

Dynamics of vortex cap solutions on the rotating unit sphere

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Abstract

In this work, we analytically study the existence of periodic vortex cap solutions for the homogeneous and incompressible Euler equations on the rotating unit 2-sphere, which was numerically conjectured in [28, 29, 60, 61]. Such solutions are piecewise constant vorticity distributions, subject to the Gauss constraint and rotating uniformly around the vertical axis. The proof is based on the bifurcation from zonal solutions given by spherical caps. For the one-interface case, the bifurcation eigenvalues correspond to Burbea’s frequencies obtained in the planar case but shifted by the rotation speed of the sphere. The two-interfaces case (also called band type or strip type solutions) is more delicate. Though, for any fixed large enough symmetry, and under some non-degeneracy conditions to avoid spectral collisions, we achieve the existence of at most two branches of bifurcation.

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1 Introduction

The purpose of this paper is to give an analytical proof of the emergence of uniformly rotating (around the z -axis) vortex caps with \mathbf{m} -fold symmetries ($\mathbf{m} \in \mathbb{N}^*$) for the incompressible Euler equations set on the rotating unit sphere \mathbb{S}^2 defined by

$$\mathbb{S}^2 \triangleq \left\{ (x, y, z) \in \mathbb{R}^3 \quad \text{s.t.} \quad x^2 + y^2 + z^2 = 1 \right\}.$$

In particular, we shall implement bifurcation techniques in order to find non-trivial vortex cap solutions close to the trivial stationary flat (spherical) ones, which was numerically conjectured in [28, 29, 61, 60]. In this introduction we present the model of interest, discuss some historical background, derive the contour dynamics equations, expose our results and give the organization of this work.

1.1 Euler equations on the rotating unit sphere

This study deals with the homogeneous incompressible Euler equations on the two dimensional unit sphere in rotation around the vertical axis. Such a model, sometimes called *barotropic*, is commonly used in geophysical fluid dynamics for meteorological predictions or to study the motion of planets' atmosphere. In order to present the model, we shall first recall some basic notions in differential calculus/geometry. The set \mathbb{S}^2 is endowed with a smooth manifold structure described by the following two charts

$$\begin{aligned} C_1 : (0, \pi) \times (0, 2\pi) &\rightarrow \mathbb{R}^3 \\ (\theta, \varphi) &\mapsto (\sin(\theta) \cos(\varphi), \sin(\theta) \sin(\varphi), \cos(\theta)), \\ C_2 : (0, \pi) \times (0, 2\pi) &\rightarrow \mathbb{R}^3 \\ (\vartheta, \phi) &\mapsto (-\sin(\vartheta) \cos(\phi), -\cos(\vartheta), -\sin(\vartheta) \sin(\phi)). \end{aligned}$$

For our purpose, we shall mainly work in the chart C_1 where the variables θ and φ are called *colatitude* and *longitude*, respectively. Notice that the physical literature mentioned above rather considers the latitude/longitude convention, but we found more convenient to work with the other one. In the colatitude/longitude chart C_1 , the Riemannian metric of \mathbb{S}^2 is given by

$$\mathbf{g}_{\mathbb{S}^2}(\theta, \varphi) \triangleq d\theta^2 + \sin^2(\theta)d\varphi^2. \quad (1.1)$$

Therefore, denoting N and S the north and south poles, we have that for any $p \in \mathbb{S}^2 \setminus \{N, S\}$, an orthonormal basis of the tangent space $T_p\mathbb{S}^2$ is given by

$$\mathbf{e}_\theta \triangleq \partial_\theta, \quad \mathbf{e}_\varphi \triangleq \frac{1}{\sin(\theta)}\partial_\varphi.$$

We have used the classical identification between tangent vectors and directional differentiations. In these coordinates, the Riemannian volume is given by

$$d\sigma = \sin(\theta)d\theta d\varphi.$$

Therefore, for any function $\mathbf{f} : \mathbb{S}^2 \rightarrow \mathbb{R}$, we define

$$f(\theta, \varphi) \triangleq \mathbf{f}(C_1(\theta, \varphi)), \quad \int_{\mathbb{S}^2} \mathbf{f}(\xi)d\sigma(\xi) \triangleq \int_0^{2\pi} \int_0^\pi f(\theta, \varphi) \sin(\theta)d\theta d\varphi. \quad (1.2)$$

In the sequel, with a small abuse of notation, we shall denote f for both \mathbf{f} or f with no possible confusion according to the context, since one is in the cartesian variables ξ and the other one is in the spherical coordinates (θ, φ) . The gradient of f is defined as follows

$$\nabla f(\theta, \varphi) \triangleq \partial_\theta f(\theta, \varphi)\mathbf{e}_\theta + \frac{\partial_\varphi f(\theta, \varphi)}{\sin(\theta)}\mathbf{e}_\varphi.$$

Similarly, we define its orthogonal as

$$\nabla^\perp f(\theta, \varphi) \triangleq J\nabla f(\theta, \varphi), \quad \text{Mat}_{(\mathbf{e}_\theta, \mathbf{e}_\varphi)}(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Laplace-Beltrami operator applied to f is defined by

$$\Delta f(\theta, \varphi) \triangleq \frac{1}{\sin(\theta)}\partial_\theta [\sin(\theta)\partial_\theta f(\theta, \varphi)] + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 f(\theta, \varphi).$$

For a vector field $U(\theta, \varphi) = U_\theta(\theta, \varphi)\mathbf{e}_\theta + U_\varphi(\theta, \varphi)\mathbf{e}_\varphi$, the divergence expresses as

$$(\nabla \cdot U)(\theta, \varphi) \triangleq \frac{1}{\sin(\theta)}\partial_\theta [\sin(\theta)U_\theta(\theta, \varphi)] + \frac{1}{\sin(\theta)}\partial_\varphi U_\varphi(\theta, \varphi).$$

The incompressible Euler equations on the rotating sphere \mathbb{S}^2 with angular rotation speed $\tilde{\gamma}$ are given by

$$(E_{\tilde{\gamma}}) \begin{cases} \partial_t \Omega(t, \theta, \varphi) + U(t, \theta, \varphi) \cdot \nabla (\Omega(t, \theta, \varphi) - 2\tilde{\gamma} \cos(\theta)) = 0, \\ U(t, \theta, \varphi) = \nabla^\perp \Psi(t, \theta, \varphi), \\ \Delta \Psi(t, \theta, \varphi) = \Omega(t, \theta, \varphi). \end{cases} \quad (1.3)$$

The reader is referred to [43, Sec. 13.4.1] and [66] (see also [67]) for a rather complete introduction to these equations. We mention that the term $-2\tilde{\gamma}U(t, \theta, \varphi) \cdot \nabla \cos(\theta)$ corresponds to the Coriolis force coming from

the rotation of the sphere. Additionally, the equations (1.3) must be completed by the physical impermeability condition

$$\forall \varphi \in [0, 2\pi], \quad U_\theta(0, \varphi) = 0 = U_\theta(\pi, \varphi).$$

In the sequel, we shall work with the following quantity called *absolute vorticity*

$$\bar{\Omega}(t, \theta, \varphi) \triangleq \Omega(t, \theta, \varphi) - 2\tilde{\gamma} \cos(\theta). \quad (1.4)$$

The second equation in (1.3) states that the velocity field U is divergence-free. Then, the divergence theorem implies the following so-called *Gauss constraint*

$$\forall t \geq 0, \quad \int_{\mathbb{S}^2} \bar{\Omega}(t, \xi) d\sigma(\xi) = \int_{\mathbb{S}^2} \Omega(t, \xi) d\sigma(\xi) = 0. \quad (1.5)$$

Notice that the first equality above is justified by

$$\int_{\mathbb{S}^2} [\bar{\Omega}(t, \xi) - \Omega(t, \xi)] d\sigma(\xi) = -4\pi\tilde{\gamma} \int_0^\pi \cos(\theta) \sin(\theta) d\theta = [\pi\tilde{\gamma} \cos(2\theta)]_0^\pi = 0.$$

According to [4], the stream function Ψ can be computed from the vorticity Ω through the following integral representation

$$\Psi(t, \xi) = \int_{\mathbb{S}^2} G(\xi, \xi') \Omega(t, \xi') d\sigma(\xi'), \quad G(\xi, \xi') \triangleq \frac{1}{2\pi} \log \left(\frac{|\xi - \xi'|_{\mathbb{R}^3}}{2} \right), \quad (1.6)$$

where $|\cdot|_{\mathbb{R}^3}$ is the usual Euclidean norm in \mathbb{R}^3 . In the colatitude/longitude coordinates, we have

$$G(\theta, \varphi, \theta', \varphi') = \frac{1}{4\pi} \log \left(1 - \cos(\theta) \cos(\theta') - \sin(\theta) \sin(\theta') \cos(\varphi - \varphi') \right) - \frac{\log(2)}{4\pi}. \quad (1.7)$$

In what follows, we shall denote for simplicity

$$D(\theta, \theta', \varphi, \varphi') \triangleq 1 - \cos(\theta) \cos(\theta') - \sin(\theta) \sin(\theta') \cos(\varphi - \varphi'). \quad (1.8)$$

Observe that we can write

$$\begin{aligned} D(\theta, \theta', \varphi, \varphi') &= 1 - \cos(\theta - \theta') + \sin(\theta) \sin(\theta') (1 - \cos(\varphi - \varphi')) \\ &= 2 \left[\sin^2 \left(\frac{\theta - \theta'}{2} \right) + \sin(\theta) \sin(\theta') \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right]. \end{aligned} \quad (1.9)$$

The above function D will play an important role since it describes the singularity of the integral operator defining the stream function. Indeed, with this last expression, we recover that

$$D(\theta, \theta', \varphi, \varphi') \geq 0 \quad \text{and} \quad D(\theta, \theta', \varphi, \varphi') = 0 \Leftrightarrow \theta = \theta' \text{ and } (\varphi = \varphi' \text{ or } \theta \in \{0, \pi\} \text{ or } \theta' \in \{0, \pi\}).$$

Notice that once one works with the colatitude/longitude coordinates instead of the physical ones, the Euclidean norm is deformed implying that the Green kernel is anisotropic and the north and south poles are degenerating points. This will generate extra complexity later when dealing with the regularity of the stream function in the new coordinates.

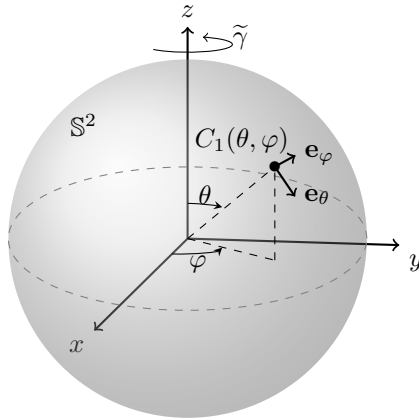


Figure 1: Convention colatitude/longitude for spherical coordinates.

1.2 Historical discussion

We shall expose here some relevant results linked to our work.

Vortex patches in the plane

Recall that the planar homogeneous incompressible Euler equations write

$$\begin{cases} \partial_t \boldsymbol{\omega}(t, x_1, x_2) + \mathbf{v}(t, x_1, x_2) \cdot \nabla \boldsymbol{\omega}(t, x_1, x_2) = 0, \\ \mathbf{v}(t, x_1, x_2) = \nabla_{\mathbb{R}^2}^{\perp} \Psi(t, x_1, x_2), \\ \Delta_{\mathbb{R}^2} \Psi(t, x_1, x_2) = \boldsymbol{\omega}(t, x_1, x_2), \end{cases} \quad \nabla_{\mathbb{R}^2}^{\perp} \triangleq \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} \end{pmatrix}, \quad \Delta_{\mathbb{R}^2} \triangleq \partial_{x_1}^2 + \partial_{x_2}^2. \quad (1.10)$$

Vortex patches are weak solutions to (1.10) in the Yudovich class and taking the form $t \mapsto \mathbf{1}_{D_t}$ where D_t is a bounded planar domain. The vorticity jump here is normalized to 1. The dynamics is described by the evolution of the boundary ∂D_t . More precisely, according to [55, p. 174], if $z(t, \cdot) : \mathbb{T} \rightarrow \partial D_t$ is a parametrization of (one connected component of) the boundary at time t , then it must solve the following contour dynamics equation

$$\operatorname{Im} \left(\partial_t z(t, x) \overline{\partial_x z(t, x)} \right) = \partial_x \left(\Psi(t, z(t, x)) \right). \quad (1.11)$$

The solutions that we construct in this study are the analogous to the V-states obtained in the planar case. These later are particular vortex patches where the dynamics is given by a uniform rotation of the initial domain around its center of mass (fixed to be the origin), namely $D_t = e^{i\Omega t} D_0$ with $\Omega \in \mathbb{R}$. As remarked by Rankine in 1858, any radial domain is a V-state rotating with any angular velocity $\Omega \in \mathbb{R}$. Later, Kirchhoff [62] discovered non-trivial explicit examples of V-states with elliptic shapes. Close to the unit disc, such solutions were first obtained numerically by Deem and Zabusky [26] and then analytically by Burbea [5] through bifurcation techniques. He constructed a countable family of local curves of V-states with \mathbf{m} -fold symmetries (i.e. invariance by $\frac{2\pi}{\mathbf{m}}$ angular rotation) bifurcating from the disc at the angular velocities

$$\Omega_{\mathbf{m}} = \frac{\mathbf{m} - 1}{2\mathbf{m}}. \quad (1.12)$$

The global continuation of the Burbea's bifurcation branches was found in [48] through global bifurcation arguments. Some rigidity results giving necessary conditions on the angular velocity to obtain non-trivial V-states can be found in [30, 51, 41]. In the last decade, there have been intensive rigorous studies on the subject, giving more properties of the V-states and exploring their existence around different topological structures. In particular, the bifurcation from the annulus

$$A_b \triangleq \{z \in \mathbb{C} \quad \text{s.t.} \quad b < |z| < 1\}, \quad b \in (0, 1),$$

has been studied by Hmidi-de la Hoz-Mateu-Verdera [25]. They proved that under the condition

$$\mathbf{p}(b, \mathbf{m}) \triangleq 1 + b^{\mathbf{m}} - \frac{1 - b^2}{2} \mathbf{m} < 0,$$

there are exactly two branches of periodic vortex patches, with two interfaces, emerging at the angular velocities

$$\Omega_{\mathbf{m}}^{\pm}(b) \triangleq \frac{1 - b^2}{4} \pm \frac{1}{2\mathbf{m}} \sqrt{\left(\frac{1 - b^2}{2} \mathbf{m} - 1 \right)^2 - b^{2\mathbf{m}}}. \quad (1.13)$$

The degenerate case $\mathbf{p}(b, \mathbf{m}) = 0$ has been discussed in [53, 69]. Periodic multiple patches were found via a desingularization of an appropriate distribution of point vortices. The first work studying the desingularization of two point vortices using contour dynamics equations is due to Hmidi-Mateu [54]. Other extensions, in the same spirit, have been achieved in [31, 32, 45, 50]. Similar results were obtained using variational arguments in [6, 7, 10, 68] or via gluing techniques in [21, 22]. Recently, the authors in [33] studied the global continuation of the desingularization of the vortex pairs. The existence of a non-uniform rotating vorticity distribution has been treated in [16, 37, 36]. In [40], Gómez-Serrano, Park and Shi have constructed stationary configurations of multi-layered patches with finite kinetic energy using Nash-Moser techniques. Some additional references can be found in [15, 24, 52, 55, 56].

Many of the results, mentioned above, apply not only to the planar Euler equations but also to other active scalar equations such as the generalized surface quasi-geostrophic equation or the quasi-geostrophic shallow-water equations, see [1, 8, 9, 11, 14, 16, 17, 23, 27, 31, 32, 33, 35, 34, 37, 38, 44, 45, 48, 50, 52, 53, 54, 56, 58, 65, 64].

We end this discussion by very recent new perspectives concerning the existence of quasi-periodic patches using the Nash-Moser scheme together with KAM theory, see [3, 39, 46, 47, 49, 57, 65].

Around stationary solutions on the rotating unit sphere

Looking for stationary solutions to (1.3) is equivalent to solving the following equation on the stream function

$$\nabla^\perp \Psi(\theta, \varphi) \cdot \nabla \left(\Delta \Psi(\theta, \varphi) - 2\tilde{\gamma} \cos(\theta) \right) = 0. \quad (1.14)$$

If $\Psi(\theta, \varphi) = \Psi(\theta)$ is longitude independent, then it automatically solves (1.14) because in this case, $\nabla^\perp \Psi(\theta)$ is colinear to \mathbf{e}_φ and $\nabla \left(\Delta \Psi(\theta) - 2\tilde{\gamma} \cos(\theta) \right)$ is colinear to \mathbf{e}_θ . Such type of solutions are called *zonal solutions*. Also, one can easily check that any solution of the semilinear elliptic problem

$$\Delta \Psi(\theta, \varphi) - 2\tilde{\gamma} \cos(\theta) = F(\Psi(\theta, \varphi)), \quad (1.15)$$

with $F \in C^1(\mathbb{R}, \mathbb{R})$, solves (1.14), but the converse is not true in general. Constantin and Germain [19] showed that any solution to (1.15) with $F' > -6$ must be zonal (modulo rotation) and stable in $H^2(\mathbb{S}^2)$ provided the additional constraint $F' < 0$. Notice that the -6 corresponds to the second eigenvalues of the Laplace-Beltrami operator. The zonal Rossby-Haurwitz stream functions of degree $n \in \mathbb{N}$ are special stationary solutions in the form

$$\Psi_n(\theta) = \beta Y_n^0(\theta) + \frac{2\tilde{\gamma}}{n(n+1)-2} \cos(\theta), \quad \beta \in \mathbb{R}^*,$$

where Y_n^0 is the spherical harmonic. We refer the reader to [2] for an introduction to spherical harmonics. In [19], the authors also discussed the local and global bifurcation of non-zonal solutions to (1.15) from Rossby-Haurwitz waves. They also proved the stability in $H^2(\mathbb{S}^2)$ of Rossby-Haurwitz zonal solutions of degree 2 as well as the instability in $H^2(\mathbb{S}^2)$ of more general non-zonal Rossby-Haurwitz type solutions. Very recently, the stability of the degree 2 Rossby-Haurwitz waves in $L^p(\mathbb{S}^2)$ spaces with $p \in (1, \infty)$ has been obtained by Cao-Wang-Zuo [12]. Recently, Nualart [63] proved the existence of non-zonal stationary solutions Gevrey-close to the zonal Rossby-Haurwitz stream functions of degree 2. The L^1 -Lyapunov stability of zonal monotone vorticities belonging to $L^p(\mathbb{S}^2)$ for $p \in (2, \infty)$ was discussed by Caprino-Marchioro [13]. We mention that stationary solutions to (1.3) correspond to traveling solutions of the non-rotating case (E_0). More generally, we have the following result, see [19, 63].

Lemma 1.1. *We consider two vorticities $\Omega, \tilde{\Omega}$ related through*

$$\Omega(t, \theta, \varphi) = -2c \cos(\theta) + \tilde{\Omega}(t, \theta, \varphi - ct),$$

with $c \in \mathbb{R}$, and associated to stream functions $\Psi, \tilde{\Psi}$ and velocity fields U, \tilde{U} :

$$\Psi(t, \theta, \varphi) = c \cos(\theta) + \tilde{\Psi}(t, \theta, \varphi - ct), \quad U(t, \theta, \varphi) = c \sin(\theta) \mathbf{e}_\varphi + \tilde{U}(t, \theta, \varphi - ct).$$

Then, the following are equivalent

- (i) (Ω, Ψ, U) is a solution to $(E_{\tilde{\gamma}})$.
- (ii) $(\tilde{\Omega}, \tilde{\Psi}, \tilde{U})$ is a solution to $(E_{\tilde{\gamma}+c})$.

For our later purposes, we shall prove that a longitude independent (absolute) vorticity generates a zonal (so stationary) flow. This is given by the following lemma, whose proof is postponed to Appendix A.4.

Lemma 1.2. *For any $\alpha \in \mathbb{R}$, we introduce the rotation of angle α around the z axis*

$$\mathcal{R}(\alpha) \triangleq \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO_3(\mathbb{R}).$$

Assume that

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \Omega(\mathcal{R}(\alpha)\xi) = \Omega(\xi),$$

or equivalently

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}(\mathcal{R}(\alpha)\xi) = \bar{\Omega}(\xi),$$

then

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \Psi(\mathcal{R}(\alpha)\xi) = \Psi(\xi).$$

This means that the flow is zonal (and thus stationary).

Patch type solutions on the rotating 2-sphere

Our analysis is strongly motivated by previous numerical works concerning the existence of patch type solutions for the rotating 2-sphere. The pioneering results are due to Dritschel-Polvani [28, 29] where they considered the sphere at rest ($\tilde{\gamma} = 0$) and found, numerically, vortex cap solutions with one and two interfaces. They also studied the numerical nonlinear stability. Later Kim [59] described the free boundary problem for patch type solutions in the rotating 2-sphere ($\tilde{\gamma} \neq 0$) by using the stereographic projection. In [61], Kim, Sakajo and Sohn numerically observed the existence of vortex caps (only one interface) and vortex bands (two interfaces) [60]. They also showed the linear stability of those solutions.

1.3 Dynamics of vortex cap solutions

Here, we introduce the notion of vortex cap solutions on the sphere, which are the analogous to vortex patches in the plane. Then, we derive the fundamental contour dynamics equations, introduced in this work to track the evolution of the caps interfaces. Our formulation (1.21) is derived by following the ideas of [55, p. 174], implemented in the context of vortex patches, but adapted to fit with the non Euclidean geometry of the sphere.

