Boundary effects on the emergence of quasi-periodic solutions for Euler equations

Zineb Hassainia* and Emeric Roulley†

Abstract

In this paper, we highlight the importance of the boundary effects on the construction of quasi-periodic vortex patches solutions close to Rankine vortices and whose existence is not known in the whole space due to the resonances of the linear frequencies. Availing of the lack of invariance by radial dilation of Euler equations in the unit disc and using a Nash-Moser implicit function iterative scheme we show the existence of such structures when the radius of the Rankine vortex belongs to a suitable massive Cantor-like set with almost full Lebesgue measure.

Contents

*NYUAD Research Institute, New York University Abu Dhabi, PO Box 129188, Abu Dhabi, United Arab Emirates. E-mail address: zh14@nyu.edu
†Univ Rennes, CNRS, IRMAR – UMR 6625, F-35000 Rennes, France
E-mail address : emeric.roulley@univ-rennes1.fr

1 Introduction

We consider the Euler system set in a domain $D \subset \mathbb{R}^2$ written in the velocity-vorticity formulation

$$\begin{cases} \partial_t \boldsymbol{\omega} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = 0 & \text{in } \mathbb{R}_+ \times D \\ \mathbf{v} = \nabla^\perp \boldsymbol{\Psi} & \\ \boldsymbol{\omega}(0, \cdot) = \boldsymbol{\omega}_0 & \text{in } D \end{cases} \tag{1.1}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$. Here, we are particularly interested in the cases where D is the full plane, $D = \mathbb{R}^2$, or the unit disc $D = \mathbb{D} := \Big\{(x_1, x_2) \in \mathbb{R}^2 \quad \text{s.t.} \quad x_1^2 + x_2^2 \leqslant 1\Big\}$. First consider the whole space dynamics. In this case, the stream function is given by

$$\forall w \in \mathbb{R}^2, \quad \boldsymbol{\Psi}(t, w) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log\big(|w - \xi|\big) \boldsymbol{\omega}(t, \xi) dA(\xi),$$

where dA is the planar Lebesgue measure. The global existence and uniqueness for weak solutions bounded and integrable follows from Yudovich's theory [64]. In particular, if the initial datum is a vortex patch, that is, the characteristic function of a smooth bounded planar domain D_0, then the solution keeps a patch form $\mathbf{1}_{D_t}$ for any time $t > 0$, where D_t is the transported domain D_0 by the flow map associated to $\mathbf{v}$, namely

$$D_t = \boldsymbol{\Phi}_t(D_0), \quad \partial_t \boldsymbol{\Phi}_t(x) = \mathbf{v}(t, \boldsymbol{\Phi}_t(x)), \quad \boldsymbol{\Phi}_0 = \mathrm{Id}_{\mathbb{R}^2}.$$

The persistence of the regularity of the boundary was proved in [17, 22]. Notice that any radial profile generates a stationary solution. It is a classical fact to look for periodic or quasi-periodic solutions close to these equilibrium state solutions for Hamiltonian systems like (1.1). A particular class of periodic solutions is given by the rigid body rotating vortex patches around the origin described by

$$D_t = e^{\mathrm{i}\Omega t} D_0,$$

where Ω is the time independent angular velocity. Such solutions are called V-states according to the terminology introduced by Deem and Zabusky in [25]. The first explicit example, discovered by Kirchhoff in [50], is the ellipse which rotates about its center of mass with the constant angular velocity

$$\Omega = \frac{ab}{(a+b)^2},$$

where a and b are the semi-axes of the ellipse. Further families of implicit solutions with higher symmetries were established by Burbea in [20] using local bifurcation tools and complex analysis. More precisely, he proved the existence of branches of $\mathbf{m}$-fold rotating solutions bifurcating from the discs at angular velocities

$$\boldsymbol{\Omega}_{\mathbf{m}} = \frac{\mathbf{m} - 1}{2\mathbf{m}}, \quad \mathbf{m} \geqslant 1. \tag{1.2}$$

