

Invariant KAM tori around annular vortex patches for 2D Euler equations

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Abstract

We construct time quasi-periodic vortex patch solutions with one hole for the planar Euler equations. These structures are captured close to any annulus provided that its modulus belongs to a massive Borel set. The proof is based on Nash-Moser scheme and KAM theory applied with a Hamiltonian system governing the radial deformations of the patch. Compared to the scalar case discussed recently in [59, 61, 69, 92], some technical issues emerge due to the interaction between the interfaces. One of them is related to a new small divisor problem in the second order Melnikov non-resonances condition coming from the transport equations advected with different velocities.

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1 Introduction

This work deals with some aspects on the vortex motion for the classical planar incompressible Euler equations that can be reformulated in the vorticity/velocity form as follows

$$\begin{cases} \partial_t \boldsymbol{\omega} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = 0, & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathbf{v} = \nabla^\perp \psi, \\ \Delta \psi = \boldsymbol{\omega}, \\ \boldsymbol{\omega}(0, \cdot) = \boldsymbol{\omega}_0. \end{cases} \quad (1.1)$$

The quantity \mathbf{v} represents the velocity field of the fluid particles which is supposed to be solenoidal according to the second equation in (1.1) where the notation $\nabla^\perp \triangleq \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}$ is used. The scalar potential $\boldsymbol{\omega}$ is called the vorticity and measures the local rotation effects inside the fluid. It is related to the velocity field by the relation

$$\boldsymbol{\omega} \triangleq \nabla^\perp \cdot \mathbf{v}.$$

From the third equation in (1.1), we can recover the stream function ψ from the vorticity $\boldsymbol{\omega}$ through the following integral operator with a logarithmic kernel

$$\psi(t, z) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|z - \xi|) \boldsymbol{\omega}(t, \xi) dA(\xi), \quad (1.2)$$

where dA is the 2-dimensional Lebesgue measure. It is well-known since the work of Yudovich [97] that any bounded and integrable initial datum $\boldsymbol{\omega}_0$ generates a unique global in time weak solution of (1.1) which is Lagrangian, namely

$$\boldsymbol{\omega}(t, x) = \boldsymbol{\omega}_0(\Phi_t^{-1}(x)), \quad \Phi_t(x) = x + \int_0^t \mathbf{v}(s, \Phi_s(x)) ds.$$

In particular, if the initial datum is the characteristic function of some bounded domain D_0 then

$$\boldsymbol{\omega}(t, \cdot) = \mathbf{1}_{D_t}, \quad D_t \triangleq \Phi_t(D_0)$$

and the resulting solution is called a *vortex patch*. The dynamics of these solutions is entirely described by the evolution of the boundary ∂D_t . The global in time persistence of the boundary regularity of type $C^{1,\alpha}$, with $\alpha \in (0, 1)$ was first proved by Chemin in [29, 30] and later by Bertozzi and Constantin in [20]. Notice that the boundary motion can be tracked from the *contour dynamics equation* of the patch. Indeed, for any parametrization $z(t) : \mathbb{T} \rightarrow \partial D_t$ of the boundary, denoting $\mathbf{n}(t, z(t, \theta)) \triangleq i\partial_\theta z(t, \theta)$ a normal vector to the boundary at the point $z(t, \theta)$, one has

$$\partial_t z(t, \theta) \cdot \mathbf{n}(t, z(t, \theta)) = \partial_\theta \left[\psi(t, z(t, \theta)) \right]. \quad (1.3)$$

We refer for instance to [68] for a complete derivation of this equation. In 1858, Rankine observed that any radial initial domain D_0 (disc, annulus, etc...) generates a stationary solution to (1.1). Then from a dynamical system point of view it is of important interest to explore the local structure of the phase portrait and to know whether periodic solutions may exist around these equilibrium states. This topic turns out to be highly rich leading to fruitful subjects connecting various areas in Mathematics. The first result in this direction is due to Kirchhoff in 1874 [75] where he proved that an ellipse $\mathcal{E}_{a,b}$ with semi-axes a and b performs a uniform rotation about its center with an angular velocity Ω if and only if

$$\Omega = \frac{ab}{(a+b)^2}.$$

Actually, the ellipses form a subclass of *relative equilibria* or *V-states* which are solutions keeping the same shape during the motion, from which we derive another subclass given by rotating patches where the domain D_t rotates uniformly about its center (due to the space invariance, we can suppose without any loss of generality that the center is the origin),

$$D_t = e^{i\Omega t} D_0, \quad \Omega \in \mathbb{R}.$$

They form a family of rigid periodic solutions where the domain is not deformed during the motion and keeps its initial shape. Then, more recently in 1978, Deem and Zabusky [40] discovered numerically 3-fold, 4-fold and 5-fold V-states living close to the unit disc. Few years after, Burbea [23] confirmed analytically these simulations using bifurcation theory. More precisely, he proved that for any integer $m \geq 1$, one can find a branch of m -fold simply-connected V-states bifurcating from the unit disc at the angular velocity

$$\Omega_m \triangleq \frac{m-1}{2m}.$$

Actually, the case $m = 1$ corresponds to a translation of the Rankine vortex whereas the branch associated with the mode $m = 2$ gives the Kirchhoff ellipses. Observe that for any $m \geq 2$, the bifurcation frequency Ω_m lives in the interval $(0, \frac{1}{2})$ and the series of works [47, 53, 63] showed that, outside this interval and in the simply-connected case, the only relative equilibria are the radial ones. The boundary regularity of the V-states and the global bifurcation were analyzed in [27, 28, 60, 68]. The second bifurcation from the ellipses has been discussed in [28, 65]. More precisely, if we consider an ellipse $\mathcal{E}_{a,b}$ ($a > b$) described by

$$\mathcal{E}_{a,b} = \left\{ \frac{a+b}{2} \left(w + \frac{\mathcal{Q}}{w} \right), \quad w \in \mathbb{T}, \quad \mathcal{Q} \triangleq \frac{a-b}{a+b} \in (0, 1) \right\},$$

then for any integer $m \geq 3$, the bifurcation occurs at the angular velocity $\Omega = \frac{1-\mathcal{Q}^2}{4}$, where \mathcal{Q} is a solution to the polynomial equation

$$f(m, \mathcal{Q}) \triangleq 1 + \mathcal{Q}^m - \frac{1-\mathcal{Q}^2}{2} m = 0. \quad (1.4)$$

The boundary effects on the emergence of V-states have been explored recently in [38] where the authors proved the existence of V-states when the fluid evolves in the unit disc \mathbb{D} . It was shown that for any integer $m \geq 1$ a family of m -fold implicit curves bifurcate from the disc $b\mathbb{D}$, $b \in (0, 1)$, at the angular velocity

$$\Omega_m(b) \triangleq \frac{m-1+b^{2m}}{2m}.$$

In contrast to the flat case \mathbb{R}^2 , the one-fold curve is no longer trivial here and moreover the numerical simulations performed in [38] show that in some regimes of b the bifurcating curves oscillate with respect to the angular velocity. In the same spirit, Hmidi, de la Hoz, Mateu and Verdera discussed in [64] the existence of rotating patches with one hole called doubly-connected V-states. They proved that for a fixed symmetry $m \geq 3$ and $b \in (0, 1)$ two m -fold curves of doubly-connected V-states bifurcate from the annulus

$$A_b \triangleq \{z \in \mathbb{C} \quad \text{s.t.} \quad b < |z| < 1\}, \quad (1.5)$$

provided that the following constraint is satisfied

$$f(m, b) < 0,$$

where f is given by (1.4). The bifurcation occurs at the angular velocities

$$\Omega_m^\pm(b) \triangleq \frac{1-b^2}{4} \pm \frac{1}{2m} \sqrt{\left(\frac{1-b^2}{2} m - 1 \right)^2 - b^{2m}}. \quad (1.6)$$

It is worthy to point out that the role played by the same function f in the two different cases (bifurcation from the ellipses and the annulus) is quite mysterious and could be explained through Joukowski transformation. As for the degenerate case where

$$f(m, b) = 0 \quad (1.7)$$

the situation turns out to be more delicate to handle. The solutions to (1.7) can be ranged in the form

$$\{(2, b), \quad b \in (0, 1)\} \quad \text{or} \quad \{(n, \underline{b}_n), \quad n \geq 3, \quad \underline{b}_n \in (0, 1)\}, \quad (1.8)$$

where the sequence $(\underline{b}_n)_{n \geq 3}$ is increasing and tends to 1. This problem has been explored by Hmidi and Mateu in [66], where they show that for $b \in (0, 1) \setminus \{b_{2p}, p \geq 2\}$ there is a trans-critical bifurcation of the 2-fold V-states. However, there is no bifurcation with the m -fold symmetry for $b = \underline{b}_m$, $m \geq 3$. Very recently, Wang, Xu and Zhou extended in [95] the 2-fold trans-critical bifurcation to the cases $b = \underline{b}_4$ and $b = \underline{b}_6$. We should also mention that over the past few years there were a lot of rich activities on the construction of V-states around more general steady shapes (multi-connected patches, Thomson polygons, von Kármán vortex streets, etc...) and for various active scalar equations (generalized quasi-geostrophic equations, quasi-geostrophic shallow-water equations, Euler- α equations). For more details, we refer to [2, 24, 25, 26, 32, 35, 36, 37, 38, 41, 49, 50, 51, 52, 54, 56, 57, 58, 62, 67, 70, 90, 91, 92, 94].

In the current work, we intend to explore the existence of time quasi-periodic vortex patches for (1.1) close to the annulus. Recall that a *quasi-periodic function* is any application $f : \mathbb{R} \rightarrow \mathbb{R}$ which can be written

$$\forall t \in \mathbb{R}, \quad f(t) = F(\omega t),$$

with $F : \mathbb{T}^d \rightarrow \mathbb{R}$, where \mathbb{T}^d denotes the flat torus of dimension $d \in \mathbb{N}^*$ and $\omega \in \mathbb{R}^d$ a non-degenerate frequency vector, namely

$$\forall l \in \mathbb{Z}^d \setminus \{0\}, \quad \omega \cdot l \neq 0. \quad (1.9)$$

Observe that the case $d = 1$ corresponds to the definition of periodic functions with frequency $\omega \in \mathbb{R}^*$. This type of functions are the natural solutions of finite dimensional integrable Hamiltonian systems where the phase space is foliated by Lagrangian invariant tori supporting quasi-periodic motion. The Kolmogorov-Arnold-Moser (KAM) theory [3, 79, 82] asserts that under suitable regularity and non-degeneracy conditions, most of these invariant tori persist, up to a smooth deformation, under a small Hamiltonian perturbation. A typical difficulty in the implementation of the KAM method is linked to the *small divisors problems* preventing some intermediate series to be convergent. The solution, proposed by Kolmogorov, is to introduce Diophantine conditions on the small denominators which lead to a fixed algebraic loss of regularity. This loss can be treated through a classical Newton method in the analytical regularity framework as proved by Kolmogorov and Arnold. However, this approach turns out to be more involved in the finitely many differentiable case (for example Sobolev or Hölder spaces). Indeed, to overcome this technical difficulty, Moser used in [83] a regularization of the Newton method in the spirit of the ideas of Nash implemented in the isometric embedding problem [84]. Now, such a method is known as Nash-Moser scheme.

The search of lower dimensional invariant tori is so relevant not only for finite dimensional Hamiltonian systems but also for Hamiltonian PDE where this query is quite natural. Actually, in the finite dimensional case, this problem has been explored for instance by Moser and Pöschel [83, 86] leading to new Diophantine conditions called *first and second order Melnikov conditions*. Later on, the theory has been extended and refined for several Hamiltonian PDE. For example, it has been implemented for the 1D semi-linear wave and Schrödinger equations in the following papers [22, 31, 33, 76, 87, 88, 96]. Several results were also obtained for semi-linear perturbations of integrable PDE [11, 12, 21, 43, 48, 74, 77, 78, 80]. However, the case of quasi-linear or fully nonlinear perturbations were only explored very recently in a series of papers [5, 6, 7, 14, 18, 46]. A typical example in this direction is given by the water-waves equations which have been the subject of rich and intensive activity over the past few years dealing with the periodic and quasi-periodic solutions, see for instance [1, 4, 15, 16, 19, 73, 85].

Concerning the emergence of quasi-periodic structures for the 2D Euler equations which is known to be Hamiltonian, few results are known in the literature and some interesting developments have been made very recently opening new perspectives around the vortex motion. One of the results on the smooth case, supplemented with periodic boundary conditions, goes back to Crouseilles and Faou in [34]. The construction of quasi-periodic solutions is founded on the superposition of localized traveling

solutions without interaction. Notice that no sophisticated tools from KAM theory are required in their approach. Very recently, their result has been extended to higher dimensions by Enciso, Peralta-Salas and Torres de Lizaur in [44]. For Euler equations on the 3-dimensional torus, Baldi and Montalto [8] were able to generate quasi-periodic solutions through small quasi-periodic forcing terms.

Another new and promising topic concerns the construction of quasi-periodic vortex patches to the system (1.1) or to various active scalar equations (generalized surface quasi-geostrophic equations, quasi-geostrophic shallow-water equations and Euler- α equations) which has been partially explored in the recent papers [59, 69, 92]. All of them deal with simply-connected quasi-periodic patches near Rankine vortices provided that the suitable external parameter is selected in a massive Cantor set. We emphasize that for Euler model there is no natural parameter anymore and one has to create an internal one. Two works have been performed in this direction. The first one is due to Berti, Hassainia and Masmoudi [17] who proved using KAM theory the existence of quasi-periodic patches close to Kirchhoff ellipses provided that the aspect ratio of the ellipse belongs to a Cantor set. The second one is obtained by Hassainia and Rouley in [61], where the fluid evolves in the unit disc, and they proved the existence of quasi-periodic patches close to Rankine vortices $\mathbf{1}_{b\mathbb{D}}$, when b belongs to a suitable Cantor set in $(0, 1)$.

Our main task here is to investigate the emergence of quasi-periodic patches (denoted by (QPP)) near the annulus A_b . The motivation behind that is the existence of time periodic patches around the annulus as stated in [64] and one may get (QPP) at the linear level by mixing a finite number of frequencies. Note that the rigidity of the frequencies (1.6) with respect to the modulus b is an essential element to get the non-degeneracy of the linear torus. One of the difficulties in the construction of (QPP) at the nonlinear level stems from the vectorial structure of the problem because we are dealing with two coupled interfaces. As we shall see, this leads to more time-space resonances coming in part from the interaction between the transport equations advected by two different speeds.

In what follows, we intend to carefully describe the situation around doubly-connected (QPP), then formulate the main result and sketch the principal ideas of the proof. First, we consider a modified polar parametrization of the two interfaces of the patch close to the annulus A_b , namely for $k \in \{1, 2\}$

$$z_k(t, \theta) \triangleq R_k(t, \theta)e^{i(\theta - \Omega t)}, \quad R_k(t, \theta) \triangleq \sqrt{b_k^2 + 2r_k(t, \theta)}, \quad b_1 \triangleq 1, \quad b_2 \triangleq b.$$

The unknown is the pair of functions $r = (r_1, r_2)$ of small radial deformations of the patch. It is worthy to point out that similarly to [17, 59, 69, 92] our parametrization is written in a rotating frame with an angular velocity $\Omega > 0$. Nevertheless, we have multiple reasons here behind the introduction of this auxiliary parameter Ω . In the previous works, we make appeal to this parameter to remedy to the degeneracy of the first equilibrium frequency leading to a trivial resonance. In our setting, this parameter is needed to avoid an exponential accumulation towards a constant of the unperturbed frequencies (eigenvalues) $\{\mathfrak{m}\Omega_{\mathfrak{m}}^{\pm}(b), \mathfrak{m} \geq 2\}$, see (1.6). This fact induces a harmful effect especially related to the second order Melnikov non-resonance condition. Therefore, thanks to the parameter Ω the eigenvalues will grow linearly with respect to the modes \mathfrak{m} . Another useful property induced by large values of Ω is the monotonicity of the eigenvalues, see Lemma 4.6, which allows in turn to get Rüssmann conditions on the diagonal part, see Lemma 4.8-(iv).

One of the major difference with [59, 61, 69, 92] is the vectorial structure of the system related to the interfaces coupling. Despite that, we are able to check the Hamiltonian structure in terms of the contour dynamics equations. In fact, we prove in Lemma 3.1 and Proposition 3.1 that the pair of radial deformations $r = (r_1, r_2)$ solves a system of two coupled nonlinear and nonlocal transport PDE admitting a Hamiltonian formulation in the form

$$\partial_t r = \mathcal{J} \nabla H(r), \quad \mathcal{J} \triangleq \begin{pmatrix} \partial_\theta & 0 \\ 0 & -\partial_\theta \end{pmatrix}, \quad (1.10)$$

where the Hamiltonian H can be recovered from the kinetic energy and the angular momentum. This Hamiltonian is reversible and invariant under space translations. The linearized operator at a general state r close to the annulus A_b is described in Lemma 4.1 and writes

$$\partial_t \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \mathcal{J} \mathbf{M}_r \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad \mathbf{M}_r \triangleq \begin{pmatrix} -V_1(r) - L_{1,1}(r) & L_{1,2}(r) \\ L_{2,1}(r) & V_2(r) - L_{2,2}(r) \end{pmatrix},$$

where $V_k(r)$ are scalar functions and $L_{k,n}(r)$ are nonlocal operators given by (4.2)–(4.4). The diagonal terms correspond to the self-induction of each interface. In particular, the operators $L_{k,k}$, for $k \in \{1, 2\}$, are of zero order and reflect the planar simply-connected Euler action. For $k \neq n \in \{1, 2\}$, the anti-diagonal operators $L_{k,n}$ describe the interaction between the two boundaries and they are smoothing at any order. In Lemma 4.2, we shall prove that at the equilibrium $r = (0, 0)$, corresponding to the annulus patch, each entry of \mathbf{M}_0 is a Fourier multiplier and the operator $\mathcal{J}\mathbf{M}_0$ can be written in Fourier expansion as a superposition of 2×2 matrices,

$$\mathcal{J}\mathbf{M}_0 \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \sum_{j \in \mathbb{Z}^*} M_j(b, \Omega) \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \quad M_j(b, \Omega) \triangleq \frac{ij}{|j|} \begin{pmatrix} -|j|(\Omega + \frac{1-b^2}{2}) + \frac{1}{2} & -\frac{b|j|}{2} \\ \frac{b|j|}{2} & -|j|\Omega - \frac{1}{2} \end{pmatrix},$$

for all ρ_1 and ρ_2 with Fourier expansion

$$\rho_k = \sum_{j \in \mathbb{m}\mathbb{Z}^*} \rho_{j,k} \mathbf{e}_j \quad \text{s.t.} \quad \rho_{-j,k} = \overline{\rho_{j,k}}, \quad \mathbf{e}_j(\theta) \triangleq e^{ij\theta}.$$

The spectrum of $M_j(b, \Omega)$ is

$$\sigma(M_j(b, \Omega)) = \{ -i\Omega_{j,1}(b), -i\Omega_{j,2}(b) \}, \quad \Omega_{j,k}(b) \triangleq \frac{j}{|j|} \left[\left(\Omega + \frac{1-b^2}{4} \right) |j| - i^{\mathbb{H}(\Delta_j(b))} \frac{(-1)^k}{2} \sqrt{|\Delta_j(b)|} \right],$$

with $\mathbb{H} \triangleq \mathbf{1}_{[0, \infty)}$ the Heaviside function and

$$\Delta_j(b) \triangleq b^{2|j|} - \left(\frac{1-b^2}{2} |j| - 1 \right)^2.$$

At this stage, we shall restrict the discussion to \mathbb{m} -fold symmetric structures for some integer $\mathbb{m} \geq 3$ large enough. This is done for several reasons. First, the mode $j = 2$ corresponds to a double root for any $b \in (0, 1)$, because $\Delta_2(b) = 0$, implying a nontrivial resonance that we cannot remove using the parameter b but simply by imposing higher symmetry for the (QPP). Second, the hyperbolic spectrum, associated to non-zero real part for the eigenvalues, that could generate instabilities and time growth emerge only for lower symmetries. We believe that with this latter configuration, one can still hope to construct (QPP) by inserting the hyperbolic modes on the normal directions as it was recently performed in [17]. We refer for instance the reader to [13, 39, 42, 55, 89, 98, 99] for an introduction to hyperbolic KAM theory in finite or infinite dimension.

Now, we fix $b^* \in (0, 1)$ and set

$$\mathbb{m}^* \triangleq \min \{ n \geq 3 \quad \text{s.t.} \quad \underline{b}_n > b^* \}, \quad (1.11)$$

where the sequence \underline{b}_n defined in (1.8). Then, for any integer $|j| \geq \mathbb{m}^*$, we have $\Delta_j(b) < 0$. Hence, the quantity $\Omega_{j,k}(b)$ is real and the matrix $M_j(b, \Omega)$ has pure imaginary spectrum. The restriction of the Fourier modes to the lattice $\mathbb{Z}_{\mathbb{m}}^* \triangleq \mathbb{m}\mathbb{Z} \setminus \{0\}$ with $\mathbb{m} \geq \mathbb{m}^*$ allows to eliminate the hyperbolic modes. At this stage, we find it convenient to work with new coordinates where the linearized operator at the equilibrium state is governed by a diagonal matrix Fourier multiplier operator. This can be done through the diagonalization of each the matrix $M_j(b, \Omega)$. To do that, we use the symplectic transformation (with respect to \mathcal{W}) \mathbf{Q} taking the form

$$\mathbf{Q} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \sum_{j \in \mathbb{Z}_{\mathbb{m}}^*} \mathbf{Q}_j \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \quad \mathbf{Q}_j \triangleq \frac{-1}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} -1 & a_j(b) \\ a_j(b) & -1 \end{pmatrix},$$

where

$$a_j(b) \triangleq \frac{b^{|j|}}{\frac{1-b^2}{2}|j| - 1 + \sqrt{\left(\frac{1-b^2}{2}|j| - 1\right)^2 - b^{2|j|}}} \in (0, 1), \quad (1.12)$$

(see Corollary 4.1 for more details on the bound of $a_j(b)$) such that

$$\mathbf{Q}^{-1} \mathcal{J}\mathbf{M}_0 \mathbf{Q} = \mathcal{J}\mathbf{L}_0, \quad \mathbf{L}_0 \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \triangleq \sum_{j \in \mathbb{Z}_{\mathbb{m}}^*} \frac{1}{j} \begin{pmatrix} -\Omega_{j,1}(b) & 0 \\ 0 & \Omega_{j,2}(b) \end{pmatrix} \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j.$$

Notice that $i\Omega_{j,1}(b)$ and $i\Omega_{j,2}(b)$ are not complex conjugate, thus the dynamics cannot be reduced to one scalar equation associated with a complex variable unlike the water-waves [4, 15, 16, 19] situation. The new Hamiltonian system through the symplectic transformation $r \mapsto \mathbf{Q}r$

$$\partial_t r = \mathcal{J}\nabla K(r), \quad K(r) \triangleq H(\mathbf{Q}r)$$

whose linearization at the trivial solution has a good normal form

$$\partial_t \rho = \mathcal{J}\nabla K_{\mathbf{L}_0}(\rho), \quad K_{\mathbf{L}_0}(\rho) \triangleq \frac{1}{2} \langle \mathbf{L}_0 \rho, \rho \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} = - \sum_{j \in \mathbb{Z}_{\mathbf{m}}^*} \left(\frac{\Omega_{j,1}(b)}{2j} |\rho_{j,1}|^2 - \frac{\Omega_{j,2}(b)}{2j} |\rho_{j,2}|^2 \right). \quad (1.13)$$

Consider two disjoint finite sets of Fourier modes

$$\mathbb{S}_1, \mathbb{S}_2 \subset \mathbf{m}\mathbb{N}^*, \quad \text{with} \quad |\mathbb{S}_1| = d_1 < \infty, \quad |\mathbb{S}_2| = d_2 < \infty \quad \text{and} \quad \mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset. \quad (1.14)$$

Then, from Lemma 4.9, we deduce that for any $0 < b^* < 1$, $\Omega > 0$ and $r_{j,1}, r_{j,2} \in \mathbb{R}^*$, for almost all $b \in [0, b^*]$, any function in the form

$$r(t, \theta) = \sum_{j \in \mathbb{S}_1} \frac{r_{j,1}}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix} \cos(j\theta - \Omega_{j,1}(b)t) + \sum_{j \in \mathbb{S}_2} \frac{r_{j,2}}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix} \cos(j\theta - \Omega_{j,2}(b)t)$$

is a quasi-periodic solution with frequency

$$\omega_{\text{Eq}}(b) \triangleq \left((\Omega_{j,1}(b))_{j \in \mathbb{S}_1}, (\Omega_{j,2}(b))_{j \in \mathbb{S}_2} \right) \quad (1.15)$$

of the original linearized equation $\partial_t r = \mathcal{J}\mathbf{M}_0 r$ which is \mathbf{m} -fold and reversible, namely $r(-t, -\theta) = r(t, \theta) = r(t, \theta + \frac{2\pi}{\mathbf{m}})$. Our main result states that these structures persist at the non-linear level. More precisely, we have the following theorem.

Theorem 1.1. *Let $0 < b_* < b^* < 1$ and fix $\mathbf{m} \in \mathbb{N}$ with $\mathbf{m} \geq \mathbf{m}^*$, where \mathbf{m}^* defined in (1.11). There exists $\Omega_{\mathbf{m}}^* \triangleq \Omega(b^*, \mathbf{m}) > 0$ satisfying*

$$\lim_{\mathbf{m} \rightarrow \infty} \Omega_{\mathbf{m}}^* = 0$$

such that for any $\Omega > \Omega_{\mathbf{m}}^$, there exists $\varepsilon_0 \in (0, 1)$ small enough with the following properties : For every amplitudes*

$$\mathbf{a} = ((\mathbf{a}_{j,1})_{j \in \mathbb{S}_1}, (\mathbf{a}_{j,2})_{j \in \mathbb{S}_2}) \in (\mathbb{R}_+^*)^{d_1+d_2} \quad \text{satisfying} \quad |\mathbf{a}| \leq \varepsilon_0,$$

there exists a Cantor-like set

$$\mathcal{C}_{\infty}^{\mathbf{a}} \subset (b_*, b^*), \quad \text{with} \quad \lim_{\mathbf{a} \rightarrow 0} |\mathcal{C}_{\infty}^{\mathbf{a}}| = b^* - b_*,$$

such that for any $b \in \mathcal{C}_{\infty}^{\mathbf{a}}$, the planar Euler equations (1.1) admit a \mathbf{m} -fold time quasi-periodic doubly-connected vortex patch solution in the form

$$\omega(t, \cdot) = \mathbf{1}_{D_t}, \quad D_t = \left\{ \ell e^{i(\theta - \Omega t)}, \quad \theta \in [0, 2\pi], \quad \sqrt{b^2 + 2r_2(t, \theta)} < \ell < \sqrt{1 + 2r_1(t, \theta)} \right\},$$

where

$$\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} (t, \theta) = \sum_{j \in \mathbb{S}_1} \frac{\mathbf{a}_{j,1} \cos(j\theta - \omega_{j,1}(b, \mathbf{a})t)}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix} + \sum_{j \in \mathbb{S}_2} \frac{\mathbf{a}_{j,2} \cos(j\theta - \omega_{j,2}(b, \mathbf{a})t)}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix} + \mathbf{p}(\omega_{\text{pe}}(b, \mathbf{a})t, \theta)$$

and $a_j(b)$ are given by (1.12). This solution is associated with a non-resonant frequency vector

$$\omega_{\text{pe}}(b, \mathbf{a}) \triangleq \left((\omega_{j,1}(b, \mathbf{a}))_{j \in \mathbb{S}_1}, (\omega_{j,2}(b, \mathbf{a}))_{j \in \mathbb{S}_2} \right) \in \mathbb{R}^{d_1+d_2}$$

satisfying the convergence

$$\omega_{\text{pe}}(b, \mathbf{a}) \xrightarrow{\mathbf{a} \rightarrow 0} \left((\Omega_{j,1}(b))_{j \in \mathbb{S}_1}, (\Omega_{j,2}(b))_{j \in \mathbb{S}_2} \right),$$

where $\Omega_{j,1}(b)$ and $\Omega_{j,2}(b)$ are the equilibrium frequencies. The perturbation $\mathbf{p} : \mathbb{T}^{d_1+d_2+1} \rightarrow \mathbb{R}^2$ is a function satisfying the symmetry properties

$$\forall(\varphi, \theta) \in \mathbb{T}^{d_1+d_2+1}, \quad \mathbf{p}(-\varphi, -\theta) = \mathbf{p}(\varphi, \theta) = \mathbf{p}(\varphi, \theta + \frac{2\pi}{\mathbf{m}})$$

and for some large index of Sobolev regularity s it satisfies the estimate

$$\|\mathbf{p}\|_{H^s(\mathbb{T}^{d_1+d_2+1}, \mathbb{R}^2)} \underset{\mathbf{a} \rightarrow 0}{=} o(|\mathbf{a}|).$$

Remark 1.1. The lower bound restriction b_* is required because the operators $L_{k,n}$, $V_{k,n}$ may become singular when $b = 0$, a situation which corresponds to the simply-connected case.

Remark 1.2. From this theorem, we obtain global non trivial solutions in the patch form which are confined around the annulus. More studies on vortex confinement can be found in [71, 72, 81].

We shall now briefly describe the main steps of the proof whose general strategy is borrowed from the Nash-Moser approach for KAM theory developed by Berti-Bolle [14] and slightly modified in [59, Sec. 6]. Recall that the Nash-Moser scheme requires to invert the linearized operator in a neighborhood of the equilibrium state and the inverse operator must satisfy suitable tame estimates in the framework of Sobolev spaces. The first step that we intend to describe now is to reformulate the problem in terms of embedded tori. Remark that the Hamiltonian system associated with K is a quasi-linear perturbation of its linearization at the equilibrium state, namely

$$\partial_t r = \mathcal{J}\mathbf{L}_0 r + X_P(r), \quad \text{with} \quad X_P(r) \triangleq \mathcal{J}\nabla K(r) - \mathcal{J}\mathbf{L}_0 r = \mathbf{Q}^{-1} X_{H \geq 3}(\mathbf{Q}r),$$

where

$$X_{H \geq 3}(r) \triangleq \mathcal{J}(\nabla H(r) - \mathbf{M}_0 r).$$

Under the rescaling $r \mapsto \varepsilon r$ and the quasi-periodic framework $\partial_t \leftrightarrow \omega \cdot \partial_\varphi$, the Hamiltonian system becomes

$$\omega \cdot \partial_\varphi r = \mathcal{J}\mathbf{L}_0 r + \varepsilon X_{P_\varepsilon}(r),$$

where X_{P_ε} is the rescaled Hamiltonian vector field defined by $X_{P_\varepsilon}(r) \triangleq \varepsilon^{-2} X_P(\varepsilon r)$. Notice that the previous equation is generated by the rescaled Hamiltonian

$$\mathcal{K}_\varepsilon(r) \triangleq \varepsilon^{-2} K(\varepsilon r) = K_{\mathbf{L}_0}(r) + \varepsilon P_\varepsilon(r),$$

with $K_{\mathbf{L}_0}$ as in (1.13) and $\varepsilon P_\varepsilon(r)$ describes all the terms of higher order more than cubic. We consider two finite sets $\mathbb{S}_1, \mathbb{S}_2$ as in (1.14) and we denote $d_1 \triangleq |\mathbb{S}_1|$, $d_2 \triangleq |\mathbb{S}_2|$, $d \triangleq d_1 + d_2$ and

$$\bar{\mathbb{S}}_k \triangleq \mathbb{S}_k \cup (-\mathbb{S}_k), \quad \bar{\mathbb{S}}_{0,k} \triangleq \bar{\mathbb{S}}_k \cup \{0\}.$$

and set

$$\mathbb{S} \triangleq \mathbb{S}_1 \cup \mathbb{S}_2, \quad \bar{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S}), \quad \bar{\mathbb{S}}_0 \triangleq \bar{\mathbb{S}} \cup \{0\}.$$

Next, we decompose the phase space into the following orthogonal sum

$$L_{\mathbf{m}}^2(\mathbb{T}) \times L_{\mathbf{m}}^2(\mathbb{T}) = \mathbf{H}_{\bar{\mathbb{S}}} \oplus \mathbf{H}_{\bar{\mathbb{S}}_0}^\perp, \quad L_{\mathbf{m}}^2(\mathbb{T}) \triangleq \left\{ f = \sum_{j \in \mathbb{Z}_{\mathbf{m}}^*} f_j \mathbf{e}_j \quad \text{s.t.} \quad f_{-j} = \bar{f}_j, \quad \sum_{j \in \mathbb{Z}_{\mathbf{m}}^*} |f_j|^2 < +\infty \right\}, \quad (1.16)$$

with

$$\mathbf{H}_{\bar{\mathbb{S}}} \triangleq \left\{ \sum_{j \in \bar{\mathbb{S}}_1} v_{j,1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{e}_j + \sum_{j \in \bar{\mathbb{S}}_2} v_{j,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{e}_j, \quad v_{j,k} \in \mathbb{C}, \quad \overline{v_{j,k}} = v_{-j,k} \right\},$$

$$\mathbf{H}_{\bar{\mathbb{S}}_0}^\perp \triangleq \left\{ \sum_{j \in \mathbb{Z}_{\mathbf{m}} \setminus \bar{\mathbb{S}}_{0,1}} z_{j,1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{e}_j + \sum_{j \in \mathbb{Z}_{\mathbf{m}} \setminus \bar{\mathbb{S}}_{0,2}} z_{j,2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{e}_j, \quad z_{j,k} \in \mathbb{C}, \quad \overline{z_{j,k}} = z_{-j,k} \right\}.$$

The sets $\mathbf{H}_{\overline{\mathbb{S}}}$ and $\mathbf{H}_{\overline{\mathbb{S}}_0}^\perp$ are called *tangential* and *normal subspaces*, respectively. The associated projections are defined through

$$\Pi_{\overline{\mathbb{S}}} \triangleq \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix} \quad \text{and} \quad \Pi_{\overline{\mathbb{S}}_0}^\perp \triangleq \begin{pmatrix} \Pi_1^\perp & 0 \\ 0 & \Pi_2^\perp \end{pmatrix} \quad (1.17)$$

namely,

$$\forall k \in \{1, 2\}, \quad \Pi_k \sum_{j \in \mathbb{Z}_m^*} v_{j,k} \mathbf{e}_j \triangleq \sum_{j \in \overline{\mathbb{S}}_k} v_{j,k} \mathbf{e}_j \quad \text{and} \quad \Pi_k^\perp \triangleq \mathbb{I}_m - \Pi_k, \quad (1.18)$$

where \mathbb{I}_m is the identity map of $L_m^2(\mathbb{T})$. On the tangential set $\mathbf{H}_{\overline{\mathbb{S}}}$, we introduce the action-angle variables

$$\vartheta \triangleq ((\vartheta_{j,1})_{j \in \mathbb{S}_1}, (\vartheta_{j,2})_{j \in \mathbb{S}_2}), \quad I \triangleq ((I_{j,1})_{j \in \mathbb{S}_1}, (I_{j,2})_{j \in \mathbb{S}_2})$$

as follows : Fix any amplitudes $(\mathbf{a}_{j,k})_{j \in \overline{\mathbb{S}}_k}$ such that for any $j \in \overline{\mathbb{S}}_k$, $\mathbf{a}_{-j,k} = \mathbf{a}_{j,k} > 0$ and set

$$\forall k \in \{1, 2\}, \quad \forall j \in \overline{\mathbb{S}}_k, \quad v_{j,k} \triangleq \sqrt{(\mathbf{a}_{j,k})^2 + |j| I_{j,k} e^{i\vartheta_{j,k}}},$$

supplemented with the symmetry properties

$$\forall k \in \{1, 2\}, \quad \forall j \in \mathbb{S}_k, \quad I_{-j,k} = I_{j,k} \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j,k} = -\vartheta_{j,k} \in \mathbb{T}.$$

Therefore, we have the following decomposition of $r = (r_1, r_2)$,

$$r = \mathbf{A}(\vartheta, I, z) \triangleq v(\vartheta, I) + z, \quad (1.19)$$

where $z \in \mathbf{H}_{\overline{\mathbb{S}}_0}^\perp$ and

$$v(\vartheta, I) \triangleq \sum_{j \in \overline{\mathbb{S}}_1} \sqrt{(\mathbf{a}_{j,1})^2 + |j| I_{j,1} e^{i\vartheta_{j,1}}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbf{e}_j + \sum_{j \in \overline{\mathbb{S}}_2} \sqrt{(\mathbf{a}_{j,2})^2 + |j| I_{j,2} e^{i\vartheta_{j,2}}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbf{e}_j \in \mathbf{H}_{\overline{\mathbb{S}}}.$$

The transformation \mathbf{A} is \mathcal{W} -symplectic and, in the new variables, the new Hamiltonian $K_\varepsilon \triangleq \mathcal{K}_\varepsilon \circ \mathbf{A}$ writes

$$K_\varepsilon = -(\mathbf{J} \omega_{\text{Eq}}(b)) \cdot I + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon, \quad \mathbf{J} \triangleq \begin{pmatrix} \mathbf{I}_{d_1} & 0 \\ 0 & -\mathbf{I}_{d_2} \end{pmatrix}, \quad \mathcal{P}_\varepsilon \triangleq P_\varepsilon \circ \mathbf{A}.$$

Observe that the Poisson structure is associated with \mathbf{J} , and will be needed later during the implementation of Berti-Bolle approach. The corresponding Hamiltonian vector field is

$$X_{K_\varepsilon} \triangleq (\mathbf{J} \partial_I K_\varepsilon, -\mathbf{J} \partial_\vartheta K_\varepsilon, \Pi_{\overline{\mathbb{S}}_0}^\perp \mathcal{J} \nabla_z K_\varepsilon).$$

Therefore, the problem is reduced to looking for embedded invariant tori

$$i : \mathbb{T}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{H}_{\overline{\mathbb{S}}_0}^\perp, \quad \varphi \mapsto i(\varphi) \triangleq (\vartheta(\varphi), I(\varphi), z(\varphi)),$$

solution of the equation

$$\omega \cdot \partial_\varphi i(\varphi) = X_{K_\varepsilon}(i(\varphi)).$$

As observed in [14, 83], it turns out to be convenient along Nash-Moser scheme to work with one degree of freedom vector-valued parameter $\alpha \in \mathbb{R}^d$ which provides at the end of the scheme a solution for the original problem when it is fixed to $-\mathbf{J} \omega_{\text{Eq}}(b)$. Therefore, we shall consider the following α -dependent family of Hamiltonians

$$K_\varepsilon^\alpha \triangleq \alpha \cdot I + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon$$

and we search for the zeros of the following functional

$$\mathcal{F}(i, \alpha, b, \omega, \varepsilon) \triangleq \omega \cdot \partial_\varphi i(\varphi) - X_{K_\varepsilon^\alpha}(i(\varphi)) = \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \mathbf{J}(\alpha - \varepsilon \partial_I \mathcal{P}_\varepsilon(i(\varphi))) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \mathbf{J} \partial_\vartheta \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \mathcal{J}[\mathbf{L}_0 z(\varphi) + \varepsilon \nabla_z \mathcal{P}_\varepsilon(i(\varphi))] \end{pmatrix}.$$

At each step of Nash-Moser scheme, we have to linearize this functional at a small reversible embedded torus $i_0 : \varphi \mapsto (\vartheta_0(\varphi), I_0(\varphi), z_0(\varphi))$ and $\alpha_0 \in \mathbb{R}^d$, then we need to construct an approximate right inverse of $d_{i_0, \alpha} \mathcal{F}(i_0, \alpha_0)$. The core of the Berti-Bolle theory is to conjugate the linearized operator by a linear diffeomorphism G_0 of the toroidal phase space $\mathbb{T}^d \times \mathbb{R}^d \times \mathbf{H}_{\mathbb{S}_0}^\perp$ in order to obtain a triangular system in the action-angle-normal variables up to error terms. Notice that in a similar way to [59], we do not use isotropic tori. Therefore, in this framework, inverting the triangular system amounts to inverting the linearized operator in the normal directions, denoted by $\widehat{\mathcal{L}}$. This latter fact is analyzed in Section 7 and uses KAM reducibility techniques similarly to [4, 19, 59, 69] that we shall now explain and extract the main new difficulties. According to Proposition 7.1, we can write

$$\widehat{\mathcal{L}} = \Pi_{\mathbb{S}_0}^\perp (\mathcal{L} - \varepsilon \partial_\theta \mathcal{R}) \Pi_{\mathbb{S}_0}^\perp, \quad \mathcal{L} \triangleq \omega \cdot \partial_\varphi \mathbf{I}_m + \mathcal{L}_{\varepsilon r}, \quad \mathbf{I}_m \triangleq \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & \mathbb{I}_m \end{pmatrix}, \quad \mathcal{R} \triangleq \begin{pmatrix} \mathcal{T}_{J_{1,1}}(r) & \mathcal{T}_{J_{1,2}}(r) \\ \mathcal{T}_{J_{2,1}}(r) & \mathcal{T}_{J_{2,2}}(r) \end{pmatrix},$$

where for any $(k, \ell) \in \{1, 2\}^2$, $\mathcal{T}_{J_{k,\ell}}$ is a m -fold and reversibility preserving integral operator with smooth kernel $J_{k,\ell}$ and $\mathcal{L}_{\varepsilon r}$ is the linearized operator

$$\mathcal{L}_{\varepsilon r} = \begin{pmatrix} \partial_\theta (\mathcal{V}_1(r) \cdot) + \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & \partial_\theta (\mathcal{V}_2(r) \cdot) - \frac{1}{2} \mathcal{H} - \partial_\theta \mathcal{Q} * \cdot \end{pmatrix} + \partial_\theta \begin{pmatrix} \mathcal{T}_{\mathcal{K}_{1,1}}(r) & \mathcal{T}_{\mathcal{K}_{1,2}}(r) \\ \mathcal{T}_{\mathcal{K}_{2,1}}(r) & \mathcal{T}_{\mathcal{K}_{2,2}}(r) \end{pmatrix},$$

where we denote by \mathcal{H} the 2π -periodic Hilbert transform and $\mathcal{V}_k(r)$, $k \in \{1, 2\}$, are scalar functions. The convolution operator $\mathcal{Q} * \cdot$ has even smooth kernel \mathcal{Q} . For $k, n \in \{1, 2\}$, the operator $\mathcal{T}_{\mathcal{K}_{k,n}}(r)$ is an integral operator with smooth, m -fold and reversibility preserving kernel $\mathcal{K}_{k,n}(r)$, see Proposition 4.1.

First, following the KAM reducibility scheme in [8, 45, 69], we can reduce the transport part and the zero order part by conjugating by a quasi-periodic symplectic change of variables in the form

$$\mathcal{B} \triangleq \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix}, \quad \forall k \in \{1, 2\}, \quad \mathcal{B}_k \rho(\mu, \varphi, \theta) = (1 + \partial_\theta \beta_k(\mu, \varphi, \theta)) \rho(\mu, \varphi, \theta + \beta_k(\mu, \varphi, \theta)).$$

More precisely, as stated in Propositions 7.2 and 7.3, we can find two functions $\mathbf{c}_1 \triangleq \mathbf{c}_1(b, \omega, i_0)$, $\mathbf{c}_2 \triangleq \mathbf{c}_2(b, \omega, i_0)$ and a Cantor set

$$\mathcal{O}_{\infty, n}^{\gamma, \tau_1}(i_0) \triangleq \bigcap_{\substack{k \in \{1, 2\} \\ (l, j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0, 0)\} \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + j \mathbf{c}_k(b, \omega, i_0)| > \frac{4\gamma^\nu(j)}{\langle l \rangle^{\tau_1}} \right\},$$

where $N_n \triangleq N_0 \left(\frac{3}{2}\right)^n$ with $N_0 \gg 1$, in which the following decomposition holds

$$\mathcal{B}^{-1}(\omega \cdot \partial_\varphi \mathbf{I}_m + \mathcal{L}_{\varepsilon r}) \mathcal{B} = \omega \cdot \partial_\varphi \mathbf{I}_m + \mathcal{D} + \mathcal{R} + \mathcal{E}_n,$$

where

$$\mathcal{D} \triangleq \begin{pmatrix} \mathbf{c}_1 \partial_\theta + \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & \mathbf{c}_2 \partial_\theta - \left(\frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot\right) \end{pmatrix},$$

and $\mathcal{R} \triangleq \mathcal{R}(\varepsilon r)$ is a real, m -fold and reversibility preserving Toeplitz in time matrix integral operator enjoying good smallness properties. The operator \mathcal{E}_n is of order one but with small coefficients decaying faster in n . The next step deals with the localization effects on the normal modes. We first introduce the operator

$$\mathcal{B}_\perp \triangleq \Pi_{\mathbb{S}_0}^\perp \mathcal{B} \Pi_{\mathbb{S}_0}^\perp = \begin{pmatrix} \Pi_1^\perp \mathcal{B}_1 \Pi_1^\perp & 0 \\ 0 & \Pi_2^\perp \mathcal{B}_2 \Pi_2^\perp \end{pmatrix}.$$

Then, according to Proposition 7.4, we prove by restricting the parameters to the set $\mathcal{O}_{\infty, n}^{\gamma, \tau_1}(i_0)$ that for any $n \in \mathbb{N}^*$ we have the identity

$$\mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp = \mathcal{L}_0 + \mathcal{E}_n^0, \quad \mathcal{L}_0 \triangleq \omega \cdot \partial_\varphi \mathbf{I}_{m, \perp} + \mathcal{D}_0 + \mathcal{R}_0,$$

where $\mathbf{I}_{m, \perp} \triangleq \Pi_{\mathbb{S}_0}^\perp \mathbf{I}_m$ and $\mathcal{D}_0 = \Pi_{\mathbb{S}_0}^\perp \mathcal{D}_0 \Pi_{\mathbb{S}_0}^\perp$ is an m -fold preserving and reversible matrix Fourier multiplier operator in the form

$$\mathcal{D}_0 \triangleq \begin{pmatrix} \mathcal{D}_{0,1} & 0 \\ 0 & \mathcal{D}_{0,2} \end{pmatrix},$$

with

$$\forall k \in \{1, 2\}, \quad \mathcal{D}_{0,k} \triangleq \left(i\mu_{j,k}^{(0)} \right)_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}, \quad \mu_{j,k}^{(0)}(b, \omega, i_0) \triangleq \Omega_{j,k}(b) + j(c_k(b, \omega) - v_k(b))$$

and $\mathcal{R}_0 = \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{R}_0 \Pi_{\bar{\mathbb{S}}_0}^\perp$ is a small real, m -fold preserving and reversible Toeplitz in time matrix remainder whose entries are integral operators with smooth kernels. The error term \mathcal{E}_n^0 plays a similar role as the previous one \mathcal{E}_n . The next goal is to implement a KAM reduction of the remainder term \mathcal{R}_0 . This is done in a new hybrid operator topology treating the diagonal and anti-diagonal terms differently. Along the scheme, the diagonal part is treated as in the scalar situation through the use of the off-diagonal Toeplitz norm, see for instance [69, Prop. 6.5], whereas the anti-diagonal part, which is smoothing at any order in the spatial variable, is studied in an isotropic topology. We refer to Section 2.2.4 for more details on this topological framework, in particular (2.34). We point out that, thanks to the nice structure of the 2D-Euler equation, the diagonal and anti-diagonal terms of the remainder term \mathcal{R}_0 are smoothing at any order in the spatial variable and therefore both can be studied using the isotropic topology. However, this fact is not true for other transport models [59, 69] where the remainders on the diagonal are not highly smoothing but of negative order. For this reason, we prefer to work in the most general framework. The Proposition 7.5 states that we can find an operator Φ_∞ such that in the following Cantor set gathering both diagonal and anti-diagonal second order Melnikov conditions

$$\begin{aligned} \mathcal{O}_{\infty,n}^{\gamma, \tau_1, \tau_2}(i_0) &\triangleq \mathcal{O}_{\infty,n}^{\gamma, \tau_1}(i_0) \\ &\bigcap_{\substack{k \in \{1, 2\} \\ j, j_0 \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}} \bigcap_{\substack{l \in \mathbb{Z}^d \\ (l, j) \neq (0, j_0) \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_0) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_0) \right| > \frac{2\gamma \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\} \\ &\bigcap_{\substack{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1} \\ j_0 \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}}} \bigcap_{\substack{l \in \mathbb{Z}^d \\ \langle l, j, j_0 \rangle \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,1}^{(\infty)}(b, \omega, i_0) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_0) \right| > \frac{2\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} \right\} \end{aligned}$$

the following decomposition holds

$$\Phi_\infty^{-1} \mathcal{L}_0 \Phi_\infty = \omega \cdot \partial_\varphi \mathbf{I}_{m,\perp} + \mathcal{D}_\infty + \mathcal{E}_n^1 \triangleq \mathcal{L}_\infty + \mathcal{E}_n^1,$$

where $\mathcal{D}_\infty = \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{D}_\infty \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{D}_\infty(b, \omega, i_0)$ is a diagonal operator with reversible Fourier multiplier entries, namely

$$\mathcal{D}_\infty \triangleq \begin{pmatrix} \mathcal{D}_{\infty,1} & 0 \\ 0 & \mathcal{D}_{\infty,2} \end{pmatrix},$$

with

$$\forall k \in \{1, 2\}, \quad \mathcal{D}_{\infty,k} \triangleq \left(i\mu_{j,k}^{(\infty)} \right)_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}, \quad \mu_{j,k}^{(\infty)}(b, \omega, i_0) \triangleq \mu_{j,k}^{(0)}(b, \omega, i_0) + r_{j,k}^{(\infty)}(b, \omega, i_0)$$

and

$$\sup_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}} |j| \left\| r_{j,k}^{(\infty)} \right\|^{q,\gamma} \lesssim \varepsilon \gamma^{-1}.$$

Notice that, according to the monotonicity of the eigenvalues the difference $\mu_{j,k}^{(\infty)} - \mu_{j_0,k}^{(\infty)}$ is not vanishing for $j \neq j_0$ and grows like $|j - j_0|$. This is no longer true for the mixed difference $\mu_{j,1}^{(\infty)} - \mu_{j_0,2}^{(\infty)}$ (coming from the mutual interactions between the interfaces) due to the different transport speeds leading to a new small divisor problem. Therefore, to handle this problem we should adjust the geometry of the Cantor sets $\mathcal{O}_{\infty,n}^{\gamma, \tau_1, \tau_2}(i_0)$ with an isotropic decay on frequency. This explains in part the introduction of the hybrid topology in (2.34) needed in the remainder reduction, Another key observation is that we have no resonances for the off-diagonal part at $j = j_0$ and consequently the associated homological equations can be solved without any residual diagonal terms. Thus, at the end of the KAM scheme

we get a diagonal Fourier multiplier operator \mathcal{D}_∞ . Now, the final operator \mathcal{L}_∞ can be easily inverted by restricting the parameters to the following first order Melnikov conditions

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) \triangleq \bigcap_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{v}_0, k) \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_0) \right| > \frac{\gamma(j)}{\langle l \rangle^{\tau_1}} \right\}.$$

As a consequence, we can construct an approximate right inverse of $\widehat{\mathcal{L}}$ provided that we choose (b, ω) in the set

$$\mathbf{G}_n^\gamma(i_0) \triangleq \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \cap \mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) \cap \Lambda_{\infty,n}^{\gamma,\tau_1}(i_0).$$

Therefore, we can perform in Proposition 8.1 and Corollary 8.1 a Nash-Moser scheme as in [19, 59, 69] with slight modifications due to our particular Poisson structure and the off-diagonal second order Melnikov conditions. Hence, we can find a non-trivial solution $(b, \omega) \mapsto (i_\infty(b, \omega), \alpha_\infty(b, \omega))$ to the equation $\mathcal{F}(i, \alpha, b, \omega, \varepsilon) = 0$ provided that we restrict the parameters (b, ω) to a Borel set \mathbf{G}_∞^γ constructed as the intersection of all the Cantor sets encountered along the different schemes of the multiple reductions. A solution to the original problem is obtained by constructing a frequency curve $b \mapsto \omega(b, \varepsilon)$ solution to the implicit equation

$$\alpha_\infty(b, \omega(b, \varepsilon)) = -\mathcal{J}\omega_{\text{Eq}}(b).$$

By this way we construct a solution for any value of b in

$$\mathcal{C}_\infty^\varepsilon \triangleq \left\{ b \in (b_*, b^*) \quad \text{s.t.} \quad (b, \omega(b, \varepsilon)) \in \mathbf{G}_\infty^\gamma \right\}.$$

The last step is to check that this final set is non-empty and massive. Actually, we prove in Proposition 8.2 the following measure bound

$$(b^* - b_*) - \varepsilon^\delta \leq |\mathcal{C}_\infty^\varepsilon| \leq (b^* - b_*) \quad \text{for some } \delta \triangleq \delta(q_0, d, \tau_1, \tau_2) > 0.$$

The proof is quite standard and based on the perturbation of Rüssmann conditions, shown to be true at the equilibrium state. We emphasize that the restriction $\Omega > \Omega_m^*$ is required by Lemma 4.8-(iv) and Lemma 8.4-(iv), and the value of Ω_m^* given in (4.76) is not necessary optimal.

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2 Function and operator spaces

This section is devoted to the presentation of the general topological framework for both functions and operators classes. In addition, we shall set some basic notations, definitions and give some technical results used in this work.

Notations. Along this paper we shall make use of the following set notations.

- The sets of numbers that will be frequently used are denoted as follows

$$\mathbb{N} = \{0, 1, 2, \dots\}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\}, \quad \mathbb{Z} = \mathbb{N} \cup (-\mathbb{N}), \quad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}, \quad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

For any $m \in \mathbb{N}^*$, we may denote

$$\mathbb{Z}_m = m\mathbb{Z}, \quad \mathbb{N}_m = m\mathbb{N}, \quad \mathbb{Z}_m^* = m\mathbb{Z}^*, \quad \mathbb{N}_m^* = m\mathbb{N}^*, \quad \mathbb{T}^m = \underbrace{\mathbb{T} \times \dots \times \mathbb{T}}_{m \text{ times}}$$

and for any $m, n \in \mathbb{Z}$, such that $m < n$,

$$\llbracket m, n \rrbracket \triangleq \{m, m+1, \dots, n-1, n\}.$$

- We fix two real numbers b_* and b^* such that

$$0 < b_* < b^* < 1.$$

The parameter b lies in the interval (b_*, b^*) and represents the radius of the annulus A_b in (1.5), corresponding to the equilibrium state and

- Consider the following parameters, that will be used to construct the Cantor set as well as the regularity of the perturbations,

$$d \in \mathbb{N}^*, \quad q \in \mathbb{N}^*, \quad (2.1)$$

$$0 < \gamma \leq 1, \quad \tau_2 > \tau_1 > d, \quad (2.2)$$

$$S \geq s \geq s_0 > \frac{d+1}{2} + q + 2. \quad (2.3)$$

- For any $n \in \mathbb{N}^*$ and any complex periodic function $\rho : \mathbb{T}^n \rightarrow \mathbb{R}$, we denote

$$\int_{\mathbb{T}^n} \rho(\eta) d\eta \triangleq \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} \rho(\eta) d\eta. \quad (2.4)$$

- Let $f : X \rightarrow Y$ be a map where X is a set and Y is a vector space. For any $r_1, r_2 \in X$, we denote

$$\Delta_{12}f = f(r_1) - f(r_2).$$

2.1 Function spaces

This section is devoted to some functional tools frequently used along this paper. First, we shall introduce the complex Sobolev space on the periodic setting $H^s(\mathbb{T}^{d+1}, \mathbb{C})$ with index regularity $s \in \mathbb{R}$. It is the set of all the complex periodic functions $\rho : \mathbb{T}^{d+1} \rightarrow \mathbb{C}$ with the Fourier expansion

$$\rho = \sum_{(l,j) \in \mathbb{Z}^{d+1}} \rho_{l,j} \mathbf{e}_{l,j}, \quad \mathbf{e}_{l,j}(\varphi, \theta) \triangleq e^{i(l \cdot \varphi + j\theta)}, \quad \rho_{l,j} \triangleq \langle \rho, \mathbf{e}_{l,j} \rangle_{L^2(\mathbb{T}^{d+1})}$$

equipped with the scalar product

$$\langle \rho, \tilde{\rho} \rangle_{H^s} \triangleq \sum_{(l,j) \in \mathbb{Z}^{d+1}} \langle l, j \rangle^{2s} \rho_{l,j} \overline{\tilde{\rho}_{l,j}}, \quad \text{with} \quad \langle l, j \rangle \triangleq \max(1, |l|, |j|),$$

where $|\cdot|$ denotes the classical ℓ^1 norm in \mathbb{R}^d . For $s = 0$ this space coincides with the standard $L^2(\mathbb{T}^{d+1}, \mathbb{C})$ space equipped with the scalar product

$$\langle \rho_1, \rho_2 \rangle_{L^2(\mathbb{T}^{d+1})} \triangleq \int_{\mathbb{T}^{d+1}} \rho_1(\varphi, \theta) \overline{\rho_2(\varphi, \theta)} d\varphi d\theta.$$

We shall make use of the product Sobolev space

$$\mathbf{H}_m^s(\mathbb{T}^{d+1}, \mathbb{C}) \triangleq H_m^s(\mathbb{T}^{d+1}, \mathbb{C}) \times H_m^s(\mathbb{T}^{d+1}, \mathbb{C}), \quad (2.5)$$

equipped with the scalar product

$$\langle (\rho_1, \rho_2), (\tilde{\rho}_1, \tilde{\rho}_2) \rangle_{\mathbf{H}_m^s(\mathbb{T}^{d+1}, \mathbb{C})} \triangleq \sum_{(l,j) \in \mathbb{Z}^{d+1}} \langle l, j \rangle^{2s} \rho_{l,j}^1 \overline{\tilde{\rho}_{l,j}^1} + \sum_{(l,j) \in \mathbb{Z}^{d+1}} \langle l, j \rangle^{2s} \rho_{l,j}^2 \overline{\tilde{\rho}_{l,j}^2}.$$

We also simply denote the real space

$$\mathbf{H}_m^s \triangleq H_m^s \times H_m^s.$$

As we shall see later, the main enemy in the construction of quasi-periodic solutions is the resonances and in particular the trivial ones which can be fortunately removed by imposing more symmetry on

the solutions. For this aim we need to work with the following subspace $H_{\mathbf{m}}^s(\mathbb{T}^{d+1}, \mathbb{C})$, with $\mathbf{m} \in \mathbb{N}^*$, whose elements enjoy the \mathbf{m} -fold symmetry in the variable θ , that is

$$H_{\mathbf{m}}^s(\mathbb{T}^{d+1}, \mathbb{C}) \triangleq \left\{ \rho \in H^s(\mathbb{T}^{d+1}, \mathbb{C}) \quad \text{s.t.} \quad \forall (\varphi, \theta) \in \mathbb{T}^{d+1}, \quad \rho(\varphi, \theta + \frac{2\pi}{\mathbf{m}}) = \rho(\varphi, \theta) \right\}.$$

Notice the \mathbf{m} -fold symmetry is equivalent to say that

$$\forall l \in \mathbb{Z}^d, \quad \forall j \in \mathbb{Z} \setminus \mathbb{Z}_{\mathbf{m}}, \quad \langle \rho, \mathbf{e}_{l,j} \rangle_{L^2(\mathbb{T}^{d+1})} = 0.$$

The real Sobolev space $H_{\mathbf{m}}^s(\mathbb{T}^{d+1}, \mathbb{R})$ is simply denoted by $H_{\mathbf{m}}^s$ and we define the subspace

$$H_{\mathbf{m}}^\infty \triangleq \bigcap_{s \in \mathbb{R}} H_{\mathbf{m}}^s.$$

For $N \in \mathbb{N}$, we define the cut-off frequency projectors Π_N and its orthogonal Π_N^\perp on $H^s(\mathbb{T}^{d+1}, \mathbb{C})$ as follows

$$\Pi_N \rho \triangleq \sum_{\substack{(l,j) \in \mathbb{Z}^{d+1} \\ \langle l,j \rangle \leq N}} \rho_{l,j} \mathbf{e}_{l,j} \quad \text{and} \quad \Pi_N^\perp \triangleq \text{Id} - \Pi_N. \quad (2.6)$$

We shall also make use of the following mixed weighted Sobolev spaces with respect to a given parameter $\gamma \in (0, 1)$. Let \mathcal{O} be an open bounded set of \mathbb{R}^{d+1} and define the Banach spaces

$$W^{q,\infty,\gamma}(\mathcal{O}, H_{\mathbf{m}}^s) \triangleq \left\{ \rho : \mathcal{O} \rightarrow H_{\mathbf{m}}^s \quad \text{s.t.} \quad \|\rho\|_s^{q,\gamma,\mathbf{m}} \triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} \|\partial_\mu^\alpha \rho(\mu, \cdot)\|_{H_{\mathbf{m}}^{s-|\alpha|}} < \infty \right\},$$

$$W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) \triangleq \left\{ \rho : \mathcal{O} \rightarrow \mathbb{C} \quad \text{s.t.} \quad \|\rho\|_s^{q,\gamma} \triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} |\partial_\mu^\alpha \rho(\mu)| < \infty \right\}.$$

Through this paper, we shall implicitly use the notation $\|\rho\|_s^{q,\gamma,\mathbf{m}}$, while the function ρ depends on more variables such as with $(\varphi, \theta) \in \mathbb{Z}^d \times \mathbb{Z}_{\mathbf{m}}^{d'} \mapsto \rho(\varphi, \theta)$, frequently encountered when we have to estimate the kernels of some operators, in which case the variables can be doubled.

In the next lemma we shall collect some useful classical results related to various actions over weighted Sobolev spaces. The proofs are standard and can be found for instance in [15, 16, 19].

Lemma 2.1. *Let $q \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^*$ and (γ, d, s_0, s) satisfy (2.2)-(2.3), then the following assertions hold true.*

(i) *Frequency growth/decay of projectors : Let $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H_{\mathbf{m}}^s)$, then for all $N \in \mathbb{N}^*$ and $t > 0$,*

$$\|\Pi_N \rho\|_{s+t}^{q,\gamma,\mathbf{m}} \leq N^t \|\rho\|_s^{q,\gamma,\mathbf{m}} \quad \text{and} \quad \|\Pi_N^\perp \rho\|_s^{q,\gamma,\mathbf{m}} \leq N^{-t} \|\rho\|_{s+t}^{q,\gamma,\mathbf{m}},$$

where the cut-off projectors are defined in (2.6).

(ii) *Product law : Let $\rho_1, \rho_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H_{\mathbf{m}}^s)$. Then $\rho_1 \rho_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H_{\mathbf{m}}^s)$ and*

$$\|\rho_1 \rho_2\|_s^{q,\gamma,\mathbf{m}} \lesssim \|\rho_1\|_{s_0}^{q,\gamma,\mathbf{m}} \|\rho_2\|_s^{q,\gamma,\mathbf{m}} + \|\rho_1\|_s^{q,\gamma,\mathbf{m}} \|\rho_2\|_{s_0}^{q,\gamma,\mathbf{m}}.$$

(iii) *Composition law 1 : Let $f \in C^\infty(\mathcal{O} \times \mathbb{R}, \mathbb{R})$ and $\rho_1, \rho_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H_{\mathbf{m}}^s)$ such that*

$$\|\rho_1\|_s^{q,\gamma,\mathbf{m}}, \|\rho_2\|_s^{q,\gamma,\mathbf{m}} \leq C_0$$

for an arbitrary constant $C_0 > 0$ and define the pointwise composition

$$\forall (\mu, \varphi, \theta) \in \mathcal{O} \times \mathbb{T}^{d+1}, \quad f(\rho)(\mu, \varphi, \theta) \triangleq f(\mu, \rho(\mu, \varphi, \theta)).$$

Then

$$\|f(\rho_1) - f(\rho_2)\|_s^{q,\gamma,\mathbf{m}} \leq C(s, d, q, f, C_0) \|\rho_1 - \rho_2\|_s^{q,\gamma,\mathbf{m}}.$$

(iv) *Composition law 2* : Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ with bounded derivatives. Let $\rho \in W^{q, \infty, \gamma}(\mathcal{O}, \mathbb{C})$. Then

$$\|f(\rho) - f(0)\|^{q, \gamma} \leq C(q, d, f) \|\rho\|^{q, \gamma} \left(1 + \|\rho\|_{L^\infty(\mathcal{O})}^{q-1}\right).$$

(v) *Interpolation inequality* : Let $q < s_1 \leq s_3 \leq s_2$ and $\bar{\theta} \in [0, 1]$, with $s_3 = \bar{\theta}s_1 + (1 - \bar{\theta})s_2$. If $\rho \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathbf{m}}^{s_2})$, then $\rho \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathbf{m}}^{s_3})$ and

$$\|\rho\|_{s_3}^{q, \gamma, \mathbf{m}} \lesssim (\|\rho\|_{s_1}^{q, \gamma, \mathbf{m}})^{\bar{\theta}} (\|\rho\|_{s_2}^{q, \gamma, \mathbf{m}})^{1-\bar{\theta}}.$$

The next result is proved in [69, Lem. 4.2] and will be useful later in the study of some regularity aspects for the linearized operator.