Fix $M \in \mathbb{N} \setminus \{0, 1\}$ and $(\omega_k)_{1 \leq k \leq M} \in \mathbb{R}^M$ such that

$$\forall k \in \llbracket 1, M-1 \rrbracket, \quad \omega_k \neq \omega_{k+1}. \quad (1.16)$$

Consider a partition of the unit sphere in the form

$$\mathbb{S}^2 = \bigsqcup_{k=1}^M \mathcal{C}_k(0),$$

where for any $k \in \llbracket 1, M-1 \rrbracket$, the boundary $\Gamma_k(0) \triangleq \partial \mathcal{C}_k(0) \cap \partial \mathcal{C}_{k+1}(0)$ is diffeomorphic to a circle. Take an initial condition in the form

$$\bar{\Omega}(0, \cdot) = \sum_{k=1}^M \omega_k \mathbf{1}_{\mathcal{C}_k(0)}. \quad (1.17)$$

The Gauss constraint (1.5) requires the following additional condition

$$\sum_{k=1}^M \omega_k \sigma(\mathcal{C}_k(0)) = 0. \quad (1.18)$$

Observe that, by virtue of (1.3)-(1.4), the absolute vorticity $\bar{\Omega}$ solves the nonlinear transport equation

$$\partial_t \bar{\Omega} + U \cdot \nabla \bar{\Omega} = 0.$$

Since the singularity of the Green function in (1.6) is logarithmic, then, similarly to Yudovich's theory [70] in the planar case, one can expect to obtain existence and uniqueness of a global in time weak solution which is Lagrangian, namely

$$\forall t \geq 0, \quad \forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}(t, \xi) = \bar{\Omega}(0, \Phi_t^{-1}(\xi)),$$

where

$$\forall \xi \in \mathbb{S}^2, \quad \partial_t \Phi_t(\xi) = U(t, \Phi_t(\xi)), \quad \Phi_0(\xi) = \xi.$$

Applying this remark with the initial condition (1.17) gives the following structure of the solution at any later time $t \geq 0$

$$\bar{\Omega}(t, \cdot) = \sum_{k=1}^M \omega_k \mathbf{1}_{\mathcal{C}_k(t)}, \quad \text{with} \quad \forall k \in \llbracket 1, M \rrbracket, \quad \mathcal{C}_k(t) \triangleq \Phi_t(\mathcal{C}_k(0)). \quad (1.19)$$

Since U is solenoidal, then the flow $t \mapsto \Phi_t$ is measure preserving

$$\forall k \in \llbracket 1, M \rrbracket, \quad \sigma(\mathcal{C}_k(t)) = \sigma(\mathcal{C}_k(0)).$$

Any solution in the form (1.19) satisfying the conditions (1.16) and (1.18) is called a *vortex cap solution*. From now on, we fix a $k \in \llbracket 1, M-1 \rrbracket$. Assume that the initial boundary $\Gamma_k(0)$ can be described as the zero level set of a certain C^1 regular function $\mathbf{h}_k : \mathbb{S}^2 \rightarrow \mathbb{R}$, namely

$$\Gamma_k(0) = \{ \xi \in \mathbb{S}^2 \quad \text{s.t.} \quad \mathbf{h}_k(\xi) = 0 \}.$$

Let us consider the following quantity

$$\forall t \geq 0, \quad \forall \xi \in \mathbb{S}^2, \quad F_k(t, \xi) \triangleq \mathbf{h}_k(\Phi_t^{-1}(\xi)). \quad (1.20)$$

Then, by construction $F_k(t, \cdot)$ describes the boundary $\Gamma_k(t) \triangleq \partial \mathcal{C}_k(t) \cap \partial \mathcal{C}_{k+1}(t)$. More precisely,

$$\Gamma_k(t) = \{ \xi \in \mathbb{S}^2 \quad \text{s.t.} \quad F_k(t, \xi) = 0 \}.$$

Differentiating the relation (1.20) with respect to time yields

$$\forall t \geq 0, \quad \forall \xi \in \mathbb{S}^2, \quad \partial_t F_k(t, \Phi_t(\xi)) + U(t, \Phi_t(\xi)) \cdot \nabla F(t, \Phi_t(\xi)) = 0.$$

Now, take a parametrization $z_k(t, \cdot) : \mathbb{T} \rightarrow \Gamma_k(t)$. We have

$$\forall t \geq 0, \quad \forall x \in \mathbb{T}, \quad F_k(t, z_k(t, x)) = 0.$$

Differentiating the previous relation with respect to time implies

$$\forall t \geq 0, \quad \forall x \in \mathbb{T}, \quad \partial_t F_k(t, z_k(t, x)) + \partial_t z_k(t, x) \cdot \nabla F_k(t, z_k(t, x)) = 0.$$

Putting together the foregoing calculations, we deduce that

$$\left[\partial_t z_k(t, x) - U(t, z_k(t, x)) \right] \cdot \nabla F_k(t, z_k(t, x)) = 0.$$

The above scalar products can be understood either as taken in the tangent space $T_{z_k(t, x)}\mathbb{S}^2 \cong \mathbb{R}^2$ or in the classical Euclidean space \mathbb{R}^3 , both are equivalent. Indeed, in the spherical coordinates the Euclidean metric writes

$$\mathbf{g}_{\mathbb{R}^3}(r, \theta, \varphi) \triangleq dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2).$$

Since the sphere is described by the equation $r = 1$, then the induced metric of $\mathbf{g}_{\mathbb{R}^3}$ on \mathbb{S}^2 is indeed $\mathbf{g}_{\mathbb{S}^2}$ as defined in (1.1). On $T_{z_k(t, x)}\mathbb{S}^2$, the operator J acts as a rotation of $-\frac{\pi}{2}$. Consequently, since $\partial_x z_k(t, x)$ is tangential to $\Gamma_k(t)$ and contained in $T_{z_k(t, x)}\mathbb{S}^2$, then $J\partial_x z_k(t, x)$ is orthogonal to $\Gamma_k(t)$ and contained in $T_{z_k(t, x)}\mathbb{S}^2$. In addition, since $\Gamma_k(t)$ is a level set of $F_k(t, \cdot)$, then $\nabla F_k(t, z_k(t, x))$ is also orthogonal to $\Gamma_k(t)$ and contained in $T_{z_k(t, x)}\mathbb{S}^2$. We deduce that $J\partial_x z_k(t, x)$ and $\nabla F_k(t, z_k(t, x))$ are proportional, which leads to

$$\forall t \geq 0, \quad \forall x \in \mathbb{T}, \quad \left[\partial_t z_k(t, x) - U(t, z_k(t, x)) \right] \cdot (J\partial_x z_k(t, x)) = 0.$$

But, using that $J^T = J^{-1} = -J$ and $\nabla^\perp = J\nabla$, we obtain

$$\begin{aligned} U(t, z_k(t, x)) \cdot (J\partial_x z_k(t, x)) &= \nabla^\perp \Psi(t, z_k(t, x)) \cdot (J\partial_x z_k(t, x)) \\ &= -\left(J\nabla^\perp \Psi(t, z_k(t, x)) \right) \cdot \partial_x z_k(t, x) \\ &= \nabla \Psi(t, z_k(t, x)) \cdot \partial_x z_k(t, x) \\ &= \partial_x \left(\Psi(t, z_k(t, x)) \right). \end{aligned}$$

Hence the contour dynamics equations for the vortex cap solutions are

$$\forall k \in \llbracket 1, M-1 \rrbracket, \quad \forall t \geq 0, \quad \forall x \in \mathbb{T}, \quad \partial_t z_k(t, x) \cdot (J\partial_x z_k(t, x)) = \partial_x \left(\Psi(t, z_k(t, x)) \right), \quad (1.21)$$

which are comparable to (1.11).

1.4 Main results

Let us now present our main results. First observe that Lemma 1.2 implies that any spherical vortex cap in the form

$$\bar{\Omega}(\theta) = \omega_1 \mathbf{1}_{0 < \theta < \theta_1} + \omega_2 \mathbf{1}_{\theta_1 \leq \theta < \theta_2} + \dots + \omega_{M-1} \mathbf{1}_{\theta_{M-2} \leq \theta < \theta_{M-1}} + \omega_M \mathbf{1}_{\theta_{M-1} \leq \theta < \pi},$$

with

$$M \in \mathbb{N} \setminus \{0, 1\}, \quad \theta_0 \triangleq 0 < \theta_1 < \dots < \theta_{M-1} < \pi \triangleq \theta_M, \quad \forall k \in \llbracket 1, M-1 \rrbracket, \quad \omega_k \neq \omega_{k+1},$$

and supplemented by the Gauss condition

$$\sum_{k=1}^M \omega_k (\cos(\theta_{k-1}) - \cos(\theta_k)) = 0,$$

generates a stationary vortex cap solution to (1.3). In the sequel, we shall focus on the cases $M = 2$ and $M = 3$ (the latter will also be referred to as *vortex bands* or *vortex strips*). More precisely, we study the existence non-trivial periodic solutions living close to these structures. Due to the symmetry of the stationary vortex caps, we will look for rotating solutions around the vertical axis at uniform velocity c , that is

$$\forall t \geq 0, \quad \forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}(t, \xi) = \bar{\Omega}(0, \mathcal{R}(ct)\xi),$$

and satisfy for some fixed $\mathbf{m} \in \mathbb{N}^*$ the \mathbf{m} -fold property

$$\forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}(0, \mathcal{R}\left(\frac{2\pi}{\mathbf{m}}\right)\xi) = \bar{\Omega}(0, \xi).$$

Our first result concerns the one-interface case $M = 2$ and reads as follows.

Theorem 1.1. Let $\tilde{\gamma} \in \mathbb{R}$, $\mathbf{m} \in \mathbb{N}^*$ and $\theta_0 \in (0, \pi)$. Consider $\omega_N, \omega_S \in \mathbb{R}$ such that

$$\frac{\omega_N + \omega_S}{\omega_N - \omega_S} = \cos(\theta_0).$$

There exists a branch of \mathbf{m} -fold uniformly rotating vortex cap solutions to (1.3) with one interface bifurcating from

$$\bar{\Omega}_{\text{FC}}(\theta) \triangleq \omega_N \mathbf{1}_{0 < \theta < \theta_0} + \omega_S \mathbf{1}_{\theta_0 \leq \theta < \pi},$$

at the velocity

$$c_{\mathbf{m}}(\tilde{\gamma}) \triangleq \tilde{\gamma} - (\omega_N - \omega_S) \frac{\mathbf{m} - 1}{2\mathbf{m}}.$$

Before moving to the second result, some remarks are in order.

Remark 1.1. 1. The bifurcation points $c_{\mathbf{m}}(\tilde{\gamma})$ correspond to a shift by the rotation speed $\tilde{\gamma}$ of Burbea's frequencies (1.12) with a vorticity jump $[[\bar{\Omega}]] \triangleq \omega_N - \omega_S = -1$. Hence, in the rotation frame of the sphere, the solutions bifurcate at Burbea's frequencies.

2. The local curve is parametrized in particular with $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto c_{\mathbf{m}}^{\varepsilon}(\tilde{\gamma}) \in \mathbb{R}$ for some small $\varepsilon_0 > 0$. Denoting $\tilde{\gamma}_{\mathbf{m}} \triangleq (\omega_N - \omega_S) \frac{\mathbf{m} - 1}{2\mathbf{m}}$, we have $c_{\mathbf{m}}^0(\tilde{\gamma}_{\mathbf{m}}) = 0$. Since the dependence of $c_{\mathbf{m}}(\tilde{\gamma})$ in $\tilde{\gamma}$ is affine, an application of the implicit function theorem allows to construct a curve $\varepsilon \in (-\varepsilon_0, \varepsilon_0) \mapsto \tilde{\gamma}_{\mathbf{m}}^{\varepsilon} \in \mathbb{R}$ such that (up to reducing the size of ε_0)

$$\forall \varepsilon \in (-\varepsilon_0, \varepsilon_0), \quad c_{\mathbf{m}}^{\varepsilon}(\tilde{\gamma}_{\mathbf{m}}^{\varepsilon}) = 0.$$

This means that we can construct a branch of non-trivial \mathbf{m} -fold solutions which are stationary in the geocentric frame. One could also obtain these solutions by implementing bifurcation theory with the parameter $\tilde{\gamma}$.

3. The bifurcation analysis is performed in Hölder spaces but, similarly to the planar case, we expect our solutions to be analytic.

Next, we shall present our second result dealing with two interfaces ($M = 3$). In this case, the computations in the spectral study are much more involved. For the sake of simplicity, we give here an informal statement of the second main theorem. More precise statement is postponed to Theorem 3.1.

Theorem 1.2. Let $\tilde{\gamma} \in \mathbb{R}$ and $0 < \theta_1 < \theta_2 < \pi$. Fix $\omega_N, \omega_C, \omega_S \in \mathbb{R}$ such that

$$\omega_N + \omega_S = (\omega_N - \omega_C) \cos(\theta_1) + (\omega_C - \omega_S) \cos(\theta_2). \quad (1.22)$$

Then, there exists $N(\theta_1, \theta_2) \triangleq N(\theta_1, \theta_2, \omega_N, \omega_S, \omega_C) \in \mathbb{N}^*$ such that for any $\mathbf{m} \in \mathbb{N}^*$ with $\mathbf{m} \geq N(\theta_1, \theta_2)$, there exists at least one branch of \mathbf{m} -fold uniformly rotating vortex strips for (1.3) bifurcating from

$$\bar{\Omega}_{\text{FC}2}(\theta) \triangleq \omega_N \mathbf{1}_{0 < \theta < \theta_1} + \omega_C \mathbf{1}_{\theta_1 \leq \theta < \theta_2} + \omega_S \mathbf{1}_{\theta_2 \leq \theta < \pi}, \quad (1.23)$$

at some value of the velocity c .

Remark 1.2. Let us make the following remarks.

1. Notice that the proof in this case reveals some non-degeneracy conditions, required to avoid spectral collisions when applying bifurcation techniques. This situation does not appear in the planar case [25], where the bifurcation eigavalues are well-separated.
2. As we shall see in Lemma 3.2 the bifurcation points are also explicit, in this case, and, a priori, are independent from the planar case (1.13), even if they globally have a similar structure. Moreover, similarly to Remark 1.1-2, we can construct at most two curves of non-trivial \mathbf{m} -fold solutions which are stationary in the geocentric frame.
3. The cases with more interfaces ($M \geq 4$) seem much more involved to study.

1.5 Organization of the paper

This work is organized as follows. In Section 2 we provide the proof of Theorem 1.1 showing the analytical existence of non-trivial vortex caps with one interface bifurcating from the trivial one. First, we characterize the existence of such solutions with the non-trivial roots of a nonlinear and nonlocal functional. Later, we give the spectral properties of such functional and conclude the proof of the first main result. In Section 3 we analyze the two-interfaces problem (Theorem 1.2), which is more delicate since one has to study a coupled nonlinear system. The spectral study is more complex in this case, and we can show the existence of non-trivial vortex strips with large \mathbf{m} -fold symmetries under non-degeneracy conditions. Finally, the Appendix gives the proof of some technical lemmas and states the Crandall-Rabinowitz theorem.

Acknowledgments

The authors would like to thank David G. Dritschel for suggesting the problem. The third author also would like to thank Vincent Duchêne and Didier Smets who also arised the question during his PhD defense. The work of Claudia García has been supported by the MINECO–Feder (Spain) research grant number RTI2018–098850–B–I00, the Junta de Andalucía (Spain) Project FQM 954, the Severo Ochoa Programme for Centres of Excellence in R&D(CEX2019-000904-S) and by the PID2021-124195NB-C32. The work of Zineb Hassainia has been supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE. The work of Emeric Roulley has been supported by PRIN 2020XB3EFL, "Hamiltonain and Dispersive PDEs".

2 The one–interface case

This section is devoted to the proof of Theorem 1.1 dealing with the case of one interface ($M = 2$). We first discuss the stationary flat cap solution. Then, by choosing a suitable ansatz, we can rewrite the vortex cap equation (1.21) which leads to reformulate our problem in looking for the zeros of a nonlinear and nonlocal functional, see (2.6). Finally, we implement bifurcation arguments in order to show the existence of non-trivial roots of such functional. Hence, the proof relies on checking all the hypothesis of Crandall-Rabinowitz Theorem, see Appendix A.3.

2.1 Equation of interest

First, we shall discuss some properties of the spherical stationary solutions with one interface.

Lemma 2.1. *Let $\theta_0 \in (0, \pi)$. For any $\omega_N, \omega_S \in \mathbb{R}$ such that*

$$\frac{\omega_N + \omega_S}{\omega_N - \omega_S} = \cos(\theta_0), \quad (2.1)$$

the following function describing the flat cap (FC)

$$\bar{\Omega}_{\text{FC}}(\theta) \triangleq \omega_N \mathbf{1}_{0 < \theta < \theta_0} + \omega_S \mathbf{1}_{\theta_0 \leq \theta < \pi},$$

is a stationary solution to Euler equations. In addition,

$$\partial_\theta \Psi_{\text{FC}}(\theta_0) = \left(\frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \sin(\theta_0). \quad (2.2)$$

Proof. ► Observe that

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}_{\text{FC}}(\mathcal{R}(\alpha)\xi) = \bar{\Omega}_{\text{FC}}(\xi).$$

Hence, Lemma 1.2 applies and proves that this is a stationary solution.

► Notice that the constraint (2.1) is required since (1.5) and (1.2) imply

$$\begin{aligned} 0 &= \int_{\mathbb{S}^2} \Omega_{\text{FC}}(t, \xi) d\sigma(\xi) = \int_0^{2\pi} \int_0^\pi \Omega_{\text{FC}}(t, \theta, \varphi) \sin(\theta) d\theta d\varphi \\ &= 2\pi \left(\omega_N \int_0^{\theta_0} \sin(\theta) d\theta + \omega_S \int_{\theta_0}^\pi \sin(\theta) d\theta \right) \\ &= 2\pi \left[\omega_N (1 - \cos(\theta_0)) + \omega_S (1 + \cos(\theta_0)) \right]. \end{aligned}$$

► The potential velocity solves the elliptic equation

$$\Delta \Psi_{\text{FC}} = \bar{\Omega}_{\text{FC}} + 2\tilde{\gamma} \cos(\theta), \quad \text{i.e.} \quad \partial_\theta [\sin(\theta) \partial_\theta \Psi_{\text{FC}}(\theta)] = \sin(\theta) (\omega_N \mathbf{1}_{0 < \theta < \theta_0} + \omega_S \mathbf{1}_{\theta_0 \leq \theta < \pi}) + \tilde{\gamma} \sin(2\theta).$$

Integrating the previous relation gives

$$\partial_\theta \Psi_{\text{FC}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta)) - \frac{\tilde{\gamma} \cos(2\theta)}{2 \sin(\theta)} + \frac{c}{\sin(\theta)}, & \text{if } \theta \in (0, \theta_0), \\ \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta_0)) + \frac{\omega_S}{\sin(\theta)} (\cos(\theta_0) - \cos(\theta)) - \frac{\tilde{\gamma} \cos(2\theta)}{2 \sin(\theta)} + \frac{c}{\sin(\theta)}, & \text{if } \theta \in [\theta_0, \pi). \end{cases}$$

Since the flow is zonal, there is no velocity at the pole. Which implies that

$$\lim_{\theta \rightarrow 0^+} \partial_\theta \Psi_{\text{FC}}(\theta) = 0.$$

As a consequence, we must take the constant of integration c as follows

$$c \triangleq \frac{\tilde{\gamma}}{2}.$$

Finally, using (2.1), we can write

$$\partial_\theta \Psi_{\text{FC}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)}(1 - \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in (0, \theta_0), \\ -\frac{\omega_S}{\sin(\theta)}(1 + \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in [\theta_0, \pi). \end{cases}$$

At $\theta = \theta_0$, using again (2.1), we find

$$\partial_\theta \Psi_{\text{FC}}(\theta_0) = \frac{1}{2 \sin(\theta_0)} \left[\omega_N(1 - \cos(\theta_0)) - \omega_S(1 + \cos(\theta_0)) \right] - \tilde{\gamma} \sin(\theta_0) = \left(\frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \sin(\theta_0).$$

The proof of Lemma 2.1 is now complete. \square

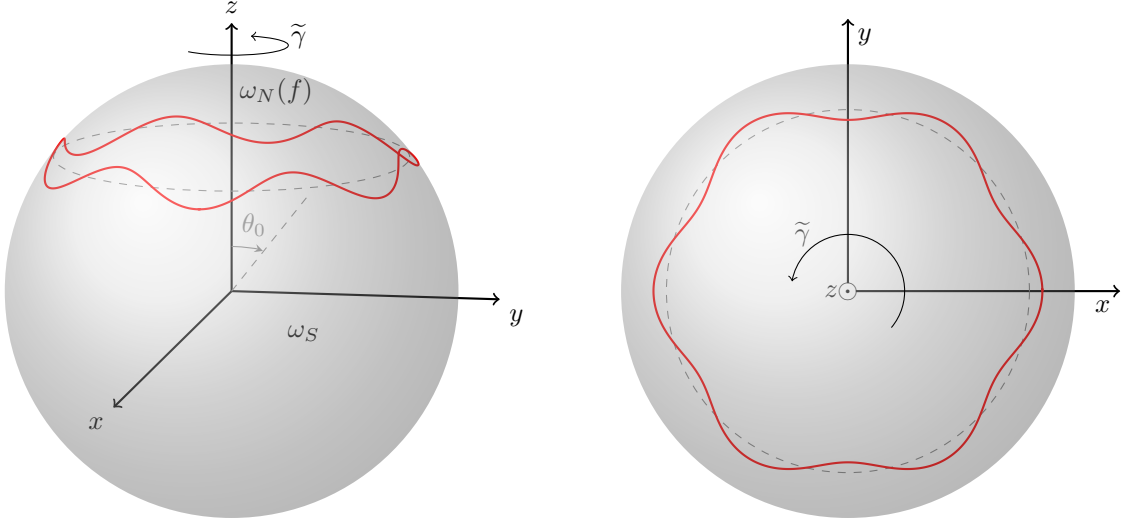


Figure 2: Representation of one interface (in red) vortex cap solutions with 6-fold symmetry.

