}$$

where the horizontal maps are given by the isomorphisms mentioned above, the left vertical map by the Higgs field and the right vertical map by the product in R .

Now, suppose $d = n + 2$. Then $H_{\mathrm{prim}}^{n,0} \cong R_0$ has dimension 1. Let $x \in M$ be a point corresponding to the Fermat hypersurface $\{X_0^{n+2} + \dots + X_{n+1}^{n+2} = 0\}$. In this case we have $R = \mathbb{C}[X_0, \dots, X_{n+1}]/(X_0^{n+1}, \dots, X_{n+1}^{n+1})$. The iterate $\theta_x^{n,0} : T_{M,x} \rightarrow \mathrm{Hom}_{\mathbb{C}}(H_{\mathrm{prim}}^{n,0}(W_x), H_{\mathrm{prim}}^{0,n}(W_x))$ of the Higgs field is non-zero. Indeed, consider $v \in T_{M,x}$ corresponding to $X_0 \dots X_{n+1} \in R_{n+2}$ by the isomorphism $T_{M,x} \cong R_{n+2}$ mentioned above. Then we have $\theta_x^{n,0}(v) \neq 0$ because $(X_0 \dots X_{n+1})^n \neq 0$ in R .

Hence we can apply Theorem 4.1.2 to $\mathbb{V}_{\mathbb{Z}} \otimes \mathbb{C}$ with $i = n$ and $j = 0$. Let m be the biggest dimension for a vector subspace $S \subset R_{n+2}$ such that for

every $g \in S$, we have $g^n = 0$. Then $\Omega_M^p(\log D)$ is Viehweg-big for every $p \in \mathbb{N}$ such that $m < p \leq \dim M$, with $(\bar{M}, \bar{D})$ a projective log-compactification of M . To compute m , we use the following lemma.

Lemma 4.8.1. *Let Y be a hypersurface of degree at least 2 in a projective space $\mathbb{P}^n$ ($n \in \mathbb{N}_{>0}$). Let c be the codimension in $\mathbb{P}^n$ of the singular locus of Y . Let c' be the minimal codimension in $\mathbb{P}^n$ for a linear subspace included in Y . Then $c \leq 2c'$.*

Proof. See [45, Remark 1.3.3] for the proof in the smooth case. The proof easily generalizes to give the general case. $\square$

Consider the hypersurface $N := \{[g], g^n = 0\} \subset \mathbb{P} R_{n+2}^\vee$. By the Jacobian criterion, the singular locus of N is the set of $[g] \in N$ such that for every $h \in R_{n+2}$, we have $g^{n-1}h = 0$. By Macaulay's theorem, it is equivalent to $g^{n-1} = 0$. Hence, if k denotes the dimension of $\{g, g^{n-1} = 0\} \subset R_{n+2}$, we have $m \leq \frac{k + \dim R_{n+2}}{2}$.

Consider the case $n = 3$ of quintic threefolds. We have $\dim M = \dim R_{n+2} = 101$. Computing in the Jacobian ring of the Fermat quintic threefold with a computer algebra system (see the SageMath code below), we get $k \leq 79$, hence $m \leq 90$. Thus we get Theorem 4.1.3: $\Omega_M^p(\log D)$ is Viehweg-big for every $p \in \mathbb{N}$ such that $91 \leq p \leq 101$.

Remarks 4.8.2. • In the statement of Theorem 4.1.3, the “91” may not be optimal. It is probable that actually $k < 79$: 79 is only the limit of our computer algebra system and our patience.

- The proof shows that the statement of Theorem 4.1.3 holds more generally for every connected smooth complex quasiprojective variety M that parametrizes a smooth polarized family $\pi : W \rightarrow M$ of quintic threefolds such that there exists a closed point $x \in M$ such that:

- the fiber $\pi^{-1}(\{x\})$ is isomorphic to the Fermat quintic threefold,
- the period map of the $\mathbb{C}$ -PVHS $R^3\pi_*\underline{\mathbb{C}}_W$ is immersive at x .

```
from sage.all import *
from itertools import product

degrees5 = [vector(t) for t in product(range(4), repeat=5) if sum(t) == 5]
# List of multidegrees of monomials generating R5
```

```

degrees10 = [vector(t) for t in product(range(4), repeat=5) if sum(
    t) == 10]
# List of multidegrees of monomials generating R10

index_map = {tuple(v): i for i, v in enumerate(degrees5)}
# Dictionary to find the index of a monomial in degrees5 from its
multidegree

S = PolynomialRing(QQ, 101, 'x')

def coordProd(i):
    """
    Computes one of the coordinates of the product R5 x R5 -> R10.

    Parameters
    -----
    i : int
        Index of the coordinate (between 0 and 100).

    Returns
    -----
    res : polynomial
        Quadratic form giving the i-th coordinate of the bilinear
        map R5 x R5 -> R10.
    """

    res = S.zero()
    monom10 = degrees10[i]
    for j in range(101):
        monom5 = degrees5[j]
        diff = monom10 - monom5
        if min(diff) >= 0:
            k = index_map.get(tuple(diff))
            if k is not None:
                res += S.gen(j) * S.gen(k)
    return res

chosenCoord = [24, 10, 45, 89, 50, 63, 9, 22, 58, 78, 29, 82, 1, 0,
    2, 3, 4, 7, 8, 100, 99, 98]
# The calculation time depends heavily on the choice of the
coordinates. This list is the best that we have found for
performing calculations in a reasonable amount of time (2-3
minutes on our computer. A few hours were not enough to compute

```

```

    the dimension with one more coordinate).

quadForms = [coordProd(i) for i in chosenCoord]

I = S.ideal(quadForms)
# Ideal included in the ideal generated by all the coordProd(i).

print(I.dimension())
# Prints "79", giving an upper bound to the dimension of the
  variety defined by the vanishing of the square  $R_5 \rightarrow R_{10}$ .

```

Chapter 5

Singularities of the Hodge metric

5.1 Introduction

In the next chapter, we will need a theorem of Kollár [51, Theorem 5.20]. In this short chapter, we rewrite and sometimes change the definitions, statements and proofs of [51, Section 5], first to complete some arguments that we were not able to follow (in particular the proof of [51, Theorem 5.20] seems incomplete), secondly because we will use in the next chapter notions of growth moderation for differential forms near a normal crossing divisor close to the ones introduced by Kollár.

The differences between this chapter and [51, Section 5] are the following. First, we change the definition of “almost bounded” so that the set of almost bounded forms is an algebra, which is used to prove [51, Theorem 5.20] (Theorem 5.3.4). Secondly, we detail the proof of [51, Proposition 5.16.(ii)] (Proposition 5.2.6.2). Thirdly, we introduce the notion of “quasipositive” to justify the integrability in [51, Corollary 5.17] (Proposition 5.2.12). Finally, we allow a constant C in the definition of “nearly bounded” and we introduce the notion of “(almost) quasibounded”. This last point is not useful for the study of PVHS, but will be important in the next chapter.

5.2 Growth moderation near singularities

Let $r \in (0, e^{-1})$. We consider the disk and the punctured disk of radius r :

$$\Delta_r := \{z \in \mathbb{C}, |z| < r\},$$

$$\Delta_r^* := \Delta_r \setminus \{0\}.$$

We denote by χ the divisor $\{z_1 \dots z_d = 0\}$ of Δ_r^d . For $1 \leq i, j \leq d, i \neq j$ and $C, \varepsilon \in \mathbb{R}_{>0}$, we denote:

$$T_{i,j}^{C,\varepsilon} := \{z \in (\Delta_r^*)^d, |z_j| \leq \exp(-C|z_i|^{-\varepsilon})\},$$

$$T_i^{C,\varepsilon} := \bigcup_{j \neq i} T_{i,j}^{C,\varepsilon}.$$

For $0 < \varepsilon < \varepsilon'$ and $0 < C < C'$, we have $T_{i,j}^{C',\varepsilon'} \subset T_{i,j}^{C,\varepsilon}$ and $T_i^{C',\varepsilon'} \subset T_i^{C,\varepsilon}$.

5.2.1 Functions with moderate growth

Definition 5.2.1. *Let f be a smooth complex-valued function over the punctured polydisk $(\Delta_r^*)^d$.*

- *f has at most log growth if there exist $A \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{N}$ such that for every $z = (z_i) \in (\Delta_r^*)^d$:*

$$|f(z)| \leq A \prod_{i=1}^d (-\ln|z_i|)^k.$$

- *f has at most log-log growth if there is $B \in \mathbb{R}_{\geq 0}$ such that for every $z = (z_i) \in (\Delta_r^*)^d$:*

$$|f(z)| \leq B \sum_{i=1}^d \ln(-\ln|z_i|).$$

- *For $C, \varepsilon \in \mathbb{R}_{>0}$, $D \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}$, f is (C, ε, D, p) -nearly bounded if for each $z \in (\Delta_r^*)^d$ one of the following cases occurs:*

- $|f(z)| \leq D$,
- *there exists $1 \leq i \leq d$ such that $z \in T_i^{C,\varepsilon}$ and $|f(z)| \leq D(-\ln|z_i|)^p$.*

- *f is nearly bounded if there exist $C, \varepsilon \in \mathbb{R}_{>0}$, $D \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}$ such that f is (C, ε, D, p) -nearly bounded.*
- *f is quasibounded if it is the product of a function with at most log-log growth by a nearly bounded function.*
- *f is integrable if it is integrable for the Lebesgue measure.*
- *f is quasipositive if it is the sum of a nonnegative real-valued function and an integrable function.*

Remark 5.2.2. Our definition of nearly bounded is more general than the one of Kollár, for whom $C = 1$.

The functions that are continuous on Δ_r^d have at most log-log growth and are nearly bounded. Functions with at most log-log growth and nearly bounded functions are quasibounded. Quasibounded functions have at most log growth and are integrable. Integrable functions are quasipositive. If f has values in $\mathbb{R}_{>0}$ and f and f^{-1} have at most log growth, then $\ln \circ f$ has at most log-log growth. The following proposition is an easy consequence of the definitions.

Proposition 5.2.3. *The set of nearly bounded functions is a complex algebra. The set of quasibounded functions is a module over this algebra.*

5.2.2 Differential forms with moderate growth on the punctured polydisk

We denote by $\mathcal{C}^\infty((\Delta_r^*)^d)$ the ring of smooth complex-valued functions over $(\Delta_r^*)^d$. We denote by $A((\Delta_r^*)^d)$ the graded $\mathcal{C}^\infty((\Delta_r^*)^d)$ -algebra of smooth complex differential forms over $(\Delta_r^*)^d$. The *Poincaré basis* is the $\mathcal{C}^\infty((\Delta_r^*)^d)$ -basis of $A((\Delta_r^*)^d)$ whose elements are the ordered wedge products of elements of $\left(\frac{dz_1}{|z_1|(-\ln|z_1|)}, \dots, \frac{dz_d}{|z_d|(-\ln|z_d|)}, \frac{d\bar{z}_1}{|z_1|(-\ln|z_1|)}, \dots, \frac{d\bar{z}_d}{|z_d|(-\ln|z_d|)} \right)$.

Definition 5.2.4. *A form of $A((\Delta_r^*)^d)$ is nearly bounded (resp. quasibounded) if its coefficients in the Poincaré basis are nearly bounded (resp. quasibounded) functions.*

As a corollary of Proposition 5.2.3, the set of nearly bounded forms is a graded complex subalgebra of $A((\Delta_r^*)^d)$.

Definition 5.2.5. *A form of $A((\Delta_r^*)^d)$ of degree $2d$ is integrable (resp. semi-positive, quasipositive) if it is the product of an integrable (resp. nonnegative real-valued, quasipositive) function by $i dz_1 \wedge d\bar{z}_1 \wedge \dots \wedge i dz_d \wedge d\bar{z}_d$.*

The integral of a quasipositive form of degree $2d$ is a well defined element of $\mathbb{C} \cup \{+\infty\}$. It is finite if and only if the form is integrable.

Proposition 5.2.6. *1. A quasibounded form of degree $2d$ is integrable.*

2. If a quasibounded form α has degree $2d - 1$, is supported on a compact of Δ_r^d and $d\alpha$ is quasipositive, then $d\alpha$ is integrable and $\int_{(\Delta_r^)^d} d\alpha = 0$.*

Proof. 1. Let us check that if g is a nearly bounded function, then

$$h := \ln(-\ln|z_1|)g(z) \prod_{i=1}^d |z_i|^{-2}(-\ln|z_i|)^{-2}$$

is integrable.

Let $C, \varepsilon \in \mathbb{R}_{>0}$, $D \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}$ be such that g is (C, ε, D, p) -nearly bounded. As $|z|^{-2}(-\ln|z|)^{-2}$ and $|z|^{-2}(-\ln|z|)^{-2} \ln(-\ln|z|)$ are integrable over Δ_r , h is integrable on $\Delta_r^d \setminus \bigcup_i T_i^{C, \varepsilon}$.

Moreover, for $1 \leq i, j \leq d, i \neq j, i \neq 1, j \neq 1$:

$$\begin{aligned} & \int_{T_{i,j}^{C, \varepsilon}} \ln(-\ln|z_1|)(-\ln|z_i|)^p \prod_{k=1}^d |z_k|^{-2}(-\ln|z_k|)^{-2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d \\ &= E \int_{\Delta_r} \min\{(-\ln r)^{-1}, C^{-1}|z_i|^\varepsilon\} |z_i|^{-2}(-\ln|z_i|)^{p-2} dz_i \wedge d\bar{z}_i \\ &\leq EC^{-1} \int_{\Delta_r} |z_i|^{-(2-\varepsilon)}(-\ln|z_i|)^{p-2} dz_i \wedge d\bar{z}_i \\ &< +\infty \end{aligned}$$

where $E \in \mathbb{R}_{>0}$ is a constant. The case $i = 1$ simply adds a factor $\ln(-\ln|z_i|)$ to the last integral, that does not change the integrability.

Finally, it remains to treat the case $j = 1$. Let $1 < i \leq d$:

$$\begin{aligned} & \int_{T_{i,1}^{C, \varepsilon}} \ln(-\ln|z_1|)(-\ln|z_i|)^p \prod_{k=1}^d |z_k|^{-2}(-\ln|z_k|)^{-2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d \\ &= F \int_{\Delta_r} \min\{(-\ln r)^{-1}(1 + \ln(-\ln r)), (C^{-1}|z_i|^\varepsilon)(1 + \ln(C|z_i|^{-\varepsilon}))\} \\ & \quad |z_i|^{-2}(-\ln|z_i|)^{p-2} dz_i \wedge d\bar{z}_i \\ &\leq FC^{-1} \int_{\Delta_r} |z_i|^{-(2-\varepsilon)}(-\ln|z_i|)^{p-2}(1 + \ln(C) + \varepsilon(-\ln(|z_i|))) dz_i \wedge d\bar{z}_i \\ &< +\infty. \end{aligned}$$

2. Let Δ be the disk of radius 1, and Δ^* the punctured disk of radius 1. Consider the following function:

$$\begin{aligned} \eta : (\Delta^*)^d &\rightarrow \mathbb{R} \\ z &\mapsto \sum_{i=1}^d \ln(-\ln|z_i|) \end{aligned}$$

with differential:

$$\omega := d\eta = \sum_{i=1}^d \frac{d|z_i|}{|z_i| \ln|z_i|}.$$

Consider the Poincaré Hermitian metric over $(\Delta^*)^d$ given by:

$$\mu := \sum_{i=1}^d \frac{|dz_i|^2}{|z_i|^2 (\ln|z_i|)^2}.$$

Let v be the unique real vector field over $(\Delta^*)^d$ such that for every tangent vector w , we have $\mu(v, w) = \frac{1}{d}\omega(w)$. Then v is complete. Indeed, its norm for the Poincaré metric is constant (of value $\frac{1}{\sqrt{d}}$ at every point). Thus integral curves of v with bounded domain are bounded for the Poincaré metric. As this one is complete, an integral curve with bounded domain is relatively compact. Hence maximal integral curves are defined on $\mathbb{R}$.

Consider an integral curve $x : \mathbb{R} \rightarrow (\Delta^*)^d$ for v . Then, for $t \in \mathbb{R}$:

$$(\eta \circ x)'(t) = \omega(v(x(t))) = 1.$$

Hence $\eta \circ x : \mathbb{R} \rightarrow \mathbb{R}$ is affine, in particular it is a diffeomorphism.

For $t \in \mathbb{R}$, let $U_t := \{\eta = t\}$. We just proved that the flow of v gives a diffeomorphism $\phi : \mathbb{R} \times U_0 \cong (\Delta^*)^d$, with $\phi(\{t\} \times U_0) = U_t$.

Now, consider the homothety $h : \Delta^d \rightarrow \Delta_r^d$ of rate r . Then $h_*\omega$ has Poincaré growth. Recall that α is a quasibounded form of degree $2d-1$ supported on a compact of Δ_r^d such that $d\alpha$ is quasipositive. Hence, thanks to Proposition 5.2.3, $h_*\omega \wedge \alpha$ is quasibounded. Thanks to the first point, it is integrable. Hence, if we denote $\alpha' := h^*\alpha$, $\omega \wedge \alpha'$ is integrable. Moreover, $\phi^*\omega = \phi^*d\eta = d\phi^*\eta = dt$. Hence the function $t \mapsto \int_{U_t} \alpha'$ is integrable on $\mathbb{R}$.

Moreover, if we denote $Z_t := \bigsqcup_{s < t} U_s$:

$$\int_{(\Delta^*)^d} d\alpha' = \lim_{t \rightarrow +\infty} \int_{Z_t} d\alpha' = \lim_{t \rightarrow +\infty} \int_{U_t} \alpha',$$

and the integrability of $\int_{U_t} \alpha'$ implies $\lim_{t \rightarrow +\infty} \int_{U_t} \alpha' = 0$.

□

5.2.3 Differential forms with moderate growth on a complex manifold

Let M be a complex manifold. Let D be a normal crossing divisor of M . Let us extend the notions of the previous subsection to this case.

Definition 5.2.7. • An admissible chart on (M, D) is a holomorphic open embedding $f : \Delta_r^d \rightarrow M$ such that $f^{-1}(D) \subset \chi$.

- A smooth differential form α over $M \setminus D$ is locally nearly bounded (resp. locally quasibounded) if for each $r \in (0, e^{-1})$ and each admissible chart $\Delta_r^d \rightarrow M$ on (M, D) , there exists $r' \in (0, r)$ such that the pull-back of α on $(\Delta_{r'}^*)^d$ is nearly bounded (resp. quasibounded).
- A permissible blow-up on (M, D) is the data of a complex manifold M' , a normal crossing divisor D' and a blow-up $b : M' \rightarrow M$ of center $Z \subset M$ such that $b^{-1}(D) = D'$ and for each $x \in Z$, there is an admissible chart $f : \Delta_r^d \rightarrow (M, D)$ such that $f(0) = x$ and $f^{-1}(Z)$ is the intersection of some of the $\{z_i = 0\}$.
- Let $E \subset M$ be a divisor. A desingularization of E in (M, D) is a composition of permissible blow-ups $g : (M', D') \rightarrow (M, D)$ such that $g^{-1}(E) \cup D'$ has normal crossings.
- A smooth differential form α on $(\Delta_r^*)^d$ is pre-almost (quasi)bounded if there exists a divisor $E \subset \Delta_r^d$ such that the pull-back of α by every desingularization of E in (Δ_r^d, χ) is locally nearly bounded (resp. locally quasibounded).
- A smooth differential form α on $M \setminus D$ is almost (quasi)bounded if for each $r \in (0, e^{-1})$ and for each admissible chart $f : \Delta_r^d \rightarrow (M, D)$, there exists $r' \in (0, r)$ such that the restriction of $f^*\alpha$ on $(\Delta_{r'}^*)^d$ is pre-almost (quasi)bounded.
- A smooth differential form α of degree $2d$ on $M \setminus D$ is locally integrable (resp. semipositive, locally quasipositive), if for every $x \in M$, there is an admissible chart f on (M, D) such that $f(0) = x$ and $f^*\alpha$ is integrable (resp. semipositive, quasipositive).

Remark 5.2.8. Our definition of almost bounded is different from Kollár's. First, our definition of nearly bounded is different (see Remark 5.2.2). Second, in the definition of “locally nearly bounded” (resp. “pre-almost bounded”, “almost bounded”), we require a condition to be verified for *every* admissible chart (resp. every desingularization of a fixed divisor, every admissible

chart), whereas Kollár only requires the *existence* of an admissible chart (resp. a proper bimeromorphism, an admissible chart) on which the condition is verified. In particular, the divisor E does not appear in Kollár's definition.

The integral of a locally quasipositive form of degree $2d$ that is supported on a compact of M is a well defined element of $\mathbb{C} \cup \{+\infty\}$. It is finite if and only if the form is locally integrable. Thanks to [81, Theorem 1.1.9], every divisor has a desingularization in (M, D) .

Proposition 5.2.9. *The set of almost bounded forms over $M \setminus D$ is a complex graded algebra. The set of almost quasibounded forms over $M \setminus D$ is a graded module over this graded algebra.*

Proof. We only prove that the set of almost bounded forms is stable by wedge product (we can prove in the same way that it is a vector space, and that almost quasibounded forms give a graded module). Let α and β be two almost bounded forms over $M \setminus D$. We prove that $\alpha \wedge \beta$ is almost bounded. Let $f : \Delta_r^d \rightarrow (M, D)$ be an admissible chart. By definition, there exists $r' \in (0, r)$ such that the restrictions of $f^*\alpha$ and $f^*\beta$ on $(\Delta_{r'}^*)^d$ are pre-almost bounded. By definition, there exists a divisor E (resp. F) of $\Delta_{r'}^d$ such that the pull back of $f^*\alpha$ (resp. $f^*\beta$) by every desingularization of E (resp. F) is locally nearly bounded. Consider now the divisor $G := E \cup F$. Then every desingularization $g : (M', D') \rightarrow (\Delta_{r'}^d, \chi)$ of G is a desingularization of both E and F . Thus $g^*f^*\alpha$ and $g^*f^*\beta$ are locally nearly bounded. By definition, for every admissible chart $\Delta_r^d \rightarrow (M', D')$, there exists $r' \in (0, r)$ such that the restrictions of $g^*f^*\alpha$ and $g^*f^*\beta$ on $(\Delta_{r'}^*)^d$ are nearly bounded. By Proposition 5.2.3, the restriction of $g^*f^*(\alpha \wedge \beta)$ on $(\Delta_{r'}^*)^d$ is nearly bounded. Thus $g^*f^*(\alpha \wedge \beta)$ is locally nearly bounded, $f^*(\alpha \wedge \beta)$ is pre-almost bounded and finally $\alpha \wedge \beta$ is almost bounded. $\square$

The following proposition is a direct consequence of Proposition 5.2.6.1.

Proposition 5.2.10. *An almost quasibounded (resp. locally quasibounded, pre-almost quasibounded) form of degree $2d$ is locally integrable.*

We also have an analog of the second point of Proposition 5.2.6. First, Propositions 5.2.3, 5.2.6.1, 5.2.9 and 5.2.10 used with Leibniz formula give us the following lemma.

Lemma 5.2.11. *Let α be an almost quasibounded (resp. quasibounded, locally quasibounded, pre-almost quasibounded) form of degree $2d - 1$ on $M \setminus D$ (resp. $(\Delta_r^*)^d$, $M \setminus D$, $(\Delta_r^*)^d$) such that $d\alpha$ is (locally) quasipositive. Let ρ be a smooth positive real function on M (resp. Δ_r^d , M , Δ_r^d). Then $\rho\alpha$ is*

almost quasibounded (resp. quasibounded, locally quasibounded, pre-almost quasibounded) and $d(\rho\alpha)$ is (locally) quasipositive.

This lemma allows us to use partitions of unity to deduce the following proposition from Proposition 5.2.6.

Proposition 5.2.12. *Let α be an almost quasibounded form of degree $2d - 1$ on $M \setminus D$ such that $d\alpha$ is locally quasipositive. Then $d\alpha$ is locally integrable. If, moreover, α is supported on a compact of M , then $\int_{M \setminus D} d\alpha = 0$.*

Corollary 5.2.13. *An almost quasibounded form α on $M \setminus D$ defines a current on M . If α is closed, then this current is closed.*

5.3 Real polarized variations of Hodge structure and singularities

Let M be a complex manifold. In the following, we identify a finite locally free $\mathcal{O}_M$ -module with the associated holomorphic vector bundle. In particular, a holomorphic vector subbundle corresponds to a $\mathcal{O}_M$ -submodule which is locally a direct factor, and a morphism between two holomorphic vector bundles corresponds to a morphism of $\mathcal{O}_M$ -modules.