Lemma 2.2. Let $q \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^*$, (γ, d, s_0, s) satisfy (2.2)-(2.3) and $f \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathbf{m}}^s)$.

We consider the function $g : \mathcal{O} \times \mathbb{T}_\varphi^d \times \mathbb{T}_\theta \times \mathbb{T}_\eta \rightarrow \mathbb{C}$ defined by

$$g(\mu, \varphi, \theta, \eta) = \begin{cases} \frac{f(\mu, \varphi, \eta) - f(\mu, \varphi, \theta)}{\sin\left(\frac{\eta - \theta}{2}\right)}, & \text{if } \theta \neq \eta, \\ 2\partial_\theta f(\mu, \varphi, \theta), & \text{if } \theta = \eta. \end{cases}$$

Then

$$\|g\|_s^{q, \gamma, \mathbf{m}} \lesssim \|f\|_{s+1}^{q, \gamma, \mathbf{m}}.$$

2.2 Operators

We intend in this section to explore some algebraic and analytical aspects on the a large class of operators that fit with our context. Firstly, we shall classify them according to their Toeplitz in time structures, real and \mathbf{m} -fold symmetry, etc... Secondly, we shall fix some specific norms, such as the off-diagonal/isotropic decay, and analyze some of their properties. This part is a crucial later in the reduction of the remainder of the linearized operator. Thirdly, a particular attention will be focused on operators with kernels by exploring the link between the different norms and the action of suitable quasi-periodic transformations. The last point concerns a short discussion on matrix operators.

2.2.1 Symmetry

Consider a smooth family of bounded linear operators acting on the Sobolev spaces $H^s(\mathbb{T}^{d+1}, \mathbb{C})$,

$$T : \mu \in \mathcal{O} \mapsto T(\mu) \in \mathcal{L}(H^s(\mathbb{T}^{d+1}, \mathbb{C})).$$

The linear operator $T(\mu)$ can be identified to the infinite dimensional matrix $\left(T_{l_0, j_0}^{l, j}(\mu)\right)_{\substack{l, l_0 \in \mathbb{Z}^d \\ j, j_0 \in \mathbb{Z}}}$ with

$$T(\mu)\mathbf{e}_{l_0, j_0} = \sum_{(l, j) \in \mathbb{Z}^d \times \mathbb{Z}} T_{l_0, j_0}^{l, j}(\mu)\mathbf{e}_{l, j}, \quad \text{where} \quad T_{l_0, j_0}^{l, j}(\mu) \triangleq \langle T(\mu)\mathbf{e}_{l_0, j_0}, \mathbf{e}_{l, j} \rangle_{L^2(\mathbb{T}^{d+1}, \mathbb{C})}.$$

Along this paper the operators and the test functions may depend on the same parameter μ and thus the action of the operator $T(\mu)$ on a scalar function $\rho \in W^{q, \infty, \gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C}))$ is by convention defined through

$$(T\rho)(\mu, \varphi, \theta) \triangleq T(\mu)\rho(\mu, \varphi, \theta).$$

We recall the following definitions of Toeplitz, real, reversible, reversibility preserving and \mathbf{m} -fold preserving operators, see for instance [5, Def. 2.2].

Definition 2.1. Let $\rho : \mathbb{T}^{d+1} \rightarrow \mathbb{R}$ be a periodic function. Define the involution

$$(\mathcal{J}_0\rho)(\varphi, \theta) \triangleq \rho(-\varphi, -\theta)$$

and for a given integer $\mathbf{m} \geq 1$ consider the transformation

$$(\mathcal{J}_{\mathbf{m}}\rho)(\varphi, \theta) \triangleq \rho\left(\varphi, \theta + \frac{2\pi}{\mathbf{m}}\right).$$

We say that an operator $T \in \mathcal{L}(L^2(\mathbb{T}^{d+1}, \mathbb{C}))$ is

- Toeplitz in time (actually in the variable φ) if its Fourier coefficients satisfy,

$$\forall (l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times \mathbb{Z}^2, \quad T_{l_0, j_0}^{l, j} = T_{j_0}^j(l - l_0), \quad \text{with} \quad T_{j_0}^j(l) \triangleq T_{0, j_0}^{l, j},$$

- real if for all $\rho \in L^2(\mathbb{T}^{d+1}, \mathbb{R})$, we have $T\rho$ is real-valued, or equivalently

$$\forall (l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times \mathbb{Z}^2, \quad T_{-l_0, -j_0}^{-l, -j} = \overline{T_{l_0, j_0}^{l, j}},$$

- reversible if $T \circ \mathcal{J}_0 = -\mathcal{J}_0 \circ T$, or equivalently,

$$\forall (l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times \mathbb{Z}^2, \quad T_{-l_0, -j_0}^{-l, -j} = -T_{l_0, j_0}^{l, j},$$

- reversibility preserving if $T \circ \mathcal{J}_0 = \mathcal{J}_0 \circ T$, or equivalently,

$$\forall (l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times \mathbb{Z}^2, \quad T_{-l_0, -j_0}^{-l, -j} = T_{l_0, j_0}^{l, j},$$

- \mathfrak{m} -fold preserving if $T \circ \mathcal{J}_{\mathfrak{m}} = \mathcal{J}_{\mathfrak{m}} \circ T$, or equivalently,

$$\forall (l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times \mathbb{Z}^2, \quad T_{l_0, j_0}^{l, j} \neq 0 \quad \Rightarrow \quad j - j_0 \in \mathbb{Z}_{\mathfrak{m}}.$$

2.2.2 Operator topologies

We shall restrict ourselves to Toeplitz operators and fix different topologies whose use will be motivated later by different applications. Given $\mathfrak{m} \in \mathbb{N}^*$, then any \mathfrak{m} -fold preserving Toeplitz operator $T(\mu)$ acting on \mathfrak{m} -fold symmetric functions $\rho = \sum_{\substack{l_0 \in \mathbb{Z}^d \\ j_0 \in \mathbb{Z}_{\mathfrak{m}}}} \rho_{l_0, j_0} \mathbf{e}_{l_0, j_0}$ is described by

$$T(\mu)\rho = \sum_{\substack{l, l_0 \in \mathbb{Z}^d \\ j, j_0 \in \mathbb{Z}_{\mathfrak{m}}}} T_{j_0}^j(\mu, l - l_0) \rho_{l_0, j_0} \mathbf{e}_{l, j}.$$

For $q \in \mathbb{N}$, $\gamma \in (0, 1]$ and $s \in \mathbb{R}$, we equip this set of operators with the off-diagonal norm given by,

$$\|T\|_{\text{O-d}, s}^{q, \gamma, \mathfrak{m}} \triangleq \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} \|\partial_{\mu}^{\alpha}(T)(\mu)\|_{\text{O-d}, s - |\alpha|}, \quad (2.7)$$

with

$$\|T\|_{\text{O-d}, s} \triangleq \sup_{(l, m) \in \mathbb{Z}^{d+1}} \langle l, m \rangle^s \sup_{j_0 - j = m} |T_{j_0}^j(\mu, l)|.$$

We define the cut-off frequency operator

$$(P_N^1 T(\mu)) \mathbf{e}_{l_0, j_0} \triangleq \sum_{\substack{(l, j) \in \mathbb{Z}^{d+1} \\ \langle l - l_0, j - j_0 \rangle \leq N}} T_{l_0, j_0}^{l, j}(\mu) \mathbf{e}_{l, j} \quad \text{and} \quad P_N^{1, \perp} T \triangleq T - P_N^1 T.$$

or equivalently

$$(P_N^1 T(\mu))_{j_0}^j(l) = \begin{cases} T_{j_0}^j(\mu, l), & \text{if } \langle l, j - j_0 \rangle \leq N, \\ 0, & \text{if not.} \end{cases} \quad (2.8)$$

Another norm that will be used together with the previous one during the reduction process of the remainder of the linearized operator, is given by the isotropic frequency decay

$$\|T\|_{\text{I-D}, s}^{q, \gamma, \mathfrak{m}} \triangleq \sup_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{\mu \in \mathcal{O}} \|\partial_{\mu}^{\alpha}(T)(\mu)\|_{\text{I-D}, s - |\alpha|}, \quad (2.9)$$

where

$$\|T(\mu)\|_{\text{I-D}, s} \triangleq \sup_{\substack{l \in \mathbb{Z}^d \\ j, j_0 \in \mathbb{Z}_{\mathfrak{m}}}} \langle l, j_0, j \rangle^s |T_{j_0}^j(\mu, l)|.$$

The associated cut-off projectors $(P_N^2)_{N \in \mathbb{N}}$ are defined as follows

$$(P_N^2 T(\mu)) \mathbf{e}_{l_0, j_0} \triangleq \begin{cases} \sum_{\substack{(l, j) \in \mathbb{Z}^d \times \mathbb{Z}^m \\ \langle l - l_0, j \rangle \leq N}} T_{j_0}^j(\mu, l - l_0) \mathbf{e}_{l, j}, & \text{if } |j_0| \leq N, \\ 0, & \text{if } |j_0| > N. \end{cases} \quad (2.10)$$

or equivalently

$$(P_N^2 T(\mu))_{j_0}^j(l) = \begin{cases} T_{j_0}^j(\mu, l), & \text{if } \langle l, j, j_0 \rangle \leq N, \\ 0, & \text{if not.} \end{cases} \quad (2.11)$$

We also define the orthogonal projector $P_N^{2, \perp} T \triangleq T - P_N^2 T$. The next lemma lists some elementary results related to the off-diagonal and the isotropic norms.

Lemma 2.3. *Let $q \in \mathbb{N}$, $\mathbf{m} \in \mathbb{N}^*$, (γ, d, s_0, s) satisfy (2.2)-(2.3), T and S be Toeplitz in time operators.*

(i) *Frequency localization : Let $N \in \mathbb{N}^*$ and $\mathfrak{t} \in \mathbb{R}_+$. Then*

$$\|P_N^1 T\|_{\text{O-d}, s+\mathfrak{t}}^{q, \gamma, \mathbf{m}} \leq N^{\mathfrak{t}} \|T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}}, \quad \|P_N^{1, \perp} T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \leq N^{-\mathfrak{t}} \|T\|_{\text{O-d}, s+\mathfrak{t}}^{q, \gamma, \mathbf{m}}$$

and

$$\|P_N^2 T\|_{\text{I-D}, s+\mathfrak{t}}^{q, \gamma, \mathbf{m}} \leq N^{\mathfrak{t}} \|T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}}, \quad \|P_N^{2, \perp} T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} \leq N^{-\mathfrak{t}} \|T\|_{\text{I-D}, s+\mathfrak{t}}^{q, \gamma, \mathbf{m}}.$$

(ii) *Link with the classical operator norm :*

$$\begin{aligned} \|T\rho\|_s^{q, \gamma, \mathbf{m}} &\lesssim \|T\|_{\text{O-d}, s_0}^{q, \gamma, \mathbf{m}} \|\rho\|_s^{q, \gamma, \mathbf{m}} + \|T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0}^{q, \gamma, \mathbf{m}}, \\ \|T\rho\|_s^{q, \gamma, \mathbf{m}} &\lesssim \|T\|_{\text{I-D}, s_0}^{q, \gamma, \mathbf{m}} \|\rho\|_s^{q, \gamma, \mathbf{m}} + \|T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0}^{q, \gamma, \mathbf{m}}. \end{aligned}$$

In particular,

$$\begin{aligned} \|T\rho\|_s^{q, \gamma, \mathbf{m}} &\lesssim \|T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \|\rho\|_s^{q, \gamma, \mathbf{m}}, \\ \|T\rho\|_s^{q, \gamma, \mathbf{m}} &\lesssim \|T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} \|\rho\|_s^{q, \gamma, \mathbf{m}}. \end{aligned}$$

(iii) *We have the embedding : for any $s \geq 0$*

$$\|T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \lesssim \|T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}}.$$

(iv) *Composition law :*

$$\|TS\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \lesssim \|T\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}} \|S\|_{\text{O-d}, s_0}^{q, \gamma, \mathbf{m}} + \|T\|_{\text{O-d}, s_0}^{q, \gamma, \mathbf{m}} \|S\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}}$$

and

$$\|TS\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} + \|ST\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} \lesssim \|T\|_{\text{I-D}, s}^{q, \gamma, \mathbf{m}} \|S\|_{\text{O-d}, s_0}^{q, \gamma, \mathbf{m}} + \|T\|_{\text{I-D}, s_0}^{q, \gamma, \mathbf{m}} \|S\|_{\text{O-d}, s}^{q, \gamma, \mathbf{m}}.$$

Proof. (i) and (ii) can be easily obtained using (2.7)-(2.11) in a similar way to [19].

(iii) We shall prove the embedding for $q = 0$ and the case $q \geq 1$ is similar. We write by definition

$$|T_{j_0}^j(\mu, l)| \leq \langle l, j_0, j \rangle^{-s} \|T(\mu)\|_{\text{I-D}, s}.$$

Hence

$$\sup_{j_0 - j = m} |T_{j_0}^j(\mu, l)| \leq \sup_j \langle l, j + m, j \rangle^{-s} \|T(\mu)\|_{\text{I-D}, s}.$$

By direct computations we infer

$$\inf_{x \in \mathbb{R}} \langle l, x + m, x \rangle = \langle l, \frac{m}{2} \rangle$$

Therefore

$$\sup_{j_0-j=m} |T_{j_0}^j(\mu, l)| \lesssim \langle l, m \rangle^{-s} \|T(\mu)\|_{\text{I-D}, s}.$$

It follows that

$$\|T\|_{\text{O-d}, s} \lesssim \|T(\mu)\|_{\text{I-D}, s}.$$

(iv) We shall prove these tame estimates for $q = 0$. The general case $q \geq 1$ can be done in a similar way using Leibniz formula. One can check that

$$(TS)_{j_0}^j(l) = \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} T_{j_0}^{j_1}(l_1) S_{j_1}^j(l - l_1).$$

Hence for $s \geq 0$ and using the norm definition and the triangle inequality we infer

$$\begin{aligned} \langle l, j_0, j \rangle^s |(TS)_{j_0}^j(l)| &\lesssim \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l_1, j_0, j_1 \rangle^s |T_{j_0}^{j_1}(l_1) S_{j_1}^j(l - l_1)| \\ &+ \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l - l_1, j - j_1 \rangle^s |T_{j_0}^{j_1}(l_1) S_{j_1}^j(l - l_1)|. \end{aligned} \quad (2.12)$$

By definition we get

$$\langle l - l_1, j - j_1 \rangle^s |S_{j_1}^j(l - l_1)| \lesssim \|S\|_{\text{O-d}, s}. \quad (2.13)$$

Consequently,

$$\langle l, j_0, j \rangle^s |(TS)_{j_0}^j(l)| \lesssim \|T\|_{\text{I-D}, s} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |S_{j_1}^j(l - l_1)| + \|S\|_{\text{O-d}, s} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |T_{j_0}^{j_1}(l_1)|.$$

Using (2.13) we deduce for $s_0 > \frac{d+1}{2}$ that

$$\begin{aligned} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |S_{j_1}^j(l - l_1)| &\leq \|S\|_{\text{O-d}, s_0} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l - l_1, j - j_1 \rangle^{-s_0} \\ &\lesssim \|S\|_{\text{O-d}, s_0}. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |T_{j_0}^{j_1}(l_1)| &\leq \|T\|_{\text{I-D}, s_0} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l_1, j_1 \rangle^{-s_0} \\ &\lesssim \|T\|_{\text{I-D}, s_0}. \end{aligned} \quad (2.14)$$

Therefore, we obtain

$$\langle l, j_0, j \rangle^s |(TS)_{j_0}^j(l)| \lesssim \|T\|_{\text{I-D}, s} \|S\|_{\text{O-d}, s_0} + \|S\|_{\text{O-d}, s} \|T\|_{\text{I-D}, s_0},$$

leading to

$$\|TS\|_{\text{I-D}, s} \lesssim \|T\|_{\text{I-D}, s} \|S\|_{\text{O-d}, s_0} + \|T\|_{\text{I-D}, s_0} \|S\|_{\text{O-d}, s}.$$

Let us now move to the estimate of ST . Proceeding as for (2.12) we get

$$\begin{aligned} \langle l, j_0, j \rangle^s |(ST)_{j_0}^j(l)| &\lesssim \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l_1, j_0 - j_1 \rangle^s |S_{j_0}^{j_1}(l_1) T_{j_1}^j(l - l_1)| \\ &+ \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l - l_1, j_1, j \rangle^s |S_{j_0}^{j_1}(l_1) T_{j_1}^j(l - l_1)|. \end{aligned}$$

Applying (2.13) together with (2.14) yields

$$\begin{aligned} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l_1, j_0 - j_1 \rangle^s |S_{j_0}^{j_1}(l_1) T_{j_1}^j(l - l_1)| &\lesssim \|S\|_{\text{O-d},s} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |T_{j_1}^j(l - l_1)| \\ &\lesssim \|S\|_{\text{O-d},s} \|T\|_{\text{I-D},s_0}. \end{aligned}$$

Similarly we get

$$\begin{aligned} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l - l_1, j_1, j \rangle^s |S_{j_0}^{j_1}(l_1) T_{j_1}^j(l - l_1)| &\lesssim \|T\|_{\text{I-D},s} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} |S_{j_0}^{j_1}(l_1)| \\ &\lesssim \|T\|_{\text{I-D},s} \|S\|_{\text{O-d},s_0} \sum_{\substack{l_1 \in \mathbb{Z}^d \\ j_1 \in \mathbb{Z}_m}} \langle l_1, j_1 - j_0 \rangle^{-s_0} \\ &\lesssim \|T\|_{\text{I-D},s} \|S\|_{\text{O-d},s_0}. \end{aligned}$$

Putting together the preceding estimates we get

$$\|ST\|_{\text{I-D},s} \lesssim \|T\|_{\text{I-D},s} \|S\|_{\text{O-d},s_0} + \|T\|_{\text{I-D},s_0} \|S\|_{\text{O-d},s}.$$

This concludes the proof of the lemma. \square

2.2.3 Integral operators

The main goal in this part is to analyze Toeplitz integral operators and connect the different norms introduced before to the regularity of the kernel. Consider a Toeplitz integral operator taking the form

$$(\mathcal{T}_K \rho)(\mu, \varphi, \theta) \triangleq \int_{\mathbb{T}} K(\mu, \varphi, \theta, \eta) \rho(\mu, \varphi, \eta) d\eta, \quad (2.15)$$

where the kernel function $K(\mu, \varphi, \theta, \eta)$ may be smooth or singular at the diagonal set $\{\theta = \eta\}$. The kernel is called \mathfrak{m} -fold preserving if

$$(\mathcal{J}_{\mathfrak{m},2} K)(\mu, \varphi, \theta, \eta) \triangleq K\left(\mu, \varphi, \theta + \frac{2\pi}{\mathfrak{m}}, \eta + \frac{2\pi}{\mathfrak{m}}\right) = K(\mu, \varphi, \theta, \eta).$$

We shall need the following lemma whose proof is a consequence of [69, Lem. 4.4].

Lemma 2.4. *Let $q \in \mathbb{N}$, $\mathfrak{m} \in \mathbb{N}^*$, (γ, d, s_0, s) satisfy (2.2)-(2.3) and \mathcal{T}_K be an integral operator with a real-valued kernel K . Then the following assertions hold true.*

- If K is even in (φ, θ, η) , then \mathcal{T}_K is reversibility preserving.
- If K is odd in (φ, θ, η) , then \mathcal{T}_K is reversible.
- If K is \mathfrak{m} -fold preserving, then \mathcal{T}_K is \mathfrak{m} -fold preserving.

In addition,

$$\|\mathcal{T}_K\|_{\text{I-D},s}^{q,\gamma,\mathfrak{m}} \lesssim \|K\|_s^{q,\gamma,\mathfrak{m}}$$

and

$$\|\mathcal{T}_K \rho\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\rho\|_{s_0}^{q,\gamma,\mathfrak{m}} \|K\|_s^{q,\gamma,\mathfrak{m}} + \|\rho\|_s^{q,\gamma,\mathfrak{m}} \|K\|_{s_0}^{q,\gamma,\mathfrak{m}}.$$

Proof. The reversibility properties have already been proved in [69, Lem. 4.4]. Now, let us prove the \mathfrak{m} -fold property. Assume that K is \mathfrak{m} -fold preserving, then

$$\begin{aligned} \mathcal{T}_K(\mathcal{J}_{\mathfrak{m}} \rho)(\varphi, \theta) &= \int_{\mathbb{T}} K(\varphi, \theta, \eta) \rho(\varphi, \eta + \frac{2\pi}{\mathfrak{m}}) d\eta \\ &= \int_{\mathbb{T}} K(\varphi, \theta + \frac{2\pi}{\mathfrak{m}}, \eta + \frac{2\pi}{\mathfrak{m}}) \rho(\varphi, \eta + \frac{2\pi}{\mathfrak{m}}) d\eta \\ &= \int_{\mathbb{T}} K(\varphi, \theta + \frac{2\pi}{\mathfrak{m}}, \eta) \rho(\varphi, \eta) d\eta \\ &= \mathcal{J}_{\mathfrak{m}}(\mathcal{T}_K \rho)(\varphi, \theta). \end{aligned}$$

Hence \mathcal{T}_K is \mathfrak{m} -fold preserving. By duality $H_{\mathfrak{m}}^{-s} - H_{\mathfrak{m}}^s$, we have

$$\begin{aligned} \left| (\mathcal{T}_K)_j^{j'}(l) \right| &= \left| \int_{\mathbb{T}^{d+2}} K(\varphi, \theta, \eta) e^{i(l \cdot \varphi + j\theta - j'\eta)} d\varphi d\theta d\eta \right| \\ &\lesssim \langle l, j, j' \rangle^{-s} \|K\|_s^{q, \gamma, \mathfrak{m}} \end{aligned}$$

proving the first estimate. The second one follows easily from Lemma 2.3-(ii)-(iii). \square

The next task is to introduce some quasi-periodic symplectic change of variables needed later in the reduction of the transport part of the linearized operator. The following lemma is proved in the scalar case $d' = 1$ in [19, Lem. 2.34]. The vectorial case $d' \geq 2$ can be obtained in a similar way, up to slight modifications.

Lemma 2.5. *Let $q \geq 0$, $\mathfrak{m}, d, d' \geq 1$, $s \geq s_0 > \frac{d+d'}{2} + q + 1$ and $\beta_1, \dots, \beta_{d'} \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathfrak{m}}^{\infty}(\mathbb{T}^{d+1}))$ such that*

$$\max_{k \in \{1, \dots, d'\}} \|\beta_k\|_{2s_0}^{q, \gamma, \mathfrak{m}} \leq \varepsilon_0, \quad (2.16)$$

with ε_0 small enough. Then the following assertions hold true.

(i) The function $\widehat{\beta}$ defined by the inverse diffeomorphism

$$y = x + \beta(\mu, \varphi, x) \quad \Leftrightarrow \quad x = y + \widehat{\beta}(\mu, \varphi, y),$$

where

$$\begin{aligned} \beta(\mu, \varphi, x) &\triangleq (\beta_1(\mu, \varphi, x_1), \dots, \beta_{d'}(\mu, \varphi, x_{d'})), \quad x = (x_1, \dots, x_{d'}), \\ \widehat{\beta}(\mu, \varphi, y) &\triangleq (\widehat{\beta}_1(\mu, \varphi, y_1), \dots, \widehat{\beta}_{d'}(\mu, \varphi, y_{d'})), \quad y = (y_1, \dots, y_{d'}), \end{aligned} \quad (2.17)$$

satisfies

$$\forall s \geq s_0, \quad \|\widehat{\beta}\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|\beta\|_s^{q, \gamma, \mathfrak{m}}. \quad (2.18)$$

(ii) The composition operator $\mathcal{B} : W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathfrak{m}}^s(\mathbb{T}^{d+d'})) \rightarrow W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathfrak{m}}^s(\mathbb{T}^{d+d'}))$, defined by

$$\mathcal{B}\rho(\mu, \varphi, x) \triangleq \rho(\mu, \varphi, x + \beta(\mu, \varphi, x)), \quad (2.19)$$

is continuous and invertible, with inverse

$$\mathcal{B}^{-1}\rho(\mu, \varphi, y) = \rho(\mu, \varphi, y + \widehat{\beta}(\mu, \varphi, y)).$$

Moreover, we have the estimates

$$\begin{aligned} \|\mathcal{B}^{\pm 1}\rho\|_s^{q, \gamma, \mathfrak{m}} &\leq \|\rho\|_s^{q, \gamma, \mathfrak{m}} (1 + C\|\beta\|_{s_0}^{q, \gamma, \mathfrak{m}}) + C\|\beta\|_s^{q, \gamma, \mathfrak{m}} \|\rho\|_{s_0}^{q, \gamma, \mathfrak{m}}, \\ \|\mathcal{B}^{\pm 1}\rho - \rho\|_s^{q, \gamma, \mathfrak{m}} &\leq C(\|\rho\|_{s+1}^{q, \gamma, \mathfrak{m}} \|\beta\|_{s_0}^{q, \gamma, \mathfrak{m}} + \|\rho\|_{s_0}^{q, \gamma, \mathfrak{m}} \|\beta\|_s^{q, \gamma, \mathfrak{m}}). \end{aligned} \quad (2.20)$$

(iii) Let $\beta^{[1]}, \beta^{[2]} \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathfrak{m}}^{\infty}(\mathbb{T}^{d+d'}))$ as in (2.17) and satisfying (2.16). If we denote

$$\Delta_{12}\beta \triangleq \beta^{[1]} - \beta^{[2]} \quad \text{and} \quad \Delta_{12}\widehat{\beta} \triangleq \widehat{\beta}^{[1]} - \widehat{\beta}^{[2]},$$

then we have

$$\forall s \geq s_0, \quad \|\Delta_{12}\widehat{\beta}\|_s^{q, \gamma, \mathfrak{m}} \leq C \left(\|\Delta_{12}\beta\|_s^{q, \gamma, \mathfrak{m}} + \|\Delta_{12}\beta\|_{s_0}^{q, \gamma, \mathfrak{m}} \max_{\ell \in \{1, 2\}} \|\beta^{[\ell]}\|_{s+1}^{q, \gamma, \mathfrak{m}} \right). \quad (2.21)$$

Next, we gather several results related to the action of the transformation (2.19) on Toeplitz integral operators.

Lemma 2.6. *Let $q \in \mathbb{N}$, $m \in \mathbb{N}^*$ and (γ, d, s_0, s) satisfy (2.2)-(2.3). Consider a smooth m -fold preserving kernel*

$$K : (\mu, \varphi, \theta_1, \theta_2) \mapsto K(\mu, \varphi, \theta_1, \theta_2).$$

Let $\beta_k : \mathcal{O} \times \mathbb{T}^{d+1} \rightarrow \mathbb{T}$, $k \in \{1, 2\}$ be odd m -fold symmetric functions and subject to the smallness condition

$$\max_{k \in \{1, 2\}} \|\beta_k\|_{2s_0}^{q, \gamma, m} \leq \varepsilon_0. \quad (2.22)$$

Consider the quasi-periodic change of variables

$$\forall k \in \{1, 2\}, \quad \mathcal{B}_k \triangleq (1 + \partial_\theta \beta_k) \mathcal{B}_k, \quad \mathcal{B}_k \rho(\mu, \varphi, \theta) = \rho(\mu, \varphi, \theta + \beta_k(\mu, \varphi, \theta)),$$

Then the following assertions hold true.

(i) *The operator $\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2$ is m -fold preserving integral operator. Moreover, we have*

$$\|\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2\|_{\text{I-D}, s}^{q, \gamma, m} \lesssim \|K\|_s^{q, \gamma, m} + \|K\|_{s_0}^{q, \gamma, m} \max_{k \in \{1, 2\}} \|\beta_k\|_{s+1}^{q, \gamma, m}. \quad (2.23)$$

and

$$\|\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2 - \mathcal{T}_K\|_{\text{I-D}, s}^{q, \gamma, m} \lesssim \|K\|_{s+1}^{q, \gamma, m} \max_{k \in \{1, 2\}} \|\beta_k\|_{s_0}^{q, \gamma, m} + \|K\|_{s_0}^{q, \gamma, m} \max_{k \in \{1, 2\}} \|\beta_k\|_{s+1}^{q, \gamma, m}. \quad (2.24)$$

(ii) *If K is even in all the variables $(\varphi, \theta_1, \theta_2)$ (resp. odd), then $\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2$ is a reversibility preserving (resp. reversible) integral operator.*

(iii) *Given smooth functionals $r \in W^{q, \infty, \gamma}(\mathcal{O}, H_m^s \times H_m^s) \mapsto K(r), \beta_k(r)$, for $k \in \{1, 2\}$.*

Consider $r^{[\ell]} = (r_1^{[\ell]}, r_2^{[\ell]}) \in W^{q, \infty, \gamma}(\mathcal{O}, H_m^s \times H_m^s)$, $\ell \in \{1, 2\}$. We denote

$$\forall k \in \{1, 2\}, \quad f^{[k]} \triangleq f(r^{[k]}) \quad \text{and} \quad \Delta_{12} f \triangleq f^{[1]} - f^{[2]}.$$

Assume that there exists $\varepsilon_0 > 0$ small enough such that

$$\max_{(k, \ell) \in \{1, 2\}^2} \|\beta_k^{[\ell]}\|_{2s_0}^{q, \gamma, m} + \max_{\ell \in \{1, 2\}} \|K^{[\ell]}\|_{s_0+1}^{q, \gamma, m} \leq \varepsilon_0. \quad (2.25)$$

Then, the following estimate holds,

$$\begin{aligned} \|\Delta_{12} \mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2\|_{\text{I-D}, s}^{q, \gamma, m} &\lesssim \|\Delta_{12} K\|_s^{q, \gamma, m} + \|\Delta_{12} K\|_{s_0}^{q, \gamma, m} \max_{(k, \ell) \in \{1, 2\}^2} \|\beta_k^{[\ell]}\|_s^{q, \gamma, m} \\ &+ \left(\max_{\ell \in \{1, 2\}} \|K^{[\ell]}\|_{s+1}^{q, \gamma, m} + \max_{(k, \ell) \in \{1, 2\}^2} \|\beta_k^{[\ell]}\|_{s+1}^{q, \gamma, m} \right) \max_{k \in \{1, 2\}} \|\Delta_{12} \beta_k\|_{s_0+1}^{q, \gamma, m} \\ &+ \max_{k \in \{1, 2\}} \|\Delta_{12} \beta_k\|_{s+1}^{q, \gamma, m}. \end{aligned} \quad (2.26)$$

Proof. (i) Straightforward computations lead to

$$\mathcal{B}_1^{-1} \rho(\mu, \varphi, y_1) = \left(1 + \partial_y \widehat{\beta}_1(\mu, \varphi, y_1)\right) \rho(\mu, \varphi, y_1 + \widehat{\beta}_1(\mu, \varphi, y_1)). \quad (2.27)$$

Thus, the conjugation of the operator \mathcal{T}_K writes

$$\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2 \rho(\mu, \varphi, \theta_1) = \int_{\mathbb{T}} \rho(\mu, \varphi, \theta_2) \widehat{K}(\mu, \varphi, \theta_1, \theta_2) d\theta_2, \quad (2.28)$$

with

$$\widehat{K}(\mu, \varphi, \theta_1, \theta_2) \triangleq (1 + \partial_{\theta_1} \widehat{\beta}_1(\mu, \varphi, \theta_1)) (\mathcal{B}^{-1} K)(\mu, \varphi, \theta_1, \theta_2)$$

and

$$\mathcal{B}^{-1} K(\mu, \varphi, \theta_1, \theta_2) = K(\mu, \varphi, \theta_1 + \widehat{\beta}_1(\mu, \varphi, \theta_1), \theta_2 + \widehat{\beta}_2(\mu, \varphi, \theta_2)).$$

Using the product laws in Lemma 2.1, Lemma 2.5 and (2.22), we get

$$\|\widehat{K}\|_s^{q,\gamma,m} \lesssim \|K\|_s^{q,\gamma,m} + \|K\|_{s_0}^{q,\gamma,m} \max_{k \in \{1,2\}} \|\beta_k\|_{s+1}^{q,\gamma,m}. \quad (2.29)$$

Consequently, we obtain the estimate (2.23) by applying Lemma 2.4. As for the difference with the original operator, we can write

$$(\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2 - \mathcal{T}_K) \rho(\mu, \varphi, \theta_1) = \int_{\mathbb{T}} \rho(\mu, \varphi, \theta_2) \widetilde{K}(\mu, \varphi, \theta_1, \theta_2) d\theta_2,$$

with

$$\widetilde{K}(\mu, \varphi, \theta_1, \theta_2) \triangleq \partial_{\theta_1} \widehat{\beta}_1(\mu, \varphi, \theta_1) (\mathcal{B}^{-1} K)(\mu, \varphi, \theta_1, \theta_2) + [\mathcal{B}^{-1} K - K](\mu, \varphi, \theta_1, \theta_2).$$

Therefore, by the product laws in Lemma 2.1, together with (2.20) and (2.22), we infer

$$\|\widetilde{K}\|_s^{q,\gamma,m} \lesssim \|K\|_{s+1}^{q,\gamma,m} \max_{k \in \{1,2\}} \|\beta_k\|_{s_0}^{q,\gamma,m} + \|K\|_{s_0}^{q,\gamma,m} \max_{k \in \{1,2\}} \|\beta_k\|_{s+1}^{q,\gamma,m}.$$

Then, the estimate (2.24) follows by applying Lemma 2.4.

(ii) The symmetry properties follow immediately from Lemma 2.4 and the symmetry assumptions.

(iii) By definition and according to (2.28) we have

$$\Delta_{12}(\mathcal{B}_1^{-1} \mathcal{T}_K \mathcal{B}_2)(\rho)(\mu, \varphi, \theta_1) = \int_{\mathbb{T}} \rho(\mu, \varphi, \theta_2) \mathbb{K}(\mu, \varphi, \theta_1, \theta_2) d\theta_2,$$

with

$$\begin{aligned} \mathbb{K}(\mu, \varphi, \theta_1, \theta_2) &\triangleq (1 + \partial_{\theta_1} \widehat{\beta}_1^{[1]}(\mu, \varphi, \theta_1)) \mathcal{B}_{[1]}^{-1} K^{[1]}(\mu, \varphi, \theta_1, \theta_2) \\ &\quad - (1 + \partial_{\theta_1} \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1)) \mathcal{B}_{[2]}^{-1} K^{[2]}(\mu, \varphi, \theta_1, \theta_2) \end{aligned}$$

and

$$\mathcal{B}_{[\ell]}^{-1} f(\mu, \varphi, y) = f(\mu, \varphi, \theta_1 + \widehat{\beta}_1^{[\ell]}(\mu, \varphi, \theta_1), \theta_2 + \widehat{\beta}_2^{[\ell]}(\mu, \varphi, \theta_2)).$$

This can also be written as

$$\begin{aligned} \mathbb{K}(\mu, \varphi, \theta_1, \theta_2) &= \partial_{\theta_1} \Delta_{12} \widehat{\beta}_1(\mu, \varphi, \theta_1) \mathcal{B}_{[1]}^{-1} K^{[1]}(\mu, \varphi, \theta_1, \theta_2) \\ &\quad + (1 + \partial_{\theta_1} \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1)) \mathcal{B}_{[1]}^{-1} (\Delta_{12} K)(\mu, \varphi, \theta_1, \theta_2) \\ &\quad + (1 + \partial_{\theta_1} \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1)) \left[\mathcal{B}_{[1]}^{-1} K^{[2]}(\mu, \varphi, \theta_1, \theta_2) - \mathcal{B}_{[2]}^{-1} K^{[2]}(\mu, \varphi, \theta_1, \theta_2) \right]. \end{aligned}$$

Then, Taylor Formula implies

$$\begin{aligned} \mathbb{K}(\mu, \varphi, \theta_1, \theta_2) &= \partial_{\theta_1} \Delta_{12} \widehat{\beta}_1(\mu, \varphi, \theta_1) \mathcal{B}_{[1]}^{-1} K^{[1]}(\mu, \varphi, \theta_1, \theta_2) \\ &\quad + (1 + \partial_{\theta_1} \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1)) \mathcal{B}_{[1]}^{-1} (\Delta_{12} K)(\mu, \varphi, \theta_1, \theta_2) \\ &\quad + (1 + \partial_{\theta_1} \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1)) \left[\Delta_{12} \widehat{\beta}_1(\mu, \varphi, \theta_1) \int_0^1 (\mathcal{B}_{[1],[2]}^\tau (\partial_{\theta_1} K^{[2]}))(\mu, \varphi, \theta_1, \theta_2) d\tau \right. \\ &\quad \left. + \Delta_{12} \widehat{\beta}_2(\mu, \varphi, \theta_2) \int_0^1 (\widetilde{\mathcal{B}}_{[1],[2]}^\tau (\partial_{\theta_2} K^{[2]}))(\mu, \varphi, \theta_1, \theta_2) d\tau \right], \end{aligned}$$

where we have used the notations

$$\begin{aligned} \mathcal{B}_{[1],[2]}^\tau f(\mu, \varphi, \theta_1, \theta_2) &= f(\mu, \varphi, \theta_1 + \tau \widehat{\beta}_1^{[1]}(\mu, \varphi, \theta_1) + (1 - \tau) \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1), \theta_2 + \widehat{\beta}_2^{[2]}(\mu, \varphi, \theta_2)), \\ \widetilde{\mathcal{B}}_{[1],[2]}^\tau f(\mu, \varphi, \theta_1, \theta_2) &= f(\mu, \varphi, \theta_1 + \widehat{\beta}_1^{[2]}(\mu, \varphi, \theta_1), \theta_2 + \tau \widehat{\beta}_2^{[1]}(\mu, \varphi, \theta_2) + (1 - \tau) \widehat{\beta}_2^{[2]}(\mu, \varphi, \theta_2)). \end{aligned}$$

By product laws, (2.21), (2.29), (2.25), we obtain

$$\begin{aligned} \|\mathbb{K}\|_s^{q,\gamma,m} &\lesssim \|\Delta_{12} K\|_s^{q,\gamma,m} + \|\Delta_{12} K\|_{s_0}^{q,\gamma,m} \max_{(k,\ell) \in \{1,2\}^2} \|\beta_k^{[\ell]}\|_s^{q,\gamma,m} + \max_{k \in \{1,2\}} \|\Delta_{12} \beta_k\|_{s+1}^{q,\gamma,m} \\ &\quad + \left(\max_{\ell \in \{1,2\}} \|K^{[\ell]}\|_{s+1}^{q,\gamma,m} + \max_{(k,\ell) \in \{1,2\}^2} \|\beta_k^{[\ell]}\|_{s+1}^{q,\gamma,m} \right) \max_{k \in \{1,2\}} \|\Delta_{12} \beta_k\|_{s_0+1}^{q,\gamma,m}. \end{aligned}$$

Finally, combining this estimate with Lemma 2.4 we conclude (2.26). This achieves the proof of Lemma 2.6. \square

Now, we recall the following result stated in [19, Lemma 2.36] and dealing with the conjugation of the Hilbert transform with the quasi-periodic change of coordinates introduced in (2.27). Here, and along the paper, \mathcal{H} denotes the standard Hilbert transform on the periodic setting acting only on the variable $\theta \in \mathbb{T}$, namely,

$$\mathcal{H}(1) = 0, \quad \forall j \in \mathbb{Z}^*, \quad \mathcal{H}e_j = -i \operatorname{sgn}(j)e_j, \quad (2.30)$$

where sgn denotes the usual sign function.

Lemma 2.7. *Let $q \in \mathbb{N}$, $\mathfrak{m} \in \mathbb{N}^*$, (γ, d, s_0, s) satisfy (2.2)-(2.3) and $\beta \in W^{q, \infty, \gamma}(\mathcal{O}, H_{\mathfrak{m}}^\infty)$ odd in the variables (φ, θ) . There exists $\varepsilon_0 > 0$ such that, if $\|\beta\|_{2s_0}^{q, \gamma, \mathfrak{m}} \leq \varepsilon_0$, then*

$$(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})\rho(\mu, \varphi, \theta) = \int_{\mathbb{T}} K(\mu, \varphi, \theta, \eta)\rho(\mu, \varphi, \eta) d\eta,$$

defines a reversible and \mathfrak{m} -fold preserving integral operator with the estimates : for all $s \geq s_0$,

$$\|K\|_s^{q, \gamma, \mathfrak{m}} \leq C(s, q)\|\beta\|_{s+2}^{q, \gamma, \mathfrak{m}}, \quad \|\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}\|_{\text{L-D}, s}^{q, \gamma, \mathfrak{m}} \leq C(s, q)\|\beta\|_{s+2}^{q, \gamma, \mathfrak{m}}$$

and

$$\|\Delta_{12}(\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H})\|_{\text{L-D}, s}^{q, \gamma, \mathfrak{m}} \leq C(s, q) \left[\|\Delta_{12}\beta\|_{s+2}^{q, \gamma, \mathfrak{m}} + \|\Delta_{12}\beta\|_{s_0+1}^{q, \gamma, \mathfrak{m}} \max_{\ell \in \{1, 2\}} \|\beta^{[\ell]}\|_{s+3}^{q, \gamma, \mathfrak{m}} \right].$$

Proof. We shall use the following classical integral representation of the Hilbert transform

$$\mathcal{H}(\rho)(\theta) = \int_{\mathbb{T}} \rho(\eta) \cot\left(\frac{\theta - \eta}{2}\right) d\eta,$$

where this integral is understood in the principal value sense. Therefore, we have

$$K(\mu, \varphi, \theta, \eta) = (1 + \partial_\theta \widehat{\beta}(\mu, \varphi, \theta)) \cot\left(\frac{\theta - \eta + \widehat{\beta}(\mu, \varphi, \theta) - \widehat{\beta}(\mu, \varphi, \eta)}{2}\right) - \cot\left(\frac{\theta - \eta}{2}\right).$$

One can easily check that

$$K(\mu, \varphi, \theta, \eta) = 2\partial_\theta \left[\log \left(\frac{\sin\left(\frac{\theta - \eta + \widehat{\beta}(\mu, \varphi, \theta) - \widehat{\beta}(\mu, \varphi, \eta)}{2}\right)}{\sin\left(\frac{\theta - \eta}{2}\right)} \right) \right].$$

This can also be written as

$$K(\mu, \varphi, \theta, \eta) = 2\partial_\theta \left[\log(1 + g(\mu, \varphi, \theta, \eta)) \right],$$

where

$$g(\mu, \varphi, \theta, \eta) = \cos\left(\frac{\widehat{\beta}(\mu, \varphi, \theta) - \widehat{\beta}(\mu, \varphi, \eta)}{2}\right) - 1 + \cos\left(\frac{\theta - \eta}{2}\right) \frac{\sin\left(\frac{\widehat{\beta}(\mu, \varphi, \theta) - \widehat{\beta}(\mu, \varphi, \eta)}{2}\right)}{\sin\left(\frac{\theta - \eta}{2}\right)}.$$

The symmetry assumptions on β (and thus $\widehat{\beta}$) implies

$$g(\mu, \varphi, \theta + \frac{2\pi}{\mathfrak{m}}, \eta + \frac{2\pi}{\mathfrak{m}}) = g(\mu, \varphi, \theta, \eta) = g(\mu, -\varphi, -\theta, -\eta),$$

that is

$$K(\mu, \varphi, \theta + \frac{2\pi}{\mathfrak{m}}, \eta + \frac{2\pi}{\mathfrak{m}}) = K(\mu, \varphi, \theta, \eta) = -K(\mu, -\varphi, -\theta, -\eta),$$

The Lemma 2.4 implies that $\mathcal{B}^{-1}\mathcal{H}\mathcal{B} - \mathcal{H}$ is a reversible and \mathfrak{m} -fold preserving integral operator. Using composition laws in Lemma 2.1, Lemma 2.2 and (2.18), we get

$$\|g\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|\beta\|_{s+1}^{q, \gamma, \mathfrak{m}}.$$

Hence, still by composition laws, we infer

$$\|K\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\beta\|_{s+2}^{q,\gamma,\mathfrak{m}}$$

and we conclude by applying Lemma 2.4. As for the difference, we have

$$\|\Delta_{12}K\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\Delta_{12}g\|_{s+1}^{q,\gamma,\mathfrak{m}}.$$

We set

$$h(\mu, \varphi, \theta, \eta) \triangleq \frac{\widehat{\beta}(\mu, \varphi, \theta) - \widehat{\beta}(\mu, \varphi, \eta)}{2}.$$

Then, using Taylor formula, we can write

$$\begin{aligned} \Delta_{12}g(\mu, \varphi, \theta, \eta) &= -\Delta_{12}h(\mu, \varphi, \theta, \eta) \int_0^1 \sin(h^{[2]}(\mu, \varphi, \theta, \eta) + t\Delta_{12}h(\mu, \varphi, \theta, \eta)) dt \\ &\quad + \cos\left(\frac{\theta-\eta}{2}\right) \frac{\Delta_{12}h(\mu, \varphi, \theta, \eta)}{\sin\left(\frac{\theta-\eta}{2}\right)} \int_0^1 \cos(h^{[2]}(\mu, \varphi, \theta, \eta) + t\Delta_{12}h(\mu, \varphi, \theta, \eta)) dt. \end{aligned}$$

From the identity $\Delta_{12}h(\mu, \varphi, \theta, \eta) = \Delta_{12}\widehat{\beta}(\mu, \varphi, \theta) - \Delta_{12}\widehat{\beta}(\mu, \varphi, \eta)$ together with the product/composition laws combined with Lemma 2.2 and (2.21), we get

$$\|\Delta_{12}g\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\Delta_{12}\beta\|_{s+1}^{q,\gamma,\mathfrak{m}} + \|\Delta_{12}\beta\|_{s_0+1}^{q,\gamma,\mathfrak{m}} \max_{\ell \in \{1,2\}} \|\beta^{[\ell]}\|_{s+2}^{q,\gamma,\mathfrak{m}}.$$

Again we conclude by invoking Lemma 2.4. \square

The following lemma deals with the kernel structure of iterative operators that will be useful later.

Lemma 2.8. *Let $q \in \mathbb{N}$, $\mathfrak{m} \in \mathbb{N}^*$, $n \in \mathbb{N}^*$ and (γ, d, s_0, s) satisfy (2.2)-(2.3) and consider a family of \mathfrak{m} -fold preserving kernel operators $(\mathcal{T}_{K_i})_{i=1}^n$ as in (2.15). Then there exists a kernel K such that*

$$\prod_{i=1}^n \mathcal{T}_{K_i} = \mathcal{T}_K, \quad \text{with} \quad \|K\|_s^{q,\gamma,\mathfrak{m}} \leq C \sum_{i=1}^n \|K_i\|_s^{q,\gamma,\mathfrak{m}} \prod_{j \neq i} \|K_j\|_{s_0}^{q,\gamma,\mathfrak{m}}.$$

In addition, if for some i_0 we have $K_{i_0}(\varphi, \theta, \eta) = f(\varphi, \theta)\delta(\theta - \eta)$, with $f(\varphi, \theta + \frac{2\pi}{\mathfrak{m}}) = f(\varphi, \theta)$, then

$$\|K\|_s^{q,\gamma,\mathfrak{m}} \leq C \|f\|_s^{q,\gamma,\mathfrak{m}} \prod_{i \neq i_0} \|K_i\|_{s_0}^{q,\gamma,\mathfrak{m}} + C \|f\|_{s_0}^{q,\gamma,\mathfrak{m}} \sum_{\substack{i=1, \\ i \neq i_0}}^n \|K_i\|_s^{q,\gamma,\mathfrak{m}} \prod_{j \neq i, i_0} \|K_j\|_{s_0}^{q,\gamma,\mathfrak{m}}.$$

Proof. The kernel K is explicit and takes the form,

$$K(\varphi, \theta, \eta) = \int_{\mathbb{T}^{n-1}} \prod_{i=1}^n K_i(\varphi, \eta_{i-1}, \eta_i) \prod_{i=1}^{n-1} d\eta_i,$$

with the convention $\eta_0 = \theta$ and $\eta_n = \eta$. The \mathfrak{m} -fold preserving property of K is inherited from the one of the K_i . Thus, to get the first result it suffices to use the products law in Lemma 2.1. In the second case, the kernel takes the form

$$K(\varphi, \theta, \eta) = \int_{\mathbb{T}^{n-2}} f(\varphi, \theta, \eta_{i_0}) K_{i_0-1}(\varphi, \eta_{i_0-2}, \eta_{i_0}) \prod_{\substack{i=1 \\ i \neq i_0, i_0-1}}^n K_i(\varphi, \eta_{i-1}, \eta_i) \prod_{\substack{i=1 \\ i \neq i_0-1}}^{n-1} d\eta_i$$

and the desired estimate follows once again from the products law detailed in Lemma 2.1. \square

2.2.4 Matrix operators

For further purposes related to the reduction of the remainder in the transport linear parts subject to a vectorial structure, we need to introduce 2×2 matrices of scalar operators taking the form

$$\mathbf{T} = \begin{pmatrix} T_1 & T_3 \\ T_4 & T_2 \end{pmatrix}, \quad (2.31)$$

acting on the product Hilbert space $\mathbf{H}_m^s(\mathbb{T}^{d+1}, \mathbb{C})$, defined in (2.5). Notice that we shall restrict our discussion to the case where all the $T_i : \mathcal{O} \rightarrow \mathcal{L}(H_m^s(\mathbb{T}^{d+1}, \mathbb{C}))$ are m -fold preserving Toeplitz kernel operators as in (2.15). The matrix operator \mathbf{T} is said to be real (resp. m -fold preserving, reversible, reversibility-preserving) if all the entries T_i enjoy this property. The diagonal part $[\mathbf{T}]$ of \mathbf{T} is defined as follows,

$$[\mathbf{T}] \triangleq \begin{pmatrix} [T_1] & 0 \\ 0 & [T_2] \end{pmatrix}, \quad (2.32)$$

where for any scalar operator T , the notation $[T]$ is its diagonal part defined by

$$\forall (l_0, j_0) \in \mathbb{Z}^d \times \mathbb{Z}_m, \quad [T] \mathbf{e}_{l_0, j_0} \triangleq T_{l_0, j_0}^{l_0, j_0} \mathbf{e}_{l_0, j_0} = \langle T \mathbf{e}_{l_0, j_0}, \mathbf{e}_{l_0, j_0} \rangle_{L^2(\mathbb{T}^{d+1}, \mathbb{C})} \mathbf{e}_{l_0, j_0}. \quad (2.33)$$

The next goal is to equip the class of matrix operators with the following hybrid norm

$$\|\|\| \mathbf{T} \|\|_s^{q, \gamma, m} \triangleq \|T_1\|_{\mathcal{O}-d, s}^{q, \gamma, m} + \|T_2\|_{\mathcal{O}-d, s}^{q, \gamma, m} + \|T_3\|_{\mathcal{I}-D, s}^{q, \gamma, m} + \|T_4\|_{\mathcal{I}-D, s}^{q, \gamma, m}. \quad (2.34)$$

The choice of this norm will be motivated later in the remainder reduction performed in Section 7.4. Actually, the off-diagonal norm used to measure the diagonal terms T_1 and T_2 is compatible with the scalar case as in the papers [17, 59, 61, 69, 92]. However the isotropic norm used to measure the anti-diagonal terms T_3 and T_4 is compatible with the smoothing effects of the operators and it is introduced to remedy to a new space resonance phenomenon in the second order Melnikov condition due to the interaction between the diagonal eigenvalues. The cut-off projectors $(\mathbf{P}_N)_{N \in \mathbb{N}}$ are defined as follows

$$\mathbf{P}_N \mathbf{T} \triangleq \begin{pmatrix} P_N^1 T_1 & P_N^2 T_3 \\ P_N^2 T_4 & P_N^1 T_2 \end{pmatrix} \quad \text{and} \quad \mathbf{P}_N^\perp \mathbf{T} \triangleq \begin{pmatrix} P_N^{1, \perp} T_1 & P_N^{2, \perp} T_3 \\ P_N^{2, \perp} T_4 & P_N^{1, \perp} T_2 \end{pmatrix}, \quad (2.35)$$

where P_N^1 is defined in (2.8) and P_N^2 in (2.10). We shall prove the following result.

Corollary 2.1. *Let $q \in \mathbb{N}$, $m \in \mathbb{N}^*$, (γ, d, s_0, s) satisfy (2.2)-(2.3) and \mathbf{T}, \mathbf{S} two matrix operators as in (2.31), then the following assertions hold true.*

(i) *Projector property : for any $\mathfrak{t} \geq 0$*

$$\|\|\| \mathbf{P}_N \mathbf{T} \|\|_{s+\mathfrak{t}}^{q, \gamma, m} \leq N^{\mathfrak{t}} \|\|\| \mathbf{T} \|\|_s^{q, \gamma, m} \quad \text{and} \quad \|\|\| \mathbf{P}_N^\perp \mathbf{T} \|\|_s^{q, \gamma, m} \leq N^{-\mathfrak{t}} \|\|\| \mathbf{T} \|\|_{s+\mathfrak{t}}^{q, \gamma, m}.$$

(ii) *Composition law :*

$$\|\|\| \mathbf{TS} \|\|_s^{q, \gamma, m} \lesssim \|\|\| \mathbf{T} \|\|_s^{q, \gamma, m} \|\|\| \mathbf{S} \|\|_{s_0}^{q, \gamma, m} + \|\|\| \mathbf{T} \|\|_{s_0}^{q, \gamma, m} \|\|\| \mathbf{S} \|\|_s^{q, \gamma, m}.$$

(iii) *Link with the operator norm : for $\rho = (\rho_1, \rho_2) \in \mathbf{H}_m^s$,*

$$\|\|\| \mathbf{T} \rho \|\|_s^{q, \gamma, m} \lesssim \|\|\| \mathbf{T} \|\|_{s_0}^{q, \gamma, m} \|\rho\|_s^{q, \gamma, m} + \|\|\| \mathbf{T} \|\|_s^{q, \gamma, m} \|\rho\|_{s_0}^{q, \gamma, m}.$$

In particular,

$$\|\|\| \mathbf{T} \rho \|\|_s^{q, \gamma, m} \lesssim \|\|\| \mathbf{T} \|\|_s^{q, \gamma, m} \|\rho\|_s^{q, \gamma, m}.$$

Proof. (i) It follows immediately from (2.35), (2.34) and Lemma 2.3-(i).

(ii) One has

$$\mathbf{TS} = \begin{pmatrix} T_1 S_1 + T_3 S_4 & T_1 S_3 + T_3 S_2 \\ T_4 S_1 + T_2 S_4 & T_2 S_2 + T_4 S_3 \end{pmatrix} \triangleq \begin{pmatrix} R_1 & R_3 \\ R_4 & R_2 \end{pmatrix}.$$

Let us estimate R_1 . One has from the law products detailed in Lemma 2.3-(iv)

$$\begin{aligned} \|R_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} &\lesssim \|T_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} \|S_1\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} + \|T_1\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} \|S_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} \\ &\quad + \|T_3\|_{\mathcal{O}-d,s}^{q,\gamma,m} \|S_4\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} + \|T_3\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} \|S_4\|_{\mathcal{O}-d,s}^{q,\gamma,m}. \end{aligned}$$

Then using the embedding estimate in Lemma 2.3-(iii) together with (2.34), we get

$$\|R_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} \lesssim \|\mathbf{T}\|_s^{q,\gamma,m} \|\mathbf{S}\|_{s_0}^{q,\gamma,m} + \|\mathbf{T}\|_{s_0}^{q,\gamma,m} \|\mathbf{S}\|_s^{q,\gamma,m}.$$

Let us now estimate R_3 . Using Lemma 2.3-(iv) and (2.34), we infer

$$\begin{aligned} \|R_3\|_{\mathcal{I}-D,s}^{q,\gamma,m} &\lesssim \|S_3\|_{\mathcal{I}-D,s}^{q,\gamma,m} \|T_1\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} + \|S_3\|_{\mathcal{I}-D,s_0}^{q,\gamma,m} \|T_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} \\ &\quad + \|T_3\|_{\mathcal{I}-D,s}^{q,\gamma,m} \|S_1\|_{\mathcal{O}-d,s_0}^{q,\gamma,m} + \|T_3\|_{\mathcal{I}-D,s_0}^{q,\gamma,m} \|S_1\|_{\mathcal{O}-d,s}^{q,\gamma,m} \\ &\lesssim \|\mathbf{T}\|_s^{q,\gamma,m} \|\mathbf{S}\|_{s_0}^{q,\gamma,m} + \|\mathbf{T}\|_{s_0}^{q,\gamma,m} \|\mathbf{S}\|_s^{q,\gamma,m}. \end{aligned}$$

The terms R_2 and R_4 can be treated in a similar way.

(iii) This point is a direct consequence of (2.34) and Lemma 2.3-(ii). \square

3 Hamiltonian reformulation

Here, we describe the contour dynamics by using polar parametrization for the two interfaces of the patch near the annulus. We end up with a system of coupled nonlinear and nonlocal transport equations satisfied by the radial deformations and that can be recast as a Hamiltonian system. This structure is crucial to establish quasi-periodic solutions near the stationary annulus patch.

3.1 Transport system for radial deformations

Let $D_0 = D_1 \setminus \overline{D_2}$ be a doubly-connected domain where D_1 and D_2 are two simply-connected domains with $\overline{D_2}$ strictly embedded in D_1 . Consider the initial datum $\omega_0 = \mathbf{1}_{D_0}$, then the Yudovich solution takes for any $t \geq 0$ the form $\omega(t) = \mathbf{1}_{D_t}$, with $D_t = D_{t,1} \setminus \overline{D_{t,2}}$ a doubly-connected domain. In addition, $D_{t,1}$ and $D_{t,2}$ are two simply-connected domains with $\overline{D_{t,2}}$ strictly embedded in $D_{t,1}$. For fixed $b \in (0, 1)$, we start with a domain D_0 close to the annulus A_b , defined in (1.5), then for a short interval of time $[0, T]$, the domain D_t will be localized around the same annulus. Therefore, we may use on this time interval the following symplectic polar parametrization of the boundary. For $k \in \{1, 2\}$,

$$\begin{aligned} z_k(t) : \mathbb{T} &\mapsto \partial D_{t,k} & \text{where} & & w_k(t, \theta) &\triangleq (b_k^2 + 2r_k(t, \theta))^{\frac{1}{2}} e^{i\theta}, \\ \theta &\mapsto e^{-i\Omega t} w_k(t, \theta), & & & b_1 &\triangleq 1, \quad b_2 \triangleq b. \end{aligned} \quad (3.1)$$

Similarly to [17, 59, 69], the introduction of the angular velocity $\Omega > 0$ is due to some technical issues and devised to circumvent the trivial resonances associated to the eigen-mode $n = 1$ and used in the current configuration to remedy to a more delicate phenomenon related to the analytic accumulation of a sequence of eigenvalues in the vectorial case, see Lemma 4.2. The radial deformations r_1 and r_2 are assumed to be small, namely

$$|r_1(t, \theta)| + |r_2(t, \theta)| \ll 1.$$

In the sequel, for more convenience, we denote

$$\forall k \in \{1, 2\}, \quad R_k(t, \theta) \triangleq (b_k^2 + 2r_k(t, \theta))^{\frac{1}{2}}. \quad (3.2)$$

Remind that in this particular case the stream function defined through (1.2) takes the form

$$\psi(t, z) = \frac{1}{2\pi} \int_{D_{t,1}} \log(|z - \xi|) dA(\xi) - \frac{1}{2\pi} \int_{D_{t,2}} \log(|z - \xi|) dA(\xi). \quad (3.3)$$

The vortex patch equation (1.3) provides a system of coupled transport-type PDE satisfied by r_1 and r_2 . This is described by the following lemma.

Lemma 3.1. *For short time $T > 0$, the radial deformations r_1 and r_2 defined through (3.1) satisfy the following nonlinear coupled system: for all $k \in \{1, 2\}$, $(t, \theta) \in [0, T] \times \mathbb{T}$,*

$$\partial_t r_k(t, \theta) + \Omega \partial_\theta r_k(t, \theta) = -\partial_\theta \left[\psi(t, z_k(t, \theta)) \right] \quad (3.4)$$

$$= (-1)^{k+1} F_{k,k}[r](t, \theta) + (-1)^k F_{k,3-k}[r](t, \theta), \quad (3.5)$$

where $r = (r_1, r_2)$ and, for all $k, n \in \{1, 2\}$,

$$F_{k,n}[r](t, \theta) \triangleq \int_{\mathbb{T}} \log(A_{k,n}(t, \theta, \eta)) \partial_{\theta\eta}^2 \left(R_k(t, \theta) R_n(t, \eta) \sin(\eta - \theta) \right) d\eta, \quad (3.6)$$

$$A_{k,n}(t, \theta, \eta) \triangleq |R_k(t, \theta) e^{i\theta} - R_n(t, \eta) e^{i\eta}|, \quad (3.7)$$

where we have used the notation (2.4).

Proof. For $k \in \{1, 2\}$, we denote by $\mathbf{n}_k(t, \cdot)$ an inward normal vector to the boundary $\partial D_{t,k}$ of the patch. According to [68, p. 174], the vortex patch equation writes

$$\forall k \in \{1, 2\}, \quad \partial_t z_k(t, \theta) \cdot \mathbf{n}_k(t, z_k(t, \theta)) = \partial_\theta \left[\psi(t, z_k(t, \theta)) \right].$$

Identifying \mathbb{C} with \mathbb{R}^2 and making the choice $\mathbf{n}_k(t, z_k(t, \theta)) = i \partial_\theta z_k(t, \theta)$ we get from (3.1)

$$\begin{aligned} \partial_t z_k(t, \theta) \cdot \mathbf{n}_k(t, z_k(t, \theta)) &= \text{Im} \left(\partial_t z_k(t, \theta) \overline{\partial_\theta z_k(t, \theta)} \right) \\ &= -\partial_t r_k(t, \theta) - \Omega \partial_\theta r_k(t, \theta). \end{aligned}$$

Combining the last two identities we obtain (3.4). Next, we intend to use Stokes theorem in order to transform the integral (3.3) into an integration on the boundary. This theorem can be recast in the complex form,

$$2i \int_D \partial_{\bar{\xi}} f(\xi, \bar{\xi}) dA(\xi) = \int_{\partial D} f(\xi, \bar{\xi}) d\xi, \quad (3.8)$$

where $f : \bar{D} \rightarrow \mathbb{C}$ is a function of class C^1 , D is a simply-connected bounded domain and ∂D is the boundary of D . To make the argument rigorous, we shall mollify the logarithmic kernel by setting,

$$\epsilon > 0, \quad f_\epsilon(\xi, \bar{\xi}) \triangleq (\bar{\xi} - \bar{z}) \left[\log(|z - \xi|^2 + \epsilon) - 1 \right].$$

Then, we have

$$\partial_{\bar{\xi}} f_\epsilon(\xi, \bar{\xi}) = \log(|z - \xi|^2 + \epsilon) - \frac{\epsilon}{|z - \xi|^2 + \epsilon}.$$

Applying (3.8) yields

$$2i \int_{D_{t,k}} \log(|z - \xi|^2 + \epsilon) dA(\xi) - 2i \int_{D_{t,k}} \frac{\epsilon}{|z - \xi|^2 + \epsilon} dA(\xi) = \int_{\partial D_{t,k}} f_\epsilon(\xi, \bar{\xi}) d\xi$$

and taking the limit $\epsilon \rightarrow 0$ together with (3.3) allow to get

$$\psi(t, z) = \frac{1}{8i\pi} \int_{\partial D_{t,1}} (\bar{\xi} - \bar{z}) \left[\log(|\xi - z|^2) - 1 \right] d\xi - \frac{1}{8i\pi} \int_{\partial D_{t,2}} (\bar{\xi} - \bar{z}) \left[\log(|\xi - z|^2) - 1 \right] d\xi.$$

Parametrizing the boundaries with (3.1) and using the notation (2.4) we infer

$$\begin{aligned} \psi(t, z) &= \frac{1}{4i} \int_{\mathbb{T}} (\bar{z}_1(t, \eta) - \bar{z}) \left[\log(|z_1(t, \eta) - z|^2) - 1 \right] \partial_\eta z_1(t, \eta) d\eta \\ &\quad - \frac{1}{4i} \int_{\mathbb{T}} (\bar{z}_2(t, \eta) - \bar{z}) \left[\log(|z_2(t, \eta) - z|^2) - 1 \right] \partial_\eta z_2(t, \eta) d\eta. \end{aligned} \quad (3.9)$$

As a consequence, we get by differentiating inside the integral

$$\partial_{\bar{z}}\psi(t, z) = -\frac{1}{4i} \int_{\mathbb{T}} \log(|z_1(t, \eta) - z|^2) \partial_{\eta} z_1(t, \eta) d\eta + \frac{1}{4i} \int_{\mathbb{T}} \log(|z_2(t, \eta) - z|^2) \partial_{\eta} z_2(t, \eta) d\eta.$$

Therefore we find through elementary computations

$$\begin{aligned} \partial_{\theta}[\psi(t, z_k(t, \theta))] &= 2\operatorname{Re}\left((\partial_{\bar{z}}\psi)(t, z_k(t, \theta))\partial_{\theta}\bar{z}_k(t, \theta)\right) \\ &= -\int_{\mathbb{T}} \log(|z_k(t, \theta) - z_1(t, \eta)|) \partial_{\theta\eta}^2 \operatorname{Im}(z_1(t, \eta)\bar{z}_k(t, \theta)) d\eta \\ &\quad + \int_{\mathbb{T}} \log(|z_k(t, \theta) - z_2(t, \eta)|) \partial_{\theta\eta}^2 \operatorname{Im}(z_2(t, \eta)\bar{z}_k(t, \theta)) d\eta. \end{aligned} \quad (3.10)$$

Using (3.1) we obtain

$$\begin{aligned} \forall k, n \in \{1, 2\}, \quad \operatorname{Im}(z_n(t, \eta)\bar{z}_k(t, \theta)) &= R_n(t, \eta)R_k(t, \theta) \sin(\eta - \theta), \\ |z_k(t, \theta) - z_n(t, \eta)| &= |R_k(t, \theta)e^{i\theta} - R_n(t, \eta)e^{i\eta}|. \end{aligned}$$

By combining the last two identities with (3.4)-(3.10) we conclude the proof of Lemma 3.1. \square

3.2 Hamiltonian structure

The main purpose is to explore the Hamiltonian structure beyond the equations described in Lemma 3.1. First, the kinetic energy associated to the vortex patch $\omega(t, \cdot) = \mathbf{1}_{D_t} = \mathbf{1}_{D_{t,1} \setminus \overline{D_{t,2}}}$ is given by

$$\begin{aligned} E(r) &\triangleq \frac{1}{2\pi} \int_{D_t} \psi(t, z) dA(z) \\ &= \frac{1}{2\pi} \int_{D_{t,1}} \psi(t, z) dA(z) - \frac{1}{2\pi} \int_{D_{t,2}} \psi(t, z) dA(z) \end{aligned} \quad (3.11)$$

and its angular impulse is defined by

$$\begin{aligned} J(r) &\triangleq \frac{1}{2\pi} \int_{D_t} |z|^2 dA(z) \\ &= \frac{1}{2\pi} \int_{D_{t,1}} |z|^2 dA(z) - \frac{1}{2\pi} \int_{D_{t,2}} |z|^2 dA(z), \end{aligned} \quad (3.12)$$

where the stream function ψ is defined in (3.3). The main result of this section reads as follows.