Now, fix $\theta_0 \in (0, \pi)$, $\omega_N, \omega_S \in \mathbb{R}$ satisfying (2.1) and let us consider a vortex cap solution close to $\bar{\Omega}_{\text{FC}}$ in the form

$$\bar{\Omega}(t, \theta, \varphi) = \omega_N(f) \mathbf{1}_{0 < \theta < \theta_0 + f(t, \varphi)} + \omega_S \mathbf{1}_{\theta_0 + f(t, \varphi) \leq \theta < \pi}, \quad |f(t, \varphi)| \ll 1.$$

The quantity $\omega_N(f)$ is defined implicitly so that the Gauss constraint is satisfied, namely

$$\begin{aligned} 0 &= \int_{\mathbb{S}^2} \bar{\Omega}(t, \xi) d\sigma(\xi) \\ &= \int_0^{2\pi} \left(\omega_N(f) \int_0^{\theta_0 + f(t, \varphi)} \sin(\theta) d\theta + \omega_S \int_{\theta_0 + f(t, \varphi)}^\pi \sin(\theta) d\theta \right) d\varphi \\ &= \int_0^{2\pi} \omega_N(f) (1 - \cos(\theta_0 + f(t, \varphi))) + \omega_S (1 + \cos(\theta_0 + f(t, \varphi))) d\varphi. \end{aligned}$$

To make the argument rigorous, we define, for any reasonable Banach algebra functional space X whose elements have zero space-average, the functional

$$\begin{aligned} \mathcal{G} : X \times \mathbb{R} &\rightarrow \mathbb{R} \\ (f, \omega) &\mapsto \int_0^{2\pi} \omega (1 - \cos(\theta_0 + f(t, \varphi))) + \omega_S (1 + \cos(\theta_0 + f(t, \varphi))) d\varphi. \end{aligned}$$

The precise function spaces will be given in the next section. The application \mathcal{G} is clearly smooth and, using that $\theta_0 \in (0, \pi)$,

$$\mathcal{G}(0, \omega_N) = 0, \quad (\partial_\omega \mathcal{G})(0, \omega_N) = 2\pi(1 - \cos(\theta_0)) \neq 0.$$

Therefore, one can apply the classical implicit function theorem to get the existence of a smooth functional $\omega_N(\cdot) : B_X(r_0) \rightarrow \mathbb{R}$ such that

$$\forall f \in B_X(r_0), \quad \mathcal{G}(f, \omega_N(f)) = 0, \quad \omega_N(0) = \omega_N. \quad (2.3)$$

We have denoted $B_X(r_0)$ the ball in X of center 0 and small enough radius $r_0 > 0$. Differentiating with respect to f in any direction $h \in X$ (which has zero average), we get

$$\begin{aligned} d_f \omega_N(0)[h] &= -\frac{(\partial_f \mathcal{G})(0, \omega_N)[h]}{(\partial_\omega \mathcal{G})(0, \omega_N)} \\ &= \frac{\sin(\theta_0)(\omega_N - \omega_S)}{2\pi(1 - \cos(\theta_0))} \int_0^{2\pi} h(t, \varphi) d\varphi = 0. \end{aligned}$$

Therefore,

$$d_f \omega_N(0) \equiv 0. \quad (2.4)$$

Let us mention that one could also fix ω_N and make ω_S vary. The evolution is given by the dynamics of the interface which can be described (in the colatitude/longitude chart) through the following parametrization

$$z(t, \varphi) = C_1(\theta_0 + f(t, \varphi), \varphi) = \begin{pmatrix} \sin(\theta_0 + f(t, \varphi)) \cos(\varphi) \\ \sin(\theta_0 + f(t, \varphi)) \sin(\varphi) \\ \cos(\theta_0 + f(t, \varphi)) \end{pmatrix}.$$

Differentiating in time amounts to

$$\partial_t z(t, \varphi) = \partial_t f(t, \varphi) \begin{pmatrix} \cos(\theta_0 + f(t, \varphi)) \cos(\varphi) \\ \cos(\theta_0 + f(t, \varphi)) \sin(\varphi) \\ -\sin(\theta_0 + f(t, \varphi)) \end{pmatrix},$$

and the derivation in the longitude variable gives

$$\partial_\varphi z(t, \varphi) = \partial_\varphi f(t, \varphi) \begin{pmatrix} \cos(\theta_0 + f(t, \varphi)) \cos(\varphi) \\ \cos(\theta_0 + f(t, \varphi)) \sin(\varphi) \\ -\sin(\theta_0 + f(t, \varphi)) \end{pmatrix} + \begin{pmatrix} -\sin(\theta_0 + f(t, \varphi)) \sin(\varphi) \\ \sin(\theta_0 + f(t, \varphi)) \cos(\varphi) \\ 0 \end{pmatrix}.$$

The vector $J\partial_\varphi z(t, \varphi)$ can be obtained using the cross product

$$\begin{aligned} J\partial_\varphi z(t, \varphi) &= \partial_\varphi z(t, \varphi) \times z(t, \varphi) \\ &= \partial_\varphi f(t, \varphi) \begin{pmatrix} \sin(\varphi) \\ -\cos(\varphi) \\ 0 \end{pmatrix} + \begin{pmatrix} \cos(\theta_0 + f(t, \varphi)) \sin(\theta_0 + f(t, \varphi)) \cos(\varphi) \\ \cos(\theta_0 + f(t, \varphi)) \sin(\theta_0 + f(t, \varphi)) \sin(\varphi) \\ -\sin^2(\theta_0 + f(t, \varphi)) \end{pmatrix}. \end{aligned}$$

Therefore,

$$\partial_t z(t, \varphi) \cdot (J\partial_\varphi z(t, \varphi)) = \sin(\theta_0 + f(t, \varphi)) \partial_t f(t, \varphi).$$

Our ansatz corresponds to a vorticity in the form

$$\Omega(t, \theta, \varphi) = \omega_N(f) \mathbf{1}_{0 < \theta < \theta_0 + f(t, \varphi)} + \omega_S \mathbf{1}_{\theta_0 + f(t, \varphi) \leq \theta < \pi} + 2\tilde{\gamma} \cos(\theta).$$

Consequently, according to (1.6), (1.7), (1.8) and (1.5), we have

$$\begin{aligned} \Psi(t, z(t, \varphi)) &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \log(D(\theta_0 + f(t, \varphi), \theta', \varphi, \varphi')) \Omega(t, \theta', \varphi') \sin(\theta') d\theta' d\varphi' \\ &= \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(t, \varphi')} \log(D(\theta_0 + f(t, \varphi), \theta', \varphi, \varphi')) \sin(\theta') d\theta' d\varphi' \\ &\quad + \frac{\omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_0 + f(t, \varphi')}^\pi \log(D(\theta_0 + f(t, \varphi), \theta', \varphi, \varphi')) \sin(\theta') d\theta' d\varphi' \\ &\quad + \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^\pi \log(D(\theta_0 + f(t, \varphi), \theta', \varphi, \varphi')) \sin(2\theta') d\theta' d\varphi'. \end{aligned}$$

Remark that the unperturbed stream function can be written as follows

$$\begin{aligned} \Psi_{\text{FC}}(\theta) &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \log(D(\theta, \theta', 0, \varphi')) \Omega_{\text{FC}}(\theta') \sin(\theta') d\theta' d\varphi' \\ &= \frac{\omega_N}{4\pi} \int_0^{2\pi} \int_0^{\theta_0} \log(D(\theta, \theta', 0, \varphi')) \sin(\theta') d\theta' d\varphi' \\ &\quad + \frac{\omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_0}^\pi \log(D(\theta, \theta', 0, \varphi')) \sin(\theta') d\theta' d\varphi' \\ &\quad + \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^\pi \log(D(\theta, \theta', 0, \varphi')) \sin(2\theta') d\theta' d\varphi'. \end{aligned}$$

Thus, making appeal to Chasles' relation, we can write

$$\begin{aligned}\Psi(t, z(t, \varphi)) &= \Psi_{\text{FC}}(\theta_0 + f(t, \varphi)) + \Psi_p^{[1]}\{f\}(\theta_0 + f(t, \varphi), \varphi) + \Psi_p^{[2]}\{f\}(\theta_0 + f(t, \varphi), \varphi) \\ &\triangleq \Psi\{f\}(\theta_0 + f(t, \varphi), \varphi),\end{aligned}$$

where

$$\begin{aligned}\Psi_p^{[1]}\{f\}(\theta, \varphi) &\triangleq \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_0 + f(t, \varphi')} \log\left(D(\theta, \theta', \varphi, \varphi')\right) \sin(\theta') d\theta' d\varphi', \\ \Psi_p^{[2]}\{f\}(\theta, \varphi) &\triangleq \frac{\omega_N(f) - \omega_N}{4\pi} \int_0^{2\pi} \int_{\theta_0}^{\theta_0 + f(t, \varphi')} \log\left(D(\theta, \theta', \varphi, \varphi')\right) \sin(\theta') d\theta' d\varphi' .\end{aligned}$$

Therefore, the vortex cap equation (1.21) becomes

$$\partial_t f(t, \varphi) = \frac{\partial_\varphi\left(\Psi\{f\}(\theta_0 + f(t, \varphi), \varphi)\right)}{\sin(\theta_0 + f(t, \varphi))}. \quad (2.5)$$

Looking for traveling solutions at speed $c \in \mathbb{R}$ leads to consider

$$f(t, \varphi) = f(\varphi - ct).$$

Inserting this into (2.5) gives

$$\mathcal{F}(c, f)(\varphi) \triangleq c \partial_\varphi f(\varphi) + \frac{\partial_\varphi\left(\Psi\{f\}(\theta_0 + f(\varphi), \varphi)\right)}{\sin(\theta_0 + f(\varphi))} = 0. \quad (2.6)$$

Observe, using in particular (2.3), that

$$\forall c \in \mathbb{R}, \quad \mathcal{F}(c, 0) = 0.$$

That corresponds to a trivial line of roots of \mathcal{F} , corresponding to the flat cap stationary solution associated with the angle θ_0 . In order to look for non-trivial roots, we shall use bifurcation arguments in terms of the Crandall-Rabinowitz Theorem. For that goal, we shall study the regularity of \mathcal{F} and the spectral properties of its linearized operator.

2.2 Bifurcation study

In this section, we shall check the hypothesis of the Crandall-Rabinowitz Theorem for the functional \mathcal{F} introduced in (2.6). Firstly, let us introduce the function spaces, in terms of Hölder regularity, that will be used in the bifurcation argument. Fix $\alpha \in (0, 1)$, then the Hölder space $C^\alpha(\mathbb{T})$ consists in 2π -periodic functions $f : \mathbb{T} \rightarrow \mathbb{R}$ such that the following norm is finite:

$$\|f\|_{C^\alpha(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\substack{(\varphi, \varphi') \in \mathbb{T}^2 \\ \varphi \neq \varphi'}} \frac{|f(\varphi) - f(\varphi')|}{|\varphi - \varphi'|^\alpha}.$$

The subspace $C^{1+\alpha}(\mathbb{T})$ of regular functions is associated with the following norm

$$\|f\|_{C^{1+\alpha}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \|\partial_\varphi f\|_{C^\alpha(\mathbb{T})}.$$

Define also the following subspaces taking into account parity and symmetries

$$\begin{aligned}X_{\mathbf{m}}^{1+\alpha} &\triangleq \left\{ f \in C^{1+\alpha}(\mathbb{T}) \quad \text{s.t.} \quad \forall \varphi \in \mathbb{T}, f(\varphi) = \sum_{n=1}^{\infty} f_n \cos(\mathbf{m}n\varphi), \quad f_n \in \mathbb{R} \right\}, \\ Y_{\mathbf{m}}^\alpha &\triangleq \left\{ g \in C^\alpha(\mathbb{T}) \quad \text{s.t.} \quad \forall \varphi \in \mathbb{T}, g(\varphi) = \sum_{n=1}^{\infty} g_n \sin(\mathbf{m}n\varphi), \quad g_n \in \mathbb{R} \right\}, \\ B_{r, \mathbf{m}}^{1+\alpha} &\triangleq \left\{ f \in X_{\mathbf{m}}^{1+\alpha} \quad \text{s.t.} \quad \|f\|_{C^{1+\alpha}(\mathbb{T})} < r \right\}, \quad r > 0.\end{aligned}$$

The next proposition gathers the regularity properties for the functional \mathcal{F} and gives the structure of its linearized operator at the flat cap.

Proposition 2.1. *Let $\alpha \in (0, 1)$ and $\mathbf{m} \in \mathbb{N}^*$. There exists $r > 0$ such that*

(i) The function $\mathcal{F} : \mathbb{R} \times B_{r, \mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^{\alpha}$ is well-defined and of class C^1 .

(ii) The partial derivative $\partial_c d_f \mathcal{F} : \mathbb{R} \times B_{r, \mathbf{m}}^{1+\alpha} \rightarrow \mathcal{L}(X_{\mathbf{m}}^{1+\alpha}, Y_{\mathbf{m}}^{\alpha})$ exists and is continuous.

(iii) At the equilibrium $f = 0$, the linearized operator admits the following Fourier representation

$$d_f \mathcal{F}(c, 0) \left[\sum_{n=1}^{\infty} h_n \cos(\mathbf{m}n\varphi) \right] = \sum_{n=1}^{\infty} \mathbf{m}n \left[-c - (\omega_N - \omega_S) \frac{\mathbf{m}n - 1}{2\mathbf{m}n} + \tilde{\gamma} \right] h_n \sin(\mathbf{m}n\varphi). \quad (2.7)$$

In addition, if $c \neq \frac{\omega_N - \omega_S}{2} - \tilde{\gamma}$, then the operator $d_f \mathcal{F}(c, 0) : X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^{\alpha}$ is of Fredholm type with index zero.

Proof. (i) First, notice that the oddness and \mathbf{m} -fold properties follow from the evenness and \mathbf{m} -fold properties of f and changes of variables in the non-local part. Now, we need to check that $\mathcal{F}(c, f)$ belongs to $C^{\alpha}(\mathbb{T})$ provided that $f \in C^{1+\alpha}(\mathbb{T})$. Then, let us write \mathcal{F} as

$$\mathcal{F}(c, f)(\varphi) = cf'(\varphi) + f'(\varphi) \frac{(\partial_{\theta} \Psi\{f\})(\theta_0 + f(\varphi), \varphi)}{\sin(\theta_0 + f(\varphi))} + \frac{(\partial_{\varphi} \Psi\{f\})(\theta_0 + f(\varphi), \varphi)}{\sin(\theta_0 + f(\varphi))}.$$

Notice that since $\theta_0 \notin \{0, \pi\}$ and $\|f\|_{C^{1+\alpha}(\mathbb{T})} < r$, hence by considering r small enough we find

$$\inf_{\varphi \in \mathbb{T}} |\sin(\theta_0 + f(\varphi))| \geq \delta_0 > 0, \quad \delta_0 \triangleq \inf_{x \in [\theta_0 - r, \theta_0 + r]} |\sin(x)| \in (0, 1). \quad (2.8)$$

Thus, in order to check that \mathcal{F} is well-defined, it is enough to prove that

$$(\partial_{\theta} \Psi\{f\})(\theta_0 + f(\cdot), \cdot) \in C^{\alpha}(\mathbb{T}) \quad \text{and} \quad (\partial_{\varphi} \Psi\{f\})(\theta_0 + f(\cdot), \cdot) \in C^{\alpha}(\mathbb{T}). \quad (2.9)$$

Note that proving (2.9) uses the same techniques as the one used to show (2.11). Thus, we shall skip the details and only check that $f \mapsto (d_f \mathcal{F})$ is continuous using the expression in (2.6). Indeed, we can compute the Gateaux derivative of \mathcal{F} and obtain

$$\begin{aligned} d_f \mathcal{F}(c, f)[h](\varphi) &= ch'(\varphi) - h(\varphi) \frac{\cos(\theta_0 + f(\varphi))}{\sin^2(\theta_0 + f(\varphi))} \partial_{\varphi} \left(\Psi\{f\}(\theta_0 + f(\varphi), \varphi) \right) \\ &\quad + \frac{1}{\sin(\theta_0 + f(\varphi))} \partial_{\varphi} \left(h(\varphi) (\partial_{\theta} \Psi\{f\})(\theta_0 + f(\varphi), \varphi) \right) \\ &\quad + \frac{1}{\sin(\theta_0 + f(\varphi))} \partial_{\varphi} \left((d_f \Psi\{f\}[h])(\theta_0 + f(\varphi), \varphi) \right), \end{aligned}$$

with

$$\begin{aligned} (d_f \Psi\{f\}[h])(\theta_0 + f(\varphi), \varphi) &= (d_f \Psi_p^{[1]}\{f\}[h])(\theta_0 + f(\varphi), \varphi) + (d_f \Psi_p^{[2]}\{f\}[h])(\theta_0 + f(\varphi), \varphi), \\ (d_f \Psi_p^{[1]}\{f\}[h])(\theta_0 + f(\varphi), \varphi) &= \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} h(\varphi') \log \left(D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') \right) \sin(\theta_0 + f(\varphi')) d\varphi', \\ (d_f \Psi_p^{[2]}\{f\}[h])(\theta_0 + f(\varphi), \varphi) &= \frac{d_f \omega_N(f)[h]}{4\pi} \int_0^{2\pi} \int_0^{\theta_0} \log \left(D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &\quad + \frac{\omega_N(f) - \omega_N}{4\pi} \int_0^{2\pi} \log \left(D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') \right) \sin(\theta_0 + f(\varphi')) d\varphi' \end{aligned}$$

and

$$\begin{aligned} (\partial_{\theta} \Psi\{f\})(\theta, \varphi) &= \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\partial_{\theta} D(\theta, \theta', \varphi, \varphi') \sin(\theta')}{D(\theta, \theta', \varphi, \varphi')} d\theta' d\varphi' \\ &\quad + \frac{\omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_0 + f(\varphi')}^{\pi} \frac{\partial_{\theta} D(\theta, \theta', \varphi, \varphi') \sin(\theta')}{D(\theta, \theta', \varphi, \varphi')} d\theta' d\varphi' \\ &\quad + \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^{\pi} \frac{\partial_{\theta} D(\theta, \theta', \varphi, \varphi') \sin(2\theta')}{D(\theta, \theta', \varphi, \varphi')} d\theta' d\varphi'. \end{aligned} \quad (2.10)$$

Notice that $d_f \mathcal{F}(c, f)$ is continuous in f provided that the functions

$$f \mapsto \partial_{\varphi} \left((\partial_{\theta} \Psi\{f\})(\theta_0 + f(\varphi), \varphi) \right) \quad \text{and} \quad f \mapsto \partial_{\varphi} \left(d_f \Psi\{f\}[h](\theta_0 + f(\varphi), \varphi) \right) \quad (2.11)$$

are continuous. Let us start with the first condition in (2.11) and, since the analysis is similar, we shall only give details for just one of the terms. Note that from (2.10), we can write

$$\begin{aligned}
(\partial_\theta \Psi\{f\})(\theta_0 + f(\varphi), \varphi) &= \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\partial_\theta D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \sin(\theta')}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} d\theta' d\varphi' \\
&\quad + \frac{\omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_0 + f(\varphi')}^\pi \frac{\partial_\theta D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \sin(\theta')}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} d\theta' d\varphi' \\
&\quad + \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{\partial_\theta D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \sin(2\theta')}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} d\theta' d\varphi' \\
&\triangleq J_1\{f\}(\varphi) + J_2\{f\}(\varphi) + J_3\{f\}(\varphi).
\end{aligned}$$

We focus on the first term J_1 . Note that to check that the first condition in (2.11) is satisfied, we need to compute $\partial_\varphi J_1\{f\}$. However, let us simplify J_1 before differentiating. Observe that taking the derivative of (1.9) leads to

$$\partial_\theta D(\theta, \theta', \varphi, \varphi') = 2 \left[\sin\left(\frac{\theta - \theta'}{2}\right) \cos\left(\frac{\theta - \theta'}{2}\right) + \cos(\theta) \sin(\theta') \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \right], \quad (2.12)$$

$$\partial_{\theta'} D(\theta, \theta', \varphi, \varphi') = 2 \left[-\sin\left(\frac{\theta - \theta'}{2}\right) \cos\left(\frac{\theta - \theta'}{2}\right) + \sin(\theta) \cos(\theta') \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \right]. \quad (2.13)$$

Hence,

$$\begin{aligned}
(\partial_\theta D + \partial_{\theta'} D)(\theta, \theta', \varphi, \varphi') &= 2 \left[\cos(\theta) \sin(\theta') + \sin(\theta) \cos(\theta') \right] \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \\
&= 2 \sin(\theta + \theta') \sin^2\left(\frac{\varphi - \varphi'}{2}\right).
\end{aligned}$$

Thus, adding and subtracting $\partial_{\theta'} D$ appropriately and integrating by parts, we find

$$\begin{aligned}
J_1\{f\}(\varphi) &= \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\sin^2\left(\frac{\varphi - \varphi'}{2}\right) \sin(\theta' + \theta_0 + f(\varphi))}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \sin(\theta') d\theta' d\varphi' \\
&\quad - \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\partial_{\theta'} D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \sin(\theta') d\theta' d\varphi' \\
&= \frac{\omega_N(f)}{2\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\sin^2\left(\frac{\varphi - \varphi'}{2}\right) \sin(\theta' + \theta_0 + f(\varphi))}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \sin(\theta') d\theta' d\varphi' \\
&\quad - \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \log\left(D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')\right) \sin(\theta_0 + f(\varphi')) d\varphi' \\
&\quad + \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \log\left(D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')\right) \cos(\theta') d\theta' d\varphi' \\
&\triangleq J_{1,1}\{f\}(\varphi) + J_{1,2}\{f\}(\varphi) + J_{1,3}\{f\}(\varphi).
\end{aligned}$$

Let us work with $J_{1,1}\{f\}$, which is the most singular term. Now, we differentiate in φ and obtain

$$\begin{aligned}
\partial_\varphi J_{1,1}\{f\}(\varphi) &= \frac{\omega_N(f)}{2\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{f'(\varphi) \cos(\theta' + \theta_0 + f(\varphi)) \sin^2\left(\frac{\varphi - \varphi'}{2}\right)}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \sin(\theta') d\theta' d\varphi' \\
&\quad + \frac{\omega_N(f)}{4\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\sin(\theta' + \theta_0 + f(\varphi)) \sin(\varphi - \varphi')}{D(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \sin(\theta') d\theta' d\varphi' \\
&\quad - \frac{\omega_N(f)}{2\pi} \int_0^{2\pi} \int_0^{\theta_0 + f(\varphi')} \frac{\sin(\theta' + \theta_0 + f(\varphi)) \sin^2\left(\frac{\varphi - \varphi'}{2}\right)}{D^2(\theta_0 + f(\varphi), \theta', \varphi, \varphi')} \partial_\varphi \left(D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\
&\triangleq J_{1,1,1}\{f\}(\varphi) + J_{1,1,2}\{f\}(\varphi) + J_{1,1,3}\{f\}(\varphi).
\end{aligned}$$

Notice that the most singular integral is $J_{1,1,3}$. Thus, we only deal with that term. Remark that

$$\begin{aligned}
\partial_\varphi \left(D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \right) &= f'(\varphi) \partial_\theta D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') + \partial_\varphi D(\theta_0 + f(\varphi), \theta', \varphi, \varphi') \\
&= 2f'(\varphi) \left[\frac{1}{2} \sin(\theta_0 + f(\varphi) - \theta') + \cos(\theta_0 + f(\varphi)) \sin(\theta') \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \right] \\
&\quad + \sin(\theta_0 + f(\varphi)) \sin(\theta') \sin(\varphi - \varphi').
\end{aligned}$$