When we denote by $(\mathbf{H}_{\mathbb{R}}, \mathbf{S}, \mathbf{F}^{\bullet})$ an $\mathbb{R}$ -PVHS, $\mathbf{H}_{\mathbb{R}}$ is the underlying real local system, $\mathbf{S}$ is the polarization form, and $\mathbf{F}^{\bullet}$ is the Hodge filtration. We denote by $\mathbf{H}_{\mathcal{O}}$ the holomorphic vector bundle associated with $\mathbf{H}_{\mathbb{R}}$. The Higgs map, which is a morphism of vector bundles $\mathbf{F}^p / \mathbf{F}^{p+1} \rightarrow \Omega_M^1 \otimes \mathbf{F}^{p-1} / \mathbf{F}^p$, is denoted by θ . We denote by $\bigoplus_{p+q=k} \mathbf{H}^{p,q}$ the Hodge decomposition of $\mathbf{H}_{\mathcal{O}}$ and by h the Hodge metric over $\mathbf{H}_{\mathcal{O}}$. We will use the same notation for the Hermitian metric induced on holomorphic vector subquotients of $\mathbf{H}_{\mathcal{O}}$.

Let $\mathbf{H}$ be a complex local system on $M \setminus D$ with unipotent monodromy around D . We denote by ${}^{\diamond}\mathbf{H}$ the Deligne extension of the holomorphic vector bundle associated with $\mathbf{H}$ on M (see [30, Proposition II.5.2]). If $(\mathbf{H}_{\mathbb{R}}, \mathbf{S}, \mathbf{F}^{\bullet})$ is an $\mathbb{R}$ -PVHS over $M \setminus D$ with unipotent monodromy around D , we denote by ${}^{\diamond}\mathbf{F}^{\bullet}$ the canonical extension of $\mathbf{F}^{\bullet}$ on ${}^{\diamond}\mathbf{H}$ given by Schmid's Nilpotent Orbit Theorem ([77, Theorem 4.12]). The definition of the Deligne extension allows one to extend the flat connection as a logarithmic connection ${}^{\diamond}\mathbf{F}^p \rightarrow \Omega_M^1(\log D) \otimes {}^{\diamond}\mathbf{F}^{p-1}$ (for $p \in \mathbb{Z}$) and the Higgs map as a morphism of holomorphic vector bundles over M : $\theta : {}^{\diamond}\mathbf{F}^p / {}^{\diamond}\mathbf{F}^{p+1} \rightarrow \Omega_M^1(\log D) \otimes {}^{\diamond}\mathbf{F}^{p-1} / {}^{\diamond}\mathbf{F}^p$.

5.3.1 Singularities of the Hodge metric

Let $(\mathbf{H}_{\mathbb{R}}, \mathbf{S}, \mathbf{F}^{\bullet})$ be an $\mathbb{R}$ -PVHS over $(\Delta_r^*)^d$ with unipotent monodromy. Let H be the matrix associated with the Hodge metric in a holomorphic local basis of ${}^{\diamond}\mathbf{H}$. By a theorem of Cattani, Kaplan and Schmid, $\det(H)^{-1}$ and the coefficients of H have at most log growth (see [29, Theorem (5.21)]). One can check that this metric property is preserved on locally split subbundles and quotients (see [39, Corollary 3.5]). Thus we get the following statement.

Theorem 5.3.1. *Let $(\mathbf{H}_{\mathbb{R}}, \mathbf{S}, \mathbf{F}^{\bullet})$ be an $\mathbb{R}$ -PVHS over $(\Delta_r^*)^d$ with unipotent monodromy. Let $p \leq q$ be two integers, let s be a non vanishing section of ${}^{\diamond}\mathbf{F}^p / {}^{\diamond}\mathbf{F}^q$. Then $h(s, s)$ and $h(s, s)^{-1}$ have at most log growth.*

In the following statement, Kollár's definition of almost bounded is different from ours, but one can easily check that the proof works for our definition.

Lemma 5.3.2 (see [51, Proposition 5.15]). *Let E be a holomorphic vector line subbundle of the canonical extension of an $\mathbb{R}$ -PVHS over $M \setminus D$ with unipotent monodromy around D , let h be its Hodge metric. Then the first Chern form $C_1(E, h)$ is almost bounded, and for every trivializing section s of $\det(E)$ on an open subset U of M , the connection form $\partial \ln \|s\|_h^2$ is almost bounded on $U \setminus D$.*

The following two statements of Kollár were originally stated with “nearly bounded” instead of “almost bounded”. However, his proof goes by reduction to the case of a line subbundle for which only the almost boundedness is known. Moreover, Kollár's proof uses the fact that the class of forms under consideration is an algebra. This is clear for the class of nearly bounded forms (Proposition 5.2.3), but not for the class of almost bounded forms in the original definition. This is why we changed the definition of almost bounded forms (second point in Remark 5.2.8), so that their set is an algebra too (see Proposition 5.2.9).

Proposition 5.3.3 (see [51, Remark 5.19]). *Let E be a holomorphic vector subquotient of the canonical extension of an $\mathbb{R}$ -PVHS over $M \setminus D$ with unipotent monodromy around D , let h be its Hodge metric. Then the first Chern form $C_1(E, h)$ is almost bounded, and for every trivializing section s of $\det(E)$ on an open subset U of M , the connection form $\partial \ln \|s\|_h^2$ is almost bounded on $U \setminus D$.*

Theorem 5.3.4 (see [51, Theorem 5.20]). *Let $E_1, \dots, E_n$ be holomorphic vector bundle subquotients of $\mathbb{R}$ -PVHSs over $M \setminus D$ with unipotent monodromy around D , let $h_1, \dots, h_n$ be their Hodge metrics, let $k_1, \dots, k_n \in \mathbb{N}_{>0}$. Let $P \in$*

$\mathbb{C}[X_1, \dots, X_n]$ be a polynomial in n variables. Then $P(C_{k_1}(E_1, h_1), \dots, C_{k_n}(E_n, h_n))$ is almost bounded and defines a closed current on M whose de Rham cohomology class is $P(c_{k_1}(E_1), \dots, c_{k_n}(E_n))$.

Proof. Same proof as [51, Theorem 5.20], using Corollary 5.2.13. □

Chapter 6

Locally symmetric varieties

6.1 Introduction

In this chapter, we study the positivity of vector bundles of logarithmic differential forms on locally symmetric varieties. Zucker gave in [90] a way to construct $\mathbb{C}$ -PVHSs on them. The results that we get if we apply Chapter 4 to those $\mathbb{C}$ -PVHSs almost reduce to an application of [49, Chapter 6.8], meaning that it is almost possible to obtain the positivity of $\Omega_{\bar{X}}^p(\log D)$ without using variational Hodge theory when X is a locally symmetric variety (see Remarks 6.4.2.4).

6.1.1 Bigness

We prove a “non-hyperbolicity” result (remark that the following statement is an equivalence). We denote by R the Riemann curvature tensor of the canonical metric of X , that is the quotients of the Bergman metric of the universal cover. The values of l are given in Remarks 6.4.2.2.

Theorem 6.1.1. *Let $(\bar{X}, D)$ be a log-compactification of a locally symmetric variety X whose universal cover is irreducible and of rank at least 2. Let $1 \leq p \leq \dim X$.*

Then the vector bundle $\Omega_{\bar{X}}^p(\log D)$ is L -big. Moreover, the following are equivalent:

- $\Omega_{\bar{X}}^p(\log D)$ is Viehweg-big,
- $\Omega_{\bar{X}}^p(\log D)$ is ample with respect to X ,
- $p > l$ with l the maximal dimension of $\ker R(v, \bar{v})$ for $v \in T_X$ with $v \neq 0$.

6.1.2 Augmented base locus

The previous theorem is a corollary of a computation of the augmented base locus. First, we define higher degree characteristic subvarieties, that generalize the (higher) characteristic subvariety $\mathcal{S}_{r-1}$ defined by Mok in [58, Appendix III] in degree 1.

Definition 6.1.2. *Let X be a locally symmetric variety whose universal cover is irreducible and of rank at least 2, let R be the Riemann curvature tensor of its canonical metric, let $1 \leq p \leq \dim X$. The higher degree characteristic subvariety $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is the set of $[\alpha] \in \mathbb{P}\Omega_X^p$ such that there exists $v \in T_X$ with $v \neq 0$ such that $\alpha \in \Lambda^p \ker R(v, \bar{v})$.*

These higher degree characteristic subvarieties can also be defined using the Lie algebra associated with G , see Definition 6.3.14. They are closely linked with $\mathbb{C}$ -PVHS because of the following proposition.

Proposition 6.1.3. *Let $\mathbb{V}$ be a non-unitary $\mathbb{C}$ -PVHS on a locally symmetric variety X whose universal cover is irreducible and of rank at least 2. Let $1 \leq p \leq \dim X$. Then $\mathcal{S}^p$ is the set of $[\alpha] \in \mathbb{P}\Omega_X^p$ such that there exists $v \in T_X$ with $v \neq 0$ such that $\alpha \in \Lambda^p C_{\mathbb{V}}(v)$ (Notation 4.1.4).*

Thus Theorem 4.1.5 gives the inclusion $\mathbb{B}_+^p(X) \subset \mathcal{S}^p$. We prove that it is actually an equality.

Theorem 6.1.4. *Let X be a locally symmetric variety whose universal cover is irreducible and of rank at least 2. Let $1 \leq p \leq \dim X$. Then $\mathbb{B}_+^p(X) = \mathcal{S}^p$.*

6.1.3 Stable base locus in degree one

In degree 1, we compute the stable base locus too. The compact case of the following theorem was proven by Mok in [58, Proposition 4 p.262].

Theorem 6.1.5. *Let X be a locally symmetric variety whose universal cover is irreducible and of rank at least 2. Then $\mathbb{B}^1(X) = \mathbb{B}_+^1(X) = \mathcal{S}$.*

As an application, we prove the generalization of [52, Corollary 1.6] to the non compact case.

Theorem 6.1.6. *A smooth projective family of curves of genus $g \geq 2$ over a locally symmetric variety X whose universal cover is irreducible and of rank at least 2 is isotrivial.*

6.1.4 Outline of the proofs

In [90], Zucker explained a procedure to construct $\mathbb{C}$ -PVHSs on a locally symmetric variety X . This procedure is recalled in Subsection 6.2.1. In Subsection 6.2.2, we explain that thanks to Margulis's superrigidity theorem, every $\mathbb{C}$ -PVHS on X comes from the construction of Zucker, up to a finite étale cover (see Proposition 6.2.1). This allows one to reduce the proof of Proposition 6.1.3 to the case of Zucker's $\mathbb{C}$ -PVHSs (see Proposition 6.2.3). However, this reduction is not necessary to prove Theorems 6.1.4 and 6.1.1 because for this, it is sufficient to know the statement of Proposition 6.1.3 for a single $\mathbb{R}$ -PVHS (with quasi-unipotent local monodromy).

In Subsection 6.3.2, we define the higher degree characteristic subvarieties $\mathcal{S}^p$ (see Definition 6.3.14). We prove that the $\mathcal{S}^p$'s are Zariski closed in $\mathbb{P}\Omega_X^p$ (see Proposition 6.3.16). The key tool used to prove the latter proposition is the o-minimal Chow's lemma [71, Corollary 4.5]. To apply it, we have to prove that the $\mathcal{S}^p$'s are definable in an o-minimal structure. It is proven as a consequence of [7, Theorem 1.1]. In Section 6.4, we prove Proposition 6.1.3. Then we prove Theorem 6.1.4 by applying Nakamaye's theorem [65, Theorem 1.1] (see Theorem 3.2.6) to them. Knowing that the $\mathcal{S}^p$'s are Zariski closed is crucial to apply this theorem. Then, Theorem 6.1.1 is an immediate corollary of Theorem 6.1.4.

Mok's proof of the compact case of Theorem 6.1.5 is based on a Finsler metric rigidity theorem. The obstruction to extending his proof in a non compact context is the use of integrals over closed subvarieties: the integrability of differential forms is obvious in the compact case but not in the non compact case. To answer this problem, we use Chapter 5. The construction of Zucker allows us to see the logarithmic tangent bundle and its metric as a holomorphic subquotient bundle of the holomorphic bundle associated with the local system of some $\mathbb{R}$ -PVHS, with the metric induced by the Hodge metric. Then, we can use the careful study of the singularities of this metric done by Cattani, Kaplan, Schmid [29] and Kollár [51].

6.2 PVHS on locally symmetric varieties

Let $d \in \mathbb{N}_{>0}$, let $\Sigma \subset \mathbb{C}^d$ be an irreducible bounded symmetric domain of rank at least 2. The group of biholomorphisms of Σ has a unique structure of real Lie group. We denote by $\mathrm{Hol}(\Sigma)^+$ its identity component. It is a connected simple real Lie group with trivial center. Let $\Gamma \subset \mathrm{Hol}(\Sigma)^+$ be a torsion-free lattice. It is arithmetic because of Margulis's arithmeticity

theorem (see [54]). Denote by X the quotient $\Gamma \backslash \Sigma$. It admits a unique structure of smooth complex algebraic variety compatible with its natural structure of complex manifold. This structure is quasiprojective (it is given by the Baily-Borel compactification, see [13, p. III.4]).

6.2.1 The construction of Zucker

In his interpretation of bounded symmetric domains as moduli spaces of Hodge structures, Deligne gives a way to construct variations of Hodge structure on Σ (see [33, Corollaire 1.1.17]). We recall this construction in this subsection, following the more explicit description given by Zucker ([90], see this article for the proofs).

Let G be a connected semisimple real Lie group. Let $\alpha : G \rightarrow \mathrm{Hol}(\Sigma)^+$ be a surjective morphism with compact kernel. Let A be a finite dimensional complex vector space and $\rho : G \rightarrow \mathrm{GL}(A)$ a representation of G . We will construct a $\mathbb{C}$ -PVHS denoted by $\tilde{V}(G, A)$ on Σ starting from these data.

Denote by G^{ad} the quotient of G by its (finite) center. As the center of $\mathrm{Hol}(\Sigma)^+$ is trivial, α factorizes by a morphism $\alpha' : G^{\mathrm{ad}} \rightarrow \mathrm{Hol}(\Sigma)^+$. As $\mathrm{Hol}(\Sigma)^+$ is simple of non compact type and $\ker \alpha'$ is semisimple compact, the surjection α' is naturally split and G^{ad} is naturally isomorphic to $\mathrm{Hol}(\Sigma)^+ \times N$ with $N := \ker(\alpha')$. In particular, the Lie algebra $\mathfrak{h}$ associated with $\mathrm{Hol}(\Sigma)^+$ naturally identifies with a direct factor of the Lie algebra $\mathfrak{g}$ associated with G . This gives a natural representation of $\mathfrak{h}$ on A .

Denote by $\mathbb{U}$ the real Lie group of complex numbers of modulus 1. The group of characters of $\mathbb{U}$ is freely generated by the inclusion $\chi : \mathbb{U} \hookrightarrow \mathbb{C}^*$. For every $x \in \Sigma$, there exists a unique real Lie group homomorphism $c_x : \mathbb{U} \hookrightarrow G$ such that:

- c_x is injective,
- $p_N \circ c_x$ is trivial, with p_N the morphism $G \rightarrow N$ induced by the decomposition $G^{\mathrm{ad}} \cong \mathrm{Hol}(\Sigma)^+ \times N$,
- the action $\alpha \circ c_x$ of $\mathbb{U}$ on Σ fixes x ,
- the holomorphic tangent vector space $T_{\Sigma, x}$ has weight χ^μ for the action $d_x \circ \alpha \circ c_x$ of $\mathbb{U}$ for a *positive* integer μ .

Suppose A is a simple representation of G . Consider the set W of weights that appear in A for the representation $\rho \circ c_x$ of $\mathbb{U}$ (W does not depend on x).

There exist $\lambda \in \mathbb{Z}$ and $m \in \mathbb{N}$ such that $W = \{\chi^{\lambda-k\mu}, 0 \leq k \leq m\}$. Denote by A_x^k the weight space of A for the weight $\chi^{\lambda-k\mu}$ under the representation $\rho \circ c_x$. This gives to A a $\mathbb{C}$ -HS A_x which depends on $x \in \Sigma$. In general, we choose $\bigoplus_i A_i$ a decomposition of A as a sum of irreducible representations. Then the $\mathbb{C}$ -HS $A_x := \bigoplus_i (A_i)_x$ does not depend on the choice of the decomposition. Those $\mathbb{C}$ -HSs vary smoothly with x . We will recall the proof that it actually gives a complex variation of Hodge structure $\tilde{V}(G, A)$ on Σ , with the constant local system of fiber A as underlying local system.

Consider $\mathfrak{h}$ equipped with the adjoint action Ad of $\text{Hol}(\Sigma)^+$. It defines $c_x : \mathbb{U} \hookrightarrow \text{Hol}(\Sigma)^+$ as above (in this case $\mu = 1$). For every $x \in \Sigma$, the kernel of the action $\text{Ad} \circ c_x$ on $\mathfrak{h}$ is the real Lie subalgebra $\mathfrak{l}$ of $\mathfrak{h}$ which is tangent to the stabilizer $K_x \subset \text{Hol}(\Sigma)^+$ of x . As $\text{Hol}(\Sigma)^+$ is simple, there exists a unique $\text{Hol}(\Sigma)^+$ -subrepresentation $\mathfrak{m}$ of $\mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{l} \oplus \mathfrak{m}$.

The smooth action of $\text{Hol}(\Sigma)^+$ on Σ induces a morphism $e : \mathfrak{h} \times \Sigma \rightarrow T_{\Sigma}^{\mathbb{R}}$ of real smooth vector bundles on Σ (where $T_{\Sigma}^{\mathbb{R}}$ is the real tangent bundle). For every $x \in \Sigma$, denote by e_x the restriction $\mathfrak{h} \rightarrow T_{\Sigma,x}^{\mathbb{R}}$. Then e_x is equivariant under the action of K_x . In particular, it is equivariant for the action of $\mathbb{U}$ by $\text{Ad} \circ c_x$ on $\mathfrak{h}$ and by scalar-vector product on $T_{\Sigma,x}^{\mathbb{R}}$. Its kernel is $\mathfrak{l}$. As $\text{Hol}(\Sigma)^+$ acts transitively on Σ , e_x is surjective. Hence it induces an isomorphism $\mathfrak{m} \xrightarrow{\sim} T_{\Sigma,x}^{\mathbb{R}}$ of representations of $\mathbb{U}$. Hence only the characters χ and χ^{-1} appears in $\mathfrak{m}_{\mathbb{C}}$, so we have $W = \{\chi, 1, \chi^{-1}\}$, $\mathfrak{h}_{\mathbb{C},x}^1 = \mathfrak{l}_{x,\mathbb{C}}$ and $\mathfrak{m}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C},x}^0 \oplus \mathfrak{h}_{\mathbb{C},x}^2$. Denote by ι_x the inverse of e_x : it induces two isomorphisms $T_{\Sigma,x} \xrightarrow{\sim} \mathfrak{h}_{\mathbb{C},x}^0$ and $T_{\Sigma,x}^{0,1} \xrightarrow{\sim} \mathfrak{h}_{\mathbb{C},x}^2$.

Let $h_x \in \mathfrak{l}$ be the unique element such that for every holomorphic tangent vector $v \in T_{\Sigma,x}$, we have $h_x.v = \mathbf{i}v$. Note that A_x^k is the eigenspace of h_x for the eigenvalue $(\lambda - k\mu)\mathbf{i}$.

Now, let $U \subset \Sigma$ be an open subset, let $k \in \mathbb{Z}$ and let $f : U \rightarrow A$ be a smooth section of A^k . That means that for every $x \in U$, we have $h_x.f(x) = (\lambda - k\mu)\mathbf{i}f(x)$. Let $v \in T_{\Sigma,x}$, let $z := \iota_x(v) \in \mathfrak{h}_{\mathbb{C},x}^0$. Consider the decomposition:

$$\partial f_x.v = a_0 + a_{-1} \quad (6.1)$$

with $a_0 = \partial f_x.v - z.f(x)$ and $a_{-1} = z.f(x)$. We have $a_{-1} \in A^{k-1}$ because $h_x.z.f(x) = [h_x, z].f(x) + z.h_x.f(x) = (\lambda - (k-1)\mu)\mathbf{i}z.f(x)$. Now we prove that $a_0 \in A^k$.

By definition of f , for $g \in \text{Hol}(\Sigma)^+$ in a neighborhood of the neutral element, we have $(\text{Ad}(g).h_x).f(g.x) = (\lambda - k\mu)\mathbf{i}f(g.x)$. Consider the

differential $\frac{d}{dg}$ of this expression at $g = 1$ and evaluate it on z : we get $[z, h_x].f(x) + h_x.d f(x).v = (\lambda - k\mu)\mathbf{i}df_x.v$. As $[z, h_x] = -\mu\mathbf{i}z$ and $v \in T_{\Sigma, x}$, this gives $-\mu\mathbf{i}z.f(x) + h_x.\partial f_x.v = (\lambda - k\mu)\mathbf{i}\partial f_x.v$. One can check that it implies that $h_x.a_0 = (\lambda - k\mu)\mathbf{i}a_0$, hence $a_0 \in A^k$.

In the same way, we can prove that for $\bar{v} \in T_{\Sigma, x}^{0,1}$, we have:

$$\bar{\partial}f_x.\bar{v} = b_0 + b_1 \quad (6.2)$$

with $b_0 = \bar{\partial}f_x.\bar{v} - \iota_x(\bar{v}).f(x) \in A^k$ and $b_1 = \iota_x(\bar{v}).f(x) \in A^{k+1}$. This proves that the $\mathbb{C}$ -HS associated with every $x \in \Sigma$ defines a complex variation of Hodge structure $\tilde{V}(G, A)$ on the constant local system defined by A on Σ . Zucker proved that it admits a polarization whose associated Hodge metric is G -equivariant.

Now we recall Zucker's construction of locally homogeneous variations of Hodge structure (see [90]). Let $\Lambda \subset G$ be a torsion-free lattice. The group Λ acts by α on Σ and by ρ on A . This gives an action of Λ on Σ equipped with the constant local system of fiber A . The $\mathbb{C}$ -PVHS $\tilde{V}(G, A)$ is equivariant under this action, and admits an equivariant polarization. Hence it descends as a $\mathbb{C}$ -PVHS on $X = \Lambda \backslash \Sigma$ that we will denote by $V(G, A, \Lambda)$.

In the case where A is of the form $A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with $A_{\mathbb{R}}$ a simple real representation of G , the set $W = \{\chi^{\lambda-k\mu}, 0 \leq k \leq m\}$ of weights defined above is stable under inversion, i.e. $m\mu = 2\lambda$. For $0 \leq k \leq m$, let $\tilde{V}(G, A_{\mathbb{R}})_x^{k, m-k} := \tilde{V}(G, A)_x^k$. This defines a real polarizable variation of Hodge structure ($\mathbb{R}$ -PVHS) $\tilde{V}(G, A_{\mathbb{R}})$ of weight m on Σ , with the constant local system of fiber $A_{\mathbb{R}}$ as underlying local system. If A is of the form $A_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ with $A_{\mathbb{R}}$ a real representation of G , we define the $\mathbb{R}$ -PVHS $\tilde{V}(G, A_{\mathbb{R}})$ as the direct sum $\bigoplus_i \tilde{V}(G, A_{i, \mathbb{R}})$ with $\bigoplus_i A_{i, \mathbb{R}}$ a decomposition of $A_{\mathbb{R}}$ as a sum of simple representations. Again, the action of G on Σ extends as an action on $\tilde{V}(G, A_{\mathbb{R}})$, and there exists an equivariant polarization. Thus, $\tilde{V}(G, A_{\mathbb{R}})$ descends as an $\mathbb{R}$ -PVHS denoted by $V(G, A_{\mathbb{R}}, \Lambda)$ on $\Lambda \backslash \Sigma$.

6.2.2 Higgs field and conjugate Higgs field

Every $\mathbb{C}$ -PVHS comes from locally homogeneous $\mathbb{C}$ -PVHSs in some sense (Proposition 6.2.1 below). In preparation for this statement, we define G - $\mathbb{C}$ -HS to define more $\mathbb{C}$ -PVHSs on X .