Proposition 3.1. *The system (3.5) is Hamiltonian and takes the form*

$$\partial_t r = \mathcal{J} \nabla H(r) \quad (3.13)$$

where $r \triangleq (r_1, r_2)$,

$$\mathcal{J} \triangleq \begin{pmatrix} \partial_{\theta} & 0 \\ 0 & -\partial_{\theta} \end{pmatrix} \quad (3.14)$$

and ∇ is the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ -gradient and the hamiltonian H is defined by

$$H(r) \triangleq -\frac{1}{2}(E(r) + \Omega J(r)), \quad (3.15)$$

where E and J are defined in (3.11) and (3.12).

Proof. We shall first compute the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ gradient of the angular impulse J . For this aim, we need to write its expression in terms of r . Using (3.8) combined with (3.12) and (3.1) yields

$$\begin{aligned} J(r) &= \frac{1}{8\pi i} \int_{\partial D_{t,1}} |z|^2 \bar{z} dz - \frac{1}{8\pi i} \int_{\partial D_{t,2}} |z|^2 \bar{z} dz \\ &= \frac{1}{4} \int_{\mathbb{T}} (1 + 2r_1(t, \theta))^2 d\theta - \frac{1}{4} \int_{\mathbb{T}} (b^2 + 2r_2(t, \theta))^2 d\theta. \end{aligned}$$

Differentiating in $r = (r_1, r_2)$ one gets for $\rho = (\rho_1, \rho_2) \in L^2(\mathbb{T}) \times L^2(\mathbb{T})$,

$$\langle \nabla J(r), \rho \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} = \int_{\mathbb{T}} (1 + 2r_1(t, \theta)) \rho_1(\theta) d\theta - \int_{\mathbb{T}} (b^2 + 2r_2(t, \theta)) \rho_2(\theta) d\theta.$$

This implies that

$$\nabla J(r) = \begin{pmatrix} 1 + 2r_1 \\ -b^2 - 2r_2 \end{pmatrix} \quad \text{and} \quad \frac{1}{2} \Omega \mathcal{J} \nabla J(r) = \Omega \partial_{\theta} r. \quad (3.16)$$

The next task is to compute the $L^2(\mathbb{T}) \times L^2(\mathbb{T})$ gradient of the kinetic energy E defined in (3.11). Combining (3.3) with (3.1) and changing ξ by $e^{-it\Omega} \xi$ we find

$$\psi(t, e^{-it\Omega} z) = \frac{1}{2\pi} \int_{\tilde{D}_{t,1}} \log(|z - \xi|) dA(\xi) - \frac{1}{2\pi} \int_{\tilde{D}_{t,2}} \log(|z - \xi|) dA(\xi),$$

where $\tilde{D}_{t,k}$ are the domains with boundaries parametrized by

$$\begin{aligned} w_k : \mathbb{T} &\mapsto \partial \tilde{D}_{t,k} \\ \theta &\mapsto R_k(t, \theta) e^{i\theta}, \quad R_k(t, \theta) = (b_k^2 + 2r_k(t, \theta))^{\frac{1}{2}}. \end{aligned}$$

Using polar change of coordinates allows to get after straightforward computations,

$$\psi(t, e^{-it\Omega} z) = \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(z, \ell_2 e^{i\eta}) \ell_2 d\ell_2 d\eta, \quad G(z, \xi) \triangleq \log(|z - \xi|). \quad (3.17)$$

Coming back to (3.11) and using once again polar change of coordinates gives

$$E(r) = \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \theta)}^{R_1(t, \theta)} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(\ell_1 e^{i\theta}, \ell_2 e^{i\eta}) \ell_1 \ell_2 d\ell_1 d\ell_2 d\theta d\eta.$$

Therefore, the Gâteaux derivative of E in a given direction $\rho = (\rho_1, \rho_2)$ takes the form

$$\begin{aligned} \frac{dE(r)\rho}{dr} &= \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(R_1(t, \theta) e^{i\theta}, \ell_2 e^{i\eta}) \rho_1(\theta) \ell_2 d\ell_2 d\theta d\eta \\ &\quad + \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \theta)}^{R_1(t, \theta)} G(\ell_1 e^{i\theta}, R_1(t, \eta) e^{i\eta}) \rho_1(\eta) \ell_1 d\ell_1 d\theta d\eta \\ &\quad - \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(R_2(t, \theta) e^{i\theta}, \ell_2 e^{i\eta}) \rho_2(\theta) \ell_2 d\ell_2 d\theta d\eta \\ &\quad - \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \theta)}^{R_1(t, \theta)} G(\ell_1 e^{i\theta}, R_2(t, \eta) e^{i\eta}) \rho_2(\eta) \ell_1 d\ell_1 d\theta d\eta. \end{aligned}$$

Since $G(z, \xi)$ is symmetric in (z, ξ) then we obtain, by exchanging $\theta \leftrightarrow \eta$ if necessary,

$$\begin{aligned} \frac{dE(r)\rho}{dr} &= 2 \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(R_1(t, \theta) e^{i\theta}, \ell_2 e^{i\eta}) \rho_1(\theta) \ell_2 d\ell_2 d\theta d\eta \\ &\quad - 2 \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(R_2(t, \theta) e^{i\theta}, \ell_2 e^{i\eta}) \rho_2(\theta) \ell_2 d\ell_2 d\theta d\eta. \end{aligned}$$

It follows from (3.17) and (3.1)

$$\nabla E(r) = \begin{pmatrix} 2 \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(w_1(t, \theta), \ell_2 e^{i\eta}) \ell_2 d\ell_2 d\eta \\ -2 \int_{\mathbb{T}} \int_{R_2(t, \eta)}^{R_1(t, \eta)} G(w_2(t, \theta), \ell_2 e^{i\eta}) \ell_2 d\ell_2 d\eta \end{pmatrix} = \begin{pmatrix} 2\psi(t, z_1(t, \theta)) \\ -2\psi(t, z_2(t, \theta)) \end{pmatrix}. \quad (3.18)$$

Finally, (3.18), (3.16) and (3.4) give the desired result. This achieves the proof of Proposition 3.1. \square

3.3 Symplectic structure and invariance

In this section, we intend to discuss the symplectic structure behind the Hamiltonian formulation already seen in Proposition 3.1. We shall also discuss some symmetry properties such as the reversibility and the m -fold persistence.

► *Symplectic structure.* We shall present the symplectic structure associated with the Hamiltonian equation (3.13). To do so, we need to fix the phase space but before that we shall use the following fact that can be derived from (3.13),

$$\frac{d}{dt} \int_{\mathbb{T}} r(t, \theta) d\theta = 0.$$

This means that the area enclosed by the boundaries is conserved in time. Therefore, we shall work with the following phase space with zero space average $L_*^2(\mathbb{T}) \times L_*^2(\mathbb{T})$ defined by

$$L_*^2(\mathbb{T}) \triangleq \left\{ f = \sum_{j \in \mathbb{Z}^*} f_j \mathbf{e}_j \quad \text{s.t.} \quad f_{-j} = \overline{f_j}, \quad \sum_{j \in \mathbb{Z}^*} |f_j|^2 < +\infty \right\}, \quad \mathbf{e}_j(\theta) \triangleq e^{ij\theta}.$$

The equation (3.13) induces on the phase space $L_*^2(\mathbb{T}) \times L_*^2(\mathbb{T})$ a symplectic structure given by the symplectic 2-form

$$\begin{aligned} \mathcal{W}(r, h) &\triangleq \langle \mathcal{J}^{-1} r, h \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\ &= \int_{\mathbb{T}} \partial_{\theta}^{-1} r_1(\theta) h_1(\theta) d\theta - \int_{\mathbb{T}} \partial_{\theta}^{-1} r_2(\theta) h_2(\theta) d\theta, \end{aligned}$$

where

$$\partial_{\theta}^{-1} f \triangleq \sum_{j \in \mathbb{Z}^*} \frac{f_j}{ij} \mathbf{e}_j \quad \text{for} \quad f = \sum_{j \in \mathbb{Z}^*} f_j \mathbf{e}_j.$$

The corresponding Hamiltonian vector field is $X_H(r) \triangleq \mathcal{J} \nabla H(r)$ (where ∇ is the $L^2 \times L^2$ -gradient). It is defined as the symplectic gradient of the Hamiltonian H with respect to the symplectic 2-form \mathcal{W} , namely

$$dH(r)[\cdot] = \mathcal{W}(X_H(r), \cdot).$$

Decomposing into Fourier series

$$r = (r_1, r_2), \quad \forall k \in \{1, 2\}, \quad r_k = \sum_{j \in \mathbb{Z}^*} r_{j,k} \mathbf{e}_j \quad \text{with} \quad r_{-j,k} = \overline{r_{j,k}},$$

then the symplectic form \mathcal{W} becomes

$$\mathcal{W}(r, h) = \sum_{j \in \mathbb{Z}^*} \frac{1}{ij} \left[r_{j,1} h_{-j,1} - r_{j,2} h_{-j,2} \right], \quad (3.19)$$

or equivalently,

$$\begin{aligned} \mathcal{W} &= \frac{1}{2} \sum_{j \in \mathbb{Z}^*} \frac{1}{ij} \left[dr_{j,1} \wedge dr_{-j,1} - dr_{j,2} \wedge dr_{-j,2} \right] \\ &= \sum_{j \in \mathbb{N}^*} \frac{1}{ij} \left[dr_{j,1} \wedge dr_{-j,1} - dr_{j,2} \wedge dr_{-j,2} \right]. \end{aligned} \quad (3.20)$$

Definition 3.1. (Symplectic) A linear transformation Φ of the phase space $L_*^2(\mathbb{T}) \times L_*^2(\mathbb{T})$ is symplectic, if Φ preserves the symplectic 2-form \mathcal{W} , i.e.

$$\mathcal{W}(\Phi u, \Phi v) = \mathcal{W}(u, v),$$

or equivalently

$$\Phi^{\top} \circ \mathcal{J}^{-1} \circ \Phi = \mathcal{J}^{-1}.$$

This allows to establish the following result which is useful later and whose proof is straightforward.

Lemma 3.2. *Let Φ be a matrix space-Fourier multiplier with the form*

$$\Phi \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \triangleq \sum_{j \in \mathbb{Z}^*} \Phi_j \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \quad \Phi_j \in M_2(\mathbb{R}),$$

and consider the symplectic 2-form \mathcal{W} defined in (3.19). Then Φ is symplectic if and only if

$$\forall j \in \mathbb{Z}^*, \quad \Phi_j^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Phi_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

► *Reversibility.* We shall analyze the reversibility property of the equation (3.13) which is crucial to reduce by symmetry the phase space and remove most of the trivial resonances. We consider the involution \mathcal{S} defined on the phase space $L_*^2(\mathbb{T}) \times L_*^2(\mathbb{T})$ by

$$(\mathcal{S}r)(\theta) \triangleq r(-\theta), \quad (3.21)$$

which satisfies

$$\mathcal{S}^2 = \text{Id} \quad \text{and} \quad \mathcal{J} \circ \mathcal{S} = -\mathcal{S} \circ \mathcal{J}. \quad (3.22)$$

Using the change of variables $\eta \mapsto -\eta$ and parity arguments, one gets from (3.6)

$$\forall k, n \in \{1, 2\}, \quad F_{k,n} \circ \mathcal{S} = -\mathcal{S} \circ F_{k,n}.$$

Then we conclude by Lemma 3.1, (3.13) and (3.22) that the Hamiltonian vector field X_H satisfies

$$X_H \circ \mathcal{S} = -\mathcal{S} \circ X_H.$$

Therefore, we will focus on quasi-periodic solutions to (3.13) satisfying the reversibility condition

$$r(-t, -\theta) = r(t, \theta). \quad (3.23)$$

► *The \mathfrak{m} -fold symmetry.* Let $\mathfrak{m} \geq 1$ be an integer and consider the transformation $\mathcal{T}_\mathfrak{m}$ on the phase space $L_*^2(\mathbb{T}) \times L_*^2(\mathbb{T})$ defined by

$$(\mathcal{T}_\mathfrak{m}r)(\theta) \triangleq r\left(\theta + \frac{2\pi}{\mathfrak{m}}\right). \quad (3.24)$$

Then it is an immediate fact that

$$\mathcal{T}_\mathfrak{m}^\mathfrak{m} = \text{Id} \quad \text{and} \quad \mathcal{J} \circ \mathcal{T}_\mathfrak{m} = \mathcal{T}_\mathfrak{m} \circ \mathcal{J}.$$

Using the change of variables $\eta \mapsto \eta + \frac{2\pi}{\mathfrak{m}}$ we easily obtain from (3.6)

$$\forall k, n \in \{1, 2\}, \quad F_{k,n} \circ \mathcal{T}_\mathfrak{m} = \mathcal{T}_\mathfrak{m} \circ F_{k,n}.$$

Therefore,

$$X_H \circ \mathcal{T}_\mathfrak{m} = \mathcal{T}_\mathfrak{m} \circ X_H.$$

Thus, the solutions that we shall be interested in satisfy the \mathfrak{m} -fold symmetry

$$r\left(t, \theta + \frac{2\pi}{\mathfrak{m}}\right) = r(t, \theta). \quad (3.25)$$

Consequently, we shall work in the closed subspace $L_\mathfrak{m}^2(\mathbb{T}) \times L_\mathfrak{m}^2(\mathbb{T})$ defined by

$$L_\mathfrak{m}^2(\mathbb{T}) \triangleq \left\{ f = \sum_{j \in \mathbb{Z}^*} f_j \mathbf{e}_j \in L_*^2(\mathbb{T}) \quad \text{s.t.} \quad f_j \neq 0 \Rightarrow j \in \mathbb{Z}_\mathfrak{m} \right\}. \quad (3.26)$$

4 Linearization and symplectic transformation

In this section we shall compute the linear Hamiltonian obtained through the linearization of the equation (3.13) at any state close to the equilibrium solution $r = 0$. It turns out that at the equilibrium state we find a matrix Fourier multiplier that can be diagonalized in a suitable basis using a linear symplectic change of coordinates, for more details we refer to Lemma 4.5. However, this procedure requires to work with higher m -fold symmetries to avoid the double eigenvalue corresponding to the mode $j = 2$ as well as potential hyperbolic directions.

4.1 Linearized operator

The main purpose is to explore the structure of the linearized operator which takes the form of a transport system with variable coefficients and subject to compact perturbations.

Lemma 4.1. *The linearized equation of (3.13) at a small state r is given by the linear Hamiltonian equation,*

$$\partial_t \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \mathcal{J} \mathbf{M}_r \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad \mathbf{M}_r \triangleq \begin{pmatrix} -V_1(r) - L_{1,1}(r) & L_{1,2}(r) \\ L_{2,1}(r) & V_2(r) - L_{2,2}(r) \end{pmatrix}, \quad (4.1)$$

where $V_k(r)$ are scalar functions and $L_{k,n}(r)$ are nonlocal operators defined by

$$V_k(r)(t, \theta) \triangleq \Omega + (-1)^k [V_{k,k}(r)(t, \theta) - V_{k,3-k}(r)(t, \theta)], \quad (4.2)$$

$$V_{k,n}(r)(t, \theta) \triangleq \int_{\mathbb{T}} \log(A_{k,n}(r)(t, \theta, \eta)) \partial_\eta \left(\frac{R_n(t, \eta)}{R_k(t, \theta)} \sin(\eta - \theta) \right) d\eta, \quad (4.3)$$

$$L_{k,n}(r)\rho(t, \theta) \triangleq \int_{\mathbb{T}} \rho(t, \eta) \log(A_{k,n}(r)(t, \theta, \eta)) d\eta \quad (4.4)$$

and $A_{k,n}(r)$ and R_k are respectively defined by (3.7) and (3.2). Moreover, if r satisfies (3.23) and (3.25) with $m \geq 1$, then the operator \mathbf{M}_r is m -fold reversibility preserving.

Proof. Throughout the proof, we shall alleviate the notation by removing the time dependence and keep r when it is relevant. In view of (3.4), it suffices to linearize the term involving the stream function. All the computations are done at a formal level, but can be rigorously justified in a classical way in the functional context introduced in Section 2. According to (3.9) we can write

$$\psi(z_k(\theta)) = (-1)^{k+1} [\tilde{\psi}(r_k, z_k(\theta)) - \tilde{\psi}(r_{3-k}, z_k(\theta))], \quad (4.5)$$

$$\tilde{\psi}(r_n, z) \triangleq \frac{1}{4i} \int_{\mathbb{T}} (\bar{z}_n(\eta) - \bar{z}) [\log(|z - z_n(\eta)|^2) - 1] \partial_\eta z_n(\eta) d\eta. \quad (4.6)$$

Applying the chain rule yields

$$\begin{aligned} d_{r_k} \left(\psi(z_k(\theta)) \right) [\rho_k](\theta) &= (-1)^{k+1} \left[d_{r_k} \tilde{\psi}(r_k, z_k(\theta)) [\rho_k](\theta) \right. \\ &\quad + 2\operatorname{Re} \left((\partial_{\bar{z}} \tilde{\psi})(r_k, z_k(\theta)) d_{r_k} \bar{z}_k(\theta) [\rho_k](\theta) \right) \\ &\quad \left. - 2\operatorname{Re} \left((\partial_{\bar{z}} \tilde{\psi})(r_{3-k}, z_k(\theta)) d_{r_k} \bar{z}_k(\theta) [\rho_k](\theta) \right) \right], \\ d_{r_{3-k}} \left(\psi(z_k(\theta)) \right) [\rho_{3-k}](\theta) &= (-1)^k d_{r_{3-k}} \tilde{\psi}(r_{3-k}, z_k(\theta)) [\rho_{3-k}](\theta). \end{aligned} \quad (4.7)$$

From (3.17), we have the following expression of $\tilde{\psi}$,

$$\tilde{\psi}(r_k, e^{-i\Omega t} z) = \int_{\mathbb{T}} \int_0^{R_k(\eta)} \log(|z - \ell_2 e^{i\eta}|) \ell_2 d\ell_2 d\eta.$$

Therefore, differentiating with respect to r_k in the direction ρ_k , we obtain

$$d_{r_k} \tilde{\psi}(r_k, e^{-i\Omega t} z) [\rho_k](\theta) = \int_{\mathbb{T}} \rho_k(\eta) \log(|z - R_k(\eta) e^{i\eta}|) d\eta.$$

It follows that, for any $k, n \in \{1, 2\}$, we have by virtue of (4.4)

$$\begin{aligned} d_{r_k} \tilde{\psi}(r_k, z_n(\theta))[\rho_k](\theta) &= \int_{\mathbb{T}} \rho_k(\eta) \log(|R_n(\theta)e^{i\theta} - R_k(\eta)e^{i\eta}|) d\eta \\ &= L_{k,n}(r) \rho_k(\theta). \end{aligned} \quad (4.8)$$

On the other hand, differentiating (3.1) leads to

$$d_{r_k} \bar{z}_k(\theta)[\rho_k](\theta) = \frac{\rho_k(\theta)}{R_k(\theta)} e^{-i(\theta - \Omega t)}.$$

In addition, by virtue of (4.6), we have

$$\partial_{\bar{z}} \tilde{\psi}(r_n, z) = -\frac{1}{4i} \int_{\mathbb{T}} \log(|z - z_n(\eta)|^2) \partial_{\eta} z_n(\eta) d\eta.$$

Combining the last two identities we infer, for $k, n \in \{1, 2\}$,

$$2\operatorname{Re}\left((\partial_{\bar{z}} \tilde{\psi})(r_n, z_k(\theta)) d_{r_k} \bar{z}_k(\theta)[\rho_k](\theta)\right) = -\frac{\rho_k(\theta)}{R_k(\theta)} \int_{\mathbb{T}} \log(|z_k(\theta) - z_n(\eta)|) \partial_{\eta} \operatorname{Im}\left(z_n(\eta) e^{-i(\theta - \Omega t)}\right) d\eta.$$

From (3.1) we find the identity

$$\operatorname{Im}\left(z_n(\eta) e^{-i(\theta - \Omega t)}\right) = R_n(\eta) \sin(\eta - \theta).$$

Then, by (4.3) we conclude that

$$2\operatorname{Re}\left((\partial_{\bar{z}} \tilde{\psi})(r_n, z_k(\theta)) d_{r_k} \bar{z}_k(\theta)[\rho_k](\theta)\right) = -V_{k,n}(r)(\theta) \rho_k(\theta). \quad (4.9)$$

Putting together (4.5), (4.7), (4.8) and (4.9) yields

$$\begin{aligned} d_r \left(\psi(z_k(\theta))\right)[\rho](\theta) &= d_{r_k} \left(\psi(z_k(\theta))\right)[\rho_k](\theta) + d_{r_{3-k}} \left(\psi(z_k(\theta))\right)[\rho_{3-k}](\theta) \\ &= (-1)^{k+1} \left[L_{k,k}(r) \rho_k(\theta) - V_{k,k}(r)(\theta) \rho_k(\theta) \right. \\ &\quad \left. + V_{k,3-k}(r)(\theta) \rho_k(\theta) - L_{k,3-k}(r) \rho_{3-k}(\theta) \right]. \end{aligned}$$

This gives the expression of \mathbf{M}_r in (4.1). Next, assume that r satisfies (3.23) and (3.25). Then, from (3.2), (4.2) and (4.3), we get the following symmetry properties

$$V_k(r)(-t, -\theta) = V_k(r)(t, \theta) = V_k(r)(t, \theta + \frac{2\pi}{m}). \quad (4.10)$$

Similarly, from (3.2) and (3.7), one has

$$A_{k,n}(r)(-t, -\theta, -\eta) = A_{k,n}(r)(t, \theta, \eta) = A_{k,n}(r)(t, \theta + \frac{2\pi}{m}, \eta + \frac{2\pi}{m}). \quad (4.11)$$

Thus, the symmetry properties of the operator \mathbf{M}_r are immediate consequences of Lemma 2.4. The proof of Lemma 4.1 is now complete. \square

The next goal is to derive the explicit structure of the linearized operator at the equilibrium state $r = 0$.

Lemma 4.2. *The linearized equation of (3.13) at $r = 0$ writes,*

$$\partial_t \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} = \mathcal{J} \mathbf{M}_0 \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \quad \mathbf{M}_0 \triangleq \begin{pmatrix} -V_1(0) - \mathcal{K}_1 * \cdot & \mathcal{K}_b * \cdot \\ \mathcal{K}_b * \cdot & V_2(0) - \mathcal{K}_1 * \cdot \end{pmatrix}, \quad (4.12)$$

where

$$\forall k \in \{1, 2\}, \quad \mathbf{v}_k(b) \triangleq V_k(0) = \Omega + (2 - k) \frac{1 - b^2}{2}, \quad (4.13)$$

$$\forall x \in (0, 1], \quad \mathcal{K}_x(\theta) \triangleq \log|1 - xe^{i\theta}|. \quad (4.14)$$

The convolution is understood in the following sense

$$\mathcal{K}_x * \rho(\theta) = \int_{\mathbb{T}} \mathcal{K}_x(\theta - \eta) \rho(\eta) d\eta.$$

Given the space Fourier expansion of the real solutions

$$\forall k \in \{1, 2\}, \quad \rho_k(t, \theta) = \sum_{j \in \mathbb{Z}^*} \rho_{j,k}(t) e^{ij\theta}, \quad \text{with} \quad \rho_{-j,k} = \overline{\rho_{j,k}},$$

the system (4.12) is equivalent to the following countable family of linear differential systems

$$\forall j \in \mathbb{Z}^*, \quad \begin{pmatrix} \dot{\rho}_{j,1} \\ \dot{\rho}_{j,2} \end{pmatrix} = M_j(b, \Omega) \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix}, \quad M_j(b, \Omega) \triangleq \frac{ij}{|j|} \begin{pmatrix} -|j|(\Omega + \frac{1-b^2}{2}) + \frac{1}{2} & -\frac{b|j|}{2} \\ \frac{b|j|}{2} & -|j|\Omega - \frac{1}{2} \end{pmatrix}. \quad (4.15)$$

Proof. We shall make use of the following formula which can be found in [27, Lem. A.3] and [91, Lem. 3.2].

$$\forall j \in \mathbb{Z}^*, \quad \forall x \in (0, 1], \quad \int_{\mathbb{T}} \log(|1 - xe^{i\theta}|) \cos(j\theta) d\theta = -\frac{x^{|j|}}{2|j|}. \quad (4.16)$$

First observe that from (3.7), one deduces for $r = 0$ that

$$A_{k,n}(0)(\theta, \eta) = |b_k e^{i\theta} - b_n e^{i\eta}| = (b_k^2 + b_n^2 - 2b_k b_n \cos(\eta - \theta))^{\frac{1}{2}}$$

leading in particular to

$$A_{1,2}(0)(\theta, \eta) = A_{2,1}(0)(\theta, \eta) = |1 - be^{i(\eta-\theta)}|.$$

Taking $r = 0$ in (4.3) and using the change of variables $\eta \mapsto \eta + \theta$ together with (4.16) imply

$$\begin{aligned} \forall k \in \{1, 2\}, \quad V_{k,k}(0)(\theta) &= \int_{\mathbb{T}} \log(|1 - e^{i\eta}|) \cos(\eta) d\eta = -\frac{1}{2}, \\ V_{k,3-k}(0)(\theta) &= \frac{b_{3-k}}{b_k} \int_{\mathbb{T}} \log(|1 - be^{i\eta}|) \cos(\eta) d\eta = -\frac{b_{3-k}b}{2b_k}. \end{aligned}$$

Combining the last identity with (4.2) yields (4.13). Substituting $r = 0$ into (4.4) gives, since the space average of ρ is zero,

$$\forall k \in \{1, 2\}, \quad L_{k,k}(0)\rho = \mathcal{K}_1 * \rho \quad \text{and} \quad L_{k,3-k}(0)\rho = \mathcal{K}_b * \rho,$$

where the kernels \mathcal{K}_1 and \mathcal{K}_b are defined by (4.14). Applying once again (4.16) allows to get

$$\forall j \in \mathbb{Z}^*, \quad \mathcal{K}_1 * \mathbf{e}_j = -\frac{1}{2|j|} \mathbf{e}_j \quad \text{and} \quad \mathcal{K}_b * \mathbf{e}_j = -\frac{b|j|}{2|j|} \mathbf{e}_j, \quad \mathbf{e}_j(\theta) = e^{ij\theta}.$$

Finally, gathering the previous computations leads to (4.15). This achieves the proof of Lemma 4.2. \square

4.2 Diagonalization at the equilibrium state

In this subsection we shall diagonalize the equilibrium matrix operator appearing in Lemma 4.2. This provides a new Hamiltonian system more adapted for the action-angles reformulation. Before that, we will establish the following result on the spectral structure of the matrix M_j introduced in (4.15).

Lemma 4.3. *Let $\Omega > 0$, $j \in \mathbb{Z}^*$ and $b \in (0, 1)$. Then the eigenvalues of the matrix $M_j(b, \Omega)$, defined in (4.15), are given by $-i\Omega_{j,k}(b)$, $k \in \{1, 2\}$, where*

$$\Omega_{j,k}(b) \triangleq \frac{j}{|j|} \left[\left(\Omega + \frac{1-b^2}{4} \right) |j| - i^{\mathbb{H}(\Delta_j(b))} \frac{(-1)^k}{2} \sqrt{|\Delta_j(b)|} \right], \quad (4.17)$$

with $\mathbb{H} \triangleq \mathbf{1}_{[0, \infty)}$ the Heaviside function and

$$\Delta_j(b) \triangleq b^{2|j|} - \left(\frac{1-b^2}{2} |j| - 1 \right)^2. \quad (4.18)$$

The corresponding eigenspaces are one dimensional and generated by the vectors

$$v_{j,1}(b) \triangleq \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix}, \quad v_{j,2}(b) \triangleq \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix}, \quad a_j(b) \triangleq \frac{b^{|j|}}{\frac{1-b^2}{2} |j| - 1 + i^{\mathbb{H}(\Delta_j(b))} \sqrt{|\Delta_j(b)|}}.$$

Proof. According to (4.15) we have

$$\forall j \in \mathbb{Z}^*, \quad M_{-j}(b, \Omega) = \overline{M_j(b, \Omega)}.$$

Thus, it suffices to consider the case $j \in \mathbb{N}^*$. The eigenvalues of the matrix $M_j(b, \Omega)$ are solutions of the following second order polynomial equation

$$X^2 + i[\mu_j + \delta_j]X - \left[\mu_j \delta_j + \frac{b^{2j}}{4} \right] = 0, \quad \text{with} \quad \mu_j \triangleq j \left(\Omega + \frac{1-b^2}{2} \right) - \frac{1}{2} \quad \text{and} \quad \delta_j \triangleq j\Omega + \frac{1}{2}. \quad (4.19)$$

The discriminant of the last equation is given by

$$\begin{aligned} -(\mu_j + \delta_j)^2 + b^{2j} + 4\mu_j \delta_j &= b^{2j} - (\mu_j - \delta_j)^2 \\ &= \Delta_j(b) \end{aligned}$$

and the solutions to the equation are $-i\Omega_{j,k}(b)$, $k \in \{1, 2\}$, where $\Omega_{j,k}(b)$ are given by (4.17). The expression of the eigenvectors $v_{j,k}(b)$ follows by direct computations. Notice that, for all $j \in \mathbb{N}^*$,

$$a_j(0) = 0, \quad a_j(1) = -1$$

and for all $b \in (0, 1)$, $a_j(b)$ is well-defined if $\Delta_j(b) > 0$. We shall prove that $a_j(b)$ is still well-defined even when $\Delta_j(b) \leq 0$. In view of (4.18), we may write for all $j \in \mathbb{N}^*$

$$\Delta_j(b) = \left(b^j - 1 + \frac{1-b^2}{2} j \right) \left(b^j + 1 - \frac{1-b^2}{2} j \right). \quad (4.20)$$

In particular we have

$$\forall b \in (0, 1), \quad \Delta_1(b) = -\frac{1}{4}(1-b)^2(1+b)^2 < 0, \quad \Delta_2(b) = 0 \quad \text{and} \quad a_2(b) = -1.$$

For $j \geq 3$, we can easily check that

$$\begin{aligned} \forall b \in (0, 1), \quad \frac{1-b^2}{2} j + b^j - 1 &= (1-b) \left(\frac{j}{2}(1+b) - (1+b) - b^2[1+b+\dots+b^{j-3}] \right) \\ &\geq (1-b) \left(\frac{j-2}{2}(1+b) - b^2(j-2) \right) \\ &\geq \frac{j-2}{2}(1-b)^2(1+2b) \\ &> 0. \end{aligned} \quad (4.21)$$

It follows that $\Delta_j(b) \leq 0$ if and only if $b^j + 1 - \frac{1-b^2}{2} j \leq 0$. In this case the denominator of $a_j(b)$ satisfies, for all $b \in (0, 1)$,

$$\frac{1-b^2}{2} |j| - 1 + \sqrt{-\Delta_j(b)} \geq \frac{1-b^2}{2} |j| - 1 \geq b^j > 0.$$

This ends the proof of Lemma 4.3. □

The next task is devoted to the study of the sign of the discriminant $\Delta_j(b)$ defined in (4.18).

Lemma 4.4. *There exists a strictly increasing sequence $(\underline{b}_j)_{j \geq 3} \subset (0, 1)$ converging to 1 such that*

$$\left\{ b \in (0, 1) \text{ s.t. } \exists j \in \mathbb{N} \setminus \{0, 1, 2\}, \Delta_j(b) = 0 \right\} = \left\{ \underline{b}_j, j \geq 3 \right\},$$

with

$$\underline{b}_3 = \frac{1}{2} \quad \text{and} \quad \underline{b}_4 = \sqrt{\sqrt{2} - 1}. \quad (4.22)$$

Moreover, for any fixed $j_0 \geq 3$ we have

$$\forall b \in (\underline{b}_{j_0}, \underline{b}_{j_0+1}), \quad \begin{cases} \Delta_j(b) > 0, & \text{if } j \leq j_0, \\ \Delta_j(b) < 0, & \text{if } j \geq j_0 + 1. \end{cases}$$

Proof. For $j \geq 3$, one gets in view of (4.20) and (4.21) that the zeros of the function $b \mapsto \Delta_j(b)$ are the zeros of the function $b \mapsto b^j + 1 - \frac{1-b^2}{2}j$. To study the zeros of the latter discrete function let us consider its continuous version

$$\forall (b, x) \in (0, 1) \times [3, \infty), \quad f(b, x) \triangleq b^x + 1 - \frac{1-b^2}{2}x. \quad (4.23)$$

Then, for fixed $x \in [3, \infty)$, one has $f(0, x) = 1 - \frac{x}{2} < 0$, $f(1, x) = 2$ and

$$\forall b \in (0, 1), \quad \partial_b f(b, x) = x(b^{x-1} + b) > 0. \quad (4.24)$$

Consequently, by the intermediate value theorem, there exists a unique $\underline{b}_x \in (0, 1)$ satisfying

$$f(\underline{b}_x, x) = 0, \quad \forall b < \underline{b}_x, \quad f(b, x) < 0 \quad \text{and} \quad \forall b > \underline{b}_x, \quad f(b, x) > 0.$$

Moreover, since

$$\forall b \in (0, 1), \quad \partial_x f(b, x) = b^x \log(b) - \frac{1-b^2}{2} < 0. \quad (4.25)$$

Then the function $x \mapsto f(b, x)$ is strictly decreasing on $[3, \infty)$, which implies that $x \mapsto \underline{b}_x$ is strictly increasing on $[3, \infty)$. It follows that for any fixed integer $j_0 \geq 3$ we have

$$f(\underline{b}_{j_0}, j_0) = 0 \quad \text{and} \quad \forall b \in (\underline{b}_{j_0}, \underline{b}_{j_0+1}), \quad \begin{cases} f(b, j) > 0, & \text{if } j \leq j_0, \\ f(b, j) < 0, & \text{if } j \geq j_0 + 1. \end{cases} \quad (4.26)$$

Combining (4.20), (4.21), (4.23) and (4.26) we conclude the desired result. Finally, (4.22) follows from the identities

$$\begin{aligned} f(b, 3) &= b^3 + \frac{3}{2}b^2 - \frac{1}{2} \\ &= (b+1)^2 \left(b - \frac{1}{2}\right) \end{aligned}$$

and

$$\begin{aligned} f(b, 4) &= b^4 + 2b^2 - 1 \\ &= (b^2 + 1 + \sqrt{2}) \left(b + \sqrt{\sqrt{2} - 1}\right) \left(b - \sqrt{\sqrt{2} - 1}\right). \end{aligned}$$

This ends the proof of Lemma 4.4. □

We shall now focus on the conditions that guarantee the ellipticity of the eigenvalues based on Lemma 4.4.

Corollary 4.1. *Let $\Omega > 0$, $b^* \in (0, 1) \setminus \{\underline{b}_n, n \geq 3\}$ and set*

$$\mathfrak{m}^* \triangleq \mathfrak{m}^*(b^*) \triangleq \min \{n \geq 3 \text{ s.t. } \underline{b}_n > b^*\}. \quad (4.27)$$

Then, for all $|j| \geq \mathfrak{m}^$ and $b \in [0, b^*]$, the eigenvalues of the matrix $M_j(b, \Omega)$, defined in (4.15), are simple and pure imaginary $-i\Omega_{j,k}(b)$, $k \in \{1, 2\}$, with*

$$\Omega_{j,k}(b) = \frac{j}{|j|} \left[\left(\Omega + \frac{1-b^2}{4} \right) |j| + \frac{(-1)^{k+1}}{2} \sqrt{\left(\frac{1-b^2}{2} |j| - 1 \right)^2 - b^2 |j|} \right] \quad (4.28)$$

$$= \frac{j}{|j|} \left[\left(\Omega + (2-k) \frac{1-b^2}{2} \right) |j| + \frac{(-1)^k}{2} + (-1)^{k+1} \mathfrak{r}_j(b) \right], \quad (4.29)$$

where

$$\forall (n, m) \in \mathbb{N}^2, \quad \forall \alpha \in \mathbb{N}^*, \quad \sup_{\substack{b \in [0, b^*] \\ |j| \geq \mathfrak{m}^*}} |j|^\alpha |\partial_j^m \partial_b^n \mathfrak{r}_j(b)| \leq C_{n,m,\alpha}. \quad (4.30)$$

The corresponding eigenspaces are real and generated by

$$v_{j,1}(b) = \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix}, \quad v_{j,2}(b) = \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix}, \quad a_j(b) = \frac{b^{|j|}}{\frac{1-b^2}{2} |j| - 1 + \sqrt{\left(\frac{1-b^2}{2} |j| - 1 \right)^2 - b^2 |j|}}. \quad (4.31)$$

Moreover, there exists $\delta \triangleq \delta(b^) > 0$ such that for all $|j| \geq \mathfrak{m}^*$ and $b \in [0, b^*]$,*

$$0 \leq a_j(b) < 1 - \delta \quad (4.32)$$

and

$$\forall n, \alpha \in \mathbb{N}^*, \quad \sup_{\substack{b \in [0, b^*] \\ |j| \geq \mathfrak{m}^*}} |j|^\alpha |\partial_b^n a_j(b)| < \infty. \quad (4.33)$$

Proof. In view of (4.27), (4.24), (4.25) and (4.26), for all $b \in [0, b^*]$ and $|j| \geq \mathfrak{m}^*$ one has

$$f(b, |j|) \leq f(b^*, \mathfrak{m}^*) < f(\underline{b}_{\mathfrak{m}^*}, \mathfrak{m}^*) = 0. \quad (4.34)$$

Combining (4.20), (4.21), (4.23) and (4.34) we find

$$\forall b \in [0, b^*], \quad \forall |j| \geq \mathfrak{m}^*, \quad \Delta_j(b) < 0.$$

Then, by Lemma 4.3, we conclude (4.28) and (4.31). On the other hand, the inequality (4.34) also implies

$$f(b^*, \mathfrak{m}^*) = (b^*)^{\mathfrak{m}^*} + 1 - \frac{1-(b^*)^2}{2} \mathfrak{m}^* < 0. \quad (4.35)$$

This gives in turn, for any $|j| \geq \mathfrak{m}^*$ and $b \in [0, b^*]$,

$$\frac{1-b^2}{2} |j| - 1 \geq \frac{1-(b^*)^2}{2} \mathfrak{m}^* - 1 > 0. \quad (4.36)$$

Consequently, for any $|j| \geq \mathfrak{m}^*$ and $b \in [0, b^*]$ we may write, by (4.28),

$$\begin{aligned} \Omega_{j,k}(b) &= \frac{j}{|j|} \left[\left(\Omega + \frac{1-b^2}{4} \right) |j| + \frac{(-1)^{k+1}}{2} \left(\frac{1-b^2}{2} |j| - 1 \right) \sqrt{1 - b^2 |j| \left(\frac{1-b^2}{2} |j| - 1 \right)^{-2}} \right] \\ &= \frac{j}{|j|} \left[\left(\Omega + \frac{1-b^2}{4} \right) |j| + \frac{(-1)^{k+1}}{2} \left(\frac{1-b^2}{2} |j| - 1 \right) + (-1)^{k+1} \mathfrak{r}_j(b) \right], \end{aligned}$$

with

$$\mathfrak{r}_j(b) \triangleq \frac{1}{2} \left(\frac{1-b^2}{2} |j| - 1 \right) \left[\sqrt{1 - b^2 |j| \left(\frac{1-b^2}{2} |j| - 1 \right)^{-2}} - 1 \right]. \quad (4.37)$$

By virtue of (4.35) and (4.36), one has for all $|j| \geq \mathfrak{m}^*$ and $b \in [0, b^*] \subset [0, 1]$,

$$b^{|j|} \left(\frac{1-b^2}{2} |j| - 1 \right)^{-1} \leq (b^*)^{|j|} \left(\frac{1-(b^*)^2}{2} \mathfrak{m}^* - 1 \right)^{-1} \leq (b^*)^{\mathfrak{m}^*} \left(\frac{1-(b^*)^2}{2} \mathfrak{m}^* - 1 \right)^{-1} < 1. \quad (4.38)$$

Thus expanding in power series the square root and using Leibniz rule we get after straightforward computations the bounds for $\mathbf{r}_j(b)$ claimed in (4.30). Next, we shall check the inequalities (4.32). Using (4.31), (4.36) and (4.38) we conclude the existence of $\delta = \delta(b^*) \in (0, 1)$ such that for all $|j| \geq \mathfrak{m}^*$ and $b \in [0, b^*]$,

$$0 \leq a_j(b) \leq b^{|j|} \left(\frac{1-b^2}{2} |j| - 1 \right)^{-1} < 1 - \delta.$$

Therefore the estimate (4.33) follows from (4.31) and Leibniz rule. This achieves the proof of Corollary 4.1. \square

As a consequence of Corollary 4.1, we may restrict the Fourier modes to the lattice $\mathbb{Z}_{\mathfrak{m}}$ with $\mathfrak{m} \geq \mathfrak{m}^*$ in order to avoid the hyperbolic spectrum. Therefore, we shall work in the phase space $L_{\mathfrak{m}}^2(\mathbb{T}) \times L_{\mathfrak{m}}^2(\mathbb{T})$ introduced in (3.26). In what follows, we introduce a suitable symplectic transformation \mathbf{Q} used in the diagonalization of the linearized operator at the equilibrium state described by Lemma 4.2. This diagonalization is required latter in order to perform the reduction of the remainder term, see Proposition 7.5. The linear transformation \mathbf{Q} is defined by its action on any element $(\rho_1, \rho_2) \in L_{\mathfrak{m}}^2(\mathbb{T}) \times L_{\mathfrak{m}}^2(\mathbb{T})$ with the Fourier expansions

$$\forall k \in \{1, 2\}, \quad \rho_k = \sum_{j \in \mathbb{Z}_{\mathfrak{m}}^*} \rho_{j,k} \mathbf{e}_j, \quad \text{with} \quad \rho_{-j,k} = \overline{\rho_{j,k}}, \quad \mathbf{e}_j(\theta) = e^{ij\theta}$$

as follows

$$\mathbf{Q} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \triangleq \sum_{j \in \mathbb{Z}_{\mathfrak{m}}^*} \mathbf{Q}_j \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \quad \mathbf{Q}_j \triangleq \frac{1}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 & -a_j(b) \\ -a_j(b) & 1 \end{pmatrix}, \quad (4.39)$$

where $a_j(b)$ is given by (4.31). We have the following properties.

Lemma 4.5. *Let $\mathfrak{m} \geq \mathfrak{m}^*$, $b \in [0, b^*]$, where \mathfrak{m}^* and b^* are defined in Corollary 4.1. Then the following assertions hold true.*

1. $\mathbf{Q} : L_{\mathfrak{m}}^2(\mathbb{T}) \times L_{\mathfrak{m}}^2(\mathbb{T}) \rightarrow L_{\mathfrak{m}}^2(\mathbb{T}) \times L_{\mathfrak{m}}^2(\mathbb{T})$ is symplectic with respect to the symplectic form (3.20). In addition $\mathbf{Q}^\top = \mathbf{Q}$.

2. \mathbf{Q} is invertible and its inverse is given by

$$\mathbf{Q}^{-1} \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \triangleq \sum_{j \in \mathbb{Z}_{\mathfrak{m}}^*} \mathbf{Q}_j^{-1} \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \quad \mathbf{Q}_j^{-1} = \frac{1}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 & a_j(b) \\ a_j(b) & 1 \end{pmatrix}. \quad (4.40)$$

3. The transformations $\mathbf{Q}^{\pm 1} \triangleq \mathbf{Q}$ and \mathbf{Q}^{-1} write

$$\mathbf{Q}^{\pm 1} = \begin{pmatrix} \mathbb{I}_{\mathfrak{m}} & 0 \\ 0 & \mathbb{I}_{\mathfrak{m}} \end{pmatrix} + \begin{pmatrix} P_1 * \cdot & \mp P_2 * \cdot \\ \mp P_2 * \cdot & P_1 * \cdot \end{pmatrix}, \quad P_k \triangleq \sum_{j \in \mathbb{Z}_{\mathfrak{m}}^*} \frac{(2-k)+(-1)^k \sqrt{(2-k)+(-1)^k a_j^2(b)}}{\sqrt{1-a_j^2(b)}} \mathbf{e}_j. \quad (4.41)$$

For any $k \in \{1, 2\}$, the kernel P_k satisfies the symmetry properties

$$P_k(-\theta) = P_k(\theta) = P_k\left(\theta + \frac{2\pi}{\mathfrak{m}}\right) \quad (4.42)$$

and the estimate

$$\forall n \in \mathbb{N}, \quad \|\partial_\theta^n P_k * \rho\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\rho\|_s^{q,\gamma,\mathfrak{m}}. \quad (4.43)$$

4. The transformation \mathbf{Q} diagonalizes the operator $\mathcal{J}\mathbf{M}_0$, where \mathbf{M}_0 is introduced in (4.12), namely

$$\mathbf{Q}^{-1} \mathcal{J}\mathbf{M}_0 \mathbf{Q} = \mathcal{J}\mathbf{L}_0, \quad \mathbf{L}_0 \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} \triangleq \sum_{j \in \mathbb{Z}_{\mathfrak{m}}^*} \frac{1}{j} \begin{pmatrix} -\Omega_{j,1}(b) & 0 \\ 0 & \Omega_{j,2}(b) \end{pmatrix} \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j. \quad (4.44)$$

5. All the real-valued solutions of the linearized contour dynamics equation (4.12) have the form

$$\rho(t, \theta) = \sum_{j \in \mathbb{Z}_m^*} \frac{A_j}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix} e^{-i(\Omega_{1,j}(b)t-j\theta)} + \frac{B_j}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix} e^{-i(\Omega_{2,j}(b)t-j\theta)} \quad (4.45)$$

with $\overline{A_j} = A_{-j}$, $\overline{B_j} = B_{-j}$.

Proof. **1.** Straightforward computations based on the definition (4.39) lead to

$$\forall j \in \mathbb{Z}_m^*, \quad \mathbf{Q}_j^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mathbf{Q}_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, using Lemma 3.2 we conclude the first point. Notice that one also has

$$\forall j \in \mathbb{Z}_m^*, \quad \mathbf{Q}_j^\top = \mathbf{Q}_j,$$

which implies $\mathbf{Q}^\top = \mathbf{Q}$.

2. The second point follows easily by direct computations.

3. In view of (4.39) and (4.40) we can write

$$\mathbf{Q}_j^{\pm 1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 - \sqrt{1-a_j^2(b)} & \mp a_j(b) \\ \mp a_j(b) & 1 - \sqrt{1-a_j^2(b)} \end{pmatrix},$$

leading to (4.41). The symmetry properties (4.42) are obtained either by the fact that $a_{-j}(b) = a_j(b)$ (see (4.31)) or by the restriction of the Fourier modes in the definition of P_k . The estimate (4.43) is obtained by applying the Leibniz and the chain rules with (4.41) and (4.33).

4. Notice, from (4.39) and (4.31), that

$$\mathbf{Q}_j = \frac{1}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} v_{j,1}(b) & v_{j,2}(b) \end{pmatrix}.$$

Then, according to Corollary 4.1, the matrices \mathbf{Q}_j diagonalize the matrices $M_j(b, \Omega)$, defined in (4.15), namely

$$\forall j \in \mathbb{Z}_m^*, \quad \mathbf{Q}_j^{-1} M_j(b, \Omega) \mathbf{Q}_j = -i \begin{pmatrix} \Omega_{j,1}(b) & 0 \\ 0 & \Omega_{j,2}(b) \end{pmatrix}.$$

Therefore we deduce from Lemma 4.2

$$\begin{aligned} (\mathbf{Q}^{-1} \mathcal{J} \mathbf{M}_0 \mathbf{Q}) \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix} &= \sum_{j \in \mathbb{Z}_m^*} \begin{pmatrix} -i\Omega_{1,j}(b) & 0 \\ 0 & -i\Omega_{2,j}(b) \end{pmatrix} \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j \\ &= \sum_{j \in \mathbb{Z}_m^*} ij \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -\frac{1}{j}\Omega_{1,j}(b) & 0 \\ 0 & \frac{1}{j}\Omega_{2,j}(b) \end{pmatrix} \begin{pmatrix} \rho_{j,1} \\ \rho_{j,2} \end{pmatrix} \mathbf{e}_j, \end{aligned} \quad (4.46)$$

which gives in turn (4.44).

5. It follows immediately from the fourth point when solving the linear differential system (4.15). This completes the proof of Lemma 4.5. \square

4.3 Symplectic change of coordinates

In this section we intend to conjugate the nonlinear Hamiltonian system (3.13) with respect to the symplectic linear transformation \mathbf{Q} introduced in (4.39). Notice that this transformation does not depend on the unknown r , it depends only on the parameter b .

Let us consider the symplectic unknown $\tilde{r} \triangleq \mathbf{Q}^{-1}r$. Then the Hamiltonian system (3.13) writes

$$\partial_t \tilde{r} = X_K(\tilde{r}) = \mathcal{J} \nabla K(\tilde{r}), \quad K(\tilde{r}) \triangleq H(\mathbf{Q}\tilde{r}), \quad \tilde{r} = (\tilde{r}_1, \tilde{r}_2) \in L_m^2(\mathbb{T}) \times L_m^2(\mathbb{T}). \quad (4.47)$$

Indeed, on one hand, we have

$$\partial_t \tilde{r} = \mathbf{Q}^{-1} \partial_t r = \mathbf{Q}^{-1} \mathcal{J} \nabla H(r) = \mathbf{Q}^{-1} \mathcal{J}(\nabla H)(\mathbf{Q} \tilde{r}). \quad (4.48)$$

On the other hand,

$$\mathcal{J} \nabla K(\tilde{r}) = \mathcal{J} \nabla (H(\mathbf{Q} \tilde{r})) = \mathcal{J} \mathbf{Q}(\nabla H)(\mathbf{Q} \tilde{r}). \quad (4.49)$$

Therefore, if $\mathbf{Q} \mathcal{J} \mathbf{Q} = \mathcal{J}$ then we have the equivalence

$$\left(\partial_t r = \mathcal{J} \nabla H(r) \quad \Leftrightarrow \quad \partial_t \tilde{r} = \mathcal{J} \nabla K(\tilde{r}) \right) \quad (4.50)$$

and this last condition is true since Lemma 4.5-1 implies that \mathbf{Q} is symplectic and $\mathbf{Q}^\top = \mathbf{Q}$.

We shall look for time quasi-periodic solutions of (4.47) in the form

$$\tilde{r}(t, \theta) = \hat{r}(\omega t, \theta),$$

where $\hat{r} : (\varphi, \theta) \in \mathbb{T}^{d+1} \rightarrow \mathbb{R}^2$ and $\omega \in \mathbb{R}^d$ is a non-resonant vector frequency. In this setting, the equation (4.47) becomes

$$\omega \cdot \partial_\varphi \hat{r} = \mathcal{J} \nabla K(\hat{r}).$$

In the sequel, we shall alleviate the notation and denote \hat{r} simply by r . Hence, the foregoing equation becomes

$$\omega \cdot \partial_\varphi r = \mathcal{J} \nabla K(r), \quad K(r) \triangleq H(\mathbf{Q}r), \quad r = (r_1, r_2) \in L_m^2(\mathbb{T}) \times L_m^2(\mathbb{T}). \quad (4.51)$$

The main result of this section reads as follows.

Proposition 4.1. *The linearized equation of (4.51) at a given small state r takes the form*

$$\omega \cdot \partial_\varphi \rho + \mathfrak{L}_r \rho = 0, \quad \mathfrak{L}_r \triangleq -d_r(\mathcal{J} \nabla K(r)),$$

with

$$\begin{aligned} \mathfrak{L}_r = & \begin{pmatrix} \partial_\theta(\mathcal{V}_1(r) \cdot) + \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & \partial_\theta(\mathcal{V}_2(r) \cdot) - \frac{1}{2} \mathcal{H} - \partial_\theta \mathcal{Q} * \cdot \end{pmatrix} \\ & + \partial_\theta \begin{pmatrix} \mathcal{T}_{\mathcal{K}_{1,1}}(r) & \mathcal{T}_{\mathcal{K}_{1,2}}(r) \\ \mathcal{T}_{\mathcal{K}_{2,1}}(r) & \mathcal{T}_{\mathcal{K}_{2,2}}(r) \end{pmatrix}, \end{aligned} \quad (4.52)$$

where

1. the scalar functions $\mathcal{V}_k(r) \triangleq V_k(\mathbf{Q}r)$, $k \in \{1, 2\}$ satisfy

$$\|\mathcal{V}_k(r) - V_k(0)\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|r\|_{s+1}^{q, \gamma, \mathfrak{m}}, \quad (4.53)$$

with $V_k(r)$ and $V_k(0)$ described in (4.2) and (4.13), respectively,

2. the convolution operator $\mathcal{Q} * \cdot$ with even kernel \mathcal{Q} is defined through

$$\forall j \in \mathbb{Z}_m^*, \quad \mathcal{Q} * \mathbf{e}_j \triangleq \frac{\mathbf{r}_j(b)}{|j|} \mathbf{e}_j, \quad (4.54)$$

with $\mathbf{r}_j(b)$ being introduced in Corollary 4.1,

3. for $k, n \in \{1, 2\}$, the operator $\mathcal{T}_{\mathcal{K}_{k,n}}(r)$ is an integral operator in the form (2.15) whose kernel $\mathcal{K}_{k,n}(r)$ is \mathfrak{m} -fold reversibility preserving and satisfies the estimates: for any $s \geq s_0$

$$\|\mathcal{K}_{k,n}(r)\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|r\|_{s+1}^{q, \gamma, \mathfrak{m}}$$

and

$$\|\Delta_{12} \mathcal{K}_{k,n}(r)\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|\Delta_{12} r\|_{s+1}^{q, \gamma, \mathfrak{m}} + \|\Delta_{12} r\|_{s_0+1}^{q, \gamma, \mathfrak{m}} \max_{\ell \in \{1, 2\}} \|r_\ell\|_{s+1}^{q, \gamma, \mathfrak{m}}.$$

Proof. According to (4.48), (4.49) and (4.50), one has

$$\mathcal{J}\nabla K(r) = \mathbf{Q}^{-1} \mathcal{J}(\nabla H)(\mathbf{Q}r). \quad (4.55)$$

Differentiating this identity with respect to r in the direction ρ and using Lemma 4.1 lead to

$$\begin{aligned} d_r(\mathcal{J}\nabla K(r))\rho &= \mathbf{Q}^{-1}(d_r(\mathcal{J}\nabla H)(\mathbf{Q}r))\mathbf{Q}\rho \\ &= \mathbf{Q}^{-1} \mathcal{J}\mathbf{M}_{\mathbf{Q}r} \mathbf{Q}\rho \\ &= \mathbf{Q}^{-1} \mathcal{J}\mathbf{M}_0 \mathbf{Q}\rho + \mathbf{Q}^{-1} \mathcal{J}(\mathbf{M}_{\mathbf{Q}r} - \mathbf{M}_0) \mathbf{Q}\rho. \end{aligned} \quad (4.56)$$

By virtue of (4.46), (4.29), (4.13) and recalling (2.30) and (4.54) we may write

$$\mathbf{Q}^{-1} \mathcal{J}\mathbf{M}_0 \mathbf{Q} = - \begin{pmatrix} V_1(0)\partial_\theta + \frac{1}{2}\mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & V_2(0)\partial_\theta - \frac{1}{2}\mathcal{H} - \partial_\theta \mathcal{Q} * \cdot \end{pmatrix}. \quad (4.57)$$

On the other hand, from Lemma 4.1 and Lemma 4.2 we deduce that

$$\mathbf{M}_{\mathbf{Q}r} - \mathbf{M}_0 = \begin{pmatrix} -f_1(r) & 0 \\ 0 & f_2(r) \end{pmatrix} + \begin{pmatrix} -L_{1,1}(\mathbf{Q}r) + \mathcal{K}_1 * \cdot & L_{1,2}(\mathbf{Q}r) - \mathcal{K}_b * \cdot \\ L_{2,1}(\mathbf{Q}r) - \mathcal{K}_b * \cdot & -L_{2,2}(\mathbf{Q}r) + \mathcal{K}_1 * \cdot \end{pmatrix} \quad (4.58)$$

with

$$\forall k \in \{1, 2\}, \quad f_k(r) \triangleq V_k(\mathbf{Q}r) - V_k(0). \quad (4.59)$$

Notice that if r satisfies (3.23) and (3.25) then, by virtue of (4.10), one gets that $f_k(r)$ itself satisfies the same symmetries,

$$f_k(r)(-\varphi, -\theta) = f_k(r)(\varphi, \theta) = f_k(r)(\varphi, \theta + \frac{2\pi}{m}). \quad (4.60)$$

We shall now turn to the quantitative estimates. For this aim, we shall first give the following decompositions. According to (3.7), we can write

$$\begin{aligned} A_{k,k}(r)(\varphi, \theta, \eta) &= \left((R_k(\varphi, \theta) - R_k(\varphi, \eta))^2 + 4R_k(\varphi, \theta)R_k(\varphi, \eta) \sin^2 \left(\frac{\eta - \theta}{2} \right) \right)^{\frac{1}{2}} \\ &= 2b_k \left| \sin \left(\frac{\eta - \theta}{2} \right) \right| \left(\left(\frac{R_k(\varphi, \theta) - R_k(\varphi, \eta)}{2b_k \sin \left(\frac{\eta - \theta}{2} \right)} \right)^2 + \frac{1}{b_k^2} R_k(\varphi, \theta)R_k(\varphi, \eta) \right)^{\frac{1}{2}} \\ &\triangleq 2b_k \left| \sin \left(\frac{\eta - \theta}{2} \right) \right| v_k(r)(\varphi, \theta, \eta). \end{aligned}$$

the function $v_k(r)$ is smooth with respect to each variables and with respect to r . In addition $v_k(0) = 1$. An application of Lemma 2.2 and Lemma 2.1-(iii) gives

$$\|v_k(r) - 1\|_s^{q,\gamma,m} \lesssim \|r\|_{s+1}^{q,\gamma,m}, \quad \|\Delta_{12}v_k\|_s^{q,\gamma,m} \lesssim \|\Delta_{12}r\|_{s+1}^{q,\gamma,m} + \|\Delta_{12}r\|_{s_0+1}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|r_\ell\|_{s+1}^{q,\gamma,m}. \quad (4.61)$$

We can also write

$$\begin{aligned} A_{k,3-k}^2(r)(\varphi, \theta, \eta) &= R_k^2(\varphi, \theta) + R_{3-k}^2(\eta) - 2R_k(\varphi, \theta)R_{3-k}(\varphi, \eta) \cos(\eta - \theta) \\ &= A_{k,3-k}^2(0)(\varphi, \theta, \eta)(1 + h_k(r)(\varphi, \theta, \eta)), \end{aligned}$$

where

$$h_k(r)(\varphi, \theta, \eta) \triangleq 2 \frac{r_k(\varphi, \theta) + r_{3-k}(\varphi, \eta) - \cos(\eta - \theta)(R_k(\varphi, \theta)R_{3-k}(\varphi, \eta) - b_k b_{3-k})}{b_k^2 + b_{3-k}^2 - 2b_k b_{3-k} \cos(\eta - \theta)}.$$

The function h_k satisfies by composition laws in Lemma 2.1-(iii)

$$\|h_k(r)\|_s^{q,\gamma,m} \lesssim \|r\|_s^{q,\gamma,m}, \quad \|\Delta_{12}h_k\|_s^{q,\gamma,m} \lesssim \|\Delta_{12}r\|_s^{q,\gamma,m}. \quad (4.62)$$

According to the foregoing decompositions and the estimates (4.61), (4.62) together with (4.59), (4.2), (4.3), the product and composition laws in Lemma 2.1-(iii) imply

$$\|f_k(r)\|_s^{q,\gamma,m} \lesssim \|r\|_{s+1}^{q,\gamma,m} \quad (4.63)$$

and

$$\|\Delta_{12}f_k\|_s^{q,\gamma,m} \lesssim \|\Delta_{12}r\|_{s+1}^{q,\gamma,m} + \|\Delta_{12}r\|_{s_0+1}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|r_\ell\|_{s+1}^{q,\gamma,m}. \quad (4.64)$$

In view of (4.4) and (4.14), we also have the following decompositions for $k \in \{1, 2\}$

$$L_{k,k}(\mathbf{Q}r) = \mathcal{K}_1 * \cdot + \mathcal{T}_{\mathbb{K}_{k,k}}(r) \quad \text{and} \quad L_{k,3-k}(\mathbf{Q}r) = \mathcal{K}_b * \cdot + \mathcal{T}_{\mathbb{K}_{k,3-k}}(r), \quad (4.65)$$

where $\mathcal{T}_{\mathbb{K}_{k,k}}(r)$ and $\mathcal{T}_{\mathbb{K}_{k,3-k}}(r)$ are integral operators with smooth kernels

$$\mathbb{K}_{k,k}(r)(\varphi, \theta, \eta) \triangleq \log(v_k(\mathbf{Q}r)(\varphi, \theta, \eta))$$

and

$$\mathbb{K}_{k,3-k}(r)(\varphi, \theta, \eta) \triangleq \frac{1}{2} \log(1 + h_k(\mathbf{Q}r)(\varphi, \theta, \eta)).$$

Moreover, if r satisfies (3.23) and (3.25) then, for all $k, n \in \{1, 2\}$, one can easily check, using in particular (4.11), that

$$\mathbb{K}_{k,n}(r)(-\varphi, -\theta, -\eta) = \mathbb{K}_{k,n}(r)(\varphi, \theta, \eta) = \mathbb{K}_{k,n}(r)(\varphi, \theta + \frac{2\pi}{m}, \eta + \frac{2\pi}{m}). \quad (4.66)$$

It is clear that (4.41)-(4.43) imply the continuity of \mathbf{Q} on \mathbf{H}_m^s . Thus, applying the composition laws in Lemma 2.1-(iii) together with (4.61) and (4.62), we infer

$$\forall (k, n) \in \{1, 2\}^2, \quad \|\mathbb{K}_{k,n}(r)\|_s^{q,\gamma,m} \lesssim \|r\|_{s+1}^{q,\gamma,m} \quad (4.67)$$

and

$$\forall (k, n) \in \{1, 2\}^2, \quad \|\Delta_{12}\mathbb{K}_{k,n}(r)\|_s^{q,\gamma,m} \lesssim \|\Delta_{12}r\|_{s+1}^{q,\gamma,m} + \|\Delta_{12}r\|_{s_0+1}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|r_\ell\|_{s+1}^{q,\gamma,m}. \quad (4.68)$$

Putting together (4.58) and (4.65) we deduce that

$$\mathbf{Q}^{-1}\mathcal{J}(\mathbf{M}_{\mathbf{Q}r} - \mathbf{M}_0)\mathbf{Q} = -\mathbf{Q}^{-1}\partial_\theta \begin{pmatrix} f_1(r) & 0 \\ 0 & f_2(r) \end{pmatrix} \mathbf{Q} - \mathbf{Q}^{-1}\partial_\theta \begin{pmatrix} \mathcal{T}_{\mathbb{K}_{1,1}}(r) & -\mathcal{T}_{\mathbb{K}_{1,2}}(r) \\ \mathcal{T}_{\mathbb{K}_{2,1}}(r) & -\mathcal{T}_{\mathbb{K}_{2,2}}(r) \end{pmatrix} \mathbf{Q}.$$

Combining the last identity with (4.56) and (4.57) we find (4.52) with

$$\begin{pmatrix} \mathcal{T}_{\mathcal{K}_{1,1}}(r) & \mathcal{T}_{\mathcal{K}_{1,2}}(r) \\ \mathcal{T}_{\mathcal{K}_{2,1}}(r) & \mathcal{T}_{\mathcal{K}_{2,2}}(r) \end{pmatrix} \triangleq \mathbf{Q}^{-1} \begin{pmatrix} f_1(r) & 0 \\ 0 & f_2(r) \end{pmatrix} \mathbf{Q} - \begin{pmatrix} f_1(r) & 0 \\ 0 & f_2(r) \end{pmatrix} + \mathbf{Q}^{-1} \begin{pmatrix} \mathcal{T}_{\mathbb{K}_{1,1}}(r) & -\mathcal{T}_{\mathbb{K}_{1,2}}(r) \\ \mathcal{T}_{\mathbb{K}_{2,1}}(r) & -\mathcal{T}_{\mathbb{K}_{2,2}}(r) \end{pmatrix} \mathbf{Q}.$$

By virtue of the preceding estimate, (4.41), (4.43), (4.63) and (4.67), together with Lemma 2.8 we find through straightforward computations that the kernels $\mathcal{K}_{k,n}(r)$ satisfy the following estimate

$$\forall (k, n) \in \{1, 2\}^2, \quad \|\mathcal{K}_{k,n}(r)\|_s^{q,\gamma,m} \lesssim \|r\|_{s+1}^{q,\gamma,m}.$$

Similar argument as before using in particular (4.68) and (4.64) implies

$$\forall (k, n) \in \{1, 2\}^2, \quad \|\Delta_{12}\mathcal{K}_{k,n}(r)\|_s^{q,\gamma,m} \lesssim \|\Delta_{12}r\|_{s+1}^{q,\gamma,m} + \|\Delta_{12}r\|_{s_0+1}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|r_\ell\|_{s+1}^{q,\gamma,m}.$$

As for the symmetry property, it can be obtained easily from the structure of the kernel. Actually, one may check from (4.66), (4.60) and (4.42) that

$$r(-\varphi, -\theta) = r(\varphi, \theta) \implies \mathcal{K}_{k,n}(r)(-\varphi, -\theta, -\eta) = \mathcal{K}_{k,n}(r)(\varphi, \theta, \eta), \quad (4.69)$$

$$r(\varphi, \theta + \frac{2\pi}{m}) = r(\varphi, \theta) \implies \mathcal{K}_{k,n}(r)(\varphi, \theta + \frac{2\pi}{m}, \eta + \frac{2\pi}{m}) = \mathcal{K}_{k,n}(r)(\varphi, \theta, \eta). \quad (4.70)$$

The proof of the desired results is now complete. \square

Remark 4.1. The linearized equation of (4.47) at $r = 0$ takes the form

$$\partial_t \rho = \mathcal{J} \nabla K_{\mathbf{L}_0}(\rho), \quad \rho = (\rho_1, \rho_2) \in L_{\mathbf{m}}^2(\mathbb{T}) \times L_{\mathbf{m}}^2(\mathbb{T}), \quad (4.71)$$

where \mathcal{J} is defined in (3.14), $K_{\mathbf{L}_0}$ is the quadratic Hamiltonian

$$K_{\mathbf{L}_0}(\rho) \triangleq \frac{1}{2} \langle \mathbf{L}_0 \rho, \rho \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} = - \sum_{j \in \mathbb{Z}_{\mathbf{m}}^*} \left(\frac{\Omega_{j,1}(b)}{2j} |\rho_{j,1}|^2 - \frac{\Omega_{j,2}(b)}{2j} |\rho_{j,2}|^2 \right) \quad (4.72)$$

and \mathbf{L}_0 is the operator defined by (4.44). Thus, real-valued oscillating solution of the linearized contour dynamics equation (4.71) are given by

$$\rho(t, \theta) = \sum_{j \in \mathbb{Z}_{\mathbf{m}}^*} A_j \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(\Omega_{1,j}(b)t - j\theta)} + B_j \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(\Omega_{2,j}(b)t - j\theta)},$$

with $\overline{A_j} = A_{-j}$, $\overline{B_j} = B_{-j}$.