We can make the change of variables $\theta' = t(\theta_0 + f(\varphi'))$ to simplify the integral obtaining

$$J_{1,1,3}\{f\}(\varphi) = -\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \mathbb{K}\{f\}(t, \varphi, \varphi') dt d\varphi',$$

where

$$\begin{aligned} \mathbb{K}\{f\}(t, \varphi, \varphi') &\triangleq \omega_N(f) \frac{\sin((1+t)\theta_0 + f(\varphi) + tf(\varphi')) \sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2(\theta_0 + f(\varphi), t(\theta_0 + f(\varphi')), \varphi, \varphi')} \sin(t(\theta_0 + f(\varphi')))(\theta_0 + f(\varphi')) \\ &\times \left\{ 2f'(\varphi) \left[\frac{1}{2} \sin((1-t)\theta_0 + f(\varphi) - tf(\varphi')) + \cos(\theta_0 + f(\varphi)) \sin(t(\theta_0 + f(\varphi'))) \sin^2\left(\frac{\varphi-\varphi'}{2}\right) \right] \right. \\ &\quad \left. + \sin(\theta_0 + f(\varphi)) \sin(t(\theta_0 + f(\varphi'))) \sin(\varphi - \varphi') \right\}. \end{aligned}$$

Our goal is to check that $f \mapsto J_{1,1,3}\{f\}$ is continuous. For this aim, we take $f_1, f_2 \in B_{r,m}^{1+\alpha}$ and we estimate the difference at those points

$$J_{1,1,3}\{f_2\}(\varphi) - J_{1,1,3}\{f_1\}(\varphi).$$

In order to simplify, let us illustrate one of the terms since the order of singularity is the same at every term. For that, define

$$\tilde{J}_{1,1,3}\{f_1, f_2\}(\varphi) \triangleq \int_0^{2\pi} \int_0^1 \mathbb{K}\{f_1, f_2\}(t, \varphi, \varphi') dt d\varphi',$$

where

$$\begin{aligned} \mathbb{K}\{f_1, f_2\}(t, \varphi, \varphi') &\triangleq \frac{\sin((1+t)\theta_0 + f_2(\varphi) + tf_2(\varphi')) \sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi')} \sin^2(t(\theta_0 + f_2(\varphi')))(\theta_0 + f_2(\varphi')) \sin(\varphi - \varphi') \\ &\times \sin(\theta_0 + f_2(\varphi)) \left[\omega_N(f_2) - \omega_N(f_1) \right]. \end{aligned}$$

To estimate the previous term in the Hölder space $C^\alpha(\mathbb{T})$, we use Proposition A.1 with kernel

$$K(\varphi, \varphi') \triangleq \int_0^1 \mathbb{K}\{f_1, f_2\}(t, \varphi, \varphi') dt.$$

Remark that, using the smoothness of $\omega_N(\cdot)$, the mean value theorem implies

$$\left| \omega_N(f_2) - \omega_N(f_1) \right| \leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})}.$$

Now, we choose

$$\begin{cases} r < \frac{1}{2} \min(\theta_0, |\theta_0 - \pi|), & \text{if } \theta_0 \neq \frac{\pi}{2}, \\ r < \frac{\pi}{4}, & \text{if } \theta_0 = \frac{\pi}{2} \end{cases}$$

and denote

$$\mathfrak{m}_{\theta_0}(r) \triangleq \begin{cases} \theta_0 - r, & \text{if } \theta_0 \leq \frac{\pi}{2}, \\ \theta_0 + r, & \text{if } \theta_0 > \frac{\pi}{2}, \end{cases} \quad \mathfrak{M}_{\theta_0}(r) \triangleq \begin{cases} \theta_0 - r, & \text{if } \theta_0 > \frac{\pi}{2}, \\ 1, & \text{if } \theta_0 = \frac{\pi}{2}, \\ \theta_0 + r, & \text{if } \theta_0 < \frac{\pi}{2}. \end{cases}$$

Notice that $\mathfrak{m}_{\theta_0}(r) \in (0, \pi)$ and $\mathfrak{M}_{\theta_0}(r) \in (0, \pi)$, thus, a convexity argument ensures that

$$\exists C_1, C_2 > 0, \quad \forall t \in [0, 1], \quad C_1 t \leq \sin(t \mathfrak{m}_{\theta_0}(r)) \leq C_2 t, \quad C_1 t \leq \sin(t \mathfrak{M}_{\theta_0}(r)) \leq C_2 t.$$

A direct estimation gives

$$\begin{aligned} |K(\varphi, \varphi')| &\leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \frac{|\sin(\varphi - \varphi')| \sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D^2(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi')} \sin^2(t \mathfrak{M}_{\theta_0}(r)) dt \\ &\leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \frac{|\sin(\varphi - \varphi')| \sin^2\left(\frac{\varphi-\varphi'}{2}\right) t^2}{D^2(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi')} dt. \end{aligned}$$

We need to control the denominator. For any $t \in [0, 1]$, we have

$$\begin{aligned}
& D^{\frac{1}{2}}(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi') \\
&= \left\{ 2 \sin^2 \left(\frac{(1-t)\theta_0 + f_2(\varphi) - t f_2(\varphi')}{2} \right) + 2 \sin(\theta_0 + f_2(\varphi)) \sin(t(\theta_0 + f_2(\varphi'))) \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right\}^{\frac{1}{2}} \\
&\geq \left\{ \sqrt{2} \left| \sin \left(\frac{(1-t)\theta_0 + f_2(\varphi) - t f_2(\varphi')}{2} \right) \right| + C\sqrt{t} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right\}.
\end{aligned} \tag{2.14}$$

Let us now estimate the first term. For r small enough, we have

$$0 < (1-t)\theta_0 - 2r \leq (1-t)\theta_0 + f_2(\varphi) - t f_2(\varphi') \leq \theta_0 + 2\|f_2\|_{L^\infty(\mathbb{T})} \leq \theta_0 + 2r < \pi.$$

Using the concavity of the function \sin on $(0, \frac{\pi}{2})$, we infer

$$\begin{aligned}
\sin \left(\frac{(1-t)\theta_0 + f_2(\varphi) - t f_2(\varphi')}{2} \right) &\geq C \left((1-t)(\theta_0 + f_2(\varphi)) + t(f_2(\varphi) - f_2(\varphi')) \right) \\
&\geq C_2(1-t) - t\|f_2\|_{\text{Lip}} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|.
\end{aligned}$$

Inserting this into (2.14) we arrive to

$$\begin{aligned}
D^{\frac{1}{2}}(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi') &\geq C \left\{ (1-t) + (\sqrt{t} - \|f_2\|_{\text{Lip}}t) \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right\} \\
&\geq C \left\{ (1-t) + \sqrt{t} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right\},
\end{aligned}$$

where in the last inequality we are using that $t \leq 1$ and that $\|f_2\|_{\text{Lip}}$ is small enough. Finally, we find

$$D(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi') \geq C \left[(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right],$$

implying

$$D^2(\theta_0 + f_2(\varphi), t(\theta_0 + f_2(\varphi')), \varphi, \varphi') \geq C \left[(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right]^2.$$

Putting together the foregoing calculations yields

$$|K(\varphi, \varphi')| \leq C\|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \frac{|\sin(\varphi - \varphi')| \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) t^2}{\left[(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right]^2} dt.$$

Now, observe that

$$\frac{t^2 \sin^2 \left(\frac{\varphi - \varphi'}{2} \right)}{(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right)} \leq Ct$$

and then

$$|K(\varphi, \varphi')| \leq C\|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \frac{t |\sin(\varphi - \varphi')|}{(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right)} dt.$$

Next, use that for any $t \in [0, 1]$,

$$\begin{aligned}
\frac{t |\sin(\varphi - \varphi')|}{(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right)} &\leq \frac{t |\sin(\varphi - \varphi')|}{\left[(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right]^{\frac{1}{2}} \sqrt{t} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|} \\
&\leq \left[(1-t)^2 + t \sin^2 \left(\frac{\varphi - \varphi'}{2} \right) \right]^{-\frac{1}{2}} \\
&\leq C \left[|1-t| + \sqrt{t} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right]^{-1}.
\end{aligned}$$

The last inequality follows from the following classical estimate

$$\forall (a, b) \in (\mathbb{R}_+)^2, \quad \sqrt{a^2 + b^2} \geq \frac{1}{\sqrt{2}}(a + b).$$

This implies in turn

$$\begin{aligned}
|K(\varphi, \varphi')| &\leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \left[|1-t| + \sqrt{t} \left| \sin\left(\frac{\varphi-\varphi'}{2}\right) \right| \right]^{-1} dt \\
&\leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})} \int_0^1 \left| \sin\left(\frac{\varphi-\varphi'}{2}\right) \right|^{-(1-\alpha)} t^{-\frac{1-\alpha}{2}} |1-t|^{-\alpha} dt \\
&\leq C \|f_1 - f_2\|_{L^\infty(\mathbb{T})} \left| \sin\left(\frac{\varphi-\varphi'}{2}\right) \right|^{-(1-\alpha)},
\end{aligned}$$

where we have used the classical interpolation estimate

$$\forall \alpha \in (0, 1), \quad \forall (a, b) \in (\mathbb{R}_+)^2, \quad (a+b)^{-1} \leq a^{-\alpha} b^{-(1-\alpha)}.$$

The above computations allow to conclude that the hypothesis (A.1) is checked for the kernel K . Similarly, we can check that (A.2) is satisfied and hence Proposition A.1 can be applied obtaining the continuity in f . Let us continue with the second condition in (2.11). We only treat the case of $\Psi_p^{[1]}$, the other one being similar. We can write

$$\partial_\varphi \left(d_f \Psi_p^{[1]} \{f\} [h] (\theta_0 + f(t, \varphi), \varphi) \right) = \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \frac{\partial_\varphi [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'))]}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'))} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi'.$$

By adding and subtracting $\partial_{\varphi'} [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'))]$ appropriately and integrating by parts, we find

$$\begin{aligned}
&\partial_\varphi \left(d_f \Psi_p^{[1]} \{f\} [h] (\theta_0 + f(\varphi), \varphi) \right) \\
&= \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \frac{\partial_\varphi [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')] + \partial_{\varphi'} [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')]}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&\quad - \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \frac{\partial_{\varphi'} [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')]}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&= \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \frac{\partial_\varphi [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')] + \partial_{\varphi'} [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')]}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&\quad + \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \log \left(D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') \right) \partial_{\varphi'} \left(\sin(\theta_0 + f(\varphi')) h(\varphi') \right) d\varphi'.
\end{aligned}$$

Using the definition of D in (1.8), we infer

$$\partial_\varphi D(\theta, \theta', \varphi, \varphi') = \sin(\theta) \sin(\theta') \sin(\varphi - \varphi') = -\partial_{\varphi'} D(\theta, \theta', \varphi, \varphi').$$

Combined with (2.12)-(2.13), we get

$$\begin{aligned}
&\partial_\varphi [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')] + \partial_{\varphi'} [D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')] \\
&= f'(\varphi) \partial_\theta D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') + f'(\varphi') \partial_{\theta'} D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') \\
&= \sin(f(\varphi) - f(\varphi')) (f'(\varphi) - f'(\varphi')) + 2f'(\varphi) \cos(\theta_0 + f(\varphi)) \sin(\theta_0 + f(\varphi')) \sin^2\left(\frac{\varphi-\varphi'}{2}\right) \\
&\quad + 2f'(\varphi') \cos(\theta_0 + f(\varphi')) \sin(\theta_0 + f(\varphi)) \sin^2\left(\frac{\varphi-\varphi'}{2}\right).
\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}
&\partial_\varphi \left(d_f \Psi_p^{[1]} \{f\} [h] (\theta_0 + f(t, \varphi), \varphi) \right) \\
&= \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \frac{\sin(f(\varphi) - f(\varphi')) (f'(\varphi) - f'(\varphi'))}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&\quad + \frac{\omega_N - \omega_S}{2\pi} \int_0^{2\pi} \frac{\cos(\theta_0 + f(\varphi)) f'(\varphi) \sin(\theta_0 + f(\varphi')) \sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&\quad + \frac{\omega_N - \omega_S}{2\pi} \int_0^{2\pi} \frac{\cos(\theta_0 + f(\varphi')) f'(\varphi') \sin(\theta_0 + f(\varphi)) \sin^2\left(\frac{\varphi-\varphi'}{2}\right)}{D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f(\varphi')) h(\varphi') d\varphi' \\
&\quad + \frac{\omega_N - \omega_S}{4\pi} \int_0^{2\pi} \log \left(D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') \right) \partial_{\varphi'} \left(\sin(\theta_0 + f(\varphi')) h(\varphi') \right) d\varphi' \\
&\triangleq \frac{\omega_N - \omega_S}{4\pi} [I_1 \{f\} h(\varphi) + I_2 \{f\} h(\varphi) + I_3 \{f\} h(\varphi) + I_4 \{f\} h(\varphi)].
\end{aligned}$$

In the following, we show that $f \mapsto I_i\{f\}$ is continuous by showing that it has a modulus of continuity. Let us just give the details for I_1 and the others follow similarly. For that, take $f_1, f_2 \in B_{r,m}^{1+\alpha}$, and estimate

$$\begin{aligned} I_1\{f_1\}h(\varphi) - I_1\{f_2\}h(\varphi) &= \int_0^{2\pi} K_1\{f_1, f_2\}(\varphi, \varphi')(f_1'(\varphi) - f_1'(\varphi'))h(\varphi')d\varphi' \\ &\quad + \int_0^{2\pi} K_2\{f_1, f_2\}(\varphi, \varphi')((f_1 - f_2)'(\varphi) - (f_1 - f_2)'(\varphi'))h(\varphi')d\varphi' \\ &\quad + \int_0^{2\pi} K_3\{f_1, f_2\}(\varphi, \varphi')(f_2'(\varphi) - f_2'(\varphi'))h(\varphi')d\varphi', \end{aligned}$$

where

$$\begin{aligned} K_1\{f_1, f_2\}(\varphi, \varphi') &\triangleq \frac{\sin(f_1(\varphi) - f_1(\varphi')) - \sin(f_2(\varphi) - f_2(\varphi'))}{D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f_1(\varphi')), \\ K_2\{f_1, f_2\}(\varphi, \varphi') &\triangleq \frac{\sin(f_2(\varphi) - f_2(\varphi'))}{D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi')} \sin(\theta_0 + f_1(\varphi')), \\ K_3\{f_1, f_2\}(\varphi, \varphi') &\triangleq \frac{\sin(f_2(\varphi) - f_2(\varphi'))}{D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi')} \left[\sin(\theta_0 + f_1(\varphi')) - \sin(\theta_0 + f_2(\varphi')) \right] \\ &\quad + \frac{\sin(f_2(\varphi) - f_2(\varphi'))}{D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi')D(\theta_0 + f_2(\varphi), \theta_0 + f_2(\varphi'), \varphi, \varphi')} \\ &\quad \times \left[D(\theta_0 + f_2(\varphi), \theta_0 + f_2(\varphi'), \varphi, \varphi') - D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi') \right] \sin(\theta_0 + f_2(\varphi')). \end{aligned}$$

Since the kernel of the integral operator has a non differentiable term, our purpose is to use Proposition A.2. For that, let us estimate each kernel K_i . First note that using (1.8) and (2.8), we get

$$\begin{aligned} D(\theta_0 + f(\varphi), \theta_0 + f(\varphi'), \varphi, \varphi') &\geq 2 \sin(\theta_0 + f(\varphi)) \sin(\theta_0 + f(\varphi')) \sin\left(\frac{\varphi - \varphi'}{2}\right) \\ &\geq 2\delta_0^2 \sin^2\left(\frac{\varphi - \varphi'}{2}\right). \end{aligned} \tag{2.15}$$

Using (2.15), it is easy to check that for K_1 we get

$$\begin{aligned} |K_1\{f_1, f_2\}(\varphi, \varphi')| &\leq C |(f_1 - f_2)(\varphi) - (f_1 - f_2)(\varphi')| \left| \sin\left(\frac{\varphi - \varphi'}{2}\right) \right|^{-2} \\ &\leq C \|f_1 - f_2\|_{C^{1+\alpha}(\mathbb{T})} \left| \sin\left(\frac{\varphi - \varphi'}{2}\right) \right|^{-1} \end{aligned}$$

and

$$|\partial_\varphi K_1\{f_1, f_2\}(\varphi, \varphi')| \leq C \|f_1 - f_2\|_{C^{1+\alpha}(\mathbb{T})} \left| \sin\left(\frac{\varphi - \varphi'}{2}\right) \right|^{-2}.$$

Similarly for K_2 we obtain

$$\begin{aligned} |K_2\{f_1, f_2\}(\varphi, \varphi')| &\leq C \left| \sin\left(\frac{\varphi - \varphi'}{2}\right) \right|^{-1}, \\ |\partial_\varphi K_2\{f_1, f_2\}(\varphi, \varphi')| &\leq C \left| \sin\left(\frac{\varphi - \varphi'}{2}\right) \right|^{-2}. \end{aligned}$$

Finally, we shall work with the last kernel K_3 . Note that

$$\begin{aligned} &D(\theta_0 + f_2(\varphi), \theta_0 + f_2(\varphi'), \varphi, \varphi') - D(\theta_0 + f_1(\varphi), \theta_0 + f_1(\varphi'), \varphi, \varphi') \\ &= 2 \sin^2\left(\frac{f_2(\varphi) - f_2(\varphi')}{2}\right) - 2 \sin^2\left(\frac{f_1(\varphi) - f_1(\varphi')}{2}\right) \\ &\quad + 2 \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \left\{ \sin(\theta_0 + f_2(\varphi)) \sin(\theta_0 + f_2(\varphi')) - \sin(\theta_0 + f_1(\varphi)) \sin(\theta_0 + f_1(\varphi')) \right\} \\ &\leq 2 \left| \sin\left(\frac{f_2(\varphi) - f_2(\varphi')}{2}\right) \right| \left| \sin\left(\frac{f_2(\varphi) - f_2(\varphi')}{2}\right) - \sin\left(\frac{f_1(\varphi) - f_1(\varphi')}{2}\right) \right| \\ &\quad + 2 \left| \sin\left(\frac{f_1(\varphi) - f_1(\varphi')}{2}\right) \right| \left| \sin\left(\frac{f_2(\varphi) - f_2(\varphi')}{2}\right) - \sin\left(\frac{f_1(\varphi) - f_1(\varphi')}{2}\right) \right| \\ &\quad + 2 \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \left| \sin(\theta_0 + f_2(\varphi)) \right| \left| \sin(\theta_0 + f_2(\varphi')) - \sin(\theta_0 + f_1(\varphi')) \right| \\ &\quad + 2 \sin^2\left(\frac{\varphi - \varphi'}{2}\right) \left| \sin(\theta_0 + f_1(\varphi')) \right| \left| \sin(\theta_0 + f_2(\varphi)) - \sin(\theta_0 + f_1(\varphi)) \right| \\ &\leq C \|f_2 - f_1\|_{C^{1+\alpha}(\mathbb{T})} \sin^2\left(\frac{\varphi - \varphi'}{2}\right). \end{aligned}$$

To get the last estimate, we have used the 1-Lipschitz property of the function \sin together with (A.5) and

$$\begin{aligned} \forall k \in \{1, 2\}, \quad \left| \sin \left(\frac{f_k(\varphi) - f_k(\varphi')}{2} \right) \right| &\leq |f_k(\varphi) - f_k(\varphi')| \\ &\leq \|f_k\|_{C^{1+\alpha}(\mathbb{T})} |\varphi - \varphi'| \\ &\leq Cr \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|. \end{aligned}$$

Hence

$$|K_3\{f_1, f_2\}(\varphi, \varphi')| \leq C \|f_2 - f_1\|_{C^{1+\alpha}(\mathbb{T})} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-1}.$$

Then, differentiating we find

$$|\partial_\varphi K_3\{f_1, f_2\}(\varphi, \varphi')| \leq C \|f_2 - f_1\|_{C^{1+\alpha}(\mathbb{T})} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-2}.$$

Hence Proposition A.2 implies

$$\|(I_1\{f_1\} - I_2\{f_2\})[h]\|_{C^\alpha(\mathbb{T})} \leq C \|f_1 - f_2\|_{C^{1+\alpha}(\mathbb{T})} \|h\|_{L^\infty(\mathbb{T})},$$

concluding that $f \mapsto I_1\{f\}$ is continuous.