Notice that the mode $\mathbf{m} = 1$ corresponds to a translation of the trivial solution and the second branch, emerging at $\boldsymbol{\Omega}_2 = \frac{1}{4}$, describes the Kirchhoff ellipses. Moreover, all the bifurcation angular velocities $\boldsymbol{\Omega}_{\mathbf{m}}$ are in the range $(0, \frac{1}{2})$. Outside this interval, the only uniformly rotating solutions are the radial ones, as proved in the series of papers [30, 34, 40]. The boundary regularity was first discussed in [21, 38, 45] and the global bifurcation diagram was studied in [38]. Note also that countable branches of rotating patches bifurcating from the ellipses at implicit angular velocities were found in [42], however, the shapes have in fact less symmetry and being at most two-folds. It is worthy to point out that Burbea's approach has been extended in the past few years to different topological structures for the V-states and for various nonlinear transport equations. For instance, we mention the existence results for the multiply-connected patches [41, 43, 46] or the multipole vortex patch patches obtained by desingularization of the point vortex system [32, 33, 35, 39, 44]. Let us now turn to the case where Euler equations (1.1) are set in the unit disc. In this setting, the stream function $\boldsymbol{\Psi}$ solves the following Dirichlet problem in the unit disc $\mathbb{D}$

$$\begin{cases} \Delta\boldsymbol{\Psi} = \boldsymbol{\omega} \\ \boldsymbol{\Psi}|_{\partial\mathbb{D}} = 0. \end{cases}$$

Thus, by using the Green function of the unit disc, we get the expression

$$\forall w \in \mathbb{D}, \quad \boldsymbol{\Psi}(t,w) = \frac{1}{4\pi}\int_{\mathbb{D}} \log\left(\left|\frac{w-\xi}{1-w\overline{\xi}}\right|^2\right)\boldsymbol{\omega}(t,\xi)dA(\xi). \tag{1.3}$$

The theory of weak solutions and vortex patches is still valid in this context and the persistence of the boundary regularity of vortex patches remains true, as proved in [26]. The existence of V-states close to the discs $b\mathbb{D}$ ($b \in (0,1)$), also called Rankine vortices, were obtained in [36]. These curves of solutions have $\mathbf{m}$-fold symmetry, perform a uniform rotation and emerge at the angular velocities

$$\boldsymbol{\Omega}_{\mathbf{m}}(b) = \frac{\mathbf{m} - 1 + b^{2\mathbf{m}}}{2\mathbf{m}}, \quad \mathbf{m} \geqslant 1. \tag{1.4}$$

It is of paramount importance to highlight different boundary effects observable at this periodic level. First, Burbea's frequencies (1.2) are shifted to the right, implying in particular that the **1**-fold patches, which are not centered at the origin, are no longer associated to the trivial solution. Second, the numerical observations in [36] show that the bifurcation curves have oscillations.

The purpose of the current paper is to prove the existence of time quasi-periodic patch structures close to Rankine vortices. Remind that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *quasi-periodic* if there exist a frequency vector $\omega \in \mathbb{R}^d$ ($d \in \mathbb{N}^*$) which is non-degenerate, that is

$$\forall\, l \in \mathbb{Z}^d \setminus \{0\}, \quad \omega \cdot l \neq 0 \tag{1.5}$$

and a function $F : \mathbb{T}^d \to \mathbb{R}$, where $\mathbb{T}^d$ denotes the flat torus of dimension d, such that

$$\forall\, t \in \mathbb{R}, \quad f(t) = F(\omega t).$$

Remark that the case $d = 1$ corresponds to the definition of periodic functions with frequency $\omega \in \mathbb{R}^*$. The variable of F living in $\mathbb{T}^d$ is denoted φ. The existence of quasi-periodic vortex patch solutions has been initiated very recently in [15, 37, 47] using KAM techniques and Nash-Moser theory. We emphasize that in the papers [37, 47], respectively devoted to the generalized surface quasi-geostrophic equations and quasi-geostrophic shallow-water equations, the authors proved the existence of quasi-periodic patches close to Rankine vortices for suitable selected values of the exterior parameters offered by the equations. The situation for Euler equations in the whole plane is quite delicate and the search of quasi-periodic solutions near the discs is not clear due to the resonances of the linear frequencies (1.2) and the absence of an exterior parameter. However, in [15], the authors show the existence of quasi-periodic solutions for Euler equations close to the ellipses and the parameter used there is the aspect ratio of the ellipse. In the same spirit, we aim here to take advantage of the lack of invariance by radial dilation to create a natural geometrical parameter b describing a family of stationary solutions.