4.4 Geometric structure of the equilibrium frequencies

This section is devoted to some useful properties of the equilibrium frequencies. We shall first discuss their monotonicity and prove some useful bounds. Then, we shall be concerned with their non-degeneracy through the study of the transversality conditions. Those latter are crucial in the measure estimates of the final Cantor set giving rise to quasi-periodic solutions emerging at the linear and nonlinear levels. We have the following lemma.

Lemma 4.6. Let $\Omega > 0$ and \mathbf{m}^*, b^* be defined as in Corollary 4.1. Then the following holds true.

1. For all $|j| \geq \mathbf{m}^*$ and $k \in \{1, 2\}$, $\Omega_{-j,k}(b) = -\Omega_{j,k}(b)$.
2. The sequence $\left(-\frac{\Delta_j(b)}{j^2} \right)_{j \geq \mathbf{m}^*}$ is positive increasing. Recall that $\Delta_j(b)$ was defined in (4.18).
3. The sequence $\left(\frac{\Omega_{j,1}(b)}{j} \right)_{j \geq \mathbf{m}^*}$ is positive increasing and the sequence $\left(\frac{\Omega_{j,2}(b)}{j} \right)_{j \geq \mathbf{m}^*}$ is positive decreasing. Moreover, for all $|j| \geq \mathbf{m}^*$ and $k \in \{1, 2\}$ we have

$$\lim_{n \rightarrow \infty} \frac{\Omega_{j,k}(b)}{j} = \Omega + (2 - k) \frac{1 - b^2}{2}. \quad (4.73)$$

and

$$|\Omega_{j,k}(b)| \geq \Omega |j|.$$

4. For all $\mathbf{m} \geq \mathbf{m}^*$, there exists $\Omega_{\mathbf{m}}^* = \Omega^*(b^*, \mathbf{m}) > 0$ satisfying

$$\lim_{\mathbf{m} \rightarrow \infty} \Omega_{\mathbf{m}}^* = 0 \quad (4.74)$$

such that for all $\Omega > \Omega_{\mathbf{m}}^*$ the sequence $(\Omega_{j,2}(b))_{j \geq \mathbf{m}}$ is increasing.

5. There exists $c > 0$ such that, for all $k \in \{1, 2\}$,

$$\forall \Omega > \Omega_{\mathbf{m}}^*, \quad \forall b \in [0, b^*], \quad \forall |j| \geq \mathbf{m}^*, \quad \forall |j'| \geq \mathbf{m}^*, \quad |\Omega_{j,k}(b) - \Omega_{j',k}(b)| \geq c|j - j'|.$$

6. Given $q_0 \in \mathbb{N}$, there exists $C > 0$ such that, for all $k \in \{1, 2\}$,

$$\forall |j| \geq \mathbf{m}^*, \quad \forall |j'| \geq \mathbf{m}^*, \quad \max_{q \in [0, q_0]} \sup_{b \in [0, b^*]} \left| \partial_b^q (\Omega_{j,k}(b) - \Omega_{j',k}(b)) \right| \leq C|j - j'|.$$

Proof. **1.** It follows immediately from (4.28).

2. In order to study the discrete function $j \mapsto -\frac{\Delta_j(b)}{j^2}$ we shall consider its continuous version

$$\forall x \geq \mathfrak{m}^*, \quad g(x) \triangleq -\frac{\Delta_x(b)}{x^2} = \frac{1}{x^2} \left(\frac{1-b^2}{2}x - 1 \right)^2 - \frac{b^{2x}}{x^2}.$$

Differentiating with respect to x and using (4.36), we conclude that

$$g'(x) = \frac{2}{x^3} \left(\frac{1-b^2}{2}x - 1 + b^{2x} \right) - \frac{2b^{2x}}{x^2} \log(b) > 0.$$

Thus, the mapping $j \mapsto -\Delta_j(b)/j^2$ is strictly increasing.

3. The monotonicity of the sequences $\left(\frac{\Omega_{j,k}(b)}{j} \right)_{j \geq \mathfrak{m}^*}$ follows from the identity

$$\frac{\Omega_{j,k}(b)}{j} = \left(\Omega + \frac{1-b^2}{4} \right) + \frac{(-1)^{k+1}}{2} \sqrt{\frac{-\Delta_j(b)}{j^2}}$$

and the second point. Moreover, from the last identity we also conclude that

$$\frac{\Omega_{j,1}(b)}{j} \geq \Omega.$$

Next, from (4.29)-(4.30) we obtain (4.73). Since $\left(\frac{\Omega_{j,2}(b)}{j} \right)_{j \geq \mathfrak{m}^*}$ is decreasing, then from (4.73) we infer that

$$\frac{\Omega_{j,2}(b)}{j} \geq \lim_{j \rightarrow \infty} \frac{\Omega_{j,2}(b)}{j} = \Omega > 0.$$

This ends the proof of the third point.

4. Consider the continuous extension of the discrete mapping $j \mapsto \Omega_{j,2}(b)$,

$$\forall (b, x) \in [0, b^*] \times (\mathfrak{m}^*, \infty), \quad h(b, x) \triangleq \Omega x + \frac{1-b^2}{4}x - \frac{1}{2} \sqrt{-\Delta_x(b)}.$$

Differentiating with respect to x and using (4.18) and (4.36) lead to

$$\begin{aligned} \partial_x h(b, x) &= \frac{b^{2x} \log(b) - \frac{1-b^2}{2} \left(\frac{1-b^2}{2}x - 1 - \sqrt{-\Delta_x(b)} \right) + 2\Omega \sqrt{-\Delta_x(b)}}{2\sqrt{-\Delta_x(b)}} \\ &= \frac{b^{2x} \log(b) + \left(\frac{1-b^2}{2}x - 1 \right) \left[2\Omega \sqrt{1 - b^{2x} \left(\frac{1-b^2}{2}x - 1 \right)^{-2}} + \frac{1-b^2}{2} \left(\sqrt{1 - b^{2x} \left(\frac{1-b^2}{2}x - 1 \right)^{-2}} - 1 \right) \right]}{2\sqrt{-\Delta_x(b)}}. \end{aligned}$$

According to (4.38) we have, for all $b < b^*$ and $x \geq \mathfrak{m} \geq \mathfrak{m}^*$,

$$0 < \sqrt{1 - (b^*)^{2\mathfrak{m}} \left(\frac{1-(b^*)^2}{2}\mathfrak{m} - 1 \right)^{-2}} \leq \sqrt{1 - b^{2x} \left(\frac{1-b^2}{2}x - 1 \right)^{-2}} < 1. \quad (4.75)$$

Thus, in view of (4.36) and (4.75) we get

$$\begin{aligned} 2\sqrt{-\Delta_x(b)} \partial_x h(b, x) &\geq b^{2x} \log(b) + \left(\frac{1-b^2}{2}x - 1 \right) \left[2\Omega \sqrt{1 - (b^*)^{2\mathfrak{m}} \left(\frac{1-(b^*)^2}{2}\mathfrak{m} - 1 \right)^{-2}} \right. \\ &\quad \left. + \frac{1-(b^*)^2}{2} \left(\sqrt{1 - (b^*)^{2\mathfrak{m}} \left(\frac{1-(b^*)^2}{2}\mathfrak{m} - 1 \right)^{-2}} - 1 \right) \right]. \end{aligned}$$

Setting

$$\Omega_{\mathfrak{m}}^* \triangleq \frac{\frac{1-(b^*)^2}{2} \left(1 - \sqrt{1 - (b^*)^{2\mathfrak{m}} \left(\frac{1-(b^*)^2}{2}\mathfrak{m} - 1 \right)^{-2}} \right) - \min_{b \in [0, b^*]} (b^{\mathfrak{m}} \log(b))}{2\sqrt{1 - (b^*)^{2\mathfrak{m}} \left(\frac{1-(b^*)^2}{2}\mathfrak{m} - 1 \right)^{-2}}} > 0 \quad (4.76)$$

gives

$$\forall \Omega > \Omega_{\mathfrak{m}}^*, \quad 2\sqrt{-\Delta_x(b)} \partial_x h(b, x) > \left(b^x + 1 - \frac{1-b^2}{2}x \right) \min_{b \in [0, b^*]} (b^{\mathfrak{m}} \log(b)) \stackrel{(4.23)}{=} f(b, x) \min_{b \in [0, b^*]} (b^{\mathfrak{m}} \log(b)).$$

Then by (4.34) we conclude that

$$\forall \Omega > \Omega_{\mathbf{m}}^*, \quad \partial_x h(b, x) > 0.$$

Taking the limit $\mathbf{m} \rightarrow \infty$ in (4.76) gives immediately (4.74). This ends the proof of the fourth point.
5. Since $j \mapsto \Omega_{j,k}(b)$, $k \in \{1, 2\}$, are odd then it is enough to check the result for $j \in \mathbb{N}^*$. The estimate on the sum $|\Omega_{j,k}(b) + \Omega_{j',k}(b)|$ easily follows from the positivity of the sequences $(\Omega_{j,k}(b))_{j \geq \mathbf{m}^*}$, $k \in \{1, 2\}$, and the third point, namely,

$$\forall b \in [0, b^*], \quad |\Omega_{j,k}(b) + \Omega_{j',k}(b)| = \Omega_{j,k}(b) + \Omega_{j',k}(b) \geq \Omega(j + j').$$

Next we shall prove the estimate on the difference. In view of (4.29), for all $j, j' \geq \mathbf{m}^*$ with $j \neq j'$, one has

$$\Omega_{j,k}(b) - \Omega_{j',k}(b) = \left(\Omega + (2 - k) \frac{1-b^2}{2} \right) (j - j') + (-1)^{k+1} (\mathbf{r}_j(b) - \mathbf{r}_{j'}(b)). \quad (4.77)$$

It follows that

$$|\Omega_{j,k}(b) - \Omega_{j',k}(b)| \geq \Omega |j - j'| - \sup_{b \in [0, b^*]} |\mathbf{r}_j(b) - \mathbf{r}_{j'}(b)|.$$

Using Taylor formula combined with (4.30) gives, for any $b \in [0, b^*]$,

$$\begin{aligned} |\mathbf{r}_j(b) - \mathbf{r}_{j'}(b)| &\leq C_0 \left| \int_{j'}^j \frac{dx}{x^2} \right| \\ &\leq C_0 \frac{|j - j'|}{jj'}. \end{aligned}$$

This implies that

$$|\Omega_{j,k}(b) - \Omega_{j',k}(b)| \geq \left(\Omega - \frac{C_0}{jj'} \right) |j - j'|. \quad (4.78)$$

Therefore, there exists N such that if $jj' > N$ the desired inequality holds. For $jj' \leq N$ we shall use the one-to-one property of $j \mapsto \Omega_{j,k}(b)$ combined with the continuity of $b \in [0, b^*] \mapsto \Omega_{j,k}(b) - \Omega_{j',k}(b)$ to get, for any $j \neq j' \in \llbracket \mathbf{m}^*, N \rrbracket$,

$$\forall \Omega > \Omega_{\mathbf{m}}^*, \quad \inf_{b \in [0, b^*]} |\Omega_{j,k}(b) - \Omega_{j',k}(b)| \triangleq c_{jj'}^k > 0.$$

Consequently

$$\inf_{\substack{j \neq j' \in \llbracket \mathbf{m}^*, N \rrbracket \\ b \in [0, b^*]}} |\Omega_{j,k}(b) - \Omega_{j',k}(b)| = \inf_{j \neq j' \in \llbracket \mathbf{m}^*, N \rrbracket} c_{jj'}^k > 0.$$

Taking

$$c \triangleq \frac{1}{N} \min \left(\inf_{j \neq j' \in \llbracket \mathbf{m}^*, N \rrbracket} c_{jj'}^k, \quad N\Omega - C_0 \right)$$

and combining the last inequality with (4.78) we get the desired result.

6. Differentiating (4.77) gives

$$\begin{aligned} \partial_b (\Omega_{j,k}(b) - \Omega_{j',k}(b)) &= -(2 - k)b(j - j') + (-1)^{k+1} \partial_b (\mathbf{r}_j(b) - \mathbf{r}_{j'}(b)), \\ \partial_b^2 (\Omega_{j,k}(b) - \Omega_{j',k}(b)) &= -(2 - k)(j - j') + (-1)^{k+1} \partial_b^2 (\mathbf{r}_j(b) - \mathbf{r}_{j'}(b)), \\ \forall q \geq 3, \quad \partial_b^q (\Omega_{j,k}(b) - \Omega_{j',k}(b)) &= (-1)^{k+1} (\partial_b^q \mathbf{r}_j(b) - \partial_b^q \mathbf{r}_{j'}(b)). \end{aligned}$$

By the mean value theorem combined with (4.30) we conclude the proof of Lemma 4.6. \square

Non-degeneracy and transversality. Through the rest of this section we shall follow the approach developed in [9, 19] to discuss the non-degeneracy and the transversality properties of the linear frequencies. Let us first recall the definition of the non-degeneracy for vector-valued functions.

Definition 4.1. Let $N \in \mathbb{N}^*$. A function $f \triangleq (f_1, \dots, f_N) : [\alpha_1, \alpha_2] \rightarrow \mathbb{R}^N$, with $\alpha_1 < \alpha_2$, is called non-degenerate if, for any vector $c \triangleq (c_1, \dots, c_N) \in \mathbb{R}^N \setminus \{0\}$, the scalar function $f \cdot c = f_1 c_1 + \dots + f_N c_N$ is not identically zero on the whole interval $[\alpha_1, \alpha_2]$.

We have the following result.

Lemma 4.7. Let $\Omega > 0$ and \mathfrak{m}^*, b^* be defined as in Corollary 4.1. Fix an integer $\mathfrak{m} \geq \mathfrak{m}^*$ and consider the finite subsets

$$\forall k \in \{1, 2\}, \quad S_k \subset \mathbb{Z}_{\mathfrak{m}} \cap \mathbb{N}^* \quad \text{with} \quad |S_k| < \infty.$$

Then the following hold true.

1. If $|S_1 \cap S_2| \leq 1$ then the vector valued function

$$[0, b^*] \ni b \mapsto \left((\Omega_{j,1}(b))_{j \in S_1}, (\Omega_{j,2}(b))_{j \in S_2} \right)$$

is non-degenerate.

2. If $|S_1 \cap S_2| = 0$ then the vector valued functions

$$\begin{aligned} [0, b^*] \ni b &\mapsto \left((\Omega_{j,1}(b))_{j \in S_1}, (\Omega_{j,2}(b))_{j \in S_2}, \mathbf{v}_1(b), \mathbf{v}_2(b) \right), \\ [0, b^*] \ni b &\mapsto \left((\Omega_{j,1}(b))_{j \in S_1}, (\Omega_{j,2}(b))_{j \in S_2}, \mathbf{v}_k(b) \right), \quad k \in \{1, 2\} \end{aligned}$$

are non-degenerate, where the \mathbf{v}_k are defined in (4.13).

Proof. We point out that the linear frequencies (4.28) are very similar to the linear frequencies close to the Kirchhoff ellipses, studied in [17]. Thus we shall use the same arguments developed in [17, Lemma 5.2] with slight modifications. According to (4.28) the functions $b \mapsto \Omega_{j,k}(b)$, $k \in \{1, 2\}$, are well defined and analytic in a full neighborhood of $b = 0$. Moreover, by (4.29) and (4.37) the frequencies $\Omega_{j,k}(b)$ write

$$\begin{aligned} \Omega_{j,k}(b) &= A_{j,k}(z) + (-1)^{k+1} B_j(z) \triangleq \tilde{\Omega}_{j,k}(z), \\ z \triangleq b^2, \quad A_{j,k}(z) &\triangleq \Omega_j + \frac{2-k}{2} j(1-z) + \frac{(-1)^k}{2}, \quad B_j(z) \triangleq \mathbf{r}_j(b) \underset{z \rightarrow 0}{=} -\frac{z^j}{2(j-2)} + O(z^{j+1}). \end{aligned} \quad (4.79)$$

1. In view of Definition 4.1 one has to prove that, for all $c \triangleq ((c_{j,1})_{j \in S_1}, (c_{j,2})_{j \in S_2}) \in \mathbb{R}^{|S_1|+|S_2|} \setminus \{0\}$, the function

$$z \mapsto \sum_{j \in S_1 \setminus (S_1 \cap S_2)} c_{j,1} \tilde{\Omega}_{j,1}(z) + \sum_{j \in S_2 \setminus (S_1 \cap S_2)} c_{j,2} \tilde{\Omega}_{j,2}(z) + \sum_{j \in S_1 \cap S_2} (c_{j,1} \tilde{\Omega}_{j,1}(z) + c_{j,2} \tilde{\Omega}_{j,2}(z))$$

is not identically zero on the interval $[0, (b^*)^2]$. By contradiction, suppose that there exists $c \triangleq ((c_{j,1})_{j \in S_1}, (c_{j,2})_{j \in S_2}) \in \mathbb{R}^{|S_1|+|S_2|} \setminus \{0\}$ such that for any $|z| \leq (b^*)^2$,

$$\sum_{j \in S_1 \setminus (S_1 \cap S_2)} c_{j,1} \tilde{\Omega}_{j,1}(z) + \sum_{j \in S_2 \setminus (S_1 \cap S_2)} c_{j,2} \tilde{\Omega}_{j,2}(z) + \sum_{j \in S_1 \cap S_2} c_{j,1} \tilde{\Omega}_{j,1}(z) + c_{j,2} \tilde{\Omega}_{j,2}(z) = 0. \quad (4.80)$$

Writing

$$S_1 \cup S_2 = \{j_1, j_2, \dots, j_d\}, \quad \text{with} \quad \mathfrak{m} \leq j_1 < j_2 < \dots < j_d,$$

then differentiating with respect to z the identity in (4.80), we find, since $\mathfrak{m} \geq 3$,

$$\begin{cases} \tilde{c}_{j_1} D_z^{(j_1)} B_{j_1}(z) + \dots + \tilde{c}_{j_d} D_z^{(j_1)} B_{j_d}(z) = 0, \\ \dots\dots\dots \\ \tilde{c}_{j_1} D_z^{(j_d)} B_{j_1}(z) + \dots + \tilde{c}_{j_d} D_z^{(j_d)} B_{j_d}(z) = 0, \end{cases}$$

where

$$\tilde{c}_j \triangleq \begin{cases} c_{j,1}, & \text{if } j \in S_1 \setminus (S_1 \cap S_2), \\ -c_{j,2}, & \text{if } j \in S_2 \setminus (S_1 \cap S_2), \\ c_{j,1} - c_{j,2}, & \text{if } j \in S_1 \cap S_2. \end{cases}$$

The latter is a linear system that can be recast as in a matricial form as

$$\mathcal{M}(z)\tilde{c} = 0, \quad \mathcal{M}(z) \triangleq \begin{pmatrix} D_z^{(j_1)} B_{j_1}(z) & \dots & D_z^{(j_1)} B_{j_d}(z) \\ \vdots & \ddots & \vdots \\ D_z^{(j_d)} B_{j_1}(z) & \dots & D_z^{(j_d)} B_{j_d}(z) \end{pmatrix}, \quad \tilde{c} \triangleq \begin{pmatrix} \tilde{c}_{j_1} \\ \vdots \\ \tilde{c}_{j_d} \end{pmatrix}.$$

Note, from (4.79), that for all $j \geq m$ we have

$$D_z^{(j)} B_j(0) = -\frac{j!}{2(j-2)} \quad \text{and} \quad \forall 2 \leq m < j, \quad D_z^{(m)} B_j(0) = 0.$$

It follows that, for some real constants $\alpha_{i,j}$, we have that

$$\mathcal{M}(0) = \begin{pmatrix} -\frac{j_1!}{2(j_1-2)} & 0 & 0 & \dots & 0 \\ \alpha_{2,1} & -\frac{j_2!}{2(j_2-2)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_{d-1,1} & \dots & \alpha_{d-1,d-2} & -\frac{j_{d-1}!}{2(j_{d-1}-2)} & 0 \\ \alpha_{d,1} & \dots & \alpha_{d,d-2} & \alpha_{d,d-1} & -\frac{j_d!}{2(j_d-2)} \end{pmatrix}$$

which is a triangular matrix whose determinant is given by

$$\det \mathcal{M}(0) = (-1)^d \prod_{i=1}^d \frac{j_i!}{2(j_i-2)} \neq 0.$$

It follows that $\tilde{c} = 0$, i.e.

$$\forall j \in (S_1 \cup S_2) \setminus (S_1 \cap S_2), \quad c_{j,k} = 0 \quad \text{and} \quad \forall j \in S_1 \cap S_2, \quad c_{j,1} = c_{j,2}. \quad (4.81)$$

Inserting (4.81) into (4.80) evaluated at $z = 0$, we get from (4.79) that

$$(2\Omega + \frac{1}{2}) \sum_{j \in S_1 \cap S_2} j c_{j,1} = 0.$$

Using the fact that $\Omega > 0$, if $S_1 \cap S_2 = \{j_0\}$ then we have

$$c_{j_0,1} = 0.$$

This with (4.81) lead to a contradiction proving the first point.

2. Next, we shall prove that the function

$$[0, b^*] \ni b \mapsto \left((\Omega_{j,1}(b))_{j \in S_1}, (\Omega_{j,2}(b))_{j \in S_2}, \mathbf{v}_1(b), \mathbf{v}_2(b) \right)$$

is non-degenerate according to the Definition 4.1 provided that $S_1 \cap S_2 = \emptyset$. Suppose, by contradiction, that there exists

$$c \triangleq ((c_{j,1})_{j \in S_1}, (c_{j,2})_{j \in S_2}, c_{0,1}, c_{0,2}) \in \mathbb{R}^{|S_1|+|S_2|+2} \setminus \{0\}$$

such that for any $|z| \leq (b^*)^2$,

$$c_{0,1} \tilde{\mathbf{v}}_1(z) + c_{0,2} \tilde{\mathbf{v}}_2(z) + \sum_{j \in S_1} c_{j,1} \tilde{\Omega}_{j,1}(z) + \sum_{j \in S_2} c_{j,2} \tilde{\Omega}_{j,2}(z) = 0, \quad (4.82)$$

where we denote

$$\forall k \in \{1, 2\}, \quad \tilde{\mathbf{v}}_k(z) \triangleq \mathbf{v}_k(b) = \Omega + \frac{2-k}{2}(1-z).$$

Arguing in a similar way to the first case we conclude by a differentiation argument that

$$\forall j \in S_1 \cup S_2, \quad \forall k \in \{1, 2\}, \quad c_{j,k} = 0.$$

Plugging these identities into (4.82) we find

$$c_{0,1}\tilde{\mathbf{v}}_1(z) + c_{0,2}\tilde{\mathbf{v}}_2(z) = 0.$$

That is

$$\Omega(c_{0,1} + c_{0,2}) + \frac{1-z}{2}c_{0,1} = 0.$$

The last expression being true for any $|z| \leq (b^*)^2$, then using the fact that $\Omega \neq 0$ we infer

$$c_{0,1} = c_{0,2} = 0.$$

Thus, the vector c is vanishing and this contradicts the assumption. The proof of the non-degeneracy of the function

$$[0, b^*] \ni b \mapsto \left((\Omega_{j,1}(b))_{j \in S_1}, (\Omega_{j,2}(b))_{j \in S_2}, \mathbf{v}_k(b) \right), \quad k \in \{1, 2\}$$

can be obtained from the previous case by choosing

$$c \triangleq ((c_{j,1})_{j \in S_1}, (c_{j,2})_{j \in S_2}, c_{0,k}, c_{0,3-k}) = ((c_{j,1})_{j \in S_1}, (c_{j,2})_{j \in S_2}, c_{0,k}, 0).$$

This ends the proof of Lemma 4.7. \square

Let $\Omega > 0$ and \mathbf{m}^*, b^* be defined as in Corollary 4.1. Fix an integer $\mathbf{m} \geq \mathbf{m}^*$ and consider the finite subsets

$$\forall k \in \{1, 2\}, \quad \mathbb{S}_k \subset \mathbb{Z}_{\mathbf{m}} \cap \mathbb{N}^*, \quad \text{with} \quad d_k \triangleq |\mathbb{S}_k| < \infty \quad \text{and} \quad \mathbb{S}_1 \cap \mathbb{S}_2 = \emptyset. \quad (4.83)$$

For all $b \in [0, b^*]$ define the tangential equilibrium frequency vector by

$$\omega_{\text{Eq}}(b) \triangleq (\omega_{\text{Eq},1}(b), \omega_{\text{Eq},2}(b)) \in \mathbb{R}^d, \quad \text{with} \quad \omega_{\text{Eq},k}(b) \triangleq (\Omega_{j,k}(b))_{j \in \mathbb{S}_k} \in \mathbb{R}^{d_k}, \quad d \triangleq d_1 + d_2 \quad (4.84)$$

and set

$$\mathbb{S} \triangleq \mathbb{S}_1 \cup \mathbb{S}_2, \quad \bar{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S}), \quad \bar{\mathbb{S}}_0 \triangleq \bar{\mathbb{S}} \cup \{0\}, \quad \bar{\mathbb{S}}_k = \mathbb{S}_k \cup (-\mathbb{S}_k) \quad \text{and} \quad \bar{\mathbb{S}}_{0,k} = \bar{\mathbb{S}}_k \cup \{0\}.$$

In the next proposition we deduce some quantitative bounds from the qualitative non-degeneracy condition of Lemma 4.7, the analyticity of the linear frequencies and their asymptotics.

Lemma 4.8. [Transversality] *There exist $q_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that the following results hold true. Recall that $\mathbf{v}_k(b)$, $\Omega_{j,k}$ and ω_{Eq} are defined in (4.13), (4.28) and (4.84) and respectively.*

1. For any $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{b \in [0, b^*]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_b^q \omega_{\text{Eq}}(b) \cdot l \right| \geq \rho_0 \langle l \rangle.$$

2. For any $k \in \{1, 2\}$ and $(l, j) \in (\mathbb{Z}^d \times \mathbb{N}_{\mathbf{m}}) \setminus \{(0, 0)\}$

$$\inf_{b \in [0, b^*]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_b^q (\omega_{\text{Eq}}(b) \cdot l + j \mathbf{v}_k(b)) \right| \geq \rho_0 \langle l \rangle.$$

3. For any $k \in \{1, 2\}$ and $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}_{\mathbf{m}}^* \setminus \mathbb{S}_k)$

$$\inf_{b \in [0, b^*]} \max_{q \in \llbracket 0, q_0 \rrbracket} \left| \partial_b^q (\omega_{\text{Eq}}(b) \cdot l + \Omega_{j,k}(b)) \right| \geq \rho_0 \langle l \rangle.$$

4. We assume the additional constraint $\Omega > \Omega_m^*$, see Lemma 4.6-4-5. For any $k \in \{1, 2\}$ and $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}_m^* \setminus \mathbb{S}_k$, satisfying the additional condition $(l, j) \neq (0, j')$, we have

$$\inf_{b \in [0, b^*]} \max_{q \in [0, q_0]} \left| \partial_b^q (\omega_{\text{Eq}}(b) \cdot l + \Omega_{j,k}(b) \pm \Omega_{j',k}(b)) \right| \geq \rho_0 \langle l \rangle.$$

5. For any $l \in \mathbb{Z}^d, j \in \mathbb{N}^* \setminus \mathbb{S}_1, j' \in \mathbb{N}^* \setminus \mathbb{S}_2$, we have

$$\inf_{b \in [0, b^*]} \max_{q \in [0, q_0]} \left| \partial_b^q (\omega_{\text{Eq}}(b) \cdot l + \Omega_{j,1}(b) \pm \Omega_{j',2}(b)) \right| \geq \rho_0 \langle l, j, j' \rangle.$$

Proof. **1.** Suppose, by contradiction, that for all $n \in \mathbb{N}$ there exist $b_n \in [0, b^*]$ and $l_n \in \mathbb{Z}^d \setminus \{0\}$ such that

$$\max_{q \in [0, n]} \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{n+1}. \quad (4.85)$$

The sequences $(b_n)_n \subset [0, b^*]$ and $(c_n)_n \triangleq \left(\frac{l_n}{\langle l_n \rangle} \right)_n \subset \mathbb{R}^d \setminus \{0\}$ are bounded. Up to an extraction we may assume that

$$\lim_{n \rightarrow \infty} \frac{l_n}{\langle l_n \rangle} = \tilde{c} \neq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} b_n = \tilde{b}.$$

Taking to the limit in (4.85) for $n \rightarrow \infty$ we deduce that

$$\forall q \in \mathbb{N}, \quad \partial_b^q (\omega_{\text{Eq}}(b) \cdot \tilde{c}) \Big|_{b=\tilde{b}} = 0, \quad \text{with} \quad \tilde{c} \neq 0.$$

Therefore, the real analytic function $b \rightarrow \omega_{\text{Eq}}(b) \cdot \tilde{c}$ is identically zero. This contradicts Lemma 4.7.

2. In the case $l = 0$ and $j \in \mathbb{N}_m^*$ we obviously have from (4.13),

$$\inf_{b \in [0, b^*]} \max_{q \in [0, q_0]} \left| \partial_b^q (j \mathbf{v}_k(b)) \right| \geq \inf_{b \in [0, b^*]} |\mathbf{v}_k(b)| \geq \Omega \geq \rho_0 \langle l \rangle,$$

for some $\rho_0 > 0$. Next, we shall consider the case $j \in \mathbb{N}_m^*, l \in \mathbb{Z}^d \setminus \{0\}$. By the triangle inequality combined with the boundedness of ω_{Eq} and $\mathbf{v}_k(b)$ we get

$$|\omega_{\text{Eq}}(b) \cdot l + j \mathbf{v}_k(b)| \geq |j| |\mathbf{v}_k(b)| - |\omega_{\text{Eq}}(b) \cdot l| \geq c|j| - C|l| \geq |l|$$

provided that $|j| \geq C_0|l|$ for some $C_0 > 0$. Hence, we shall only consider indices j and l satisfying

$$|j| \leq C_0|l|, \quad j \in \mathbb{N}_m, \quad l \in \mathbb{Z}^d \setminus \{0\}. \quad (4.86)$$

By contradiction, assume the existence of sequences $\{l_n\} \subset \mathbb{Z}^d \setminus \{0\}, \{j_n\} \subset \mathbb{N}_m$ satisfying (4.86) and $\{b_n\} \subset [0, b^*]$ such that

$$\forall q \in \mathbb{N}, \quad \forall n \geq q, \quad \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n \rangle} + \frac{j_n \mathbf{v}_k(b)}{\langle l_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}. \quad (4.87)$$

The sequences $\{b_n\}, \{d_n\} \triangleq \left\{ \frac{j_n}{\langle l_n \rangle} \right\}$ and $\{c_n\} \triangleq \left\{ \frac{l_n}{\langle l_n \rangle} \right\}$ are bounded. Thus, up to an extraction, we may assume that

$$\lim_{n \rightarrow \infty} b_n = \tilde{b}, \quad \lim_{n \rightarrow \infty} d_n = \tilde{d} \geq 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n = \tilde{c} \neq 0.$$

Hence, letting $n \rightarrow +\infty$ in (4.87) and using the fact that $b \mapsto \mathbf{v}_k(b)$ is smooth we obtain

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Consequently, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b)$ with $(\tilde{c}, \tilde{d}) \neq (0, 0)$ is identically zero and this is in contradiction with Lemma 4.7.

3. Let $k \in \{1, 2\}$ and consider $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}_m^* \setminus \mathbb{S}_k)$. By the triangle inequality and Lemma 4.6-3, we get

$$|\omega_{\text{Eq}}(b) \cdot l + \Omega_{j,k}(b)| \geq |\Omega_j(b)| - |\omega_{\text{Eq}}(b) \cdot l| \geq \Omega_j - C|l| \geq \frac{\Omega}{2} \langle l \rangle$$

provided that $j \geq C_0|l|$ for some $C_0 > 0$. Therefore, we shall restrict the proof to integers j with

$$0 \leq j < C_0|l|, \quad j \in \mathbb{N}_m^* \setminus \mathbb{S}_k \quad \text{and} \quad l \in \mathbb{Z}^d \setminus \{0\}. \quad (4.88)$$

By contradiction, for all $n \in \mathbb{N}$, we assume the existence of sequences $\{l_n\} \subset \mathbb{Z}^d \setminus \{0\}$, $\{j_n\} \subset \mathbb{N}_m^* \setminus \mathbb{S}_k$ and $\{b_n\} \subset [0, b^*]$ such that

$$\forall q \in \mathbb{N}, \quad \forall n \geq q, \quad \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n \rangle} + \frac{\Omega_{j_n, k}(b)}{\langle l_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}. \quad (4.89)$$

Since the sequences $\{b_n\}$ and $\{c_n\} \triangleq \left\{ \frac{l_n}{\langle l_n \rangle} \right\}$ are bounded, then by compactness we can assume that

$$\lim_{n \rightarrow \infty} b_n = \tilde{b} \quad \text{and} \quad \lim_{n \rightarrow \infty} c_n = \tilde{c} \neq 0.$$

We shall distinguish two cases.

• *Case 1:* $\{l_n\}$ is bounded. From (4.88) and up to an extraction the sequences $\{l_n\}$ and $\{j_n\}$ are stationary. Thus, we can assume that for any $n \in \mathbb{N}$, we have $l_n = \tilde{l} \in \mathbb{Z}^d \setminus \{0\}$ and $j_n = \tilde{j} \in \mathbb{N}_m^* \setminus \mathbb{S}_k$. Taking the limit as $n \rightarrow +\infty$ in (4.89) yields

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{l} + \Omega_{\tilde{j}, k}(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Consequently, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{l} + \Omega_{\tilde{j}, k}(b)$ with $(\tilde{l}, 1) \neq (0, 0)$ is identically zero and this contradicts Lemma 4.7.

• *Case 2:* $\{l_n\}$ is unbounded. Up to a subsequence, we assume that $\lim_{n \rightarrow \infty} |l_n| = \infty$ and $\lim_{n \rightarrow \infty} \frac{l_n}{\langle l_n \rangle} = \tilde{c} \in \mathbb{R}^d \setminus \{0\}$. We shall distinguish two sub-cases.

• *Sub-case ①.* The sequence $\{j_n\}$ is bounded. Up to an extraction we may assume that this sequence of integers is stationary. Taking the limit $n \rightarrow +\infty$ in (4.89), we get

$$\forall q \in \mathbb{N}, \quad \partial_b^q \omega_{\text{Eq}}(b) \Big|_{b=\tilde{b}} \cdot \tilde{c} = 0.$$

Thus, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c}$, with $\tilde{c} \neq 0$, is identically zero and this is a contradiction with the Lemma 4.7.

• *Sub-case ②.* The sequence $\{j_n\}$ is unbounded. Then up to an extraction we can assume that $\lim_{n \rightarrow \infty} j_n = \infty$. According to (4.29) we have

$$\frac{\Omega_{j_n, k}(b)}{\langle l_n \rangle} = \frac{j_n}{\langle l_n \rangle} \left(\Omega + (2-k) \frac{1-b^2}{2} \right) + \frac{(-1)^k}{2\langle l_n \rangle} + (-1)^{k+1} \frac{\mathbf{r}_{j_n}(b)}{\langle l_n \rangle}. \quad (4.90)$$

By (4.88), the sequence $\left\{ \frac{j_n}{\langle l_n \rangle} \right\}$ is bounded. Up to a subsequence, it converges to \tilde{d} . Differentiating then taking the limit in (4.90) we obtain

$$\lim_{n \rightarrow +\infty} \frac{\partial_b^q \Omega_{j_n, k}(b_n)}{\langle l_n \rangle} = \partial_b^q \left(\tilde{d} \mathbf{v}_k(b) \right) \Big|_{b=\tilde{b}},$$

having used in the last identity the estimate (4.30). Hence, taking the limit $j \rightarrow +\infty$ in (4.89) gives

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Thus, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b)$ is identically zero. This contradicts Lemma 4.7 as $(\tilde{c}, \tilde{d}) \neq 0$.

4. Let $l \in \mathbb{Z}^d$, $j, j' \in \mathbb{N}_m^* \setminus \mathbb{S}_k$ with $(l, j) \neq (0, j')$. By the triangle inequality and Lemma 4.6-5, since $\Omega > \Omega_m^*$, we infer that

$$\left| \omega_{\text{Eq}}(b) \cdot l + \Omega_{j, k}(b) \pm \Omega_{j', k}(b) \right| \geq \left| \Omega_{j, k}(b) \pm \Omega_{j', k}(b) \right| - \left| \omega_{\text{Eq}}(b) \cdot l \right| \geq c|j \pm j'| - C|l| \geq \langle l \rangle$$

provided $|j - j'| \geq C_0|l|$ for some $C_0 > 0$. In this case the desired estimate is trivial. So we shall restrict the proof to integers such that

$$|j \pm j'| < C_0\langle l \rangle, \quad l \in \mathbb{Z}^d \setminus \{0\}, \quad j, j' \in \mathbb{N}_m^* \setminus \mathbb{S}_k. \quad (4.91)$$

Arguing by contradiction, assume that for all $n \in \mathbb{N}$, there exists $(l_n, j_n) \neq (0, j'_n) \in \mathbb{Z}^{d+1}$ satisfying (4.91) and $b_n \in [0, b^*]$ such that

$$\forall q \in \mathbb{N}, \quad \forall n \geq q, \quad \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n \rangle} + \frac{\Omega_{j_n, k}(b) \pm \Omega_{j'_n, k}(b)}{\langle l_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}. \quad (4.92)$$

Since the sequences $\left\{ \frac{l_n}{\langle l_n \rangle} \right\}_n$ and $\{b_n\}_n$ are bounded, then up to an extraction we can assume that

$\lim_{n \rightarrow \infty} \frac{l_n}{\langle l_n \rangle} = \tilde{c} \neq 0$ and $\lim_{n \rightarrow \infty} b_n = \tilde{b}$. We distinguish two cases :

- *Case 1:* $(l_n)_n$ is bounded. We shall only focus on the most delicate case associated to the difference $\Omega_{j_n, k} - \Omega_{j'_n, k}$. Up to an extraction we may assume that this sequence of integers is stationary, that is, $l_n = \tilde{l}$. Looking at (4.91) we have two sub-cases.

- Sub-case ① : $(j_n)_n$ and $(j'_n)_n$ are bounded. Up to an extraction we can assume that they are stationary, that is, $j_n = \tilde{j}$, $j'_n = \tilde{j}'$. Moreover, by assumption we also have $(\tilde{l}, \tilde{j}) \neq (0, \tilde{j}')$ and $\tilde{j}, \tilde{j}' \notin \mathbb{S}_k$. Hence taking the limit $n \rightarrow +\infty$ in (4.92), we get

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{l} + \Omega_{\tilde{j}, k}(b) - \Omega_{\tilde{j}', k}(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Therefore, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{l} + \Omega_{\tilde{j}, k}(b) - \Omega_{\tilde{j}', k}(b)$ is identically zero. If $\tilde{j} = \tilde{j}'$ then this contradicts Lemma 4.7 since $\tilde{l} \neq 0$. In the case $\tilde{j} \neq \tilde{j}' \in \mathbb{N}_m^* \setminus \mathbb{S}_k$ this still contradicts Lemma 4.7, applied with the vector frequency $(\omega_{\text{Eq}}, \Omega_{\tilde{j}, k}, \Omega_{\tilde{j}', k})$.

- Sub-case ② : $(j_n)_n$ and $(j'_n)_n$ are unbounded. Up to an extraction, we assume that $\lim_{n \rightarrow \infty} j_n = \lim_{m \rightarrow \infty} j'_m = \infty$. Assume, without loss of generality, that for a given n we have $j_n \geq j'_n$. In view of (4.29) we may write

$$\frac{\partial_b^q (\Omega_{j_n, k}(b) - \Omega_{j'_n, k}(b))}{\langle l_n \rangle} = \partial_b^q \mathbf{v}_k(b) \frac{j_n - j'_n}{\langle l_n \rangle} + \frac{(-1)^{k+1}}{\langle l_n \rangle} \partial_b^k (\mathbf{r}_{j_n}(b) - \mathbf{r}_{j'_n}(b)). \quad (4.93)$$

According to (4.91), up to an extraction, we can assume that $\lim_{n \rightarrow \infty} \frac{j_n - j'_n}{\langle l_n \rangle} = \tilde{d}$. Therefore, combining (4.93) and (4.30), we find

$$\lim_{n \rightarrow \infty} \partial_b^q \left(\frac{\Omega_{j_n, k}(b) - \Omega_{j'_n, k}(b)}{\langle l_n \rangle} \right) \Big|_{b=b_n} = \tilde{d} \partial_b^q (\mathbf{v}_k(b)) \Big|_{b=\tilde{b}}.$$

Taking the limit $n \rightarrow +\infty$ in (4.92) gives

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Then, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b)$ with $(\tilde{c}, \tilde{d}) \neq (0, 0)$ is identically zero. This contradicts Lemma 4.7.

- *Case 2:* $(l_n)_n$ is unbounded. Up to an extraction we can assume that $\lim_{n \rightarrow \infty} |l_n| = \infty$. We shall distinguish three sub-cases.

- Sub-case ①. The sequences $(j_n)_n$ and $(j'_n)_n$ are bounded. Thus, up to an extraction they will converge. Taking the limit in (4.92) leads to

$$\forall q \in \mathbb{N}, \quad \partial_b^q \omega_{\text{Eq}}(\tilde{b}) \cdot \tilde{c} = 0.$$

which gives a contradiction with Lemma 4.7.

- Sub-case ②. The sequences $(j_n)_n$ and $(j'_n)_n$ are both unbounded. This case is similar to the sub-case

② of the case 1.

• Sub-case ③. The sequence $(j_n)_n$ is unbounded and $(j'_n)_n$ is bounded. Without loss of generality, we can assume that $\lim_{n \rightarrow \infty} j_n = \infty$ and $j'_n = \tilde{j}$. By (4.91) and up to an extraction one gets $\lim_{n \rightarrow \infty} \frac{j_n \pm j'_n}{|l_n|} = \tilde{d}$. Using Taylor formula combined with (4.30) gives for any $b \in [0, b^*]$,

$$\begin{aligned} |\partial_b^q \mathbf{r}_{j'_n}(b) - \partial_b^q \mathbf{r}_{j_n}(b)| &\leq C \left| \int_{j'_n}^{j_n} \frac{dx}{x^2} \right| \\ &\leq C |j_n - j'_n| (j_n j'_n)^{-1}. \end{aligned} \quad (4.94)$$

Using (4.29) combined with (4.94) and (4.30) we get, for any $q \in \mathbb{N}$,

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle l_n \rangle^{-1} \partial_b^q \left(\Omega_{j_n, k}(b) \pm \Omega_{j'_n, k}(b) - (j_n \pm j'_n) \mathbf{v}_k(b) \right) \Big|_{b=b_n} &= \\ (-1)^k \lim_{n \rightarrow \infty} \partial_b^q \left(\frac{1 \pm 1}{2 \langle l_n \rangle} - \frac{\mathbf{r}_{j_n}(b) \pm \mathbf{r}_{j'_n}(b)}{\langle l_n \rangle} \right) \Big|_{b=b_n} &= 0. \end{aligned}$$

Hence, taking the limit in (4.92) implies

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Thus, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c} + \tilde{d} \mathbf{v}_k(b)$ is identically zero with $(\tilde{c}, \tilde{d}) \neq 0$ leading to a contradiction with Lemma 4.7.

5. Arguing by contradiction, suppose that for all $n \in \mathbb{N}$, there exist $b_n \in [0, b^*]$ and $(l_n, j_n, j'_n) \in \mathbb{Z}^{d+2} \setminus \{0\}$, with $j_n \in (\mathbb{N}^* \cap \mathbb{Z}_m) \setminus \mathbb{S}_1$, and $j'_n \in (\mathbb{N}^* \cap \mathbb{Z}_m) \setminus \mathbb{S}_2$, such that

$$\max_{q \in \llbracket 0, n \rrbracket} \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n, j_n, j'_n \rangle} + \frac{\Omega_{j_n, 1}(b) \pm \Omega_{j'_n, 2}(b)}{\langle l_n, j_n, j'_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}$$

and therefore

$$\forall q \in \mathbb{N}, \quad \forall n \geq q, \quad \left| \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \frac{l_n}{\langle l_n, j_n, j'_n \rangle} + \frac{\Omega_{j_n, 1}(b) \pm \Omega_{j'_n, 2}(b)}{\langle l_n, j_n, j'_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}. \quad (4.95)$$

The sequence $(b_n)_n \subset [0, b^*]$ is bounded. Up to an extraction we may assume that

$$\lim_{n \rightarrow \infty} b_n = \tilde{b} \in [0, b^*].$$

We distinguish two cases.

• *Case 1:* The sequence $\{\langle l_n, j_n, j'_n \rangle\}_n$ is bounded. Then up to an extraction we may assume that

$$\lim_{n \rightarrow \infty} l_n = \tilde{c} \in \mathbb{Z}^d, \quad \lim_{n \rightarrow \infty} j_n = \tilde{j} \in (\mathbb{N}^* \cap \mathbb{Z}_m) \setminus \mathbb{S}_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} j'_n = \tilde{j}' \in (\mathbb{N}^* \cap \mathbb{Z}_m) \setminus \mathbb{S}_2.$$

Taking the limit in (4.95) we find

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left(\omega_{\text{Eq}}(b) \cdot \tilde{c} + \Omega_{\tilde{j}, 1}(b) \pm \Omega_{\tilde{j}', 2}(b) \right) \Big|_{b=\tilde{b}} = 0.$$

Thus, the real analytic function $b \mapsto \omega_{\text{Eq}}(b) \cdot \tilde{c} + \Omega_{\tilde{j}, 1}(b) \pm \Omega_{\tilde{j}', 2}(b)$ is identically zero on the interval $[0, b^*]$. This contradicts Lemma 4.7 if one of the following holds:

$$\tilde{j} \notin \mathbb{S}_2 \quad \text{and} \quad \tilde{j}' \notin \mathbb{S}_1,$$

or

$$\tilde{j} \in \mathbb{S}_2 \quad \text{and} \quad \tilde{j}' \notin \mathbb{S}_1,$$

or

$$\tilde{j} \notin \mathbb{S}_2 \quad \text{and} \quad \tilde{j}' \in \mathbb{S}_1.$$

Thus, it remain to check the case where

$$\tilde{j} \in \mathbb{S}_2 \quad \text{and} \quad \tilde{j}' \in \mathbb{S}_1. \quad (4.96)$$

Denoting $\tilde{c} = \left((\tilde{c}_{j,1})_{j \in \mathbb{S}_1}, (\tilde{c}_{j,1})_{j \in \mathbb{S}_2} \right)$, then we have for any $z \in [0, (b^*)^2]$,

$$\sum_{j \in \mathbb{S}_1 \setminus \{\tilde{j}'\}} \tilde{c}_{j,1} \tilde{\Omega}_{j,1}(z) + \sum_{j \in \mathbb{S}_2 \setminus \{\tilde{j}\}} \tilde{c}_{j,2} \tilde{\Omega}_{j,2}(z) + \tilde{c}_{\tilde{j}',1} \tilde{\Omega}_{\tilde{j}',1}(z) \pm \tilde{\Omega}_{\tilde{j},2}(z) + \tilde{c}_{\tilde{j},2} \tilde{\Omega}_{\tilde{j},2}(z) + \tilde{\Omega}_{\tilde{j},1}(z) = 0. \quad (4.97)$$

Arguing as in the proof of Lemma 4.7 we conclude by a differentiation argument that

$$\forall j \in (\mathbb{S}_1 \cup \mathbb{S}_2) \setminus \{\tilde{j}', \tilde{j}\}, \quad \forall k \in \{1, 2\}, \quad c_{j,k} = 0, \quad c_{\tilde{j},2} = 1 \quad \text{and} \quad c_{\tilde{j},1} = \pm 1.$$

Substituting these identities into (4.97) evaluated at $z = 0$ and using (4.79) we get

$$(2\Omega + \frac{1}{2})(\tilde{j}' \pm \tilde{j}) = 0.$$

This implies that $\tilde{j}' = \mp \tilde{j}$ contradicting (4.96) and (4.83).

• *Case 2:* The sequence $\{\langle l_n, j_n, j'_n \rangle\}_n$ is unbounded. Using (4.29) we may write

$$\forall q \in \mathbb{N}, \quad \forall n \geq q, \quad \left| \partial_b^q \left((\omega_{\text{Eq}}(b), \Omega + \frac{1-b^2}{2}, \pm \Omega) \cdot \frac{\langle l_n, j_n, j'_n \rangle}{\langle l_n, j_n, j'_n \rangle} + \frac{-\frac{1}{2} \pm \frac{1}{2} + \mathbf{r}_{j_n}(b) \mp \mathbf{r}_{j'_n}(b)}{\langle l_n, j_n, j'_n \rangle} \right) \Big|_{b=b_n} \right| < \frac{1}{1+n}. \quad (4.98)$$

The sequence $(c_n)_n \triangleq \left(\frac{\langle l_n, j_n, j'_n \rangle}{\langle l_n, j_n, j'_n \rangle} \right)_n \subset \mathbb{R}^d \setminus \{0\}$ is bounded. By compactness and up to an extraction we may assume that

$$\lim_{n \rightarrow \infty} \frac{\langle l_n, j_n, j'_n \rangle}{\langle l_n, j_n, j'_n \rangle} = \tilde{c} \neq 0.$$

Taking the limit in (4.98) and using (4.30) we get

$$\forall q \in \mathbb{N}, \quad \partial_b^q \left((\omega_{\text{Eq}}(b), \Omega + \frac{1-b^2}{2}, \pm \Omega) \cdot \tilde{c} \right)_{b=\tilde{b}} = 0.$$

Thus, the real analytic function $b \mapsto (\omega_{\text{Eq}}(b), \Omega + \frac{1-b^2}{2}, \pm \Omega) \cdot \tilde{c}$ is identically zero with $\tilde{c} \neq 0$ which contradicts Lemma 4.7. This completes the proof of the lemma. \square

Linear quasi-periodic solution. Notice that by selecting only a finite number of frequencies, the sum in (4.45) gives rise to quasi-periodic solutions of the linearized equation (4.12), provided that the parameter b belongs to a suitable Cantor-like set of full measure. The following result follows in a similar way to [59, Lem 3.3], based on Lemma 4.8-(i) and Lemma 8.1.

Lemma 4.9. *Let $\Omega > 0$, $\mathbb{S}_1, \mathbb{S}_2 \subset \mathbb{N}^*$, as in (4.83) and b^* as in Corollary 4.1. Then, there exists a Cantor-like set $\mathcal{C} \subset [0, b^*]$ satisfying $|\mathcal{C}| = b^*$ and such that for all $b \in \mathcal{C}$, every function in the form*

$$\rho(t, \theta) = \sum_{j \in \mathbb{S}_1} \frac{\rho_{j,1}}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} 1 \\ -a_j(b) \end{pmatrix} \cos(j\theta - \Omega_{j,1}(b)t) + \sum_{j \in \mathbb{S}_2} \frac{\rho_{j,2}}{\sqrt{1-a_j^2(b)}} \begin{pmatrix} -a_j(b) \\ 1 \end{pmatrix} \cos(j\theta - \Omega_{j,2}(b)t),$$

with $\rho_{j,1}, \rho_{j,2} \in \mathbb{R}^*$, is a time quasi-periodic reversible solution to the equation (4.12) with the vector frequency $\omega_{\text{Eq}}(b)$, defined in (4.84).

5 Hamiltonian toolkit

The main scope of this section is to relate the existence of quasi-periodic solutions to the Hamiltonian equation (4.47) to the construction of invariant tori in a suitable phase space. More precisely, we shall reformulate the problem in terms of embedded tori through the introduction of action-angle variables. Note that, according to Remark 4.1, (4.55) and (4.44), the equation (4.47) can be seen as a quasilinear perturbation of its linear part at the equilibrium state, namely,

$$\partial_t r = \mathcal{J} \mathbf{L}_0 r + X_P(r), \quad \text{with} \quad X_P(r) \triangleq \mathcal{J} \nabla K(r) - \mathcal{J} \mathbf{L}_0 r = \mathbf{Q}^{-1} X_{H \geq 3}(\mathbf{Q}r), \quad (5.1)$$

where

$$X_{H \geq 3}(r) \triangleq \mathcal{J}(\nabla H(r) - \mathbf{M}_0 r)$$

and \mathbf{Q} , H , \mathbf{M}_0 , \mathbf{L}_0 are defined in (4.39), (3.15), (4.12), (4.44), respectively. The following lemma summarizes some tame estimates satisfied by the vector field X_P . Notice that the structure of the two components of vector field $X_{H \geq 3}(r)$ are very similar to the one obtained in the setting of Euler equations in the unit disc [61, eq. (5.1)]. Moreover, the symplectic change of variables $\mathbf{Q}^{\pm 1}$ depends only on the parameter b (and not on r) and it acts continuously from \mathbf{H}_m^s into itself for any s . Therefore, one gets in a similar way to [61, Lemma 5.2] the following estimates.

Lemma 5.1. *Let b^*, m^* as in Corollary 4.1, $m \geq m^*$, and (γ, q, s_0, s) satisfy (2.1), (2.2) and (2.3). There exists $\varepsilon_0 \in (0, 1]$ such that if*

$$\|r\|_{s_0+2}^{q, \gamma, m} \leq \varepsilon_0,$$

then the vector field X_P , defined in (5.1) satisfies the following estimates

$$(i) \|X_P(r)\|_s^{q, \gamma, m} \lesssim \|r\|_{s+2}^{q, \gamma, m} \|r\|_{s_0+1}^{q, \gamma, m}.$$

$$(ii) \|d_r X_P(r)[\rho]\|_s^{q, \gamma, m} \lesssim \|\rho\|_{s+2}^{q, \gamma, m} \|r\|_{s_0+1}^{q, \gamma, m} + \|r\|_{s+2}^{q, \gamma, m} \|\rho\|_{s_0+1}^{q, \gamma, m}.$$

$$(iii) \|d_r^2 X_P(r)[\rho_1, \rho_2]\|_s^{q, \gamma, m} \lesssim \|\rho_1\|_{s_0+1}^{q, \gamma, m} \|\rho_2\|_{s+2}^{q, \gamma, m} + (\|\rho_1\|_{s+2}^{q, \gamma, m} + \|r\|_{s+2}^{q, \gamma, m} \|\rho_1\|_{s_0+1}^{q, \gamma, m}) \|\rho_2\|_{s_0+1}^{q, \gamma, m}.$$

Since we shall look for small amplitude quasi-periodic solutions then it is more convenient to rescale the solution as follows $r \mapsto \varepsilon r$ with r bounded in a suitable functions space. Hence, the Hamiltonian equation (4.51) takes the form

$$\omega \cdot \partial_\varphi r = \partial_\theta \mathbf{L}_0 r + \varepsilon X_{P_\varepsilon}(r), \quad (5.2)$$

where \mathbf{L}_0 is the operator defined by (4.44) and X_{P_ε} is the rescaled Hamiltonian vector field defined by $X_{P_\varepsilon}(r) \triangleq \varepsilon^{-2} X_P(\varepsilon r)$. Notice that (5.2) is the Hamiltonian system generated by the rescaled Hamiltonian

$$\begin{aligned} \mathcal{K}_\varepsilon(r) &\triangleq \varepsilon^{-2} K(\varepsilon r) \\ &= K_{\mathbf{L}_0}(r) + \varepsilon P_\varepsilon(r), \end{aligned} \quad (5.3)$$

with $K_{\mathbf{L}_0}$ the quadratic Hamiltonian defined in Remark 4.1 and $\varepsilon P_\varepsilon(r)$ describes all the terms of higher order more than cubic.