(ii) It follows from

$$\partial_c d_f \mathcal{F}(c, f)[h] = \partial_\varphi h. \quad (2.16)$$

(iii) We assume now that $f = 0$. First notice that (2.3)-(2.4) imply $d_f \Psi_p^{[2]}(0) = 0$. In addition, one readily has $\Psi_p^{[1]}\{0\} = 0$. Hence, we have

$$d_f \mathcal{F}(c, 0)[h](\varphi) = c \partial_\varphi h(\varphi) + \frac{1}{\sin(\theta_0)} \partial_\varphi \left(\partial_\theta \Psi_{\text{FC}}(\theta_0) h(\varphi) + (d_f \Psi_p^{[1]}\{0\}[h])(\theta_0, \varphi) \right).$$

From (2.2), we deduce

$$\frac{1}{\sin(\theta_0)} \partial_\varphi \left(\partial_\theta \Psi_{\text{FC}}(\theta_0) h \right) = \left(\frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \partial_\varphi h = - \left(\frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \sum_{n=1}^{\infty} \mathbf{m} n h_n \sin(\mathbf{m} n \varphi). \quad (2.17)$$

After straightforward simplifications and using Lemma A.1, we get

$$\begin{aligned} (d_f \Psi_p^{[1]}\{0\}[h])(\theta_0, \varphi) &= \frac{\omega_N - \omega_S}{4\pi} \sin(\theta_0) \int_0^{2\pi} h(\varphi') \log \left(1 - \cos^2(\theta_0) - \sin^2(\theta_0) \cos(\varphi - \varphi') \right) d\varphi' \\ &= \frac{\omega_N - \omega_S}{4\pi} \sin(\theta_0) \sum_{n=1}^{\infty} h_n \int_0^{2\pi} \log \left(1 - \cos^2(\theta_0) - \sin^2(\theta_0) \cos(\varphi') \right) \cos(\mathbf{m} n (\varphi - \varphi')) d\varphi' \\ &= \frac{\omega_N - \omega_S}{2} \sin(\theta_0) \sum_{n=1}^{\infty} h_n I_{\mathbf{m} n}(\theta_0, \theta_0) \cos(\mathbf{m} n \varphi) \\ &= - \frac{\omega_N - \omega_S}{2} \sin(\theta_0) \sum_{n=1}^{\infty} \frac{h_n}{\mathbf{m} n} \cos(\mathbf{m} n \varphi). \end{aligned}$$

Therefore,

$$\frac{1}{\sin(\theta_0)} \partial_\varphi \left((d_f \Psi_p^{[1]}\{0\}[h])(\theta_0, \varphi) \right) = \frac{\omega_N - \omega_S}{2} \sum_{n=1}^{\infty} h_n \sin(\mathbf{m} n \varphi). \quad (2.18)$$

Introducing the classical 2π -periodic Hilbert transform \mathcal{H} defined by

$$\mathcal{H}h(\varphi) \triangleq \frac{1}{2\pi} \int_0^{2\pi} h(\varphi') \cot \left(\frac{\varphi - \varphi'}{2} \right) d\varphi',$$

which acts on the cosine basis as

$$\forall n \in \mathbb{N}^*, \quad \mathcal{H} \cos(n\varphi) = \sin(n\varphi),$$

we have

$$\frac{1}{\sin(\theta_0)} \partial_\varphi \left((d_f \Psi_p^{[1]}\{0\}[h])(\theta_0, \varphi) \right) = \frac{\omega_N - \omega_S}{2} \mathcal{H}h(\varphi). \quad (2.19)$$

Combining (2.17), (2.18) and (2.19), we obtain the Fourier representation (2.7) or equivalently, the following structure for the linearized operator

$$d_f \mathcal{F}(c, 0) = \left(c + \frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \partial_\varphi + \frac{\omega_N - \omega_S}{2} \mathcal{H}. \quad (2.20)$$

Clearly, if $c \neq \tilde{\gamma} - \frac{\omega_N - \omega_S}{2}$, the operator $\left(c + \frac{\omega_N - \omega_S}{2} - \tilde{\gamma} \right) \partial_\varphi : X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha$ is an isomorphism. We shall prove the compactness of the Hilbert transform in the Hölder spaces. For that, we come back to the integral expression which can be rewritten as

$$\mathcal{H}h(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} K(\varphi, \varphi') \partial_{\varphi'} h(\varphi') d\varphi', \quad K(\varphi, \varphi') \triangleq \log \left(\left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \right).$$

For any $\delta \in (0, 1)$, we have

$$|K(\varphi, \varphi')| \lesssim \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-\delta}, \quad |\partial_\varphi K(\varphi, \varphi')| \lesssim \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-(1+\delta)}.$$

Thus, we can apply Lemma A.1 (with $\delta = 1 - \beta$) and get

$$\forall \beta \in (\alpha, 1), \quad \|\mathcal{H}h\|_{C^\beta(\mathbb{T})} \lesssim \|\partial_\varphi h\|_{L^\infty(\mathbb{T})} \lesssim \|h\|_{C^{1+\alpha}(\mathbb{T})}.$$

Since, for $\beta \in (\alpha, 1)$, the injection $C^\beta(\mathbb{T}) \hookrightarrow C^\alpha(\mathbb{T})$ is compact, we deduce that the operator $\mathcal{H} : C^{1+\alpha}(\mathbb{T}) \rightarrow C^\alpha(\mathbb{T})$ is compact. Thus, (2.20) together with [18, Cor. 5.9] implies, for $c \neq \tilde{\gamma} - \frac{\omega_N - \omega_S}{2}$, the desired Fredholmness property. This proves Proposition 2.1. \square

According to (2.7), the candidates for bifurcation points define the following singular set

$$\mathcal{S}_c \triangleq \left\{ c_{\mathbf{m}}(\tilde{\gamma}) \triangleq \tilde{\gamma} - (\omega_N - \omega_S) \frac{\mathbf{m}-1}{2\mathbf{m}}, \quad \mathbf{m} \in \mathbb{N}^* \right\}. \quad (2.21)$$

Finally, in the following proposition, we gather all the remaining conditions required to apply the Crandall-Rabinowitz Theorem. Then, Theorem 1.1 follows immediately from this proposition.

Proposition 2.2. *Let $\alpha \in (0, 1)$, $\tilde{\gamma} \in \mathbb{R}$ and $\mathbf{m} \in \mathbb{N}^*$.*

- (i) *The linear operator $d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0) : X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha$ is of Fredholm type with index zero.*
- (ii) *The kernel of $d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0)$ is one dimensional. More precisely,*

$$\ker \left(d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0) \right) = \text{span}(\varphi \mapsto \cos(\mathbf{m}\varphi)). \quad (2.22)$$

- (iii) *The transversality condition is satisfied, namely*

$$\partial_c d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0)[\varphi \mapsto \cos(\mathbf{m}\varphi)] \notin \text{Im} \left(d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), \tilde{\gamma}, 0) \right). \quad (2.23)$$

Proof. (i) By construction (2.21), our bifurcation points satisfy

$$c_{\mathbf{m}}(\tilde{\gamma}) = \tilde{\gamma} - (\omega_N - \omega_S) \frac{\mathbf{m}-1}{2\mathbf{m}} \neq \tilde{\gamma} - \frac{\omega_N - \omega_S}{2}.$$

Hence, Proposition 2.1-(iii) gives the desired Fredholmness property.

(ii) The sequence $(c_{\mathbf{m}n}(\tilde{\gamma}))_{n \in \mathbb{N}^*}$ being strictly monotone, then (2.7) and (2.21) give that the kernel is one dimensional and generated by $\varphi \mapsto \cos(\mathbf{m}\varphi)$.

(iii) To prove the transversality condition, we first need to describe the range. For this aim, we introduce on $Y_{\mathbf{m}}^\alpha$ the scalar product

$$\left(\sum_{n=1}^{\infty} a_n \sin(\mathbf{m}n\varphi) \middle| \sum_{n=1}^{\infty} b_n \sin(\mathbf{m}n\varphi) \right) \triangleq \sum_{n=1}^{\infty} a_n b_n.$$

Now we claim that

$$\text{Im} \left(d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0) \right) = \text{span}^{\perp(\cdot, \cdot)}(\varphi \mapsto \sin(\mathbf{m}\varphi)). \quad (2.24)$$

Indeed, the first inclusion is obvious from (2.7)-(2.21). The converse inclusion is obtained because the range is closed and of codimension 1, which results from the Fredholmness property with zero index and the one dimensional kernel condition. Now, it remains to check the transversality condition. In view of (2.16), we infer

$$\partial_c d_f \mathcal{F}(c_{\mathbf{m}}(\tilde{\gamma}), 0)[\cos(\mathbf{m}\varphi)] = -\mathbf{m} \sin(\mathbf{m}\varphi) \in \text{span}(\varphi \mapsto \sin(\mathbf{m}\varphi)). \quad (2.25)$$

Combining (2.24) and (2.25), the condition (2.23) follows. This achieves the proof of Proposition 2.2. \square

3 The two–interfaces case: vorticity bands

This section is devoted to the proof of Theorem 1.2 dealing with the case of two interfaces ($M = 3$). As before, we shall reformulate the problem with a suitable functional and implement bifurcation techniques. The computations are more involved due to the interactions between the boundaries which leads to a vectorial analysis.

3.1 Equations of interest

We start again by some remarks on the flat solution.

Lemma 3.1. *Let $0 < \theta_1 < \theta_2 < \pi$. For any $\omega_N, \omega_C, \omega_S \in \mathbb{R}$ such that*

$$\omega_N + \omega_S = (\omega_N - \omega_C) \cos(\theta_1) + (\omega_C - \omega_S) \cos(\theta_2), \quad (3.1)$$

the following function describing the flat cap (FC2)

$$\bar{\Omega}_{\text{FC2}}(\theta) \triangleq \omega_N \mathbf{1}_{0 < \theta < \theta_1} + \omega_C \mathbf{1}_{\theta_1 \leq \theta < \theta_2} + \omega_S \mathbf{1}_{\theta_2 \leq \theta < \pi}$$

is a stationary solution to Euler equations. In addition,

$$\partial_\theta \Psi_{\text{FC2}}(\theta_1) = \omega_N \tan\left(\frac{\theta_1}{2}\right) - \tilde{\gamma} \sin(\theta_1), \quad \Psi_{\text{FC2}}(\theta_2) = -\omega_S \cot\left(\frac{\theta_2}{2}\right) - \tilde{\gamma} \sin(\theta_2). \quad (3.2)$$

Proof. ► Observe that

$$\forall \alpha \in \mathbb{R}, \quad \forall \xi \in \mathbb{S}^2, \quad \bar{\Omega}_{\text{FC2}}(\mathcal{R}(\alpha)\xi) = \bar{\Omega}_{\text{FC2}}(\xi).$$

Hence, Lemma 1.2 applies and proves that this is a stationary solution.

► The constraint (3.1) follows again from (1.5) and (1.2), namely

$$\begin{aligned} 0 &= \int_{\mathbb{S}^2} \Omega_{\text{FC2}}(t, \xi) d\sigma(\xi) = \int_0^{2\pi} \int_0^\pi \Omega_{\text{FC2}}(t, \theta, \varphi) \sin(\theta) d\theta d\varphi \\ &= 2\pi \left(\omega_N \int_0^{\theta_1} \sin(\theta) d\theta + \omega_C \int_{\theta_1}^{\theta_2} \sin(\theta) d\theta + \omega_S \int_{\theta_2}^\pi \sin(\theta) d\theta \right) \\ &= 2\pi \left[\omega_N (1 - \cos(\theta_1)) + \omega_C (\cos(\theta_1) - \cos(\theta_2)) + \omega_S (1 + \cos(\theta_2)) \right]. \end{aligned}$$

► The potential velocity solves the elliptic equation

$$\Delta \Psi_{\text{FC2}} = \Omega_{\text{FC2}}, \quad \text{i.e.} \quad \partial_\theta [\sin(\theta) \partial_\theta \Psi_{\text{FC2}}(\theta)] = \sin(\theta) \left(\omega_N \mathbf{1}_{0 < \theta < \theta_1} + \omega_C \mathbf{1}_{\theta_1 \leq \theta < \theta_2} + \omega_S \mathbf{1}_{\theta_2 \leq \theta < \pi} \right) + \tilde{\gamma} \sin(2\theta).$$

Integrating the previous relation and choosing the constant of integration as in Lemma 2.1 gives

$$\partial_\theta \Psi_{\text{FC2}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in (0, \theta_1), \\ \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta_1)) + \frac{\omega_C}{\sin(\theta)} (\cos(\theta_1) - \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in [\theta_1, \theta_2), \\ \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta_1)) + \frac{\omega_C}{\sin(\theta)} (\cos(\theta_1) - \cos(\theta_2)) + \frac{\omega_S}{\sin(\theta)} (\cos(\theta_2) - \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in [\theta_2, \pi). \end{cases}$$

Finally, using (3.1), we can write

$$\partial_\theta \Psi_{\text{FC2}}(\theta) = \begin{cases} \frac{\omega_N}{\sin(\theta)} (1 - \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in (0, \theta_1), \\ \frac{\omega_C}{\sin(\theta)} (\cos(\theta_2) - \cos(\theta)) - \frac{\omega_S}{\sin(\theta)} (1 + \cos(\theta_2)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in [\theta_1, \theta_2), \\ -\frac{\omega_S}{\sin(\theta)} (1 + \cos(\theta)) - \tilde{\gamma} \sin(\theta), & \text{if } \theta \in [\theta_2, \pi). \end{cases}$$

At $\theta = \theta_1$, we find

$$\begin{aligned} \partial_\theta \Psi_{\text{FC2}}(\theta_1) &= \frac{\omega_N}{\sin(\theta_1)} (1 - \cos(\theta_1)) - \tilde{\gamma} \sin(\theta_1) \\ &= \omega_N \tan\left(\frac{\theta_1}{2}\right) - \tilde{\gamma} \sin(\theta_1). \end{aligned}$$

At $\theta = \theta_2$, we find

$$\begin{aligned} \partial_\theta \Psi_{\text{FC2}}(\theta_2) &= -\frac{\omega_S}{\sin(\theta_2)} (1 + \cos(\theta_2)) - \tilde{\gamma} \sin(\theta_2) \\ &= -\omega_S \cot\left(\frac{\theta_2}{2}\right) - \tilde{\gamma} \sin(\theta_2). \end{aligned}$$

□

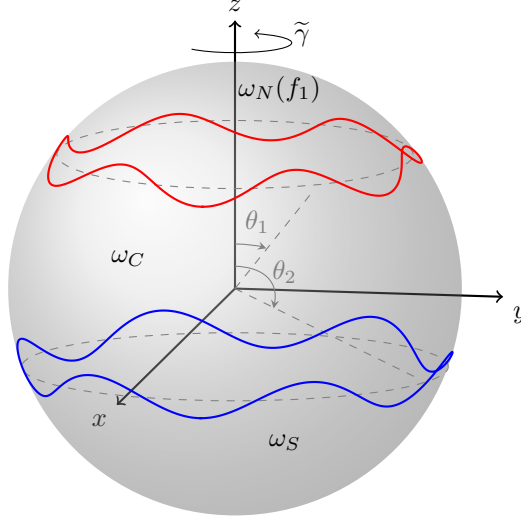


Figure 3: Representation of two interfaces (in red and blue) vortex cap solutions with 6-fold symmetry.

From now on, we fix

$$0 < \theta_1 < \theta_2 < \pi, \quad (3.3)$$

$\omega_N, \omega_C, \omega_S \in \mathbb{R}$ satisfying (3.1) and consider a vortex cap solution close to $\bar{\Omega}_{\text{FC2}}$ in the form

$$\bar{\Omega}(t, \theta, \varphi) = \omega_N(f_1) \mathbf{1}_{0 < \theta < \theta_1 + f_1(t, \varphi)} + \omega_C \mathbf{1}_{\theta_1 + f_1(t, \varphi) \leq \theta < \theta_2 + f_2(t, \varphi)} + \omega_S \mathbf{1}_{\theta_2 + f_2(t, \varphi) \leq \theta < \pi},$$

with

$$\forall k \in \{1, 2\}, \quad |f_k(t, \varphi)| \ll 1.$$

The quantity $\omega_N(f_1)$ is defined implicitly so that the Gauss constraint is satisfied, namely

$$\begin{aligned} 0 &= \int_{\mathbb{S}^2} \bar{\Omega}(t, \xi) d\sigma(\xi) \\ &= \int_0^{2\pi} \left(\omega_N(f_1) \int_0^{\theta_1 + f_1(t, \varphi)} \sin(\theta) d\theta + \omega_C \int_{\theta_2 + f_2(t, \varphi)}^{\theta_1 + f_1(t, \varphi)} \sin(\theta) d\theta + \omega_S \int_{\theta_2 + f_2(t, \varphi)}^{\pi} \sin(\theta) d\theta \right) d\varphi \\ &= \int_0^{2\pi} \left[\omega_N(f_1) (1 - \cos(\theta_1 + f_1(t, \varphi))) + \omega_C (\cos(\theta_2 + f_2(t, \varphi)) - \cos(\theta_1 + f_1(t, \varphi))) \right. \\ &\quad \left. + \omega_S (1 + \cos(\theta_2 + f_2(t, \varphi))) \right] d\varphi. \end{aligned}$$

As in Section 2.1, by means of the implicit function theorem, we can rigorously justify the definition of $\omega_N(f_1)$ and obtain

$$\omega_N(0) = \omega_N, \quad d_{f_1} \omega_N(0) = 0. \quad (3.4)$$

For $k \in \{1, 2\}$, the interface oscillating around $\theta = \theta_k$ can be parametrized by

$$z_k(t, \varphi) \triangleq \begin{pmatrix} \sin(\theta_k + f_k(t, \varphi)) \cos(\varphi) \\ \sin(\theta_k + f_k(t, \varphi)) \sin(\varphi) \\ \cos(\theta_k + f_k(t, \varphi)) \end{pmatrix}.$$

In view of (1.21), the parametrizations z_1 and z_2 must satisfy the following equations

$$\forall k \in \{1, 2\}, \quad \partial_t z_k(t, \varphi) \cdot (J \partial_\varphi z_k(t, \varphi)) = \partial_\varphi (\Psi(t, z_k(t, \varphi))).$$

Proceeding as in Section 2.1, we obtain

$$\partial_t z_k(t, \varphi) \cdot (J \partial_\varphi z_k(t, \varphi)) = \sin(\theta_k + f_k(t, \varphi)) \partial_t f_k(t, \varphi).$$

Consequently, the unknowns f_1 and f_2 have to solve the following (coupled) system

$$\forall k \in \{1, 2\}, \quad \partial_t f_k(t, \varphi) = \frac{\partial_\varphi (\Psi(t, z_k(t, \theta)))}{\sin(\theta_0 + f_k(t, \varphi))}.$$

Now, the stream function writes

$$\begin{aligned}\Psi(t, z_k(t, \varphi)) &= \frac{\omega_N(f_1)}{4\pi} \int_0^{2\pi} \int_0^{\theta_1+f_1(t, \varphi')} \log \left(D(\theta_k + f_k(t, \varphi), \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_C}{4\pi} \int_0^{2\pi} \int_{\theta_1+f_1(t, \varphi')}^{\theta_2+f_2(t, \varphi')} \log \left(D(\theta_k + f_k(t, \varphi), \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_2+f_2(t, \varphi')}^{\pi} \log \left(D(\theta_k + f_k(t, \varphi), \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^{\pi} \log \left(D(\theta_k + f_k(t, \varphi), \theta', \varphi, \varphi') \right) \sin(2\theta') d\theta' d\varphi'.\end{aligned}$$

Remark that the unperturbed stream function can be written as follows

$$\begin{aligned}\Psi_{\text{FC2}}(\theta) &= \frac{\omega_N}{4\pi} \int_0^{2\pi} \int_0^{\theta_1} \log \left(D(\theta, \theta', 0, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_C}{4\pi} \int_{\theta_1}^{\theta_2} \int_{\theta_0}^{\pi} \log \left(D(\theta, \theta', 0, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_S}{4\pi} \int_{\theta_2}^{\pi} \int_{\theta_0}^{\pi} \log \left(D(\theta, \theta', 0, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\tilde{\gamma}}{4\pi} \int_0^{2\pi} \int_0^{\pi} \log \left(D(\theta, \theta', 0, \varphi') \right) \sin(2\theta') d\theta' d\varphi'.\end{aligned}$$

Making appeal to Chasles' relation, we can write

$$\begin{aligned}\Psi(t, z_k(t, \varphi)) &= \Psi_{\text{FC2}}(\theta_k + f_k(t, \varphi)) + \Psi_{p,2}^{[1]} \{f_1, f_2\}(\theta_k + f_k(t, \varphi), \varphi) + \Psi_{p,2}^{[2]} \{f_1\}(\theta_k + f_k(t, \varphi), \varphi) \\ &\triangleq \Psi \{f_1, f_2\}(\theta_k + f_k(t, \varphi), \varphi), \\ \Psi_{p,2}^{[1]} \{f_1, f_2\}(\theta, \varphi) &\triangleq \frac{\omega_N - \omega_C}{4\pi} \int_0^{2\pi} \int_{\theta_1}^{\theta_1+f_1(t, \varphi')} \log \left(D(\theta, \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_C - \omega_S}{4\pi} \int_0^{2\pi} \int_{\theta_2}^{\theta_2+f_2(t, \varphi')} \log \left(D(\theta, \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi', \\ \Psi_{p,2}^{[2]} \{f_1\}(\theta, \varphi) &\triangleq \frac{\omega_N(f_1) - \omega_N}{4\pi} \int_0^{2\pi} \int_0^{\theta_1+f_1(t, \varphi')} \log \left(D(\theta, \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi'.\end{aligned}$$

Therefore, the vortex cap equation (1.21) becomes

$$\forall k \in \{1, 2\}, \quad \partial_t f_k(t, \varphi) = \frac{\partial_\varphi \left(\Psi \{f_1, f_2\}(\theta_k + f_k(t, \varphi), \varphi) \right)}{\sin(\theta_k + f_k(t, \varphi))}. \quad (3.5)$$

We look for traveling solutions at speed $c \in \mathbb{R}$

$$\forall k \in \{1, 2\}, \quad f_k(t, \varphi) = f_k(\varphi - ct).$$

Thus, we shall solve

$$\mathcal{G}(c, f_1, f_2) = 0, \quad \mathcal{G} \triangleq (\mathcal{G}_1, \mathcal{G}_2),$$

where

$$\mathcal{G}_k(c, f_1, f_2)(\varphi) \triangleq c \partial_\varphi f_k(\varphi) + \frac{\partial_\varphi \left(\Psi \{f_1, f_2\}(\theta_k + f_k(\varphi), \varphi) \right)}{\sin(\theta_k + f_k(\varphi))}.$$

Observe that

$$\forall c \in \mathbb{R}, \quad \mathcal{G}(c, 0, 0) = 0.$$

This leads again to implement bifurcation theory.

3.2 Spectral properties

We check here the hypothesis of Crandall-Rabinowitz Theorem.

Proposition 3.1. *Let $\alpha \in (0, 1)$ and $\mathbf{m} \in \mathbb{N}^*$. There exists $r > 0$ such that*

(i) The function $\mathcal{G} : \mathbb{R} \times B_{r,\mathbf{m}}^{1+\alpha} \times B_{r,\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha$ is well-defined and of class C^1 .

(ii) The partial derivative $\partial_c d_{(f_1, f_2)} \mathcal{G} : \mathbb{R} \times B_{r,\mathbf{m}}^{1+\alpha} \times B_{r,\mathbf{m}}^{1+\alpha} \rightarrow \mathcal{L}(X_{\mathbf{m}}^{1+\alpha} \times X_{\mathbf{m}}^{1+\alpha}, Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha)$ exists and is continuous.

(iii) At the equilibrium $(f_1, f_2) = (0, 0)$, the linearized operator admits the following Fourier representation

$$\begin{aligned} & d_{(f_1, f_2)} \mathcal{G}(c, 0, 0) \left[\sum_{n=1}^{\infty} h_n^{(1)} \cos(\mathbf{m}n\varphi), \sum_{n=1}^{\infty} h_n^{(2)} \cos(\mathbf{m}n\varphi) \right] \\ &= \sum_{n=1}^{\infty} \mathbf{m}n M_{\mathbf{m}n}(c, \theta_1, \theta_2) \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} \sin(\mathbf{m}n\varphi), \end{aligned} \quad (3.6)$$

with

$$M_n(c, \theta_1, \theta_2) \triangleq \begin{pmatrix} -c + \frac{\omega_N - \omega_C}{2n} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} + \tilde{\gamma} & \frac{\omega_C - \omega_S}{2n} \frac{\sin(\theta_2)}{\sin(\theta_1)} \tan^n(\frac{\theta_1}{2}) \cot^n(\frac{\theta_2}{2}) \\ \frac{\omega_N - \omega_C}{2n} \frac{\sin(\theta_1)}{\sin(\theta_2)} \tan^n(\frac{\theta_1}{2}) \cot^n(\frac{\theta_2}{2}) & -c + \frac{\omega_C - \omega_S}{2n} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \end{pmatrix}. \quad (3.7)$$

In addition, if $c \notin \left\{ \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})}, \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} \right\}$, then the operator $d_{(f_1, f_2)} \mathcal{G}(c, 0, 0) : X_{\mathbf{m}}^{1+\alpha} \times X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha$ is of Fredholm type with index zero.