Let G be a Lie group. We call G - $\mathbb{C}$ -HS a $\mathbb{C}$ -HS equipped with an action of G such that G acts by automorphisms of $\mathbb{C}$ -HS. A $\mathbb{C}$ -HS A with an action

of an algebraic group G is a G - $\mathbb{C}$ -HS if and only if for every $i \in \mathbb{Z}$, A^i is a subrepresentation of G . Now, let G be a connected semisimple real Lie group with a surjective morphism with compact kernel $\alpha : G \rightarrow \mathrm{Hol}(\Sigma)^+$ (as in Section 6.2.1), and let A be a G - $\mathbb{C}$ -HS. Consider the $\mathbb{C}$ -PVHS $\tilde{V}(G, A^i)$ on Σ , for every $i \in \mathbb{Z}$. Shift the grading of $\tilde{V}(G, A^i)$ by i . Sum the obtained $\mathbb{C}$ -PVHS for every i : we get a $\mathbb{C}$ -PVHS $\tilde{V}(G, A)$ on Σ .

Let $\Lambda \subset G$ be a torsion-free lattice. Then $\tilde{V}(G, A)$ descends as a $\mathbb{C}$ -PVHS $V(G, A, \Lambda)$ on $\Lambda \backslash \Sigma$. The monodromy representation of the local system underlying $V(G, A, \Lambda)$ is A with the action of G restricted to Λ . Up to a finite étale covering map, every $\mathbb{C}$ -PVHS on X has the form $V(G, A, \Lambda)$ for some group G , some G - $\mathbb{C}$ -HS A and some lattice Λ .

Proposition 6.2.1. *Let Σ be a bounded symmetric domain of rank at least 2. Let $\Gamma \subset \mathrm{Hol}(\Sigma)^+$ be an irreducible torsion-free lattice. Let $\mathbb{V}$ be a $\mathbb{C}$ -PVHS on $\Gamma \backslash \Sigma$.*

Then there exists a connected semisimple real Lie group G , a surjective morphism with compact kernel $\alpha : G \rightarrow \mathrm{Hol}(\Sigma)^+$, a G - $\mathbb{C}$ -HS A and a torsion-free lattice $\Lambda \subset G$ such that $\alpha(\Lambda)$ is a finite index subgroup of Γ and such that the pull-back of $\mathbb{V}$ on $\Lambda \backslash \Sigma$ is isomorphic to $V(G, A, \Lambda)$.

Proof. As Σ has rank at least 2 and Γ is irreducible, Γ is arithmetic by Margulis's arithmeticity theorem (see [54, Theorem (1') p.4]). There exist a connected $\mathbb{Q}$ -simple $\mathbb{Q}$ -group $G_{\mathbb{Q}}$, a surjective morphism of real Lie groups $\alpha : G_{\mathbb{Q}}(\mathbb{R}) \rightarrow \mathrm{Hol}(\Sigma)^+$ with compact kernel and an arithmetic lattice (in the sense of $\mathbb{Q}$ -groups) $\Lambda' \subset G_{\mathbb{Q}}(\mathbb{Q})$ such that $\alpha(\Lambda')$ is a finite index subgroup of Γ . By the existence of algebraic universal covers for algebraic groups, we can suppose $G_{\mathbb{Q}}$ simply connected. By Malcev's theorem, we can suppose that Λ' is torsion-free. Denote by π the natural finite étale covering map $\Lambda' \backslash \Sigma \rightarrow \Gamma \backslash \Sigma$.

By [32, Proposition 1.13], a $\mathbb{C}$ -PVHS decomposes as a direct sum of $\mathbb{C}$ -PVHSs with irreducible underlying local system. Hence one can suppose the local system underlying $\pi^* \mathbb{V}$ irreducible. We can replace Λ' by a finite index subgroup to suppose moreover that it remains irreducible after pull-back by every finite étale cover. By [32, Proposition 1.13] again, it is sufficient to prove that up to a finite étale covering map, the local system underlying $\pi^* \mathbb{V}$ is isomorphic to the local system underlying $V(G, A, \Lambda)$ for some representation A of G and some finite index subgroup $\Lambda \subset \Lambda'$. It is equivalent to prove that the representation of Λ' given by the monodromy of $\pi^* \mathbb{V}$ coincides with a representation of G on a finite index subgroup $\Lambda \subset \Lambda'$. As Λ' is an arithmetic lattice of a simply connected semisimple $\mathbb{Q}$ -group of rank at least 2, it is

given by the superrigidity theorem of Margulis for arithmetic lattices (see [54, Theorem (6)(iii) p.5]). $\square$

The Higgs field and the conjugate Higgs field of $\tilde{V}(G, A)$ are transparent in the construction of Zucker.

Proposition 6.2.2. *Let A be a G - $\mathbb{C}$ -HS. Let $x \in \Sigma$. Let $\theta_x : T_{\Sigma, x} \rightarrow \text{End}(A)$ be the Higgs field of $\tilde{V}(G, A)$ at x and θ_x^* the conjugate Higgs field. Let $v \in T_{\Sigma, x}$, let $a \in A$. Then $\theta_x(v).a = \iota_x(v).a$ and $\theta_x^*(\bar{v}).a = \iota_x(\bar{v}).a$.*

Proof. The Higgs field is given by a_{-1} in (6.1) and the conjugate Higgs field by b_1 in (6.2). $\square$

Recall that the Hodge filtration of a $\mathbb{C}$ -PVHS $\mathbb{V}$ is flat if and only if the Higgs field of $\mathbb{V}$ vanishes. In the case where the $\mathbb{C}$ -PVHS is defined on a smooth complex quasiprojective variety, this is also equivalent to the relative compactness of the image of the monodromy representation of $\mathbb{V}$ in the linear group of a fiber (it is an easy consequence of [32, Proposition 1.13.(ii)]). In this case, $\mathbb{V}$ is said to be *unitary*.

Proposition 6.2.3. *Let $\mathbb{V}$ be a non-unitary $\mathbb{C}$ -PVHS on a locally symmetric variety X whose universal cover $p : \Sigma \rightarrow X$ is irreducible and of rank at least 2. Let $x \in \Sigma$, let $v, w \in T_{\Sigma, x}$. Denote $y := p(x)$, $v' := dp_x.v$ and $w' := dp_x.w$. Let $\theta_y : T_{X, y} \rightarrow \text{End}(\mathbb{V}_y)$ be the Higgs field of $\mathbb{V}$ at y , let θ_y^* be the conjugate Higgs field. Then $[\theta_y(v'), \theta_y^*(\bar{w}')] = 0$ (in $\text{End}_{\mathbb{C}}(\mathbb{V}_y)$) if and only if $[\iota_x(v), \iota_x(\bar{w})] = 0$ (in $\mathfrak{h}_{\mathbb{C}}$).*

Proof. It is sufficient to prove the statement on a finite étale covering of X . Hence, thanks to Proposition 6.2.1, we can assume $\mathbb{V} = V(G, A, \Gamma)$ for a connected semisimple real Lie group G , a surjective morphism with compact kernel $\alpha : G \rightarrow \text{Hol}(\Sigma)^+$ and a G - $\mathbb{C}$ -HS A .

As $\mathbb{V}$ is non-unitary, the action of G on A is nontrivial. By simplicity of $\mathfrak{h}$, the associated action of $\mathfrak{h}$ on A is faithful. Hence $\iota_x(v)$ and $\iota_x(\bar{w})$ commute in $\mathfrak{h}_{\mathbb{C}}$ if and only if their actions on A commute. Thus the conclusion results from Proposition 6.2.2. $\square$

6.2.3 A useful PVHS

We will need a PVHS with good properties on locally symmetric varieties.

Proposition 6.2.4. *Let X be a locally symmetric variety whose universal cover is irreducible and of rank at least 2. Then there exists:*

- a finite étale covering map $X' \rightarrow X$,

- a projective log-compactification $(\bar{X}', D')$ of X' ,
- a real polarized variation of Hodge structure $\mathbb{V}$ on X' with unipotent local monodromy (on $\bar{X}'$),

such that:

- if (E, θ) denote the Higgs bundle defined by $\mathbb{V}$ on X' , then the pull-back by θ of the Hodge metric on $\text{End}(E)$ is a constant multiple of the canonical metric g on $T_{X'}$,
- if $({}^\diamond E, {}^\diamond \theta)$ denote the log-Higgs bundle defined by $\mathbb{V}$ on $\bar{X}'$, then the injection ${}^\diamond \theta : T_{\bar{X}'}(-\log D') \hookrightarrow \text{End}({}^\diamond E)$ is locally split.

Proof. Consider the $\mathbb{R}$ -PVHS $V(\text{Hol}(\Sigma)^+, \mathfrak{h}, \Gamma)$ defined over X by the adjoint representation of $\text{Hol}(\Sigma)^+$ over $\mathfrak{h}$. As Γ is an arithmetic lattice, it has a neat (see [11, §17.1]) finite index subgroup Γ' . We set $X' := \Gamma' \backslash \Sigma$ and $\mathbb{V} := V(\text{Hol}(\Sigma)^+, \mathfrak{h}, \Gamma')$ on X' .

As X' is a locally symmetric variety and Γ' is arithmetic and neat, X' admits a toroidal compactification $(\bar{X}', D')$ (see [3]). A small loop around an irreducible component of D' corresponds to a unipotent element of G (see [62, p.255-256]), thus the local monodromy is unipotent.

We can easily check with Proposition 6.2.2 that the Higgs field $\theta : T_{X'} \rightarrow \text{End}(E)$ of $\mathbb{V}$ is locally split injective. On the one hand $T_{\bar{X}'}(-\log D')$ is the good extension of $T_{X'}$ in the sense of [62]. On the other hand the Hodge metric has at most log growth on the sections of $\text{End}(E)$ thanks to Theorem 5.3.1. Thus, thanks to [62, Proposition 1.3], the log-Higgs field ${}^\diamond \theta$ is also locally split injective over D' .

Choose a polarization of $\mathbb{V}$: the pull-back $\theta^* h$ of the Hodge metric h on $\text{End}(E)$ is an Hermitian metric on X' . Finally, the Higgs map of $\tilde{V}(\text{Hol}(\Sigma)^+, \mathfrak{h})$ is equivariant with respect to the action of G . As G acts transitively on Σ and the isotropy group K_x acts irreducibly on $T_{\Sigma, x}$ (for $x \in \Sigma$), $\theta^* h$ is a constant multiple of g . Hence we can change the polarization by a constant multiple so that over X' , $\theta^* h = g$. $\square$

6.3 Characteristic subvarieties

We recall the definition and main properties of Mok's characteristic subvarieties. Then we define and prove the main properties of the higher degree characteristic subvarieties.

6.3.1 Characteristic subvarieties

The proofs of the results of this subsection can be found in [58, Chapter 10, Appendices III & IV]. Let $r \in \mathbb{N}$ be the rank of Σ . Let $x \in \Sigma$, let K_x be the isotropy group of x . We equip Σ with the Bergman metric g . The starting point is the polydisk theorem.

Theorem 6.3.1. *There is a totally geodesic complex submanifold $D \subset \Sigma$ biholomorphically isometric to the Poincaré polydisk Δ^r such that $\Sigma = \bigcup_{\gamma \in K_x} \gamma \cdot D$ and $T_{\Sigma, x} = \bigcup_{\gamma \in K_x} d\gamma \cdot T_{D, x}$.*

Proposition 6.3.2. *Let $v \in T_{\Sigma, x}$. Let D, D' be two totally geodesic complex submanifolds of Σ with biholomorphic isometries $f : D \rightarrow \Delta^r$ and $f' : D' \rightarrow \Delta^r$, such that $x \in D \cap D'$ and $v \in T_{D, x} \cap T_{D', x}$. Then $df_x(v)$ has as many nonzero coordinates as $df'_x(v)$.*

Definition 6.3.3. *Let $v \in T_{\Sigma, x}$. Let D be a totally geodesic complex submanifold of Σ with a biholomorphic isometry $f : D \rightarrow \Delta^r$ such that $x \in D$ and $v \in T_{D, x}$. The rank of v is the number of nonzero coordinates of $df_x(v)$.*

The rank exists thanks to Theorem 6.3.1 and is unique thanks to the previous proposition. As the rank is invariant under the action of $\mathbb{C}^*$, we can set the following definition.

Definition 6.3.4. *Let $0 \leq k \leq r$. We call k -th characteristic subvariety at x the subset $\mathcal{S}_k^x \subset \mathbb{P}\Omega_{\Sigma, x}^1$ of lines of vectors of rank at most k . We call k -th characteristic bundle of Σ the union $\mathcal{S}_k(\Sigma) := \bigcup_{x \in \Sigma} \mathcal{S}_k^x \subset \mathbb{P}\Omega_{\Sigma}^1$.*

The characteristic subvarieties are Zariski closed complex subvarieties of $\mathbb{P}\Omega_{\Sigma, x}^1$. The characteristic bundles are closed holomorphic subbundles of $\mathbb{P}\Omega_{\Sigma}^1$ (over Σ). Notice that $\mathcal{S}_0^x = \emptyset$ and $\mathcal{S}_0(\Sigma) = \emptyset$.

Fix $1 \leq k \leq r$. We can describe characteristic subvarieties in terms of group actions: $\mathcal{S}_k^x$ is stable under the action of K_x over $\mathbb{P}\Omega_{\Sigma, x}^1$. Moreover, this action induces an action of the complexified group $K_{x, \mathbb{C}}$ over $\mathbb{P}\Omega_{\Sigma, x}^1$. Hence, if we denote $\mathcal{S}_k^{\text{reg}, x} := \mathcal{S}_k^x \setminus \mathcal{S}_{k-1}^x$ and $\mathcal{S}_k(\Sigma)^{\text{reg}} := \mathcal{S}_k(\Sigma) \setminus \mathcal{S}_{k-1}(\Sigma)$, according to the first section of [57], the set $\mathcal{S}_k^{\text{reg}, x}$ is an orbit of $\mathbb{P}\Omega_{\Sigma, x}^1$ under the action of $K_{x, \mathbb{C}}$. The set $\mathcal{S}_k^{\text{reg}, x}$ (resp. $\mathcal{S}_k(\Sigma)^{\text{reg}}$) is a smooth open subset of $\mathcal{S}_k^x$ (resp. $\mathcal{S}_k(\Sigma)$). Remark that it is dense (by definition of characteristic subvarieties).

We can make a link between characteristic subvarieties and curvature. For a holomorphic vector v tangent to Σ at x , we denote by $\mathcal{N}_v$ the complex

vector space of holomorphic vectors u such that the antilinear form $w \mapsto g(R(v, \bar{v})u, \bar{w})$ is zero. There is a unique decreasing map $n : \llbracket 1, r \rrbracket \rightarrow \llbracket 0, d \rrbracket$ such that for each $v \in T_{\Sigma, x}$ of rank k , $\dim \mathcal{N}_v$ is $n(k)$. This number determines the dimension of characteristic subvarieties.

Proposition 6.3.5. *We have $\dim \mathcal{S}_k^x = d - n(k) - 1$ and $\dim \mathcal{S}_k(\Sigma) = 2d - n(k) - 1$.*

Using the equation (4.2), we see that for $y = (x, [v]) \in \mathbb{P}\Omega_\Sigma^1$ of rank k , the projection $\pi : \mathbb{P}\Omega_\Sigma^1 \rightarrow \Sigma$ induces an isomorphism between the kernel of $C_1(\mathcal{O}(1), \hat{g})(y)$ and $\mathcal{N}_v$. Moreover, the kernel of $C_1(\mathcal{O}(1), \hat{g})(y)$ is tangent to $\mathcal{S}_k(\Sigma)$. This proves the following proposition.

Proposition 6.3.6. *The pull-back of $C_1(\mathcal{O}(1), \hat{g})(y)$ on $\mathcal{S}_k(\Sigma)^{\text{reg}}$ has rank $2d - 2n(k) - 1$.*

An explicit computation gives the following result on dimensions.

Proposition 6.3.7. *If $1 \leq k < r - 1$, $\mathcal{S}_k(\Sigma)$ has codimension at least 2 in $\mathcal{S}_{k+1}(\Sigma)$.*

Let X be a smooth complex algebraic variety whose universal cover is isomorphic to Σ . Let $\Gamma \subset \text{Aut}(\Sigma)$ be its fundamental group. Let $0 \leq k \leq r$. Remark that $\mathcal{S}_k(\Sigma)$ and $\mathcal{S}_k(\Sigma)^{\text{reg}}$ are stable under the action of Γ . We denote $\mathcal{S}_k := \Gamma \backslash \mathcal{S}_k(\Sigma) \subset \mathbb{P}\Omega_X^1$ and $\mathcal{S}_k^{\text{reg}} := \Gamma \backslash \mathcal{S}_k(\Sigma)^{\text{reg}}$. We will call $\mathcal{S}_k$ a *characteristic subvariety* of X (though it is a subvariety of $\mathbb{P}\Omega_X^1$). We will denote by $\mathcal{S}$ the characteristic subvariety $\mathcal{S}_{r-1}$.

Proposition 6.3.8. *For every $0 \leq k \leq r$, $\mathcal{S}_k$ is a closed algebraic subvariety of $\mathbb{P}\Omega_X^1$.*

To prove the previous proposition, one can use [79, Definition 3.1 & Theorem 3.3]. This article gives a characterization of characteristic subvarieties in terms of PVHS. Then the algebraicity results from Schmid's nilpotent orbit theorem (see [77]) and from Serre's GAGA theorem. A proof similar to the one of Proposition 6.3.16 below also works.

6.3.2 Higher degree characteristic subvarieties

In this subsection, we define the higher degree characteristic subvarieties and prove that they are Zariski closed (that we will need to prove Theorems 6.1.4 and 6.1.5). To do this, we will need the notion of Borel dual (see [88] for the proofs). For every $x \in \Sigma$, let $K_{x, \mathbb{C}}$ be the connected complex Lie subgroup of $\text{GL}(T_{\Sigma, x})$ whose associated Lie algebra is $\mathfrak{l}_{\mathbb{C}}$. Let $H_{\mathbb{C}}$ be the adjoint complex

Lie group of the complex Lie algebra $\mathfrak{h}_{\mathbb{C}}$. The faithful action of $\mathrm{Hol}(\Sigma)^+$ on $\mathfrak{h}_{\mathbb{C}}$ by the adjoint action induces an injective real Lie group homomorphism $\mathrm{Hol}(\Sigma)^+ \hookrightarrow H_{\mathbb{C}}$. Moreover, the restriction $K_x \hookrightarrow H_{\mathbb{C}}$ extends as an injective complex Lie group homomorphism $i_x : K_{x,\mathbb{C}} \hookrightarrow H_{\mathbb{C}}$. Define the *Borel dual* $\check{\Sigma}$ of Σ as the orbit of the morphism i_x under the action of $H_{\mathbb{C}}$ (this orbit does not depend on x). It admits a unique structure of complex manifold such that $H_{\mathbb{C}}$ acts biholomorphically. It is a smooth projective variety. The map $i : x \mapsto i_x$ is a holomorphic open embedding of Σ in $\check{\Sigma}$, equivariant under the action of G . Denote by P_x the stabilizer of i_x under the action of $H_{\mathbb{C}}$. Then the image of P_x in $\mathrm{Aut}(T_{\Sigma,x})$ is equal to $K_{x,\mathbb{C}}$.

We will also need some o-minimal geometry (see [85]). In this theory, some subsets of $V(\mathbb{R})$ of any real algebraic variety V are said to be *semi-algebraic*, and some subsets are said *definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$* . Those two classes of subsets are stable by finite unions, finite intersections, complement, direct and inverse images by a morphism of real algebraic varieties, and they contain the “trivial” subset $V(\mathbb{R})$ of $V(\mathbb{R})$ for every real algebraic variety V . The class of semi-algebraic subsets can be defined as the smallest class that verifies these properties. The class of subsets definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$ is the smallest class of subsets that verifies the previous properties and that contains the following subsets:

- the graph of the real exponential (as a subset of $\mathbb{A}^2(\mathbb{R})$),
- the subset $\{(x, f(x)), x \in [0, 1]^n\}$ of $\mathbb{A}^{n+1}(\mathbb{R})$ for every $n \in \mathbb{N}_{>0}$ and every real analytic function f defined in a Euclidean open neighborhood of $[0, 1]^n$.

A map between semi-algebraic (resp. definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$) subsets is said to be *semi-algebraic* (resp. *definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$*) if its graph is. For every complex algebraic variety V , the set of closed points of V canonically identifies with the set of real points of its real Weil restriction of scalars. Hence, the notions of semi-algebraic subsets and subsets definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$ of $V(\mathbb{C})$ are well-defined.

In the following, we will only need the following facts.

- Constructible subsets and morphisms of complex algebraic varieties are semi-algebraic. Semi-algebraic subsets and maps are definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$.
- The class of semi-algebraic (resp. definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$) subsets is stable under finite union, finite intersection, complement, direct and inverse image by a semi-algebraic (resp. definable in $\mathbb{R}_{\mathrm{an}, \mathrm{exp}}$) map.

- o-minimal Chow's lemma: an analytic closed subset definable in $\mathbb{R}_{\text{an}, \exp}$ is Zariski closed (see [71, Corollary 4.5], that we can apply here because $\mathbb{R}_{\text{an}, \exp}$ is an o-minimal structure by [35]).
- The open subset $i(\Sigma) \subset \check{\Sigma}$ is semi-algebraic.
- There exists a semi-algebraic subset $\mathfrak{S} \subset \Sigma$ such that the restriction to $\mathfrak{S}$ of the quotient map $\Sigma \rightarrow X$ is surjective and definable in $\mathbb{R}_{\text{an}, \exp}$ (see [7, Theorem 1.1] and the paragraph just below it).

Let $x \in \Sigma$, let $v \in T_{\Sigma, x}$. Consider the vector space:

$$C(v) := \{w \in T_{\Sigma, x}, [\iota_x(\bar{v}), \iota_x(w)] = 0\} \subset T_{\Sigma, x}.$$

Remark 6.3.9. The domain Σ admits a canonical Kähler metric called the Bergman metric. Up to a constant multiple, it is the only Hermitian metric equivariant under the action of G . Denote by $\tilde{R}$ its Riemann curvature tensor. It is seminegative in the sense of Griffiths. By [49, (6.8.8) and (6.8.5)], we have $C(v) = \ker \tilde{R}(v, \bar{v})$.

As ι_x is equivariant under the action of K_x , for $g \in K_{x, \mathbb{C}}$, we have:

$$C(g.v) = \bar{g}.C(v). \quad (6.3)$$

Denote:

$$\mathcal{S}_x^{\mathbf{Gr}} := \{C(v) | v \in T_{\Sigma, x} \setminus \{0\}\} \subset \mathbf{Gr}(T_{\Sigma, x})$$

with $\mathbf{Gr}(T_{\Sigma, x}) := \bigsqcup_{k=0}^{\dim \Sigma} \mathbf{Gr}_k(T_{\Sigma, x})$ where $\mathbf{Gr}_k(T_{\Sigma, x})$ is the Grassmannian variety of vector subspaces of dimension k of $T_{\Sigma, x}$.

Lemma 6.3.10. *The set $\mathcal{S}_x^{\mathbf{Gr}}$ is a finite union of orbits of $\mathbf{Gr}(T_{\Sigma, x})$ under the algebraic action of the algebraic group $K_{x, \mathbb{C}}$. In particular, it is a constructible subset of $\mathbf{Gr}(T_{\Sigma, x})$.*

Proof. The vector space $T_{\Sigma, x}$ has finitely many orbits under the action of the algebraic group $K_{x, \mathbb{C}} \times \mathbb{C}^*$, where $K_{x, \mathbb{C}}$ acts by adjunction and $\mathbb{C}^*$ by homothety (see [57, (1.1)]). Hence, the first statement results from (6.3). Then the second statement results from Chevalley's theorem. $\square$

Let $1 \leq p \leq \dim \Sigma$. Denote:

$$\mathcal{S}_x^p := \bigcup_{C \in \mathcal{S}_x^{\mathbf{Gr}}} \{[\alpha] | \alpha \in \Lambda^p C \setminus \{0\}\} \subset \mathbb{P}\Omega_{\Sigma, x}^p.$$

Lemma 6.3.11. *Let V, W be two vector spaces. Let $1 \leq p \leq \dim V$. Let F be the subset of $\mathbb{P} \Lambda^p V^\vee \times \text{Hom}(V, W)$ defined by:*

$$F := \{([\alpha], f) | \alpha \in \Lambda^p \ker f \setminus \{0\}\}.$$

Then F is Zariski closed.