Action-angle-normal variables Recalling the notations introduced in (3.26), (4.83)–(4.84). Given the decomposition (1.16) of the phase space $L_m^2(\mathbb{T}) \times L_m^2(\mathbb{T})$ and the decomposition in action-angle-normal variables (1.19), the symplectic 2-form in (3.20) becomes

$$\mathcal{W} = \sum_{j \in \mathbb{S}_1} d\vartheta_{j,1} \wedge dI_{j,2} - \sum_{j \in \mathbb{S}_2} d\vartheta_{j,2} \wedge dI_{j,2} + \frac{1}{2i} \sum_{j \in \mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,1}} \frac{1}{j} dr_{j,1} \wedge dr_{-j,1} - \frac{1}{2i} \sum_{j \in \mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,2}} \frac{1}{j} dr_{j,2} \wedge dr_{-j,2}. \quad (5.4)$$

The Poisson bracket is given by

$$\{F, G\} \triangleq \mathcal{W}(X_F, X_G) = \langle \nabla F, \mathbf{J} \nabla G \rangle, \quad (5.5)$$

where $\langle \cdot, \cdot \rangle$ is the inner product, defined by

$$\langle (\vartheta, I, z), (\underline{\vartheta}, \underline{I}, \underline{z}) \rangle \triangleq \vartheta \cdot \underline{\vartheta} + I \cdot \underline{I} + \langle z, \underline{z} \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}.$$

The Poisson structure \mathbf{J} corresponding to \mathcal{W} , defined by the identity (5.5), is the unbounded operator

$$\mathbf{J} : (\vartheta, I, z) \mapsto (JI, -\mathbf{J}\vartheta, \mathcal{J}z),$$

where \mathcal{J} is given by (3.14) and

$$\mathbf{J} \triangleq \begin{pmatrix} \mathbf{I}_{d_1} & 0 \\ 0 & -\mathbf{I}_{d_2} \end{pmatrix},$$

with I_{d_k} the identity matrix of size d_k . Now we shall study the Hamiltonian system generated by the Hamiltonian \mathcal{K}_ε in (5.3), in the action-angle-normal variables $(\vartheta, I, z) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbf{H}_{\mathbb{S}_0}^\perp$. We consider the Hamiltonian $K_\varepsilon(\vartheta, I, z)$ defined by

$$K_\varepsilon \triangleq \mathcal{K}_\varepsilon \circ \mathbf{A} \quad (5.6)$$

where \mathbf{A} is the map defined in (1.19). Since \mathbf{L}_0 in (4.44) stabilizes the subspace $\mathbf{H}_{\mathbb{S}_0}^\perp$ then the quadratic Hamiltonian $K_{\mathbf{L}_0}$ in (4.72) in the variables (ϑ, I, z) reads, up to a constant,

$$\begin{aligned} K_{\mathbf{L}_0} \circ \mathbf{A} &= - \sum_{j \in \mathbb{S}_1} \Omega_{j,1}(b) I_{j,1} + \sum_{j \in \mathbb{S}_2} \Omega_{j,2}(b) I_{j,2} + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\ &= -(\mathbf{J} \omega_{\text{Eq}}(b)) \cdot I + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \end{aligned} \quad (5.7)$$

where $\omega_{\text{Eq}}(b) \in \mathbb{R}^d$ is the unperturbed tangential frequency vector. By (5.3) and (5.7), the Hamiltonian K_ε in (5.6) reads

$$\begin{aligned} K_\varepsilon &= \mathcal{N} + \varepsilon \mathcal{P}_\varepsilon, \quad \text{with} \\ \mathcal{N} &\triangleq -(\mathbf{J} \omega_{\text{Eq}}(b)) \cdot I + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}, \quad \mathcal{P}_\varepsilon \triangleq P_\varepsilon \circ \mathbf{A}. \end{aligned} \quad (5.8)$$

We look for an embedded invariant torus

$$i : \mathbb{T}^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d \times \mathbf{H}_{\mathbb{S}_0}^\perp, \quad \varphi \mapsto i(\varphi) \triangleq (\vartheta(\varphi), I(\varphi), z(\varphi)),$$

where $\vartheta(\varphi) - \varphi$ is a $(2\pi)^d$ -periodic function, of the Hamiltonian vector field

$$X_{K_\varepsilon} \triangleq (\mathbf{J} \partial_I K_\varepsilon, -\mathbf{J} \partial_\vartheta K_\varepsilon, \Pi_{\mathbb{S}_0}^\perp \mathcal{J} \nabla_z K_\varepsilon)$$

filled by quasi-periodic solutions with Diophantine frequency vector ω . Note that for the value $\varepsilon = 0$, the Hamiltonian system

$$\omega \cdot \partial_\varphi i(\varphi) = (X_{\mathcal{N}} + \varepsilon X_{\mathcal{P}_\varepsilon})(i(\varphi)) \quad (5.9)$$

possesses, for any value of the parameter $b \in [0, b^*]$, the invariant torus

$$i_{\text{flat}}(\varphi) \triangleq (\varphi, 0, 0),$$

provided that $\omega = -\omega_{\text{Eq}}(b)$. Now, in order to construct an invariant torus to the Hamiltonian system (5.9) which supports a quasi-periodic motion with frequency vector ω , close to $-\omega_{\text{Eq}}(b)$, we shall formulate the problem as a "Nash-Moser Theorem of hypothetical conjugation" established in [19]. It consists in using the frequencies $\omega \in \mathbb{R}^d$ as parameters and introducing "counter-terms" $\alpha \in \mathbb{R}^d$ in the family of Hamiltonians

$$K_\varepsilon^\alpha \triangleq \mathcal{N}_\alpha + \varepsilon \mathcal{P}_\varepsilon, \quad \mathcal{N}_\alpha \triangleq \alpha \cdot I + \frac{1}{2} \langle \mathbf{L}_0 z, z \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}. \quad (5.10)$$

The value of α will be adjusted along the iteration in order to control the average of the I -component at the linear level of the Hamiltonian equation

$$\begin{aligned} \mathcal{F}(i, \alpha) &\triangleq \mathcal{F}(i, \alpha, \omega, b, \varepsilon) \triangleq \omega \cdot \partial_\varphi i(\varphi) - X_{K_\varepsilon^\alpha}(i(\varphi)) = \omega \cdot \partial_\varphi i(\varphi) - (X_{\mathcal{N}_\alpha} + \varepsilon X_{\mathcal{P}_\varepsilon})(i(\varphi)) \\ &= \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \mathbf{J}(\alpha - \varepsilon \partial_I \mathcal{P}_\varepsilon(i(\varphi))) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \mathbf{J} \partial_\vartheta \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \mathcal{J} \mathbf{L}_0 z(\varphi) - \varepsilon \mathcal{J} \nabla_z \mathcal{P}_\varepsilon(i(\varphi)) \end{pmatrix} = 0. \end{aligned} \quad (5.11)$$

This degree of freedom through a parameter α will provides at the end of the scheme a solution (ω, i) for the original problem when it is fixed to $\alpha = -\mathbf{J} \omega_{\text{Eq}}(b)$ for any value of b in a suitable Cantor set. Note that the involution \mathcal{S} , described in (3.21), becomes

$$\mathcal{S} : (\vartheta, I, z) \mapsto (-\vartheta, I, \mathcal{S}z) \quad (5.12)$$

and the operator \mathfrak{T}_m in (3.24) becomes

$$\mathfrak{T}_m : (\vartheta, I, z) \mapsto (\vartheta, I, \mathfrak{T}_m z). \quad (5.13)$$

Moreover, we can easily check that the Hamiltonian vector field $X_{K_\varepsilon^\alpha}$ is reversible with respect \mathfrak{S} and m -fold preserving with respect to \mathfrak{T}_m . Thus, it is natural to look for m -fold reversible solutions of $\mathcal{F}(i, \alpha) = 0$, namely satisfying

$$\vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathcal{S}z(\varphi)) \quad \text{and} \quad \mathfrak{T}_m z(\varphi) = z(\varphi). \quad (5.14)$$

In the sequel, we shall denote by

$$\mathfrak{J}(\varphi) \triangleq i(\varphi) - (\varphi, 0, 0) = (\vartheta(\varphi) - \varphi, I(\varphi), z(\varphi))$$

the periodic component of the torus $\varphi \mapsto i(\varphi)$. We end this section by summarizing some tame estimates satisfied by the Hamiltonian vector field

$$X_{\mathcal{P}_\varepsilon} \triangleq (\mathcal{J}\partial_I \mathcal{P}_\varepsilon, -\mathcal{J}\partial_\vartheta \mathcal{P}_\varepsilon, \Pi_{\mathbb{S}_0}^\perp \mathcal{J}\nabla_z \mathcal{P}_\varepsilon),$$

where \mathcal{P}_ε is defined in (5.8). The proof of the next lemma follows in a similar way to [19, Lem. 5.1] using Lemma 5.1.

Lemma 5.2. *Let b^*, m^* as in Corollary 4.1, $m \geq m^*$ and (γ, q, s_0, s) satisfy (2.1), (2.2) and (2.3). There exists $\varepsilon_0 \in (0, 1)$ such that if*

$$\varepsilon \leq \varepsilon_0 \quad \text{and} \quad \|\mathfrak{J}\|_{s_0+2}^{q, \gamma, m} \leq 1,$$

then the perturbed Hamiltonian vector field $X_{\mathcal{P}_\varepsilon}$ satisfies the following tame estimates,

$$(i) \quad \|X_{\mathcal{P}_\varepsilon}(i)\|_s^{q, \gamma, m} \lesssim 1 + \|\mathfrak{J}\|_{s+2}^{q, \gamma, m}.$$

$$(ii) \quad \|d_i X_{\mathcal{P}_\varepsilon}(i)[\widehat{i}]\|_s^{q, \gamma, m} \lesssim \|\widehat{i}\|_{s+2}^{q, \gamma, m} + \|\mathfrak{J}\|_{s+2}^{q, \gamma, m} \|\widehat{i}\|_{s_0+1}^{q, \gamma, m}.$$

$$(iii) \quad \|d_i^2 X_{\mathcal{P}_\varepsilon}(i)[\widehat{i}, \widehat{i}]\|_s^{q, \gamma, m} \lesssim \|\widehat{i}\|_{s+2}^{q, \gamma, m} \|\widehat{i}\|_{s_0+1}^{q, \gamma, m} + \|\mathfrak{J}\|_{s+2}^{q, \gamma, m} (\|\widehat{i}\|_{s_0+1}^{q, \gamma, m})^2.$$

6 Approximate inverse

In order to prove the Theorem 1.1 using a Nash-Moser scheme, we have to construct an approximate right inverse of the linearized operator associated to the functional \mathcal{F} , defined in (5.11), at any m -fold and reversible state (i_0, α_0) close to the flat torus,

$$d_{(i, \alpha)} \mathcal{F}(i_0, \alpha_0) = \omega \cdot \partial_\varphi i_0 - d_i X_{K_\varepsilon^{\alpha_0}}(i_0) - \begin{pmatrix} \mathcal{J}\widehat{\alpha} \\ 0 \\ 0 \end{pmatrix}. \quad (6.1)$$

For this aim, we shall use the Berti-Bolle approach for the approximate inverse developed in [12] and which "approximately" decouples the linearized equations through a triangular system in the action-angle components and the normal ones. This strategy was slightly simplified in [59, Section 6] bypassing the introduction of an intermediate isotropic torus and directly working with the original one i_0 . Here, we shall closely follow this latter procedure with giving close attention to the difference in the Hamiltonian structure, which is due to the vectorial framework. Thus, for completeness sake, we shall reproduce all the algebraic computations and refer the reader to [59, Section 6] for more details on the analysis, which is very similar.

We first introduce the diffeomorphism $G_0 : (\phi, y, w) \mapsto (\vartheta, I, z)$ of the phase space $\mathbb{R}^d \times \mathbb{R}^d \times \mathbf{H}_{\mathbb{S}_0}^\perp$ given by

$$\begin{pmatrix} \vartheta \\ I \\ z \end{pmatrix} \triangleq G_0 \begin{pmatrix} \phi \\ y \\ w \end{pmatrix} \triangleq \begin{pmatrix} \vartheta_0(\phi) \\ I_0(\phi) + L_1(\phi)y + L_2(\phi)w \\ z_0(\phi) + w \end{pmatrix}, \quad (6.2)$$

where

$$L_1(\phi) \triangleq \mathcal{J}[\partial_\varphi \vartheta_0(\phi)]^{-\top}, \quad L_2(\phi) \triangleq \mathcal{J}[(\partial_\vartheta \tilde{z}_0)(\vartheta_0(\phi))]^\top \mathcal{J}^{-1}, \quad \tilde{z}_0(\vartheta) \triangleq z_0(\vartheta_0^{-1}(\vartheta)) \quad (6.3)$$

and the transposed operator is defined through the following duality relation : Given a Hilbert space \mathbb{H} equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ and a linear operator $A \in \mathcal{L}(\mathbb{R}^d, \mathbb{H})$,

$$\forall u \in \mathbb{H}, \quad \forall v \in \mathbb{R}^d, \quad \langle A^\top u, v \rangle_{\mathbb{R}^d} \triangleq \langle u, Av \rangle_{\mathbb{H}}.$$

Note that, in the new coordinates, i_0 becomes the trivial embedded torus $(\phi, y, w) = (\varphi, 0, 0)$, namely

$$G_0(\varphi, 0, 0) = i_0(\varphi).$$

In what follows we shall use the following notations

- We denote by $\mathbf{u} = (\phi, y, w)$ the coordinates induced by G_0 in (6.2).

- The mapping

$$\mathbf{u}_0(\varphi) \triangleq G_0^{-1}(i_0)(\varphi) = (\varphi, 0, 0)$$

refers to the trivial torus

- We shall denote by

$$\tilde{G}_0(\mathbf{u}, \alpha) \triangleq (G_0(\mathbf{u}), \alpha)$$

the diffeomorphism with the identity on the α -component.

- We quantify how an embedded torus $i_0(\mathbb{T})$ is approximately invariant for the Hamiltonian vector field $X_{K_\varepsilon^{\alpha_0}}$ in terms of the "error function"

$$Z(\varphi) \triangleq (Z_1, Z_2, Z_3)(\varphi) \triangleq \mathcal{F}(i_0, \alpha_0)(\varphi) = \omega \cdot \partial_\varphi i_0(\varphi) - X_{K_\varepsilon^{\alpha_0}}(i_0(\varphi)). \quad (6.4)$$

6.1 Linear change of variables and defect of the symplectic structure

In this subsection we shall conjugate the linearized operator $d_{i_0, \alpha} \mathcal{F}(i_0, \alpha_0)$ in (6.1), via the linear change of variables

$$DG_0(\mathbf{u}_0(\varphi)) \begin{pmatrix} \hat{\phi} \\ \hat{y} \\ \hat{w} \end{pmatrix} = \begin{pmatrix} \partial_\varphi \vartheta_0(\varphi) & 0 & 0 \\ \partial_\varphi I_0(\varphi) & L_1(\varphi) & L_2(\varphi) \\ \partial_\varphi z_0(\varphi) & 0 & I \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \hat{y} \\ \hat{w} \end{pmatrix}, \quad (6.5)$$

to a triangular system with small errors of size $Z = \mathcal{F}(i_0, \alpha_0)$. Our main result is the following.

Proposition 6.1. *Under the linear change of variables $DG_0(\mathbf{u}_0)$ the linearized operator $d_{i_0, \alpha} \mathcal{F}(i_0, \alpha_0)$ is transformed into*

$$[DG_0(\mathbf{u}_0)]^{-1} d_{(i_0, \alpha)} \mathcal{F}(i_0, \alpha_0) D\tilde{G}_0(\mathbf{u}_0)[\hat{\phi}, \hat{y}, \hat{w}, \hat{\alpha}] = \mathbb{D}[\hat{\phi}, \hat{y}, \hat{w}, \hat{\alpha}] + \mathbb{E}[\hat{\phi}, \hat{y}, \hat{w}] \quad (6.6)$$

where

1. the operator \mathbb{D} has the triangular form

$$\mathbb{D}[\hat{\phi}, \hat{y}, \hat{w}, \hat{\alpha}] \triangleq \begin{pmatrix} \omega \cdot \partial_\varphi \hat{\phi} - [K_{20}(\varphi)\hat{y} + K_{11}^\top(\varphi)\hat{w} + L_1^\top(\varphi)\hat{\alpha}] \\ \omega \cdot \partial_\varphi \hat{y} + \mathcal{B}(\varphi)\hat{\alpha} \\ \omega \cdot \partial_\varphi \hat{w} - \mathcal{J}[K_{11}(\varphi)\hat{y} + K_{02}(\varphi)\hat{w} + L_2^\top(\varphi)\hat{\alpha}] \end{pmatrix},$$

$\mathcal{B}(\varphi)$ and $K_{20}(\varphi)$ are $d \times d$ real matrices,

$$\begin{aligned} \mathcal{B}(\varphi) &\triangleq L_1^{-1}(\varphi) \partial_\varphi I_0(\varphi) L_1^\top(\varphi) + [\partial_\varphi z_0(\varphi)]^\top L_2^\top(\varphi), \\ K_{20}(\varphi) &\triangleq \varepsilon L_1^\top(\varphi) (\partial_{II} \mathcal{P}_\varepsilon)(i_0(\varphi)) L_1(\varphi), \end{aligned}$$

$K_{02}(\varphi)$ is a linear self-adjoint operator of $\mathbf{H}_{\mathbb{S}_0}^\perp$, given by

$$\begin{aligned} K_{02}(\varphi) \triangleq & (\partial_z \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) + \varepsilon L_2^\top(\varphi)(\partial_{II} \mathcal{P}_\varepsilon)(i_0(\varphi)) L_2(\varphi) \\ & + \varepsilon L_2^\top(\varphi)(\partial_{zI} \mathcal{P}_\varepsilon)(i_0(\varphi)) + \varepsilon(\partial_I \nabla_z \mathcal{P}_\varepsilon)(i_0(\varphi)) L_2(\varphi), \end{aligned} \quad (6.7)$$

and $K_{11}(\varphi) \in \mathcal{L}(\mathbb{R}^d, \mathbf{H}_{\mathbb{S}_0}^\perp)$,

$$K_{11}(\varphi) \triangleq \varepsilon L_2^\top(\varphi)(\partial_{II} \mathcal{P}_\varepsilon)(i_0(\varphi)) L_1(\varphi) + \varepsilon(\partial_I \nabla_z \mathcal{P}_\varepsilon)(i_0(\varphi)) L_1(\varphi),$$

2. the remainder \mathbb{E} is given by

$$\begin{aligned} \mathbb{E}[\widehat{\phi}, \widehat{y}, \widehat{w}] \triangleq & [DG_0(\mathbf{u}_0)]^{-1} \partial_\varphi Z(\varphi) \widehat{\phi} \\ & + \begin{pmatrix} 0 \\ \mathcal{A}(\varphi) [K_{20}(\varphi) \widehat{y} + K_{11}^\top(\varphi) \widehat{w}] - R_{10}(\varphi) \widehat{y} - R_{01}(\varphi) \widehat{w} \\ 0 \end{pmatrix} \end{aligned}$$

where $\mathcal{A}(\varphi)$ and $R_{10}(\varphi)$ are $d \times d$ real matrices,

$$\begin{aligned} \mathcal{A}(\varphi) \triangleq & [\partial_\varphi \vartheta_0(\varphi)]^\top \mathbf{J} \partial_\varphi I_0(\varphi) - [\partial_\varphi I_0(\varphi)]^\top \mathbf{J} \partial_\varphi \vartheta_0(\varphi) - [\partial_\varphi z_0(\varphi)]^\top \mathcal{J}^{-1} \partial_\varphi z_0(\varphi), \\ R_{10}(\varphi) \triangleq & [\partial_\varphi Z_1(\varphi)]^\top [\partial_\varphi \vartheta_0(\varphi)]^{-\top}, \end{aligned}$$

and $R_{01}(\varphi) \in \mathcal{L}(\mathbf{H}_{\mathbb{S}_0}^\perp, \mathbb{R}^d)$,

$$R_{01}(\varphi) \triangleq [\partial_\varphi Z_1(\varphi)]^\top [(\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\varphi))]^\top \mathcal{J}^{-1} - [\partial_\varphi Z_3(\varphi)]^\top \mathcal{J}^{-1}.$$

Proof. The composition of the nonlinear operator \mathcal{F} , in (5.11), with the map G_0 is given by

$$\mathcal{F}(G_0(\mathbf{u}(\varphi)), \alpha) = \omega \cdot \partial_\varphi (G_0(\mathbf{u}(\varphi))) - X_{K_\varepsilon^\alpha}(G_0(\mathbf{u}(\varphi))). \quad (6.8)$$

Then, by differentiating (6.8) at (\mathbf{u}_0, α_0) in the direction $(\widehat{\mathbf{u}}, \widehat{\alpha})$ we obtain

$$\begin{aligned} d_{(\mathbf{u}, \alpha)}(\mathcal{F} \circ G_0)(\mathbf{u}_0, \alpha_0)[(\widehat{\mathbf{u}}, \widehat{\alpha})](\varphi) = & \omega \cdot \partial_\varphi (DG_0(\mathbf{u}_0) \widehat{\mathbf{u}}) - \partial_\phi [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{\phi} \\ & - \partial_y [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{y} - \partial_w [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{w} - \begin{pmatrix} \mathbf{J} \widehat{\alpha} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (6.9)$$

In view of (6.5), one has

$$\omega \cdot \partial_\varphi (DG_0(\mathbf{u}_0) [\widehat{\mathbf{u}}](\varphi)) = DG_0(\mathbf{u}_0) \omega \cdot \partial_\varphi \widehat{\mathbf{u}} + \partial_\varphi (\omega \cdot \partial_\varphi i_0) \widehat{\phi} + \begin{pmatrix} 0 \\ (\omega \cdot \partial_\varphi L_1(\varphi)) \widehat{y} + (\omega \cdot \partial_\varphi L_2(\varphi)) \widehat{w} \\ 0 \end{pmatrix}, \quad (6.10)$$

and from (6.3)–(6.4) we get

$$\begin{aligned} \omega \cdot \partial_\varphi L_1(\varphi) = & -\mathbf{J} [\partial_\varphi \vartheta_0(\varphi)]^{-\top} (\omega \cdot \partial_\varphi [\partial_\varphi \vartheta_0(\varphi)]^\top) [\partial_\varphi \vartheta_0(\varphi)]^{-\top} \\ = & -\mathbf{J} [\partial_\varphi \vartheta_0(\varphi)]^{-\top} \left([\partial_\varphi Z_1(\varphi)]^\top + [\partial_\varphi ((\partial_I K_\varepsilon^{\alpha_0})(i_0(\varphi)))]^\top \mathbf{J} \right) [\partial_\varphi \vartheta_0(\varphi)]^{-\top}. \end{aligned} \quad (6.11)$$

Observe, from (6.3), that we have the identity

$$\partial_\varphi z_0(\varphi) = (\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\varphi)) \partial_\varphi \vartheta_0(\varphi). \quad (6.12)$$

Therefore the operator $L_2(\varphi)$ can be written as

$$L_2(\varphi) = \mathbf{J} [\partial_\varphi \vartheta_0(\varphi)]^{-\top} [\partial_\varphi z_0(\varphi)]^\top \mathcal{J}^{-1} = L_1(\varphi) [\partial_\varphi z_0(\varphi)]^\top \mathcal{J}^{-1}. \quad (6.13)$$

From the last two identities we find

$$\begin{aligned}\omega \cdot \partial_\varphi L_2(\varphi) &= -\mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} (\omega \cdot \partial_\varphi [\partial_\varphi \vartheta_0(\varphi)]^\top) [\partial_\varphi \vartheta_0(\varphi)]^{-\top} [\partial_\varphi z_0(\varphi)]^\top \mathcal{J}^{-1} \\ &\quad + \mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} [\partial_\varphi (\omega \cdot \partial_\varphi z_0)(\varphi)]^\top \mathcal{J}^{-1}\end{aligned}$$

and by (6.4) we obtain

$$\begin{aligned}\omega \cdot \partial_\varphi L_2(\varphi) &= -\mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} \left([\partial_\varphi Z_1(\varphi)]^\top + [\partial_\varphi ((\partial_I K_\varepsilon^{\alpha_0})(i_0(\varphi)))]^\top \right) L_2(\varphi) \\ &\quad + \mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} \left([\partial_\varphi Z_3(\varphi)]^\top - [\partial_\varphi ((\nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)))]^\top \right).\end{aligned}\quad (6.14)$$

Gathering (6.10), (6.11) and (6.14) gives

$$\begin{aligned}\omega \cdot \partial_\varphi (DG_0(\mathbf{u}_0)[\widehat{\mathbf{u}}](\varphi)) &= DG_0(\mathbf{u}_0) \omega \cdot \partial_\varphi \widehat{\mathbf{u}} + \partial_\varphi (\omega \cdot \partial_\varphi i_0) \widehat{\phi} \\ &\quad - \begin{pmatrix} 0 \\ \mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} \left([\mathcal{C}_I(\varphi)L_1(\varphi) + R_{10}(\varphi)] \widehat{\mathbf{y}} + [\mathcal{C}_I(\varphi)L_2(\varphi) + \mathcal{C}_z(\varphi) + R_{01}(\varphi)] \widehat{\mathbf{w}} \right) \\ 0 \end{pmatrix},\end{aligned}\quad (6.15)$$

where $R_{10}(\varphi)$ and $R_{01}(\varphi)$ are given by (ii) and

$$\begin{aligned}\mathcal{C}_I(\varphi) &\triangleq [\partial_\varphi ((\partial_I K_\varepsilon^{\alpha_0})(i_0(\varphi)))]^\top \\ &= [\partial_\varphi I_0(\varphi)]^\top (\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) + [\partial_\varphi \vartheta_0(\varphi)]^\top (\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) + [\partial_\varphi z_0(\varphi)]^\top (\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)),\end{aligned}\quad (6.16)$$

$$\begin{aligned}\mathcal{C}_z(\varphi) &\triangleq [\partial_\varphi ((\nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)))]^\top \\ &= [\partial_\varphi I_0(\varphi)]^\top (\partial_{zI} K_\varepsilon^{\alpha_0})(i_0(\varphi)) + [\partial_\varphi \vartheta_0(\varphi)]^\top (\partial_{z\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) + [\partial_\varphi z_0(\varphi)]^\top (\partial_z \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)).\end{aligned}\quad (6.17)$$

According (5.11) and (6.2), one may writes

$$\begin{aligned}\partial_\phi [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{\phi} &= \partial_\varphi [X_{K_\varepsilon^{\alpha_0}}(i_0(\varphi))] \widehat{\phi}, \\ \partial_{\mathbf{y}} [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{\mathbf{y}} &= \begin{pmatrix} \mathbf{J}(\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}} \\ -\mathbf{J}(\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}} \\ \mathcal{J}[(\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}}] \end{pmatrix}, \\ \partial_{\widehat{\mathbf{w}}} [X_{K_\varepsilon^{\alpha_0}}(G_0(\mathbf{u}(\varphi)))]_{\mathbf{u}=\mathbf{u}_0} \widehat{\mathbf{w}} &= \begin{pmatrix} (\mathbf{J} \partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) \widehat{\mathbf{w}} + \mathbf{J}(\partial_{zI} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \widehat{\mathbf{w}} \\ -\mathbf{J}(\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) \widehat{\mathbf{w}} - \mathbf{J}(\partial_{z\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \widehat{\mathbf{w}} \\ \mathcal{J}[(\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) \widehat{\mathbf{w}} + (\partial_z \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \widehat{\mathbf{w}}] \end{pmatrix}.\end{aligned}$$

Therefore inserting (6.15) and the last three identities into (6.9) we get

$$\begin{aligned}d_{(\mathbf{u}, \alpha)}(\mathcal{F} \circ G_0)(\mathbf{u}_0, \alpha_0)[(\widehat{\mathbf{u}}, \widehat{\alpha})] &= DG_0(\mathbf{u}_0) \omega \cdot \partial_\varphi \widehat{\mathbf{u}} + \partial_\varphi [\mathcal{F}(i_0(\varphi))] \widehat{\phi} \\ &\quad + \begin{pmatrix} -\mathbf{J}(\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}} \\ \mathbf{J}(\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}} - \mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} [\mathcal{C}_I(\varphi)L_1(\varphi) + R_{10}(\varphi)] \widehat{\mathbf{y}} \\ -\mathcal{J}(\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) \widehat{\mathbf{y}} \end{pmatrix} \\ &\quad + \begin{pmatrix} -\mathbf{J}(\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) \widehat{\mathbf{w}} - \mathbf{J}(\partial_{zI} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \widehat{\mathbf{w}} \\ [\mathbf{J}(\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) + \mathbf{J}(\partial_{z\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi))] \widehat{\mathbf{w}} \\ -\mathcal{J}[(\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) \widehat{\mathbf{w}} + (\partial_z \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \widehat{\mathbf{w}}] \end{pmatrix} \\ &\quad - \begin{pmatrix} 0 \\ \mathbf{J}[\partial_\varphi \vartheta_0(\varphi)]^{-\top} [\mathcal{C}_I(\varphi)L_2(\varphi) + \mathcal{C}_z(\varphi) + R_{01}(\varphi)] \widehat{\mathbf{w}} \\ 0 \end{pmatrix} - \begin{pmatrix} \mathbf{J} \widehat{\alpha} \\ 0 \\ 0 \end{pmatrix}.\end{aligned}\quad (6.18)$$

From (6.5), (6.12) and (6.13), one may easily check that

$$[DG_0(\mathbf{u}_0)]^{-1} = \begin{pmatrix} [\partial_\varphi \vartheta_0(\varphi)]^{-1} & 0 & 0 \\ -\mathcal{B}(\varphi) \mathbf{J} & [\partial_\varphi \vartheta_0(\varphi)]^\top \mathbf{J} & -[\partial_\varphi z_0(\varphi)]^\top \mathcal{J}^{-1} \\ -(\partial_{\vartheta} \widetilde{z}_0)(\vartheta_0(\varphi)) & 0 & I \end{pmatrix}$$

where $\mathcal{B}(\varphi)$ is given by (i). Finally, applying $[DG_0(\mathbf{u}_0)]^{-1}$ to (6.18) and using (6.16), (6.17) we obtain

$$[DG_0(\mathbf{u}_0)]^{-1}d_{(\mathbf{u},\alpha)}(\mathcal{F} \circ G_0)(\mathbf{u}_0, \alpha_0)[\widehat{\mathbf{u}}, \widehat{\alpha}] = \omega \cdot \partial_\varphi \widehat{\mathbf{u}} + [DG_0(\mathbf{u}_0)]^{-1} \partial_\varphi [\mathcal{F}(i_0(\varphi))] \widehat{\phi} \\ + \begin{pmatrix} -K_{20}(\varphi) \widehat{\mathbf{y}} \\ \mathcal{A}(\varphi) K_{20}(\varphi) \widehat{\mathbf{y}} - R_{10}(\varphi) \widehat{\mathbf{y}} \\ -\mathcal{J} K_{11}(\varphi) \widehat{\mathbf{y}} \end{pmatrix} + \begin{pmatrix} -K_{11}^\top(\varphi) \widehat{\mathbf{w}} \\ \mathcal{A}(\varphi) K_{11}^\top(\varphi) \widehat{\mathbf{w}} - R_{01}(\varphi) \widehat{\mathbf{w}} \\ -\mathcal{J} K_{02}(\varphi) \widehat{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} -[\partial_\varphi \vartheta_0(\varphi)]^{-1} \mathbf{J} \widehat{\alpha} \\ \mathcal{B}(\varphi) \widehat{\alpha} \\ [(\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\varphi))] \mathbf{J} \widehat{\alpha} \end{pmatrix},$$

where $\mathcal{A}(\varphi)$ is defined in (ii) and satisfies $\mathcal{B}(\varphi) = \mathcal{A}(\varphi) L_1^\top(\varphi) + [\partial_\varphi I_0(\varphi)]^\top$, and

$$K_{20}(\varphi) \triangleq L_1^\top(\varphi) (\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi), \\ K_{11}(\varphi) \triangleq L_2^\top(\varphi) (\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi) + (\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_1(\varphi), \\ K_{02}(\varphi) \triangleq (\partial_z \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) + L_2^\top(\varphi) (\partial_{II} K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi) + L_2^\top(\varphi) (\partial_{zI} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \\ + (\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) L_2(\varphi).$$

This together with (5.10) give the desired identity, concluding the proof of Proposition 6.1. \square

Next, in order to prove that the remainder \mathbb{E} is of size Z , we shall prove that the matrix \mathcal{A} , defined in Proposition 6.1-(ii), is zero at an exact solution on some Cantor like set, up to an exponentially small remainder. In particular, we shall prove the following lemma.

Lemma 6.1. *The coefficients of the matrix \mathcal{A} , given by*

$$\mathcal{A}_{kj}(\varphi) \triangleq [\mathbf{J} \partial_{\varphi_j} I_0(\varphi)] \cdot \partial_{\varphi_k} \vartheta_0(\varphi) - [\mathbf{J} \partial_{\varphi_j} \vartheta_0(\varphi)] \cdot \partial_{\varphi_k} I_0(\varphi) - \langle \mathcal{J}^{-1} \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})},$$

satisfy for all $\varphi \in \mathbb{T}^d$, are

$$\omega \cdot \partial_\varphi \mathcal{A}_{kj}(\varphi) = [\mathbf{J} \partial_{\varphi_j} Z_2(\varphi)] \cdot \partial_{\varphi_k} \vartheta_0(\varphi) - [\mathbf{J} \partial_{\varphi_j} Z_1(\varphi)] \cdot \partial_{\varphi_k} I_0(\varphi) - \langle \mathcal{J}^{-1} \partial_{\varphi_j} Z_3(\varphi), \partial_{\varphi_k} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\ + [\mathbf{J} \partial_{\varphi_j} I_0(\varphi)] \cdot \partial_{\varphi_k} Z_2(\varphi) - [\mathbf{J} \partial_{\varphi_j} \vartheta_0(\varphi)] \cdot \partial_{\varphi_k} Z_1(\varphi) - \langle \mathcal{J}^{-1} \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} Z_3(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}.$$

where $(\underline{e}_1, \dots, \underline{e}_d)$ denotes the canonical basis of \mathbb{R}^d .

Proof. From the expression of the coefficients \mathcal{A}_{kj} one has

$$\omega \cdot \partial_\varphi \mathcal{A}_{kj}(\varphi) = \langle \mathbf{J} \partial_{\varphi_j} \omega \cdot \partial_\varphi I_0(\varphi), \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \mathbf{J} \partial_{\varphi_j} I_0(\varphi), \partial_{\varphi_k} \omega \cdot \partial_\varphi \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} \\ - \langle \mathbf{J} \partial_{\varphi_j} \omega \cdot \partial_\varphi \vartheta_0(\varphi), \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} - \langle \mathbf{J} \partial_{\varphi_j} \vartheta_0(\varphi), \partial_{\varphi_k} \omega \cdot \partial_\varphi I_0(\varphi) \rangle_{\mathbb{R}^d} \\ - \langle \mathcal{J}^{-1} \partial_{\varphi_j} \omega \cdot \partial_\varphi z_0(\varphi), \partial_{\varphi_k} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} - \langle \mathcal{J}^{-1} \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} \omega \cdot \partial_\varphi z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}$$

In view of (6.4) we get

$$\omega \cdot \partial_\varphi \mathcal{A}_{kj}(\varphi) = \langle \mathbf{J} \partial_{\varphi_j} Z_2(\varphi), \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \mathbf{J} \partial_{\varphi_j} I_0(\varphi), \partial_{\varphi_k} Z_1(\varphi) \rangle_{\mathbb{R}^d} \tag{6.19} \\ - \langle \mathbf{J} \partial_{\varphi_j} Z_1(\varphi), \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} - \langle \mathbf{J} \partial_{\varphi_j} \vartheta_0(\varphi), \partial_{\varphi_k} Z_2(\varphi) \rangle_{\mathbb{R}^d} \\ - \langle \mathcal{J}^{-1} \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} Z_3(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} - \langle \mathcal{J}^{-1} \partial_{\varphi_j} Z_3(\varphi), \partial_{\varphi_k} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\ + \mathcal{B}_{kj}^1(\varphi) + \mathcal{B}_{kj}^2(\varphi) + \mathcal{B}_{kj}^3(\varphi),$$

where

$$\mathcal{B}_{kj}^1(\varphi) \triangleq -\langle \partial_{\varphi_j} (\partial_I K_\varepsilon^{\alpha_0})(i_0(\varphi)), \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \partial_{\varphi_j} I_0(\varphi), \partial_{\varphi_k} (\partial_I K_\varepsilon^{\alpha_0})(i_0(\varphi)) \rangle_{\mathbb{R}^d}, \\ \mathcal{B}_{kj}^2(\varphi) \triangleq -\langle \partial_{\varphi_j} (\partial_\vartheta K_\varepsilon^{\alpha_0})(i_0(\varphi)), \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \partial_{\varphi_j} \vartheta_0(\varphi), \partial_{\varphi_k} (\partial_\vartheta K_\varepsilon^{\alpha_0})(i_0(\varphi)) \rangle_{\mathbb{R}^d}, \\ \mathcal{B}_{kj}^3(\varphi) \triangleq \langle \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} (\nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} - \langle \partial_{\varphi_j} (\nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)), \partial_{\varphi_k} z_0(\varphi) \rangle_{L^2 \times L^2}.$$

Straightforward computations leads to

$$\begin{aligned}
\mathcal{B}_{kj}^1(\varphi) &= -\langle (\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} \vartheta_0(\varphi), \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} - \langle (\partial_{zI} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} \\
&\quad + \langle \partial_{\varphi_j} I_0(\varphi), (\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \partial_{\varphi_j} I_0(\varphi), (\partial_{Iz} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} z_0(\varphi) \rangle_{\mathbb{R}^d}, \\
\mathcal{B}_{kj}^2(\varphi) &= -\langle (\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} I_0(\varphi), \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} - \langle (\partial_{z\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} z_0(\varphi), \partial_{\varphi_k} \vartheta_0(\varphi) \rangle_{\mathbb{R}^d} \\
&\quad + \langle \partial_{\varphi_j} \vartheta_0(\varphi), (\partial_{I\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} I_0(\varphi) \rangle_{\mathbb{R}^d} + \langle \partial_{\varphi_j} \vartheta_0(\varphi), (\partial_{z\vartheta} K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} z_0(\varphi) \rangle_{\mathbb{R}^d}, \\
\mathcal{B}_{kj}^3(\varphi) &= \langle (\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} I_0(\varphi), \partial_{\varphi_j} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\
&\quad - \langle \partial_{\varphi_k} z_0(\varphi), (\partial_I \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} I_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\
&\quad + \langle (\partial_{I\vartheta} \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_k} \vartheta_0(\varphi), \partial_{\varphi_j} z_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})} \\
&\quad - \langle \partial_{\varphi_k} z_0(\varphi), (\partial_{I\vartheta} \nabla_z K_\varepsilon^{\alpha_0})(i_0(\varphi)) \partial_{\varphi_j} \vartheta_0(\varphi) \rangle_{L^2(\mathbb{T}) \times L^2(\mathbb{T})}.
\end{aligned}$$

Combining the last three identities we obtain

$$\mathcal{B}_{kj}^1(\varphi) + \mathcal{B}_{kj}^2(\varphi) + \mathcal{B}_{kj}^3(\varphi) = 0.$$

This with (6.19) concludes the proof of the lemma. \square

We define the sequence $(N_n)_{n \in \mathbb{N} \cup \{-1\}}$ as

$$N_{-1} \triangleq 1, \quad \forall n \in \mathbb{N}, \quad N_n \triangleq N_0^{\left(\frac{3}{2}\right)^n}, \quad \text{with} \quad N_0 \geq 2. \quad (6.20)$$

The following lemma is proved in [19, Lemma 5.3] and [59, Lemma 6.2].

Lemma 6.2. *The coefficients \mathcal{A}_{kj} , defined in Lemma 6.1, decomposes as*

$$\mathcal{A}_{kj} = \mathcal{A}_{kj}^{(n)} + \mathcal{A}_{kj}^{(n),\perp}, \quad \text{with} \quad \mathcal{A}_{kj}^{(n)} \triangleq \Pi_{N_n} \mathcal{A}_{kj} \quad \text{and} \quad \mathcal{A}_{kj}^{(n),\perp} \triangleq \Pi_{N_n}^\perp \mathcal{A}_{kj}. \quad (6.21)$$

In addition, the following properties hold true.

1. The function $\mathcal{A}_{kj}^{(n),\perp}$ satisfies for any $s \in \mathbb{R}$,

$$\forall \mathbf{b} \geq 0, \quad \|\mathcal{A}_{kj}^{(n),\perp}\|_s^{q,\gamma,\mathbf{m}} \lesssim N_n^{-\mathbf{b}} \|\mathcal{J}_0\|_{s+1+\mathbf{b}}^{q,\gamma,\mathbf{m}}.$$

2. There exist functions $\mathcal{A}_{kj}^{(n),\text{ext}}$ defined for any $(b, \omega) \in \mathcal{O}$ and satisfying, for any $s \geq s_0$, the estimate

$$\|\mathcal{A}_{kj}^{(n),\text{ext}}\|_s^{q,\gamma,\mathbf{m}} \lesssim \gamma^{-1} (\|Z\|_{s+\tau_1(q+1)+1}^{q,\gamma,\mathbf{m}} + \|Z\|_{s_0+1}^{q,\gamma,\mathbf{m}} \|\mathcal{J}_0\|_{s+\tau_1(q+1)+1}^{q,\gamma,\mathbf{m}}).$$

Moreover, $\mathcal{A}_{kj}^{(n),\text{ext}}$ coincides with $\mathcal{A}_{kj}^{(n)}$ on the Cantor set

$$\text{DC}_{N_n}(\gamma, \tau_1) \triangleq \bigcap_{\substack{l \in \mathbb{Z}^d \setminus \{0\} \\ |l| \leq N_n}} \left\{ \omega \in \mathbb{R}^d \quad \text{s.t.} \quad |\omega \cdot l| \geq \frac{\gamma}{|l|^{\tau_1}} \right\}. \quad (6.22)$$

6.2 Construction of an approximate inverse

According to Proposition 6.1-(ii) and Lemma 6.2, the error term \mathbb{E} is zero at an exact solution, up to an exponentially small remainder on the Cantor set $\text{DC}_{N_n}(\gamma, \tau_1)$. Therefore, in order to find an approximate inverse of the linear operator in (6.6) it is sufficient to almost invert the operator \mathbb{D} , which is triangular. More precisely, we first invert the action-component equation, in the linear system $\mathbb{D}[\hat{u}] = (g_1, g_2, g_3)$, which is decoupled from the other equations,

$$\omega \cdot \partial_\varphi \hat{y} = g_2 - \mathcal{B}(\varphi) \hat{\alpha}.$$

Then, we shall solve the last normal-component equation

$$\omega \cdot \partial_\varphi \hat{w} - \mathcal{J} K_{02}(\varphi) \hat{w} = g_3 + \mathcal{J} [K_{11}(\varphi) \hat{y} + L_2^\top(\varphi) \hat{\alpha}].$$

For this aim we need to find an approximate right inverse of the linearized operator in the normal direction

$$\widehat{\mathcal{L}} \triangleq \Pi_{\mathbb{S}_0}^\perp (\omega \cdot \partial_\varphi - \mathcal{J}K_{02}(\varphi)) \Pi_{\mathbb{S}_0}^\perp \quad (6.23)$$

when the set of parameters is restricted to a Cantor-like set. Here the projector $\Pi_{\mathbb{S}_0}^\perp$ is the one defined in (1.17). Finally, we shall solve the first equation in $\mathbb{D}[\widehat{u}] = (g_1, g_2, g_3)$ after choosing $\widehat{\alpha}$ in such way we get zero average in the equation.

The following proposition gives a brief statement about the invertibility in the normal direction; the construction of an approximate right inverse of the operator $\widehat{\mathcal{L}}$ is the subject of Section 7 and a precise statement with a detailed description of Cantor like sets, see Proposition 7.6.

Proposition 6.2. *Given the conditions (2.2), (2.3), (7.42) and (7.43). There exists $\sigma_5 \triangleq \sigma_5(\tau_1, \tau_2, q, d) > 0$ such that if*

$$\|\mathfrak{J}_0\|_{s_h + \sigma_5}^{q, \gamma, \mathfrak{m}} \leq 1,$$

then there exists a family of linear operators $(\widehat{\mathcal{T}}_n)_{n \in \mathbb{N}}$ defined in \mathcal{O} and satisfying the estimate

$$\forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\widehat{\mathcal{T}}_n \rho\|_s^{q, \gamma, \mathfrak{m}} \lesssim \gamma^{-1} (\|\rho\|_{s + \sigma_5}^{q, \gamma, \mathfrak{m}} + \|\mathfrak{J}_0\|_{s + \sigma_5}^{q, \gamma, \mathfrak{m}} \|\rho\|_{s_0 + \sigma_5}^{q, \gamma, \mathfrak{m}})$$

and, for any $n \in \mathbb{N}$, we have the following splitting

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}_n + \widehat{\mathcal{R}}_n, \quad \text{with} \quad \widehat{\mathcal{L}}_n \widehat{\mathcal{T}}_n = \text{Id},$$

in a Cantor set $\mathbf{G}_n \triangleq \mathbf{G}_n(\gamma, \tau_1, \tau_2, i_0) \subset \text{DC}_{N_n}(\gamma, \tau_1) \times (b_*, b^*)$, where the operators $\widehat{\mathcal{L}}_n$ and $\widehat{\mathcal{R}}_n$ are defined in \mathcal{O} and satisfy

$$\begin{aligned} \forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\widehat{\mathcal{L}}_n \rho\|_s^{q, \gamma, \mathfrak{m}} &\lesssim \|\rho\|_{s+1}^{q, \gamma, \mathfrak{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s+\sigma_5}^{q, \gamma, \mathfrak{m}} \|\rho\|_{s_0+1}^{q, \gamma, \mathfrak{m}}, \\ \forall s \in [s_0, S], \quad \|\widehat{\mathcal{R}}_n \rho\|_{s_0}^{q, \gamma, \mathfrak{m}} &\lesssim N_n^{s_0-s} \gamma^{-1} (\|\rho\|_{s+\sigma_5}^{q, \gamma, \mathfrak{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s+\sigma_5}^{q, \gamma, \mathfrak{m}} \|\rho\|_{s_0+\sigma_5}^{q, \gamma, \mathfrak{m}}) \\ &\quad + \varepsilon \gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0+\sigma_5}^{q, \gamma, \mathfrak{m}}. \end{aligned}$$

The main goal is to find an approximate inverse to the operator $[DG_0(\mathbf{u}_0)]^{-1} d_{(i, \alpha)} \mathcal{F}(i_0, \alpha_0) D\widetilde{G}_0(\mathbf{u}_0)$ in (6.6). For this aim, since we require only finitely many non-resonance conditions (6.22), for any $\omega \in \mathbb{R}^d$, we decompose $\omega \cdot \partial_\varphi$ as

$$\omega \cdot \partial_\varphi = \mathcal{D}_{(n)} + \mathcal{D}_{(n)}^\perp, \quad \mathcal{D}_{(n)} \triangleq \omega \cdot \partial_\varphi \Pi_{N_n} + \Pi_{N_n, \mathbf{g}}^\perp, \quad \mathcal{D}_{(n)}^\perp \triangleq \omega \cdot \partial_\varphi \Pi_{N_n}^\perp - \Pi_{N_n, \mathbf{g}}^\perp, \quad (6.24)$$

where

$$\Pi_{N_n, \mathbf{g}}^\perp \sum_{l \in \mathbb{Z}^d \setminus \{0\}} h_l \mathbf{e}_l \triangleq \sum_{\substack{l \in \mathbb{Z}^d \setminus \{0\} \\ |l| > N_n}} \mathbf{g}(l) h_l \mathbf{e}_l.$$

and the function $\mathbf{g} : \mathbb{Z}^d \setminus \{0\} \rightarrow \{-1, 1\}$ is defined, for all $l = (l_1, \dots, l_d) \in \mathbb{Z}^d \setminus \{0\}$, as the sign of the first non-zero component in the vector l . Thus, it satisfies

$$\forall l \in \mathbb{Z}^d \setminus \{0\}, \quad \mathbf{g}(-l) = -\mathbf{g}(l).$$

The projector $\Pi_{N_n, \mathbf{g}}^\perp$ is used here instead of $\Pi_{N_n}^\perp$ in order to preserve the reversibility property. Then, according to Proposition 6.1, the identities (6.21)-(6.24) and Proposition 6.2 we have the following decomposition

$$[DG_0(\mathbf{u}_0)]^{-1} d_{(i, \alpha)} \mathcal{F}(i_0, \alpha_0) D\widetilde{G}_0(\mathbf{u}_0) = \mathbb{D}_n + \mathbb{E}_n + \mathcal{F}_n + \mathcal{Q}_n, \quad (6.25)$$

with

$$\begin{aligned}\mathbb{D}_n[\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}] &\triangleq \begin{pmatrix} \mathcal{D}_{(n)}\widehat{\phi} - K_{20}\widehat{y} - K_{11}^\top\widehat{w} - L_1^\top\widehat{\alpha} \\ \mathcal{D}_{(n)}\widehat{y} + \mathcal{B}\widehat{\alpha} \\ \widehat{L}_{\omega,n}\widehat{w} - \mathcal{J}[K_{11}\widehat{y} + L_2^\top\widehat{\alpha}] \end{pmatrix}, \\ \mathbb{E}_n[\widehat{\phi}, \widehat{y}, \widehat{w}] &\triangleq [DG_0(\mathbf{u}_0)]^{-1}[\partial_\varphi Z]\widehat{\phi} + \begin{pmatrix} 0 \\ \mathcal{A}^{(n)}[K_{20}\widehat{y} + K_{11}^\top\widehat{w}] - R_{10}\widehat{y} - R_{01}\widehat{w} \\ 0 \end{pmatrix} \\ \mathcal{P}_n[\widehat{\phi}, \widehat{y}, \widehat{w}] &\triangleq \begin{pmatrix} \mathcal{D}_{(n)}^\perp\widehat{\phi} \\ \mathcal{D}_{(n)}^\perp\widehat{y} + \mathcal{A}^{(n),\perp}[K_{20}\widehat{y} + K_{11}^\top\widehat{w}] \\ 0 \end{pmatrix}, \quad \mathcal{Q}_n[\widehat{\phi}, \widehat{y}, \widehat{w}] \triangleq \begin{pmatrix} 0 \\ 0 \\ \widehat{R}_n[\widehat{w}] \end{pmatrix},\end{aligned}$$

where $\mathcal{A}^{(n)}$ and $\mathcal{A}^{(n),\perp}$ are the matrices with coefficients $\mathcal{A}_{kj}^{(n)}$ and $\mathcal{A}_{kj}^{(n),\perp}$ respectively, see (6.21). We define the linear operator \mathbb{L}_{ext} as

$$\mathbb{L}_{\text{ext}} \triangleq \mathbb{D}_n + \mathbb{E}_n^{\text{ext}} + \mathcal{P}_n + \mathcal{Q}_n, \quad (6.26)$$

where the operator $\mathbb{E}_n^{\text{ext}}$ vanishes at exact solutions on the whole set of parameters \mathcal{O} and it is given by

$$\mathbb{E}_n^{\text{ext}}[\widehat{\phi}, \widehat{y}, \widehat{w}] \triangleq [DG_0(\mathbf{u}_0)]^{-1}[\partial_\varphi Z]\widehat{\phi} + \begin{pmatrix} 0 \\ \mathcal{A}^{(n),\text{ext}}[K_{20}(\varphi)\widehat{y} + K_{11}^\top\widehat{w}] - R_{10}\widehat{y} - R_{01}\widehat{w} \\ 0 \end{pmatrix},$$

with $\mathcal{A}^{(n),\text{ext}}$ is the matrix with coefficients $\mathcal{A}_{kj}^{(n),\text{ext}}$, see (6.21). The operator \mathbb{L}_{ext} is defined on the whole set \mathcal{O} and, by construction, coincides with the linear operator in (6.25) on the Cantor set \mathbf{G}_n ,

$$\forall (b, \omega) \in \mathbf{G}_n, \quad \mathbb{L}_{\text{ext}} = [DG_0(\mathbf{u}_0)]^{-1}d_{(i,\alpha)}\mathcal{F}(i_0, \alpha_0)D\widetilde{G}_0(\mathbf{u}_0). \quad (6.27)$$

The following proposition shows that the principal term \mathbb{D}_n has an exact inverse. Its proof can be found in [59, Prop. 6.3]

Proposition 6.3. *Given the conditions (2.2), (2.3), (7.42) and (7.43). There exists $\sigma_6 \triangleq \sigma_6(\tau_1, \tau_2, q, d) > 0$ such that if*

$$\|\mathfrak{J}_0\|_{s_h + \sigma_6}^{q, \gamma, \mathbf{m}} \leq 1,$$

then there exist a family of operators $([\mathbb{D}_n]_{\text{ext}}^{-1})_n$ such that for all $g \triangleq (g_1, g_2, g_3)$ satisfying the reversibility and \mathbf{m} -fold symmetry properties

$$g_1(\varphi) = g_1(-\varphi), \quad g_2(\varphi) = -g_2(-\varphi), \quad g_3(\varphi) = -(\mathcal{S}g_3)(\varphi), \quad (\mathcal{T}_m g_3)(\varphi) = g_3(\varphi), \quad (6.28)$$

the function $[\mathbb{D}_n]_{\text{ext}}^{-1}g$ satisfies the estimate, for all $s_0 \leq s \leq S$,

$$\|[\mathbb{D}_n]_{\text{ext}}^{-1}g\|_s^{q, \gamma, \mathbf{m}} \lesssim \gamma^{-1}(\|g\|_{s + \sigma_6}^{q, \gamma, \mathbf{m}} + \|\mathfrak{J}_0\|_{s + \sigma_6}^{q, \gamma, \mathbf{m}}\|g\|_{s_0 + \sigma_6}^{q, \gamma, \mathbf{m}})$$

and for all $(b, \omega) \in \mathbf{G}_n$ one has

$$\mathbb{D}_n[\mathbb{D}_n]_{\text{ext}}^{-1} = \text{Id}.$$

Coming back to the linear operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$, according to (6.26) and (6.27), on the Cantor set \mathbf{G}_n , we have the decomposition

$$\begin{aligned}d_{i,\alpha}\mathcal{F}(i_0, \alpha_0) &= DG_0(\mathbf{u}_0)\mathbb{D}_n[D\widetilde{G}_0(\mathbf{u}_0)]^{-1} + DG_0(\mathbf{u}_0)\mathbb{E}_n^{\text{ext}}[D\widetilde{G}_0(\mathbf{u}_0)]^{-1} \\ &\quad + DG_0(\mathbf{u}_0)\mathcal{P}_n[D\widetilde{G}_0(\mathbf{u}_0)]^{-1} + DG_0(\mathbf{u}_0)\mathcal{Q}_n[D\widetilde{G}_0(\mathbf{u}_0)]^{-1}.\end{aligned}$$

Applying the operator

$$\mathbb{T}_0 \triangleq \mathbb{T}_0(i_0) \triangleq D\widetilde{G}_0(\mathbf{u}_0)[\mathbb{D}_n]_{\text{ext}}^{-1}[DG_0(\mathbf{u}_0)]^{-1} \quad (6.29)$$

to the right of the last identity we get for all $(b, \omega) \in \mathbf{G}_n$,

$$d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)\mathsf{T}_0 - \text{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)} \quad \text{with} \quad \begin{aligned} \mathcal{E}_1^{(n)} &\triangleq DG_0(\mathbf{u}_0) \mathbb{E}_n^{\text{ext}} [D\tilde{G}_0(\mathbf{u}_0)]^{-1}\mathsf{T}_0, \\ \mathcal{E}_2^{(n)} &\triangleq DG_0(\mathbf{u}_0) \mathcal{P}_n [D\tilde{G}_0(\mathbf{u}_0)]^{-1}\mathsf{T}_0, \\ \mathcal{E}_3^{(n)} &\triangleq DG_0(\mathbf{u}_0) \mathcal{Q}_n [D\tilde{G}_0(\mathbf{u}_0)]^{-1}\mathsf{T}_0. \end{aligned}$$

Consequently, the operator T_0 is an approximate right inverse for $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$. In particular, we have the following result, whose proof is similar to [69, Theorem 5.1].

Theorem 6.1. (Approximate inverse) *Let $(\gamma, q, d, \tau_1, s_0, \mu_2, s_h, S)$ satisfy (2.2)–(2.3) and (7.42)–(7.43). There exists $\bar{\sigma} = \bar{\sigma}(\tau_1, \tau_2, d, q) > 0$ such that if*

$$\|\mathfrak{J}_0\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \leq 1, \quad (6.30)$$

then for smooth $g = (g_1, g_2, g_3)$, satisfying (6.28), the operator T_0 defined in (6.29) is reversible, \mathfrak{m} -fold preserving and satisfies

$$\forall s \in [s_0, S], \quad \|\mathsf{T}_0 g\|_s^{q,\gamma,\mathfrak{m}} \lesssim \gamma^{-1} (\|g\|_{s+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} + \|\mathfrak{J}_0\|_{s+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|g\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}}). \quad (6.31)$$

Moreover T_0 is an almost-approximate right inverse of $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ on the Cantor set \mathbf{G}_n . More precisely, for all $(b, \omega) \in \mathbf{G}_n$ one has

$$d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)\mathsf{T}_0 - \text{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)}, \quad (6.32)$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ are defined in the whole set \mathcal{O} with the estimates

$$\|\mathcal{E}_1^{(n)}\rho\|_{s_0}^{q,\gamma,\mathfrak{m}} \lesssim \gamma^{-1} \|\mathcal{F}(i_0, \alpha_0)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|\rho\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}}, \quad (6.33)$$

$$\forall \mathbf{b} \geq 0, \quad \|\mathcal{E}_2^{(n)}\rho\|_{s_0}^{q,\gamma,\mathfrak{m}} \lesssim \gamma^{-1} N_n^{-\mathbf{b}} (\|\rho\|_{s_0+\bar{\sigma}+\mathbf{b}}^{q,\gamma,\mathfrak{m}} + \|\mathfrak{J}_0\|_{s_0+\bar{\sigma}+\mathbf{b}}^{q,\gamma,\mathfrak{m}} \|\rho\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}}), \quad (6.34)$$

$$\begin{aligned} \forall \mathbf{b} \in [0, S], \quad \|\mathcal{E}_3^{(n)}\rho\|_{s_0}^{q,\gamma,\mathfrak{m}} &\lesssim N_n^{-\mathbf{b}} \gamma^{-2} \left(\|\rho\|_{s_0+\mathbf{b}+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s_0+\mathbf{b}+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|\rho\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \right) \\ &\quad + \varepsilon \gamma^{-4} N_0^{\mu_2} N_n^{-\mu_2} \|\rho\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}}. \end{aligned} \quad (6.35)$$

7 Reduction

This section is devoted to the reducibility of the linearized operator associated to the nonlinear equation (4.51), whose structure is detailed in Proposition 4.1. The first main step is to conjugate it into a diagonal matrix Fourier multiplier using a suitable quasi-periodic symplectic change of coordinates as in [59, 69]. The second part deals with the asymptotic structure of the operator localized on the normal directions. In the last part, we focus on the remainder reduction. To formulate our statements we need to introduce the following parameters.

$$\begin{aligned} s_l &\triangleq s_0 + \tau_1 q + \tau_1 + 2, & \bar{\mu}_2 &\triangleq 4\tau_1 q + 6\tau_1 + 3, \\ \bar{s}_l &\triangleq s_l + \tau_2 q + \tau_2, & \bar{s}_h &\triangleq \frac{3}{2}\bar{\mu}_2 + s_l + 1 \end{aligned} \quad (7.1)$$

and

$$\sigma_1 \triangleq s_0 + \tau_1 q + 2\tau_1 + 4, \quad \sigma_2 \triangleq \sigma_1 + 3. \quad (7.2)$$

Throughout this section and we shall work under the following assumption

$$\mathcal{O} \triangleq (b_*, b^*) \times \mathcal{U}, \quad \text{with} \quad 0 < b_* < b^* < 1 \quad \text{and} \quad \mathfrak{m} \geq \mathfrak{m}^*, \quad (7.3)$$

where \mathfrak{m}^* is defined in Corollary 4.1. The set \mathcal{U} is an open subset of \mathbb{R}^d containing the equilibrium frequency vector curve, namely, we choose

$$\mathcal{U} \triangleq B(0, R_0) \quad \text{s.t.} \quad \omega_{\text{Eq}}([b_*, b^*]) \subset B(0, \frac{R_0}{2}), \quad R_0 > 0. \quad (7.4)$$

We denote

$$\mathbf{H}_{\perp, \mathfrak{m}}^s \triangleq \mathbf{H}_{\mathfrak{m}}^s \cap \mathbf{H}_{\bar{s}_0}^{\perp}$$

and equip this space with the same norm as Sobolev spaces.

7.1 Structure of the linearized operator restricted to the normal directions

Here, we present the structure of the linearized operator in the normal directions

$$\widehat{\mathcal{L}} = \widehat{\mathcal{L}}(i_0) = \Pi_{\mathbb{S}_0}^\perp (\omega \cdot \partial_\varphi - \mathcal{J}K_{02}(\varphi)) \Pi_{\mathbb{S}_0}^\perp$$

defined through (6.23) and (6.7), where $i_0 = (\vartheta_0, I_0, z_0)$ is a m -fold reversible torus (satisfying (5.14)) and whose periodic component \mathfrak{J}_0 satisfy the smallness condition

$$\|\mathfrak{J}_0\|_{s_0+2}^{q,\gamma,m} \leq 1,$$

given in Lemma (5.2). The linear operator $\widehat{\mathcal{L}}$ decomposes as a finite rank perturbation of the linearized operator associated with the original problem, as the following shows. We refer the reader to [69, Prop. 6.1] for a detailed proof that one can adapt to our matrix case. We mention that the m -fold symmetry property can also be easily tracked.

Proposition 7.1. *Let (γ, q, d, s_0) satisfy (2.3). Then the operator $\widehat{\mathcal{L}}$ defined in (6.23) takes the form*

$$\widehat{\mathcal{L}} = \Pi_{\mathbb{S}_0}^\perp (\mathcal{L} - \varepsilon \partial_\theta \mathcal{R}) \Pi_{\mathbb{S}_0}^\perp, \quad \mathcal{L} \triangleq \omega \cdot \partial_\varphi \mathbf{I}_m + \mathfrak{L}_{\varepsilon r}, \quad \mathcal{R} \triangleq \begin{pmatrix} \mathcal{T}_{J_{1,1}}(r) & \mathcal{T}_{J_{1,2}}(r) \\ \mathcal{T}_{J_{2,1}}(r) & \mathcal{T}_{J_{2,2}}(r) \end{pmatrix},$$

where

$$\mathbf{I}_m \triangleq \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & \mathbb{I}_m \end{pmatrix}$$

denotes the identity map of $L_m^2(\mathbb{T}^{d+1}) \times L_m^2(\mathbb{T}^{d+1})$. The operator $\mathfrak{L}_{\varepsilon r}$ is defined in Proposition 4.1 and from (1.19) we have

$$r(\varphi) = \mathbf{A}(\vartheta_0(\varphi), I_0(\varphi), z_0(\varphi)),$$

with \mathbf{A} as in (1.19), supplemented with the reversibility and m -fold properties

$$r(-\varphi, -\theta) = r(\varphi, \theta) = r(\varphi, \theta + \frac{2\pi}{m}).$$

Moreover, for any $k, n \in \{1, 2\}$, the operator $\mathcal{T}_{J_{k,n}}(r)$ is an integral operator in the form (2.15), whose kernel $J_{k,n}(r)$ is m -fold reversibility preserving. In addition, under the assumption

$$\|\mathfrak{J}_0\|_{s_0}^{q,\gamma,m} \leq 1, \tag{7.5}$$

we have for all $s \geq s_0$,

(i) the function r satisfies the estimates,

$$\|r\|_s^{q,\gamma,m} \lesssim 1 + \|\mathfrak{J}_0\|_s^{q,\gamma,m} \tag{7.6}$$

and

$$\|\Delta_{12} r\|_s^{q,\gamma,m} \lesssim \|\Delta_{12} i\|_s^{q,\gamma,m} + \|\Delta_{12} i\|_{s_0}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|\mathfrak{J}_\ell\|_s^{q,\gamma,m}. \tag{7.7}$$

(ii) for any $k, n \in \{1, 2\}$, the kernel $J_{k,n}$ satisfies the following estimates

$$\|J_{k,n}\|_s^{q,\gamma,m} \lesssim 1 + \|\mathfrak{J}_0\|_{s+3}^{q,\gamma,m} \tag{7.8}$$

and

$$\|\Delta_{12} J_{k,n}\|_s^{q,\gamma,m} \lesssim \|\Delta_{12} i\|_{s+3}^{q,\gamma,m} + \|\Delta_{12} i\|_{s_0+3}^{q,\gamma,m} \max_{\ell \in \{1,2\}} \|\mathfrak{J}_\ell\|_{s+3}^{q,\gamma,m}. \tag{7.9}$$

Here $\mathfrak{J}_\ell(\varphi) = i_\ell(\varphi) - (\varphi, 0, 0)$, and for any function f , $\Delta_{12} f \triangleq f(i_1) - f(i_2)$ refers to the difference of f taken at two different states i_1 and i_2 satisfying (7.5).

7.2 Reduction of the transport part

The main purpose is to reduce to constant coefficients the transport parts in the linearized operator, described in Proposition 4.1. Notice that the transport operator is diagonal, therefore we shall reduce each scalar component apart. This was done by a KAM iterative scheme in [59, 69], in the same spirit of the papers [4, 8, 15, 45]. We skip the proof of the following proposition since it is the same as in [69, Prop. 6.2], where the scheme is initialized by (4.59), (4.53) and (7.12). Moreover, the persistence of the m -fold symmetry property can be easily checked along the scheme.

Proposition 7.2. *Given the conditions (2.3)–(2.2) and (7.1)–(7.3). Let $v \in (0, \frac{1}{q+1}]$. For any (μ_2, \mathbf{p}, s_h) satisfying*

$$\mu_2 \geq \bar{\mu}_2, \quad \mathbf{p} \geq 0, \quad s_h \geq \max\left(\frac{3}{2}\mu_2 + s_l + 1, \bar{s}_h + \mathbf{p}\right), \quad (7.10)$$

there exists $\varepsilon_0 > 0$ such that if

$$\varepsilon\gamma^{-1}N_0^{\mu_2} \leq \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{s_h+\sigma_1}^{q,\gamma,m} \leq 1, \quad (7.11)$$

then for all $k \in \{1, 2\}$ there exist $\mathbf{c}_k \triangleq \mathbf{c}_k(b, \omega, i_0) \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C})$ and $\beta_k \triangleq \beta_k(b, \omega, i_0) \in W^{q,\infty,\gamma}(\mathcal{O}, H_m^S)$ such that the following results hold true.

(i) The constants \mathbf{c}_k satisfy the following estimate,

$$\|\mathbf{v}_k - \mathbf{c}_k\|^{q,\gamma} \lesssim \varepsilon, \quad (7.12)$$

where \mathbf{v}_k is defined in (4.13).