Before stating our main theorem, we shall briefly recall some results related to the use of KAM theory in PDE. Notice that KAM theory is named after Kolmogorov [54], Arnold [2] and Moser [56] works where they proved, for both the analytic and the finitely many differentiable cases, the persistence of invariant tori supporting quasi-periodic motions under a small perturbation of integrable finite dimensional Hamiltonian systems. We mention that in the differentiable case, Moser used a modified version of a regularizing Newton method developed by Nash for the isometric embedding problem [57]; commonly known as Nash-Moser scheme. KAM theory was extended and refined for several Hamiltonian PDE with small divisors problems. For instance, it has been implemented for the 1-d semilinear wave and Schrödinger equations in several papers [19, 23, 24, 51, 58, 59, 63]. Many results were also obtained for semi-linear perturbations of PDE [9, 10, 18, 27, 31, 49, 52, 53, 55]. However the case of quasi-linear or fully nonlinear perturbations were explored in [4, 5, 6, 11, 12, 29]. Many interesting results have also been obtained in the past few years on the periodic and quasi-periodic settings for the water-waves equations as in [1, 3, 13, 14, 16, 48, 60]. Very recently, a quasi-periodic forcing term was used in [7] to generate quasi-periodic solutions for 3D Euler equations. In the current context, we have no any forcing to use, and we rather use the internal structure to find quasi-periodic solutions.

We shall now present the main result of this work and discuss the key ideas of its proof. We first consider a polar parametrization of a patch boundary close to the stationary solution $b\mathbb{D}$, namely

$$z(t,\theta)=R(b,t,\theta)e^{\mathrm{i}\theta},\quad R(b,t,\theta)=\sqrt{b^2+2r(t,\theta)}.$$

The quantity of interest is the radial deformation r assumed to be of small size. We emphasize that our ansatz is slightly different from the one in the papers [15, 37, 47] where the parametrization is written in a rotating frame with an angular velocity Ω to remedy to the degeneracy of the first frequency. This is not the case in our context due to the non-degeneracy of the first frequency according to (1.4). As explained in Lemma 2.1 and Proposition 2.1, the radial deformation solves a non-linear and non-local transport PDE which admits a Hamiltonian formulation in the form

$$\partial_t r=-\tfrac{1}{2}\partial_\theta\nabla E(r), \tag{1.6}$$

where E is the kinetic energy related to the stream function given by (1.3). In view of Lemma 3.1, the linearized operator at a state r close to the Rankine patch $b\mathbb{D}$ takes the form

$$\mathcal{L}_r=\partial_t+\partial_\theta\Big(V_r\cdot+\mathbf{L}_r-\mathbf{S}_r\Big), \tag{1.7}$$

where

$$\begin{aligned}V_r(b,t,\theta)=&-\tfrac{1}{2}\int_{\mathbb{T}}\tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)}d\eta-\tfrac{1}{R(b,t,\theta)}\int_{\mathbb{T}}\log\big(A_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta\\&-\tfrac{1}{R^3(b,t,\theta)}\int_{\mathbb{T}}\log\big(B_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta,\end{aligned}$$

$\mathbf{L}_r$ is a non-local operator in the form

$$\mathbf{L}_r(\rho)(b,t,\theta)=\int_{\mathbb{T}}\rho(t,\eta)\log\left(A_r(b,t,\theta,\eta)\right)d\eta,\quad A_r(b,t,\theta,\eta)=\big|R(b,t,\theta)e^{\mathrm{i}\theta}-R(b,t,\eta)e^{\mathrm{i}\eta}\big|$$

and $\mathbf{S}_r$ is a smoothing non-local operator in the form

$$\mathbf{S}_r(\rho)(b,t,\theta)=\int_{\mathbb{T}}\rho(t,\eta)\log\left(B_r(b,t,\theta,\eta)\right)d\eta,\quad B_r(b,t,\theta,\eta)=\big|1-R(b,t,\theta)R(b,t,\eta)e^{\mathrm{i}(\eta-\theta)}\big|.$$