Proof. For $f \in \text{Hom}(V, W)$, consider the linear map:

$$\begin{aligned} \Lambda^p V &\rightarrow W \otimes \Lambda^{p-1} V \\ v_1 \wedge \cdots \wedge v_p &\mapsto \sum_{i=1}^p (-1)^{i-1} f(v_i) \otimes v_1 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_p. \end{aligned}$$

Its kernel is $\Lambda^p \ker f$. This procedure defines a linear map $\chi : \text{Hom}(V, W) \rightarrow \text{Hom}(V', W')$ with $V' := \Lambda^p V$ and $W' := W \otimes \Lambda^{p-1} V$. The following subset of $V' \times \text{Hom}(V', W')$ is closed:

$$G := \{(\beta, g) | g(\beta) = 0\}.$$

Pulling-back by χ , we obtain that the following subset of $\Lambda^p V \times \text{Hom}(V, W)$ is closed:

$$F' := \{(\alpha, f) | \alpha \in \Lambda^p \ker f\}.$$

Hence F is closed in $\mathbb{P} \Lambda^p V^\vee \times \text{Hom}(V, W)$. $\square$

Lemma 6.3.12. *The set $\mathcal{S}_x^p$ is a Zariski closed subset of $\mathbb{P} \Omega_{\Sigma, x}^p$. It is stable under the action of $K_{x, \mathbb{C}}$.*

Proof. Let $S \subset \mathbb{P} \Omega_{\Sigma, x}^p \times \mathbf{Gr}(T_{\Sigma, x})$ be defined as the set of points $([\alpha], C)$ such that $\alpha \in \Lambda^p C$ and $C \in \mathcal{S}_x^{\mathbf{Gr}}$. By Lemma 6.3.10, it is a constructible subset stable under the action of $K_{x, \mathbb{C}}$. Its projection on $\mathbb{P} \Omega_{\Sigma, x}^p$ is $\mathcal{S}_x^p$. Hence, $\mathcal{S}_x^p$ is stable under the action of $K_{x, \mathbb{C}}$ and, by Chevalley's theorem, it is constructible.

Then, consider the Euclidean continuous map:

$$\begin{aligned} c : T_{\Sigma, x} &\rightarrow \text{End}_{\mathbb{C}}(T_{\Sigma, x}) \\ v &\mapsto w \mapsto [\iota_x(\bar{v}), \iota_x(w)]. \end{aligned}$$

It induces a Euclidean continuous map $c' : \mathbb{P} \Omega_{\Sigma, x}^p \times T_{\Sigma, x} \rightarrow \mathbb{P} \Omega_{\Sigma, x}^p \times \text{End}_{\mathbb{C}}(T_{\Sigma, x})$. Now, let $S' \subset \mathbb{P} \Omega_{\Sigma, x}^p \times T_{\Sigma, x}$ be defined as the set of points $([\alpha], v)$ such that $\alpha \in \Lambda^p C(v)$. By definition, it is the pull-back by c' of the set F of Lemma 6.3.11 (with $V = W = T_{\Sigma, x}$). Hence, by this lemma, S' is a Euclidean closed subset of $\mathbb{P} \Omega_{\Sigma, x}^p \times T_{\Sigma, x}$. As it is stable under the action of $\mathbb{C}^*$ on the second factor, it defines a Euclidean closed subset S'' of $\mathbb{P} \Omega_{\Sigma, x}^p \times \mathbb{P} \Omega_{\Sigma, x}^1$. By definition, $\mathcal{S}_x^p$ is the image of S'' by the projection on $\mathbb{P} \Omega_{\Sigma, x}^p$. As $\mathbb{P} \Omega_{\Sigma, x}^1$ is Euclidean compact, $\mathcal{S}_x^p$ is Euclidean closed. As $\mathcal{S}_x^p$ is both constructible and Euclidean closed, it is Zariski closed. $\square$

Now, define $\mathcal{S}_\Sigma^p \subset \mathbb{P}\Omega_\Sigma^p$ as the disjoint union of each $\mathcal{S}_x^p$ for every $x \in \Sigma$. Define $\mathcal{S}_\Sigma^p$ as $H_{\mathbb{C}} \cdot \mathcal{S}_x^p \subset \mathbb{P}\Omega_\Sigma^p$. We have $\mathcal{S}_\Sigma^p = \mathcal{S}_\Sigma^p \cap \mathbb{P}\Omega_\Sigma^p$. Indeed, by Lemma 6.3.12, $\mathcal{S}_x^p$ is stable under the action of $K_{x,\mathbb{C}}$, hence under the action of P_x .

Lemma 6.3.13. *The subset $\mathcal{S}_\Sigma^p$ is a Zariski closed subset and a holomorphic subbundle of $\mathbb{P}\Omega_\Sigma^p$ over $\check{\Sigma}$. In particular, the subset $\mathcal{S}_\Sigma^p$ is a semi-algebraic subset and a closed holomorphic subbundle of $\mathbb{P}\Omega_\Sigma^p$ over Σ .*

Proof. Let $x \in \check{\Sigma}$: we prove that in a neighborhood of x , $\mathcal{S}_\Sigma^p$ is a closed holomorphic subbundle of $\mathbb{P}\Omega_\Sigma^p$ over $\check{\Sigma}$. By applying an element of $H_{\mathbb{C}}$, we can suppose $x \in \Sigma$. Let $\mathfrak{p}_x$ be complex Lie subalgebra of $\mathfrak{h}_{\mathbb{C}}$ defined by the complex Lie subgroup $P_x \subset H_{\mathbb{C}}$. Let $\mathfrak{m}^+$ be a complement of $\mathfrak{p}_x$ in $\mathfrak{h}_{\mathbb{C}}$. There exists an open neighborhood U of 0 in $\mathfrak{m}^+$ such that $v \mapsto \exp(v) \cdot x$ is a biholomorphism from U to an open neighborhood V of x in Σ . This biholomorphism induces a local trivialization $\mathbb{P}\Omega_{\Sigma,x}^p \times U \cong \mathbb{P}\Omega_V^p$ of the holomorphic bundle $\mathbb{P}\Omega_\Sigma^p$ over $\check{\Sigma}$. This trivialization restricts as a bijection $\mathcal{S}_x^p \times U \cong \mathcal{S}_\Sigma^p \times_{\check{\Sigma}} V$. As $\mathcal{S}_x^p$ is Zariski closed in $\mathbb{P}\Omega_{\Sigma,x}^p$ by Lemma 6.3.12, this proves that $\mathcal{S}_\Sigma^p$ is a closed holomorphic subbundle of $\mathbb{P}\Omega_\Sigma^p$.

In particular, it is a closed analytic subset. Moreover, as $H_{\mathbb{C}}$ and $\mathcal{S}_x^p$ are algebraic varieties, $\mathcal{S}_\Sigma^p = H_{\mathbb{C}} \cdot \mathcal{S}_x^p$ is constructible by Chevalley's theorem. Hence it is Zariski closed.

As $i : \Sigma \hookrightarrow \check{\Sigma}$ is a holomorphic open embedding with semi-algebraic image, it is also true for the induced injection $\mathbb{P}\Omega_\Sigma^p \hookrightarrow \mathbb{P}\Omega_{\check{\Sigma}}^p$. Hence the second statement is an immediate consequence of the first one. $\square$

Definition 6.3.14. *The higher degree characteristic subvariety $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is the set of $[\alpha] \in \mathbb{P}\Omega_X^p$ such that there exists $x \in \Sigma$, $v \in T_{\Sigma,x} \setminus \{0\}$ and $\beta \in \Lambda^p T_{\Sigma,x}$ such that α is the image of β in $\Lambda^p T_X$ and such that $\beta \in \Lambda^p C(v)$.*

Recall that $C(v) = \{w \in T_{\Sigma,x}, [\iota_x(\bar{v}), \iota_x(w)] = 0\}$ by definition and $C(v) = \ker \tilde{R}(v, \bar{v})$ by Remark 6.3.9. Remark that $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is the quotient of $\mathcal{S}_\Sigma^p$ by the action of Γ .

Remark 6.3.15. As the Bergman metric is equivariant for the action of G , it descends as Kähler metric g on X . Denote by R its curvature: then $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is the set of $[\alpha] \in \mathbb{P}\Omega_X^p$ such that there exists $v \in T_X$ with $v \neq 0$ such that $\alpha \in \Lambda^p \ker R(v, \bar{v})$. This proves that $\mathcal{S}^1 = \mathcal{S}$ (Mok's characteristic subvariety $\mathcal{S}_{r-1}$).

Proposition 6.3.16. *The subset $\mathcal{S}^p \subset \mathbb{P}\Omega_X^p$ is Zariski closed. It is a holomorphic subbundle of $\mathbb{P}\Omega_X^p$ on X .*

Proof. The second statement and the analytic closure are immediate consequences of Lemma 6.3.13. By o-minimal Chow's lemma, it is sufficient to prove that $\mathcal{S}^p$ is definable in $\mathbb{R}_{\text{an}, \text{exp}}$ to conclude. By Lemma 6.3.13, $\mathcal{S}_\Sigma^p \cap (\mathbb{P}\Omega_\Sigma^p \times_\Sigma \mathfrak{S})$ is semi-algebraic. By definition of $\mathfrak{S}$, its image in $\mathbb{P}\Omega_X^p$ is definable in $\mathbb{R}_{\text{an}, \text{exp}}$ and is equal to the image of $\mathcal{S}_\Sigma^p$. Hence, $\mathcal{S}^p$ is definable in $\mathbb{R}_{\text{an}, \text{exp}}$. $\square$

6.4 Augmented base locus

In this section, we prove Proposition 6.1.3 and Theorems 6.1.4 and 6.1.1.

Proof of Proposition 6.1.3. By definition of the higher degree characteristic variety, the statement of the proposition is an immediate consequence of Proposition 6.2.3. $\square$

Lemma 6.4.1. *In the setting given by Proposition 6.2.4, the tautological line bundle $\mathcal{O}_p(1)$ on $\mathbb{P}\Omega_{\bar{X}'}^p(\log D')$ is nef.*

Proof. The log-Higgs field of $\mathbb{V}$ gives a locally split injection $\Lambda^p T_{\bar{X}'}(-\log D') \hookrightarrow \Lambda^p \text{End}({}^\diamond E)$. Hence it induces a locally split injection $\mathcal{O}_p(-1) \hookrightarrow \pi^* \Lambda^p \text{End}({}^\diamond E)$ with π the projection $\mathbb{P}\Omega_{\bar{X}}^p(\log D') \rightarrow \bar{X}'$. Because of Lemma 4.5.4 (with $b = \text{id}$ and $L = \mathcal{O}_p(-1)$), the Higgs field vanishes on $\mathcal{O}_p(-1)$. Now Theorem 4.5.5 gives our statement. $\square$

Proof of Theorem 6.1.4. Because of Proposition 3.4.6.5, it is sufficient to prove Theorem 6.1.4 up to a finite étale covering map. Hence, we can apply Proposition 6.2.4 and suppose without loss of generality that $X' = X$. The inclusion $\mathbb{B}_+^p(X) \subset \mathcal{S}^p$ is given by Theorem 4.1.5 and Proposition 6.1.3 applied to $\mathbb{V}$.

Denote by $\mathcal{S}^{p, \text{reg}}$ the smooth locus of $\mathcal{S}^p$. Let $x = [\alpha] \in \mathcal{S}^{p, \text{reg}}$. Let $y := \pi(x)$ with π the projection $\mathbb{P}\Omega_X^p(\log D) \rightarrow \bar{X}$. By Proposition 6.3.16, there exists a Euclidean open neighborhood U of y in X and a biholomorphism $b : \mathbb{P}\Omega_U^p \cong \mathbb{P}\Omega_{X,y}^p \times U$ such that $b(\mathcal{S}^p \cap \mathbb{P}\Omega_U^p) = \mathcal{S}_y^p \times U$. This biholomorphism gives decompositions $T_{\mathbb{P}\Omega_X^p, x} \cong T_{\mathbb{P}\Omega_{X,y}^p, x} \oplus T_{X,y}$ and $T_{\mathcal{S}^{p, \text{reg}}, x} \cong T_{\mathcal{S}_y^{p, \text{reg}}, x} \oplus T_{X,y}$.

Consider formula (3.1): the kernel of the alternating bilinear form $\omega := C_1(\mathcal{O}_p(1), \hat{g})_x$ on $T_{\mathbb{P}\Omega_X^p, x}$ is included in $T_{X,y}$. Hence, the dimension of this kernel stays the same after restriction to $\mathcal{S}^{p, \text{reg}}$. In particular, $\omega|_{\mathcal{S}^{p, \text{reg}}}$ is not positive definite at x if and only if ω is not positive definite at x . By Lemma 3.2.12, this is equivalent to the existence of $v \in T_{X,y} \setminus \{0\}$ such that $\alpha \in \Lambda^p \ker R(v, \bar{v})$. By Remark 6.3.9, this is always true, because $x \in \mathcal{S}^p$: thus $\omega|_{\mathcal{S}^{p, \text{reg}}}$ does not have maximal rank at x . Therefore, $\omega|_{\mathcal{S}^{p, \text{reg}}}^{\dim \mathcal{S}^p} = 0$.

Consider a desingularization $d : \hat{\mathcal{S}}^p \rightarrow \overline{\mathcal{S}}^p$, with $\overline{\mathcal{S}}^p$ the Zariski closure of $\mathcal{S}^p$ in $\mathbb{P}\Omega_{\hat{X}}^p(\log D)$. The properties of the (log-) Higgs field given by Proposition 6.2.4 give the hypothesis of [51, Theorem 5.20] (Theorem 5.3.4 above). This theorem gives $\int_{\hat{\mathcal{S}}^p} d^*C_1(\mathcal{O}_p(1), \hat{g})^{\dim \mathcal{S}^p} = \int_{\hat{\mathcal{S}}^p} c_1(d^*\mathcal{O}_p(1))^{\dim \mathcal{S}^p}$. With the foregoing, this implies $\int_{\hat{\mathcal{S}}^p} c_1(d^*\mathcal{O}_p(1))^{\dim \mathcal{S}^p} = 0$, so $\int_{\overline{\mathcal{S}}^p} c_1(\mathcal{O}_p(1))^{\dim \mathcal{S}^p} = 0$. By Lemma 6.4.1 $\mathcal{O}_p(1)$ is nef, so we can apply Theorem 3.2.6, which gives $\mathcal{S}^p \subset \mathbb{B}_+(\mathcal{O}_p(1))$. $\square$

Proof of Theorem 6.1.1. It is an immediate consequence of Theorem 6.1.4 and Subsection 3.2.5. $\square$

- Remarks 6.4.2.* 1. As a consequence of Theorem 6.1.4, we have that for every $x \in X$ and for every $\alpha \in \mathbb{B}_+(\mathcal{O}_p(1))_x$, there exists a vector subspace $E \subset T_{X,x}$ such that $\alpha \in \Lambda^p E$ and the subset $\mathbb{P}E^\vee$ of $\mathbb{P}\Omega_{X,x}^1$ is included in $\mathbb{B}_+(\mathcal{O}_1(1))$.
2. The l of Theorem 6.1.1 is also the maximum dimension of boundary components of Σ (see [49, Theorem 6.8.17]). The values of l are computed in [89] (the following table comes from [49, p.329]).

	Domain	Dimension	l
I	$SU(p, q)/S(U(p) \times U(q))$	pq	$(p-1)(q-1)$
II	$Sp(m, \mathbb{R})/U(m)$	$\frac{m(m+1)}{2}$	$\frac{(m-1)m}{2}$
III	$SO^*(2m)/U(m)$	$\frac{m(m-1)}{2}$	$\frac{(m-2)(m-3)}{2}$
IV	$SO_0(m, 2)/(SO(m) \times SO(2))$	m	1
V	$E_6/(SO(10) \cdot SO(2))$	16	1
VI	$E_7/(E_6 \cdot SO(2))$	27	8

3. For the compact case, the fact that Ω_X^p is ample if $p > l$ is a consequence of [68, Theorem 3.4], formula (3.1) and Kodaira's embedding theorem.
4. In the computation of the augmented base loci, it is possible not to use Theorem 4.1.5 and to use the canonical metric instead of the Hodge metric (see [49, Chapter 6.8]). However, for the non compact case, there are two points where we do not know how to avoid the use of variational Hodge theory: to prove the nefness of the tautological line bundle, and to use Chern-Weil theory in the proof of Theorem 6.1.4.

6.5 Stable base locus

In this section, we compute $\mathbb{B}^1(X)$ (Theorem 6.1.5). We begin with a technical lemma.

Lemma 6.5.1. *Let $r \in (0, e^{-1})$.*

- *Let f be a smooth positive real-valued function over $(\Delta_r^*)^d$ such that f^{-1} has at most log growth.*
- *Let a be a holomorphic function over Δ_r^d such that $\{a = 0\} \subset \chi$.*

Let $g := |a|^2$. Then there exists $r' \in (0, r)$ such that $\frac{\partial g}{f+g}$ is nearly bounded on $(\Delta_{r'}^)^d$.*

Proof. Let (z_i) be the coordinates of Δ_r^d . Because $\{a = 0\} \subset \chi$, there exist nonnegative integers $(n_i) \in \mathbb{N}^d$ and a non-vanishing holomorphic function α over Δ_r^d such that:

$$a = \alpha \prod_{i=1}^d z_i^{n_i}. \quad (6.4)$$

Then:

$$\frac{\partial g}{g} = \frac{\partial \alpha}{\alpha} + \sum_{i=1}^d n_i \frac{\partial z_i}{z_i},$$

hence:

$$\frac{\partial g}{f+g} = \frac{g}{f+g} \frac{\partial \alpha}{\alpha} + \sum_{i=1}^d n_i \frac{g}{f+g} \frac{\partial z_i}{z_i}. \quad (6.5)$$

Consider the first term of (6.5). As f and g are positive real-valued on $(\Delta_r^*)^d$, $\frac{g}{f+g}$ is bounded (by 1) on it. As α is non-vanishing and continuous over Δ_r^d and $\partial \alpha$ is continuous over it, for each $r' \in (0, r)$, $\frac{\partial \alpha}{\alpha}$ is bounded on $(\Delta_{r'}^*)^d$. Hence $\frac{g}{f+g} \frac{\partial \alpha}{\alpha}$ is bounded on it too.

So, going back to (6.5), it remains to check that for each i such that $n_i > 0$, there exists $r' \in (0, r)$ such that on $(\Delta_{r'}^*)^d$, $\frac{g}{f+g} \frac{\partial z_i}{z_i}$ is a nearly bounded form. We have:

$$\frac{g}{f+g} \frac{\partial z_i}{z_i} = \frac{g(-\ln|z_i|)}{f+g} \frac{\partial z_i}{z_i(-\ln|z_i|)}.$$

As the factor $\frac{\partial z_i}{z_i(-\ln|z_i|)}$ has Poincaré growth, it remains to check that there exists $r' \in (0, r)$ such that on $(\Delta_{r'}^*)^d$, the positive real-valued function

$$b := \frac{g(-\ln|z_i|)}{f+g}$$

is nearly bounded.

To do this, we first need to give the values of the constants C and ε that appear in Definition 5.2.1 of nearly bounded functions, and the radius $r' \in (0, r)$ of the polydisk which we restrict ourselves to. Without loss of

generality, we consider the index $i = 1$ and suppose $n_1 > 0$. Because f^{-1} has at most log growth, there are $A \in \mathbb{R}_{>0}, k \in \mathbb{N}_{>0}$ such that for every $z \in (\Delta_r^*)^d$:

$$f(z) \geq A \prod_{i=1}^d (-\ln|z_i|)^{-k}. \quad (6.6)$$

Let:

$$\varepsilon := \frac{n_1}{dk} > 0. \quad (6.7)$$

The map:

$$\begin{aligned} \xi : (0, r] &\rightarrow \mathbb{R}_{>0} \\ h &\mapsto (-\ln(h) - 1)^{-\frac{1}{dk}} h^{-\varepsilon} \end{aligned}$$

is continuous, and $\lim_{h \rightarrow 0} \xi(h) = +\infty$. Hence ξ is bounded below by a constant > 0 . Moreover, choose any $r'' \in (0, r)$. As $|\alpha(z)|^2$ is positive real-valued continuous on Δ_r^d , $|\alpha(z)|^{-\frac{2}{dk}}$ is bounded below by a constant > 0 on $\Delta_{r''}^d$. Therefore, we can define the following positive real constant.

$$C := \inf_{z \in (\Delta_{r''}^*)^d} (A^{-1}|\alpha(z)|^2(-\ln|z_1| - 1))^{-\frac{1}{dk}} |z_1|^{-\varepsilon} > 0. \quad (6.8)$$

Finally, we choose $r' \in (0, r'')$ small enough so that for every $h \in (0, r')$, $Ch^{-\varepsilon} > -\ln h$.

Now that we have our two constants C and ε and our radius r' , we can check that b is $(C, \varepsilon, 1, 1)$ -nearly bounded on $(\Delta_{r'}^*)^d$. First, for each $z \in (\Delta_{r'}^*)^d$, we have $b(z) \leq -\ln|z_1|$, in particular it is true for $z \in T_1^{C, \varepsilon}$.

Now, let $z \in (\Delta_{r'}^*)^d \setminus T_1^{C, \varepsilon}$, which means that for every $1 \leq i \leq d$, $\ln|z_i| > -C|z_1|^{-\varepsilon}$. Moreover, we chose r' so that we have $\ln|z_1| > -C|z_1|^{-\varepsilon}$ too. Then, for $1 \leq i \leq d$:

$$\ln|z_i| > -C|z_1|^{-\varepsilon} \quad (6.9)$$

$$\geq -\left(A^{-1}|\alpha(z)|^2(-\ln|z_1| - 1)|z_1|^{2n_1}\right)^{-\frac{1}{dk}} \quad (6.10)$$

$$\geq -\left(A^{-1}|\alpha(z)|^2(-\ln|z_1| - 1) \prod_{j=1}^d |z_j|^{2n_j}\right)^{-\frac{1}{dk}}, \quad (6.11)$$

using the definition (6.8) of C and the definition (6.7) of ε for the line (6.10), and the fact that for every $1 \leq j \leq d$, $|z_j| < 1$ for the line (6.11). Therefore:

$$(-\ln|z_i|)^{-k} > \left(A^{-1}|\alpha(z)|^2(-\ln|z_1| - 1) \prod_{j=1}^d |z_j|^{2n_j}\right)^{\frac{1}{d}} \quad (6.12)$$

Hence:

$$f(z) \geq A \prod_{i=1}^d (-\ln|z_i|)^{-k} \quad (6.13)$$

$$> (-\ln|z_1| - 1) |\alpha(z)|^2 \prod_{i=1}^d |z_i|^{2n_i} \quad (6.14)$$

$$= (-\ln|z_1| - 1) g(z), \quad (6.15)$$

using (6.6) for the line (6.13), (6.12) for the line (6.14) and (6.4) for the line (6.15). Eventually, we get $f(z) + g(z) > (-\ln|z_1|)g(z)$. Hence by definition, b is bounded (by 1) outside of $T_1^{C,\varepsilon}$. $\square$

Let X be a locally symmetric variety whose universal cover is irreducible and of rank at least 2. Let Σ be its universal cover. We denote by $\pi : \mathbb{P}\Omega_X^1 \rightarrow X$ the canonical projection. Let $\mathcal{O}(1)$ be the tautological line bundle over $\mathbb{P}\Omega_X^1$, $\mathcal{O}(p) := \mathcal{O}(1)^{\otimes p}$ its tensor power (for $p \in \mathbb{Z}$), $\hat{g}$ the Hermitian metric induced on $\mathcal{O}(-1)$ by the canonical Kähler metric g on X , quotient of the Bergman metric of Σ . Let ν be a Kähler form on $\mathbb{P}\Omega_X^1$ (for example, the sum of $C_1(\mathcal{O}(-1), \hat{g})$ and of the canonical Kähler form over X). In the following, we will omit $\mathcal{O}(-1)$ in the notations of first Chern forms C_1 .