(ii) The transformation $\mathcal{B}_k^{\pm 1}$, related to the functions β_k and $\widehat{\beta}_k$ through (2.19)–(2.27), are m -fold reversibility preserving and satisfying the following estimates: for all $s \in [s_0, S]$

$$\|\mathcal{B}_k^{\pm 1}\rho\|_s^{q,\gamma,m} \lesssim \|\rho\|_s^{q,\gamma,m} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{s+\sigma_1}^{q,\gamma,m}\|\rho\|_{s_0}^{q,\gamma,m}, \quad (7.13)$$

$$\|\widehat{\beta}_k\|_s^{q,\gamma,m} \lesssim \|\beta_k\|_s^{q,\gamma,m} \lesssim \varepsilon\gamma^{-1}(1 + \|\mathfrak{I}_0\|_{s+\sigma_1}^{q,\gamma,m}). \quad (7.14)$$

(iii) In the Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \triangleq \bigcap_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0,0)\} \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + j\mathbf{c}_k(b, \omega)| > \frac{4\gamma^v \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\} \quad (7.15)$$

we have

$$\mathcal{B}_k^{-1} \left(\omega \cdot \partial_\varphi \mathbb{I}_m + \partial_\theta (\mathcal{V}_k(\varepsilon r) \cdot) \right) \mathcal{B}_k = \omega \cdot \partial_\varphi \mathbb{I}_m + \mathbf{c}_k \partial_\theta + \mathcal{E}_{n,k}, \quad (7.16)$$

where \mathcal{V}_k are defined in Proposition 4.1-1 and $\mathcal{E}_{n,k} \triangleq \mathcal{E}_{n,k}(b, \omega, i_0)$ are linear operators satisfying

$$\|\mathcal{E}_{n,k}\rho\|_{s_0}^{q,\gamma,m} \lesssim \varepsilon N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0+2}^{q,\gamma,m}. \quad (7.17)$$

(iv) Given two tori i_1 and i_2 both satisfying (7.11) (replacing \mathfrak{I}_0 by \mathfrak{I}_1 or \mathfrak{I}_2), we have

$$\|\Delta_{12}\mathbf{c}_k\|^{q,\gamma} \lesssim \varepsilon \|\Delta_{12}i\|_{\bar{s}_h+\sigma_1}^{q,\gamma,m}, \quad (7.18)$$

$$\|\Delta_{12}\beta_k\|_{\bar{s}_h+\mathbf{p}}^{q,\gamma,m} + \|\Delta_{12}\widehat{\beta}_k\|_{\bar{s}_h+\mathbf{p}}^{q,\gamma,m} \lesssim \varepsilon\gamma^{-1} \|\Delta_{12}i\|_{\bar{s}_h+\mathbf{p}+\sigma_1}^{q,\gamma,m}. \quad (7.19)$$

Define the matrix operators

$$\mathcal{B} \triangleq \begin{pmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2 \end{pmatrix} \quad \text{and} \quad \mathcal{E}_n \triangleq \begin{pmatrix} \mathcal{E}_{n,1} & 0 \\ 0 & \mathcal{E}_{n,2} \end{pmatrix}, \quad (7.20)$$

where \mathcal{B}_k and $\mathcal{E}_{n,k}$ have been defined in Proposition 7.2. Next, we plan to describe the action of the transformation \mathcal{B} on the linearized operator introduced in Proposition 4.1 and derive some useful estimates.

Proposition 7.3. *Given the conditions (2.2)–(2.3), (7.1), (7.3) and (7.10). Then, there exists $\varepsilon_0 > 0$ such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2} \leq \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_2}^{\gamma,\mathfrak{m}} \leq 1, \quad (7.21)$$

where σ_2 is given by (7.2), then by restricting the parameters to the Cantor set defined in (7.15) we get

$$\mathcal{L} \triangleq \mathcal{B}^{-1}(\omega \cdot \partial_\varphi \mathbf{I}_m + \mathfrak{L}_{\varepsilon r})\mathcal{B} = \omega \cdot \partial_\varphi \mathbf{I}_m + \mathcal{D} + \mathcal{R} + \mathcal{E}_n, \quad (7.22)$$

where

$$\mathcal{D} \triangleq \begin{pmatrix} \mathfrak{c}_1 \partial_\theta \cdot + \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & \mathfrak{c}_2 \partial_\theta \cdot - (\frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot) \end{pmatrix},$$

and the operator $\mathcal{R} \triangleq \mathcal{R}(\varepsilon r)$ is a real, \mathfrak{m} -fold and reversibility preserving matricial integral operator satisfying

$$\forall s \in [s_0, S], \quad \|\mathcal{R}\|_s^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \left(1 + \|\mathfrak{I}_0\|_{s+\sigma_2}^{q,\gamma,\mathfrak{m}}\right). \quad (7.23)$$

Moreover, given two tori i_1 and i_2 both satisfying (7.21) (replacing \mathfrak{I}_0 by \mathfrak{I}_1 or \mathfrak{I}_2), we have

$$\|\Delta_{12}\mathcal{R}\|_{\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \|\Delta_{12}i\|_{\bar{s}_h+\mathfrak{p}+\sigma_2}^{q,\gamma,\mathfrak{m}}. \quad (7.24)$$

Proof. From (7.20) and (4.52) we may write

$$\begin{aligned} \mathcal{B}^{-1}(\omega \cdot \partial_\varphi \mathbf{I}_m + \mathfrak{L}_{\varepsilon r})\mathcal{B} &= \begin{pmatrix} \mathcal{B}_1^{-1}(\omega \cdot \partial_\varphi \mathbf{I}_m + \partial_\theta(\mathcal{V}_1(\varepsilon r) \cdot))\mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_2^{-1}(\omega \cdot \partial_\varphi \mathbf{I}_m + \partial_\theta(\mathcal{V}_2(\varepsilon r) \cdot))\mathcal{B}_2 \end{pmatrix} \\ &+ \begin{pmatrix} \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & -\frac{1}{2} \mathcal{H} - \partial_\theta \mathcal{Q} * \cdot \end{pmatrix} + \begin{pmatrix} \mathcal{R}_{1,1} & \mathcal{R}_{1,2} \\ \mathcal{R}_{2,1} & \mathcal{R}_{2,2} \end{pmatrix}, \end{aligned}$$

where

$$\mathcal{R}_{k,k'} \triangleq \mathcal{B}_k^{-1} \partial_\theta \mathcal{T}_{\mathcal{K}_{k,k'}(\varepsilon r)} \mathcal{B}_{k'} + (-1)^{k+1} \delta_{k,k'} \left(\frac{1}{2} \mathcal{B}_k^{-1} \mathcal{H} \mathcal{B}_k - \frac{1}{2} \mathcal{H} + \mathcal{B}_k^{-1} (\partial_\theta \mathcal{Q} * \cdot) \mathcal{B}_k - \partial_\theta \mathcal{Q} * \cdot \right)$$

and $\delta_{k,k'}$ denotes the usual Kronecker symbol. Putting together (7.16) and (4.52) allows to get in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ the decomposition (7.22) with

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_{1,1} & \mathcal{R}_{1,2} \\ \mathcal{R}_{2,1} & \mathcal{R}_{2,2} \end{pmatrix}.$$

The symmetry properties of \mathcal{Q} , β and $\widehat{\beta}$, given by Proposition 4.1-2 and Proposition 7.2-2, together with Lemma 2.4 and Lemma 2.7 imply that $\mathcal{B}_k^{-1} \mathcal{H} \mathcal{B}_k - \mathcal{H}$ and $\mathcal{B}_k^{-1} \mathcal{Q} \mathcal{B}_k - \mathcal{Q}$ are real, reversible and \mathfrak{m} -fold preserving integral operators. In view of Lemma 2.7, (7.14), (7.19), (7.21) and (7.2) the operator $\mathcal{B}_k^{-1} \mathcal{H} \mathcal{B}_k - \mathcal{H}$ is an integral operator and satisfies

$$\|\mathcal{B}_k^{-1} \mathcal{H} \mathcal{B}_k - \mathcal{H}\|_{\text{I-D},s}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \left(1 + \|\mathfrak{I}_0\|_{s+2+\sigma_1}^{q,\gamma,\mathfrak{m}}\right), \quad (7.25)$$

$$\|\Delta_{12}(\mathcal{B}_k^{-1} \mathcal{H} \mathcal{B}_k - \mathcal{H})\|_{\text{I-D},\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \|\Delta_{12}i\|_{\bar{s}_h+\mathfrak{p}+2+\sigma_1}^{q,\gamma,\mathfrak{m}}. \quad (7.26)$$

As for the term in \mathcal{Q} , we observe according to the notation (2.15), (4.54) and (4.30) that

$$\mathcal{Q} * \rho = \mathcal{T}_{\widehat{Q}} \rho, \quad \text{with} \quad \widehat{Q}(b, \varphi, \theta, \eta) \triangleq \mathcal{Q}(b, \theta - \eta)$$

and

$$\|\widehat{Q}\|_s^{q,\gamma,\mathfrak{m}} \leq C(q, s),$$

for some constant $C(q, s) > 0$. Therefore, applying Lemma 2.6 together with (7.14), (7.19) and the smallness condition (7.21) yield to

$$\|\mathcal{B}_k^{-1} (\partial_\theta \mathcal{Q} * \cdot) \mathcal{B}_k - \partial_\theta \mathcal{Q} * \cdot\|_{\text{I-D},s}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \left(1 + \|\mathfrak{I}_0\|_{s+\sigma_1+1}^{q,\gamma,\mathfrak{m}}\right), \quad (7.27)$$

$$\|\Delta_{12}(\mathcal{B}_k^{-1} (\partial_\theta \mathcal{Q} * \cdot) \mathcal{B}_k - \partial_\theta \mathcal{Q} * \cdot)\|_{\text{I-D},\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon\gamma^{-1} \|\Delta_{12}i\|_{\bar{s}_h+\mathfrak{p}+\sigma_1+1}^{q,\gamma,\mathfrak{m}}. \quad (7.28)$$

As for the operator $\mathcal{B}_k^{-1} \partial_\theta \mathcal{T}_{\mathcal{K}_{k,k'}(\varepsilon r)} \mathcal{B}_{k'}$, first it is real, \mathfrak{m} -fold preserving and reversible Toeplitz in time operator according to Lemma 2.4 together with (4.69), (4.70) and the symmetry properties of $\beta, \widehat{\beta}$, given by Proposition 7.2-2. In addition, using (2.23), Proposition 4.1-3, (7.14), (7.6), (7.21) and (7.2), we deduce that

$$\begin{aligned} \|\mathcal{B}_k^{-1} \partial_\theta \mathcal{T}_{\mathcal{K}_{k,k'}(\varepsilon r)} \mathcal{B}_{k'}\|_{\text{I-D},s}^{q,\gamma,\mathfrak{m}} &\lesssim \|\mathcal{K}_{k,k'}(\varepsilon r)\|_{s+1}^{q,\gamma,\mathfrak{m}} + \|\mathcal{K}_{k,k'}(\varepsilon r)\|_{s_0}^{q,\gamma,\mathfrak{m}} \max_{\ell \in \{1,2\}} \|\beta_\ell\|_{s+2}^{q,\gamma,\mathfrak{m}} \\ &\lesssim \varepsilon \gamma^{-1} (1 + \|\mathcal{J}_0\|_{s+\sigma_1+2}^{q,\gamma,\mathfrak{m}}). \end{aligned} \quad (7.29)$$

Using Proposition 4.1-3 supplemented by (7.6) and (7.7), we obtain

$$\begin{aligned} \|\Delta_{12} \mathcal{K}_{k,k'}(\varepsilon r)\|_s^{q,\gamma,\mathfrak{m}} &\lesssim \varepsilon \|\Delta_{12} r\|_{s+1}^{q,\gamma,\mathfrak{m}} + \varepsilon \|\Delta_{12} r\|_{s_0+1}^{q,\gamma,\mathfrak{m}} \max_{\ell \in \{1,2\}} \|r_\ell\|_{s+1}^{q,\gamma,\mathfrak{m}} \\ &\lesssim \varepsilon \left(\|\Delta_{12} i\|_{s+1}^{q,\gamma,\mathfrak{m}} + \|\Delta_{12} i\|_{s_0+1}^{q,\gamma,\mathfrak{m}} \max_{\ell \in \{1,2\}} \|\mathcal{J}_\ell\|_{s+1}^{q,\gamma,\mathfrak{m}} \right). \end{aligned} \quad (7.30)$$

Therefore, applying (2.26) together with (7.30), (7.14), (7.19) and (7.21), we get

$$\|\Delta_{12} \mathcal{B}_k^{-1} \partial_\theta \mathcal{T}_{\mathcal{K}_{k,k'}(\varepsilon r)} \mathcal{B}_{k'}\|_{\text{I-D},\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-1} \|\Delta_{12} i\|_{\bar{s}_h+\mathfrak{p}+2+\sigma_1}^{q,\gamma,\mathfrak{m}}. \quad (7.31)$$

Combining (7.25), (7.27) and (7.29), we find

$$\|\mathcal{R}_{k,k'}\|_{\text{I-D},s}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-1} (1 + \|\mathcal{J}_0\|_{s+2+\sigma_1}^{q,\gamma,\mathfrak{m}}).$$

Moreover, putting together (7.26), (7.28) and (7.31) implies

$$\|\Delta_{12} \mathcal{R}_{k,k'}\|_{\text{I-D},\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-1} \|\Delta_{12} i\|_{\bar{s}_h+\mathfrak{p}+2+\sigma_1}^{q,\gamma,\mathfrak{m}}.$$

This proves the Proposition 7.3. \square

7.3 Localization into the normal directions

We shall focus in this section on the localization effects in the normal directions for the reduction of the transport part. For this aim, we consider the localized quasi-periodic symplectic change of coordinates defined by

$$\mathcal{B}_\perp \triangleq \Pi_{\bar{S}_0}^\perp \mathcal{B} \Pi_{\bar{S}_0}^\perp = \begin{pmatrix} \Pi_1^\perp \mathcal{B}_1 \Pi_1^\perp & 0 \\ 0 & \Pi_2^\perp \mathcal{B}_2 \Pi_2^\perp \end{pmatrix},$$

where the projectors are defined in (1.17)-(1.18). Then, the main result of this section reads as follows.

Proposition 7.4. *Let $(\gamma, q, d, \tau_1, s_0, S, \mathfrak{m})$ satisfy (2.2)–(2.3) and (7.3). Let $(\bar{\mu}_2, s_l, \bar{s}_h, \mu_2, \mathfrak{p}, s_h)$ satisfy (7.1) and (7.10). There exist $\varepsilon_0 > 0$ and $\sigma_3 \triangleq \sigma_3(\tau_1, q, d, s_0) \geq \sigma_2$, where σ_2 is given by (7.2), such that if*

$$\varepsilon \gamma^{-1} N_0^{\mu_2} \leq \varepsilon \quad \text{and} \quad \|\mathcal{J}_0\|_{s_h+\sigma_3}^{q,\gamma,\mathfrak{m}} \leq 1, \quad (7.32)$$

then the following assertions hold true.

(i) *The operators $\mathcal{B}_\perp^{\pm 1}$ satisfy the following estimate*

$$\|\mathcal{B}_\perp^{\pm 1} \rho\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\rho\|_s^{q,\gamma,\mathfrak{m}} + \varepsilon \gamma^{-1} \|\mathcal{J}_0\|_{s+\sigma_3}^{q,\gamma,\mathfrak{m}} \|\rho\|_{s_0}^{q,\gamma,\mathfrak{m}}. \quad (7.33)$$

(ii) *For any $n \in \mathbb{N}^*$, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ introduced in (7.15), we have*

$$\mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp = \mathcal{L}_0 + \mathcal{E}_n^0, \quad \mathcal{L}_0 \triangleq \omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m},\perp} + \mathcal{D}_0 + \mathcal{R}_0, \quad (7.34)$$

where $\mathbf{I}_{\mathfrak{m},\perp} \triangleq \Pi_{\bar{S}_0}^\perp \mathbf{I}_{\mathfrak{m}}$ and $\mathcal{D}_0 = \Pi_{\bar{S}_0}^\perp \mathcal{D}_0 \Pi_{\bar{S}_0}^\perp$ is a reversible Fourier multiplier operator given by

$$\mathcal{D}_0 \triangleq \begin{pmatrix} \mathcal{D}_{0,1} & 0 \\ 0 & \mathcal{D}_{0,2} \end{pmatrix}, \quad \mathcal{D}_{0,k} \triangleq \left(i \mu_{j,k}^{(0)} \right)_{j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{S}_{0,k}}, \quad \mu_{-j,k}^{(0)}(b, \omega) = -\mu_{j,k}^{(0)}(b, \omega),$$

with

$$\mu_{j,k}^{(0)}(b, \omega, i_0) \triangleq \Omega_{j,k}(b) + jr_k^{(0)}(b, \omega, i_0), \quad r_k^{(0)}(b, \omega, i_0) \triangleq \mathbf{c}_k(b, \omega) - \mathbf{v}_k(b) \quad (7.35)$$

and such that

$$\|r_k^{(0)}\|^{q,\gamma} \lesssim \varepsilon \quad \text{and} \quad \|\Delta_{12}r_k^{(0)}\|^{q,\gamma} \lesssim \varepsilon \|\Delta_{12}i\|_{\bar{s}_h+\sigma_1}^{q,\gamma,\mathfrak{m}}. \quad (7.36)$$

Notice that the frequencies $\Omega_{j,k}(b)$ are defined in (4.17).

(iii) The operator \mathcal{E}_n^0 satisfies the following estimate

$$\|\mathcal{E}_n^0\rho\|_{s_0}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0+2}^{q,\gamma,\mathfrak{m}}. \quad (7.37)$$

(iv) The operator \mathcal{R}_0 is an \mathfrak{m} -fold preserving and reversible Toeplitz in time matricial operator satisfying

$$\forall s \in [s_0, S], \quad \|\mathcal{R}_0\|_s^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-1} (1 + \|\mathcal{J}_0\|_{s+\sigma_3}^{q,\gamma,\mathfrak{m}}) \quad (7.38)$$

and

$$\|\Delta_{12}\mathcal{R}_0\|_{\bar{s}_h+\mathfrak{p}}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-1} \|\Delta_{12}i\|_{\bar{s}_h+\mathfrak{p}+\sigma_3}^{q,\gamma,\mathfrak{m}}. \quad (7.39)$$

(v) The operator \mathcal{L}_0 satisfies

$$\forall s \in [s_0, S], \quad \|\mathcal{L}_0\rho\|_s^{q,\gamma,\mathfrak{m}} \lesssim \|\rho\|_{s+1}^{q,\gamma,\mathfrak{m}} + \varepsilon \gamma^{-1} \|\mathcal{J}_0\|_{s+\sigma_3}^{q,\gamma,\mathfrak{m}} \|\rho\|_{s_0}^{q,\gamma,\mathfrak{m}}. \quad (7.40)$$

Proof. (i) It is obtained using (7.13) and Lemma 2.1-(i).

(ii) The first estimate of (7.36) follows from (7.12) and the second one from (7.18). On the other hand, using the expression of $\widehat{\mathcal{L}}$ detailed in Proposition 7.1, combined with the decomposition $\text{Id} = \Pi_{\mathbb{S}_0} + \Pi_{\bar{\mathbb{S}}_0}^\perp$ we write

$$\begin{aligned} \mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp &= \mathcal{B}_\perp^{-1} \Pi_{\bar{\mathbb{S}}_0}^\perp (\mathcal{L} - \varepsilon \partial_\theta \mathcal{R}) \mathcal{B}_\perp \\ &= \mathcal{B}_\perp^{-1} \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{L} \mathcal{B} \Pi_{\bar{\mathbb{S}}_0}^\perp - \mathcal{B}_\perp^{-1} \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{L} \Pi_{\mathbb{S}_0} \mathcal{B} \Pi_{\bar{\mathbb{S}}_0}^\perp - \varepsilon \mathcal{B}_\perp^{-1} \Pi_{\bar{\mathbb{S}}_0}^\perp \partial_\theta \mathcal{R} \mathcal{B}_\perp. \end{aligned}$$

By virtue of Proposition 7.3 one has in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma;\tau_1}(i_0)$,

$$\mathcal{L} \mathcal{B} = \mathcal{B} \mathcal{L}$$

and therefore, using also that $\mathcal{B}_\perp^{-1} \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{B}_\perp^{-1}$, we get

$$\mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp = \mathcal{B}_\perp^{-1} \mathcal{B} \mathcal{L} \Pi_{\bar{\mathbb{S}}_0}^\perp - \mathcal{B}_\perp^{-1} \mathcal{L} \Pi_{\mathbb{S}_0} \mathcal{B} \Pi_{\bar{\mathbb{S}}_0}^\perp - \varepsilon \mathcal{B}_\perp^{-1} \partial_\theta \mathcal{R} \mathcal{B}_\perp.$$

Thus, using (7.22) we deduce that

$$\mathcal{B}_\perp^{-1} \mathcal{B} \mathcal{L} \Pi_{\bar{\mathbb{S}}_0}^\perp = (\omega \cdot \partial_\varphi \mathbf{I}_\mathfrak{m} + \mathcal{D}) \Pi_{\bar{\mathbb{S}}_0}^\perp + \mathcal{E}_n^0 + \mathcal{B}_\perp^{-1} \mathcal{B} \mathcal{R} \Pi_{\bar{\mathbb{S}}_0}^\perp,$$

with

$$\mathcal{E}_n^0 \triangleq \mathcal{B}_\perp^{-1} \mathcal{B} \mathcal{E}_n \Pi_{\bar{\mathbb{S}}_0}^\perp. \quad (7.41)$$

Consequently, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma;\tau_1}(i_0)$, one has the following reduction

$$\mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp = \omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m},\perp} + \mathcal{D}_0 + \mathcal{R}_0 + \mathcal{E}_n^0,$$

where we set

$$\mathcal{D}_0 \triangleq \begin{pmatrix} \mathbf{c}_1 \partial_\theta \cdot + \frac{1}{2} \mathcal{H} + \partial_\theta \mathcal{Q} * \cdot & 0 \\ 0 & \mathbf{c}_2 \partial_\theta \cdot - \frac{1}{2} \mathcal{H} - \partial_\theta \mathcal{Q} * \cdot \end{pmatrix} \Pi_{\bar{\mathbb{S}}_0}^\perp,$$

$$\mathcal{R}_0 \triangleq -\mathcal{B}_\perp^{-1} \mathcal{L} \Pi_{\mathbb{S}_0} \mathcal{B} \Pi_{\bar{\mathbb{S}}_0}^\perp - \varepsilon \mathcal{B}_\perp^{-1} \partial_\theta \mathcal{R} \mathcal{B}_\perp + \mathcal{B}_\perp^{-1} \mathcal{B} \mathcal{R} \Pi_{\bar{\mathbb{S}}_0}^\perp.$$

(iii) It can be obtained from (7.41), (7.33), (7.13) and (7.17).

(iv) To get the estimates (7.38) and (7.39), we may refer to Lemma [69, Prop 6.3 and Lem. 6.3] up to very slight modifications corresponding to the hybrid topology introduced in Section 2. The computations are long and based on a duality representations of $\mathcal{B}_\perp^{\pm 1}$ and $\mathcal{B}^{\pm 1}$. In particular, one may use Lemma 2.4, (7.23), (7.24), (7.8), (7.9) and (7.19).

(v) This estimate follows from (7.34), (7.12), (7.38), (7.32) and Corollary 2.1-(iii). \square

7.4 Reduction of the remainder

This section is devoted to the conjugation of the operator \mathcal{L}_0 defined in Proposition 7.4 to a diagonal one, up to a fast decaying small remainders. This will be achieved through a standard KAM reducibility techniques well-adapted to the operators setting. This will be implemented by taking advantage of the exterior parameters which are restricted to a suitable Cantor set that prevents the resonances in the second order Melnikov assumption. Notice that one gets from this study some estimates on the distribution of the eigenvalues and their stability with respect to the torus parametrization. This is considered as the key step not only to get an approximate inverse but also to achieve the Nash-Moser scheme with a final massive Cantor set. We may refer for instance to [5, 10, 46, 59, 69] for some implementations of this KAM strategy to PDEs.

Proposition 7.5. *Let $(\gamma, q, d, \tau_1, \tau_2, s_0, \bar{s}_l, \bar{\mu}_2, S, \mathfrak{m})$ satisfy (2.3), (2.2), (7.1) and (7.3). For any (μ_2, s_h) satisfying*

$$\mu_2 \geq \bar{\mu}_2 + 2\tau_2 q + 2\tau_2 \quad \text{and} \quad s_h \geq \frac{3}{2}\mu_2 + \bar{s}_l + 1, \quad (7.42)$$

there exist $\varepsilon_0 \in (0, 1)$ and $\sigma_4 \triangleq \sigma_4(\tau_1, \tau_2, q, d) \geq \sigma_3$, with σ_3 defined in Proposition 7.4, such that if

$$\varepsilon \gamma^{-2-q} N_0^{\mu_2} \leq \varepsilon_0 \quad (7.43)$$

and

$$\|\mathfrak{J}_0\|_{s_h + \sigma_4}^{q, \gamma, \mathfrak{m}} \leq 1, \quad (7.44)$$

then the following assertions hold true.

(i) *There exists a family of invertible linear operator $\Phi_\infty : \mathcal{O} \rightarrow \mathcal{L}(\mathbf{H}_{\perp, \mathfrak{m}}^s)$ satisfying the estimates*

$$\forall s \in [s_0, S], \quad \|\Phi_\infty^{\pm 1} \rho\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|\rho\|_s^{q, \gamma, \mathfrak{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s + \sigma_4}^{q, \gamma, \mathfrak{m}} \|\rho\|_{s_0}^{q, \gamma, \mathfrak{m}}. \quad (7.45)$$

There exists a diagonal operator $\mathcal{L}_\infty \triangleq \mathcal{L}_\infty(b, \omega, i_0)$ taking the form

$$\mathcal{L}_\infty = \omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m}, \perp} + \mathcal{D}_\infty$$

where $\mathcal{D}_\infty = \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{D}_\infty \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{D}_\infty(b, \omega, i_0)$ is a diagonal operator with reversible Fourier multiplier entries, namely

$$\mathcal{D}_\infty \triangleq \begin{pmatrix} \mathcal{D}_{\infty, 1} & 0 \\ 0 & \mathcal{D}_{\infty, 2} \end{pmatrix}, \quad \mathcal{D}_{\infty, k} \triangleq \left(i \mu_{j, k}^{(\infty)} \right)_{j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, k}}, \quad \mu_{-j, k}^{(\infty)}(b, \omega) = -\mu_{j, k}^{(\infty)}(b, \omega),$$

with

$$\forall j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, k}, \quad \mu_{j, k}^{(\infty)}(b, \omega, i_0) \triangleq \mu_{j, k}^{(0)}(b, \omega, i_0) + r_{j, k}^{(\infty)}(b, \omega, i_0) \quad (7.46)$$

and

$$\sup_{j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, k}} |j| \left\| r_{j, k}^{(\infty)} \right\|^{q, \gamma} \lesssim \varepsilon \gamma^{-1}, \quad (7.47)$$

such that in the Cantor set

$$\begin{aligned} \mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) &\triangleq \mathcal{O}_{\infty, n}^{\gamma, \tau_1}(i_0) \\ &\bigcap_{\substack{k \in \{1, 2\} \\ j, j_0 \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, k}}} \bigcap_{\substack{l \in \mathbb{Z}^d \\ (l, j) \neq (0, j_0) \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + \mu_{j, k}^{(\infty)}(b, \omega, i_0) - \mu_{j_0, k}^{(\infty)}(b, \omega, i_0)| > \frac{2\gamma \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\} \\ &\bigcap_{\substack{j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, 1} \\ j_0 \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0, 2}}} \bigcap_{\substack{l \in \mathbb{Z}^d \\ (l, j, j_0) \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + \mu_{j, 1}^{(\infty)}(b, \omega, i_0) - \mu_{j_0, 2}^{(\infty)}(b, \omega, i_0)| > \frac{2\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} \right\} \end{aligned}$$

we have

$$\Phi_\infty^{-1} \mathcal{L}_0 \Phi_\infty = \mathcal{L}_\infty + \mathcal{E}_n^1,$$

and the linear operator \mathcal{E}_n^1 satisfies the estimate

$$\|\mathcal{E}_n^1 \rho\|_{s_0}^{q,\gamma,\mathfrak{m}} \lesssim \varepsilon \gamma^{-2} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0+1}^{q,\gamma,\mathfrak{m}}. \quad (7.48)$$

We refer to (7.15), (7.34) and (7.35) for the definition of $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, \mathcal{L}_0 and $(\mu_{j,k}^{(0)}(b, \omega, i_0))_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}$, respectively.

(ii) Given two tori i_1 and i_2 both satisfying (7.43)-(7.44), then for $k \in \{1, 2\}$ we have

$$\forall j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}, \quad \left\| \Delta_{12} r_{j,k}^{(\infty)} \right\|^{q,\gamma} \lesssim \varepsilon \gamma^{-1} \|\Delta_{12} i\|_{\bar{s}_h + \sigma_4}^{q,\gamma,\mathfrak{m}} \quad (7.49)$$

and

$$\forall j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}, \quad \left\| \Delta_{12} \mu_{j,k}^{(\infty)} \right\|^{q,\gamma} \lesssim \varepsilon \gamma^{-1} |j| \|\Delta_{12} i\|_{\bar{s}_h + \sigma_4}^{q,\gamma,\mathfrak{m}}. \quad (7.50)$$

Proof. (i) First recall that Proposition 7.4 states that in restriction to the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ the following identity holds

$$\mathcal{B}_\perp^{-1} \widehat{\mathcal{L}} \mathcal{B}_\perp = \mathcal{L}_0 + \mathcal{E}_n^0,$$

where the operator \mathcal{L}_0 decomposes as follows

$$\mathcal{L}_0 = \omega \cdot \partial_\varphi \mathbf{I}_{\mathbb{m},\perp} + \mathcal{D}_0 + \mathcal{R}_0,$$

with

$$\mathcal{D}_0 = \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{D}_0 \Pi_{\bar{\mathbb{S}}_0}^\perp = \begin{pmatrix} \mathcal{D}_{0,1} & 0 \\ 0 & \mathcal{D}_{0,2} \end{pmatrix}, \quad \mathcal{D}_{0,k} = \left(i \mu_{j,k}^{(0)} \right)_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}, \quad \mu_{-j,k}^{(0)}(b, \omega) = -\mu_{j,k}^{(0)}(b, \omega)$$

and \mathcal{R}_0 a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{R}_0 \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{R}_0$. Let us define the quantity

$$\delta_0(s) = \gamma^{-1} \|\mathcal{R}_0\|_s^{q,\gamma,\mathfrak{m}},$$

By virtue of (7.38), we find

$$\delta_0(s) \leq C \varepsilon \gamma^{-2} (1 + \|\mathcal{J}_0\|_{s+\sigma_3}^{q,\gamma,\mathfrak{m}}). \quad (7.51)$$

Thus, combining (7.42), (7.43) and the fact that $\sigma_4 \geq \sigma_3$ yields

$$\begin{aligned} N_0^{\mu_2} \delta_0(s_h) &\leq C N_0^{\mu_2} \varepsilon \gamma^{-2} \\ &\leq C \varepsilon_0. \end{aligned} \quad (7.52)$$

The smallness conditions (7.51) and (7.52) allow to start a KAM reduction procedure similarly to the scalar case [69, Prop. 6.5]. Nevertheless the following KAM iteration is done at the matricial level. For this aim, we need to consider the hybrid norm (2.34) to overcome spatial resonances coming from the anti-diagonal entries when solving the homological equations. To clarify this point, let us first discuss a general KAM step of the procedure.

► **KAM step.** Now, we explain the typical KAM step used in the reduction of the remainder. Assume that we have a linear operator \mathcal{L} taking the following form when the parameters are restricted to some Cantor set \mathcal{O}

$$\mathcal{L} = \omega \cdot \partial_\varphi \mathbf{I}_{\mathbb{m},\perp} + \mathcal{D} + \mathcal{R},$$

with

$$\mathcal{D} = \Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{D} \Pi_{\bar{\mathbb{S}}_0}^\perp = \begin{pmatrix} \mathcal{D}_1 & 0 \\ 0 & \mathcal{D}_2 \end{pmatrix}, \quad \mathcal{D}_k = \left(i \mu_{j,k} \right)_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}, \quad \mu_{-j,k}(b, \omega) = -\mu_{j,k}(b, \omega). \quad (7.53)$$

In addition we assume that the matrix operator

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_3 \\ \mathcal{R}_4 & \mathcal{R}_2 \end{pmatrix}$$

is real, reversible Toeplitz in time of zero order and satisfies

$$\Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{R} \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{R}.$$

One may check from (1.17) that this latter assumption is equivalent to

$$\begin{pmatrix} \Pi_1^\perp \mathcal{R}_1 \Pi_1^\perp & \Pi_1^\perp \mathcal{R}_3 \Pi_2^\perp \\ \Pi_2^\perp \mathcal{R}_4 \Pi_1^\perp & \Pi_2^\perp \mathcal{R}_2 \Pi_2^\perp \end{pmatrix} = \begin{pmatrix} \mathcal{R}_1 & \mathcal{R}_3 \\ \mathcal{R}_4 & \mathcal{R}_2 \end{pmatrix}. \quad (7.54)$$

According to Definition 2.1, the real and reversibility properties of \mathcal{R}_k are equivalent to say

$$(\mathcal{R}_k)_{l_0, j_0}^{l, j} \triangleq i r_{j_0, k}^j(b, \omega, l - l_0) \in i\mathbb{R} \quad \text{and} \quad (\mathcal{R}_k)_{-l_0, -j_0}^{-l, -j} = -(\mathcal{R}_k)_{l_0, j_0}^{l, j}. \quad (7.55)$$

Moreover, the condition (7.54) is equivalent to

$$\forall k \in \{1, 2\}, \quad \forall l \in \mathbb{Z}^d, \quad \forall j \text{ or } j_0 \in \bar{\mathbb{S}}_{0, k}, \quad r_{j_0, k}^j(b, \omega, l) = 0, \quad (7.56)$$

$$\forall l \in \{3, 4\}, \quad \forall l \in \mathbb{Z}^d, \quad \forall j \in \bar{\mathbb{S}}_{0, \ell-2} \text{ or } j_0 \in \bar{\mathbb{S}}_{0, 5-\ell}, \quad r_{j_0, \ell}^j(b, \omega, l) = 0. \quad (7.57)$$

Now, consider a linear invertible transformation close to the identity

$$\Phi = \mathbf{I}_{\mathbf{m}, \perp} + \Psi : \mathcal{O} \rightarrow \mathcal{L}(\mathbf{H}_{\perp, \mathbf{m}}^s), \quad \Psi = \Pi_{\bar{\mathbb{S}}_0}^\perp \Psi \Pi_{\bar{\mathbb{S}}_0}^\perp = \begin{pmatrix} \Psi_1 & \Psi_3 \\ \Psi_4 & \Psi_2 \end{pmatrix}, \quad (7.58)$$

with Ψ depending on \mathcal{R} and *small* in a suitable sense related to the hybrid norm (2.34). Then, one readily obtains, in restriction to \mathcal{O} , the following decomposition

$$\begin{aligned} \Phi^{-1} \mathcal{L} \Phi &= \Phi^{-1} \left(\Phi \left(\omega \cdot \partial_\varphi \Pi_{\bar{\mathbb{S}}_0}^\perp + \mathcal{D} \right) + \left[\omega \cdot \partial_\varphi \Pi_{\bar{\mathbb{S}}_0}^\perp + \mathcal{D}, \Psi \right] + \mathcal{R} + \mathcal{R} \Psi \right) \\ &= \omega \cdot \partial_\varphi \Pi_{\bar{\mathbb{S}}_0}^\perp + \mathcal{D} + \Phi^{-1} \left(\left[\omega \cdot \partial_\varphi \Pi_{\bar{\mathbb{S}}_0}^\perp + \mathcal{D}, \Psi \right] + \mathbf{P}_\mathbf{N} \mathcal{R} + \mathbf{P}_\mathbf{N}^\perp \mathcal{R} + \mathcal{R} \Psi \right), \end{aligned}$$

where $\mathbf{P}_\mathbf{N} \mathcal{R}$ and $\mathbf{P}_\mathbf{N}^\perp \mathcal{R}$ are defined as in (2.35). We shall select Ψ such that the above expression contains a new remainder $\mathcal{R}_{\text{next}}$ quadratically smaller than the previous one \mathcal{R} up to modify the diagonal part \mathcal{D} into a new one $\mathcal{D}_{\text{next}}$ with the same structure (7.53). Therefore, we choose Ψ such that it solves the following *matricial homological equation*

$$\left[\omega \cdot \partial_\varphi \mathbf{I}_{\mathbf{m}, \perp} + \mathcal{D}, \Psi \right] + \mathbf{P}_\mathbf{N} \mathcal{R} = \lfloor \mathbf{P}_\mathbf{N} \mathcal{R} \rfloor, \quad (7.59)$$

where $\lfloor \mathbf{P}_\mathbf{N} \mathcal{R} \rfloor$ is the diagonal part of the matrix operator $\mathbf{P}_\mathbf{N} \mathcal{R}$ as defined by (2.32)-(2.33), namely

$$\lfloor \mathbf{P}_\mathbf{N} \mathcal{R} \rfloor = \begin{pmatrix} \lfloor P_N^1 \mathcal{R}_1 \rfloor & 0 \\ 0 & \lfloor P_N^1 \mathcal{R}_2 \rfloor \end{pmatrix}.$$

The matricial equation (7.59) is equivalent to the following set of four scalar homological equations

$$\begin{cases} \left[\omega \cdot \partial_\varphi \Pi_1^\perp + \mathcal{D}_1, \Psi_1 \right] = \lfloor P_N^1 \mathcal{R}_1 \rfloor - P_N^1 \mathcal{R}_1, \\ \left[\omega \cdot \partial_\varphi \Pi_2^\perp + \mathcal{D}_2, \Psi_2 \right] = \lfloor P_N^1 \mathcal{R}_2 \rfloor - P_N^1 \mathcal{R}_2, \\ (\omega \cdot \partial_\varphi \Pi_1^\perp + \mathcal{D}_1) \Psi_3 - \Psi_3 (\omega \cdot \partial_\varphi \Pi_2^\perp + \mathcal{D}_2) = -P_N^2 \mathcal{R}_3, \\ (\omega \cdot \partial_\varphi \Pi_2^\perp + \mathcal{D}_2) \Psi_4 - \Psi_4 (\omega \cdot \partial_\varphi \Pi_1^\perp + \mathcal{D}_1) = -P_N^2 \mathcal{R}_4. \end{cases} \quad (7.60)$$

As we shall see these equations can be solved modulo the selection of suitable parameters (b, ω) among a Cantor-type set connected to non-resonance conditions. Let us begin with the diagonal equations

on Ψ_1 and Ψ_2 which can be treated in a similar way. Fix $k \in \{1, 2\}$, then we are interested in solving the equation

$$[\omega \cdot \partial_\varphi \Pi_k^\perp + \mathcal{D}_k, \Psi_k] = [P_N^1 \mathcal{R}_k] - P_N^1 \mathcal{R}_k.$$

This will be done by using the Fourier expansion of our operators. First notice that similarly to (7.54)-(7.56), the condition $\Psi_k = \Pi_k^\perp \Psi_k \Pi_k^\perp$ is equivalent to say that the Fourier coefficients of Ψ_k satisfy

$$\forall (l, l_0) \in (\mathbb{Z}^d)^2, \quad \forall j \text{ or } j_0 \in \bar{\mathbb{S}}_{0,k}, \quad (\Psi_k)_{l_0, j_0}^{l, j} = 0. \quad (7.61)$$

Straightforward computations lead to

$$\forall (l_0, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}), \quad [\omega \cdot \partial_\varphi \Pi_k^\perp, \Psi_k] \mathbf{e}_{l_0, j_0} = i \sum_{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})} (\Psi_k)_{l_0, j_0}^{l, j} \omega \cdot (l - l_0) \mathbf{e}_{l, j}$$

and using (7.53)

$$\forall (l_0, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}), \quad [\mathcal{D}_k, \Psi_k] \mathbf{e}_{l_0, j_0} = i \sum_{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})} (\Psi_k)_{l_0, j_0}^{l, j} (\mu_{j, k}(b, \omega) - \mu_{j_0, k}(b, \omega)) \mathbf{e}_{l, j}.$$

Consequently Ψ_k is a solution of (7.59) if and only if for any $(l_0, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})$,

$$\sum_{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})} (\Psi_k)_{l_0, j_0}^{l, j} \left(\omega \cdot (l - l_0) + \mu_{j, k}(b, \omega) - \mu_{j_0, k}(b, \omega) \right) \mathbf{e}_{l, j} = - \sum_{\substack{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \\ \langle l - l_0, j - j_0 \rangle \leq N \\ (l, j) \neq (l_0, j_0)}} r_{j_0, k}^j(b, \omega, l - l_0) \mathbf{e}_{l, j}.$$

By identification, we deduce that for any $(l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$, if $\langle l - l_0, j - j_0 \rangle > N$, then $(\Psi_k)_{l_0, j_0}^{l, j} = 0$, otherwise we have

$$(\Psi_k)_{l_0, j_0}^{l, j} \left(\omega \cdot (l - l_0) + \mu_{j, k}(b, \omega) - \mu_{j_0, k}(b, \omega) \right) = \begin{cases} -r_{j_0, k}^j(b, \omega, l - l_0), & \text{if } (l, j) \neq (l_0, j_0), \\ 0, & \text{if } (l, j) = (l_0, j_0). \end{cases}$$

As a consequence, we have that Ψ_k is a Toeplitz in time operator with $(\Psi_k)_{j_0}^j(l - l_0) \triangleq (\Psi_k)_{l_0, j_0}^{l, j}$. In addition, for $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$ with $\langle l, j - j_0 \rangle \leq N$, one gets

$$(\Psi_k)_{j_0}^j(b, \omega, l) = \begin{cases} \frac{-r_{j_0, k}^j(b, \omega, l)}{\omega \cdot l + \mu_{j, k}(b, \omega) - \mu_{j_0, k}(b, \omega)}, & \text{if } (l, j) \neq (0, j_0), \\ 0, & \text{if } (l, j) = (0, j_0), \end{cases} \quad (7.62)$$

provided that the denominator is non zero. This latter fact is imposed by selecting suitable values of the parameters (b, ω) among the following set

$$\mathcal{O}_k = \bigcap_{\substack{(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2 \\ |l| \leq N \\ (l, j) \neq (0, j_0)}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad |\omega \cdot l + \mu_{j, k}(b, \omega) - \mu_{j_0, k}(b, \omega)| > \frac{\gamma \langle j - j_0 \rangle}{\langle l \rangle^{\gamma/2}} \right\}.$$

This restriction avoids the resonances and implies that the identity (7.62) is well defined. Now, we shall extend Ψ_k to the whole set \mathcal{O} by using the cut-off function $\chi \in C^\infty(\mathbb{R}, [0, 1])$ defined by

$$\chi(x) = \begin{cases} 1 & \text{if } |x| \geq \frac{1}{2} \\ 0 & \text{if } |x| \leq \frac{1}{3}. \end{cases} \quad (7.63)$$

Then, the extension of Ψ_k , still denoted Ψ_k , is obtained by defining the Fourier coefficients by (7.61) and for $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$ with $\langle l, j - j_0 \rangle \leq N$,

$$(\Psi_k)_{j_0}^j(b, \omega, l) = \begin{cases} -\varrho_{j_0, k}^j(b, \omega, l) r_{j_0, k}^j(b, \omega, l), & \text{if } (l, j) \neq (0, j_0), \\ 0, & \text{if } (l, j) = (0, j_0), \end{cases} \quad (7.64)$$

with

$$\varrho_{j_0,k}^j(b, \omega, l) \triangleq \frac{\chi((\omega \cdot l + \mu_{j,k}(b, \omega) - \mu_{j_0,k}(b, \omega))(\gamma \langle j - j_0 \rangle)^{-1} \langle l \rangle^{\tau_2})}{\omega \cdot l + \mu_{j,k}(b, \omega) - \mu_{j_0,k}(b, \omega)}. \quad (7.65)$$

The extension (7.64) is smooth and coincides with (7.62) for the parameters taken in \mathcal{O}_k . In addition, putting together (7.64), (7.65), (7.55) and (7.53) gives

$$(\Psi_k)_{j_0}^j(l) \in \mathbb{R} \quad \text{and} \quad (\Psi_k)_{-j_0}^{-j}(-l) = (\Psi_k)_{j_0}^j(l).$$

Therefore Definition 2.1 implies that Ψ_k is a real and reversibility preserving Toeplitz in time operator. We now turn to the anti-diagonal equations satisfied by Ψ_3 and Ψ_4 in (7.60) which can be unified in the following form. Fix $\ell \in \{3, 4\}$, then both equations of interest write

$$(\omega \cdot \partial_\varphi \Pi_{\ell-2}^\perp + \mathcal{D}_{\ell-2})\Psi_\ell - \Psi_\ell(\omega \cdot \partial_\varphi \Pi_{5-\ell}^\perp + \mathcal{D}_{5-\ell}) = -P_N^2 \mathcal{R}_\ell. \quad (7.66)$$

First notice that similarly to (7.54)-(7.57), the condition $\Psi_\ell = \Pi_{\ell-2}^\perp \Psi_\ell \Pi_{5-\ell}^\perp$ implies that the Fourier coefficients of Ψ_ℓ satisfy

$$\forall (l, l_0) \in (\mathbb{Z}^d)^2, \quad \forall j \in \bar{\mathbb{S}}_{0, \ell-2} \text{ or } j_0 \in \bar{\mathbb{S}}_{0, 5-\ell}, \quad (\Psi_\ell)_{l_0, j_0}^{l, j} = 0. \quad (7.67)$$

One readily has that Ψ_ℓ is a solution of (7.66) if and only if for any $(l_0, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 5-\ell})$,

$$\begin{aligned} \sum_{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, \ell-2})} (\Psi_\ell)_{l_0, j_0}^{l, j} \left(\omega \cdot (l - l_0) + \mu_{j, \ell-2}(b, \omega) - \mu_{j_0, 5-\ell}(b, \omega) \right) \mathbf{e}_{l, j} \\ = - \sum_{\substack{(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, \ell-2}) \\ \langle l - l_0, j, j_0 \rangle \leq N}} r_{j_0, \ell}^j(b, \omega, l - l_0) \mathbf{e}_{l, j}. \end{aligned}$$

By identification, we deduce that for any $(l, l_0, j, j_0) \in (\mathbb{Z}^d)^2 \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, \ell-2}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 5-\ell})$, if $\langle l - l_0, j, j_0 \rangle > N$, then $(\Psi_\ell)_{l_0, j_0}^{l, j} = 0$, otherwise we have

$$(\Psi_\ell)_{l_0, j_0}^{l, j} \left(\omega \cdot (l - l_0) + \mu_{j, \ell-2}(b, \omega) - \mu_{j_0, 5-\ell}(b, \omega) \right) = -r_{j_0, \ell}^j(b, \omega, l - l_0).$$

As a consequence, we have that Ψ_ℓ is a Toeplitz in time operator with $(\Psi_\ell)_{j_0}^j(l - l_0) \triangleq (\Psi_\ell)_{l_0, j_0}^{l, j}$. In addition, for $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, \ell-2}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 5-\ell})$ with $\langle l, j, j_0 \rangle \leq N$, one gets

$$(\Psi_\ell)_{j_0}^j(b, \omega, l) = \frac{-r_{j_0, \ell}^j(b, \omega, l)}{\omega \cdot l + \mu_{j, \ell-2}(b, \omega) - \mu_{j_0, 5-\ell}(b, \omega)} \quad (7.68)$$

provided that the denominator is non zero. This latter fact is imposed by selecting suitable values of the parameters (b, ω) among the following set

$$\mathcal{O}_{1,2} = \bigcap_{\substack{(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 2}) \\ \langle l, j, j_0 \rangle \leq N_n}} \left\{ (b, \omega) \in \mathcal{O}_{\infty, n}^{\gamma, \tau_1}(i_0) \quad \text{s.t.} \quad |\omega \cdot l + \mu_{j, 1}(b, \omega) - \mu_{j_0, 2}(b, \omega)| > \frac{\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} \right\}.$$

This implies that the identity (7.68) is well defined. Now, the extension of Ψ_ℓ , still denoted Ψ_ℓ , is obtained by defining the Fourier coefficients by (7.67) and for $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, \ell-2}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0, 5-\ell})$ with $\langle l, j, j_0 \rangle \leq N$,

$$(\Psi_\ell)_{j_0}^j(b, \omega, l) = -\varrho_{j_0, \ell}^j(b, \omega, l) r_{j_0, \ell}^j(b, \omega, l) \quad (7.69)$$

with

$$\varrho_{j_0, \ell}^j(b, \omega, l) \triangleq \frac{\chi((\omega \cdot l + \mu_{j, \ell-2}(b, \omega) - \mu_{j_0, 5-\ell}(b, \omega))\gamma^{-1} \langle l, j, j_0 \rangle^{\tau_2})}{\omega \cdot l + \mu_{j, \ell-2}(b, \omega) - \mu_{j_0, 5-\ell}(b, \omega)}, \quad (7.70)$$

where χ is the cut-off function introduced in (7.63). The extension (7.69) is smooth and coincides with (7.68) for the parameters taken in $\mathcal{O}_{1,2}$. In addition, putting together (7.69), (7.70), (7.55) and (7.53) gives

$$(\Psi_\ell)_{j_0}^j(l) \in \mathbb{R} \quad \text{and} \quad (\Psi_\ell)_{-j_0}^{-j}(-l) = (\Psi_\ell)_{j_0}^j(l).$$

Therefore Definition 2.1 implies that Ψ_ℓ is a real and reversibility preserving Toeplitz in time operator. Now consider,

$$\mathcal{D}_{\text{next}} \triangleq \mathcal{D} + [\mathbf{P}_\mathbf{N} \mathcal{R}], \quad \mathcal{R}_{\text{next}} \triangleq \Phi^{-1}(-\Psi[\mathbf{P}_\mathbf{N} \mathcal{R}] + \mathbf{P}_\mathbf{N}^\perp \mathcal{R} + \mathcal{R} \Psi) \quad (7.71)$$

and

$$\mathcal{L}_{\text{next}} \triangleq \omega \cdot \partial_\varphi \mathbf{I}_{\mathbf{m}, \perp} + \mathcal{D}_{\text{next}} + \mathcal{R}_{\text{next}}.$$

Recall that \mathcal{D} , \mathcal{R} and Ψ satisfy the localizations properties (7.53), (7.54) and (7.58), respectively. One can easily check that this property is stable under composition/addition and therefore obtains

$$\Pi_{\mathbb{S}_0}^\perp \mathcal{D}_{\text{next}} \Pi_{\mathbb{S}_0}^\perp = \mathcal{D}_{\text{next}} \quad \text{and} \quad \Pi_{\mathbb{S}_0}^\perp \mathcal{R}_{\text{next}} \Pi_{\mathbb{S}_0}^\perp = \mathcal{R}_{\text{next}}.$$

Therefore, in restriction to the Cantor set

$$\mathcal{O}_{\text{next}}^\gamma \triangleq \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_{1,2},$$

the above construction implies that

$$\mathcal{L}_{\text{next}} = \Phi^{-1} \mathcal{L} \Phi.$$

To end this KAM step, we shall now give some quantitative estimates in order to prove the convergence of the scheme. For this aim, we assume that the following estimates hold true.

$$\forall k \in \{1, 2\}, \quad \forall (j, j_0) \in (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,k})^2, \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,k}(b, \omega) - \mu_{j_0,k}(b, \omega) \right) \right| \leq C |j - j_0| \quad (7.72)$$

and

$$\forall (j, j_0) \in (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,2}), \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,1}(b, \omega) - \mu_{j_0,2}(b, \omega) \right) \right| \leq C \langle j, j_0 \rangle. \quad (7.73)$$

We denote

$$A_{l,j,j_0}^{k,k'}(b, \omega) \triangleq \omega \cdot l + \mu_{j,k}(b, \omega) - \mu_{j_0,k'}(b, \omega), \quad a_{l,j,j_0} \triangleq (\gamma \langle j - j_0 \rangle)^{-1} \langle l \rangle^{\tau_2}, \quad \tilde{a}_{l,j,j_0} \triangleq \gamma^{-1} \langle l, j, j_0 \rangle^{\tau_2}.$$

Then, we can write

$$\begin{aligned} \forall k \in \{1, 2\}, \quad \varrho_{j_0,k}^j(b, \omega, l) &= a_{l,j,j_0} \widehat{\chi} \left(a_{l,j,j_0} A_{l,j,j_0}^{k,k}(b, \omega) \right), \\ \forall \ell \in \{3, 4\}, \quad \varrho_{j_0,\ell}^j(b, \omega, l) &= \tilde{a}_{l,j,j_0} \widehat{\chi} \left(\tilde{a}_{l,j,j_0} A_{l,j,j_0}^{\ell-2,5-\ell}(b, \omega) \right), \end{aligned}$$

where $\widehat{\chi}(x) \triangleq \frac{\chi(x)}{x}$ is C^∞ with bounded derivatives. The assumptions (7.72)-(7.73) imply

$$\forall (l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,k})^2, \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha A_{l,j,j_0}^{k,k}(b, \omega) \right| \leq C \langle l, j - j_0 \rangle$$

and

$$\forall (l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,\ell-2}) \times (\mathbb{Z}_\mathbf{m} \setminus \overline{\mathbb{S}}_{0,5-\ell}), \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha A_{l,j,j_0}^{\ell-2,5-\ell}(b, \omega) \right| \leq C \langle l, j, j_0 \rangle.$$

Applying Lemma 2.1-(iv), we obtain for any $\alpha \in \mathbb{N}^{d+1}$ with $|\alpha| \in \llbracket 0, q \rrbracket$,

$$\begin{aligned} \forall k \in \{1, 2\}, \quad \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \varrho_{j_0,k}^j(b, \omega, l) \right| &\leq C \gamma^{-(|\alpha|+1)} \langle l, j - j_0 \rangle^{\tau_2 |\alpha| + \tau_2 + |\alpha|}, \\ \forall \ell \in \{3, 4\}, \quad \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \varrho_{j_0,\ell}^j(b, \omega, l) \right| &\leq C \gamma^{-(|\alpha|+1)} \langle l, j, j_0 \rangle^{\tau_2 |\alpha| + \tau_2 + |\alpha|}. \end{aligned}$$

In similar way to [69, Prop. 6.5], making use Leibniz rule implies

$$\forall k \in \{1, 2\}, \quad \|\Psi_k\|_{\mathcal{O}\text{-d}, s}^{q, \gamma, \mathfrak{m}} \leq C\gamma^{-1} \|P_N^1 \mathcal{R}_k\|_{\mathcal{O}\text{-d}, s+\tau_2 q+\tau_2}^{q, \gamma, \mathfrak{m}}, \quad (7.74)$$

$$\forall \ell \in \{3, 4\}, \quad \|\Psi_\ell\|_{\mathcal{I}\text{-D}, s}^{q, \gamma, \mathfrak{m}} \leq C\gamma^{-1} \|P_N^2 \mathcal{R}_\ell\|_{\mathcal{I}\text{-D}, s+\tau_2 q+\tau_2}^{q, \gamma, \mathfrak{m}}. \quad (7.75)$$

Combining (7.74), (7.75), (2.34) and Corollary 2.1, we get

$$\begin{aligned} \|\Psi\|_s^{q, \gamma, \mathfrak{m}} &\leq C\gamma^{-1} \|\mathbf{P}_N \mathcal{R}\|_{s+\tau_2 q+\tau_2}^{q, \gamma, \mathfrak{m}} \\ &\leq C\gamma^{-1} N^{\tau_2 q+\tau_2} \|\mathcal{R}\|_s^{q, \gamma, \mathfrak{m}}. \end{aligned} \quad (7.76)$$

Now assume the following smallness condition

$$\gamma^{-1} N^{\tau_2 q+\tau_2} \|\mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} \leq C\varepsilon_0. \quad (7.77)$$

Putting together (7.76) and (7.77), we obtain

$$\|\Psi\|_{s_0}^{q, \gamma, \mathfrak{m}} \leq C\varepsilon_0. \quad (7.78)$$

We deduce that, for ε_0 small enough, the operator Φ is invertible and its inverse is given by

$$\Phi^{-1} = \sum_{n=0}^{\infty} (-1)^n \Psi^n \triangleq \text{Id} + \Sigma.$$

According to Corollary 2.1-(ii), (7.76) and (7.78), one obtains

$$\|\Sigma\|_s^{q, \gamma, \mathfrak{m}} \leq \|\Psi\|_s^{q, \gamma, \mathfrak{m}} \left(1 + \sum_{n=1}^{\infty} (C \|\Psi\|_{s_0}^{q, \gamma, \mathfrak{m}})^n \right) \quad (7.79)$$

$$\leq C\gamma^{-1} N^{\tau_2 q+\tau_2} \|\mathcal{R}\|_s^{q, \gamma, \mathfrak{m}}. \quad (7.80)$$

In particular, (7.77) implies

$$\|\Sigma\|_{s_0}^{q, \gamma, \mathfrak{m}} \leq C\gamma^{-1} N^{\tau_2 q+\tau_2} \|\mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} \leq C\varepsilon_0. \quad (7.81)$$

The second identity in (7.71) also writes

$$\mathcal{R}_{\text{next}} = \mathbf{P}_N^\perp \mathcal{R} + \Phi^{-1} \mathcal{R} \Psi - \Psi [\mathbf{P}_N \mathcal{R}] + \Sigma (\mathbf{P}_N^\perp \mathcal{R} - \Psi [\mathbf{P}_N \mathcal{R}]).$$

Hence, one gets from Corollary 2.1-(ii), (7.79), (7.78) and (7.81),

$$\begin{aligned} \|\mathcal{R}_{\text{next}}\|_s^{q, \gamma, \mathfrak{m}} &\leq \|\mathbf{P}_N^\perp \mathcal{R}\|_s^{q, \gamma, \mathfrak{m}} + C \|\Sigma\|_s^{q, \gamma, \mathfrak{m}} \left(\|\mathbf{P}_N^\perp \mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} + \|\Psi\|_{s_0}^{q, \gamma, \mathfrak{m}} \|\mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} \right) \\ &\quad + C (1 + \|\Sigma\|_{s_0}^{q, \gamma, \mathfrak{m}}) \left(\|\Psi\|_s^{q, \gamma, \mathfrak{m}} \|\mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} + \|\Psi\|_{s_0}^{q, \gamma, \mathfrak{m}} \|\mathcal{R}\|_s^{q, \gamma, \mathfrak{m}} \right). \end{aligned} \quad (7.82)$$

Using Corollary 2.1-(i), (7.76), (7.77), (7.78), (7.80), (7.81) and (7.82), we have for all $S \geq \bar{s} \geq s \geq s_0$,

$$\|\mathcal{R}_{\text{next}}\|_s^{q, \gamma, \mathfrak{m}} \leq N^{s-\bar{s}} \|\mathcal{R}\|_{\bar{s}}^{q, \gamma, \mathfrak{m}} + C\gamma^{-1} N^{\tau_2 q+\tau_2} \|\mathcal{R}\|_{s_0}^{q, \gamma, \mathfrak{m}} \|\mathcal{R}\|_s^{q, \gamma, \mathfrak{m}}. \quad (7.83)$$

One also has

$$\begin{aligned} \partial_\theta \mathcal{R}_{\text{next}} &= \Phi^{-1} \left(\mathbf{P}_N^\perp \partial_\theta \mathcal{R} + \partial_\theta \mathcal{R} \Psi - \Psi \partial_\theta [\mathbf{P}_N \mathcal{R}] - [\partial_\theta, \Psi] [\mathbf{P}_N \mathcal{R}] \right) \\ &\quad + [\partial_\theta, \Sigma] \left(\mathbf{P}_N^\perp \mathcal{R} + \mathcal{R} \Psi - \Psi [\mathbf{P}_N \mathcal{R}] \right). \end{aligned}$$

Using the fact that for any scalar operator T ,

$$\|[\partial_\theta, T]\|_{\mathcal{O}\text{-d}, s}^{q, \gamma, \mathfrak{m}} \lesssim \|T\|_{\mathcal{O}\text{-d}, s+1}^{q, \gamma, \mathfrak{m}}, \quad \|[\partial_\theta, T]\|_{\mathcal{I}\text{-D}, s}^{q, \gamma, \mathfrak{m}} \lesssim \|T\|_{\mathcal{I}\text{-D}, s+1}^{q, \gamma, \mathfrak{m}},$$

one has for any matricial operator \mathbf{T} ,

$$\|[\partial_\theta, \mathbf{T}]\|_s^{q, \gamma, \mathfrak{m}} \lesssim \|\mathbf{T}\|_{s+1}^{q, \gamma, \mathfrak{m}}.$$

Thus, in a similar way to (7.83), one obtains for any $s_0 \leq s \leq \bar{s} \leq S$,

$$\widehat{\delta}_{\text{next}}(s) \leq N^{s-\bar{s}} \widehat{\delta}(\bar{s}) + C\gamma^{-1} N^{\tau_2 q + \tau_2 + 1} \widehat{\delta}(s_0) \widehat{\delta}(s), \quad (7.84)$$

where

$$\widehat{\delta}(s) \triangleq \max \left(\gamma^{-1} \|\partial_\theta \mathcal{R}\|_{s}^{q, \gamma, \mathfrak{m}}, \delta(s) \right).$$

► **Initialization** Now, we shall check the validity of the assumptions (7.72), (7.73) and (7.77) for the initial operator $\mathcal{L} = \mathcal{L}_0$ in (7.34). It is clear from (4.29)-(4.30) that

$$\forall k \in \{1, 2\}, \quad \forall (j, j_0) \in \mathbb{Z}^2, \quad \max_{q' \in \llbracket 0, q \rrbracket} \sup_{b \in (b_*, b^*)} \left| \partial_b^{q'} (\Omega_{j,k}(b) - \Omega_{j_0,k}(b)) \right| \leq C |j - j_0|$$

and

$$\forall (j, j_0) \in \mathbb{Z}^2, \quad \max_{q' \in \llbracket 0, q \rrbracket} \sup_{b \in (b_*, b^*)} \left| \partial_b^{q'} (\Omega_{j,1}(b) - \Omega_{j_0,2}(b)) \right| \leq C \langle j, j_0 \rangle.$$

Consequently, we infer from (7.35)-(7.36),

$$\forall k \in \{1, 2\}, \quad \forall (j, j_0) \in \mathbb{Z}^2, \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,k}^{(0)}(b, \omega) - \mu_{j_0,k}^{(0)}(b, \omega) \right) \right| \leq C |j - j_0|$$

and

$$\forall (j, j_0) \in \mathbb{Z}^2, \quad \max_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \in \llbracket 0, q \rrbracket}} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,1}^{(0)}(b, \omega) - \mu_{j_0,2}^{(0)}(b, \omega) \right) \right| \leq C \langle j, j_0 \rangle.$$

This proves the initial assumptions (7.72)-(7.73). Now let us focus on the assumption (7.77). This latter is obtained by gathering (7.38), (7.43) and (7.42). Indeed,

$$\begin{aligned} \gamma^{-1} N_0^{\tau_2 q + \tau_2} \|\mathcal{R}_0\|_{s_0}^{q, \gamma, \mathfrak{m}} &\leq C \varepsilon \gamma^{-2} N_0^{\mu_2} (1 + \|\mathfrak{J}_0\|_{s_h + \sigma_4}^{q, \gamma, \mathfrak{m}}) \\ &\leq C \varepsilon_0. \end{aligned}$$

► **KAM iteration.** Now, we shall implement the complete KAM reduction scheme. Given $m \in \mathbb{N}$ we assume that we have constructed a linear operator

$$\mathcal{L}_m \triangleq \omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m}, \perp} + \mathcal{D}_m + \mathcal{R}_m, \quad (7.85)$$

with

$$\mathcal{D}_m = \begin{pmatrix} \mathcal{D}_{m,1} & 0 \\ 0 & \mathcal{D}_{m,2} \end{pmatrix}, \quad \mathcal{D}_{m,k} = \left(i \mu_{j,k}^{(m)} \right)_{j \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0,k}}, \quad \mu_{-j,k}^{(m)}(b, \omega) = -\mu_{j,k}^{(m)}(b, \omega)$$

and \mathcal{R}_m a real and reversible Toeplitz in time matrix operator of zero order satisfying $\Pi_{\bar{\mathbb{S}}_0}^\perp \mathcal{R}_m \Pi_{\bar{\mathbb{S}}_0}^\perp = \mathcal{R}_m$. In addition, we assume that the assumptions (7.72), (7.73) and (7.77) hold for \mathcal{D}_m and \mathcal{R}_m . Notice that for $m = 0$ we take the operator \mathcal{L}_0 defined in (7.34). Applying the KAM step we can construct a linear invertible operator $\Phi_m = \mathbf{I}_{\mathfrak{m}, \perp} + \Psi_m$ with Ψ_m living in \mathcal{O} such that in restriction to the Cantor set

$$\begin{aligned} \mathcal{O}_{m+1}^\gamma &= \bigcap_{\substack{k \in \{1, 2\} \\ j, j_0 \in \mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0,k}}} \bigcap_{\substack{l \in \mathbb{Z}^d \\ (l, j) \neq (0, j_0) \\ |l| \leq N_m}} \left\{ (b, \omega) \in \mathcal{O}_m^\gamma \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,k}^{(m)}(b, \omega, i_0) - \mu_{j_0,k}^{(m)}(b, \omega, i_0) \right| > \frac{\gamma \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\} \\ &\quad \bigcap_{\substack{(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_{\mathfrak{m}} \setminus \bar{\mathbb{S}}_{0,2}) \\ (l, j, j_0) \leq N_m}} \left\{ (b, \omega) \in \mathcal{O}_m^\gamma \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,1}^{(m)}(b, \omega) - \mu_{j_0,2}^{(m)}(b, \omega) \right| > \frac{\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} \right\}, \quad (7.86) \end{aligned}$$

the operator Ψ_m satisfies the following homological equation

$$[\omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m}, \perp} + \mathcal{D}_m, \Psi_m] + \mathbf{P}_{N_m} \mathcal{R}_m = [\mathbf{P}_{N_m} \mathcal{R}_m]$$

and consequently, the following identity holds in \mathcal{O}_{m+1}^γ

$$\Phi_m^{-1} \mathcal{L}_m \Phi_m = \omega \cdot \partial_\varphi \mathbf{I}_{m,\perp} + \mathcal{D}_{m+1} + \mathcal{R}_{m+1}, \quad (7.87)$$

with

$$\mathcal{D}_{m+1} \triangleq \mathcal{D}_m + [\mathbf{P}_{N_m} \mathcal{R}_m] \quad \text{and} \quad \mathcal{R}_{m+1} \triangleq \Phi_m^{-1} \left(-\Psi_m [\mathbf{P}_{N_m} \mathcal{R}_m] + \mathbf{P}_{N_m}^\perp \mathcal{R}_m + \mathcal{R}_m \Psi_m \right). \quad (7.88)$$