Proof. (i) The proof is very close to Proposition 2.1-(i). Indeed, the functional involves terms corresponding to the self-interaction of each boundary with itself, which correspond to the one-interface analysis. It also involves new terms corresponding to the interaction between both boundaries, but which are non-singular (smooth kernels). Therefore, we omit the proof and just give the expression of the linearized operator. For $k \in \{1, 2\}$,

$$\begin{aligned} & d_{f_k} \mathcal{G}_k(c, f_1, f_2)[h_k](\varphi) \\ &= c \partial_\varphi h_k(\varphi) + h_k(\varphi) \frac{\cos(\theta_k + f_k(\varphi))}{\sin^2(\theta_k + f_k(\varphi))} \partial_\varphi \left(\Psi_{\text{FC}}(\theta_k + f_k(\varphi)) + (\Psi_{p,2}^{[1]} \{f_1, f_2\} + \Psi_{p,2}^{[2]} \{f_1\}) (\theta_k + f_k(\varphi), \varphi) \right) \\ &+ \frac{1}{\sin(\theta_k + f_k(\varphi))} \partial_\varphi \left(h_k(\varphi) \left[\partial_\theta \Psi_{\text{FC}}(\theta_k + f_k(\varphi)) + (\partial_\theta \Psi_{p,2}^{[1]} \{f_1, f_2\} + \partial_\theta \Psi_{p,2}^{[2]} \{f_1\}) (\theta_k + f_k(\varphi), \varphi) \right] \right) \\ &+ \frac{1}{\sin(\theta_k + f_k(\varphi))} \partial_\varphi \left((d_{f_k} \Psi_{p,2}^{[1]} \{f_1, f_2\} [h_k]) (\theta_k + f_k(\varphi), \varphi) + \delta_{k,1} d_{f_k} (\Psi_{p,2}^{[2]} \{f_1\} [h_k]) (\theta_k + f_k(\varphi), \varphi) \right), \end{aligned}$$

where $\delta_{k,1}$ is the kronecker symbol and

$$\begin{aligned} d_{f_{3-k}} \mathcal{G}_k(c, f_1, f_2)[h_{3-k}](\varphi) &= \frac{1}{\sin(\theta_k + f_k(\varphi))} \partial_\varphi \left((d_{f_{3-k}} \Psi_{p,2}^{[1]} \{f_1, f_2\} [h_{3-k}]) (\theta_k + f_k(\varphi), \varphi) \right) \\ &+ \frac{\delta_{3-k,1}}{\sin(\theta_k + f_k(\varphi))} \partial_\varphi \left((d_{f_{3-k}} \Psi_{p,2}^{[2]} \{f_1\} [h]) (\theta_k + f_k(\varphi), \varphi) \right). \end{aligned}$$

If we denote $(\omega_N, \omega_C, \omega_S) = (\omega_1, \omega_2, \omega_3)$, then for $k \in \{1, 2\}$, we have,

$$(d_{f_k} \Psi_{p,2}^{[1]} \{f_1, f_2\} [h_k]) (\theta, \varphi) = \frac{\omega_k - \omega_{k+1}}{4\pi} \int_0^{2\pi} h_k(\varphi') \log \left(D(\theta, \theta_k + f_k(\varphi'), \varphi, \varphi') \right) \sin(\theta_k + f_k(\varphi')) d\varphi'.$$

Moreover,

$$\begin{aligned} (d_{f_1} \Psi_{p,2}^{[2]} \{f_1\} [h_1]) (\theta, \varphi) &= \frac{d_{f_1} \omega_N(f_1) [h_1]}{4\pi} \int_0^{2\pi} \int_0^{\theta_1 + f_1(\varphi')} \log \left(D(\theta, \theta', \varphi, \varphi') \right) \sin(\theta') d\theta' d\varphi' \\ &+ \frac{\omega_N(f_1) - \omega_N}{4\pi} \int_0^{2\pi} h_1(\varphi') \log \left(D(\theta, \theta_1 + f_1(\varphi'), \varphi, \varphi') \right) \sin(\theta_1 + f_1(\varphi')) d\varphi'. \end{aligned}$$

(ii) Immediate since

$$\partial_c f_{(f_1, f_2)} \mathcal{G}(c, f_1, f_2)[h_1, h_2] = \begin{pmatrix} \partial_\varphi h_1 & 0 \\ 0 & \partial_\varphi h_2 \end{pmatrix}. \quad (3.8)$$

(iii) We assume now that $(f_1, f_2) = (0, 0)$. Using (3.4) and $\Psi_{p,2}^{[1]} \{0, 0\} = 0$, we have

$$\begin{aligned} d_{f_k} \mathcal{G}_k(c, 0, 0)[h_k](\varphi) &= c \partial_\varphi h_k(\varphi) + \frac{1}{\sin(\theta_k)} \partial_\varphi \left(\partial_\theta \Psi_{\text{FC}}(\theta_k) h_k(\varphi) + (d_{f_k} \Psi_{p,2}^{[1]} \{0, 0\} [h_k]) (\theta_k, \varphi) \right), \\ d_{f_{3-k}} \mathcal{G}_k(c, 0, 0)[h_{3-k}](\varphi) &= \frac{1}{\sin(\theta_k)} \partial_\varphi \left((d_{f_{3-k}} \Psi_{p,2}^{[1]} \{0, 0\} [h_{3-k}]) (\theta_k, \varphi) \right). \end{aligned}$$

From (3.2), we deduce

$$\begin{aligned}
\frac{1}{\sin(\theta_1)} \partial_\varphi \left(\partial_\theta \Psi_{\text{FC}}(\theta_1) h_1 \right) &= \left(\frac{\omega_N \tan\left(\frac{\theta_1}{2}\right)}{\sin(\theta_1)} - \tilde{\gamma} \right) \partial_\varphi h_1 \\
&= \left(\frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} - \tilde{\gamma} \right) \partial_\varphi h_1 \\
&= \left(\tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} \right) \sum_{n=1}^{\infty} \mathbf{m} n h_n^{(1)} \sin(\mathbf{m} n \varphi)
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
\frac{1}{\sin(\theta_2)} \partial_\varphi \left(\partial_\theta \Psi_{\text{FC}}(\theta_2) h_2 \right) &= - \left(\frac{\omega_S \cot\left(\frac{\theta_2}{2}\right)}{\sin(\theta_2)} + \tilde{\gamma} \right) \partial_\varphi h_2 \\
&= - \left(\frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} + \tilde{\gamma} \right) \partial_\varphi h_2 \\
&= \left(\frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} + \tilde{\gamma} \right) \sum_{n=1}^{\infty} \mathbf{m} n h_n^{(2)} \sin(\mathbf{m} n \varphi).
\end{aligned} \tag{3.10}$$

After straightforward simplifications and using Lemma A.1, we get for $k, \ell \in \{1, 2\}$,

$$\begin{aligned}
(d_{f_\ell} \Psi_{p,2}^{[1]} \{0, 0\} [h_\ell]) (\theta_k, \varphi) &= \frac{\omega_\ell - \omega_{\ell+1}}{4\pi} \sin(\theta_\ell) \int_0^{2\pi} h_\ell(\varphi') \log \left(D(\theta_k, \theta_\ell, \varphi, \varphi') \right) d\varphi' \\
&= \frac{\omega_\ell - \omega_{\ell+1}}{2} \sin(\theta_\ell) \sum_{n=1}^{\infty} h_n^{(\ell)} I_{\mathbf{m}n}(\theta_k, \theta_\ell) \cos(\mathbf{m} n \varphi) \\
&= - \frac{\omega_\ell - \omega_{\ell+1}}{2} \sin(\theta_\ell) \sum_{n=1}^{\infty} \frac{h_n^{(\ell)}}{\mathbf{m}n} \tan^{\mathbf{m}n} \left(\frac{\min(\theta_k, \theta_\ell)}{2} \right) \cot^{\mathbf{m}n} \left(\frac{\max(\theta_k, \theta_\ell)}{2} \right) \cos(\mathbf{m} n \varphi).
\end{aligned}$$

Therefore,

$$\frac{\partial_\varphi \left((d_{f_\ell} \Psi_{p,2}^{[1]} \{0, 0\} [h_\ell]) (\theta_k, \varphi) \right)}{\sin(\theta_k)} = \frac{\omega_\ell - \omega_{\ell+1}}{2} \frac{\sin(\theta_\ell)}{\sin(\theta_k)} \sum_{n=1}^{\infty} h_n^{(\ell)} \tan^{\mathbf{m}n} \left(\frac{\min(\theta_k, \theta_\ell)}{2} \right) \cot^{\mathbf{m}n} \left(\frac{\max(\theta_k, \theta_\ell)}{2} \right) \sin(\mathbf{m} n \varphi).$$

Putting together the foregoing calculations, we get (3.6)-(3.7). Now, denoting

$$\mathcal{Q}(\varphi) \triangleq \log \left(1 - \cos(\theta_1) \cos(\theta_2) - \sin(\theta_1) \sin(\theta_2) \cos(\varphi) \right),$$

we have

$$\begin{aligned}
d_{(f_1, f_2)} \mathcal{G}(c, 0, 0) &= I + K, \\
I &\triangleq \begin{pmatrix} \left(c + \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} - \tilde{\gamma} \right) \partial_\varphi & 0 \\ 0 & \left(c - \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} - \tilde{\gamma} \right) \partial_\varphi \end{pmatrix}, \\
K &\triangleq \begin{pmatrix} \frac{\omega_N - \omega_C}{2} \mathcal{H} & \frac{\omega_C - \omega_S}{2} \frac{\sin(\theta_2)}{\sin(\theta_1)} \partial_\varphi \mathcal{Q} * \cdot \\ \frac{\omega_N - \omega_C}{2} \frac{\sin(\theta_1)}{\sin(\theta_2)} \partial_\varphi \mathcal{Q} * \cdot & \frac{\omega_C - \omega_S}{2} \mathcal{H} \end{pmatrix}.
\end{aligned} \tag{3.11}$$

If $c \notin \left\{ \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)}, \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} \right\}$, then $I : X_{\mathbf{m}}^{1+\alpha} \times X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha$ is an isomorphism. We have already studied the compact property of the Hilbert transform in the proof of Proposition 2.2, so we are left with the anti-diagonal terms. Actually, the corresponding symbol decays exponentially fast in n , which implies that $\partial_\varphi \mathcal{Q} * \cdot$ is smoothing at every order (so a fortiori compact in the considered functional framework). Thus, the operator $K : X_{\mathbf{m}}^{1+\alpha} \times X_{\mathbf{m}}^{1+\alpha} \rightarrow Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha$ is compact. We deduce the desired Fredholmness property. \square

We shall now study the spectrum.

Lemma 3.2. Let $\tilde{\gamma} \in \mathbb{R}$. There exists $N(\theta_1, \theta_2) \triangleq N(\theta_1, \theta_2, \omega_N, \omega_S, \omega_C) \in \mathbb{N}^*$ such that for any $n \in \mathbb{N}^*$ with $n \geq N(\theta_1, \theta_2)$, there exist two velocities

$$c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2) \triangleq \tilde{\gamma} + \frac{\omega_S}{4 \sin^2(\frac{\theta_2}{2})} - \frac{\omega_N}{4 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_S}{4n} \quad (3.12)$$

$$\pm \frac{1}{4} \sqrt{\left(\frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right)^2 + \frac{1}{n^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n}(\frac{\theta_1}{2}) \cot^{2n}(\frac{\theta_2}{2})}$$

for which the matrix $M_n(c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2)$ is singular. The sequences $(c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2))_{n \geq N(\theta_1, \theta_2)}$ are strictly monotone and

$$\mathbb{L} \triangleq \left\{ \lim_{n \rightarrow \infty} c_n^+(\tilde{\gamma}, \theta_1, \theta_2), \lim_{n \rightarrow \infty} c_n^-(\tilde{\gamma}, \theta_1, \theta_2) \right\} = \left\{ \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})}, \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} \right\}. \quad (3.13)$$

Moreover,

1. If

$$\omega_S \cos^2(\frac{\theta_1}{2}) + \omega_N \sin^2(\frac{\theta_2}{2}) \neq 0. \quad (3.14)$$

then $|\mathbb{L}| = 2$ and the following equations have no solution

$$c_p^+(\tilde{\gamma}, \theta_1, \theta_2) = c_q^-(\tilde{\gamma}, \theta_1, \theta_2), \quad p, q \geq N(\theta_1, \theta_2). \quad (3.15)$$

2. If

$$\omega_S \cos^2(\frac{\theta_1}{2}) + \omega_N \sin^2(\frac{\theta_2}{2}) = 0. \quad (3.16)$$

then

$$|\mathbb{L}| = 1, \quad \omega_N + \omega_S = \omega_C \neq 0, \quad \omega_N \omega_S < 0.$$

In particular, (3.12) simplifies into

$$c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_N - \omega_S}{4n} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} \pm \frac{1}{4n} \sqrt{\omega_C^2 - \omega_N \omega_S \tan^{2n}(\frac{\theta_1}{2}) \cot^{2n}(\frac{\theta_2}{2})}. \quad (3.17)$$

In addition, for any $\mathbf{m} \in \mathbb{N}$ with $\mathbf{m} \geq N(\theta_1, \theta_2)$, under one of the additional constraints

$$\begin{aligned} (\mathbf{H1+}) \quad & \omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0, \\ (\mathbf{H2+}) \quad & \omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0 \quad \text{and} \quad 2 \cos^2(\frac{\theta_1}{2}) > \sin^2(\frac{\theta_2}{2}), \\ (\mathbf{H3+}) \quad & \omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0, \\ (\mathbf{H4+}) \quad & \omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0 \quad \text{and} \quad 2 \sin^2(\frac{\theta_2}{2}) > \cos^2(\frac{\theta_1}{2}), \end{aligned}$$

the following equations have no solution

$$c_{\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{k\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*.$$

And under one of the additional constraints

$$\begin{aligned} (\mathbf{H1-}) \quad & \omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0, \\ (\mathbf{H2-}) \quad & \omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0 \quad \text{and} \quad 2 \sin^2(\frac{\theta_2}{2}) > \cos^2(\frac{\theta_1}{2}), \\ (\mathbf{H3-}) \quad & \omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0, \\ (\mathbf{H4-}) \quad & \omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0 \quad \text{and} \quad 2 \cos^2(\frac{\theta_1}{2}) > \sin^2(\frac{\theta_2}{2}), \end{aligned}$$

the following equations have no solution

$$c_{k\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*.$$

This means that there is no \mathbf{m} -fold spectral collision, i.e. the \mathbf{m} -fold spectrum is simple.

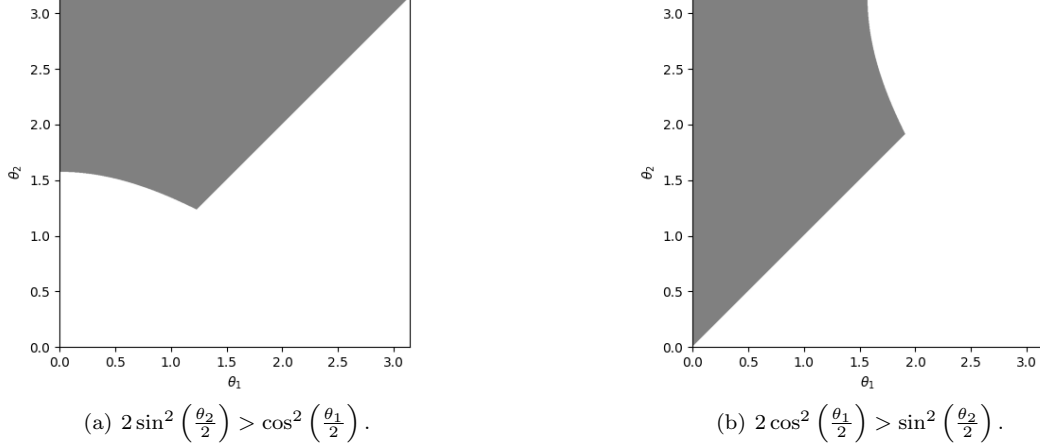


Figure 4: The shaded regions represent the admissible couples $(\theta_1, \theta_2) \in (0, \pi)^2$ with $\theta_1 < \theta_2$.

Proof. From (3.7), we have that the determinant of $M_n(c, \theta_1, \theta_2)$ is

$$\det(M_n(c, \theta_1, \theta_2)) \triangleq c^2 - \beta_n(\tilde{\gamma}, \theta_1, \theta_2)c + \gamma_n(\tilde{\gamma}, \theta_1, \theta_2) \in \mathbb{R}_2[c],$$

$$\beta_n(\tilde{\gamma}, \theta_1, \theta_2) \triangleq \frac{\omega_N - \omega_S}{2n} - \frac{\omega_N}{2 \cos^2 \left(\frac{\theta_1}{2} \right)} + \frac{\omega_S}{2 \sin^2 \left(\frac{\theta_2}{2} \right)} + 2\tilde{\gamma}, \quad (3.18)$$

$$\gamma_n(\tilde{\gamma}, \theta_1, \theta_2) \triangleq \left(\frac{\omega_C - \omega_S}{2n} + \frac{\omega_S}{2 \sin^2 \left(\frac{\theta_2}{2} \right)} + \tilde{\gamma} \right) \left(\frac{\omega_N - \omega_C}{2n} - \frac{\omega_N}{2 \cos^2 \left(\frac{\theta_1}{2} \right)} + \tilde{\gamma} \right)$$

$$- \frac{1}{4n^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n} \left(\frac{\theta_1}{2} \right) \cot^{2n} \left(\frac{\theta_2}{2} \right). \quad (3.19)$$

The discriminant of the previous polynomial is

$$\Delta_n(\theta_1, \theta_2) \triangleq \beta_n^2(\tilde{\gamma}, \theta_1, \theta_2) - 4\gamma_n(\tilde{\gamma}, \theta_1, \theta_2)$$

$$= \frac{1}{4} \left[\left(\frac{\omega_S}{\sin^2 \left(\frac{\theta_2}{2} \right)} + \frac{\omega_N}{\cos^2 \left(\frac{\theta_1}{2} \right)} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right)^2 + \frac{1}{n^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n} \left(\frac{\theta_1}{2} \right) \cot^{2n} \left(\frac{\theta_2}{2} \right) \right].$$

Notice that $\Delta_n(\theta_1, \theta_2)$ is independent of $\tilde{\gamma}$. We shall now prove that

$$\exists N(\theta_1, \theta_2) \in \mathbb{N}^*, \quad \forall n \in \mathbb{N}, \quad n \geq N(\theta_1, \theta_2) \Rightarrow \Delta_n(\theta_1, \theta_2) > 0. \quad (3.20)$$

Assuming that (3.20) is true, then we conclude that, for n large enough, we have two distinct real roots

$$c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2) \triangleq \frac{1}{2} \beta_n(\tilde{\gamma}, \theta_1, \theta_2) \pm \frac{1}{2} \sqrt{\Delta_n(\theta_1, \theta_2)}$$

$$= \tilde{\gamma} + \frac{\omega_S}{4 \sin^2 \left(\frac{\theta_2}{2} \right)} - \frac{\omega_N}{4 \cos^2 \left(\frac{\theta_1}{2} \right)} + \frac{\omega_N - \omega_S}{4n}$$

$$\pm \frac{1}{4} \sqrt{\left(\frac{\omega_S}{\sin^2 \left(\frac{\theta_2}{2} \right)} + \frac{\omega_N}{\cos^2 \left(\frac{\theta_1}{2} \right)} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right)^2 + \frac{1}{n^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n} \left(\frac{\theta_1}{2} \right) \cot^{2n} \left(\frac{\theta_2}{2} \right)}.$$

1. First assume that (3.14) holds. From the proof of Lemma A.1, we know that $\tan \left(\frac{\theta_1}{2} \right) \cot \left(\frac{\theta_2}{2} \right) < 1$, then

$$\forall k \in \mathbb{N}, \quad \frac{1}{n^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n} \left(\frac{\theta_1}{2} \right) \cot^{2n} \left(\frac{\theta_2}{2} \right) \underset{n \rightarrow \infty}{=} O_{\theta_1, \theta_2} \left(\frac{1}{n^k} \right). \quad (3.21)$$

Then,

$$\Delta_\infty(\theta_1, \theta_2) = \frac{1}{4} \left(\frac{\omega_S}{\sin^2 \left(\frac{\theta_2}{2} \right)} + \frac{\omega_N}{\cos^2 \left(\frac{\theta_1}{2} \right)} \right)^2 > 0, \quad (3.22)$$

and (3.20) is true. Factorizing, we can write for any n sufficiently large

$$c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{4 \sin^2(\frac{\theta_2}{2})} - \frac{\omega_N}{4 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_S}{4n} \pm \frac{1}{4} \left| \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right| \pm \mathbf{r}_n(\theta_1, \theta_2),$$

with

$$\mathbf{r}_n(\theta_1, \theta_2) \triangleq \frac{1}{4} \left| \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right| \left[\sqrt{1 + \frac{(\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2n}(\frac{\theta_1}{2}) \cot^{2n}(\frac{\theta_2}{2})}{\left(\left[\frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} \right] n - (\omega_N + \omega_S - 2\omega_C) \right)^2}} - 1 \right].$$

Notice that (3.21) implies

$$\forall k \in \mathbb{N}, \quad \mathbf{r}_n(\theta_1, \theta_2) \underset{n \rightarrow \infty}{=} O_{\theta_1, \theta_2} \left(\frac{1}{n^k} \right). \quad (3.23)$$

We have the following dichotomy.