The operator $\mathbf{L}_r$ is of order zero and reflects the planar Euler action. Moreover, we observe two boundary effects of $\mathbb{D}$. The first one is quasi-linear in the transport part through the last term of V_r, but with a smoothing action. The second one is given by the operator $\mathbf{S}_r$ which is smoothing since it involves a smooth kernel. At the equilibrium state $r=0$, the linearized operator is a Fourier multiplier given by

$$\mathcal{L}_0=\partial_t+\tfrac{1}{2}\partial_\theta+\partial_\theta\mathcal{K}_{1,b}*\cdot-\partial_\theta\mathcal{K}_{2,b}*\cdot,$$

where

$$\mathcal{K}_{1,b}(\theta)=\tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right)\quad\text{and}\quad\mathcal{K}_{2,b}(\theta)=\log\left(|1-b^2e^{\mathrm{i}\theta}|\right).$$

Notice that the convolution with the kernel $\partial_\theta\mathcal{K}_{1,b}$ is exactly the Hilbert transform in the periodic setting. From direct computations, we may show that the kernel of $\mathcal{L}_0$ is given by the set of functions in the form

$$(t,\theta)\mapsto\sum_{j\in\mathbb{Z}^*}r_je^{\mathrm{i}(j\theta-\Omega_j(b)t)},$$

where

$$\forall j \in \mathbb{Z}\backslash\{0\}, \quad \Omega_j(b) = \tfrac{sgn(j)}{2}\big(|j| - 1 + b^{2|j|}\big), \tag{1.8}$$

where we denote by sgn the sign function. Notice that here and in the sequel, we shall use the notation

$$\mathbb{N} = \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathbb{N}^* = \{1, 2, \ldots\}.$$

Consider a finite number of Fourier modes

$$\mathbb{S} = \big\{j_1, \ldots, j_d\big\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < \ldots < j_d, \quad (d \in \mathbb{N}^*).$$

Then, from Proposition 3.1, we deduce that, for any $0 < b_0 < b_1 < 1$, for almost all $b \in [b_0, b_1]$, any function in the form

$$r : (t, \theta) \mapsto \sum_{j \in \mathbb{S}} r_j \cos(j\theta - \Omega_j(b)t), \quad r_j \in \mathbb{R}$$

is a quasi-periodic solution with frequency $\omega_{\mathrm{Eq}}(b) = (\Omega_j(b))_{j\in\mathbb{S}}$ of the equation $\mathcal{L}_0 r = 0$ which is reversible, namely $r(-t, -\theta) = r(t, \theta)$. The measure of the Cantor set in b generating these solutions is estimated using Rüssemann Lemma 3.6 requiring a lower bound on the maximal derivative of a given function up to order q_0. In our case, the value of q_0 is explicit, namely $q_0 = 2j_d + 2$ which is due to the polynomial structure of the $\Omega_j(b)$. The aim of this work is to prove that these structures persist at the non-linear level, more precisely, our main result reads as follows.

Theorem 1.1. *Let $0 < b_0 < b_1 < 1$, $d \in \mathbb{N}^*$ and $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. There exists $\varepsilon_0 \in (0, 1)$ small enough with the following properties : For every amplitudes $\mathtt{a} = (\mathtt{a}_j)_{j\in\mathbb{S}} \in (\mathbb{R}_+^*)^d$ satisfying*

$$|\mathtt{a}| \leqslant \varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty \subset (b_0, b_1)$ with asymptotically full Lebesgue measure as $\mathtt{a} \to 0$, i.e.

$$\lim_{\mathtt{a}\to 0} |\mathcal{C}_\infty| = b_1 - b_0,$$

such that for any $b \in \mathcal{C}_\infty$, the equation (1.6) admits a time quasi-periodic solution with diophantine frequency vector $\omega_{\mathrm{pe}}(b, \mathtt{a}) := (\omega_j(b, \mathtt{a}))_{j\in\mathbb{S}} \in \mathbb{R}^d$ and taking the form

$$r(t, \theta) = \sum_{j\in\mathbb{S}} \mathtt{a}_j \cos\big(j\theta + \omega_j(b, \mathtt{a})t\big) + \mathtt{p}\big(\omega_{\mathrm{pe}}(b, \mathtt{a})t, \theta\big),$$

with

$$\omega_{\mathrm{pe}}(b, \mathtt{a}) \underset{\mathtt{a}\to 0}{\longrightarrow} (-\Omega_j(b))_{j\in\mathbb{S}},$$

where $\Omega_j(b)$ are the equilibrium frequencies defined in (1.8) and the perturbation $\mathtt{p} : \mathbb{T}^{d+1} \to \mathbb{R}$ is an even function satisfying

$$\|\mathtt{p}\|_{H^s(\mathbb{T}^{d+1},\mathbb{R})} \underset{\mathtt{a}\to 0}{=} o(|\mathtt{a}|)$$

for some large index of regularity s.