Recall that the operator $[\mathbf{P}_{N_m} \mathcal{R}_m]$ is defined by

$$[\mathbf{P}_{N_m} \mathcal{R}_m] = \begin{pmatrix} [P_{N_m}^1 \mathcal{R}_{m,1}] & 0 \\ 0 & [P_{N_m}^1 \mathcal{R}_{m,2}] \end{pmatrix},$$

with

$$[P_{N_m}^1 \mathcal{R}_{m,k}] = \left(i r_{j,k}^{(m)} \right)_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}}, \quad r_{-j,k}^{(m)}(b, \omega) = -r_{j,k}^{(m)}(b, \omega).$$

Observe that the symmetry condition for $r_{j,k}^{(m)}$ is a consequence of the reversibility of \mathcal{R}_m . By construction, we find

$$\mu_{j,k}^{(m+1)} = \mu_{j,k}^{(m)} + r_{j,k}^{(m)}. \quad (7.89)$$

We point out that working with this extension for Ψ_m allows to extend both \mathcal{D}_{m+1} and the remainder \mathcal{R}_{m+1} provided that the operators \mathcal{D}_m and \mathcal{R}_m are defined in the whole range of parameters. Thus the operator defined by the right-hand side in (7.87) can be extended to the whole set \mathcal{O} and we denote this extension by \mathcal{L}_{m+1} . that is,

$$\omega \cdot \partial_\varphi \mathbf{I}_{m,\perp} + \mathcal{D}_{m+1} + \mathcal{R}_{m+1} \triangleq \mathcal{L}_{m+1}.$$

This enables to construct by induction the sequence of operators (\mathcal{L}_{m+1}) in the full set \mathcal{O} . Similarly the operator $\Phi_m^{-1} \mathcal{L}_m \Phi_m$ admits an extension in \mathcal{O} induced by the extension of $\Phi_m^{\pm 1}$. However, by construction the identity $\mathcal{L}_{m+1} = \Phi_m^{-1} \mathcal{L}_m \Phi_m$ in (7.87) occurs in the Cantor set \mathcal{O}_{m+1}^γ and may fail outside this set. Define

$$\delta_m(s) \triangleq \gamma^{-1} \|\| \mathcal{R}_m \|\|_s^{q,\gamma,m} \quad \text{and} \quad \widehat{\delta}_m(s) \triangleq \max \left(\delta_m(s), \gamma^{-1} \|\| \partial_\theta \mathcal{R}_m \|\|_s^{q,\gamma,m} \right). \quad (7.90)$$

Assume that the following estimates hold

$$\forall (j, j_0) \in (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2, \quad \max_{|\alpha| \in [0, q]} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,k}^{(m)}(b, \omega) - \mu_{j_0,k}^{(m)}(b, \omega) \right) \right| \leq C |j - j_0| \quad (7.91)$$

and

$$\forall (j, j_0) \in (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}), \quad \max_{|\alpha| \in [0, q]} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,1}^{(m)}(b, \omega) - \mu_{j_0,2}^{(m)}(b, \omega) \right) \right| \leq C \langle j, j_0 \rangle. \quad (7.92)$$

Applying the KAM step, we deduce from (7.83) and (7.84) the following induction formulae true for any $s_0 \leq s \leq \bar{s} \leq S$,

$$\begin{aligned} \delta_{m+1}(s) &\leq N_m^{s-\bar{s}} \delta_m(\bar{s}) + C N_m^{\tau_2 q + \tau_2} \delta_m(s_0) \delta_m(s), \\ \widehat{\delta}_{m+1}(s) &\leq N_m^{s-\bar{s}} \widehat{\delta}_m(\bar{s}) + C N_m^{\tau_2 q + \tau_2 + 1} \widehat{\delta}_m(s_0) \widehat{\delta}_m(s). \end{aligned}$$

Hence, in a similar way to [69, Prop. 6.5], our choice of parameters (7.1) allow to prove by induction on $m \in \mathbb{N}$ that

$$\forall m \in \mathbb{N}, \quad \delta_m(\bar{s}_l) \leq \delta_0(s_h) N_0^{\mu_2} N_m^{-\mu_2} \quad \text{and} \quad \delta_m(s_h) \leq \left(2 - \frac{1}{m+1} \right) \delta_0(s_h), \quad (7.93)$$

and

$$\forall m \in \mathbb{N}, \quad \widehat{\delta}_m(s_0) \leq \widehat{\delta}_0(s_h) N_0^{\mu_2} N_m^{-\mu_2} \quad \text{and} \quad \widehat{\delta}_m(s_h) \leq \left(2 - \frac{1}{m+1} \right) \widehat{\delta}_0(s_h). \quad (7.94)$$

Observe that the first condition in (7.93) together with (7.52) and (7.42) implies that the smallness condition (7.77) is satisfied for any m (replacing \mathcal{R} by \mathcal{R}_m and N by N_m). Using the Toeplitz structure of $\mathcal{R}_{m,k}$ and an integration by parts, we get from (7.89)

$$\begin{aligned} \left\| \mu_{j,k}^{(m+1)} - \mu_{j,k}^{(m)} \right\|^{q,\gamma} &= \left\| \langle P_{N_m}^1 \mathcal{R}_{m,k} \mathbf{e}_{l,j}, \mathbf{e}_{l,j} \rangle_{L^2(\mathbb{T}^{d+1})} \right\|^{q,\gamma} \\ &= \left\| \langle P_{N_m}^1 \mathcal{R}_{m,k} \mathbf{e}_{0,j}, \mathbf{e}_{0,j} \rangle_{L^2(\mathbb{T}^{d+1})} \right\|^{q,\gamma} \\ &= \frac{1}{|j|} \left\| \langle P_{N_m}^1 \mathcal{R}_{m,k} \mathbf{e}_{0,j}, \partial_\theta \mathbf{e}_{0,j} \rangle_{L^2(\mathbb{T}^{d+1})} \right\|^{q,\gamma} \\ &= \frac{1}{|j|} \left\| \langle P_{N_m}^1 \partial_\theta \mathcal{R}_{m,k} \mathbf{e}_{0,j}, \mathbf{e}_{0,j} \rangle_{L^2(\mathbb{T}^{d+1})} \right\|^{q,\gamma}. \end{aligned}$$

Therefore, a duality argument together with (7.90), (7.94), (7.51), (7.43) and Corollary 2.1-(iii) imply

$$\begin{aligned} |j| \left\| \mu_{j,k}^{(m+1)} - \mu_{j,k}^{(m)} \right\|^{q,\gamma} &\leq \left\| \partial_\theta \mathcal{R}_m \mathbf{e}_{0,j} \right\|_{s_0}^{q,\gamma,m} \langle j \rangle^{-s_0} \\ &\leq C \left\| \partial_\theta \mathcal{R}_m \right\|_{s_0}^{q,\gamma,m} \left\| \mathbf{e}_{0,j} \right\|_{H^{s_0}} \langle j \rangle^{-s_0} \\ &\leq C \gamma \widehat{\delta}_0(s_h) N_0^{\mu_2} N_m^{-\mu_2} \\ &\leq C \varepsilon \gamma^{-1} N_0^{\mu_2} N_m^{-\mu_2}. \end{aligned} \tag{7.95}$$

Now we shall check that the assumptions (7.91) and (7.92) are satisfied for the next step. Combining (7.95) with (7.91) we infer that for $k \in \{1, 2\}$ and $(j, j_0) \in (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k})^2$,

$$\max_{|\alpha| \in \llbracket 0, q \rrbracket} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,k}^{(m+1)}(b, \omega) - \mu_{j_0,k}^{(m+1)}(b, \omega) \right) \right| \leq C (1 + \varepsilon \gamma^{-1-q} N_0^{\mu_2} N_m^{-\mu_2}) |j - j_0|.$$

Now putting together (7.95) and (7.92), we get for $(j, j_0) \in (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,2})$,

$$\max_{|\alpha| \in \llbracket 0, q \rrbracket} \sup_{(b, \omega) \in \mathcal{O}} \left| \partial_{b, \omega}^\alpha \left(\mu_{j,1}^{(m+1)}(b, \omega) - \mu_{j_0,2}^{(m+1)}(b, \omega) \right) \right| \leq C (1 + \varepsilon \gamma^{-1-q} N_0^{\mu_2} N_m^{-\mu_2}) \langle j, j_0 \rangle.$$

The convergence of the series $\sum N_m^{-\mu_2}$ implies the desired result with a constant C uniform in m . This achieves the induction argument. Observe that the bound (7.95) implies the convergence of the sequence $\left(\mu_{j,k}^{(m)} \right)_{m \in \mathbb{N}}$ toward some $\mu_{j,k}^{(\infty)} \in W^{q, \infty, \gamma}(\mathcal{O}, \mathbb{C})$ given by

$$\mu_{j,k}^{(\infty)} = \mu_{j,k}^{(0)} + \sum_{m=0}^{\infty} \left(\mu_{j,k}^{(m+1)} - \mu_{j,k}^{(m)} \right) \triangleq \mu_{j,k}^{(0)} + r_{j,k}^{(\infty)}, \tag{7.96}$$

where $(\mu_{j,k}^{(0)})$ was introduced in Proposition 7.4, writes

$$\mu_{j,k}^{(0)}(b, \omega, i_0) = \Omega_{j,k}(b) + j(c_k(b, \omega, i_0) - \mathbf{v}_k(b)).$$

The estimate (7.47) follows immediately from (7.96) and (7.95). Define the diagonal operator $\mathcal{D}_{\infty,k}$ defined on the normal modes by

$$\forall (l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k}), \quad \mathcal{D}_{\infty,k} \mathbf{e}_{l,j} = i \mu_{j,k}^{(\infty)} \mathbf{e}_{l,j}. \tag{7.97}$$

By definition of the off-diagonal norm and (7.95), we obtain

$$\left\| \mathcal{D}_{m,k} - \mathcal{D}_{\infty,k} \right\|_{\mathcal{O}\text{-d}, s_0}^{q,\gamma,m} = \sup_{j \in \mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k}} \left\| \mu_{j,k}^{(m)} - \mu_{j,k}^{(\infty)} \right\|^{q,\gamma} \leq C \gamma \delta_0(s_h) N_0^{\mu_2} N_m^{-\mu_2}. \tag{7.98}$$

Consider the diagonal operator $\mathcal{L}_\infty \triangleq \omega \cdot \partial_\varphi \mathbf{I}_{m,\perp} + \mathcal{D}_\infty$, where \mathcal{D}_∞ is introduced in (7.97). For any $m \in \mathbb{N}$, applying (7.98) and (7.93) yields

$$\begin{aligned} \left\| \mathcal{L}_m - \mathcal{L}_\infty \right\|_{s_0}^{q,\gamma,m} &\leq 2 \max_{k \in \{1,2\}} \left\| \mathcal{D}_{m,k} - \mathcal{D}_{\infty,k} \right\|_{\mathcal{O}\text{-d}, s_0}^{q,\gamma,m} + \left\| \mathcal{R}_m \right\|_{s_0}^{q,\gamma,m} \\ &\leq C \gamma \delta_0(s_h) N_0^{\mu_2} N_m^{-\mu_2}, \end{aligned}$$

where \mathcal{L}_m is given in (7.85). As a consequence,

$$\lim_{m \rightarrow \infty} \|\mathcal{L}_m - \mathcal{L}_\infty\|_{s_0}^{q, \gamma, \mathfrak{m}} = 0.$$

Now we define the sequence $(\widehat{\Phi}_m)_{m \in \mathbb{N}}$ of the successive transformations as follows

$$\widehat{\Phi}_0 \triangleq \Phi_0 \quad \text{and} \quad \forall m \geq 1, \quad \widehat{\Phi}_m \triangleq \Phi_0 \circ \Phi_1 \circ \dots \circ \Phi_m. \quad (7.99)$$

The identity $\Phi_m = \text{Id} + \Psi_m$ gives

$$\widehat{\Phi}_{m+1} = \widehat{\Phi}_m + \widehat{\Phi}_m \Psi_{m+1}.$$

Using (7.76) and (7.93), a completeness argument implies that the series $\sum (\widehat{\Phi}_{m+1} - \widehat{\Phi}_m)$ converges to an element Φ_∞ still close to the identity, so invertible and which satisfies

$$\|\Phi_\infty^{-1} - \widehat{\Phi}_n^{-1}\|_{s_0+1}^{q, \gamma, \mathfrak{m}} + \|\Phi_\infty - \widehat{\Phi}_n\|_{s_0+1}^{q, \gamma, \mathfrak{m}} \lesssim \delta_0(s_h) N_0^{\mu_2} N_{n+1}^{-\mu_2} \quad (7.100)$$

and (7.45). We refer the reader to [69, Prop. 6.5] for the complete computations up to slight modifications corresponding to the hybrid norm. By construction (7.99) and (7.87), we have in \mathcal{O}_{n+1}^γ the following identity

$$\begin{aligned} \widehat{\Phi}_n^{-1} \mathcal{L}_0 \widehat{\Phi}_n &= \omega \cdot \partial_\varphi \mathbf{I}_{\mathfrak{m}, \perp} + \mathcal{D}_{n+1} + \mathcal{R}_{n+1} \\ &= \mathcal{L}_\infty + \mathcal{D}_{n+1} - \mathcal{D}_\infty + \mathcal{R}_{n+1}. \end{aligned}$$

Assume for a while that the set $\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0)$ described in Proposition 7.5 satisfies the following inclusion property with respect to the intermediate Cantor sets given by (7.86),

$$\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) \subset \bigcap_{m=0}^{n+1} \mathcal{O}_m^\gamma = \mathcal{O}_{n+1}^\gamma. \quad (7.101)$$

Hence, in restriction to $\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) \subset \mathcal{O}_{n+1}^\gamma$, we obtain

$$\begin{aligned} \widehat{\Phi}_\infty^{-1} \mathcal{L}_0 \widehat{\Phi}_\infty &= \mathcal{L}_\infty + (\mathcal{D}_{n+1} - \mathcal{D}_\infty + \mathcal{R}_{n+1}) \Pi_{\mathbb{S}_0}^\perp \\ &\quad + \widehat{\Phi}_\infty^{-1} \mathcal{L}_0 (\Phi_\infty - \widehat{\Phi}_n) + (\Phi_\infty^{-1} - \widehat{\Phi}_n^{-1}) \mathcal{L}_0 \widehat{\Phi}_n \\ &\triangleq \mathcal{L}_\infty + \mathcal{E}_n^1. \end{aligned}$$

The estimate (7.48) is obtained by using (2.34), Lemma 2.3-(ii)-(iii), (7.98), (7.40) and (7.100) combined with (7.90), (7.93), (7.51), (7.45) and (7.43). Now it remains to prove (7.101). This is done by a finite induction on m with n fixed. First, by definition we have $\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) \subset \mathcal{O} \triangleq \mathcal{O}_0^\gamma$. Now suppose that $\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) \subset \mathcal{O}_m^\gamma$ for $m \leq n$ and let us prove that

$$\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0) \subset \mathcal{O}_{m+1}^\gamma. \quad (7.102)$$

Let $(b, \omega) \in \mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0)$. For $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,2})$ such that $0 \leq \langle l, j, j_0 \rangle \leq N_m$, the triangle inequality, (7.98), (7.42) and (7.52) imply

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,1}^{(m)}(b, \omega) - \mu_{j_0,2}^{(m)}(b, \omega) \right| &\geq \left| \omega \cdot l + \mu_{j,1}^{(\infty)}(b, \omega) - \mu_{j_0,2}^{(\infty)}(b, \omega) \right| - 2 \max_{k \in \{1,2\}} \sup_{j \in \mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k}} \left\| \mu_{j,k}^{(m)} - \mu_{j,k}^{(\infty)} \right\|^{q, \gamma} \\ &\geq \frac{2\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} - 2C\gamma\delta_0(s_h) N_0^{\mu_2} N_m^{-\mu_2} \\ &\geq \frac{\gamma}{\langle l, j, j_0 \rangle^{\tau_2}} (2 - 2C\gamma\varepsilon_0 \langle l, j, j_0 \rangle^{\tau_2 - \mu_2}). \end{aligned}$$

Thus for ε_0 small enough and by (7.42) (implying that $\mu_2 \geq \tau_2$) we get

$$\left| \omega \cdot l + \mu_{j,1}^{(m)}(b, \omega) - \mu_{j_0,2}^{(m)}(b, \omega) \right| > \frac{\gamma}{\langle l, j, j_0 \rangle^{\tau_2}}.$$

Now for $k \in \{1, 2\}$ and $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$ with $(l, j) \neq (0, j_0)$ and $|l| \leq N_m$, we get

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,k}^{(m)}(b, \omega) - \mu_{j_0,k}^{(m)}(b, \omega) \right| &\geq \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega) - \mu_{j_0,k}^{(\infty)}(b, \omega) \right| - 2 \sup_{j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}} \left\| \mu_{j,k}^{(m)} - \mu_{j,k}^{(\infty)} \right\|^{q,\gamma} \\ &\geq \frac{2\gamma \langle j-j_0 \rangle}{\langle l \rangle^{\tau_2}} - 2C\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2} \\ &\geq \frac{\gamma \langle j-j_0 \rangle}{\langle l \rangle^{\tau_2}} (2 - 2C\gamma\varepsilon_0 \langle l \rangle^{\tau_2 - \mu_2}). \end{aligned}$$

Hence, taking ε_0 small enough, we obtain $k \in \{1, 2\}$,

$$\left| \omega \cdot l + \mu_{j,k}^{(m)}(b, \omega) - \mu_{j_0,k}^{(m)}(b, \omega) \right| > \frac{\gamma \langle j-j_0 \rangle}{\langle l \rangle^{\tau_2}}.$$

Hence, $(b, \omega) \in \mathcal{O}_{m+1}^\gamma$ which proves (7.102).

(ii) One can get the estimates (7.49) and (7.50) by a similar induction procedure as above starting with (7.18) and (7.39) applied with $\mathbf{p} = 4\tau_2q + 4\tau_2$. For more details, we refer the reader to [69, Prop. 6.5]. \square

We end this section with the effective construction of the approximate right inverse of the linearized operator in the normal directions. Since we have constructed a diagonal operator \mathcal{L}_∞ with Fourier multiplier entries, the situation is brought back to two decoupled scalar studies. Therefore, we can copy the proof done in [69, Prop. 6.6] with small adaptations and obtain the following result.

Proposition 7.6. *Let $(\gamma, q, d, \tau_1, s_0, \mu_2, s_h, S, \mathbf{m})$ satisfying (2.2)–(2.3), (7.3) and (7.42)–(7.43). There exists $\sigma_5 \triangleq \sigma_5(\tau_1, \tau_2, q, d) \geq \sigma_4$ such that if*

$$\|\mathcal{J}_0\|_{s_h + \sigma_5}^{q,\gamma,\mathbf{m}} \leq 1, \quad (7.103)$$

then the following assertions hold true.

(i) Consider the operator \mathcal{L}_∞ defined in Proposition 7.5, then there exists a family of linear operators $(\mathbf{T}_n)_{n \in \mathbb{N}}$ defined in \mathcal{O} satisfying the estimate

$$\forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\mathbf{T}_n \rho\|_s^{q,\gamma,\mathbf{m}} \lesssim \gamma^{-1} \|\rho\|_{s+\tau_1q+\tau_1}^{q,\gamma,\mathbf{m}}$$

and such that for any $n \in \mathbb{N}$, in the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) \triangleq \bigcap_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \bar{\mathbb{S}}_0^c \\ |l| \leq N_n}} \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_0) \right| > \frac{\gamma \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\},$$

we have

$$\mathcal{L}_\infty \mathbf{T}_n = \mathbf{I}_{\mathbf{m},\perp} + \mathcal{E}_n^2,$$

with

$$\forall s_0 \leq s \leq \bar{s} \leq S, \quad \|\mathcal{E}_n^2 \rho\|_s^{q,\gamma,\mathbf{m}} \lesssim N_n^{s-\bar{s}} \gamma^{-1} \|\rho\|_{\bar{s}+1+\tau_1q+\tau_1}^{q,\gamma,\mathbf{m}}.$$

(ii) There exists a family of linear operators $(\widehat{\mathbf{T}}_n)_{n \in \mathbb{N}}$ satisfying

$$\forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\widehat{\mathbf{T}}_n \rho\|_s^{q,\gamma,\mathbf{m}} \lesssim \gamma^{-1} (\|\rho\|_{s+\sigma_5}^{q,\gamma,\mathbf{m}} + \|\mathcal{J}_0\|_{s+\sigma_5}^{q,\gamma,\mathbf{m}} \|\rho\|_{s_0+\sigma_5}^{q,\gamma,\mathbf{m}})$$

and such that in the Cantor set

$$\mathbf{G}_n(\gamma, \tau_1, \tau_2, i_0) \triangleq \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \cap \mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) \cap \Lambda_{\infty,n}^{\gamma,\tau_1}(i_0), \quad (7.104)$$

we have

$$\widehat{\mathcal{L}} \widehat{\mathbf{T}}_n = \mathbf{I}_{\mathbf{m},\perp} + \mathbf{E}_n,$$

where \mathbf{E}_n satisfies the following estimate

$$\forall s \in [s_0, S], \quad \|\mathbf{E}_n \rho\|_{s_0}^{q, \gamma, \mathbf{m}} \lesssim N_n^{s_0 - s} \gamma^{-1} \left(\|\rho\|_{s + \sigma_5}^{q, \gamma, \mathbf{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s + \sigma_5}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0 + \sigma_5}^{q, \gamma, \mathbf{m}} \right) \\ + \varepsilon \gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0 + \sigma_5}^{q, \gamma, \mathbf{m}}.$$

Recall that $\widehat{\mathcal{L}}, \mathcal{O}_{\infty, n}^{\gamma, \tau_1}(i_0)$ and $\mathcal{O}_{\infty, n}^{\gamma, \tau_1, \tau_2}(i_0)$ are given in Propositions 7.1, 7.2 and 7.5, respectively.

(iii) In the Cantor set $\mathbf{G}_n(\gamma, \tau_1, \tau_2, i_0)$, we have the following splitting

$$\widehat{\mathcal{L}} = \widehat{\mathbf{L}}_n + \widehat{\mathbf{R}}_n, \quad \text{with} \quad \widehat{\mathbf{L}}_n \widehat{\mathbf{T}}_n = \text{Id} \quad \text{and} \quad \widehat{\mathbf{R}}_n = \mathbf{E}_n \widehat{\mathbf{L}}_n,$$

where the operators $\widehat{\mathbf{L}}_n$ and $\widehat{\mathbf{R}}_n$ are defined in \mathcal{O} and satisfy the following estimates

$$\forall s \in [s_0, S], \quad \sup_{n \in \mathbb{N}} \|\widehat{\mathbf{L}}_n \rho\|_s^{q, \gamma, \mathbf{m}} \lesssim \|\rho\|_{s+1}^{q, \gamma, \mathbf{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s + \sigma_5}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0 + 1}^{q, \gamma, \mathbf{m}}, \\ \forall s \in [s_0, S], \quad \|\widehat{\mathbf{R}}_n \rho\|_{s_0}^{q, \gamma, \mathbf{m}} \lesssim N_n^{s_0 - s} \gamma^{-1} \left(\|\rho\|_{s + \sigma_5}^{q, \gamma, \mathbf{m}} + \varepsilon \gamma^{-2} \|\mathfrak{J}_0\|_{s + \sigma_5}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0 + \sigma_5}^{q, \gamma, \mathbf{m}} \right) \\ + \varepsilon \gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{s_0 + \sigma_5}^{q, \gamma, \mathbf{m}}.$$

8 Construction of quasi-periodic solutions

We provide, in this last section, a construction of a non-trivial solution to the equation (5.11). This is done in two steps. First, we implement a Nash-Moser iteration, where we find a solution provided that the parameters (b, ω) belong to a suitable Borel set. The latter is constructed as the intersection of the Cantor sets required to invert the linearized operator in the normal modes for all the steps of the procedure. Then we rigidified the frequencies in order to get a solution for the original problem where $\alpha = -\mathbf{J}\omega_{\text{Eq}}(b)$. This gives rise to a final set described in terms of b that we should estimate its Lebesgue measure. Actually, we prove that it has asymptotically full measure as the parameter ε vanishes.

8.1 Nash-Moser iteration

Here, we perform the Nash-Moser scheme which allows to find a solution of

$$\mathcal{F}(i, \alpha, b, \omega) \triangleq \mathcal{F}(i, \alpha, b, \omega, \varepsilon) = 0,$$

with \mathcal{F} as in (5.11). This method is classical and has been used in several papers, see for instance [4, 5, 14]. The iterative construction of the approximate solutions is summarized in the following proposition. The proof is a slight modification of the one exposed in [19, 59, 69].

Proposition 8.1. (Nash-Moser scheme)

Let $(\tau_1, \tau_2, q, d, s_0)$ satisfy (2.2)–(2.3) and $\mathbf{m} \geq \mathbf{m}^*$, where \mathbf{m}^* is defined in Corollary 4.1. We consider the following parameters

$$\begin{cases} \bar{a} &= \tau_2 + 3 \\ \mu_1 &= 3q(\tau_2 + 3) + 6\bar{\sigma} + 6 \\ a_1 &= 6q(\tau_2 + 3) + 12\bar{\sigma} + 15 \\ a_2 &= 3q(\tau_2 + 3) + 6\bar{\sigma} + 9 \\ \mu_2 &= 2q(\tau_2 + 3) + 5\bar{\sigma} + 7 \\ s_h &= s_0 + 4q(\tau_2 + 3) + 9\bar{\sigma} + 11 \\ \kappa_1 &= 2s_h - s_0 \end{cases} \quad (8.1)$$

where the number $\bar{\sigma} = \bar{\sigma}(\tau_1, \tau_2, d)$ is the total loss of regularity given by Theorem 6.1. There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ we impose the constraint relating γ and N_0 to ε ,

$$0 < a < \frac{1}{\mu_2 + q + 2}, \quad \gamma \triangleq \varepsilon^a, \quad N_0 \triangleq \gamma^{-1}. \quad (8.2)$$

Let $n \in \mathbb{N}$. We introduce the finite dimensional subspace $E_{n, \mathbf{m}}$ defined by

$$E_{n, \mathbf{m}} \triangleq \left\{ \mathfrak{J} = (\Theta, I, z) \in \mathbb{T}^d \times \mathbb{R}^d \times \mathbf{H}_{\perp, \mathbf{m}}^\infty \quad \text{s.t.} \quad \Theta = \Pi_{N_n} \Theta, \quad I = \Pi_{N_n} I \quad \text{and} \quad z = \Pi_{N_n} z \right\},$$

where Π_{N_n} is the projector defined through (2.6). Then, the following properties hold true.

(P1)_n There exists a q -times differentiable application

$$\begin{aligned} \mathbb{W}_n : \quad \mathcal{O} &\rightarrow E_{n-1, \mathfrak{m}} \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ (b, \omega) &\mapsto (\mathfrak{J}_n, \alpha_n - \mathbb{J}\omega, 0) \end{aligned}$$

satisfying $\mathbb{W}_0 = 0$ and for $n \in \mathbb{N}^*$,

$$\|\mathbb{W}_n\|_{s_0+\bar{\sigma}}^{q, \gamma, \mathfrak{m}} \leq C_* \varepsilon \gamma^{-1} N_0^{q\bar{a}}. \quad (8.3)$$

We set

$$\mathbb{U}_0 \triangleq ((\varphi, 0, 0), \mathbb{J}\omega, (b, \omega)) \quad (8.4)$$

and for $n \in \mathbb{N}^*$,

$$\mathbb{U}_n \triangleq \mathbb{U}_0 + \mathbb{W}_n \quad \text{and} \quad \mathbb{H}_n \triangleq \mathbb{U}_n - \mathbb{U}_{n-1}, \quad (8.5)$$

Then

$$\forall s \in [s_0, S], \quad \|\mathbb{H}_1\|_s^{q, \gamma, \mathfrak{m}} \leq \frac{1}{2} C_* \varepsilon \gamma^{-1} N_0^{q\bar{a}}, \quad (8.6)$$

$$\forall 2 \leq m \leq n, \quad \|\mathbb{H}_m\|_{s_0+\bar{\sigma}}^{q, \gamma, \mathfrak{m}} \leq C_* \varepsilon \gamma^{-1} N_{m-1}^{-a_2}, \quad (8.7)$$

$$\forall n \geq 2, \quad \|\mathbb{H}_n\|_{s_n+\bar{\sigma}}^{q, \gamma, \mathfrak{m}} \leq C_* \varepsilon \gamma^{-1} N_{n-1}^{-a_2}. \quad (8.8)$$

(P2)_n Set

$$i_n \triangleq (\varphi, 0, 0) + \mathfrak{J}_n, \quad \gamma_n \triangleq \gamma(1 + 2^{-n}) \in [\gamma, 2\gamma]. \quad (8.9)$$

The torus i_n is reversible and \mathfrak{m} -fold, that is

$$\mathfrak{S}i_n(\varphi) = i_n(-\varphi) \quad \text{and} \quad \mathfrak{T}_{\mathfrak{m}}i_n(\varphi) = i_n(\varphi), \quad (8.10)$$

with \mathfrak{S} and $\mathfrak{T}_{\mathfrak{m}}$ as in (5.12)-(5.13). Define also

$$\mathbb{A}_0^\gamma \triangleq \mathcal{O} \quad \text{and} \quad \mathbb{A}_{n+1}^\gamma \triangleq \mathbb{A}_n^\gamma \cap \mathbb{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$$

where $\mathbb{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$ is given through (7.104). Consider the open sets

$$\forall \mathfrak{v} > 0, \quad \mathbb{O}_n^\mathfrak{v} \triangleq \left\{ (b, \omega) \in \mathcal{O} \quad \text{s.t.} \quad \text{dist}((b, \omega), \mathbb{A}_n^{2\gamma}) < \mathfrak{v} N_n^{-\bar{a}} \right\}, \quad \text{dist}(x, \mathbb{A}) \triangleq \inf_{y \in \mathbb{A}} \|x - y\|.$$

Then we have the following estimate

$$\|\mathcal{F}(\mathbb{U}_n)\|_{s_0}^{q, \gamma, \mathfrak{m}, \mathbb{O}_n^{2\gamma}} \triangleq \sum_{\substack{\alpha \in \mathbb{N}^{d+1} \\ |\alpha| \leq q}} \gamma^{|\alpha|} \sup_{(b, \omega) \in \mathbb{O}_n^{2\gamma}} \|\partial_{(b, \omega)}^\alpha \mathcal{F}(\mathbb{U}_n)(b, \omega, \cdot)\|_{H_{\mathfrak{m}}^{s_0-|\alpha|}} \leq C_* \varepsilon N_{n-1}^{-a_1}. \quad (8.11)$$

(P3)_n We have the following growth in high regularity norm

$$\|\mathbb{W}_n\|_{\kappa_1+\bar{\sigma}}^{q, \gamma, \mathfrak{m}} \leq C_* \varepsilon \gamma^{-1} N_{n-1}^{\mu_1}. \quad (8.12)$$

Proof. We follow closely [69, Prop. 7.1]. First notice that the initial guess \mathbb{U}_0 is associated to a reversible flat torus and satisfies by virtue of (5.11) and Lemma 5.2 the following estimate for some large enough constant C_*

$$\forall s \geq 0, \quad \|\mathcal{F}(\mathbb{U}_0)\|_s^{q, \gamma, \mathfrak{m}} \leq C_* \varepsilon. \quad (8.13)$$

The properties (P1)₀, (P2)₀ and (P3)₀ follow immediately since $N_{-1} = 1$ and $\mathbb{O}_0^\gamma = \mathcal{O}$ and by setting $\mathbb{W}_0 = 0$. Now, let us turn to the induction step. Fix $n \in \mathbb{N}^*$ and suppose that (P1)_ℓ, (P2)_ℓ and (P3)_ℓ hold for any $\ell \in \llbracket 0, n \rrbracket$. The purpose is to verify that these properties also hold at the order $n+1$. We denote by

$$L_n \triangleq L_n(b, \omega) \triangleq d_{i, \alpha} \mathcal{F}(i_n(b, \omega), \alpha_n(b, \omega), (b, \omega))$$

the linearized operator of \mathcal{F} at the state (i_n, α_n) . As we shall see later, the next approximation \mathbb{U}_{n+1} can be obtained through the construction of a reversible and \mathfrak{m} -fold preserving approximate right

inverse for L_n , which is the subject of Theorem 6.1. To apply this result and get some bounds on \mathbf{U}_{n+1} we need to establish first some intermediate results connected to the smallness condition and to some Cantor set inclusions.

► **Smallness/boundedness properties.** First observe that (8.1) implies (7.42). Thus, to apply Theorem 6.1, we need to check the smallness (7.43) and boundedness (6.30) properties. According to (8.2), a small enough choice of ε leads, for some a priori fixed $\varepsilon_0 > 0$, to

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2} = \varepsilon^{1-a(\mu_2+q+2)} \leq \varepsilon_0, \quad (8.14)$$

which is exactly (7.43). Now, since from (8.1) $\kappa_1 = 2s_h - s_0$, then by interpolation inequality in Lemma 2.1-(v), we have for $n \geq 2$,

$$\|\mathbf{H}_n\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \lesssim (\|\mathbf{H}_n\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}})^{\frac{1}{2}} (\|\mathbf{H}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}})^{\frac{1}{2}}. \quad (8.15)$$

The property (8.12) applied with the indices n and $n-1$ gives

$$\begin{aligned} \|\mathbf{H}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &= \|\mathbf{U}_n - \mathbf{U}_{n-1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &= \|\mathbf{W}_n - \mathbf{W}_{n-1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq \|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} + \|\mathbf{W}_{n-1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq 2C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}. \end{aligned}$$

Inserting the last estimate together with (8.7) into (8.15) leads to

$$\forall n \geq 2, \quad \|\mathbf{H}_n\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \leq CC_*\varepsilon\gamma^{-1}N_{n-1}^{\frac{1}{2}(\mu_1-a_2)}. \quad (8.16)$$

Observe that (8.1) implies in particular $a_2 \geq \mu_1 + 2$. Hence, by (8.2), (8.6) and (8.16), we infer

$$\begin{aligned} \|\mathbf{W}_n\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq \|\mathbf{H}_1\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} + \sum_{k=2}^n \|\mathbf{H}_k\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq \frac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{\bar{q}a} + CN_0^{-1}C_*\varepsilon\gamma^{-1} \\ &\leq C_*\varepsilon^{1-a(1+\bar{q}a)}. \end{aligned}$$

Remark that (8.1) and (8.2) provide $a \leq \frac{1}{2(1+\bar{q}a)}$. Thus, taking ε small enough and $\bar{\sigma} \geq \sigma_5$ with σ_5 as in Proposition (7.6), we get

$$\begin{aligned} \|\mathcal{J}_n\|_{s_h+\sigma_5}^{q,\gamma,\mathbf{m}} &\leq \|\mathbf{W}_n\|_{s_h+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq C_*\varepsilon^{\frac{1}{2}} \\ &\leq 1, \end{aligned} \quad (8.17)$$

which corresponds to (7.103). Up to increase the value of $\bar{\sigma}$, we can always assume that $s_0+\bar{\sigma} \geq \bar{s}_h+\sigma_4$ where \bar{s}_h and σ_4 are respectively given by (7.1) and Proposition 7.5. Consequently (8.7) gives (8.8).

► **Set inclusions.** The properties (8.14) and (8.17) allow to apply Theorem 6.1. Hence, we can reduce the linearized operator L_n at the current step. Therefore, the sets \mathbf{A}_ℓ^γ for $\ell \leq n+1$ and $\gamma \in (0,1)$ are well-defined. Our next purpose is to check some suitable inclusions required later for defining the extensions of our quantities outside the constructed Cantor sets. More precisely, we shall verify the following

$$\mathbf{A}_{n+1}^{2\gamma} \subset \mathbf{O}_{n+1}^{4\gamma} \subset (\mathbf{A}_{n+1}^\gamma \cap \mathbf{O}_n^{2\gamma}).$$

Obviously, by construction, the first inclusion is trivial. Hence, we are left to prove the last one. Observe that by construction $\mathbf{A}_{\ell+1}^{2\gamma} \subset \mathbf{A}_\ell^{2\gamma}$. Then for $(b, \omega) \in \mathbf{O}_{\ell+1}^{4\gamma}$, we have

$$\begin{aligned} \text{dist}((b, \omega), \mathbf{A}_\ell^{2\gamma}) &\leq \text{dist}((b, \omega), \mathbf{A}_{\ell+1}^{2\gamma}) \\ &< 4\gamma N_{\ell+1}^{-\bar{a}} = 4\gamma N_\ell^{-\bar{a}} N_0^{-\frac{1}{2}\bar{a}} \\ &< 2\gamma N_\ell^{-\bar{a}}. \end{aligned}$$

The last estimate is true provided that $2N_0^{-\frac{1}{2}\bar{a}} < 1$, which is obtained taking ε small enough according to (8.2). Thus, we have proved

$$\forall \ell \in \llbracket 0, n \rrbracket, \quad \mathbf{O}_{\ell+1}^{4\gamma} \subset \mathbf{O}_\ell^{2\gamma}. \quad (8.18)$$

Now we prove by induction in ℓ the following inclusion

$$\forall \ell \in \llbracket 0, n+1 \rrbracket, \quad \mathbf{O}_\ell^{4\gamma} \subset \mathbf{A}_\ell^\gamma. \quad (8.19)$$

The case $\ell = 0$ is obvious because $\mathbf{O}_0^{4\gamma} = \mathcal{O} = \mathbf{A}_0^\gamma$. Now suppose that (8.19) is true for some $\ell \in \llbracket 0, n \rrbracket$ and let us check the inclusion property (8.19) at the next order $\ell + 1$. Putting together (8.18) and (8.19), we get

$$\mathbf{O}_{\ell+1}^{4\gamma} \subset \mathbf{O}_\ell^{2\gamma} \subset \mathbf{O}_\ell^{4\gamma} \subset \mathbf{A}_\ell^\gamma.$$

Hence, it remains to verify that

$$\mathbf{O}_{\ell+1}^{4\gamma} \subset \mathbf{G}_\ell(\gamma_{\ell+1}, \tau_1, \tau_2, i_\ell).$$

Let $(b, \omega) \in \mathbf{O}_{\ell+1}^{4\gamma}$, then by construction, one can find $(b', \omega') \in \mathbf{A}_{\ell+1}^{2\gamma}$ with

$$\mathbf{dist}((b, \omega), (b', \omega')) < 4\gamma N_{\ell+1}^{-\bar{a}}. \quad (8.20)$$

Let us start proving that $(b, \omega) \in \mathcal{O}_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1}(i_\ell)$. For all $k \in \{1, 2\}$ and $(l, j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0, 0)\}$ with $|l| \leq N_\ell$, we have by triangle and Cauchy-Schwarz inequalities together with (8.20) and the fact $(b', \omega') \in \mathcal{O}_{\infty, \ell}^{2\gamma_{\ell+1}, \tau_1}(i_\ell)$,

$$\begin{aligned} |\omega \cdot l + jc_k(b, \omega, i_\ell)| &\geq |\omega' \cdot l + jc_k(b', \omega', i_\ell)| - |\omega - \omega'| |l| - |j| |c_k(b, \omega, i_\ell) - c_k(b', \omega', i_\ell)| \\ &> \frac{4\gamma_{\ell+1}^v 2^v \langle j \rangle}{\langle l \rangle^{\tau_1}} - 4\gamma N_{\ell+1}^{1-\bar{a}} - \langle j \rangle |c_k(b, \omega, i_\ell) - c_k(b', \omega', i_\ell)|. \end{aligned}$$

Now the Mean Value Theorem and the definition of $\mathbf{O}_{\ell+1}^{4\gamma}$ imply

$$|c_k(b, \omega, i_\ell) - c_k(b', \omega', i_\ell)| \leq CN_{\ell+1}^{-\bar{a}} \|c_k(i_\ell)\|^{q, \gamma}.$$

From (7.12), we deduce

$$\begin{aligned} \|c_k(i_\ell)\|^{q, \gamma} &\leq \|c_k(i_\ell) - \mathbf{v}_k\|^{q, \gamma} + \|\mathbf{v}_k\|^{q, \gamma} \\ &\leq C. \end{aligned}$$

Combining the last two estimates gives

$$|c_k(b, \omega, i_\ell) - c_k(b', \omega', i_\ell)| \leq C\gamma\gamma^{-1}N_{\ell+1}^{-\bar{a}} \leq C\gamma N_{\ell+1}^{1-\bar{a}}.$$

Consequently, using the facts that $\gamma_\ell \geq \gamma$ and $v \in (0, 1)$, we get

$$\begin{aligned} |\omega \cdot l + jc_k(b, \omega, i_\ell)| &> \frac{4\gamma_{\ell+1}^v 2^v \langle j \rangle}{\langle l \rangle^{\tau_1}} - C\gamma \langle j \rangle N_{\ell+1}^{1-\bar{a}} \\ &\geq \frac{4\gamma_{\ell+1}^v \langle j \rangle}{\langle l \rangle^{\tau_1}} \left(2^v - CN_{\ell+1}^{\tau_1+1-\bar{a}} \right). \end{aligned}$$

Our choice of parameters (8.1) and (2.2) implies in particular

$$\bar{a} = \tau_2 + 3 \geq \tau_1 + 2. \quad (8.21)$$

Therefore, taking N_0 sufficiently large, we obtain

$$2^v - CN_{\ell+1}^{\tau_1+1-\bar{a}} \geq 2^v - CN_0^{-1} > 1,$$

which implies in turn

$$|\omega \cdot l + jc_k(b, \omega, i_\ell)| > \frac{4\gamma_{\ell+1}^v \langle j \rangle}{\langle l \rangle^{\tau_1}}.$$

This proves that $(b, \omega) \in \mathcal{O}_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1}(i_\ell)$. Now, let us check that $(b, \omega) \in \mathcal{O}_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1, \tau_2}(i_\ell)$. For all $k \in \{1, 2\}$ and $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k})^2$ with $|l| \leq N_\ell$, using the triangle and Cauchy-Schwarz inequalities together with (8.20) and the fact that $(b', \omega') \in \mathcal{O}_{\infty, \ell}^{2\gamma_{\ell+1}, \tau_1, \tau_2}(i_\ell)$

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) \right| &\geq \left| \omega' \cdot l + \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) - \mu_{j_0,k}^{(\infty)}(b', \omega', i_\ell) \right| - |\omega - \omega'| |l| \\ &\quad - \left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) + \mu_{j_0,k}^{(\infty)}(b', \omega', i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \\ &> \frac{4\gamma_{\ell+1} \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} - 4\gamma N_{\ell+1}^{1-\bar{a}} \\ &\quad - \left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) + \mu_{j_0,k}^{(\infty)}(b', \omega', i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right|. \end{aligned}$$

We remind from (7.46) that the perturbed eigenvalues admit the following structure

$$\mu_{j,k}^{(\infty)}(b, \omega, i_\ell) = \mu_{j,k}^{(0)}(b, \omega, i_\ell) + r_{j,k}^{(\infty)}(b, \omega, i_\ell). \quad (8.22)$$

Hence,

$$\begin{aligned} &\left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) + \mu_{j_0,k}^{(\infty)}(b', \omega', i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \\ &\leq \left| \mu_{j,k}^{(0)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(0)}(b, \omega, i_\ell) + \mu_{j_0,k}^{(0)}(b', \omega', i_\ell) - \mu_{j,k}^{(0)}(b', \omega', i_\ell) \right| \\ &\quad + \left| r_{j,k}^{(\infty)}(b, \omega, i_\ell) - r_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| + \left| r_{j_0,k}^{(\infty)}(b, \omega, i_\ell) - r_{j_0,k}^{(\infty)}(b', \omega', i_\ell) \right|. \end{aligned}$$

The Mean Value Theorem, (8.20) and the definition of $\mathcal{O}_{\ell+1}^{4\gamma}$ allow to write

$$\left| \mu_{j,k}^{(0)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(0)}(b, \omega, i_\ell) + \mu_{j_0,k}^{(0)}(b', \omega', i_\ell) - \mu_{j,k}^{(0)}(b', \omega', i_\ell) \right| \leq \gamma C N_{\ell+1}^{1-\bar{a}} \langle j - j_0 \rangle. \quad (8.23)$$

Similarly, using in particular (7.47), (8.14) and the definition of $\mathcal{O}_{\ell+1}^{4\gamma}$ we get

$$\left| r_{j,k}^{(\infty)}(b, \omega, i_\ell) - r_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \leq C \gamma N_{\ell+1}^{-\bar{a}} \varepsilon \gamma^{-2} \leq \gamma C N_{\ell+1}^{1-\bar{a}} \langle j - j_0 \rangle. \quad (8.24)$$

Gathering the previous inequalities and using the facts that $|l| \leq N_\ell$ and $\gamma_{\ell+1} \geq \gamma$ we deduce

$$\left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) \right| \geq \frac{\gamma_{\ell+1} \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}} \left(4 - C N_{\ell+1}^{\tau_2+1-\bar{a}} \right).$$

By the choice (8.21), if N_0 is large enough, then we obtain

$$C N_{\ell+1}^{\tau_2+1-\bar{a}} \leq C N_0^{-1} < 1,$$

which implies in turn

$$\left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,k}^{(\infty)}(b, \omega, i_\ell) \right| > \frac{2\gamma_{\ell+1} \langle j - j_0 \rangle}{\langle l \rangle^{\tau_2}}.$$

For all $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,2})$ with $\langle l, j, j_0 \rangle \leq N_\ell$, using the triangle and Cauchy-Schwarz inequalities together with (8.20) and the fact that $(b', \omega') \in \mathcal{O}_{\infty, \ell}^{2\gamma_{\ell+1}, \tau_1, \tau_2}(i_\ell)$

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,1}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_\ell) \right| &\geq \left| \omega' \cdot l + \mu_{j,1}^{(\infty)}(b', \omega', i_\ell) - \mu_{j_0,2}^{(\infty)}(b', \omega', i_\ell) \right| - |\omega - \omega'| |l| \\ &\quad - \left| \mu_{j,1}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_\ell) + \mu_{j_0,2}^{(\infty)}(b', \omega', i_\ell) - \mu_{j,1}^{(\infty)}(b', \omega', i_\ell) \right| \\ &> \frac{4\gamma_{\ell+1}}{\langle l, j, j_0 \rangle^{\tau_2}} - 4\gamma N_{\ell+1}^{1-\bar{a}} \\ &\quad - \left| \mu_{j,1}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_\ell) + \mu_{j_0,2}^{(\infty)}(b', \omega', i_\ell) - \mu_{j,1}^{(\infty)}(b', \omega', i_\ell) \right|. \end{aligned}$$

Similarly to (8.23) and (8.24), we get

$$\begin{aligned} & \left| \mu_{j,1}^{(0)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(0)}(b, \omega, i_\ell) + \mu_{j_0,2}^{(0)}(b', \omega', i_\ell) - \mu_{j,1}^{(0)}(b', \omega', i_\ell) \right| \leq \gamma C N_{\ell+1}^{1-\bar{a}} \langle j, j_0 \rangle \leq \gamma C N_{\ell+1}^{2-\bar{a}}, \\ & \left| r_{j,1}^{(\infty)}(b, \omega, i_\ell) - r_{j,1}^{(\infty)}(b', \omega', i_\ell) \right| + \left| r_{j_0,2}^{(\infty)}(b, \omega, i_\ell) - r_{j_0,2}^{(\infty)}(b', \omega', i_\ell) \right| \leq \gamma C N_{\ell+1}^{1-\bar{a}} \langle j, j_0 \rangle \leq \gamma C N_{\ell+1}^{2-\bar{a}}. \end{aligned}$$

Therefore,

$$\left| \omega \cdot l + \mu_{j,1}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_\ell) \right| \geq \frac{\gamma_{\ell+1}}{\langle l, j, j_0 \rangle^{\tau_2}} \left(4 - C N_{\ell+1}^{\tau_2+2-\bar{a}} \right).$$

By the choice (8.21), if N_0 is large enough, then we obtain

$$C N_{\ell+1}^{\tau_2+2-\bar{a}} \leq C N_0^{-1} < 1,$$

and then

$$\left| \omega \cdot l + \mu_{j,1}^{(\infty)}(b, \omega, i_\ell) - \mu_{j_0,2}^{(\infty)}(b, \omega, i_\ell) \right| > \frac{2\gamma_{\ell+1}}{\langle l, j, j_0 \rangle^{\tau_2}}.$$

This proves that $(b, \omega) \in \mathcal{O}_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1, \tau_2}(i_\ell)$. It remains to check that $(b, \omega) \in \Lambda_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1}(i_\ell)$. For all $k \in \{1, 2\}$ and $(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})$ with $|l| \leq N_\ell$, we have by left triangle and Cauchy-Schwarz inequalities together with (8.20) and the fact $(b', \omega') \in \Lambda_{\infty, \ell}^{2\gamma_{\ell+1}, \tau_1}(i_\ell)$

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) \right| & \geq \left| \omega' \cdot l + \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| - |\omega - \omega'| |l| - \left| \mu_{j,k}^{(\infty)}(b, \omega, i_k) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \\ & > \frac{2\gamma_{\ell+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} - 4\gamma N_\ell N_{\ell+1}^{-\bar{a}} - \left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \\ & > \frac{2\gamma_{\ell+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} - 4\gamma N_{\ell+1}^{1-\bar{a}} - \left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right|. \end{aligned}$$

The Mean Value Theorem and the definition of $\mathcal{O}_{\ell+1}^{4\gamma}$ give

$$\begin{aligned} \left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| & \leq |(b, \omega) - (b', \omega')| \gamma^{-1} \|\mu_{j,k}^{(\infty)}(i_\ell)\|^{q, \gamma} \\ & \leq 4N_{\ell+1}^{-\bar{a}} \|\mu_{j,k}^{(\infty)}(i_\ell)\|^{q, \gamma}. \end{aligned}$$

Now, by triangle inequality

$$\forall j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}, \quad \|\mu_{j,k}^{(\infty)}(i_\ell)\|^{q, \gamma} \leq \|\mu_{j,k}^{(\infty)}(i_\ell) - \Omega_{j,k}\|^{q, \gamma} + \|\Omega_{j,k}\|^{q, \gamma}.$$

From (4.73) one has for all $|j| \geq m^*$,

$$\|\Omega_{j,k}\|^{q, \gamma} \leq C|j|.$$

Besides, (7.46), (7.35), (7.36) and (7.47) imply

$$\forall j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}, \quad \|\mu_{j,k}^{(\infty)}(i_\ell) - \Omega_{j,k}\|^{q, \gamma} \leq C|j|.$$

Putting together the preceding three estimates gives

$$\forall j \in \mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}, \quad \|\mu_{j,k}^{(\infty)}(i_\ell)\|^{q, \gamma} \leq C|j|.$$

As a consequence, we have

$$\left| \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) - \mu_{j,k}^{(\infty)}(b', \omega', i_\ell) \right| \leq C \langle j \rangle N_{\ell+1}^{-\bar{a}} \leq C \gamma \langle j \rangle N_{\ell+1}^{1-\bar{a}}.$$

Usint that $|l| \leq N_\ell \leq N_{\ell+1}$ and $\gamma_{\ell+1} \geq \gamma$, we get

$$\begin{aligned} \left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) \right| & \geq \frac{2\gamma_{\ell+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} - C \gamma \langle j \rangle N_{\ell+1}^{1-\bar{a}} \\ & \geq \frac{\gamma_{\ell+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} \left(2 - C N_{\ell+1}^{\tau_1+1-\bar{a}} \right). \end{aligned}$$

Now, we choose N_0 sufficiently large so that

$$CN_{\ell+1}^{\tau_1+1-\bar{a}} \leq CN_0^{-1} < 1,$$

and then

$$\left| \omega \cdot l + \mu_{j,k}^{(\infty)}(b, \omega, i_\ell) \right| > \frac{\gamma_{\ell+1}(j)}{\langle l \rangle^{\tau_1}}.$$

This shows that, $(b, \omega) \in \Lambda_{\infty, \ell}^{\gamma_{\ell+1}, \tau_1}(i_\ell)$ and finally $(b, \omega) \in \mathbf{G}_\ell(\gamma_{\ell+1}, \tau_1, \tau_2, i_\ell)$. Therefore $(b, \omega) \in \mathbf{A}_{\ell+1}^\gamma$. This achieves the induction proof of (8.19).

► **Construction of the next approximation.** Our next task is to construct the next approximate solution denoted \mathbf{U}_{n+1} . Observe that according to Theorem 6.1, the properties (8.14) and (8.17) allow to construct a reversible approximate right inverse $\mathbf{T}_n \triangleq \mathbf{T}_n(b, \omega)$ of the linearized operator L_n . Recall that the operator \mathbf{T}_n is well-defined on the whole set of parameters \mathcal{O} and satisfies, by virtue of (6.31), the following tame estimate

$$\forall s \in [s_0, S], \quad \|\mathbf{T}_n \rho\|_s^{q, \gamma, \mathbf{m}} \lesssim \gamma^{-1} (\|\rho\|_{s+\bar{\sigma}}^{q, \gamma, \mathbf{m}} + \|\mathcal{J}_n\|_{s+\bar{\sigma}}^{q, \gamma, \mathbf{m}} \|\rho\|_{s_0+\bar{\sigma}}^{q, \gamma, \mathbf{m}}). \quad (8.25)$$

In addition it is an approximate right inverse of L_n when restricted to $\mathbf{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$. More precisely, according to (6.32) we have in $\mathbf{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$

$$L_n \mathbf{T}_n - \text{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)}, \quad (8.26)$$

where the error terms in the right hand-side satisfy the estimates (6.33), (6.34) and (6.35). The next approximation is defined as follows,

$$\tilde{\mathbf{U}}_{n+1} \triangleq \mathbf{U}_n + \tilde{\mathbf{H}}_{n+1}, \quad \tilde{\mathbf{H}}_{n+1} \triangleq (\tilde{\mathcal{J}}_{n+1}, \hat{\alpha}_{n+1}, 0) \triangleq -\mathbf{\Pi}_{N_n} \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) \in E_{n, \mathbf{m}} \times \mathbb{R}^d \times \mathbb{R}^{d+1},$$

where the projector $\mathbf{\Pi}_{N_n}$ and its orthogonal are defined by

$$\mathbf{\Pi}_{N_n}(\mathcal{J}, \alpha, 0) = (\mathbf{\Pi}_{N_n} \mathcal{J}, \alpha, 0) \quad \text{and} \quad \mathbf{\Pi}_{N_n}^\perp(\mathcal{J}, \alpha, 0) = (\mathbf{\Pi}_{N_n}^\perp \mathcal{J}, 0, 0). \quad (8.27)$$

Then, applying Taylor formula yields

$$\begin{aligned} \mathcal{F}(\tilde{\mathbf{U}}_{n+1}) &= \mathcal{F}(\mathbf{U}_n) - L_n \mathbf{\Pi}_{N_n} \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + Q_n \\ &= \mathcal{F}(\mathbf{U}_n) - L_n \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + L_n \mathbf{\Pi}_{N_n}^\perp \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + Q_n \\ &= \mathcal{F}(\mathbf{U}_n) - \mathbf{\Pi}_{N_n} L_n \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + (L_n \mathbf{\Pi}_{N_n}^\perp - \mathbf{\Pi}_{N_n}^\perp L_n) \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + Q_n \\ &= \mathbf{\Pi}_{N_n}^\perp \mathcal{F}(\mathbf{U}_n) - \mathbf{\Pi}_{N_n} (L_n \mathbf{T}_n - \text{Id}) \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + (L_n \mathbf{\Pi}_{N_n}^\perp - \mathbf{\Pi}_{N_n}^\perp L_n) \mathbf{T}_n \mathbf{\Pi}_{N_n} \mathcal{F}(\mathbf{U}_n) + Q_n, \end{aligned} \quad (8.28)$$

where Q_n denotes the quadratic part given by

$$Q_n = \mathcal{F}(\mathbf{U}_n + \tilde{\mathbf{H}}_{n+1}) - \mathcal{F}(\mathbf{U}_n) - L_n \tilde{\mathbf{H}}_{n+1}. \quad (8.29)$$

Now, we shall prove (8.11) at the order $n+1$ for a suitable extension \mathbf{U}_{n+1} of $\tilde{\mathbf{U}}_{n+1}|_{\mathcal{O}_{n+1}^{2\gamma}}$. This is done in two steps. The first one is to prove that

$$\|\mathcal{F}(\tilde{\mathbf{U}}_{n+1})\|_{s_0}^{q, \gamma, \mathbf{m}, \mathcal{O}_{n+1}^{4\gamma}} \leq C_* \varepsilon N_n^{-a_1}. \quad (8.30)$$

The second step is to construct the classical extension of \mathbf{U}_{n+1} which fulfills the desired estimate (8.11). ► *Proof of (8.30).* We estimate each one of the four terms in the right hand-side of (8.28). Let us begin with the first one. Applying Lemma 2.1-(i) and using the inclusion (8.18), we obtain

$$\|\mathbf{\Pi}_{N_n}^\perp \mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q, \gamma, \mathbf{m}, \mathcal{O}_{n+1}^{4\gamma}} \leq N_n^{s_0 - \kappa_1} \|\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1}^{q, \gamma, \mathbf{m}, \mathcal{O}_n^{2\gamma}}. \quad (8.31)$$

Now, Taylor formula together with (5.11), Lemma 5.2, (8.13) and (8.5) imply

$$\begin{aligned} \forall s \geq s_0, \quad \|\mathcal{F}(\mathbf{U}_n)\|_s^{q, \gamma, \mathbf{m}, \mathcal{O}_n^{2\gamma}} &\leq \|\mathcal{F}(\mathbf{U}_0)\|_s^{q, \gamma, \mathbf{m}} + \|\mathcal{F}(\mathbf{U}_n) - \mathcal{F}(\mathbf{U}_0)\|_s^{q, \gamma, \mathbf{m}, \mathcal{O}_n^{2\gamma}} \\ &\lesssim \varepsilon + \|\mathbf{W}_n\|_{s+\bar{\sigma}}^{q, \gamma, \mathbf{m}}. \end{aligned} \quad (8.32)$$

Besides, (8.12), (6.20) and (8.2) together give

$$\begin{aligned}\varepsilon + \|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq \varepsilon \left(1 + C_*\gamma^{-1}N_{n-1}^{\mu_1}\right) \\ &\leq 2C_*\varepsilon N_n^{\frac{2}{3}\mu_1+1}.\end{aligned}\tag{8.33}$$

Inserting (8.32) and (8.33) into (8.31) yields

$$\|\Pi_{N_n}^\perp \mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_{n+1}^{4\gamma}} \lesssim C_*\varepsilon N_n^{s_0+\frac{2}{3}\mu_1+1-\kappa_1}.\tag{8.34}$$

Let us move on to the second term. According to (8.19), we have the following inclusions

$$\mathbf{O}_{n+1}^{4\gamma} \subset \mathbf{A}_{n+1}^\gamma \subset \mathbf{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n).$$

Hence, the decomposition (8.26) holds on $\mathbf{O}_{n+1}^{4\gamma}$ and we can write

$$\Pi_{N_n}(L_n T_n - \text{Id})\Pi_{N_n}\mathcal{F}(\mathbf{U}_n) = \mathfrak{E}_{1,n} + \mathfrak{E}_{2,n} + \mathfrak{E}_{3,n},$$

with for any $k \in \{1, 2, 3\}$,

$$\mathfrak{E}_{k,n} \triangleq \Pi_{N_n}\mathcal{E}_k^{(n)}\Pi_{N_n}\mathcal{F}(\mathbf{U}_n).$$

Thus, we need to estimate each one of the error terms $\mathfrak{E}_{k,n}$. We begin with $\mathfrak{E}_{1,n}$ for which we need the following interpolation-type inequality

$$\begin{aligned}\|\mathcal{F}(\mathbf{U}_n)\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} &\leq \|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + \|\Pi_{N_n}^\perp \mathcal{F}(\mathbf{U}_n)\|_{q,s_0+\bar{\sigma}}^{\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} \\ &\leq N_n^{\bar{\sigma}}\|\mathcal{F}(\mathbf{U}_n)\|_{q,s_0}^{\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + N_n^{s_0-\kappa_1}\|\mathcal{F}(\mathbf{U}_n)\|_{q,\kappa_1+\bar{\sigma}}^{\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}.\end{aligned}\tag{8.35}$$

Now, putting together (8.32) and (8.33), we infer

$$\|\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} \leq C_*\varepsilon N_n^{\bar{\sigma}+\frac{2}{3}\mu_1+1}.\tag{8.36}$$

Combining (6.33), (8.35), (8.3), (8.14) and (8.36), we obtain

$$\begin{aligned}\|\mathfrak{E}_{1,n}\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} &\lesssim \gamma^{-1}\|\mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}(1 + \|\mathfrak{J}_n\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}}) \\ &\lesssim \gamma^{-1}N_n^{\bar{\sigma}}\left(N_n^{\bar{\sigma}}\|\mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + N_n^{s_0-\kappa_1}\|\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}\right)\|\mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}(1 + \|\mathbf{W}_n\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}}) \\ &\lesssim C_*\varepsilon\left(N_n^{2\bar{\sigma}-\frac{4}{3}a_1} + N_n^{s_0+2\bar{\sigma}+\frac{2}{3}\mu_1+1-\frac{2}{3}a_1-\kappa_1}\right).\end{aligned}\tag{8.37}$$

As for $\mathfrak{E}_{2,n}$, we apply (6.34) with $\mathbf{b} = \kappa_1 - s_0$ and use (8.14), (8.11) and (8.12) in order to find

$$\begin{aligned}\|\mathfrak{E}_{2,n}\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} &\lesssim \gamma^{-1}N_n^{s_0-\kappa_1}\left(\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + \varepsilon\|\mathfrak{J}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}}\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}\right) \\ &\lesssim \gamma^{-1}N_n^{s_0-\kappa_1}\left(\|\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + \varepsilon N_n^{\bar{\sigma}}\|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}}\|\mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}\right) \\ &\lesssim C_*\varepsilon N_n^{s_0+\bar{\sigma}+\frac{2}{3}\mu_1+2-\kappa_1} + C_*\varepsilon N_n^{s_0+\bar{\sigma}+\frac{2}{3}\mu_1+2-\frac{2}{3}a_1-\kappa_1} \\ &\lesssim C_*\varepsilon N_n^{s_0+\bar{\sigma}+\frac{2}{3}\mu_1+2-\kappa_1}.\end{aligned}\tag{8.38}$$

Similarly, putting together (6.35), (6.20), (8.2) and (8.14), we infer

$$\begin{aligned}\|\mathfrak{E}_{3,n}\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} &\lesssim N_n^{s_0-\kappa_1}\gamma^{-2}\left(\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} + \varepsilon\gamma^{-2}\|\mathfrak{J}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}}\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}}\right) \\ &\quad + \varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m},\mathbf{O}_n^{2\gamma}} \\ &\lesssim C_*\varepsilon\left(N_n^{s_0+\bar{\sigma}+\frac{2}{3}\mu_1+2-\kappa_1} + N_n^{\bar{\sigma}+1-\mu_2-\frac{2}{3}a_1}\right).\end{aligned}\tag{8.39}$$

Gathering, (8.37), (8.38) and (8.39), we deduce

$$\|\Pi_{N_n}(L_n T_n - \text{Id})\Pi_{N_n}\mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_{n+1}^{4\gamma}} \leq CC_*\varepsilon \left(N_n^{2\bar{\sigma}-\frac{4}{3}a_1} + N_n^{s_0+2\bar{\sigma}+\frac{2}{3}\mu_1+1-\kappa_1} + N_n^{\bar{\sigma}+1-\mu_2-\frac{2}{3}a_1} \right). \quad (8.40)$$

For $n = 0$, we deduce from (8.13), (8.14) and by slight modifications of the preceding computations

$$\begin{aligned} \|\Pi_{N_0}(L_0 T_0 - \text{Id})\Pi_{N_0}\mathcal{F}(\mathbf{U}_0)\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_1^{4\gamma}} &\leq \|\mathfrak{E}_{1,0}\|_{s_0}^{q,\gamma,\mathfrak{m}} + \|\mathfrak{E}_{2,0}\|_{s_0}^{q,\gamma,\mathfrak{m}} + \|\mathfrak{E}_{3,0}\|_{s_0}^{q,\gamma,\mathfrak{m}} \\ &\lesssim \varepsilon^2\gamma^{-1} + \varepsilon\gamma^{-1} + (\varepsilon\gamma^{-2}N_0^{s_0-\kappa_1} + \varepsilon^2\gamma^{-4}) \\ &\lesssim \varepsilon\gamma^{-2}. \end{aligned} \quad (8.41)$$