- If $\omega_S \cos^2(\frac{\theta_1}{2}) + \omega_N \sin^2(\frac{\theta_2}{2}) > 0$, then for n large enough we have

$$\left| \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right| = \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n},$$

and therefore

$$\begin{cases} c_n^+(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \frac{\omega_C - \omega_S}{2n} + \mathbf{r}_n(\theta_1, \theta_2), \\ c_n^-(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_C}{2n} - \mathbf{r}_n(\theta_1, \theta_2). \end{cases}$$

As a consequence,

$$\begin{cases} c_{n+1}^+(\tilde{\gamma}, \theta_1, \theta_2) - c_n^+(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_S - \omega_C}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right), \\ c_{n+1}^-(\tilde{\gamma}, \theta_1, \theta_2) - c_n^-(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_C - \omega_N}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right). \end{cases}$$

Since $\omega_N \neq \omega_C$ and $\omega_C \neq \omega_S$, then we conclude the asymptotic (strict) monotonicity of $n \mapsto c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2)$. In addition,

$$\lim_{n \rightarrow \infty} c_n^+(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})}, \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n^-(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})}.$$

- If $\omega_S \cos^2(\frac{\theta_1}{2}) + \omega_N \sin^2(\frac{\theta_2}{2}) < 0$, then for n large enough we have

$$\left| \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} + \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{n} \right| = \frac{\omega_N + \omega_S - 2\omega_C}{n} - \frac{\omega_S}{\sin^2(\frac{\theta_2}{2})} - \frac{\omega_N}{\cos^2(\frac{\theta_1}{2})}$$

and therefore

$$\begin{cases} c_n^+(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_C}{2n} - \mathbf{r}_n(\theta_1, \theta_2), \\ c_n^-(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \frac{\omega_C - \omega_S}{2n} + \mathbf{r}_n(\theta_1, \theta_2). \end{cases}$$

As a consequence,

$$\begin{cases} c_{n+1}^+(\tilde{\gamma}, \theta_1, \theta_2) - c_n^+(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_C - \omega_N}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right), \\ c_{n+1}^-(\tilde{\gamma}, \theta_1, \theta_2) - c_n^-(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_S - \omega_C}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right). \end{cases}$$

The monotonicity conclusion is still valid. In addition, in this case, the limits are exchanged

$$\lim_{n \rightarrow \infty} c_n^+(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n^-(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})}.$$

The condition (3.14) implies that the above limits are well-separated. Together with the strict monotonicity property, we conclude (3.15).

2. Conversely, we assume that (3.16) holds. This condition can also be written

$$\omega_S = -\omega_N \frac{1 - \cos(\theta_2)}{1 + \cos(\theta_1)}. \quad (3.24)$$

Notice that the Gauss constraint (3.1) can be written as follows

$$\omega_C = \frac{\omega_N(1 - \cos(\theta_1)) + \omega_S(1 + \cos(\theta_2))}{\cos(\theta_2) - \cos(\theta_1)}. \quad (3.25)$$

Plugging (3.24) into (3.25) gives

$$\begin{aligned} \omega_C &= \omega_N \frac{(1 - \cos(\theta_1))(1 + \cos(\theta_1)) - (1 - \cos(\theta_2))(1 + \cos(\theta_2))}{(\cos(\theta_2) - \cos(\theta_1))(1 + \cos(\theta_1))} \\ &= \omega_N \frac{\cos^2(\theta_2) - \cos^2(\theta_1)}{(\cos(\theta_2) - \cos(\theta_1))(1 + \cos(\theta_1))} \\ &= \omega_N \frac{\cos(\theta_1) + \cos(\theta_2)}{1 + \cos(\theta_1)}. \end{aligned} \quad (3.26)$$

From (3.24) and (3.26), we get

$$\omega_N + \omega_S = \frac{\omega_N}{1 + \cos(\theta_1)} \left[1 + \cos(\theta_1) - (1 - \cos(\theta_2)) \right] = \omega_N \frac{\cos(\theta_1) + \cos(\theta_2)}{1 + \cos(\theta_1)} = \omega_C.$$

This last expression implies that $\omega_N \neq 0$ (resp. $\omega_S \neq 0$) otherwise $\omega_S = \omega_C$ (resp. $\omega_N = \omega_C$) which is excluded by construction. We also deduce

$$\omega_N - \omega_C = -\omega_S, \quad \omega_C - \omega_S = \omega_N.$$

Now, using (3.24), we infer

$$\omega_N \omega_S = -\omega_N^2 \frac{1 - \cos(\theta_2)}{1 + \cos(\theta_1)} < 0. \quad (3.27)$$

In particular $\omega_N \neq \omega_S$ and have opposite sign. Now, assume for the sake of contradiction that $\omega_C = 0$, i.e. $\omega_N = -\omega_S$. Combined with (3.16) and the fact that $\frac{\theta_1}{2}, \frac{\theta_2}{2} \in (0, \frac{\pi}{2})$, we deduce

$$\cos^2\left(\frac{\theta_1}{2}\right) = \sin^2\left(\frac{\theta_2}{2}\right), \quad \text{i.e.} \quad \theta_1 = \theta_2.$$

This enters in contradiction with (3.3). Thus,

$$\omega_C \neq 0 \quad \text{and} \quad \omega_N + \omega_S - 2\omega_C = -\omega_C \neq 0. \quad (3.28)$$

In this case, the discriminant becomes

$$\forall n \in \mathbb{N}^*, \quad \Delta_n(\theta_1, \theta_2) = \frac{1}{4n^2} \left[\omega_C^2 - \omega_N \omega_S \tan^{2n}\left(\frac{\theta_1}{2}\right) \cot^{2n}\left(\frac{\theta_2}{2}\right) \right] > 0.$$

This implies in particular (3.20). Factorizing, we can write

$$\begin{aligned} c_n^\pm(\tilde{\gamma}, \theta_1, \theta_2) &= \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N - \omega_S}{4n} \pm \frac{|\omega_C|}{4n} \pm \mathbf{r}_n(\theta_1, \theta_2) \\ &= \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N - \omega_S}{4n} \pm \frac{|\omega_C|}{4n} \pm \mathbf{r}_n(\theta_1, \theta_2), \end{aligned}$$

with

$$\mathbf{r}_n(\theta_1, \theta_2) \triangleq \frac{|\omega_C|}{4n} \left[\sqrt{1 - \frac{\omega_N \omega_S}{\omega_C^2} \tan^{2n}\left(\frac{\theta_1}{2}\right) \cot^{2n}\left(\frac{\theta_2}{2}\right)} - 1 \right].$$

We have the following dichotomy.

- If $\omega_C > 0$, then $|\omega_C| = \omega_C = \omega_N + \omega_S$ and therefore

$$\begin{cases} c_n^+(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N}{2n} + \mathbf{r}_n(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{2n} + \mathbf{r}_n(\theta_1, \theta_2), \\ c_n^-(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_S}{2n} - \mathbf{r}_n(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_S}{2n} - \mathbf{r}_n(\theta_1, \theta_2). \end{cases} \quad (3.29)$$

As a consequence,

$$\begin{cases} c_{n+1}^+(\tilde{\gamma}, \theta_1, \theta_2) - c_n^+(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} -\frac{\omega_N}{2n(n+1)} + O_{\theta_1, \theta_2}\left(\frac{1}{n^3}\right), \\ c_{n+1}^-(\tilde{\gamma}, \theta_1, \theta_2) - c_n^-(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_S}{2n(n+1)} + O_{\theta_1, \theta_2}\left(\frac{1}{n^3}\right). \end{cases}$$

This is sufficient to conclude the asymptotic strict monotonicity. In addition, since $\omega_N \omega_S < 0$, then both sequences have the same monotonicity asymptotically. Nevertheless, in this case, both part of the spectrum accumulate at the same point

$$\lim_{n \rightarrow \infty} c_n^+(\tilde{\gamma}, \theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} = \lim_{n \rightarrow \infty} c_n^-(\tilde{\gamma}, \theta_1, \theta_2).$$

Therefore, one must avoid the spectral collisions by a more careful analysis.

► Let us first study the equation

$$c_{\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{k\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*.$$

According to (3.29), this equation is equivalent to

$$-k(\omega_N + \tilde{\mathbf{r}}_{\mathbf{m}}) = \omega_S + \tilde{\mathbf{r}}_{k\mathbf{m}}, \quad \tilde{\mathbf{r}}_n \triangleq 2n\mathbf{r}_n(\theta_1, \theta_2) > 0. \quad (3.30)$$

Observe that $n \mapsto \tilde{\mathbf{r}}_n$ is asymptotically decreasing and satisfies (3.23).

► If $\omega_N > 0$, then the equation (3.30) can be written

$$0 = \omega_C + 2(k-1)\omega_N + k\tilde{\mathbf{r}}_{\mathbf{m}} + \tilde{\mathbf{r}}_{k\mathbf{m}}.$$

Each term in the right hand-side is non-negative and $\omega_C > 0$, then this equation has no solution.

► Assume now that $\omega_N < 0$. By virtue of (3.27), we have $\omega_S > 0$. According to (3.23) and (3.30), we can select \mathbf{m} large enough to ensure

$$\tilde{\mathbf{r}}_{\mathbf{m}} < |\omega_N|.$$

Added to the asymptotic decay property of $n \mapsto \tilde{\mathbf{r}}_n$ we get

$$\forall k \in \mathbb{N} \setminus \{0, 1\}, \quad -k(\omega_N + \tilde{\mathbf{r}}_{\mathbf{m}}) \geq 2(|\omega_N| - \tilde{\mathbf{r}}_{\mathbf{m}}) \quad \text{and} \quad \omega_S + \tilde{\mathbf{r}}_{2\mathbf{m}} \geq \omega_S + \tilde{\mathbf{r}}_{k\mathbf{m}}.$$

Hence, it suffices to impose

$$2(|\omega_N| - \tilde{\mathbf{r}}_{\mathbf{m}}) > \omega_S + \tilde{\mathbf{r}}_{2\mathbf{m}}, \quad \text{i.e.} \quad 2|\omega_N| > \omega_S + 2\tilde{\mathbf{r}}_{\mathbf{m}} + \tilde{\mathbf{r}}_{2\mathbf{m}},$$

so that the equations (3.30) admit no solution for any $k \in \mathbb{N}^*$ (recall that $c_{\mathbf{m}}^+ \neq c_{\mathbf{m}}^-$). Using (3.23), we deduce that, up to taking \mathbf{m} large enough, the following condition is sufficient

$$2|\omega_N| > \omega_S. \quad (3.31)$$

But, according to (3.16), the constraint (3.31) is equivalent to

$$2 \cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right).$$

► Now, we turn to the study of the equation

$$c_{k\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*.$$

Using again (3.29), this equation is equivalent to

$$\omega_N + \tilde{\mathbf{r}}_{k\mathbf{m}} = -k(\omega_S + \tilde{\mathbf{r}}_{\mathbf{m}}).$$

This is basically the same equation as (3.30) where ω_S and ω_N have been exchanged. So either $\omega_S > 0$ and there is no solution, or $\omega_S < 0$ and there is no solution provided that \mathbf{m} is large enough and

$$2|\omega_S| > \omega_N, \quad \text{i.e.} \quad 2 \sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right).$$

- If $\omega_C < 0$, then $|\omega_C| = -\omega_C = -\omega_N - \omega_S$ and therefore

$$\begin{cases} c_n^+(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} - \frac{\omega_S}{2n} + \mathbf{r}_n(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_S}{2n} + \mathbf{r}_n(\theta_1, \theta_2), \\ c_n^-(\theta_1, \theta_2) = \tilde{\gamma} - \frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_N}{2n} - \mathbf{r}_n(\theta_1, \theta_2) = \tilde{\gamma} + \frac{\omega_N}{2 \sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_N}{2n} - \mathbf{r}_n(\theta_1, \theta_2). \end{cases} \quad (3.32)$$

As a consequence,

$$\begin{cases} c_{n+1}^+(\tilde{\gamma}, \theta_1, \theta_2) - c_n^+(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} \frac{\omega_S}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right), \\ c_{n+1}^-(\tilde{\gamma}, \theta_1, \theta_2) - c_n^-(\tilde{\gamma}, \theta_1, \theta_2) \underset{n \rightarrow \infty}{=} -\frac{\omega_N}{2n(n+1)} + O_{\theta_1, \theta_2} \left(\frac{1}{n^3} \right). \end{cases}$$

As before, we can conclude the asymptotic strict monotonicity with the same limit.

► According to (3.32), the equation

$$c_{\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{k\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*$$

is equivalent to

$$-k(-\omega_S + \tilde{\mathbf{r}}_{\mathbf{m}}) = -\omega_N + \tilde{\mathbf{r}}_{k\mathbf{m}}$$

which is (3.30) with (ω_N, ω_S) replaced by $(-\omega_S, -\omega_N)$. So there is no solution for either $\omega_S < 0$ or $\omega_S > 0$ and $2 \sin^2 \left(\frac{\theta_2}{2} \right) > \cos^2 \left(\frac{\theta_1}{2} \right)$.

► The equation

$$c_{k\mathbf{m}}^+(\tilde{\gamma}, \theta_1, \theta_2) = c_{\mathbf{m}}^-(\tilde{\gamma}, \theta_1, \theta_2), \quad k \in \mathbb{N}^*$$

is equivalent to

$$-\omega_S + \tilde{\mathbf{r}}_{k\mathbf{m}} = -k(-\omega_N + \tilde{\mathbf{r}}_{\mathbf{m}})$$

which is (3.30) with (ω_N, ω_S) replaced by $(-\omega_N, -\omega_S)$. So there is no solution for either $\omega_N < 0$ or $\omega_N > 0$ and $2 \cos^2 \left(\frac{\theta_1}{2} \right) > \sin^2 \left(\frac{\theta_2}{2} \right)$. □

3.3 Proof of the main result

In the following proposition, we gather all the remaining conditions required to apply the Crandall-Rabinowitz Theorem.

Proposition 3.2. *Let $\alpha \in (0, 1)$, $\kappa \in \{+, -\}$ and $\mathbf{m} \in \mathbb{N}^*$ with $\mathbf{m} \geq N(\theta_1, \theta_2)$. Assume that (3.14) holds or assume that (3.16) with $(\mathbf{H}\kappa)$ for some $k \in \llbracket 1, 4 \rrbracket$ holds.*

(i) *The linear operator $d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0)$ is of Fredholm type with index zero.*

(ii) *The kernel of $d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0)$ is one dimensional. More precisely,*

$$\ker \left(d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0) \right) = \text{span}(u_0), \quad (3.33)$$

with

$$u_0 : \varphi \mapsto \begin{pmatrix} -c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2 \left(\frac{\theta_2}{2} \right)} + \tilde{\gamma} \\ \frac{\omega_C - \omega_N}{2\mathbf{m}} \frac{\sin(\theta_1)}{\sin(\theta_2)} \tan^{\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{\mathbf{m}} \left(\frac{\theta_2}{2} \right) \end{pmatrix} \cos(\mathbf{m}\varphi).$$

(iii) *The transversality condition is satisfied, namely*

$$\partial_c d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0)[u_0] \notin \text{Im} \left(d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0) \right). \quad (3.34)$$

Proof. (i) The strict monotonicity of $n \mapsto c_n^\kappa(\tilde{\gamma}, \theta_1, \theta_2)$ gives $c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) \notin \mathbb{L}$, where \mathbb{L} is defined in (3.13). Together with Proposition 3.1-(iii) this implies the desired Fredholmness property.

(ii) The non-degeneracy conditions imply that

$$\forall n \in \mathbb{N} \setminus \{0, 1\}, \quad \det \left(M_{\mathbf{m}n}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \right) \neq 0.$$

Together with the fact that the matrix $M_{\mathbf{m}}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2)$ is singular and non-zero, we obtain from (3.6) the desired result because

$$\ker \left(M_{\mathbf{m}}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \right) = \text{span} \left(\begin{pmatrix} -c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2 \left(\frac{\theta_2}{2} \right)} + \tilde{\gamma} \\ \frac{\omega_C - \omega_N}{2\mathbf{m}} \frac{\sin(\theta_1)}{\sin(\theta_2)} \tan^{\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{\mathbf{m}} \left(\frac{\theta_2}{2} \right) \end{pmatrix} \right).$$

(iii) The next step is to describe the range. For this aim, we introduce on $Y_{\mathbf{m}}^\alpha \times Y_{\mathbf{m}}^\alpha$ the scalar product

$$\left(\left(\sum_{n=1}^{\infty} a_n \sin(\mathbf{m}n\varphi), \sum_{n=1}^{\infty} c_n \sin(\mathbf{m}n\varphi) \right) \middle| \left(\sum_{n=1}^{\infty} b_n \sin(\mathbf{m}n\varphi), \sum_{n=1}^{\infty} d_n \sin(\mathbf{m}n\varphi) \right) \right)_2 \triangleq \sum_{n=1}^{\infty} a_n b_n + c_n d_n.$$

Now we claim that

$$\text{Im} \left(d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0) \right) = \text{span}^{\perp(\cdot)_2} \left(g_0 : \varphi \mapsto \begin{pmatrix} -c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_S - \omega_C}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \\ \frac{\omega_S - \omega_C}{2\mathbf{m}} \frac{\sin(\theta_2)}{\sin(\theta_1)} \tan^{\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{\mathbf{m}} \left(\frac{\theta_2}{2} \right) \end{pmatrix} \sin(\mathbf{m}\varphi) \right). \quad (3.35)$$

Indeed, as in the proof of Proposition 2.2-(iii), we shall prove the first inclusion and the second one is obtained by the Fredholmness property and the previous point. First observe that

$$v_{\mathbf{m}}^\kappa(\theta_1, \theta_2) \triangleq \begin{pmatrix} -c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \\ \frac{\omega_S - \omega_C}{2\mathbf{m}} \frac{\sin(\theta_2)}{\sin(\theta_1)} \tan^{\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{\mathbf{m}} \left(\frac{\theta_2}{2} \right) \end{pmatrix} \in \ker \left(M_{\mathbf{m}}^\top(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \right).$$

Now, consider

$$g : \varphi \mapsto \sum_{n=1}^{\infty} \mathbf{m}n M_{\mathbf{m}n}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \begin{pmatrix} h_n^{(1)} \\ h_n^{(2)} \end{pmatrix} \sin(\mathbf{m}n\varphi) \in \text{Im} \left(d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0) \right).$$

Then, denoting \cdot the usual scalar product on \mathbb{R}^2 , we have

$$\begin{aligned} (g | g_0)_2 &= \left(\mathbf{m} M_{\mathbf{m}}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \begin{pmatrix} h_1^{(1)} \\ h_1^{(2)} \end{pmatrix} \right) \cdot v_{\mathbf{m}}^\kappa(\theta_1, \theta_2) \\ &= \mathbf{m} \begin{pmatrix} h_1^{(1)} \\ h_1^{(2)} \end{pmatrix} \cdot \left(M_{\mathbf{m}}^\top(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) v_{\mathbf{m}}^\kappa(\theta_1, \theta_2) \right) \\ &= 0. \end{aligned}$$

This proves the claim. Now, we turn to the transversality condition. We shall prove that the following quantity does not vanish

$$\begin{aligned} & \left(\partial_c d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), 0, 0) [u_0] | g_0 \right)_2 \\ &= \mathbf{m} \left[\left(-c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \right)^2 + \frac{1}{4\mathbf{m}^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{2\mathbf{m}} \left(\frac{\theta_2}{2} \right) \right]. \end{aligned}$$

Using the fact that $\det \left(M_{\mathbf{m}}(c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2), \theta_1, \theta_2) \right) = 0$, we obtain

$$\begin{aligned} & \left(-c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \right)^2 + \frac{1}{4\mathbf{m}^2} (\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2\mathbf{m}} \left(\frac{\theta_1}{2} \right) \cot^{2\mathbf{m}} \left(\frac{\theta_2}{2} \right) \\ &= \left(-c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} \right) \left(-2c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_S}{2\mathbf{m}} + 2\tilde{\gamma} \right). \end{aligned}$$

According to the definition of $c_{\mathbf{m}}^\kappa$, the equation

$$-2c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} - \frac{\omega_N}{2 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_N - \omega_S}{2\mathbf{m}} + 2\tilde{\gamma} = 0$$

is equivalent to $\Delta_{\mathbf{m}}(\theta_1, \theta_2) = 0$, which corresponds to a double eigenvalue. This is excluded by hypothesis. Besides, we can write

$$-c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} = \frac{\omega_N}{4 \cos^2(\frac{\theta_1}{2})} + \frac{\omega_S}{4 \sin^2(\frac{\theta_2}{2})} - \frac{\omega_N + \omega_S - 2\omega_C}{4\mathbf{m}} + \kappa \frac{1}{2} \sqrt{\Delta_{\mathbf{m}}(\theta_1, \theta_2)}.$$

Assume for the sake of contradiction that

$$-c_{\mathbf{m}}^\kappa(\tilde{\gamma}, \theta_1, \theta_2) + \frac{\omega_C - \omega_S}{2\mathbf{m}} + \frac{\omega_S}{2 \sin^2(\frac{\theta_2}{2})} + \tilde{\gamma} = 0.$$

This equation is equivalent to

$$\frac{\omega_N}{2 \cos^2\left(\frac{\theta_1}{2}\right)} + \frac{\omega_S}{2 \sin^2\left(\frac{\theta_2}{2}\right)} - \frac{\omega_N + \omega_S - 2\omega_C}{2\mathbf{m}} = -\kappa \sqrt{\Delta_{\mathbf{m}}(\theta_1, \theta_2)}.$$

Taking the square, we end up with

$$\frac{1}{\mathbf{m}^2}(\omega_N - \omega_C)(\omega_C - \omega_S) \tan^{2\mathbf{m}}\left(\frac{\theta_1}{2}\right) \cot^{2\mathbf{m}}\left(\frac{\theta_2}{2}\right) = 0.$$

But, by construction $\omega_N \neq \omega_C$ and $\omega_C \neq \omega_S$. In addition $\tan\left(\frac{\theta_1}{2}\right) \cot\left(\frac{\theta_2}{2}\right) \in (0, 1)$. Contradiction. Consequently,

$$\left(\partial_c d_{(f_1, f_2)} \mathcal{G}(c_{\mathbf{m}}^{\kappa}(\tilde{\gamma}, \theta_1, \theta_2), 0, 0)[u_0] | g_0\right)_2 \neq 0.$$

This ends the proof of Proposition 3.2. \square

Finally, from the previous analysis, we have proved the following theorem, which gives a refined version of Theorem 1.2.