Let $p \in \mathbb{N}_{>0}$, let σ be a global section of $\mathcal{O}(p)$ over $\mathbb{P}\Omega_X^1$, suppose that it comes from a global section of $\text{Sym}^p \Omega_{\bar{X}}^1(\log D)$ for some log-compactification $(\bar{X}, D)$ of X (it does not depend on the choice of the compactification). We have to prove that σ vanishes on $\mathcal{S}$ (recall that $\mathcal{S} := \mathcal{S}_{r-1}$). Remark that $\mu := (\hat{g}^p + \sigma \otimes \bar{\sigma})^{\frac{1}{p}}$ defines an Hermitian metric of seminegative curvature on $\mathcal{O}(-1)$ (see [58, Proof of Proposition 2 p.260]). Let $u := \ln\left(\frac{\mu}{\hat{g}}\right)$. Then $\varphi := \partial u$ is the difference between the Chern connections of μ and $\hat{g}$, it is a differential form of type $(1, 0)$ defined over $\mathbb{P}\Omega_X^1$ (which does not depend on the choice of local trivializations). Fix a k such that $1 \leq k < r$, let $\iota : \mathcal{S}_k^{\text{reg}} \hookrightarrow \mathbb{P}\Omega_X^1$ be the inclusion. Let $\beta := C_1(\hat{g})^{2d-2n(k)-1} \wedge \nu^{n(k)-1}$, with $n(k)$ defined in Subsection 6.3.1.

Lemma 6.5.2. *We have:*

$$\iota^*(\bar{\varphi} \wedge \varphi \wedge \beta) = 0.$$

Proof. It is obviously sufficient to prove the lemma for a finite étale covering of X . Then we can apply Proposition 6.2.4: we have a PVHS $(\mathbf{H}_{\mathbb{R}}, \mathbf{S}, \mathbf{F}^\bullet)$ over X and a log-compactification $(\bar{X}, D)$ of X such that $\mathbf{H}$ has unipotent monodromy and there is a locally split injective morphism of holomorphic

complex vector bundles $T_{\bar{X}}(-\log D) \rightarrow {}^\circ\mathbf{F}^{-1}/{}^\circ\mathbf{F}^0$ such that over X , the pull-back of the Hodge metric is the metric induced by g .

We extend π as the canonical projection $\mathbb{P}\Omega_{\bar{V}}^1(\log D) \rightarrow \bar{X}$ and $\mathcal{O}(1)$ as the tautological line bundle over it. By definition, σ is (the restriction of) a global section of $\mathcal{O}(p)$ over $\mathbb{P}\Omega_{\bar{V}}^1(\log D)$. We denote $D' := \pi^{-1}D$, hence $(\mathbb{P}\Omega_{\bar{V}}^1(\log D), D')$ is a log-compactification of $\mathbb{P}\Omega_X^1$.

Let $\bar{\mathcal{S}}_k$ be the closure of $\mathcal{S}_k$ in $\mathbb{P}\Omega_{\bar{V}}^1(\log D)$ for the Euclidean topology. Thanks to Proposition 6.3.8, $\bar{\mathcal{S}}_k$ is a Zariski closed subvariety of $\mathbb{P}\Omega_{\bar{V}}^1(\log D)$, and its intersection with $\mathbb{P}\Omega_X^1$ is $\mathcal{S}_k$. Hironaka's desingularization theorem gives us a smooth projective variety A and a proper birational morphism $b : A \rightarrow \bar{\mathcal{S}}_k$ such that $D_1 := b^{-1}(\{\sigma = 0\} \cup D')$ is a normal crossing divisor. We now work on A and, in order to lighten notations, we will forget the mention of “ $b^*\iota^*$ ” before $\mathcal{O}_{\mathbb{P}\Omega_{\bar{V}}^1(\log D)}$ -modules and differential forms.

Let us show that $\varphi \wedge \beta$ is almost bounded and $u\varphi \wedge \beta$ is almost quasibounded. Let $r \in (0, e^{-1})$, let $f : \Delta_r^d \rightarrow A$ be an admissible chart on (A, D_1) . Let s be a trivializing section of $\mathcal{O}(-1)$ defined in a neighborhood of $f(0)$. Thanks to Proposition 5.3.3 (“Connection form of the Hodge metric on a subquotient of a PVHS with unipotent monodromy near a normal crossing divisor is almost bounded”), there exists $r' \in (0, r)$ such that s is defined on $f(\Delta_{r'}^d)$ and the pull-backs of $\partial \ln \|s\|_{\hat{g}}^2$ and $C_1(\hat{g})$ on $(\Delta_{r'}^d)^d$ are pre-almost bounded. There exists a divisor E on $\Delta_{r'}^d$, such that the pull-backs of $f^*\partial \ln \|s\|_{\hat{g}}^2$ and $f^*C_1(\hat{g})$ by any desingularization of E are locally nearly bounded. Let $g : (M, D_M) \rightarrow (\Delta_{r'}^d, \chi)$ be a desingularization of E . Let $t \in (0, e^{-1})$, let $f_1 : \Delta_t^d \rightarrow M$ be an admissible chart on (M, D_M) . Then, there is $t' \in (0, t)$ such that the restrictions of $f_1^*g^*f^*\partial \ln \|s\|_{\hat{g}}^2$ and $f_1^*g^*f^*C_1(\hat{g})$ to $(\Delta_{t'}^d)^d$ are nearly bounded. It remains to show that $f_1^*g^*f^*(\varphi \wedge \beta)$ is nearly bounded and $f_1^*g^*f^*(u\varphi \wedge \beta)$ is quasibounded on $(\Delta_{t'}^d)^d$. Now, we work on $\Delta_{t'}^d$, and we forget to mention “ $f_1^*g^*f^*$ ” before differential forms.

We have:

$$\begin{aligned} \varphi &= \partial \ln \left(\frac{\|s\|_{\mu}^2}{\|s\|_{\hat{g}}^2} \right) \\ &= \frac{1}{p} \partial \ln (\|s\|_{\hat{g}}^{2p} + |\sigma(s^p)|^2) - \partial \ln (\|s\|_{\hat{g}}^2) \end{aligned}$$

The right term is nearly bounded. Hence, it remains to show that the left term is nearly bounded. We have:

$$\partial \ln (\|s\|_{\hat{g}}^{2p} + |\sigma(s^p)|^2) = \frac{\|s\|_{\hat{g}}^{2p}}{\|s\|_{\hat{g}}^{2p} + |\sigma(s^p)|^2} \partial \ln (\|s\|_{\hat{g}}^{2p}) + \frac{\partial (|\sigma(s^p)|^2)}{\|s\|_{\hat{g}}^{2p} + |\sigma(s^p)|^2}.$$

The first term is nearly bounded because $\frac{\|s\|_{\hat{g}}^{2p}}{\|s\|_{\hat{g}}^{2p} + |\sigma(s^p)|^2}$ is bounded by 1 and $\partial \ln \|s\|_{\hat{g}}^2$ is nearly bounded. Moreover, we can pull back the PVHS and apply Theorem 5.3.1 (“Hodge metric has at most log growth at singularities of PVHS with unipotent monodromy”) to obtain that for each $i \in \mathbb{Z}$, $\|s\|_{\hat{g}}^i$ has at most log growth. Then Lemma 6.5.1 shows that the second term is nearly bounded. Therefore φ is nearly bounded and, as β too, $\varphi \wedge \beta$ is nearly bounded. Moreover, as for each $i \in \mathbb{Z}$, $\|s\|_{\hat{g}}^i$ has at most log growth, so u has at most log-log growth. Hence, on A , $f^*(\varphi \wedge \beta)$ is almost bounded and $f^*(u\varphi \wedge \beta)$ is almost quasibounded.

Because of $d\varphi = C_1(\mu) - C_1(\hat{g})$ and Proposition 6.3.6 (“on $\mathcal{S}_k^{\text{reg}}$, $C_1(\hat{g})$ has rank $2d - 2n(k) - 1$ ”), we have $df^*(\varphi \wedge \beta) = f^*(C_1(\mu) \wedge \beta)$, which is a semipositive form. Indeed, as the Bergman metric g has a seminegative Griffiths curvature, the formula (4.2) implies that the first Chern form $C_1(\mathcal{O}(1), \hat{g})$ is semipositive. Hence, Proposition 5.2.12 gives us $\int_A df^*(\varphi \wedge \beta) = 0$, and by positivity, $df^*(\varphi \wedge \beta) = 0$. Hence by Leibniz formula: $df^*(u\varphi \wedge \beta) = f^*(\bar{\varphi} \wedge \varphi \wedge \beta)$. As the first term is almost quasibounded and the second term is clearly semipositive, we obtain in the same way $f^*(\bar{\varphi} \wedge \varphi \wedge \beta) = 0$. Thus $\iota^*(\bar{\varphi} \wedge \varphi \wedge \beta) = 0$. $\square$

In the previous proof, we could not simply use the fact that g is good (in the sense of [62]) because this property is not necessarily shared by $\hat{g}$ (see [39, Example 3.9]).

Now that we have the previous lemma, the proof of the inclusion $\mathcal{S} \subset \mathbb{B}^1(X)$ is exactly the same as Mok’s one in the compact case. First [58, Chapter 6 (3.2)] gives the following lemma.

Lemma 6.5.3. *The pull-back of u over $\mathcal{S}_{r-1}(\Sigma) \subset \mathbb{P}\Omega_{\Sigma}^1$ is invariant under the action of G .*

Then the proof of [58, Appendix 4 Proposition 4] gives the inclusion we want.

Lemma 6.5.4. *We have $\mathcal{S} \subset \mathbb{B}^1(X)$.*

We finally have all the inclusions of Theorem 6.1.5, which is the equality $\mathbb{B}^1(X) = \mathbb{B}_+^1(X) = \mathcal{S}$.

Proof of Theorem 6.1.5. The inclusion $\mathcal{S} \subset \mathbb{B}^1(X)$ is the previous lemma. The inclusion $\mathbb{B}^1(X) \subset \mathbb{B}_+^1(X)$ is true for every smooth complex algebraic variety. The inclusion $\mathbb{B}_+^1(X) \subset \mathcal{S}$ is given by Theorem 6.1.4. $\square$

6.6 Application: isotriviality of families of curves

In this section, we prove Theorem 6.1.6. Consider an integer $g \geq 2$. Let V be a smooth complex quasiprojective variety with a projective log-compactification (W, D) . The ampleness of $\Omega_W^1(\log D)$ with respect to V is equivalent to the emptiness of $\mathbb{B}_+^1(V)$, so we have the following proposition.

Proposition 6.6.1 (see [87, Proposition 2.6]). *Suppose there is a smooth projective family $f : C \rightarrow V$ of curves of genus g such that the Kodaira-Spencer map $T_V \rightarrow R^1 f_* T_{C/V}$ is a locally split injection. Then $\mathbb{B}_+^1(V)$ is empty.*

Lemma 6.6.2. *Let M be a smooth complex quasiprojective variety with a finite étale covering map $M \rightarrow \mathcal{M}_g \otimes \mathbb{C}$. Then $\mathbb{B}_+^1(M)$ is empty. In particular $\mathbb{B}^1(M)$ is empty.*

Proof. According to Proposition 3.4.6.3, the conclusion does not depend on the choice of M , hence we can choose the moduli space of connected smooth projective curves of genus g with a 3-level structure to suppose M quasiprojective. Then the previous proposition with C the pull-back of the universal family gives $\mathbb{B}_+^1(M) = \emptyset$, and we have $\mathbb{B}^1(M) \subset \mathbb{B}_+^1(M)$. $\square$

Proof of Theorem 6.1.6. Let M be a quasiprojective smooth complex algebraic variety with a finite étale covering map $M \rightarrow \mathcal{M}_g(\mathbb{C})$. A smooth projective family of curves of genus $g \geq 2$ over X induces by base change a map $h : X' \rightarrow M$ where X' is a finite étale covering of X . Hence X' verifies the same hypotheses as X . Theorem 6.1.5 and the definition of $\mathcal{S}$ give that $\mathbb{B}^1(X')$ is linearly non-degenerate (see Definition 3.4.2). Moreover the previous lemma gives that $\mathbb{B}^1(M)$ is empty. Hence Proposition 3.4.3 applied to h (in characteristic 0, the separability hypothesis is always verified) gives the conclusion. $\square$

This theorem has been proved in the compact case in [52, Corollary 1.6]. Their proof relies on the use of Mok's metric rigidity theorem ([60, Theorem 1]). There actually does not seem to be any obstruction to apply the generalized version of To [82, Theorem 1] of the metric rigidity theorem to keep the same proof for the non compact case.

Hence, the usefulness of Theorem 6.1.5 and Proposition 3.4.3 over $\mathbb{C}$ would be revealed in cases where we consider morphisms $X \rightarrow V$ whose target has an empty stable (or augmented) base locus, but does not have an obvious metric of Griffiths seminegative curvature.

Chapter 7

Applications in positive characteristic

7.1 Introduction

We give some applications of the computations of $\mathbb{B}_+^p$ and $\mathbb{B}^1$ of the previous chapters in positive characteristic. For $g, n \in \mathbb{N}$ with $n \geq 3$, the moduli space $\mathcal{A}_g(n)$ of principally polarized Abelian varieties of dimension g with a level- n structure is a scheme of finite type over $\mathrm{Spec}(R_n)$ with $R_n := \mathbb{Z}[1/n, e^{2i\pi/n}]$. The complex variety $\mathcal{A}_g(n) \otimes_{R_n} \mathbb{C}$ is locally symmetric: its universal cover is biholomorphic to the g -th Siegel upper half-space. In particular, this universal cover is irreducible. If moreover $g \geq 2$, its rank is at least 2. Thus, we can apply the results of Chapter 6 to $\mathcal{A}_g(n) \otimes_{R_n} \mathbb{C}$ and apply the spreading-out properties of Chapter 3 to compute the base loci of $\mathcal{A}_g(n) \otimes_{R_n} k$ with k a field of positive characteristic.

Using the (“non-hyperbolicity”) inclusion $\mathbb{B}^1(X) \subset \mathcal{S}$ given by Theorem 6.1.5 and the functoriality of $\mathbb{B}^1$, we will obtain the following generalization of Theorem 6.1.6 for $\mathcal{A}_g(n)$. Consider a field k and a k -variety V . For $g \geq 2$ an integer, we call *family of smooth projective curves* of genus g over V the data of a k -variety W and a smooth projective morphism $W \rightarrow V$ whose geometric fibers are connected smooth projective curves of genus g . Let $\mathcal{M}_g$ be the moduli stack of connected smooth projective curves of genus g . The data of an isomorphism class of families of smooth projective curves over V is equivalent to the data of a k -morphism $V \rightarrow \mathcal{M}_g \otimes_{\mathbb{Z}} k$.

If V is geometrically integral, a morphism f from V to a Deligne-Mumford stack $\mathcal{M}$ of finite type over k is *separable* if for every integral k -variety V' (resp. M), for every dominant étale morphism $p : V' \rightarrow V$ (resp. $q : M \rightarrow$

$\mathcal{M}$) and every lift $f' : V' \rightarrow M$ of f , the corestriction $f' : V' \rightarrow \overline{\text{Im } f'}$ is separable. This is equivalent to the existence of such V', M, p, q and f' that realize this condition. A family of smooth projective curves of genus g over V is *separable* if the associated morphism $f : V \rightarrow \mathcal{M}_g \otimes_{\mathbb{Z}} k$ is separable.

Let M_g be the coarse moduli space associated with $\mathcal{M}_g$. A family of smooth projective curves of genus g over V is *isotrivial* if the composition of the morphism $V \rightarrow \mathcal{M}_g \otimes_{\mathbb{Z}} k$ associated with the family of smooth projective curves and of the natural morphism $\mathcal{M}_g \otimes_{\mathbb{Z}} k \rightarrow M_g \otimes_{\mathbb{Z}} k$ is constant.

Theorem 7.1.1. *For every integers $g, h \geq 2, n \geq 3$, for all but finitely many prime numbers p , for every field k of characteristic 0 or p with a primitive n -th root of unity, every separable family of smooth projective curves of genus h over $\mathcal{A}_g(n) \otimes_{R_n} k$ is isotrivial.*

Using the (“hyperbolicity”) inclusion $\mathcal{S}^p \subset \mathbb{B}_+^p$ given by Theorem 6.1.4 and the finiteness Theorem 3.5.3, we obtain the theorem below. If V is a smooth k -variety, consider a principally polarized Abelian V -scheme $\pi : A \rightarrow V$ of relative dimension $g \in \mathbb{N}$. In this case, there is a canonical isomorphism $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\text{Lie}}_{A/V}^{\otimes 2}$ with $\underline{\text{Lie}}_{A/V}$ the vector bundle of rank g defined by the Lie algebra of fibers of π . Moreover, if the characteristic of k is not 2, if we denote by KS the Kodaira-Spencer map $T_V \rightarrow R^1\pi_*T_{A/V}$, then the image of $\beta \circ \text{KS}$ is in the direct factor $\text{Sym}^2 \underline{\text{Lie}}_{A/V}$ of $\underline{\text{Lie}}_{A/V}^{\otimes 2}$ (see [2, Section 1]).

Theorem 7.1.2. *Let $g \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

*Let V be a smooth k -variety. Then for all but finitely many principally polarized Abelian V -schemes $\pi : A \rightarrow V$ of relative dimension g , for every closed point $x \in V$, there exists a vector subspace $S \subset \text{Lie } A_x$ of dimension $g-1$ such that $\text{Im}(\beta_x \circ \text{KS}_x) \subset \text{Sym}^2 S$, with $\text{KS} : T_V \rightarrow R^1\pi_*T_{A/V}$ the Kodaira-Spencer map and $\beta : R^1\pi_*T_{A/V} \xrightarrow{\sim} \underline{\text{Lie}}_{A/V}^{\otimes 2}$ the canonical isomorphism.*

The previous property that we obtain (almost surely) for A gives in particular a bound on the rank of KS. This bound is also the last degree p for which $\Omega_{\mathcal{A}_g(n)}^p(\log D)$ (for a projective log-compactification $(\bar{\mathcal{A}}_g(n), D)$ of $\mathcal{A}_g(n)$) is not ample with respect to $\mathcal{A}_g(n)$: see Theorem 6.1.1 and Remarks 6.4.2.2 (type II). We denote by $\mathcal{A}_g$ the Deligne-Mumford stack of principally polarized Abelian varieties of dimension g .

Corollary 7.1.3. *Let $g \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

Let V be a smooth k -variety. Then for all but finitely many principally polarized Abelian V -schemes $\pi : A \rightarrow V$ of relative dimension g , the rank of the Kodaira-Spencer map is bounded above by $\frac{g(g-1)}{2}$.

As the dimension of the image of a separable morphism is equal to the generic rank of the differential, we obtain the following corollary.

Corollary 7.1.4. *Let $g \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

Let V be a reduced k -variety. Then there exist only finitely many principally polarized Abelian V -schemes $\pi : A \rightarrow V$ of relative dimension g such that the morphism $V \rightarrow \mathcal{A}_g \otimes_{\mathbb{Z}} k$ associated with A is separable and the dimension of the closure of its image is more than $\frac{g(g-1)}{2}$.

In this last corollary, the $\frac{g(g-1)}{2}$ is optimal. Indeed, by taking the product of Abelian schemes, we obtain a family of finite morphisms $\mathcal{A}_{g-1}(3) \rightarrow \mathcal{A}_g$ parameterized by $\mathcal{A}_1$ ($\mathcal{A}_{g-1}(3)$ has dimension $\frac{g(g-1)}{2}$).

Also with $\mathcal{S}^p \subset \mathbb{B}_+^p$ and Theorem 3.5.3, applied this time to the period domain of K3 surfaces, we get analogous theorem and corollaries for families of K3 surfaces (see Subsection 7.3.2 for the definitions).

Theorem 7.1.5. *Let $d \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

*Let V be a connected smooth k -variety. Then for all but finitely many polarized families $\pi : S \rightarrow V$ of K3 surfaces such that the polarizations of the fibers of π are primitive of degree $2d$, either $\text{KS}_{S/V}$ is zero or there exists an isotropic invertible $\mathcal{O}_V$ -submodule L of the primitive part $P_{S/V}^{1,1}$ of $R^1\pi_*T_{S/V}$ such that $\text{Im}(\text{KS}_{S/V}) \subset L$.*

Corollary 7.1.6. *Let $d \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

Let V be a smooth k -variety. Then for all but finitely many polarized families $\pi : S \rightarrow V$ of K3 surfaces such that the polarizations of the fibers of π are primitive of degree $2d$, the rank of the Kodaira-Spencer map is bounded above by 1.

Denote by $\mathcal{M}_d$ the fine moduli space of K3 surfaces with a primitive polarization of degree $2d$ ($d \in \mathbb{N}_{>0}$). It is defined over $R_d := \mathbb{Z}[1/(2d)]$.

Corollary 7.1.7. *Let $d \in \mathbb{N}_{>0}$. In all but finitely many positive characteristics and in characteristic 0, for every algebraically closed field k of this characteristic, the following holds.*

Let V be a reduced k -variety. Then there exist only finitely many separable morphisms $f : V \rightarrow \mathcal{M}_d \otimes_{R_d} k$ such that the dimension of the closure of the image of f is at least 2.

The proof of Theorems 7.1.2 and 7.1.5 is based on a spreading-out argument. To apply this argument, it is necessary to have a moduli space defined in mixed characteristic. Moreover, to apply the previous chapters, it is necessary to have an infinitesimal Torelli theorem over $\mathbb{C}$. We give a general theorem for families of varieties that satisfy these two conditions (see Theorem 7.3.14 for the statement). This theorem generalizes Theorems 7.1.2 and 7.1.5.

7.2 Isotriviality of families of curves parametrized by the moduli space of Abelian varieties

We prove Theorem 7.1.1. For $h \geq 2$, recall that $\mathcal{M}_h$ is the moduli stack of connected smooth projective curves of genus h . There exist a subring R' of $\mathbb{Q}$ finite over $\mathbb{Z}$, a scheme $\bar{M}$ smooth and proper over $\mathrm{Spec}(R')$, the complement $M \subset \bar{M}$ of a normal crossing divisor and a finite étale covering map $M \rightarrow \mathcal{M}_h \otimes_{\mathbb{Z}} R'$. For example, we can take $R' = \mathbb{Z}[1/3]$ and M the moduli space of connected smooth projective curves of genus h with a convenient non-Abelian level structure, see [72].

Proof (of Theorem 7.1.1). Thanks to Lemma 6.6.2, $\mathbb{B}^1(M \otimes_{R'} \mathbb{C}) = \emptyset$, hence $\mathrm{Bs}_m^1(M \otimes_{R'} \mathbb{C}) = \emptyset$ for some $m \in \mathbb{N}_{>0}$. Thanks to Proposition 3.4.7 and Corollary 3.4.10, for all but finitely many prime numbers p , for every field k of characteristic 0 or p , $\mathrm{Bs}_m^1(M \otimes_{R'} k) = \emptyset$.

As the universal covering of the smooth complex algebraic variety $\mathcal{A}_g(n) \otimes_{R_n} \mathbb{C}$ is biholomorphic to the g -th Siegel upper half-space, a symmetric variety of type II, Theorem 6.1.5 implies that $\mathbb{B}^1(\mathcal{A}_g(n) \otimes_{R_n} \mathbb{C})$ is linearly non-degenerate. Thus $\mathrm{Bs}_a^1(\mathcal{A}_g(n) \otimes_{R_n} \mathbb{C})$ is linearly non-degenerate for every $a \leq md!$, with d the degree of the finite étale covering map $M \rightarrow \mathcal{M}_h \otimes_{\mathbb{Z}} R'$. Thanks to Proposition 3.4.7 and Corollary 3.4.10, there exists a finite set P of prime numbers such that for every field k of characteristic not in P with a primitive n -th root of unity, $\mathrm{Bs}_a^1(\mathcal{A}_g(n) \otimes_{R_n} k)$ is linearly non-degenerate for every $a \leq md!$.

Let k be a field of characteristic not in P with a primitive n -th root of unity (which defines a morphism $R_n \rightarrow k$). Consider a separable family

of smooth projective curves over $\mathcal{A}_g(n) \otimes_{R_n} k$ associated with a morphism $\mathcal{A}_g(n) \otimes_{R_n} k \rightarrow \mathcal{M}_h \otimes_{\mathbb{Z}} k$. It lifts as a morphism of k -varieties $f : C \rightarrow M \otimes_{R'} k$ with C a finite étale covering of $\mathcal{A}_g(n) \otimes_{R_n} k$ of degree at most d . By Proposition 3.4.6, $\text{Bs}_m^p(C)$ is linearly non-degenerate. We conclude with Proposition 3.4.3. $\square$

7.3 Infinitesimal Arakelov's theorems

7.3.1 Families of Abelian varieties

We prove Theorem 7.1.2 by applying Corollary 3.5.7. We start by looking at $\mathbb{B}_+^p$ in the complex case.