Now, we turn to the estimate corresponding to the third term in (8.28). In view of (6.1), we have for $\mathbf{H} = (\widehat{\mathcal{J}}, \widehat{\alpha})$ with $\widehat{\mathcal{J}} = (\widehat{\Theta}, \widehat{I}, \widehat{z})$,

$$L_n \mathbf{H} = \omega \cdot \partial_\varphi \widehat{\mathcal{J}} - (0, 0, \mathcal{J}\mathbf{L}_0(b)\widehat{z}) - \varepsilon d_i X_{\mathcal{P}_\varepsilon}(i_n)\widehat{\mathcal{J}} - (\mathcal{J}\widehat{\alpha}, 0, 0). \quad (8.42)$$

Now, (8.27) and the fact that $\omega \cdot \partial_\varphi$ and $\mathcal{J}\mathbf{L}_0(b)$ are diagonal yield

$$\left(L_n \Pi_{N_n}^\perp - \Pi_{N_n}^\perp L_n \right) \mathbf{H} = -\varepsilon [d_i X_{\mathcal{P}_\varepsilon}(i_n), \Pi_{N_n}^\perp] \widehat{\mathcal{J}}.$$

Applying Lemma 5.2-(ii) together with Lemma 2.1-(i) and (8.18), we infer

$$\left\| \left(L_n \Pi_{N_n}^\perp - \Pi_{N_n}^\perp L_n \right) \mathbf{H} \right\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_{n+1}^{4\gamma}} \lesssim \varepsilon N_n^{s_0-\kappa_1} \left(\|\widehat{\mathcal{J}}\|_{\kappa_1+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} + \|\mathcal{J}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|\widehat{\mathcal{J}}\|_{s_0+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \right).$$

Hence,

$$\begin{aligned} \left\| \left(L_n \Pi_{N_n}^\perp - \Pi_{N_n}^\perp L_n \right) T_n \Pi_n \mathcal{F}(\mathbf{U}_n) \right\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_{n+1}^{4\gamma}} &\lesssim \varepsilon N_n^{s_0-\kappa_1} \|T_n \Pi_n \mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \\ &\quad + \varepsilon N_n^{s_0-\kappa_1} \|\mathcal{J}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|T_n \Pi_n \mathcal{F}(\mathbf{U}_n)\|_{s_0+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}}. \end{aligned}$$

Now, using (8.25), (8.18), Lemma 2.1-(i), Sobolev embeddings, (8.14) and (8.2), we get

$$\begin{aligned} &\left\| \left(L_n \Pi_{N_n}^\perp - \Pi_{N_n}^\perp L_n \right) T_n \Pi_n \mathcal{F}(\mathbf{U}_n) \right\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_{n+1}^{4\gamma}} \\ &\lesssim \varepsilon\gamma^{-1} N_n^{s_0-\kappa_1} \|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} + \|\mathcal{J}_n\|_{\kappa_1+\bar{\sigma}+1}^{q,\gamma,\mathfrak{m}} \|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \\ &\quad + \varepsilon\gamma^{-1} N_n^{s_0-\kappa_1} \|\mathcal{J}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \left(\|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}+1}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} + \|\mathcal{J}_n\|_{s_0+\bar{\sigma}+1}^{q,\gamma,\mathfrak{m}} \|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \right) \\ &\lesssim \varepsilon N_n^{s_0+2-\kappa_1} \left(\|\mathcal{F}(\mathbf{U}_n)\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} + \|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \right). \end{aligned}$$

Besides, from Lemma 2.1-(ii), (8.11) and (6.20), we obtain

$$\begin{aligned} \|\Pi_{N_n} \mathcal{F}(\mathbf{U}_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} &\leq N_n^{\bar{\sigma}} \|\mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_n^{2\gamma}} \\ &\leq C_* \varepsilon N_n^{\bar{\sigma}} N_n^{-a_1} \\ &\leq C_* \varepsilon N_n^{\bar{\sigma}-\frac{2}{3}a_1}. \end{aligned}$$

Added to (8.1), (8.36) and (8.12), we deduce that

$$\|(L_n \Pi_n^\perp - \Pi_n^\perp L_n) T_n \Pi_n \mathcal{F}(\mathbf{U}_n)\|_{s_0}^{q,\gamma,\mathfrak{m},\mathbf{O}_{n+1}^{4\gamma}} \leq CC_* \varepsilon N_n^{s_0+\bar{\sigma}+\frac{2}{3}\mu_1+3-\kappa_1}. \quad (8.43)$$

We are left with the quadratic term in (8.28). Another application of Taylor formula with (8.29) leads to

$$Q_n = \int_0^1 (1-t) d_{i,\alpha}^2 \mathcal{F}(\mathbf{U}_n + t\widetilde{\mathbf{H}}_{n+1}) [\widetilde{\mathbf{H}}_{n+1}, \widetilde{\mathbf{H}}_{n+1}] dt.$$

Now, (8.42) and Lemma 5.2-(iii) give

$$\|Q_n\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \lesssim \varepsilon \left(1 + \|W_n\|_{s_0+2}^{q,\gamma,\mathfrak{m}} + \|\tilde{H}_{n+1}\|_{s_0+2}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \right) \left(\|\tilde{H}_{n+1}\|_{s_0+2}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \right)^2. \quad (8.44)$$

Observe that, (8.32), (8.2) and (8.3) imply

$$\gamma^{-1} \|\mathcal{F}(U_n)\|_{s_0}^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} \leq 1. \quad (8.45)$$

Gathering (8.18), (8.19), (8.25), (8.32) and (8.45), we obtain for all $s \in [s_0, S]$

$$\begin{aligned} \|\tilde{H}_{n+1}\|_s^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} &= \|\Pi_{N_n} T_n \Pi_{N_n} \mathcal{F}(U_n)\|_s^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \\ &\lesssim \gamma^{-1} \left(\|\Pi_{N_n} \mathcal{F}(U_n)\|_{s+\bar{\sigma}}^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} + \|\mathcal{J}_n\|_{s+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \|\Pi_{N_n} \mathcal{F}(U_n)\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} \right) \\ &\lesssim \gamma^{-1} \left(N_n^{\bar{\sigma}} \|\mathcal{F}(U_n)\|_s^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} + N_n^{2\bar{\sigma}} \|\mathcal{J}_n\|_s^{q,\gamma,\mathfrak{m}} \|\mathcal{F}(U_n)\|_{s_0}^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} \right) \\ &\lesssim \gamma^{-1} N_n^{2\bar{\sigma}} (\varepsilon + \|W_n\|_s^{q,\gamma,\mathfrak{m}}). \end{aligned} \quad (8.46)$$

Similarly, (8.25), (8.18), (8.3), (8.14) and (8.11), we infer

$$\begin{aligned} \|\tilde{H}_{n+1}\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} &\lesssim \gamma^{-1} N_n^{\bar{\sigma}} \|\mathcal{F}(U_n)\|_{s_0}^{q,\gamma,\mathfrak{m},O_n^{2\gamma}} \\ &\lesssim C_* \varepsilon \gamma^{-1} N_n^{\bar{\sigma}} N_{n-1}^{-a_1}. \end{aligned} \quad (8.47)$$

For ε sufficiently small, (8.3) and (8.47) imply

$$\begin{aligned} \|W_n\|_{s_0+2}^{q,\gamma,\mathfrak{m}} + \|\tilde{H}_{n+1}\|_{s_0+2}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} &\leq C_* \varepsilon \gamma^{-1} N_0^{q\bar{a}} + N_n^2 \|\tilde{H}_{n+1}\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \\ &\leq 1 + C \varepsilon \gamma^{-1} N_n^{\bar{\sigma}+2} N_{n-1}^{-a_1} \\ &\leq 1 + C \varepsilon \gamma^{-1} N_{n-1}^{3+\frac{3}{2}\bar{\sigma}-a_1}. \end{aligned}$$

But (8.1) gives in particular $a_1 \geq 3 + \frac{3}{2}\bar{\sigma}$. Thus,

$$\|W_n\|_{s_0+2}^{q,\gamma,\mathfrak{m}} + \|\tilde{H}_{n+1}\|_{s_0+2}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \leq 2. \quad (8.48)$$

Therefore, inserting (8.48) and (8.47) into (8.44) and using (8.2) and (8.14), we get

$$\begin{aligned} \|Q_n\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} &\lesssim \varepsilon \left(\|\tilde{H}_{n+1}\|_{s_0+2}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \right)^2 \\ &\leq \varepsilon N_n^4 \left(\|\tilde{H}_{n+1}\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \right)^2 \\ &\lesssim \varepsilon C_* N_n^{2\bar{\sigma}+4} N_{n-1}^{-2a_1}. \end{aligned}$$

Using (6.20), we finally obtain for $n \in \mathbb{N}^*$,

$$\|Q_n\|_{s_0}^{q,\gamma,\mathfrak{m},O_{n+1}^{4\gamma}} \leq C C_* \varepsilon N_n^{2\bar{\sigma}+4-\frac{4}{3}a_1}. \quad (8.49)$$

As for $n = 0$, we come back to (8.46) and (8.13) to get for all $s \in [s_0, S]$

$$\begin{aligned} \|\tilde{H}_1\|_s^{q,\gamma,\mathfrak{m},O_1^{4\gamma}} &\lesssim \gamma^{-1} \|\Pi_0 \mathcal{F}(U_0)\|_{s+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \\ &\lesssim C_* \varepsilon \gamma^{-1}. \end{aligned} \quad (8.50)$$

Finally, the inequality (8.49) becomes for $n = 0$,

$$\|Q_0\|_{s_0}^{q,\gamma,\mathfrak{m},O_0^{4\gamma}} \lesssim C_* \varepsilon^3 \gamma^{-2}. \quad (8.51)$$

Plugging (8.34), (8.40), (8.43) and (8.49), into (8.28) gives for $n \in \mathbb{N}^*$,

$$\|\mathcal{F}(\tilde{\mathbf{U}}_{n+1})\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_{n+1}^{4\gamma}} \leq CC_*\varepsilon \left(N_n^{s_0+2\bar{\sigma}+\frac{2}{3}\mu_1+1-\kappa_1} + N_n^{\bar{\sigma}+1-\mu_2-\frac{2}{3}a_1} + N_n^{2\bar{\sigma}+4-\frac{4}{3}a_1} \right).$$

Now, our choice of parameters in (8.1) implies

$$s_0 + 2\bar{\sigma} + \frac{2}{3}\mu_1 + 2 + a_1 \leq \kappa_1, \quad \bar{\sigma} + \frac{1}{3}a_1 + 2 \leq \mu_2, \quad 2\bar{\sigma} + 5 \leq \frac{1}{3}a_1.$$

Consequently, for N_0 large enough, that is ε small enough, we can obtain for any $n \in \mathbb{N}$,

$$\max \left(CN_n^{s_0+2\bar{\sigma}+\frac{2}{3}\mu_1+1-\kappa_1}, CN_n^{\bar{\sigma}+1-\mu_2-\frac{2}{3}a_1}, CN_n^{2\bar{\sigma}+4-\frac{4}{3}a_1} \right) \leq \frac{1}{3}N_n^{-a_1}, \quad (8.52)$$

which implies in turn (8.30) for $n \in \mathbb{N}^*$. As for the case $n = 0$, we insert (8.34), (8.41), (8.43) and (8.51) into (8.28) to get

$$\|\mathcal{F}(\tilde{\mathbf{U}}_1)\|_{q,s_0}^{\gamma,\mathbf{m},\mathbf{O}_1^{4\gamma}} \leq CC_*\varepsilon \left(N_0^{s_0+2\bar{\sigma}+\frac{3}{2}\mu_1+1-\kappa_1} + \varepsilon\gamma^{-2} + \varepsilon^2\gamma^{-2} \right).$$

Hence, using (8.52) and the fact that (8.2) and (8.1) imply $0 < a < \frac{1}{2+a_1}$, then taking ε small enough, we infer

$$C(\varepsilon\gamma^{-2} + \varepsilon^2\gamma^{-2}) \leq \frac{2}{3}N_0^{-a_1}.$$

As a consequence, (8.30) occurs also for $n = 0$.

➤ *Construction of the extension.* The next goal is to construct an extension of $\tilde{\mathbf{H}}_{n+1}$ to the whole set of parameters \mathcal{O} and still satisfying nice decay properties. For this aim, we introduce the following cut-off function $\chi_{n+1} \in C^\infty(\mathcal{O}, [0, 1])$ given by

$$\chi_{n+1} = \begin{cases} 1 & \text{in } \mathbf{O}_{n+1}^{2\gamma} \\ 0 & \text{in } \mathcal{O} \setminus \mathbf{O}_{n+1}^{4\gamma} \end{cases}$$

and such that

$$\forall \alpha \in \mathbb{N}^d, \quad |\alpha| \in \llbracket 0, q \rrbracket, \quad \|\partial_{b,\omega}^\alpha \chi_{n+1}\|_{L^\infty(\mathcal{O})} \lesssim (\gamma^{-1}N_n^{\bar{a}})^{|\alpha|}. \quad (8.53)$$

Therefore, we can define the extension \mathbf{H}_{n+1} of $\tilde{\mathbf{H}}_{n+1}$ as follows

$$\mathbf{H}_{n+1} \triangleq \begin{cases} \chi_{n+1} \tilde{\mathbf{H}}_{n+1} & \text{in } \mathbf{O}_{n+1}^{4\gamma}, \\ 0 & \text{in } \mathcal{O} \setminus \mathbf{O}_{n+1}^{4\gamma}. \end{cases} \quad (8.54)$$

We also define

$$\mathbf{W}_{n+1} \triangleq \mathbf{W}_n + \mathbf{H}_{n+1}, \quad \mathbf{U}_{n+1} \triangleq \mathbf{U}_0 + \mathbf{W}_{n+1} = \mathbf{U}_n + \mathbf{H}_{n+1}. \quad (8.55)$$

We can observe that in restriction to $\mathbf{O}_{n+1}^{2\gamma}$, we have

$$\mathbf{H}_{n+1} = \tilde{\mathbf{H}}_{n+1}, \quad \mathbf{U}_{n+1} = \tilde{\mathbf{U}}_{n+1} \quad \text{and} \quad \mathcal{F}(\mathbf{U}_{n+1}) = \mathcal{F}(\tilde{\mathbf{U}}_{n+1}).$$

The last identity together with (8.30) and the fact that $\mathbf{O}_{n+1}^{2\gamma} \subset \mathbf{O}_{n+1}^{4\gamma}$ imply (8.11). Now, the product laws in Lemma 2.1 together with (8.54) and (8.53) provide the following estimate

$$\forall s \geq s_0, \quad \|\mathbf{H}_{n+1}\|_s^{q,\gamma,\mathbf{m}} \lesssim N_n^{q\bar{a}} \|\tilde{\mathbf{H}}_{n+1}\|_s^{q,\gamma,\mathbf{m},\mathbf{O}_{n+1}^{4\gamma}}. \quad (8.56)$$

Then, gathering (8.56) and (8.47), implies for any $n \in \mathbb{N}^*$,

$$\begin{aligned} \|\mathbf{H}_{n+1}\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq CN_n^{q\bar{a}+\bar{\sigma}} \|\tilde{\mathbf{H}}_{n+1}\|_{s_0}^{q,\gamma,\mathbf{m},\mathbf{O}_{n+1}^{4\gamma}} \\ &\leq CC_*\varepsilon\gamma^{-1} N_n^{q\bar{a}+2\bar{\sigma}-\frac{2}{3}a_1}. \end{aligned}$$

The constraint (8.1) implies in particular $a_2 = \frac{2}{3}a_1 - q\bar{a} - 2\bar{\sigma} - 1 \geq 1$, we get for ε small enough

$$\begin{aligned} \|\mathbf{H}_{n+1}\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{-a_2} \\ &\leq C_*\varepsilon\gamma^{-1}N_n^{-a_2}. \end{aligned} \quad (8.57)$$

As for the case $n = 0$, we combine (8.56) and (8.50) to get, up to take C_* sufficiently large,

$$\|\mathbf{H}_1\|_s^{q,\gamma,\mathbf{m}} \leq \frac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\bar{a}}. \quad (8.58)$$

Putting together (8.5), (8.58) and (8.57), we deduce

$$\begin{aligned} \|\mathbf{W}_{n+1}\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq \|\mathbf{H}_1\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}} + \sum_{k=2}^{n+1} \|\mathbf{H}_k\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq \frac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\bar{a}} + CN_0^{-1}C_*\varepsilon\gamma^{-1} \\ &\leq C_*\varepsilon\gamma^{-1}N_0^{q\bar{a}}. \end{aligned}$$

This proves (8.3) at the order $n + 1$. Now (8.46), (8.56) and (8.12) all together yield

$$\begin{aligned} \|\mathbf{W}_{n+1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq \|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} + CN_n^{q\bar{a}}\|\mathbf{H}_{n+1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} \\ &\leq C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1} + CC_*\gamma^{-1}N_n^{q\bar{a}+2\bar{\sigma}}(\varepsilon + \|\mathbf{W}_n\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}}) \\ &\leq CC_*\varepsilon\gamma^{-1}N_n^{q\bar{a}+2\bar{\sigma}+1+\frac{2}{3}\mu_1}. \end{aligned}$$

By (8.1) we get $q\bar{a} + 2\bar{\sigma} + 2 = \frac{\mu_1}{3}$, which implies that for ε small enough we have

$$\begin{aligned} \|\mathbf{W}_{n+1}\|_{\kappa_1+\bar{\sigma}}^{q,\gamma,\mathbf{m}} &\leq CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{\mu_1} \\ &\leq C_*\varepsilon\gamma^{-1}N_n^{\mu_1}. \end{aligned}$$

This proves (8.12) at the order $n + 1$.

➤ *Reversibility preserving property of the scheme.* Form $(\mathcal{P}2)_n$, we know that the torus i_n is reversible. Observe that the projectors Π_{N_n} are reversibility preserving thanks to the symmetry with respect to the Fourier modes. Now, using the reversibility property of the operators \mathbb{T}_n and Π_{N_n} we have that the torus component of $\widehat{\mathcal{J}}_{n+1}$ of $\widetilde{\mathbf{H}}_{n+1}$ is reversible. Since the cut-off function χ_{n+1} only depends on the variables (b, ω) , then the reversibility property is also preserved for the torus component \mathcal{J}_{n+1} of \mathbf{H}_{n+1} . Looking at the first component of (8.55), we have

$$i_{n+1} = i_n + \mathcal{J}_{n+1}.$$

Hence, the reversibility property (8.10) also holds at the order $n + 1$. \square

The previous iteration procedure converges and allows to find a non trivial reversible quasi-periodic solution of our problem provided some restriction on the internal radius b . More precisely, we have the following result.

Corollary 8.1. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the following assertions hold true. There exists a q -times differentiable function*

$$\begin{aligned} \mathbf{U}_\infty : \quad \mathcal{O} &\rightarrow \left(\mathbb{T}^d \times \mathbb{R}^d \times \mathbf{H}_{\perp, \mathbf{m}}^{s_0} \right) \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ (b, \omega) &\mapsto (i_\infty(b, \omega), \alpha_\infty(b, \omega), (b, \omega)) \end{aligned}$$

such that in restriction to the Cantor set \mathbf{G}_∞^γ defined by

$$\mathbf{G}_\infty^\gamma \triangleq \bigcap_{n \in \mathbb{N}} \mathbf{A}_n^\gamma, \quad (8.59)$$

we have

$$\forall (b, \omega) \in \mathbf{G}_\infty^\gamma, \quad \mathcal{F}(\mathbf{U}_\infty(b, \omega)) = 0. \quad (8.60)$$

The torus i_∞ is \mathfrak{m} -fold and reversible. The vector $\alpha_\infty \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{R}^d)$ satisfies

$$\alpha_\infty(b, \omega) = \mathbf{J}\omega + \mathbf{r}_\varepsilon(b, \omega), \quad \|\mathbf{r}_\varepsilon\|^{q,\gamma} \lesssim \varepsilon\gamma^{-1}N_0^{q\bar{a}}. \quad (8.61)$$

In addition, there exists a q -times differentiable function $b \in (b_*, b^*) \mapsto \omega(b, \varepsilon)$ implicitly defined by

$$\alpha_\infty(b, \omega(b, \varepsilon)) = -\mathbf{J}\omega_{\text{Eq}}(b) \quad (8.62)$$

with

$$\omega(b, \varepsilon) = -\omega_{\text{Eq}}(b) + \bar{r}_\varepsilon(b), \quad \|\bar{r}_\varepsilon\|^{q,\gamma} \lesssim \varepsilon\gamma^{-1}N_0^{q\bar{a}}, \quad (8.63)$$

such that

$$\forall b \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\left(\mathbf{U}_\infty(b, \omega(b, \varepsilon))\right) = 0, \quad (8.64)$$

where

$$\mathcal{C}_\infty^\varepsilon \triangleq \left\{ b \in (b_*, b^*) \text{ s.t. } (b, \omega(b, \varepsilon)) \in \mathbf{G}_\infty^\gamma \right\}. \quad (8.65)$$

Proof. We deduce from (8.5) and (8.57) that

$$\|\mathbf{W}_{n+1} - \mathbf{W}_n\|_{s_0}^{q,\gamma,\mathfrak{m}} = \|\mathbf{H}_{n+1}\|_{s_0}^{q,\gamma,\mathfrak{m}} \leq \|\mathbf{H}_{n+1}\|_{s_0+\bar{\sigma}}^{q,\gamma,\mathfrak{m}} \leq C_*\varepsilon\gamma^{-1}N_n^{-a_2}.$$

Consequently, the sequence $(\mathbf{W}_n)_{n \in \mathbb{N}}$ converges and we denote

$$\mathbf{W}_\infty \triangleq \lim_{n \rightarrow \infty} \mathbf{W}_n \triangleq (\mathcal{I}_\infty, \alpha_\infty - \mathbf{J}\omega, 0, 0), \quad \mathbf{U}_\infty \triangleq (i_\infty, \alpha_\infty, (b, \omega)) = \mathbf{U}_0 + \mathbf{W}_\infty.$$

The reversibility and \mathfrak{m} -fold properties of i_∞ are obtained as the pointwise limit in (8.10). Now, for ε small enough, we get (8.60) from (8.11). The identity (8.61) follows from the previous construction and the corresponding estimate is obtained by taking the limit in (8.3). We recall that the open set \mathcal{O} is defined in (7.3)-(7.4) by

$$\mathcal{O} = (b_*, b^*) \times \mathcal{U}, \quad \mathcal{U} = B(0, R_0), \quad \omega_{\text{Eq}}([b_*, b^*]) \subset B(0, \frac{R_0}{2}).$$

According to (8.61), we have that for any $b \in (b_*, b^*)$, the mapping $\omega \in \mathcal{U} \mapsto \alpha_\infty(b, \omega) \in \alpha_\infty(b, \mathcal{U})$ is invertible, implying that

$$\hat{\omega} = \alpha_\infty(b, \omega) = \mathbf{J}\omega + \mathbf{r}_\varepsilon(b, \omega) \quad \Leftrightarrow \quad \omega = \alpha_\infty^{-1}(b, \hat{\omega}) = \mathbf{J}\hat{\omega} + \hat{\mathbf{r}}_\varepsilon(b, \hat{\omega}).$$

In particular,

$$\hat{\mathbf{r}}_\varepsilon(b, \hat{\omega}) = -\mathbf{J}\mathbf{r}_\varepsilon(b, \omega).$$

Then, differentiating the previous relation and using (8.61), we get

$$\|\hat{\mathbf{r}}_\varepsilon\|^{q,\gamma} \lesssim \varepsilon\gamma^{-1}N_0^{q\bar{a}}. \quad (8.66)$$

Finally, if we denote

$$\omega(b, \varepsilon) \triangleq \alpha_\infty^{-1}(b, -\mathbf{J}\omega_{\text{Eq}}(b)) = -\omega_{\text{Eq}}(b) + \bar{r}_\varepsilon(b), \quad \bar{\mathbf{r}}_\varepsilon(b) \triangleq \hat{\mathbf{r}}_\varepsilon(b, -\mathbf{J}\omega_{\text{Eq}}(b)),$$

then we have in particular (8.62) and the estimate (8.63) follows from (8.66). In addition, combining (8.60), (8.62) and (8.65), the identity (8.64) holds. The proof of Corollary 8.1 is now complete. \square

8.2 Measure of the final Cantor set

In this last section, we check that the final Cantor set $\mathcal{C}_\infty^\varepsilon$ in the variable b given by (8.65) is massive set, which proves the existence of non-trivial quasi-periodic solution to our problem. Actually, we prove that the measure of $\mathcal{C}_\infty^\varepsilon$ is ε -close to $(b^* - b_*)$. One of the main technical ingredient is the following Rüssmann Lemma [93, Thm. 17.1].

Lemma 8.1. Let $q_0 \in \mathbb{N}^*$ and $\alpha, \beta \in \mathbb{R}_+^*$. Let $f \in C^{q_0}([a, b], \mathbb{R})$ such that

$$\inf_{x \in [a, b]} \max_{k \in [0, q_0]} |f^{(k)}(x)| \geq \beta.$$

Then, there exists $C = C(a, b, q_0, \|f\|_{C^{q_0}([a, b], \mathbb{R})}) > 0$ such that

$$\left| \{x \in [a, b] \text{ s.t. } |f(x)| \leq \alpha \} \right| \leq C \frac{\alpha^{\frac{1}{q_0}}}{\beta^{1 + \frac{1}{q_0}}},$$

where the notation $|A|$ corresponds to the Lebesgue measure of a given measurable set A .

Our main result is stated in the next proposition.

Proposition 8.2. Let q_0 be defined as in Lemma 4.8 and assume that (8.1) and (8.2) hold with $q = q_0 + 1$. Assume the additional conditions

$$\tau_1 > dq_0, \quad \tau_2 > \tau_1 + dq_0, \quad v \triangleq \frac{1}{q_0 + 3}. \quad (8.67)$$

Then there exists $C > 0$ such that

$$(b^* - b_*) - C\varepsilon^{\frac{av}{q_0}} \leq |C_\infty^\varepsilon| \leq b^* - b_*.$$

Proof. The identities (8.65) and (8.59) provide the following decomposition of the final Cantor set

$$C_\infty^\varepsilon = \bigcap_{n \in \mathbb{N}} C_n^\varepsilon, \quad \text{where} \quad C_n^\varepsilon \triangleq \left\{ b \in (b_*, b^*) \text{ s.t. } (b, \omega(b, \varepsilon)) \in A_n^\gamma \right\}, \quad (8.68)$$

with A_n^γ and $\omega(b, \varepsilon)$ as in Proposition 8.1 and (8.63). We can write

$$(b_*, b^*) \setminus C_\infty^\varepsilon = ((b_*, b^*) \setminus C_0^\varepsilon) \sqcup \bigsqcup_{n=0}^{\infty} (C_n^\varepsilon \setminus C_{n+1}^\varepsilon). \quad (8.69)$$

First, let us prove that

$$(b_*, b^*) \setminus C_0^\varepsilon = \emptyset, \quad \text{i.e.} \quad C_0^\varepsilon = (b_*, b^*). \quad (8.70)$$

For this purpose, notice that (8.63) and (8.2) imply

$$\sup_{b \in (b_*, b^*)} |\omega(b, \varepsilon) + \omega_{\text{Eq}}(b)| \leq \|\bar{r}_\varepsilon\|^{q, \gamma} \leq C\varepsilon\gamma^{-1}N_0^{q\bar{a}} = C\varepsilon^{1-a(1+q\bar{a})}.$$

But (8.1) and (8.2) give in particular

$$0 < a < \frac{1}{1 + q\bar{a}}.$$

Hence, in view of (7.4), for ε sufficiently small we can ensure

$$\forall b \in (b_*, b^*), \quad \omega(b, \varepsilon) \in \mathcal{U} = B(0, R_0).$$

By construction of A_0^γ and \mathcal{O} , we deduce (8.70). Coming back to (8.69), we find

$$\begin{aligned} \left| (b_*, b^*) \setminus C_\infty^\varepsilon \right| &\leq \sum_{n=0}^{\infty} \left| C_n^\varepsilon \setminus C_{n+1}^\varepsilon \right| \\ &\triangleq \sum_{n=0}^{\infty} S_n. \end{aligned} \quad (8.71)$$

Using the notations of Propositions 7.5 and 7.4, we denote the perturbed frequencies associated with the reduced linearized operator at state i_n in the following way

$$\begin{aligned} \mu_{j,k,n}^{(\infty)}(b, \varepsilon) &\triangleq \mu_{j,k,n}^{(\infty)}(b, \omega(b, \varepsilon), i_n) \\ &= \Omega_{j,k}(b) + jr_{k,n}^{(0)}(b, \varepsilon) + r_{j,k,n}^{(\infty)}(b, \varepsilon), \end{aligned} \quad (8.72)$$

where

$$r_{k,n}^{(0)}(b, \varepsilon) \triangleq \mathbf{c}_{k,n}(b, \varepsilon) - \mathbf{v}_k(b), \quad \mathbf{c}_{k,n}(b, \varepsilon) \triangleq \mathbf{c}_k(b, \omega(b, \varepsilon), i_n), \quad r_{j,k,n}^{(\infty)}(b, \varepsilon) \triangleq r_{j,k}^{(\infty)}(b, \omega(b, \varepsilon), i_n).$$

Now, according to (8.68) and Propositions 7.2, 7.5 and 7.6, one has by construction that for any $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = & \bigcup_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0,0)\} \\ |l| \leq N_n}} \mathcal{R}_{l,j}^{(0,k)}(i_n) \cup \bigcup_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \\ |l| \leq N_n}} \mathcal{R}_{l,j}^{(1,k)}(i_n) \\ & \cup \bigcup_{\substack{k \in \{1,2\} \\ (l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2 \\ |l| \leq N_n \\ (l,j) \neq (0,j_0)}} \mathcal{R}_{l,j,j_0}^k(i_n) \cup \bigcup_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}) \\ (l,j,j_0) \leq N_n}} \mathcal{R}_{l,j,j_0}^{1,2}(i_n), \end{aligned} \quad (8.73)$$

where we denote for $k \in \{1, 2\}$,

$$\begin{aligned} \mathcal{R}_{l,j}^{(0,k)}(i_n) & \triangleq \left\{ b \in \mathcal{C}_n^\varepsilon \text{ s.t. } \left| \omega(b, \varepsilon) \cdot l + j \mathbf{c}_{k,n}(b, \varepsilon) \right| \leq \frac{4\gamma_{n+1}^v \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\}, \\ \mathcal{R}_{l,j,j_0}^k(i_n) & \triangleq \left\{ b \in \mathcal{C}_n^\varepsilon \text{ s.t. } \left| \omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon) \right| \leq \frac{2\gamma_{n+1} \langle j-j_0 \rangle}{\langle l \rangle^{\tau_2}} \right\}, \\ \mathcal{R}_{l,j,j_0}^{1,2}(i_n) & \triangleq \left\{ b \in \mathcal{C}_n^\varepsilon \text{ s.t. } \left| \omega(b, \varepsilon) \cdot l + \mu_{j,1,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n}^{(\infty)}(b, \varepsilon) \right| \leq \frac{2\gamma_{n+1}}{\langle l, j, j_0 \rangle^{\tau_2}} \right\}, \\ \mathcal{R}_{l,j}^{(1,k)}(i_n) & \triangleq \left\{ b \in \mathcal{C}_n^\varepsilon \text{ s.t. } \left| \omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) \right| \leq \frac{\gamma_{n+1} \langle j \rangle}{\langle l \rangle^{\tau_1}} \right\}. \end{aligned}$$

Since

$$W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) \hookrightarrow C^{q-1}(\mathcal{O}, \mathbb{C}) \quad \text{and} \quad q = q_0 + 1,$$

one obtains for any $n \in \mathbb{N}$ and for any $(k, \ell) \in \{1, 2\}^2$, the C^{q_0} regularity of the curves

$$\begin{aligned} b & \mapsto \omega(b, \varepsilon) \cdot l + j \mathbf{c}_{k,n}(b, \varepsilon), & (l, j) & \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0, 0)\}, \\ b & \mapsto \omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,\ell,n}^{(\infty)}(b, \varepsilon), & (l, j, j_0) & \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,\ell}), \\ b & \mapsto \omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon), & (l, j) & \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}). \end{aligned}$$

Therefore, applying Lemma 8.1 together with Lemma 8.4 yields that for all $n \in \mathbb{N}$,

$$\begin{aligned} \forall |j| \leq C_0 \langle l \rangle, & \quad \left| \mathcal{R}_{l,j}^{(0,k)}(i_n) \right| \lesssim \gamma^{\frac{v}{q_0}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}}, \\ \forall |j| \leq C_0 \langle l \rangle, & \quad \left| \mathcal{R}_{l,j}^{(1,k)}(i_n) \right| \lesssim \gamma^{\frac{1}{q_0}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}}, \\ \forall |j - j_0| \leq C_0 \langle l \rangle, & \quad \left| \mathcal{R}_{l,j,j_0}^k(i_n) \right| \lesssim \gamma^{\frac{1}{q_0}} \langle j - j_0 \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_2 + 1}{q_0}}, \\ & \quad \left| \mathcal{R}_{l,j,j_0}^{1,2}(i_n) \right| \lesssim \gamma^{\frac{1}{q_0}} \langle l, j, j_0 \rangle^{-1 - \frac{\tau_2 + 1}{q_0}}. \end{aligned} \quad (8.74)$$

We first estimate the measure of \mathcal{S}_0 and \mathcal{S}_1 defined in (8.71). Their estimation cannot be done in a similar way to the other terms due to the range of validity of the estimate (8.80) obtained later in the proof of Lemma 8.2. From Lemma 8.3, we have some trivial inclusions allowing us to write for $n \in \{0, 1\}$,

$$\begin{aligned} \mathcal{S}_n & \lesssim \sum_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0,0)\} \\ |j| \leq C_0 \langle l \rangle, |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(0,k)}(i_n) \right| + \sum_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \\ |j| \leq C_0 \langle l \rangle, |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(1,k)}(i_n) \right| \\ & + \sum_{\substack{k \in \{1,2\} \\ (l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2 \\ |j-j_0| \leq C_0 \langle l \rangle, |l| \leq N_n \\ (l,j) \neq (0,j_0), \min(|j|, |j_0|) \leq c_2 \gamma_1^{-v} \langle l \rangle^{\tau_1}}} \left| \mathcal{R}_{l,j,j_0}^k(i_n) \right| + \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}) \\ (l,j,j_0) \leq N_n}} \left| \mathcal{R}_{l,j,j_0}^{1,2}(i_n) \right|. \end{aligned} \quad (8.75)$$

Inserting (8.74) into (8.75) implies that for $n \in \{0, 1\}$,

$$\begin{aligned} \mathcal{S}_n &\lesssim \gamma^{\frac{v}{q_0}} \sum_{\substack{(l,j) \in \mathbb{Z}^{d+1} \\ |j| \leq C_0 \langle l \rangle}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}} + \gamma^{\frac{1}{q_0}} \sum_{\substack{(l,j) \in \mathbb{Z}^{d+1} \\ |j| \leq C_0 \langle l \rangle}} \langle j \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}} \\ &+ \gamma^{\frac{1}{q_0}} \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^{d+2} \\ |j-j_0| \leq C_0 \langle l \rangle \\ \min(|j|, |j_0|) \leq c_2 \gamma_1^v \langle l \rangle^{\tau_1}}} \langle j-j_0 \rangle^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_2 + 1}{q_0}} + \gamma^{\frac{1}{q_0}} \sum_{(l,j,j_0) \in \mathbb{Z}^{d+2}} \langle l, j, j_0 \rangle^{-1 - \frac{\tau_2 + 1}{q_0}}. \end{aligned}$$

Notice that the conditions $|j - j_0| \leq C_0 \langle l \rangle$ and $\min(|j|, |j_0|) \leq c_2 \gamma_1^{-v} \langle l \rangle^{\tau_1}$ imply

$$\max(|j|, |j_0|) \leq \min(|j|, |j_0|) + |j - j_0| \leq c_2 \gamma_1^{-v} \langle l \rangle^{\tau_1} + C_0 \langle l \rangle \lesssim \gamma^{-v} \langle l \rangle^{\tau_1}. \quad (8.76)$$

Consequently, we have

$$\max_{n \in \{0,1\}} \mathcal{S}_n \lesssim \gamma^{\frac{1}{q_0}} \left(\sum_{l \in \mathbb{Z}^d} \langle l \rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-v} \sum_{l \in \mathbb{Z}^d} \langle l \rangle^{\tau_1 - 1 - \frac{\tau_2}{q_0}} + \sum_{(l,j,j_0) \in \mathbb{Z}^{d+2}} \langle l, j, j_0 \rangle^{-1 - \frac{\tau_2 + 1}{q_0}} \right) + \gamma^{\frac{v}{q_0}} \sum_{l \in \mathbb{Z}^d} \langle l \rangle^{-\frac{\tau_1}{q_0}}. \quad (8.77)$$

Observe that (8.67) implies

$$\min\left(\frac{v}{q_0} \frac{1}{q_0} - v\right) = \frac{v}{q_0}.$$

Now the constraints on τ_1 and τ_2 listed in (8.67) allow to make the series in (8.77) convergent and we get

$$\max_{n \in \{0,1\}} \mathcal{S}_n \lesssim \gamma^{\min\left(\frac{v}{q_0}, \frac{1}{q_0} - v\right)} = \gamma^{\frac{v}{q_0}}. \quad (8.78)$$

Let us now move to the estimate of \mathcal{S}_n for $n \geq 2$ defined by (8.71). Using Lemma 8.2 and Lemma 8.3, we infer

$$\begin{aligned} \mathcal{S}_n &\lesssim \sum_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0,0)\} \\ |j| \leq C_0 \langle l \rangle, N_{n-1} < |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(0,k)}(i_n) \right| + \sum_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \\ |j| \leq C_0 \langle l \rangle, N_{n-1} < |l| \leq N_n}} \left| \mathcal{R}_{l,j}^{(1,k)}(i_n) \right| \\ &+ \sum_{\substack{k \in \{1,2\} \\ (l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2 \\ |j-j_0| \leq C_0 \langle l \rangle, N_{n-1} < |l| \leq N_n \\ (l,j) \neq (0,j_0), \min(|j|, |j_0|) \leq c_2 \gamma_{n+1}^{-v} \langle l \rangle^{\tau_1}}} \left| \mathcal{R}_{l,j,j_0}^k(i_n) \right| + \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}) \\ N_{n-1} < (l,j,j_0) \leq N_n}} \left| \mathcal{R}_{l,j,j_0}^{1,2}(i_n) \right|. \end{aligned}$$

Similarly to (8.76), we have the implication

$$\left(\min(|j|, |j_0|) \leq c_2 \gamma_{n+1}^{-v} \langle l \rangle^{\tau_1} \quad \text{and} \quad |j - j_0| \leq C_0 \langle l \rangle \right) \Rightarrow \max(|j|, |j_0|) \lesssim \gamma^{-v} \langle l \rangle^{\tau_1}.$$

Hence, we deduce from (8.74) that for any $n \geq 2$

$$\mathcal{S}_n \lesssim \gamma^{\frac{1}{q_0}} \left(\sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-v} \sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{\tau_1 - 1 - \frac{\tau_2}{q_0}} + \sum_{\substack{(l,j,j_0) \in \mathbb{Z}^{d+2} \\ (l,j,j_0) > N_{n-1}}} \langle l, j, j_0 \rangle^{-1 - \frac{\tau_2 + 1}{q_0}} \right) + \gamma^{\frac{v}{q_0}} \sum_{\substack{l \in \mathbb{Z}^d \\ |l| > N_{n-1}}} \langle l \rangle^{-\frac{\tau_1}{q_0}}.$$

We deduce that the series of general term \mathcal{S}_n converges and

$$\sum_{n=2}^{\infty} \mathcal{S}_n \lesssim \gamma^{\frac{v}{q_0}} = \varepsilon^{\frac{av}{q_0}}. \quad (8.79)$$

Inserting (8.78) and (8.79) into (8.71) yields

$$\left| (b_*, b^*) \setminus \mathcal{C}_{\infty}^{\varepsilon} \right| \lesssim \varepsilon^{\frac{av}{q_0}}.$$

This proves the Proposition 8.2. □

We shall now prove Lemma 8.2 and Lemma 8.3 used in the proof of Proposition 8.2.

Lemma 8.2. *Let $n \in \mathbb{N} \setminus \{0, 1\}$ and $k \in \{1, 2\}$. Then the following assertions hold true.*

- (i) For $(l, j) \in \mathbb{Z}^d \times \mathbb{Z}_m$ with $(l, j) \neq (0, 0)$ and $|l| \leq N_{n-1}$, we get $\mathcal{R}_{l,j}^{(0,k)}(i_n) = \emptyset$.
- (ii) For $(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})$ with $|l| \leq N_{n-1}$, we get $\mathcal{R}_{l,j}^{(1,k)}(i_n) = \emptyset$.
- (iii) For $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$ with $|l| \leq N_{n-1}$ and $(l, j) \neq (0, j_0)$, we get $\mathcal{R}_{l,j,j_0}^k(i_n) = \emptyset$.
- (iv) For $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2})$ with $\langle l, j, j_0 \rangle \leq N_{n-1}$, we get $\mathcal{R}_{l,j,j_0}^{1,2}(i_n) = \emptyset$.
- (v) For any $n \in \mathbb{N} \setminus \{0, 1\}$,

$$\begin{aligned} \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = & \bigcup_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0,0)\} \\ N_{n-1} < |l| \leq N_n}} \mathcal{R}_{l,j}^{(0,k)}(i_n) \cup \bigcup_{\substack{k \in \{1,2\} \\ (l,j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k}) \\ N_{n-1} < |l| \leq N_n}} \mathcal{R}_{l,j}^{(1,k)}(i_n) \\ & \cup \bigcup_{\substack{k \in \{1,2\} \\ (l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2 \\ N_{n-1} < |l| \leq N_n \\ (l,j) \neq (0,j_0)}} \mathcal{R}_{l,j,j_0}^k(i_n) \cup \bigcup_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2}) \\ N_{n-1} < \langle l,j,j_0 \rangle \leq N_n}} \mathcal{R}_{l,j,j_0}^{1,2}(i_n). \end{aligned}$$

Proof. Observe that the point (v) follows immediately from (8.73) and the points (i), (ii), (iii) and (iv). The points (i), (ii) and (iii) can be proved similarly to [69, Lem. 7.1-(i)-(ii)-(iii)] based on the following estimate, obtained from (8.8).

$$\begin{aligned} \forall n \geq 2, \quad \|i_n - i_{n-1}\|_{\bar{s}_h + \bar{\sigma}}^{q,\gamma,\mathfrak{m}} & \leq \|U_n - U_{n-1}\|_{\bar{s}_h + \bar{\sigma}}^{q,\gamma,\mathfrak{m}} \\ & \leq \|H_n\|_{\bar{s}_h + \bar{\sigma}}^{q,\gamma,\mathfrak{m}} \\ & \leq C_* \varepsilon \gamma^{-1} N_{n-1}^{-a_2}. \end{aligned} \tag{8.80}$$

We mention that the required constraint on v stated in (8.67) appears in the skipped proofs. Now it remains to prove the point (iv).

(iv) Let $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2})$ such that $\langle l, j, j_0 \rangle \leq N_{n-1}$. It is sufficient to prove that

$$\mathcal{R}_{l,j,j_0}^{1,2}(i_n) \subset \mathcal{R}_{l,j,j_0}^{1,2}(i_{n-1}).$$

Indeed, if this inclusion holds, then by construction

$$\mathcal{R}_{l,j,j_0}^{1,2}(i_n) \subset (\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon) \cap (\mathcal{C}_{n-1}^\varepsilon \setminus \mathcal{C}_n^\varepsilon) = \emptyset.$$

Take $b \in \mathcal{R}_{l,j,j_0}^{1,2}(i_n) \subset \mathcal{C}_n^\varepsilon \subset \mathcal{C}_{n-1}^\varepsilon$. Then coming back to (8.73), we deduce from the triangle inequality that

$$\left| \omega(b, \varepsilon) \cdot l + \mu_{j,1,n-1}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right| \leq \frac{2\gamma_{n+1}}{\langle l,j,j_0 \rangle^{\tau_2}} + \varrho_{j,j_0,n}(b, \varepsilon), \tag{8.81}$$

where

$$\varrho_{j,j_0,n}(b, \varepsilon) \triangleq \left| \mu_{j,1,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n}^{(\infty)}(b, \varepsilon) - \mu_{j,1,n-1}^{(\infty)}(b, \varepsilon) + \mu_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right|.$$

Using the decomposition (8.72), we infer

$$\begin{aligned} \varrho_{j,j_0,n}(b, \varepsilon) & \leq |j| \left| r_{1,n}^{(0)}(b, \varepsilon) - r_{1,n-1}^{(0)}(b, \varepsilon) \right| + |j_0| \left| r_{2,n}^{(0)}(b, \varepsilon) - r_{2,n-1}^{(0)}(b, \varepsilon) \right| \\ & \quad + \left| r_{j,1,n}^{(\infty)}(b, \varepsilon) - r_{j,1,n-1}^{(\infty)}(b, \varepsilon) \right| + \left| r_{j_0,2,n}^{(\infty)}(b, \varepsilon) - r_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right|. \end{aligned} \tag{8.82}$$

Applying (7.36) together with (8.80), (8.2) and the fact that $\sigma_4 \geq \sigma_3$, we obtain

$$\begin{aligned} \left| r_{1,n}^{(0)}(b, \varepsilon) - r_{1,n-1}^{(0)}(b, \varepsilon) \right| + \left| r_{2,n}^{(0)}(b, \varepsilon) - r_{2,n-1}^{(0)}(b, \varepsilon) \right| & \lesssim \varepsilon \|i_n - i_{n-1}\|_{\bar{s}_h + \sigma_3}^{q,\gamma,\mathfrak{m}} \\ & \lesssim \varepsilon^2 \gamma^{-1} N_{n-1}^{-a_2} \\ & \lesssim \varepsilon^{2-a} N_{n-1}^{-a_2}. \end{aligned} \tag{8.83}$$

In the same way, we can apply (7.47) together with (8.80) and (8.2) to deduce

$$\begin{aligned} \left| r_{j,1,n}^{(\infty)}(b, \varepsilon) - r_{j,1,n-1}^{(\infty)}(b, \varepsilon) \right| + \left| r_{j_0,2,n}^{(\infty)}(b, \varepsilon) - r_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right| &\lesssim \varepsilon \gamma^{-1} \|i_n - i_{n-1}\|_{\frac{q}{s_h} + \sigma_4}^{\gamma, \mathfrak{m}} \\ &\lesssim \varepsilon^2 \gamma^{-2} N_{n-1}^{-a_2} \\ &\lesssim \varepsilon^{2(1-a)} \langle l, j, j_0 \rangle N_{n-1}^{-a_2}. \end{aligned} \quad (8.84)$$

Inserting (8.83) and (8.84) into (8.82) gives

$$q_{j,j_0,n}(b, \varepsilon) \lesssim \varepsilon^{2(1-a)} \langle l, j, j_0 \rangle N_{n-1}^{-a_2}. \quad (8.85)$$

Now putting together (8.81), (8.85) and the fact that $\gamma_{n+1} = \gamma_n - \varepsilon^a 2^{-n-1}$, we get

$$\left| \omega(b, \varepsilon) \cdot l + \mu_{j,1,n-1}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right| \leq \frac{2\gamma_n}{\langle l, j, j_0 \rangle^{\tau_2}} - \varepsilon^a 2^{-n} \langle l, j, j_0 \rangle^{-\tau_2} + C \varepsilon^{2(1-a)} \langle l, j, j_0 \rangle N_{n-1}^{-a_2}.$$

Added to the constraint $\langle l, j, j_0 \rangle \leq N_{n-1}$, we obtain

$$-\varepsilon^a 2^{-n} \langle l, j, j_0 \rangle^{-\tau_2} + C \varepsilon^{2(1-a)} N_{n-1}^{-a_2} \leq \varepsilon^a 2^{-n} \langle l, j, j_0 \rangle^{-\tau_2} \left(-1 + C \varepsilon^{2-3a} 2^n N_{n-1}^{-a_2 + \tau_2 + 1} \right).$$

Observe that our choice of parameters (8.1) and (8.2) gives in particular

$$a_2 > \tau_2 + 1 \quad \text{and} \quad a < \frac{2}{3}.$$

Hence, up to taking ε small enough, we infer

$$\forall n \in \mathbb{N}, \quad -1 + C \varepsilon^{2-3a} 2^n N_{n-1}^{-a_2 + \tau_2 + 1} \leq 0.$$

Consequently,

$$\left| \omega(b, \varepsilon) \cdot l + \mu_{j,1,n-1}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n-1}^{(\infty)}(b, \varepsilon) \right| \leq \frac{2\gamma_n}{\langle l, j, j_0 \rangle^{\tau_2}}.$$

This means that $b \in \mathcal{R}_{l,j,j_0}^{1,2}(i_{n-1})$. This concludes the proof of Lemma 8.2. \square

The following lemma provides necessary constraints between time and space Fourier modes so that the sets in (8.73) are not void.

Lemma 8.3. *Let $k \in \{1, 2\}$. There exists ε_0 such that for any $\varepsilon \in [0, \varepsilon_0]$ and $n \in \mathbb{N}$ the following assertions hold true.*

(i) *Let $(l, j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0, 0)\}$. If $\mathcal{R}_{l,j}^{(0,k)}(i_n) \neq \emptyset$, then $|j| \leq C_0 \langle l \rangle$.*

(ii) *Let $(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k})$. If $\mathcal{R}_{l,j}^{(1,k)}(i_n) \neq \emptyset$, then $|j| \leq C_0 \langle l \rangle$.*

(iii) *Let $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k})^2$. If $\mathcal{R}_{l,j,j_0}^k(i_n) \neq \emptyset$, then $|j - j_0| \leq C_0 \langle l \rangle$.*

(iv) *Let $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \overline{\mathbb{S}}_{0,k})^2$. There exists $c_2 > 0$ such that if $\min(|j|, |j_0|) \geq c_2 \gamma_{n+1}^{-v} \langle l \rangle^{\tau_1}$, then*

$$\mathcal{R}_{l,j,j_0}^k(i_n) \subset \mathcal{R}_{l,j-j_0}^{(0,k)}(i_n).$$

Proof. (i) Observe that the case $j = 0$ is trivial. Now, for $j \neq 0$ we assume that $\mathcal{R}_{l,j}^{(0,k)}(i_n) \neq \emptyset$. Then, there exists $b \in (b_*, b^*)$ such that

$$|\omega(b, \varepsilon) \cdot l + j \mathbf{c}_{k,n}(b, \varepsilon)| \leq \frac{4\gamma_{n+1}^v |j|}{\langle l \rangle^{\tau_1}}.$$

From triangle and Cauchy-Schwarz inequalities, (8.9), (8.2) and the fact that $(b, \varepsilon) \mapsto \omega(b, \varepsilon)$ is bounded, we deduce

$$\begin{aligned} |\mathbf{c}_{k,n}(b, \varepsilon)| |j| &\leq 4|j| \gamma_{n+1}^v \langle l \rangle^{-\tau_1} + |\omega(b, \varepsilon) \cdot l| \\ &\leq 4|j| \gamma_{n+1}^v + C \langle l \rangle \\ &\leq 8\varepsilon^{av} |j| + C \langle l \rangle. \end{aligned} \quad (8.86)$$

Now by construction (7.35), we can write

$$\mathbf{c}_{k,n}(b, \varepsilon) = \mathbf{v}_k(b) + r_{k,n}^{(0)}(b, \varepsilon).$$

Remark that by definition (4.13), we have

$$\inf_{k \in \{1,2\}} \inf_{b \in (b_*, b^*)} |\mathbf{v}_k(b)| \geq \Omega.$$

Together with (7.12) and Proposition 8.1-(P1)_n, this implies

$$\begin{aligned} \forall q' \in \llbracket 0, q \rrbracket, \quad \sup_{n \in \mathbb{N}} \sup_{b \in (b_*, b^*)} |\partial_b^{q'} r_{k,n}^{(0)}(b, \varepsilon)| &\leq \gamma^{-q'} \sup_{n \in \mathbb{N}} \|r_{k,n}^{(0)}\|^{q, \gamma} \\ &\lesssim \varepsilon \gamma^{-q'} \\ &\lesssim \varepsilon^{1-aq'}. \end{aligned} \quad (8.87)$$

Hence, choosing ε small enough implies

$$\inf_{k \in \{1,2\}} \inf_{n \in \mathbb{N}} \inf_{b \in (b_*, b^*)} |\mathbf{c}_{k,n}(b, \varepsilon)| \geq \frac{\Omega}{2}. \quad (8.88)$$

Inserting (8.88) into (8.86) yields

$$\left(\frac{\Omega}{2} - 8\varepsilon^{av}\right)|j| \leq C\langle l \rangle.$$

Thus, selecting ε small enough ensures that $|j| \leq C_0\langle l \rangle$ for some $C_0 > 0$.

(ii) The case $j = 0$ is obvious so we may treat the case where $j \neq 0$. Assume that $\mathcal{R}_{l,j}^{(1,k)}(i_n) \neq \emptyset$. Then, we can find $b \in (b_*, b^*)$ such that

$$|\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon)| \leq \frac{\gamma_{n+1}|j|}{\langle l \rangle^{\tau_1}}.$$

Applying the triangle and Cauchy-Schwarz inequalities together with (8.9) and (8.2) yields

$$\begin{aligned} |\mu_{j,k,n}^{(\infty)}(b, \varepsilon)| &\leq \gamma_{n+1}|j|\langle l \rangle^{-\tau_1} + |\omega(b, \varepsilon) \cdot l| \\ &\leq 2\varepsilon^a|j| + C\langle l \rangle. \end{aligned} \quad (8.89)$$

Now coming back to the structure of the eigenvalues in (8.72), then using the triangle inequality, we infer

$$|\mu_{j,k,n}^{(\infty)}(b, \varepsilon)| \geq |\Omega_{j,k}(b)| - |j| |r_{k,n}^{(0)}(b, \varepsilon)| - |r_{j,k,n}^{(\infty)}(b, \varepsilon)|. \quad (8.90)$$

Notice that (7.47) implies

$$\begin{aligned} \forall q' \in \llbracket 0, q \rrbracket, \quad \sup_{n \in \mathbb{N}} \sup_{b \in (b_*, b^*)} \sup_{j \in \bar{\mathcal{S}}_{0,k}} |\partial_b^{q'} r_{j,k,n}^{(\infty)}(b, \varepsilon)| &\leq \gamma^{-q'} \sup_{n \in \mathbb{N}} \sup_{j \in \bar{\mathcal{S}}_{0,k}} \|r_{j,k,n}^{(\infty)}\|^{q, \gamma} \\ &\lesssim \varepsilon \gamma^{-q'-1} \\ &\lesssim \varepsilon^{1-a(q'+1)}. \end{aligned} \quad (8.91)$$

Gathering (8.90), Lemma 4.6-3, (8.87) and (8.91), we obtain

$$|\mu_{j,k,n}^{(\infty)}(b, \varepsilon)| \geq \Omega|j| - C\varepsilon^{1-a}|j|. \quad (8.92)$$

Inserting (8.92) into (8.89) yields

$$(\Omega - C\varepsilon^{1-a} - 2\varepsilon^a)|j| \leq C\langle l \rangle.$$

Hence, taking ε small enough we obtain $|j| \leq C_0\langle l \rangle$, for some $C_0 > 0$.

(iii) Notice that for $j = j_0$ we have $\mathcal{R}_{l,j_0,j_0}^k(i_n) = \mathcal{R}_{l,0}^{(0,k)}(i_n)$. Hence this situation has already been

studied in the first point. Therefore, we shall consider $j \neq j_0$. Let us assume that $\mathcal{R}_{l,j,j_0}(i_n) \neq \emptyset$. We can find $b \in (b_*, b^*)$ such that

$$|\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)| \leq \frac{2\gamma_{n+1}|j-j_0|}{\langle l \rangle^{\tau_2}}.$$

Using one more time the triangle and Cauchy-Schwarz inequalities together with (8.9) and (8.2), we infer

$$\begin{aligned} |\mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)| &\leq 2\gamma_{n+1}|j-j_0|\langle l \rangle^{-\tau_2} + |\omega(b, \varepsilon) \cdot l| \\ &\leq 2\gamma_{n+1}|j-j_0| + C\langle l \rangle \\ &\leq 4\varepsilon^a|j-j_0| + C\langle l \rangle. \end{aligned}$$

On the other hand, the triangle inequality, Lemma 4.6-5, (8.87) and (8.91) give for ε small enough

$$\begin{aligned} |\mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)| &\geq |\Omega_{j,k}(b) - \Omega_{j_0,k}(b)| - |r_{k,n}^{(0)}(b, \varepsilon)||j-j_0| - |r_{j,k,n}^{(\infty)}(b, \varepsilon)| - |r_{j_0,k,n}^{(\infty)}(b, \varepsilon)| \\ &\geq (c - C\varepsilon^{1-a})|j-j_0| \\ &\geq \frac{c}{2}|j-j_0|. \end{aligned}$$

Putting together the foregoing inequalities yields

$$\left(\frac{c}{2} - 4\varepsilon^a\right)|j-j_0| \leq C\langle l \rangle.$$

Thus, for ε sufficiently small, we get $|j-j_0| \leq C_0\langle l \rangle$, for some $C_0 > 0$.

(iv) Observe that the case $j = j_0$ is trivial, so we may restrict our discussion to the case $j \neq j_0$. Now, according to the symmetry property $\mu_{-j,k,n}^{(\infty)}(b, \varepsilon) = -\mu_{j,k,n}^{(\infty)}(b, \varepsilon)$ we can assume, without loss of generality that $0 < j < j_0$. Take $b \in \mathcal{R}_{l,j,j_0}^k(i_n)$. Then by definition, we have

$$|\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) \pm \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)| \leq \frac{2\gamma_{n+1}\langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_2}}. \quad (8.93)$$

Recall from (4.13) and (4.29) the decompositions for $j > 0$,

$$\begin{aligned} \mathbf{v}_k(b) &= \Omega + (2-k)\frac{1-b^2}{2}, \\ \Omega_{j,k}(b) &= j\mathbf{v}_k(b) + \frac{(-1)^k}{2} + (-1)^{k+1}\mathbf{r}_j(b). \end{aligned}$$

Therefore, by the triangle inequality, we get

$$\begin{aligned} |\omega(b, \varepsilon) \cdot l + (j \pm j_0)\mathbf{c}_{k,n}(b, \varepsilon)| &\leq |\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) \pm \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)| + \frac{1}{2}(1 \pm 1) \\ &\quad + |\mathbf{r}_j(b) \pm \mathbf{r}_{j_0}(b)| + |r_{j,k,n}^{(\infty)}(b, \varepsilon) \pm r_{j_0,k,n}^{(\infty)}(b, \varepsilon)|. \end{aligned} \quad (8.94)$$

First, it is obvious that

$$1 \pm 1 \leq \frac{\langle j \pm j_0 \rangle}{j}. \quad (8.95)$$

Second, the estimate (4.30) implies in particular

$$\begin{aligned} |\mathbf{r}_j(b) \pm \mathbf{r}_{j_0}(b)| &\leq C(|j|^{-1} + |j_0|^{-1}) \\ &\leq C\frac{\langle j \pm j_0 \rangle}{j}. \end{aligned} \quad (8.96)$$

Third, the estimate (7.47) together with (8.2) give

$$\begin{aligned} |r_{j,k,n}^{(\infty)}(b, \varepsilon) \pm r_{j_0,k,n}^{(\infty)}(b, \varepsilon)| &\leq C\varepsilon^{1-a}(|j|^{-1} + |j_0|^{-1}) \\ &\leq C\frac{\langle j \pm j_0 \rangle}{j}. \end{aligned} \quad (8.97)$$

Gathering (8.93), (8.94), (8.95), (8.96) and (8.97) yields

$$|\omega(b, \varepsilon) \cdot l + (j \pm j_0)c_{k,n}(b, \varepsilon)| \leq \frac{2\gamma_{n+1}\langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_2}} + C \frac{\langle j \pm j_0 \rangle}{j}.$$

Hence, using the assumptions $j \geq \frac{C}{2}\gamma_{n+1}^{-\nu}\langle l \rangle^{\tau_1}$ and $\tau_2 > \tau_1$, we infer

$$|\omega(b, \varepsilon) \cdot l + (j \pm j_0)c_{k,n}(b, \varepsilon)| \leq \frac{4\gamma_{n+1}^{\nu}\langle j \pm j_0 \rangle}{\langle l \rangle^{\tau_1}}.$$

This gives the desired result and ends the proof of Lemma 8.3, up to defining $c_2 \triangleq \frac{C}{2}$. \square

The next and last lemma is concerned by the transversality property of the perturbed frequency vector $\omega(b, \varepsilon)$ given by Corollary 8.1.

Lemma 8.4. *Let q_0 , C_0 and ρ_0 as in Lemma 4.8. There exist $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in [0, \varepsilon_0]$ the following assertions hold true.*

(i) *For all $l \in \mathbb{Z}^d \setminus \{0\}$, we have*

$$\inf_{b \in [b_*, b^*]} \max_{q' \in [0, q_0]} |\partial_b^{q'} (\omega(b, \varepsilon) \cdot l)| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

(ii) *For all $(l, j) \in \mathbb{Z}^d \times \mathbb{Z}_m \setminus \{(0, 0)\}$ such that $|j| \leq C_0 \langle l \rangle$, we have*

$$\forall n \in \mathbb{N}, \quad \inf_{b \in [b_*, b^*]} \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega(b, \varepsilon) \cdot l + jc_{k,n}(b, \varepsilon)) \right| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

(iii) *For all $(l, j) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})$ such that $|j| \leq C_0 \langle l \rangle$, we have*

$$\forall n \in \mathbb{N}, \quad \inf_{b \in [b_*, b^*]} \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon)) \right| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

(iv) *For all $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,k})^2$ such that $|j - j_0| \leq C_0 \langle l \rangle$, we have*

$$\forall n \in \mathbb{N}, \quad \inf_{b \in [b_*, b^*]} \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega(b, \varepsilon) \cdot l + \mu_{j,k,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,k,n}^{(\infty)}(b, \varepsilon)) \right| \geq \frac{\rho_0 \langle l \rangle}{2}.$$

(v) *For all $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,1}) \times (\mathbb{Z}_m \setminus \bar{\mathbb{S}}_{0,2})$, we have*

$$\forall n \in \mathbb{N}, \quad \inf_{b \in [b_*, b^*]} \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega(b, \varepsilon) \cdot l + \mu_{j,1,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n}^{(\infty)}(b, \varepsilon)) \right| \geq \frac{\rho_0 \langle l, j, j_0 \rangle}{2}.$$

Proof. The points (i), (ii), (iii) and (iv) are obtained following closely [69, Lem. 7.3]. The estimates are obtained by a perturbative argument from the equilibrium transversality conditions proved in Lemma 4.8-1-2-3-4. Therefore, it remains to prove the point (v).

(v) Using the decompositions (7.35)-(7.46)-(8.63) together with (8.87), (8.91) and Lemma 4.8-5, we get for ε sufficiently small

$$\begin{aligned} \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega(b, \varepsilon) \cdot l + \mu_{j,1,n}^{(\infty)}(b, \varepsilon) - \mu_{j_0,2,n}^{(\infty)}(b, \varepsilon)) \right| &\geq \max_{q' \in [0, q_0]} \left| \partial_b^{q'} (\omega_{\text{Eq}}(b) \cdot l + \Omega_{j,1}(b) - \Omega_{j_0,2}(b)) \right| \\ &\quad - \max_{q' \in [0, q]} \left| \partial_b^{q'} (\bar{\Gamma}_\varepsilon(b) \cdot l + jr_{1,n}^{(0)}(b, \varepsilon) - j_0 r_{2,n}^{(0)}(b, \varepsilon) + r_{j,1,n}^{(\infty)}(b, \varepsilon) - r_{j_0,2,n}^{(\infty)}(b, \varepsilon)) \right| \\ &\geq \rho_0 \langle l, j, j_0 \rangle - C\varepsilon^{1-a(1+q+\bar{q}a)} \langle l, j, j_0 \rangle - C\varepsilon^{1-a(1+q)} \langle l, j, j_0 \rangle \\ &\geq \frac{\rho_0 \langle l, j, j_0 \rangle}{2}. \end{aligned}$$

This ends the proof of Lemma 8.4. \square

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