Theorem 3.1. *Let $\tilde{\gamma} \in \mathbb{R}$ and $0 < \theta_1 < \theta_2 < \pi$. Fix $\omega_N, \omega_C, \omega_S \in \mathbb{R}$ satisfying (1.22). Then, there exists $N(\theta_1, \theta_2) \triangleq N(\theta_1, \theta_2, \omega_N, \omega_S, \omega_C) \in \mathbb{N}^*$ such that for any $\mathbf{m} \in \mathbb{N}^*$ with $\mathbf{m} \geq N(\theta_1, \theta_2)$, the following hold true.*

1. *If the condition (3.14) is satisfied then there exist two branches of \mathbf{m} -fold uniformly rotating vortex strips for (1.3) bifurcating from the absolute vorticity distribution $\overline{\Omega}_{\text{FC2}}$, given by (1.23) at the velocities $c_{\mathbf{m}}^{\pm}(\tilde{\gamma}, \theta_1, \theta_2)$, defined in (3.17).*
2. *Fix $\kappa \in \{+, -\}$ and assume that the condition (3.16), supplemented with $(\mathbf{H}\mathbf{k}\kappa)$ for some $k \in \llbracket 1, 4 \rrbracket$, holds. Then, there exist one branch of \mathbf{m} -fold uniformly rotating vortex strips for (1.3) bifurcating from $\overline{\Omega}_{\text{FC2}}$ at the velocity $c_{\mathbf{m}}^{\kappa}(\tilde{\gamma}, \theta_1, \theta_2)$.*

A Appendix

A.1 An integral

In this appendix, we give an explicit value for an integral with parameters oftenly used in this work.

Lemma A.1. *Let $n \in \mathbb{N}^*$ and $a, b \in (0, \pi)$. We define*

$$I_n(a, b) \triangleq \frac{1}{2\pi} \int_0^{2\pi} \cos(nx) \log(1 - \cos(a) \cos(b) - \sin(a) \sin(b) \cos(x)) dx.$$

Then

$$I_n(a, b) = I_n(b, a) = -\frac{1}{n} \tan^n\left(\frac{\min(a, b)}{2}\right) \cot^n\left(\frac{\max(a, b)}{2}\right).$$

Proof. \blacktriangleright Let us first begin with the case $a = b$. Observe that

$$\begin{aligned} \log(1 - \cos^2(a) - \sin^2(a) \cos(x)) &= \log(1 - \cos(x)) + \log(\sin^2(a)) \\ &= \log\left(\sin^2\left(\frac{x}{2}\right)\right) + \log(2) + \log(\sin^2(a)). \end{aligned}$$

Hence, using [15, Lem. A.3], we get

$$I_n(a, a) = \frac{1}{2\pi} \int_0^{2\pi} \log\left(\sin^2\left(\frac{x}{2}\right)\right) \cos(nx) dx = -\frac{1}{n}.$$

\blacktriangleright Now, we assume $a \neq b$ with, without loss of generality, $a < b$. We can write

$$I_n(a, b) = \frac{1}{\pi} \int_0^{\pi} \cos(nx) \log(1 - \mu_{a,b} \cos(x)) dx, \quad \mu_{a,b} \triangleq \frac{\sin(a) \sin(b)}{1 - \cos(a) \cos(b)} > 0.$$

Notice that

$$\begin{aligned} a \neq b &\Leftrightarrow \cos(a - b) < 1 \Leftrightarrow \cos(a) \cos(b) + \sin(a) \sin(b) < 1 \\ &\Leftrightarrow \sin(a) \sin(b) < 1 - \cos(a) \cos(b) \\ &\Leftrightarrow \mu_{a,b} < 1. \end{aligned}$$

Performing an integration by parts yields

$$I_n(a, b) = -\frac{\mu_{a,b}}{n\pi} \int_0^\pi \frac{\sin(nx) \sin(x)}{1 - \mu_{a,b} \cos(x)} dx.$$

Now we shall use the following result which can be found in [42, p. 391]

$$\frac{1}{1 + \alpha^2} \int_0^\pi \frac{\sin(nx) \sin(x)}{1 - \frac{2\alpha}{1 + \alpha^2} \cos(x)} dx = \int_0^\pi \frac{\sin(nx) \sin(x)}{1 - 2\alpha \cos(x) + \alpha^2} dx = \begin{cases} \frac{\pi}{2} \alpha^{n-1}, & \text{if } \alpha^2 < 1, \\ \frac{\pi}{2\alpha^{n+1}}, & \text{if } \alpha^2 > 1. \end{cases}$$

We apply it with

$$\frac{2\alpha}{1 + \alpha^2} = \mu_{a,b}, \quad \text{i.e.} \quad \mu_{a,b} \alpha^2 - 2\alpha + \mu_{a,b} = 0.$$

The discriminant of the previous second order polynomial equation is $\Delta = 4(1 - \mu_{a,b}^2) > 0$, so we can take

$$\alpha = \alpha_{a,b} \triangleq \frac{1 - \sqrt{1 - \mu_{a,b}^2}}{\mu_{a,b}} \in (0, 1).$$

Observe that

$$\begin{aligned} \alpha_{a,b} < 1 &\Leftrightarrow 1 - \mu_{a,b} < \sqrt{1 - \mu_{a,b}^2} \\ &\Leftrightarrow (1 - \mu_{a,b})^2 < 1 - \mu_{a,b}^2 \\ &\Leftrightarrow \mu_{a,b} < 1. \end{aligned}$$

Hence,

$$I_n(a, b) = -\frac{\mu_{a,b}(1 + \alpha_{a,b}^2)\alpha_{a,b}^{n-1}}{2n} = -\frac{\alpha_{a,b}^n}{n}.$$

We can also write

$$\begin{aligned} \alpha_{a,b} &= \frac{1}{\mu_{a,b}} - \sqrt{\frac{1}{\mu_{a,b}^2} - 1} \\ &= \frac{1 - \cos(a) \cos(b) - \sqrt{(1 - \cos(a) \cos(b))^2 - \sin^2(a) \sin^2(b)}}{\sin(a) \sin(b)}. \end{aligned}$$

But

$$\begin{aligned} (1 - \cos(a) \cos(b))^2 - \sin^2(a) \sin^2(b) &= 1 + \cos^2(a) \cos^2(b) - 2 \cos(a) \cos(b) - (1 - \cos^2(a))(1 - \cos^2(b)) \\ &= \cos^2(a) + \cos^2(b) - 2 \cos(a) \cos(b) \\ &= (\cos(a) - \cos(b))^2. \end{aligned}$$

Since $a, b \in (0, \pi)$ with $a < b$, then $\cos(a) > \cos(b)$. Consequently,

$$\begin{aligned} \alpha_{a,b} &= \frac{1 - \cos(a) \cos(b) - (\cos(a) - \cos(b))}{\sin(a) \sin(b)} \\ &= \frac{(1 - \cos(a))(1 + \cos(b))}{\sin(a) \sin(b)} \\ &= \frac{2 \sin^2\left(\frac{a}{2}\right) 2 \cos^2\left(\frac{b}{2}\right)}{2 \sin\left(\frac{a}{2}\right) \cos\left(\frac{a}{2}\right) 2 \sin\left(\frac{b}{2}\right) \cos\left(\frac{b}{2}\right)} \\ &= \tan\left(\frac{a}{2}\right) \cot\left(\frac{b}{2}\right). \end{aligned}$$

Finally, we have

$$\forall n \in \mathbb{N}^*, \quad \forall 0 < a < b < \pi, \quad I_n(a, b) = -\frac{\tan^n\left(\frac{a}{2}\right) \cot^n\left(\frac{b}{2}\right)}{n}.$$

□

A.2 Potential theory

This section is devoted to some results on the continuity of specific operators with singular kernels. The proof of the next result can be found in [58, Lem. 2.6].

Proposition A.1. *Let $\alpha \in (0, 1)$. Consider a kernel $K : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ smooth out of the diagonal and satisfying, for some $C_0 > 0$,*

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad |K(\varphi, \varphi')| \leq C_0 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-(1-\alpha)}, \quad (\text{A.1})$$

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad |\partial_\varphi K(\varphi, \varphi')| \leq C_0 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-(2-\alpha)}. \quad (\text{A.2})$$

Then, the integral operator \mathcal{K} defined by

$$\forall \varphi \in \mathbb{T}, \quad \mathcal{K}(f)(\varphi) \triangleq \int_0^{2\pi} K(\varphi, \varphi') f(\varphi') d\varphi'$$

is bounded from $L^\infty(\mathbb{T})$ into $C^\alpha(\mathbb{T})$. More precisely, we have the following estimate

$$\forall f \in L^\infty(\mathbb{T}), \quad \|\mathcal{K}(f)\|_{C^\alpha(\mathbb{T})} \leq CC_0 \|f\|_{L^\infty(\mathbb{T})},$$

with $C > 0$ an absolute constant.

In some cases, the above proposition cannot be applied directly because the kernel K has a non differentiable term, and thus the condition (A.2) does not make sense. In those cases, let us give the alternative result.

Proposition A.2. *Let $\alpha \in (0, 1)$ and $g \in C^\alpha(\mathbb{T})$, consider a kernel $K : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{R}$ smooth out of the diagonal and satisfying*

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad |K(\varphi, \varphi')| \leq C_0 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-1}, \quad (\text{A.3})$$

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad |\partial_\varphi K(\varphi, \varphi')| \leq C_0 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-2}. \quad (\text{A.4})$$

Then, the integral operator $\tilde{\mathcal{K}}_g$ defined by

$$\forall \varphi \in \mathbb{T}, \quad \tilde{\mathcal{K}}_g(f)(\varphi) \triangleq \int_0^{2\pi} K(\varphi, \varphi') (g(\varphi) - g(\varphi')) f(\varphi') d\varphi'$$

is bounded from $L^\infty(\mathbb{T})$ into $C^\alpha(\mathbb{T})$. More precisely, we have the following estimate

$$\forall f \in L^\infty(\mathbb{T}), \quad \|\tilde{\mathcal{K}}_g(f)\|_{C^\alpha(\mathbb{T})} \leq CC_0 \|g\|_{C^\alpha(\mathbb{T})} \|f\|_{L^\infty(\mathbb{T})},$$

with $C > 0$ an absolute constant.

Proof. The L^∞ norm of $\tilde{\mathcal{K}}_g(f)$ can be estimated as

$$\begin{aligned} \left| \tilde{\mathcal{K}}_g(f)(\varphi) \right| &\leq C \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_0^{2\pi} |K(\varphi, \varphi')| |\varphi - \varphi'|^\alpha d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \sup_{\varphi \in \mathbb{T}} \int_0^{2\pi} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-1} |\varphi - \varphi'|^\alpha d\varphi'. \end{aligned}$$

Since we work on the torus, we can always assume that

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad 0 < |\varphi - \varphi'| \leq \pi.$$

As a consequence, we have the following classical convexity estimate

$$\forall \varphi \neq \varphi' \in \mathbb{T}, \quad \frac{2}{\pi} |\varphi - \varphi'| \leq 2 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| \leq |\varphi - \varphi'|. \quad (\text{A.5})$$

Therefore, by using (A.5), a change of variable and the fact that $\alpha \in (0, 1)$, we get

$$\forall \varphi \in \mathbb{T}, \quad \int_0^{2\pi} \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right|^{-1} |\varphi - \varphi'|^\alpha d\varphi' \leq C \int_0^{2\pi} \left| \sin \left(\frac{\varphi'}{2} \right) \right|^{-(1-\alpha)} d\varphi' < \infty.$$

Hence, we obtain

$$\|\tilde{\mathcal{K}}_g(f)\|_{L^\infty(\mathbb{T})} \leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})}.$$

For the Hölder regularity, take $\varphi_1 \neq \varphi_2 \in \mathbb{T}$. Define

$$d \triangleq 2 \left| \sin \left(\frac{\varphi_1 - \varphi_2}{2} \right) \right| = |e^{i\varphi_1} - e^{i\varphi_2}|$$

and for $\varphi \in \mathbb{T}$ and $r > 0$,

$$B_\varphi(r) \triangleq \left\{ \varphi' \in \mathbb{T} \quad \text{s.t.} \quad 2 \left| \sin \left(\frac{\varphi - \varphi'}{2} \right) \right| < r \right\}, \quad B_\varphi^c(r) \triangleq \mathbb{T} \setminus B_\varphi(r).$$

Hence

$$\begin{aligned} \tilde{\mathcal{K}}_g(f)(\varphi_1) - \tilde{\mathcal{K}}_g(f)(\varphi_2) &= \int_0^{2\pi} K(\varphi_1, \varphi') (g(\varphi_1) - g(\varphi')) f(\varphi') d\varphi' - \int_0^{2\pi} K(\varphi_2, \varphi') (g(\varphi_2) - g(\varphi')) f(\varphi') d\varphi' \\ &= \int_{B_{\varphi_1}(3d)} K(\varphi_1, \varphi') (g(\varphi_1) - g(\varphi')) f(\varphi') d\varphi' \\ &\quad - \int_{B_{\varphi_1}(3d)} K(\varphi_2, \varphi') (g(\varphi_2) - g(\varphi')) f(\varphi') d\varphi' \\ &\quad + \int_{B_{\varphi_1}^c(3d)} (K(\varphi_1, \varphi') - K(\varphi_2, \varphi')) (g(\varphi_1) - g(\varphi')) f(\varphi') d\varphi' \\ &\quad + \int_{B_{\varphi_1}^c(3d)} K(\varphi_2, \varphi') (g(\varphi_1) - g(\varphi_2)) f(\varphi') d\varphi' \\ &\triangleq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Using (A.1), (A.5) and a change of variables, we arrive at

$$\begin{aligned} |I_1| &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_1}(3d)} \left| \sin \left(\frac{\varphi_1 - \varphi'}{2} \right) \right|^{-1} |\varphi_1 - \varphi'|^\alpha d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_1}(3d)} \left| \sin \left(\frac{\varphi_1 - \varphi'}{2} \right) \right|^{-(1-\alpha)} d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_0^{\frac{3}{2}d} \frac{dw}{|w|^{1-\alpha} \sqrt{1-w^2}} dw \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} d^\alpha \\ &= CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2|^\alpha. \end{aligned}$$

In order to work with I_2 , note that $B_{\varphi_1}(3d) \subset B_{\varphi_2}(4d)$. Thus, proceeding as before, we infer

$$\begin{aligned} |I_2| &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_1}(3d)} \left| \sin \left(\frac{\varphi_2 - \varphi'}{2} \right) \right|^{-1} |\varphi_2 - \varphi'|^\alpha d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_2}(4d)} \left| \sin \left(\frac{\varphi_2 - \varphi'}{2} \right) \right|^{-1} |\varphi_2 - \varphi'|^\alpha d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_2}(4d)} \left| \sin \left(\frac{\varphi_2 - \varphi'}{2} \right) \right|^{-(1-\alpha)} d\varphi' \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2|^\alpha. \end{aligned}$$

For the third term I_3 we use the mean value theorem and (A.2) achieving

$$\begin{aligned} |I_3| &\leq C \left| (\varphi_1 - \varphi_2) \int_0^1 \int_{B_{\varphi_1}^c(3d)} (\partial_x K)(\varphi_1 + (1-s)(\varphi_2 - \varphi_1), \varphi') (g(\varphi_1) - g(\varphi')) f(\varphi') d\varphi' ds \right| \\ &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2| \int_0^1 \int_{B_{\varphi_1}^c(3d)} \left| \sin \left(\frac{\varphi_1 + (1-s)(\varphi_2 - \varphi_1) - \varphi'}{2} \right) \right|^{-2} |\varphi_1 - \varphi'|^\alpha d\varphi' ds. \end{aligned}$$

Note that if $\varphi' \in B_{\varphi_1}^c(3d)$ and $s \in [0, 1]$, then

$$\begin{aligned} 2 \left| \sin \left(\frac{\varphi_1 + (1-s)(\varphi_2 - \varphi_1) - \varphi'}{2} \right) \right| &= \left| e^{i(\varphi_1 - \varphi')} - e^{i(1-s)(\varphi_1 - \varphi_2)} \right| \\ &\geq \left| e^{i(\varphi_1 - \varphi')} - 1 \right| - \left| e^{i(1-s)(\varphi_1 - \varphi_2)} - 1 \right| \\ &\geq \left| e^{i(\varphi_1 - \varphi')} - 1 \right| - \left| e^{i(\varphi_1 - \varphi_2)} - 1 \right| = \left| e^{i\varphi_1} - e^{i\varphi'} \right| - \left| e^{i\varphi_1} - e^{i\varphi_2} \right| \\ &\geq \frac{2}{3} \left| e^{i\varphi_1} - e^{i\varphi'} \right| = \frac{4}{3} \left| \sin \left(\frac{\varphi_1 - \varphi'}{2} \right) \right| \end{aligned} \tag{A.6}$$

which implies, through (A.5) and a change of variables,

$$\begin{aligned}
|I_3| &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2| \int_{B_{\varphi_1}^c(3d)} \left| \sin\left(\frac{\varphi_1 - \varphi'}{2}\right) \right|^{-2} |\varphi_1 - \varphi'|^\alpha d\varphi' \\
&\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2| \int_{B_{\varphi_1}^c(3d)} \left| \sin\left(\frac{\varphi_1 - \varphi'}{2}\right) \right|^{-(2-\alpha)} d\varphi' \\
&\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2| \int_d^1 \frac{dw}{|w|^{2-\alpha} \sqrt{1-w^2}} \\
&\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2| \frac{1}{|\varphi_1 - \varphi_2|^{1-\alpha}} \\
&\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2|^\alpha.
\end{aligned}$$

Let us finish with I_4 , using that $g \in C^\alpha(\mathbb{T})$,

$$|I_4| \leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_1}^c(3d)} \left| \sin\left(\frac{\varphi_2 - \varphi'}{2}\right) \right|^{-1} |\varphi_1 - \varphi_2|^\alpha d\varphi'.$$

Applying (A.6) with $s = 0$, we get

$$2 \left| \sin\left(\frac{\varphi_2 - \varphi'}{2}\right) \right| \geq \frac{4}{3} \left| \sin\left(\frac{\varphi_1 - \varphi'}{2}\right) \right|.$$

Besides, for $\varphi' \in B_{\varphi_1}^c(3d)$, we have

$$\left| \sin\left(\frac{\varphi_1 - \varphi_2}{2}\right) \right| \leq \frac{1}{3} \left| \sin\left(\frac{\varphi_1 - \varphi'}{2}\right) \right|.$$

Combining the foregoing facts, we end up with

$$\begin{aligned}
|I_4| &\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} \int_{B_{\varphi_1}^c(3d)} \left| \sin\left(\frac{\varphi_1 - \varphi'}{2}\right) \right|^{-(1-\alpha)} d\varphi' \\
&\leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2|^\alpha.
\end{aligned}$$

Putting together the preceding estimates yields

$$\left| \tilde{\mathcal{K}}_g(f)(\varphi_1) - \tilde{\mathcal{K}}_g(f)(\varphi_2) \right| \leq CC_0 \|f\|_{L^\infty(\mathbb{T})} \|g\|_{C^\alpha(\mathbb{T})} |\varphi_1 - \varphi_2|^\alpha,$$

concluding the proof. \square

A.3 Crandall-Rabinowitz theorem

In this last appendix, we recall the classical Crandall-Rabinowitz Theorem whose proof can be found in [20].

Theorem A.1 (Crandall-Rabinowitz Theorem). *Let $\lambda_0 \in \mathbb{R}$, X, Y be two Banach spaces, V be a neighborhood of 0 in X and $\mathcal{F} : \mathbb{R} \times V \rightarrow Y$ be a function with the properties,*

1. $\mathcal{F}(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$.
2. The partial derivatives $\partial_\lambda \mathcal{F}$, $d_f \mathcal{F}$ and $\partial_\lambda d_f \mathcal{F}$ exist and are continuous.
3. The operator $d_f \mathcal{F}(\lambda_0, 0)$ is Fredholm of zero index and $\ker(d_f \mathcal{F}(\lambda_0, 0)) = \text{span}(f_0)$ is one-dimensional.
4. Transversality assumption: $\partial_\lambda d_f \mathcal{F}(\lambda_0, 0)[f_0] \notin \text{Im}(d_f \mathcal{F}(\lambda_0, 0))$.

If Z is any complement of $\ker(d_f \mathcal{F}(\lambda_0, 0))$ in X , then there is a neighborhood U of $(\lambda_0, 0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$ with $a > 0$, and two continuous functions $\Phi : (-a, a) \rightarrow \mathbb{R}$, $\beta : (-a, a) \rightarrow Z$ such that $\Phi(0) = \lambda_0$ and $\beta(0) = 0$ and

$$\mathcal{F}^{-1}(0) \cap U = \{(\Phi(s), s f_0 + s \beta(s)) : |s| < a\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}.$$

A.4 Proof of Lemma 1.2

Fix $\alpha \in \mathbb{R}$. Since $\mathcal{R}(\alpha) \in SO_3(\mathbb{R})$, then it preserves the Euclidean norm $|\cdot|_{\mathbb{R}^3}$, i.e.

$$\forall \xi \in \mathbb{R}^3, \quad |\mathcal{R}(\alpha)\xi|_{\mathbb{R}^3} = |\xi|_{\mathbb{R}^3}.$$

As a consequence,

$$\begin{aligned} \forall (\xi, \xi') \in (\mathbb{S}^2)^2, \quad \forall \alpha \in \mathbb{R}, \quad G(\mathcal{R}(\alpha)\xi, \mathcal{R}(\alpha)\xi') &= \frac{1}{2\pi} \log \left(\frac{|\mathcal{R}(\alpha)\xi - \mathcal{R}(\alpha)\xi'|_{\mathbb{R}^3}}{2} \right) \\ &= \frac{1}{2\pi} \log \left(\frac{|\mathcal{R}(\alpha)(\xi - \xi')|_{\mathbb{R}^3}}{2} \right) \\ &= \frac{1}{2\pi} \log \left(\frac{|\xi - \xi'|_{\mathbb{R}^3}}{2} \right) \\ &= G(\xi, \xi'). \end{aligned}$$

Hence, using the change of variables $\xi' \mapsto \mathcal{R}(\alpha)\xi' \in SO(\mathbb{R}^3)$ (which preserves \mathbb{S}^2), we get for any $\xi \in \mathbb{S}^2$,

$$\begin{aligned} \Psi(\mathcal{R}(\alpha)\xi) &= \int_{\mathbb{S}^2} G(\mathcal{R}(\alpha)\xi, \xi') \Omega(\xi') d\xi' \\ &= \int_{\mathbb{S}^2} G(\mathcal{R}(\alpha)\xi, \mathcal{R}(\alpha)\xi') \Omega(\mathcal{R}(\alpha)\xi') d\xi' \\ &= \int_{\mathbb{S}^2} G(\xi, \xi') \Omega(\xi') d\xi' \\ &= \Psi(\xi). \end{aligned} \tag{A.7}$$

This achieves the proof of Lemma 1.2.

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