Lemma 7.3.1. *Let $g \in \mathbb{N}_{>0}$. Let V be a smooth complex quasiprojective variety. Let $\pi : A \rightarrow V$ be a principally polarized Abelian V -scheme of relative dimension g . Assume the period map of the canonical $\mathbb{Z}$ -PVHS on $R^1\pi_*\mathbb{Z}_A$ is immersive. Then for every $p \in \mathbb{N}$ with $1 \leq p \leq \dim V$, for every closed point $x \in V$, for every $\alpha \in \mathbb{B}_+^p(V)_x$, there exists a vector subspace $S \subset \text{Lie } A_x$ of dimension $g - 1$ such that $\Lambda^p(\beta \circ \text{KS})(\alpha) \in \Lambda^p \text{Sym}^2 S$.*

Proof. The Higgs bundle associated with the canonical $\mathbb{C}$ -PVHS on the $\mathbb{C}$ -local system $R^1\pi_*\mathbb{C}_A$ is $\pi_*\Omega_{A/V}^1 \oplus R^1\pi_*\mathcal{O}_A$. The Higgs field θ is given by the cup-product with the Kodaira-Spencer map $T_V \rightarrow R^1\pi_*T_{A/V}$. Let $x \in V$ be a closed point, let $\alpha \in \mathbb{B}_+^p(V)$ be above x . As the period map of $R^1\pi_*\mathbb{Z}_A$ is immersive, we can apply Theorem 4.1.5: there exists a non-zero linear map $h : (R^1\pi_*\mathcal{O}_A)_x \rightarrow (\pi_*\Omega_{A/V}^1)_x$ and a vector subspace $H \subset T_{V,x}$ such that $\alpha \in \Lambda^p H$ and for every $v \in H$, $\theta(v)$ and h commute as endomorphisms of $(\pi_*\Omega_{A/V}^1)_x \oplus (R^1\pi_*\mathcal{O}_A)_x$.

There are canonical isomorphisms:

- $\underline{\text{Lie}}_{A/V}^\vee \cong \pi_*\Omega_{A/V}^1$,
- $\underline{\text{Lie}}_{A/V} \cong R^1\pi_*\mathcal{O}_A$,
- $\beta^{-1} : \underline{\text{Lie}}_{A/V}^{\otimes 2} \xrightarrow{\sim} R^1\pi_*T_{A/V}$,

that identify the tensor contraction $\underline{\text{Lie}}_{A/V}^\vee \otimes_{\mathcal{O}_V} \underline{\text{Lie}}_{A/V}^{\otimes 2} \rightarrow \underline{\text{Lie}}_{A/V}$ with the cup-product $\pi_*\Omega_{A/V}^1 \otimes_{\mathcal{O}_V} R^1\pi_*T_{A/V} \rightarrow R^1\pi_*\mathcal{O}_A$ (see [2, Section 1]).

Using these isomorphisms, h gives a non-zero element $f : \text{Lie } A_x \rightarrow (\text{Lie } A_x)^\vee$. Denote by $F \subset \text{Sym}^2 \text{Lie } A_x$ the image of H by $\beta_x \circ \text{KS}_x$. Then we have $\Lambda^p(\beta \circ \text{KS})(\alpha) \in \Lambda^p F$ and for every $s \in F$, if we denote by $\tilde{s}$ the contraction $(\text{Lie } A_x)^\vee \rightarrow \text{Lie } A_x$ defined by s , then $\tilde{s} \circ f = 0$ and $f \circ \tilde{s} = 0$.

In particular the image of $\tilde{s}$ is in $\ker f$. Let S be a vector space of dimension $g - 1$ such that $\ker f \subset S \subsetneq \operatorname{Lie} A_x$. Then $s \in \operatorname{Sym}^2 \ker f \subset \operatorname{Sym}^2 S$. $\square$

Denote by $\mathcal{A}_g(3)$ the moduli space of principally polarized Abelian varieties with a level-3 structure. It is a smooth R -scheme of relative dimension $\frac{g(g+1)}{2}$ with $R := R_3 = \mathbb{Z}[1/3, e^{2i\pi/3}]$. Let $\pi : \mathbb{A}_g \rightarrow \mathcal{A}_g(3)$ be the universal family of principally polarized Abelian varieties on $\mathcal{A}_g(3)$. The morphism $\beta \circ \text{KS}$ (defined by $\mathbb{A}_g$ as above) is an isomorphism $T_{\mathcal{A}_g(3)/R} \xrightarrow{\sim} \operatorname{Sym}^2 \underline{\operatorname{Lie}}_{\mathbb{A}_g/\mathcal{A}_g(3)}$.

Proposition 7.3.2. *Let $g \in \mathbb{N}_{>0}$. There exists an integer $N \in \mathbb{N}_{>0}$ such that for every algebraically closed field k whose characteristic does not divide N , for every morphism $R \rightarrow k$, for every $p \in \mathbb{N}$ with $1 \leq p \leq \frac{g(g+1)}{2}$, for every closed point $x \in \mathcal{A}_g(3) \otimes_R k$, for every $\alpha \in {}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R k)_x$, there exists a vector subspace $S \subset \operatorname{Lie}(\mathbb{A}_g)_x$ of dimension $g - 1$ such that $\Lambda^p(\beta \circ \text{KS})(\alpha) \in \Lambda^p \operatorname{Sym}^2 S$.*

Proof. We can fix $p \in \mathbb{N}$ with $1 \leq p \leq \frac{g(g+1)}{2}$ and take at the end the product of the N 's obtained for each value of p . By Proposition 3.4.9, there exists an integer $N' \in \mathbb{N}_{>0}$ such that for every prime ideal $\mathfrak{p} \subset R[1/N']$, if we denote by k the residue field $\operatorname{Frac}(R[1/N']/\mathfrak{p})$, then the set ${}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R k)$ is included in the closure $\bar{B}_+$ of ${}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R \operatorname{Frac}(R))$ in $\Lambda^p T_{\mathcal{A}_g(3)/R}$. Moreover, as ${}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R \operatorname{Frac}(R))$ is closed in $\Lambda^p T_{\mathcal{A}_g(3) \otimes_R \operatorname{Frac}(R)}$, we have ${}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R \operatorname{Frac}(R)) = \bar{B}_+ \otimes_R \operatorname{Frac}(R)$. As $\mathbb{B}_+^p$ is partially compatible with base change (see Proposition 3.4.7), then for every field k and every morphism $f : R[1/N'] \rightarrow k$, we have ${}^T\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R k) \subset \bar{B}_+ \otimes_R k$, with equality if f is injective.

Let $Y \rightarrow \mathcal{A}_g(3)$ be the Grassmannian bundle where for every $x \in \mathcal{A}_g(3)$, the fiber of Y above x parametrizes the $(g - 1)$ -dimensional vector subspaces of $\operatorname{Lie}(\mathbb{A}_g)_x$. Consider E the tautological vector bundle (of rank $g - 1$) on Y . We have a canonical $\mathcal{A}_g(3)$ -morphism from the total space of E to the total space of $\underline{\operatorname{Lie}}_{\mathbb{A}_g/\mathcal{A}_g(3)}$. It induces a $\mathcal{A}_g(3)$ -morphism $\xi : \Lambda^p \operatorname{Sym}^2 E \rightarrow \Lambda^p \operatorname{Sym}^2 \underline{\operatorname{Lie}}_{\mathbb{A}_g/\mathcal{A}_g(3)}$ (we use the same notation for a vector bundle and for its total space). Denote $\gamma := \Lambda^p((\beta \circ \text{KS})^{-1}) \circ \xi : \Lambda^p \operatorname{Sym}^2 E \rightarrow \Lambda^p T_{\mathcal{A}_g(3)/R}$.

We have to prove that there exists $N'' \in \mathbb{N}_{>0}$ such that for every algebraically closed field k and every morphism $R[1/N''] \rightarrow k$, the following holds:

(*) $\bar{B}_+ \otimes_R k$ is in the image of the morphism $\gamma \otimes_R k$.

Indeed, when k is algebraically closed, (*) holds if and only if $\bar{B}_+(k)$ is included in the image of $\Lambda^p \operatorname{Sym}^2 E(k)$ by γ . Every k -point of $\Lambda^p \operatorname{Sym}^2 E$ is above a k -point of Y , that gives S as in the statement of the proposition. Thus we obtain the proposition with $N := N'N''$.

We prove more generally that there exists $N'' \in \mathbb{N}_{>0}$ such that $(*)$ holds for every field k (not necessarily algebraically closed) with a morphism $R[1/N''] \rightarrow k$. The case $k = \mathbb{C}$ of $(*)$ is given by Lemma 7.3.1 applied to $\mathcal{A}_g(3) \otimes_R \mathbb{C}$. It is clear that if k'/k is a field extension, then $(*)$ is true for k if and only if it is true for k' . Hence $(*)$ holds for $k = \text{Frac}(R)$.

By Chevalley's theorem, the image I of γ is a constructible subset of $\Lambda^p T_{\mathcal{A}_g(3)/R}$. Thus $\bar{B}_+ \setminus I$ is constructible, so by Chevalley's theorem again, its projection J on $\text{Spec } R$ is constructible. As $(*)$ holds for $k = \text{Frac}(R)$, J does not contain the generic point. As R has Krull dimension one, J contains only finitely many points. Let N'' be the product of the characteristic of their residue field. Then $(*)$ holds for every $k = \text{Frac}(R[1/N'']/\mathfrak{p})$ with $\mathfrak{p} \subset R[1/N'']$ a prime ideal. As $(*)$ is compatible with extension and every field k with a morphism $f : R[1/N''] \rightarrow k$ is an extension of $\text{Frac}(R[1/N'']/\ker(f))$, $(*)$ holds for every field k and every morphism $R[1/N''] \rightarrow k$. $\square$

Proof of Theorem 7.1.2. Consider the integer N given by Proposition 7.3.2. Let k be an algebraically closed field whose characteristic is not 3 and divides neither N nor the order n of the group $\text{Sp}_{2g}(\mathbb{F}_3)$. Then there exists a morphism $R \rightarrow k$. Let V be a smooth k -variety. Consider $\mathcal{A}_g$ the moduli stack of principally polarized Abelian varieties of dimension g . There is a canonical finite étale covering map $\mathcal{A}_g(3) \otimes_R k \rightarrow \mathcal{A}_g \otimes_{\mathbb{Z}} k$ of degree n . Moreover, by Faltings and Chai [38, Section IV.6] (or by Lemma 3.4.8 and a change of N), $\mathcal{A}_g(3) \otimes_R k$ is projectively log-compactifiable. Thus, we can apply Corollary 3.5.7: for all but finitely many morphisms $f : V \rightarrow \mathcal{A}_g$, for every $p \in \mathbb{N}$ with $1 \leq p \leq \frac{g(g+1)}{2}$, we have $\text{Im}(\Lambda^p df) \subset {}^{\text{T}}\mathbb{B}_+^p(\mathcal{A}_g(3) \otimes_R k)$ with $\tilde{f} : V \times_{\mathcal{A}_g} \mathcal{A}_g(3) \otimes_R k \rightarrow \mathcal{A}_g(3) \otimes_R k$ the pull-back of f .

Choose such a morphism f . Denote by A the pull-back of the universal family by f , by $\text{KS} : T_V \rightarrow R^1\pi_* T_{A/V}$ the Kodaira-Spencer map and by $\beta : R^1\pi_* T_{A/V} \xrightarrow{\sim} \underline{\text{Lie}}_{A/V}^{\otimes 2}$ the canonical isomorphism. Let x be a closed point of V . Let p be the rank of $\beta_x \circ \text{KS}_x$. Let $v_1, \dots, v_p \in T_{V,x}$ be a basis of a complement of $\ker \beta_x \circ \text{KS}_x$. Apply Proposition 7.3.2: there exists a vector subspace $S \subset \text{Lie } A_x$ of dimension $g - 1$ such that $\Lambda^p(\beta \circ \text{KS})(v_1 \wedge \dots \wedge v_p) \in \Lambda^p \text{Sym}^2 S$. As $(v_1, \dots, v_p)$ is free, we have $\beta(\text{KS}(v_i)) \in \text{Sym}^2 S$ for every $i \in \mathbb{N}_{>0}$ with $i \leq p$. As the $\beta(\text{KS}(v_i))$'s generate the image of $\beta \circ \text{KS}$, we have $\text{Im}(\beta_x \circ \text{KS}_x) \subset \text{Sym}^2 S$. $\square$

Remarks 7.3.3. 1. In the case $k = \mathbb{C}$, Theorem 7.1.2 is a consequence of “Arakelov-Faltings's” Theorem 2.1.18. Indeed, let $\pi : A \rightarrow V$ be a family of principally polarized Abelian varieties on a smooth complex algebraic variety V . Assume there exists a skew-symmetric endomorphism f of $R^1\pi_* \underline{\mathbb{Z}}_A$ that does not come from an endomorphism of A . Then in

the Hodge decomposition $f = f^{1,-1} + f^{0,0} + f^{-1,1}$, we have $f^{1,-1} \neq 0$ and $f^{-1,1} \neq 0$. By Deligne's fixed part theorem [31, Corollaire 4.1.2], $f^{1,-1}$ and $f^{-1,1}$ are (nonzero) endomorphisms of the complex local system $R^1\pi_*\mathbb{C}_A$. Thus $\mathbb{V} := \text{Im}(f^{1,-1}) \oplus \text{Im}(f^{-1,1})$ is a nontrivial flat sub- $\mathbb{R}$ -PVHS of $R^1\pi_*\mathbb{R}_A$. Consider the orthogonal for a polarization $\mathbb{V}^\perp$ of $\mathbb{V}$ in $R^1\pi_*\mathbb{R}_A$. For every $x \in V$, $(\mathbb{V}_x^\perp)^{0,1}$ is a proper vector subspace of $H^{0,1}(A_x)$. By the canonical isomorphism $H^{0,1}(A_x) \cong \text{Lie } A_x$, it corresponds to a proper vector subspace S of $\text{Lie } A_x$ for which the statement of Theorem 7.1.2 holds.

2. Remark that in the proof of the previous point, the vector spaces S combine to form a direct factor of $\underline{\text{Lie}}_{A/V}$. In the general case (k not necessarily $\mathbb{C}$), we can choose the spaces S as fibers of a coherent $\mathcal{O}_V$ -submodule F of $\underline{\text{Lie}}_{A/V}$. Indeed, if σ is a section of $\text{Sym}^2 \underline{\text{Lie}}_{A/V}$ and τ is a section of $\underline{\text{Lie}}_{A/V}^\vee$, denote by $\sigma \cdot \tau$ their tensor contraction, that is a section of $\underline{\text{Lie}}_{A/V}$. Then define F as the image of the following morphism of $\mathcal{O}_V$ -modules.

$$\begin{aligned} T_V \otimes_{\mathcal{O}_V} \underline{\text{Lie}}_{A/V}^\vee &\rightarrow \underline{\text{Lie}}_{A/V} \\ v \otimes \phi &\mapsto \beta(\text{KS}(v)) \cdot \phi. \end{aligned}$$

We do not know if in general, we can realize F as a direct factor of $\underline{\text{Lie}}_{A/V}$, or even as a locally free module.

3. In the case $k = \mathbb{C}$, we can also deduce from Faltings's Theorem 2.1.17 that for every smooth algebraic complex variety V , there exist only finitely many isomorphism classes of Higgs bundles on V of the form $\pi_*\Omega_{A/V}^1 \oplus R^1\pi_*\mathcal{O}_A$, with the Higgs field given by the Kodaira-Spencer map, with A a principally polarized Abelian scheme on V (see the beginning of the proof of Lemma 7.3.1). Indeed, the Higgs field associated with a $\mathbb{C}$ -PVHS on V only depends on the underlying local system (it is for instance an application of [32, Proposition 1.13]).

The analogous finiteness statement is not true in positive characteristic. Indeed, let k be an algebraically closed field of characteristic at least 5. Moret-Bailly proved that there are rational curves in $\mathcal{A}_g(3) \otimes_R k$ (see [61]). Choose one of them $f : \mathbb{P}_k^1 \rightarrow \mathcal{A}_g(3)$. We can freely suppose that its Kodaira-Spencer map is nonzero. Consider a finite surjective separable morphism $h : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ of degree 2. Then the Kodaira-Spencer map of $(f \circ h)^*\mathbb{A}_g$ is nonzero, but vanishes at the ramification points of h . Thus we obtain infinitely many non-isomorphic Higgs bundles by making $\text{PSL}(2, k)$ act on $\mathbb{P}_k^1$.

7.3.2 Families of K3 surfaces

In this subsection, we prove Theorem 7.1.5. Let k be a field, let S be a K3 surface over k . The canonical line bundle is trivial, thus $H^0(S, \omega_S)$ has dimension 1 and its nonzero elements vanish nowhere (they are symplectic forms on S). Let $\phi \in H^0(S, \omega_S) \setminus \{0\}$ be a symplectic form. The interior product defines an isomorphism of $\mathcal{O}_S$ -modules:

$$\begin{aligned} T_S &\xrightarrow{\sim} \Omega_S^1 \\ v &\mapsto (v^\phi : w \mapsto \phi(v, w)). \end{aligned}$$

It induces an isomorphism of cohomology groups:

$$\begin{aligned} H^1(S, T_S) &\xrightarrow{\sim} H^1(S, \Omega_S^1) \\ \mu &\mapsto \mu^\phi. \end{aligned}$$

Denote by b the product $\text{Sym}^2 H^1(S, \Omega_S^1) \rightarrow k$ induced in cohomology by the cup product $\Omega_S^1 \wedge \Omega_S^1 \rightarrow \omega_S$ and the trace isomorphism $H^2(S, \omega_S) \cong k$.

Definition 7.3.4. *An element $\mu \in H^1(S, T_S)$ is isotropic if μ^ϕ is b -isotropic, i.e. $b(\mu^\phi, \mu^\phi) = 0$.*

Fix a polarization $\omega \in H^1(S, \Omega_S^1)$ of S (i.e. the Atiyah class of an ample line bundle). An element $\mu \in H^1(S, T_S)$ is primitive if μ^ϕ is b -orthogonal to ω , i.e. $b(\mu^\phi, \omega) = 0$.

These notions do not depend on the choice of ϕ , but the primitive part depends on the choice of ω . The polarization ω is never b -isotropic. The primitive elements form a vector subspace $P^{1,1}(S)$ of dimension 19 of $H^1(S, \Omega_S^1)$ (that has dimension 20). We can consider these notions for families of K3 surfaces.

Definition 7.3.5. *Let V be a smooth k -variety. Let $\pi : S \rightarrow V$ be a family of K3 surfaces over V (i.e. a smooth proper V -variety whose fibers are K3 surfaces). A local section of $R^1\pi_*T_{S/V}$ is isotropic if it gives an isotropic element of the fiber $R^1\pi_*T_{S/V,x} \cong H^1(S_x, T_{S_x})$ for every $x \in V$. A $\mathcal{O}_V$ -submodule of $R^1\pi_*T_{S/V}$ is isotropic if all its local sections are isotropic.*

*If $\pi : S \rightarrow V$ is projective, consider a polarization. The primitive part of $R^1\pi_*T_{S/V}$ is the direct factor of rank 19 denoted by $P_{S/V}^{1,1}$ such that for every $x \in V$, $P_{S/V,x}^{1,1}$ is the subspace $P^{1,1}(S_x)$ of primitive elements of $H^1(S_x, T_{S_x})$.*

Denote by $\text{KS}_{S/V}$ the Kodaira-Spencer map $T_V \rightarrow R^1\pi_*T_{S/V}$. If the family is polarized, then the image of $\text{KS}_{S/V}$ is in the primitive part $P_{S/V}^{1,1}$.

For a K3 surface S , the tensor contraction $T_S \otimes_{\mathcal{O}_S} \Omega_S^1 \rightarrow \omega_S$ with the cup-product gives a linear map $c : H^1(S, T_S) \otimes_k H^1(S, \Omega_S^1) \rightarrow H^2(S, \mathcal{O}_S)$.

Lemma 7.3.6. *Let k be a field, let S be a K3 surface over k , let $\phi \in H^0(S, \omega_S) \setminus \{0\}$ be a symplectic form, let $\mu \in H^1(S, T_S)$. Then μ is isotropic if and only if $c(\mu \otimes \mu^\phi) = 0$.*

Proof. Let v be a local section of T_S and α a local section of Ω_S^1 . As the interior product is an antiderivation, we have $\alpha(v)\phi = \alpha \wedge v^\phi$. This means that the following square of $\mathcal{O}_S$ -modules commutes up to a sign:

$$\begin{array}{ccc} T_S \otimes_{\mathcal{O}_S} \Omega_S^1 & \longrightarrow & \mathcal{O}_S \\ \bullet^\phi \otimes \text{id} \downarrow \wr & & \phi \downarrow \wr \\ (\Omega_S^1)^{\otimes 2} & \xrightarrow{\wedge} & \omega_S \end{array}$$

where the upper map is the tensor contraction. Thus the following square commutes (up to a sign):

$$\begin{array}{ccc} H^1(S, T_S) \otimes_k H^1(S, \Omega_S^1) & \xrightarrow{c} & H^2(S, \mathcal{O}_S) \\ \bullet^\phi \otimes \text{id} \downarrow \wr & & \downarrow \wr \\ H^1(S, \Omega_S^1)^{\otimes 2} & \xrightarrow{\gamma} & H^2(S, \omega_S) \end{array}$$

where the lower map γ is the composition of the cup and wedge products. For $\alpha, \alpha' \in H^1(S, \Omega_S^1)$, $\gamma(\alpha \otimes \alpha')$ identifies with $b(\alpha, \alpha')$ via the trace isomorphism $H^2(S, \omega_S) \cong k$. Thus we have $c(\mu \otimes \alpha) = 0$ if and only if $b(\mu^\phi, \alpha) = 0$. The statement of the lemma is the case $\alpha = \mu^\phi$. $\square$

Lemma 7.3.7. *Let V be a smooth complex quasiprojective variety. Let $\pi : S \rightarrow V$ be a polarized family of K3 surfaces over V . Assume the period map of the natural $\mathbb{Z}$ -PVHS on the primitive part $R_{\text{prim}}^2 \pi_* \mathbb{Z}_S$ of $R^2 \pi_* \mathbb{Z}_S$ is immersive.*

For every closed point $x \in V$, for every $v \in \mathbb{T}\mathbb{B}_+^1(V)_x$, the primitive vector $\text{KS}_{S/V}(v) \in P_{S/V,x}^{1,1}$ is isotropic.

Proof. The primitive part $R_{\text{prim}}^2 \pi_* \mathbb{Z}_S$ of $R^2 \pi_* \mathbb{Z}_S$ defines a $\mathbb{Z}$ -PVHS of weight 2 on V . The Hodge decomposition of $R_{\text{prim}}^2 \pi_* \mathbb{Z}_S$ at x is $H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ with $H^{2,0} = \pi_* \omega_{S/V}$, $H^{1,1} \subset R^1 \pi_* \Omega_{S/V}^1$ and $H^{0,2} = R^2 \pi_* \mathcal{O}_S$. Let $\theta_x : T_{V,x} \hookrightarrow \text{Hom}(H^{2,0}, H^{1,1}) \oplus \text{Hom}(H^{1,1}, H^{0,2})$ be the Higgs field at x . It is injective by hypothesis. As the PVHS is real, the induced linear map $\theta_x : T_{V,x} \hookrightarrow \text{Hom}(H^{2,0}, H^{1,1})$ is injective too. As S_x is a K3 surface, we have $\dim H^{2,0} = 1$. Thus we can apply Proposition 4.7.4 and Proposition 4.7.5, with $i = 2$: either

$\theta_x(v)$ vanishes on $H^{2,0}$ (case $j = 1$), or the iterate $\theta_x(v)^2$ vanishes (case $j = 0$). Obviously, the first case implies the second one, so $\theta_x(v)^2 = 0$.

The Higgs field is given by cup-product and contraction by the Kodaira-Spencer class. Hence, for $\phi \in H^0(S_x, \omega_{S_x}) \setminus \{0\}$, $\theta_x(v)^2 = 0$ is equivalent to $c(\text{KS}_{S/V}(v) \otimes \text{KS}_{S/V}(v)^\phi) = 0$. By Lemma 7.3.6, $\text{KS}_{S/V}(v)$ is isotropic. $\square$

Let $d \in \mathbb{N}_{>0}$, let $R_d := \mathbb{Z}[1/(2d)]$. There exists a Deligne-Mumford stack $\mathcal{M}_d$ that is a fine moduli space for families K3 surfaces with a primitive polarization of degree $2d$ over R_d -schemes (see [9, Subsection 1.2], with $\mathcal{M}_d = \mathfrak{M}_{2d} \otimes_{\mathbb{Z}[1/2]} R_d$). It is smooth and of finite type over R_d . Let $\pi : \mathcal{S} \rightarrow \mathcal{M}_d$ be the universal family of K3 surfaces. There exists a finite étale covering map $\mathcal{M}'_d \rightarrow \mathcal{M}_d$, a smooth R_d -scheme of finite type X and an open embedding $q : \mathcal{M}'_d \hookrightarrow X$ (in particular $\mathcal{M}'_d$ is a scheme) such that $X_{\mathbb{C}} := X \otimes_{R_d} \mathbb{C}$ is a finite étale cover of the period domain and $q \otimes_{R_d} \mathbb{C}$ is a lift of the period map for the $\mathbb{Z}$ -PVHS defined by the primitive cohomology of degree 2 of $(\mathcal{S} \otimes_{\mathcal{M}_d} \mathcal{M}'_d) \otimes_{R_d} \mathbb{C}$ above $\mathcal{M}'_d \otimes_{R_d} \mathbb{C}$ (see [9, Théorème 1.4]). In particular, the universal cover of $X_{\mathbb{C}}$ is biholomorphic to the bounded symmetric domain $SO_0(19, 2)/(SO(19) \times SO(2))$ (type IV).

Proposition 7.3.8. *Let $d \in \mathbb{N}_{>0}$. There exists an integer $N \in \mathbb{N}_{>0}$ such that for every field k whose characteristic does not divide N :*

- for every $x \in \mathcal{M}'_d \otimes_{R_d} k$ and every $v \in {}^{\text{T}}\mathbb{B}_+^1(\mathcal{M}'_d \otimes_{R_d} k)_x$, $\text{KS}_x(v)$ is isotropic,
- for every $p \in \mathbb{N}$ with $2 \leq p \leq \dim \mathcal{M}'_d \otimes_{R_d} k = 19$, $\mathbb{B}_+^p(\mathcal{M}'_d \otimes_{R_d} k)$ is empty.

Proof. As in the proof of Proposition 7.3.2, we reduce the proof to the case $k = \mathbb{C}$. In this case, the first point is given by Lemma 7.3.7. Moreover, for every $p \in \mathbb{N}$ with $2 \leq p \leq 19$, $\mathbb{B}_+^p(X_{\mathbb{C}})$ is empty thanks to Theorem 6.1.1 with Remarks 6.4.2.2. As there is an open embedding $\mathcal{M}'_d \otimes_{R_d} \mathbb{C} \hookrightarrow X_{\mathbb{C}}$, the second point is given by Proposition 3.4.5. $\square$

Proof of Theorem 7.1.5. Consider an integer N as given by Proposition 7.3.8. Let k be an algebraically closed field whose characteristic divides neither N nor the degree n of the finite étale covering map $\mathcal{M}'_d \rightarrow \mathcal{M}_d$. Then there exists a morphism $R_d[1/N] \rightarrow k$. Let V be a smooth k -variety. By Lemma 3.4.8, we can freely change N to suppose $\mathcal{M}_d \otimes_{R_d} k$ projectively log-compactifiable. Thus, we can apply Corollary 3.5.7: for all but finitely many morphisms $f : V \rightarrow \mathcal{M}_d$, for every p , we have $\text{Im}(\Lambda^p d\tilde{f}) \subset {}^{\text{T}}\mathbb{B}_+^p(\mathcal{M}'_d \otimes_{R_d} k)$ with $\tilde{f} : V \times_{\mathcal{M}_d} \mathcal{M}'_d \rightarrow \mathcal{M}'_d \otimes_{R_d} k$ the pull-back of f .

Choose such a morphism f . Denote by $\pi : S \rightarrow V$ the pull-back by f of the universal family, by $\text{KS} : T_V \rightarrow R^1\pi_*T_{S/V}$ the Kodaira-Spencer map and by L the image of KS . Let $x \in V \times_{\mathcal{M}_d} \mathcal{M}'_d$ and $v, v' \in T_{V \times_{\mathcal{M}_d} \mathcal{M}'_d, x}$. By hypothesis on f , we have $\tilde{d}f_x(v) \in {}^T\mathbb{B}_+^1(\mathcal{M}'_d \otimes_{R_d} k)$. By Proposition 7.3.8, this implies that L is isotropic. Moreover, we have $\tilde{d}f_x(v) \wedge \tilde{d}f_x(v') \in {}^T\mathbb{B}_+^2(\mathcal{M}'_d \otimes_{R_d} k)$. By Proposition 7.3.8, $\mathbb{B}_+^2(\mathcal{M}'_d \otimes_{R_d} k) = \emptyset$, thus $\tilde{d}f_x(v) \wedge \tilde{d}f_x(v') = 0$. Hence for every $x \in V$, the dimension of the fiber L_x is at most 1.

As V is connected, if there exists $x \in V$ with $\dim L_x = 0$, then L has torsion. As $R^1\pi_*T_{S/V}$ is locally free, this implies that L is zero. Otherwise, for every $x \in V$, $\dim L_x = 1$, hence L is invertible. $\square$

Remark 7.3.9. For $k = \mathbb{C}$, the corollaries of Theorem 7.1.5 (i.e. the fact that the image of the Kodaira-Spencer map is almost surely zero or included in a line bundle) were proven by Miyano and Noguchi (see [56, (4.11) & (4.12)]). The fact that the image is almost surely isotropic can be deduced from [70, Theorem 4.2]. See also [76] about non-rigid families of K3 surfaces.

7.3.3 Families of polarized varieties in mixed characteristic

The crucial points in the proof of the two previous theorems are that principally polarized Abelian varieties and K3 surfaces have a moduli space in mixed characteristic, and that over $\mathbb{C}$, there is an infinitesimal Torelli theorem. The fact that the moduli spaces over $\mathbb{C}$ are embedded in locally symmetric varieties is not important. In this subsection, we give a general theorem that applies for every type of polarized varieties that admit a moduli space in mixed characteristic and an infinitesimal Torelli theorem.

We begin by the definition of a polarization.

Definition 7.3.10. *Let $\mathcal{M}$ and $\mathcal{F}$ be two Deligne-Mumford stacks and $\pi : \mathcal{F} \rightarrow \mathcal{M}$ a smooth proper representable morphism. A polarization of $\pi : \mathcal{F} \rightarrow \mathcal{M}$ is a global section ω of $R^1\pi_*\Omega_{\mathcal{F}/\mathcal{M}}^1$ such that there exists a scheme M and a surjective étale morphism $q : M \rightarrow \mathcal{M}$ such that $q^*\omega \in H^0(M, R^1\pi'_*\Omega_{\mathcal{F} \times_{\mathcal{M}} M/M}^1)$ is given by the Atiyah class (relatively to M) of a line bundle on $\mathcal{F} \times_{\mathcal{M}} M$ that is relatively ample over M (with $\pi' : \mathcal{F} \times_{\mathcal{M}} M \rightarrow M$ the pullback of π by q).*

In particular, $\mathcal{F} \times_{\mathcal{M}} M$ is projective over M . Now we can define families of polarized varieties in mixed characteristic.

Definition 7.3.11. A family of polarized varieties in mixed characteristic is the data of:

- an integral ring R of characteristic 0, quasi-finite over $\mathbb{Z}$,
- a Deligne-Mumford stack $\mathcal{M}$ of finite type over R ,
- a Deligne-Mumford stack $\mathcal{F}$ with a smooth proper representable morphism $\pi : \mathcal{F} \rightarrow \mathcal{M}$ with equidimensional fibers,
- a polarization ω of $\pi : \mathcal{F} \rightarrow \mathcal{M}$,

such that there exists a smooth quasiprojective $\text{Frac}(R)$ -variety M and a finite étale covering $\text{Frac}(R)$ -map $M \rightarrow \mathcal{M} \otimes_R \text{Frac}(R)$.

To state the theorem, we define the primitive cohomology in a similar way as over $\mathbb{C}$.

Definition 7.3.12. Let k be a field, V a smooth projective k -variety of dimension $d \in \mathbb{N}$, $\omega \in H^1(V, \Omega_V^1)$ the Atiyah class of an ample line bundle on V , $i, j \in \mathbb{N}$ with $i + j \leq d$. The primitive part P_V^{ij} of $H^j(V, \Omega_V^i)$ is the cokernel of the k -linear map $H^{j-1}(V, \Omega_V^{i-1}) \rightarrow H^j(V, \Omega_V^i)$ given by cup product and wedge product with ω if $i, j > 0$, and is $H^j(V, \Omega_V^i)$ if $i = 0$ or $j = 0$.

The primitive part depends on the choice of ω . If $k = \mathbb{C}$, by the hard Lefschetz theorem, the linear map $H^{j-1}(V, \Omega_V^{i-1}) \rightarrow H^j(V, \Omega_V^i)$ of the previous definition is injective.

If $\mathcal{M}$ and $\mathcal{F}$ are two Deligne-Mumford stacks and $\pi : \mathcal{F} \rightarrow \mathcal{M}$ is a smooth proper representable morphism with fiber of dimension $d \in \mathbb{N}$ and with a polarization ω , then for every $i, j \in \mathbb{N}$ with $i + j \leq d$, the primitive parts of the fibers define a submodule $P_{\mathcal{F}/\mathcal{M}}^{ij}$ of $R^j \pi_* \Omega_{\mathcal{F}/\mathcal{M}}^i$.

Now, let k be a field, V and F two smooth k -varieties, $\pi : F \rightarrow V$ a smooth proper morphism with fibers of dimension $d \in \mathbb{N}$ and with a polarization ω . Denote by $\text{KS}_{F/V} : T_V \rightarrow R^1 \pi_* T_{F/V}$ the Kodaira-Spencer map. The cup-product and tensor contraction with ω gives a morphism of $\mathcal{O}_V$ -modules:

$$R^1 \pi_* T_{F/V} \rightarrow R^2 \pi_* \mathcal{O}_F.$$

This morphism vanishes on the image of $\text{KS}_{F/V}$ (see [78, Theorem 3.3.11.(iii)]). Thus, cup-product/tensor contraction by an element of the image of $\text{KS}_{F/V}$ commutes with cup-product/tensor contraction by ω . In particular, the product by an element of the image of $\text{KS}_{F/V}$ stabilizes primitive parts. Hence

for every $i, j \in \mathbb{N}$ with $i \geq 1$ and $i + j \leq d$, we have a canonical morphism of $\mathcal{O}_V$ -modules:

$$\begin{aligned} T_V \otimes_{\mathcal{O}_V} P_{F/V}^{ij} &\rightarrow P_{F/V}^{i-1, j+1} \\ v \otimes \alpha &\mapsto \text{KS}_{F/V}(v) \cdot \alpha. \end{aligned}$$

For every point $x \in V$ and every $n \in \mathbb{N}$ with $n \leq d$, that gives to the graded vector space $P_{F/V, x}^n := \bigoplus_{i=0}^n P_{F/V, x}^{i, n-i}$ a structure of infinitesimal system of Hodge bundles on $T_{V, x}$ in the following sense.

Definition 7.3.13. *Let k be a field. An infinitesimal system of Hodge bundles on a k -vector space T is the data of a finite dimensional graded k -vector space $E = \bigoplus_{i \in \mathbb{Z}} E^i$ with a linear map $\delta : T \rightarrow \text{End}(E)^{-1}$ such that for every $v, w \in T$, $\delta(v)$ and $\delta(w)$ commute.*

The direct sum of two infinitesimal systems of Hodge bundles on a same vector space has a structure of infinitesimal system of Hodge bundles. Moreover, the tensor product of an infinitesimal system of Hodge bundles with a finite dimensional graded k -vector space has a structure of infinitesimal system of Hodge bundles (make $v \in T$ act by $\delta(v) \otimes \text{id}$). Now we can state the theorem.

Theorem 7.3.14. *Let $(R, \mathcal{M}, \mathcal{F}, \omega)$ be a family of polarized varieties in mixed characteristic. Let d be the dimension of the fibers of $\mathcal{F}$ over $\mathcal{M}$. Let $n \in \mathbb{N}$ be such that $n \leq d$. Suppose that for a morphism of rings $R \rightarrow \mathbb{C}$, a complex scheme M and a finite étale covering map $M \rightarrow \mathcal{M} \otimes_R \mathbb{C}$, the period map of the $\mathbb{Z}$ -PVHS defined by the primitive cohomology of degree n of $\mathcal{F} \otimes_{\mathcal{M}} M$ over M is immersive. Then there exists $N \in \mathbb{N}_{>0}$ such that for every algebraically closed field k and every morphism of rings $R[1/N] \rightarrow k$, the following holds.*

Let V be a smooth k -variety. Then for all but finitely many R -morphisms $f : V \rightarrow \mathcal{M}$, for every closed point $x \in V$, the following holds.

Denote by F_x the fiber of $\mathcal{F} \times_{\mathcal{M}} V$ over V at x . Denote $P_x^n := P_{\mathcal{F} \times_{\mathcal{M}} V/V, x}^n$. Then there exists:

- *a nonzero $\delta \in \text{End}(P_x^n)^{-1}$ given by the Kodaira-Spencer map of a deformation of F_x given by an R -morphism $\text{Spec}(k[\varepsilon]/(\varepsilon^2)) \rightarrow \mathcal{M}$,*
- *a finite set A ,*
- *for every $a \in A$, an infinitesimal system of Hodge bundles E_a on $T_{V, x}$,*
- *for every $a \in A$, a finite-dimensional graded k -vector space H_a ,*
- *for every $a \in A$, an element $\delta_a \in \text{End}(H_a)^{-1}$,*

- an isomorphism $\alpha : \bigoplus_{a \in A} E_a \otimes H_a \xrightarrow{\sim} P_x^n$ of infinitesimal systems of Hodge bundles on $T_{V,x}$,

such that via the isomorphism α , δ corresponds to the sum of the δ_a 's (i.e. $\delta = \alpha \circ (\bigoplus_{a \in A} \text{id}_{E_a} \otimes \delta_a) \circ \alpha^{-1}$).

Remarks 7.3.15. 1. We can also give a nontrivial statement in the case where the period map is only generically immersive. In this case, we can define the “degeneracy locus” of the primitive cohomology (with the similar definition as for $\mathbb{C}$ -PVHS in the complex case). For a convenient choice of N , a spreading-out argument gives that it is a closed subset of $\mathcal{M}$. The statement of the theorem remains true (with the same proof) if we replace “for every closed point $x \in V$, the following holds” by “for every closed point $x \in V$, $f(x)$ is in the degeneracy locus or the following holds”.

2. Theorems 7.1.2 and 7.1.5 can be deduced from Theorem 7.3.14.

The crucial point of the theorem is that δ is nonzero, making the decomposition nontrivial (in the sense that there exists $a \in A$ such that $\dim E_a \geq 1$ and $\dim H_a \geq 2$). Here are two consequences of the existence of the decomposition.

Proposition 7.3.16. *Let $(R, \mathcal{M}, \mathcal{F}, \omega), n, N, k, V, f, x$ be as in the previous theorem.*

1. *In the category of infinitesimal systems of Hodge bundles on $T_{V,x}$, there exists a decomposition $P_x^n = B \oplus C \oplus C'$ where C and C' are nonzero and are isomorphic up to a shift of 1 in the grading.*
2. *There exists $i, j \in \mathbb{Z}$ with $j < i$ and a nonzero $\gamma \in \text{End}(P_x^n)^{-1}$ given by the Kodaira-Spencer map of a deformation of F_x given by a morphism $\text{Spec}(k[\varepsilon]/(\varepsilon^2)) \rightarrow \mathcal{M}$ such that for every $w \in T_{V,x}$, the rank of the linear map $(P_x^n)^i \rightarrow (P_x^n)^j$ given by iterating the action of w is strictly less than the rank of the linear map $(P_x^n)^i \rightarrow (P_x^n)^j$ given by iterating γ .*

Proof. For the first point: as $\delta \neq 0$, there exists $a \in A$ such that $\delta_a \neq 0$ and $\dim E_a > 0$. Thus there exists a nonzero homogeneous element $h \in H_a$ such that $\delta_a(h) \neq 0$. Let D_a be a graded vector subspace of H_a such that $H_a = k.h \oplus k.\delta_a(h) \oplus D_a$. Let $D := E_a \otimes_k D_a \oplus \bigoplus_{b \in A \setminus \{a\}} E_b \otimes_k H_b$. Then we set $B := \alpha(D)$, $C := \alpha(E_a \otimes_k h)$ and $C' := \alpha(E_a \otimes_k \delta_a(h))$. The restriction of δ to C gives an isomorphism of degree -1 with C' .

The second point has the same proof as Proposition 4.7.2. $\square$

To prove the theorem, we need two lemmas.

Lemma 7.3.17. *Let $(R, \mathcal{M}, \mathcal{F}, \omega)$ be a family of polarized varieties in mixed characteristic. Let d be the dimension of the fibers of $\mathcal{F}$ over $\mathcal{M}$. Then there exists $N \in \mathbb{N}_{>0}$, a smooth and quasiprojective $R[1/N]$ -scheme M and a finite étale covering $R[1/N]$ -map $M \rightarrow \mathcal{M} \otimes_R R[1/N]$ such that for every $i, j \in \mathbb{N}$ with $i + j \leq d$, the $\mathcal{O}_M$ -module $P_{\mathcal{F} \times_{\mathcal{M}} M/M}^{ij}$ is locally free.*

Proof. By definition of a family of polarized varieties in mixed characteristic, there exists a smooth and quasiprojective $\text{Frac}(R)$ -scheme $M_{\text{Frac}(R)}$ and a finite étale covering $\text{Frac}(R)$ -map $M_{\text{Frac}(R)} \rightarrow \mathcal{M} \otimes_R \text{Frac}(R)$. We first need to prove that this holds with $R[1/N]$ instead of $\text{Frac}(R)$ for some $N \in \mathbb{N}_{>0}$. By definition of a Deligne-Mumford stack, there exists a scheme D and a surjective étale morphism $D \rightarrow \mathcal{M}$, and the datum of a finite étale cover of $\mathcal{M}$ is equivalent to the data of a finite étale cover of D with descent data. Thus, by classical spreading-out arguments for schemes (see EGA or [73, Theorem 3.2.1]), there exists a nonempty open subset $U \subset \text{Spec}(R)$, a smooth and quasiprojective U -scheme M_U and a finite étale covering U -map $M_U \rightarrow \mathcal{M} \times_R U$. As R is quasi-finite over $\mathbb{Z}$, U contains $\text{Spec}(R[1/N])$ for some $N \in \mathbb{N}_{>0}$.

Let C be the set of points of M at which for some i, j , the rank of $P_{\mathcal{F} \times_{\mathcal{M}} M/M}^{ij}$ is not maximal. As the $\mathcal{O}_M$ -modules $P_{\mathcal{F} \times_{\mathcal{M}} M/M}^{ij}$ are coherent, C is closed. Moreover, as the construction of $P_{\mathcal{F} \times_{\mathcal{M}} M/M}^{ij}$ is compatible with field extension, the hard Lefschetz theorem over $\mathbb{C}$ gives that C does not intersect $M \otimes_R \text{Frac}(R)$. By Chevalley's theorem, the projection of C on $\text{Spec}(R[1/N])$ is constructible. As it does not contain the generic point and R has Krull dimension 1, this projection is a finite set. Thus we can change N so that C is empty, meaning that the $P_{\mathcal{F} \times_{\mathcal{M}} M/M}^{ij}$'s are locally free. $\square$

Lemma 7.3.18. *Let (R, M, F, ω) be a family of polarized varieties in mixed characteristic, where M (and then F) is a scheme. Let d be the dimension of the fibers of F over M . Let $n \in \mathbb{N}$ be such that $n \leq d$. Suppose that for a morphism of rings $R \rightarrow \mathbb{C}$, the period map of the $\mathbb{Z}$ -PVHS defined by the primitive cohomology of degree n of F over M is immersive. Then there exists $N \in \mathbb{N}_{>0}$ such that for every algebraically closed field k , every morphism of rings $R[1/N] \rightarrow k$, every k -point x of M , every $p \in \mathbb{N}$ with $1 \leq p \leq d$, every $\beta \in {}^{\text{TB}}_+(M \otimes_R k)_x$, the following holds.*

Denote by F_x the fiber of F over M at x . Denote by P_x^n the infinitesimal system of Hodge bundles $P_{F \otimes_R k/M \otimes_R k, x}^n$ on $T_{M \otimes_R k, x}$. Then there exists:

- a vector subspace $S \subset T_{M \otimes_R k, x}$ such that $\beta \in \Lambda^p S$,
- a vector $v \in T_{M \otimes_R k, x}$ whose action $\delta \in \text{End}(P_x^n)^{-1}$ on P_x^n is nonzero,

- a finite set A ,
- for every $a \in A$, an infinitesimal system of Hodge bundles E_a on S ,
- for every $a \in A$, a finite dimensional graded k -vector space H_a ,
- for every $a \in A$, an element $\delta_a \in \text{End}(H_a)^{-1}$,
- an isomorphism $\alpha : \bigoplus_{a \in A} E_a \otimes H_a \xrightarrow{\sim} P_x^n$ of infinitesimal systems of Hodge bundles on S ,

such that via the isomorphism α , δ corresponds to the sum of the δ_a 's (i.e. $\delta = \alpha \circ (\bigoplus_{a \in A} \text{id}_{E_a} \otimes \delta_a) \circ \alpha^{-1}$).

Proof. By Lemma 7.3.17, we can suppose that for every $i, j \in \mathbb{N}$ with $i + j = n$, the $\mathcal{O}_M$ -module $P_{F/M}^{ij}$ is locally free. For $p' \in \mathbb{N}$, denote by $\mathbf{Gr}_{p'}$ the smooth proper M -scheme that parametrizes vector subspaces of dimension p' of $T_{M/R}$ (in particular $\mathbf{Gr}_1 = \mathbb{P}\Omega_{M/R}^1$). Denote $B := \mathbf{Gr}_p \times_M \mathbf{Gr}_1$ and by q (resp. q_p) the projection of B on M (resp. $\mathbf{Gr}_p$). There is a tautological locally free $\mathcal{O}_B$ -submodule T_p (resp. T_1) of $q^*T_{M/R}$ of rank p (resp. 1).

Let $e = (e_a^i)_{\substack{0 \leq a \\ 0 \leq i \leq n}}$ and $h = (h_a^i)_{\substack{0 \leq a \\ 0 \leq i \leq n}}$ be two elements of $(\mathbb{N}^{n+1})^{\mathbb{N}}$ such that the sum $\sum_{\substack{0 \leq a \\ 0 \leq j \leq i}} e_a^j h_a^{i-j}$ is the rank of $P_{F/M}^{i, n-i}$ if $0 \leq i \leq n$ and 0 if $i > n$ and

such that the sequence $\left(\sum_{0 \leq j \leq i} e_a^j h_a^{i-j} \right)_{0 \leq a}$ is non-increasing. The families of

numbers e and h describe possible dimensions for the vector spaces E_a^i and H_a^i . Consider now the following locally free $\mathcal{O}_B$ -module:

$$\begin{aligned} \Xi := & \bigoplus_{\substack{0 \leq a \\ 1 \leq i \leq n}} \text{Hom}_{\mathcal{O}_B} \left(T_p, \text{Hom}_{\mathcal{O}_B} \left(\mathcal{O}_B^{\oplus e_a^i}, \mathcal{O}_B^{\oplus e_a^{i-1}} \right) \right) \\ & \oplus \bigoplus_{\substack{0 \leq a \\ 1 \leq i \leq n}} \text{Hom}_{\mathcal{O}_B} \left(T_1, \text{Hom}_{\mathcal{O}_B} \left(\mathcal{O}_B^{\oplus h_a^i}, \mathcal{O}_B^{\oplus h_a^{i-1}} \right) \right) \\ & \oplus \bigoplus_{0 \leq i \leq n} \text{Hom}_{\mathcal{O}_B} \left(\bigoplus_{\substack{0 \leq a \\ 0 \leq j \leq i}} \mathcal{O}_B^{\oplus e_a^j} \otimes_{\mathcal{O}_B} \mathcal{O}_B^{\oplus h_a^{i-j}}, q^* P_{F/M}^{i, n-i} \right). \end{aligned}$$

Denote by C the total space of Ξ and by r the projection $C \rightarrow B$. There are tautological morphisms of $\mathcal{O}_C$ -modules:

- $\delta'_i : r^*T_p \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{O}_C^{\oplus e_a^i}, \mathcal{O}_C^{\oplus e_a^{i-1}})$ for $0 \leq a, 1 \leq i \leq n$,

- $\delta_a^i : r^*T_1 \rightarrow \text{Hom}_{\mathcal{O}_C}(\mathcal{O}_C^{\oplus h_a^i}, \mathcal{O}_C^{\oplus h_a^{i-1}})$ for $0 \leq a, 1 \leq i \leq n$,
- $\alpha^i : \bigoplus_{\substack{0 \leq a \\ 0 \leq j \leq i}} \mathcal{O}_C^{\oplus e_a^j} \otimes_{\mathcal{O}_C} \mathcal{O}_C^{\oplus h_a^{i-j}} \rightarrow r^*q^*P_{F/M}^{i,n-i}$ for $0 \leq i \leq n$.

Moreover, for every i with $1 \leq i \leq n$, denote by θ^i the morphism of $\mathcal{O}_C$ -modules $r^*q^*T_{M/R} \rightarrow \text{Hom}_{\mathcal{O}_C}(r^*q^*P_{F/M}^{i,n-i}, r^*q^*P_{F/M}^{i-1,n-i+1})$ given by the product by the Kodaira-Spencer map of the family $F \rightarrow M$.

Now, denote by $D_{e,h}$ the set of points x (with residue field denoted by $\kappa(x)$) of C such that in the fibre over x :

- there exists $0 \leq a, 0 \leq i \leq n$ and $1 \leq j \leq n$ such that $h_a^i \neq 0$ and $\delta_a^j \neq 0$,
- every α^i is an isomorphism,
- for every $w \in (r^*T_p)_x \subset (r^*q^*T_{M/R})_x$, every $a \in \mathbb{N}$, every i with $1 \leq i \leq n$, every j with $0 \leq j \leq i$, every $\sigma \in \kappa(x)^{\oplus e_a^j}$, every $\tau \in \kappa(x)^{\oplus h_a^{i-j}}$, we have $(\theta^i(w) \circ \alpha^i)(\sigma \otimes \tau) = (\alpha^{i-1} \circ (\delta_a^j(w) \otimes \text{id}))(\sigma \otimes \tau)$,
- for every $v \in (r^*T_1)_x \subset (r^*q^*T_{M/R})_x$, every $a \in \mathbb{N}$, every i with $1 \leq i \leq n$, every j with $0 \leq j \leq i$, every $\sigma \in \kappa(x)^{\oplus e_a^j}$, every $\tau \in \kappa(x)^{\oplus h_a^{i-j}}$, we have $(\theta^i(v) \circ \alpha^i)(\sigma \otimes \tau) = (\alpha^{i-1} \circ (\text{id} \otimes \delta_a^{i-j}(v)))(\sigma \otimes \tau)$.

Then $D_{e,h}$ is a locally closed subset of C . There are finitely many possible choices for e and h , thus the union D of the $D_{e,h}$'s is a constructible subset of C . By Chevalley's theorem, its projection $D' := (q_p \circ r)(D)$ on $\mathbf{Gr}_p$ is constructible. Consider the inverse image Y of D in (the total space of) the vector bundle $\Lambda^p T_p$ on $\mathbf{Gr}_p$. We have a canonical M -morphism $\Lambda^p T_p \rightarrow \Lambda^p T_{M/R}$. Denote by Z the image of Y : again, it is constructible by Chevalley's theorem.

To prove the statement, it is now sufficient to prove that there exists $N \in \mathbb{N}_{>0}$ such that for every algebraically closed field k and every morphism of rings $R[1/N] \rightarrow k$, the following statement holds:

(*) for every $p \in \mathbb{N}$ with $1 \leq p \leq d$, the image of $\mathbb{T}\mathbb{B}_+^p(M \otimes_R k)$ in $\Lambda^p T_{M/R}$ is included in Z .

Indeed, in this case, as k is algebraically closed, every element β of $\mathbb{T}\mathbb{B}_+^p(M \otimes_R k)$ is the image of a k -point y in Y . Its image in $\mathbf{Gr}_p$ is a k -point s' in D' . This s' gives the vector space S of the statement of the lemma. Consider a k -point s in D above s' : then the image of s in $\mathbf{Gr}_1$ gives a line of tangent vectors, the v of the statement of the lemma can be chosen

in this line. The statement of the lemma is verified by definition of $D_{e,h}$, with $E_a^i := k^{\oplus e_a^i}$ and $H_a^i := k^{\oplus h_a^i}$.

Actually, (*) is true even when k is not algebraically closed. Using Proposition 3.4.9 and Proposition 3.4.7 as in the proof of Proposition 7.3.2, it reduces to the case $k = \mathbb{C}$. Now (*) is given by Theorem 4.1.5 and Proposition 4.6.5. $\square$

Now we can prove the theorem.

Proof of Theorem 7.3.14. By Lemma 7.3.17, there exists $N' \in \mathbb{N}_{>0}$, a smooth and quasiprojective $R[1/N']$ -scheme M and a finite étale covering $R[1/N']$ -map $M \rightarrow \mathcal{M} \otimes_R R[1/N']$. Then Lemma 7.3.18 applied to the family $(R[1/N'], M, \mathcal{F} \times_{\mathcal{M}} M, \omega \times_{\mathcal{M}} M)$ gives information on $\mathbb{B}_+^p(M \otimes_{R[1/N']} k)$ for every algebraically closed field k with a morphism $R[1/N''] \rightarrow k$ for some $N'' \in \mathbb{N}_{>0}$. Thanks to Lemma 3.4.8, for some $N \in \mathbb{N}_{>0}$ divisible by N'' , for every field k with a morphism $R[1/N] \rightarrow k$, $M \otimes_{R[1/N']} k$ is projectively log-compactifiable. Now apply Corollary 3.5.7: for every smooth k -variety V and all but finitely many k -morphisms $f : V \rightarrow \mathcal{M} \otimes_R k$, there exists a lift $\tilde{f} : \tilde{V} \rightarrow M \otimes_{R[1/N]} k$ of f with $\tilde{V}$ a finite étale covering of V such that for every $p \in \mathbb{N}$ with $1 \leq p \leq \dim M \otimes_{R[1/N]} k$, we have $\text{Im}(\Lambda^p d\tilde{f}) \subset {}^t\mathbb{B}_+^p(M \otimes_{R[1/N]} k)$.

Let x be a k -point of $\tilde{V}$. Let p be the rank of $d\tilde{f}_x$, let $v_1, \dots, v_p \in T_{\tilde{V},x}$ be a basis of a complement of $\ker d\tilde{f}_x$. Then $\Lambda^p d\tilde{f}(v_1 \wedge \dots \wedge v_p)$ is in $\Lambda^p S$ with $S \subset T_{M \otimes_{R[1/N]} k, \tilde{f}(x)}$ as in the statement of Lemma 7.3.18. As $(d\tilde{f}(v_1), \dots, d\tilde{f}(v_p))$ is free, we have $d\tilde{f}(v_i) \in S$ for every $i \in \mathbb{N}_{>0}$ with $i \leq p$. As the $d\tilde{f}(v_i)$'s generate the image of $d\tilde{f}_x$, we have $\text{Im}(d\tilde{f}_x) \subset S$. As the Kodaira-Spencer map is compatible with pull-backs (here by f), this gives the theorem. $\square$

Bibliography

- [1] Dan Abramovich. “Subvarieties of semiabelian varieties”. In: *Compositio Mathematica* 90.1 (1994), pp. 37–52.
- [2] Yves André. “On the Kodaira-Spencer map of abelian schemes”. In: *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*. 5th ser. 17 (2017), pp. 1397–1416.
- [3] Avner Ash, David Mumford, Michael Rapoport, and Yung-sheng Tai. *Smooth compactifications of locally symmetric varieties*. Cambridge University Press, 2010.
- [4] Walter L Baily and Armand Borel. “Compactification of arithmetic quotients of bounded symmetric domains”. In: *Annals of mathematics* (1966), pp. 442–528.
- [5] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. “o-minimal GAGA and a conjecture of Griffiths”. In: *Inventiones mathematicae* 232.1 (2023), pp. 163–228.
- [6] Benjamin Bakker, Yohan Brunebarbe, and Jacob Tsimerman. *The linear Shafarevich conjecture for quasiprojective varieties and algebraicity of Shafarevich morphisms*. 2024. arXiv: 2408.16441 [math.AG]. URL: <https://arxiv.org/abs/2408.16441>.
- [7] Benjamin Bakker, Bruno Klingler, and Jacob Tsimerman. “Tame topology of arithmetic quotients and algebraicity of Hodge loci”. In: *Journal of the American Mathematical Society* 33.4 (2020), pp. 917–939.
- [8] Thomas Bauer, Sándor J Kovács, Alex Küronya, Ernesto C Mistretta, Tomasz Szemberg, and Stefano Urbinati. “On positivity and base loci of vector bundles”. In: *European Journal of Mathematics* 1.2 (2015), pp. 229–249.
- [9] Olivier Benoist. “Construction de courbes sur les surfaces K3”. In: *Séminaire Bourbaki* (2014), p. 66.
- [10] Caucher Birkar. “The augmented base locus of real divisors over arbitrary fields”. In: *Mathematische Annalen* 368.3 (2017), pp. 905–921.

- [11] Armand Borel. “Introduction aux groupes arithmétiques. Publications de l’Institut de Mathématique de l’Université de Strasbourg, XV”. In: *Actualités scientifiques et industrielles* 1341 (1969).
- [12] Armand Borel. “Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem”. In: *Journal of Differential Geometry* 6.4 (1972), pp. 543–560.
- [13] Armand Borel and Lizhen Ji. *Compactifications of symmetric and locally symmetric spaces*. Springer, 2006.
- [14] Armand Borel and Raghavan Narasimhan. “Uniqueness conditions for certain holomorphic mappings”. In: *Inventiones mathematicae* 2.4 (1967), pp. 247–255.
- [15] Sébastien Boucksom. “Cônes positifs des variétés complexes compactes”. PhD thesis. Université Joseph-Fourier-Grenoble I, 2002.
- [16] Sébastien Boucksom. “Divisorial Zariski decompositions on compact complex manifolds”. In: *Annales scientifiques de l’Ecole normale supérieure*. Vol. 37. 1. 2004, pp. 45–76.
- [17] Sébastien Boucksom. “On the volume of a line bundle”. In: *International Journal of Mathematics* 13.10 (2002), pp. 1043–1063.
- [18] Sébastien Boucksom, Amaël Broustet, and Gianluca Pacienza. “Uniruledness of stable base loci of adjoint linear systems via Mori theory”. In: *Mathematische Zeitschrift* 275 (2013), pp. 499–507.
- [19] Sébastien Boucksom, Salvatore Cacciola, and Angelo Felice Lopez. “Augmented base loci and restricted volumes on normal varieties”. In: *Mathematische Zeitschrift* 278.3 (2014), pp. 979–985.
- [20] Yohan Brunebarbe. “A strong hyperbolicity property of locally symmetric varieties”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 53.6 (2020), pp. 1545–1560.
- [21] Yohan Brunebarbe. *Semi-positivity from Higgs bundles*. 2017. arXiv: 1707.08495 [math.AG]. URL: <https://arxiv.org/abs/1707.08495>.
- [22] Yohan Brunebarbe. “Symmetric differentials and variations of Hodge structures”. In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2018.743 (2018), pp. 133–161.
- [23] Yohan Brunebarbe and Benoit Cadorel. “Hyperbolicity of varieties supporting a variation of Hodge structure”. In: *International Mathematics Research Notices* 2020.6 (2020), pp. 1601–1609.
- [24] Yohan Brunebarbe, Bruno Klingler, and Burt Totaro. “Symmetric differentials and the fundamental group”. In: (2013).

- [25] Benoît Cadorel. “Subvarieties of quotients of bounded symmetric domains”. In: *Mathematische Annalen* 384.1 (2022), pp. 1–38.
- [26] Benoît Cadorel. “Symmetric differentials on complex hyperbolic manifolds with cusps”. In: *Journal of Differential Geometry* 118.3 (2021), pp. 373–398.
- [27] Benoît Cadorel and Ya Deng. *Picard hyperbolicity of manifolds admitting nilpotent harmonic bundles*. 2021. arXiv: 2107.07550 [math.AG]. URL: <https://arxiv.org/abs/2107.07550>.
- [28] James Carlson, Mark Green, Phillip Griffiths, and Joe Harris. “Infinitesimal variations of Hodge structure (I)”. In: *Compositio Mathematica* 50.2-3 (1983), pp. 109–205.
- [29] Eduardo Cattani, Aroldo Kaplan, and Wilfried Schmid. “Degeneration of Hodge structures”. In: *Annals of Mathematics* 123.3 (1986), pp. 457–535.
- [30] Pierre Deligne. *Équations différentielles à points singuliers réguliers*. Vol. 163. Springer, 2006.
- [31] Pierre Deligne. “Théorie de hodge: II”. In: *Publications Mathématiques de l’IHÉS* 40 (1971), pp. 5–57.
- [32] Pierre Deligne. “Un théoreme de finitude pour la monodromie”. In: *Discrete groups in geometry and analysis: papers in honor of GD Mostow on his sixtieth birthday*. Springer, 1987, pp. 1–19.
- [33] Pierre Deligne. “Variétés de Shimura: interprétation modulaire, et techniques de construction de modeles canoniques”. In: *Proceedings of Symposia in Pure Mathematics*. Vol. 33. American Mathematical Society, 1979, pp. 247–289.
- [34] Ya Deng. “Big Picard theorems and algebraic hyperbolicity for varieties admitting a variation of Hodge structures”. In: *Épjournal de Géométrie Algébrique* 7 (2023).
- [35] Lou van den Dries and Chris Miller. “On the real exponential field with restricted analytic functions”. In: *Israel Journal of Mathematics* 85 (1994), pp. 19–56.
- [36] Lawrence Ein, Robert Lazarsfeld, Mircea Mustață, Michael Nakamaye, and Mihnea Popa. “Asymptotic invariants of base loci”. In: *Annales de l’institut Fourier*. Vol. 56. 6. 2006, pp. 1701–1734.
- [37] Philip Engel, Alice Lin, and Salim Tayou. “Shafarevich’s Conjecture for Families of Hypersurfaces over Function Fields”. In: *International Mathematics Research Notices* 2025.15 (2025), rnaf225.

- [38] Gerd Faltings and Ching-Li Chai. *Degeneration of Abelian Varieties*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1990. ISBN: 978-3-662-02632-8.
- [39] Gérard Freixas i Montplet. “Généralisations de la théorie de l’intersection arithmétique”. PhD thesis. Paris 11, 2007.
- [40] Phillip Griffiths and Wilfried Schmid. “Locally homogeneous complex manifolds”. In: (1969).
- [41] Alexandre Grothendieck. “Revêtement étales et groupe fondamental (SGA1)”. In: *Lecture Notes in Math.* 288 (1971).
- [42] Robin Hartshorne. *Algebraic geometry*. Vol. 52. Springer Science & Business Media, 2013.
- [43] Robin Hartshorne. “Stable reflexive sheaves”. In: *Mathematische annalen* 254.2 (1980), pp. 121–176.
- [44] Sigurdur Helgason. *Differential geometry and symmetric spaces*. Vol. 341. American Mathematical Soc., 2001.
- [45] Daniel Huybrechts. *The geometry of cubic hypersurfaces*. Vol. 206. Cambridge University Press, 2023.
- [46] Ariyan Javanpeykar and Robert Kucharczyk. “Algebraicity of analytic maps to a hyperbolic variety”. In: *Mathematische Nachrichten* 293.8 (2020), pp. 1490–1504.
- [47] Ariyan Javanpeykar and Daniel Litt. “Integral points on algebraic subvarieties of period domains: from number fields to finitely generated fields”. In: *manuscripta mathematica* 173.1 (2024), pp. 23–44.
- [48] Ariyan Javanpeykar and Daniel Loughran. “The moduli of smooth hypersurfaces with level structure”. In: *manuscripta mathematica* 154.1 (2017), pp. 13–22.
- [49] Shoshichi Kobayashi. *Hyperbolic complex spaces*. Vol. 318. Springer Science & Business Media, 2013.
- [50] Shoshichi Kobayashi and Takushiro Ochiai. “Meromorphic mappings onto compact complex spaces of general type”. In: *Inventiones mathematicae* 31.1 (1975), pp. 7–16.
- [51] János Kollár. “Subadditivity of the Kodaira dimension: fibers of general type”. In: *Algebraic geometry, Sendai, 1985*. Vol. 10. Mathematical Society of Japan, 1987, pp. 361–399.
- [52] Kefeng Liu, Xiaofeng Sun, Xiaokui Yang, and Shing-Tung Yau. “Curvatures of moduli space of curves and applications”. In: *Asian Journal of Mathematics* 21.5 (2017), pp. 841–854.

- [53] Zhiqin Lu. “On the geometry of classifying spaces and horizontal slices”. In: *American Journal of Mathematics* 121.1 (1999), pp. 177–198.
- [54] Gregori A Margulis. *Discrete subgroups of semisimple Lie groups*. Vol. 17. Springer Science & Business Media, 1991.
- [55] Mireille Martin-Deschamps and Renée Lewin-Ménégaux. “Applications rationnelles séparables dominantes sur une variété de type général”. In: *Bulletin de la Société mathématique de France* 106 (1978), pp. 279–287.
- [56] Toshiki Miyano and Junjiro Noguchi. “Moduli spaces of harmonic and holomorphic mappings and Diophantine geometry”. In: *Prospects in Complex Geometry: Proceedings of the 25th Taniguchi International Symposium held in Katata, and the Conference held in Kyoto, July 31–August 9, 1989*. Springer. 2006, pp. 227–253.
- [57] Ngaiming Mok. “Characterization of certain holomorphic geodesic cycles on quotients of bounded symmetric domains in terms of tangent subspaces”. In: *Compositio Mathematica* 132.3 (2002), pp. 289–309.
- [58] Ngaiming Mok. *Metric rigidity theorems on Hermitian locally symmetric manifolds*. Vol. 6. World Scientific, 1989.
- [59] Ngaiming Mok. “Rigidity problems on compact quotients of bounded symmetric domains”. In: *AMS IP STUDIES IN ADVANCED MATHEMATICS* 39 (2007), p. 201.
- [60] Ngaiming Mok. “Uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domains”. In: *Annals of Mathematics* 125.1 (1987), pp. 105–152.
- [61] Laurent Moret-Bailly. “Familles de courbes et de variétés abéliennes sur P^1 - II. Exemples”. In: *Séminaire sur les pinceaux de courbes de genre au moins deux*. Astérisque, 1981, pp. 125–140.
- [62] David Bryant Mumford. “Hirzebruch’s proportionality theorem in the non-compact case”. In: *Inventiones mathematicae* (1977).
- [63] Alan Michael Nadel. “The nonexistence of certain level structures on abelian varieties over complex function fields”. In: *Annals of Mathematics* 129.1 (1989), pp. 161–178.
- [64] Masayoshi Nagata. “Imbedding of an abstract variety in a complete variety”. In: *Journal of Mathematics of Kyoto University* 2.1 (1962), pp. 1–10.
- [65] Michael Nakamaye. “Stable base loci of linear series”. In: *Mathematische Annalen* 318 (2000), pp. 837–847.

- [66] Junjiro Noguchi. “Logarithmic jet spaces and extensions of de Franchis’ theorem”. In: *Contributions to Several Complex Variables: In Honour of Wilhelm Stoll*. Springer, 1986, pp. 227–249.
- [67] Junjiro Noguchi. “Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and locally symmetric spaces”. In: *Inventiones mathematicae* 93.1 (1988), pp. 15–34.
- [68] Junjiro Noguchi and Toshikazu Sunada. “Finiteness of the family of rational and meromorphic mappings into algebraic varieties”. In: *American Journal of Mathematics* 104.4 (1982), pp. 887–900.
- [69] Fabrice Orgogozo. “Altérations et groupe fondamental premier à p ”. In: *Bulletin de la société mathématique de France* 131.1 (2003), pp. 123–147.
- [70] CAM Peters. “Rigidity for variations of Hodge structure and Arakelov-type finiteness theorems”. In: *Compositio Mathematica* 75.1 (1990), pp. 113–126.
- [71] Ya’acov Peterzil and Sergei Starchenko. “Complex analytic geometry and analytic-geometric categories”. In: *J. Reine Angew. Math.* 626 (2009), pp. 39–74.
- [72] Martin Pikaart and A Johan de Jong. “Moduli of curves with non-abelian level structure”. In: *The moduli space of curves*. Springer, 1995, pp. 483–509.
- [73] Bjorn Poonen. *Rational points on varieties*. Vol. 186. American Mathematical Society, 2023.
- [74] Erwan Rousseau. “Hyperbolicity, automorphic forms and Siegel modular varieties”. In: *Annales Scientifiques de l’École Normale Supérieure*. Vol. 49. 1. 2016, pp. 249–255.
- [75] Claude Sabbah and Christian Schnell. *The MHM project (version 2)*. book in progress, 2019. URL: <http://www.cmls.polytechnique.fr/perso/sabbah/MHMPProject/mhm.html>.
- [76] Masa-Hiko Saito and Steven Zucker. “Classification of non-rigid families of K3 surfaces and a finiteness theorem of Arakelov type”. In: *Mathematische Annalen* 289.1 (1991), pp. 1–31.
- [77] Wilfried Schmid. “Variation of Hodge structure: the singularities of the period mapping”. In: *Inventiones mathematicae* 22.3 (1973), pp. 211–319.
- [78] Edoardo Sernesi. *Deformations of algebraic schemes*. Springer, 2006.

- [79] Mao Sheng and Kang Zuo. “Polarized variation of Hodge structures of Calabi–Yau type and characteristic subvarieties over bounded symmetric domains”. In: *Mathematische Annalen* 348.1 (2010), pp. 211–236.
- [80] The Stacks Project Authors. *Stacks Project*. <https://stacks.math.columbia.edu>. 2018.
- [81] Michael Temkin. “Functorial desingularization over $\mathbb{Q}$: boundaries and the embedded case”. In: *Israel Journal of Mathematics* 224 (2018), pp. 455–504.
- [82] Wing-Keung To. “Hermitian metrics of semi-negative curvature on quotients of bounded symmetric domains”. In: *Inventiones mathematicae* 95.3 (1989), pp. 559–578.
- [83] Valentino Tosatti. “Nakamaye’s theorem on complex manifolds”. In: *Algebraic Geometry: Salt Lake City 2015*. American Mathematical Society, June 2018. ISBN: 9781470446789. DOI: 10.1090/pspum/097.1. URL: <http://dx.doi.org/10.1090/pspum/097.1>.
- [84] Ryuji Tsushima. “Rational maps to varieties of hyperbolic type”. In: (1979).
- [85] Lou Van den Dries. *Tame topology and o-minimal structures*. Vol. 248. Cambridge university press, 1998.
- [86] Eckart Viehweg. “Arakelov inequalities”. In: *Surveys in Differential Geometry* 13 (Jan. 2008). DOI: 10.4310/SDG.2008.v13.n1.a8.
- [87] Eckart Viehweg. “Positivity of direct image sheaves and applications to families of higher dimensional manifolds”. In: (2001).
- [88] Joseph A Wolf. “Fine structure of Hermitian symmetric spaces”. In: *Symmetric spaces, Pure Appl. Math.* 8 (1972), pp. 271–357.
- [89] Joseph A Wolf and Adam Korányi. “Generalized Cayley transformations of bounded symmetric domains”. In: *American Journal of Mathematics* 87.4 (1965), pp. 899–939.
- [90] Steven Zucker. “Locally homogeneous variations of Hodge structure”. In: *Enseign. Math.(2)* 27.3-4 (1981), pp. 243–276.
- [91] Kang Zuo. “On the negativity of kernels of Kodaira-Spencer maps on Hodge bundles and applications”. In: *Asian Journal of Mathematics* 4 (2000), pp. 279–301.