We shall now sketch the main steps used to prove the previous theorem. First remark that small divisors problems already appear in the proof of Proposition 3.1 to find quasi-periodic structures at the linear level from the equilibrium. We can invert the linearized operator at the equilibrium with some fixed loss of regularity. Hence, we need to use a Nash-Moser scheme to find quasi-periodic solutions for the non-linear model. To do so, we must invert the linearized operator in a neighborhood of the equilibrium state. Since $\mathcal{L}_r$ has non constant coefficients, the task is more delicate. The basic idea consists in diagonalizing, namely to conjugate to constant coefficients operator. Actually, we may follow the procedure presented in [11], slightly modified in [37, 47], where the dynamics is decoupled into tangential and normal parts. On the tangential modes, we introduce action-angles variables (I, ϑ)

allowing to reformulate the problem in terms of embedded tori. More precisely, we shall look for the zeros of the following functional

$$\mathcal{F}(i,\alpha,b,\omega,\varepsilon)=\begin{pmatrix}\omega\cdot\partial_\varphi\vartheta(\varphi)-\alpha-\varepsilon\partial_I\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi I(\varphi)+\varepsilon\partial_\vartheta\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi z(\varphi)-\partial_\theta\big[\mathrm{L}(b)z(\varphi)+\varepsilon\nabla_z\mathcal{P}_\varepsilon\big(i(\varphi)\big)\big]\end{pmatrix}.$$

It turns out that it is more convenient to introduce one degree of freedom through the parameter α which provides at the end of the scheme a solution for the original problem when it is fixed to $-\omega_{\mathrm{Eq}}(b)$. Given any small reversible embedded torus $i_0:\varphi\mapsto(\vartheta_0(\varphi),I_0(\varphi),z_0(\varphi))$ and any $\alpha_0\in\mathbb{R}^d$, conjugating the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ via a suitable linear diffeomorphism of the toroidal phase space $\mathbb{T}^d\times\mathbb{R}^d\times L^2_\perp$, we obtain a triangular system in the action-angle-normal variables up to error terms. To solve the triangular system, we only have to invert the linearized operator in the normal directions, which is denoted by $\widehat{\mathcal{L}}_\omega$. This is done using KAM reducibility techniques in a similar way to [3, 16, 37, 47]. According to Proposition 6.1, we can write

$$\widehat{\mathcal{L}}_\omega=\Pi_{\mathbb{S}_0}^\perp\Big(\mathcal{L}_{\varepsilon r}-\varepsilon\partial_\theta\mathcal{R}\Big)\Pi_{\mathbb{S}_0}^\perp,$$

where $\Pi_{\mathbb{S}_0}^\perp$ is the projector in the normal directions, $\mathcal{R}$ is an integral operator and $\mathcal{L}_{\varepsilon r}$ is defined by (1.7). First, following the KAM reducibility scheme in [7, 28, 47], we can reduce the transport part and the zero order part by conjugating by a quasi-periodic symplectic invertible change of variables in the form

$$\mathscr{B}\rho(\mu,\varphi,\theta)=\Big(1+\partial_\theta\beta(\mu,\varphi,\theta)\Big)\rho\big(\mu,\varphi,\theta+\beta(\mu,\varphi,\theta)\big).$$

More precisely, as stated in Proposition 6.2, we can find a function $V_{i_0}^\infty=V_{i_0}^\infty(b,\omega)$ and a Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|l|\leqslant N_n}}\Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+jV_{i_0}^\infty(b,\omega)\big|>\tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}$$

in which the following decomposition holds

$$\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_{1,b}*\cdot-\partial_\theta\mathcal{K}_{2,b}*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0,$$

where $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving Toeplitz in time integral operator enjoying good smallness properties. The operator $\mathtt{E}_n^0$ is an error term of order one associated to the time troncation of the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$. Notice that N_n is defined by

$$N_n=N_0^{(\frac{3}{2})^n}\quad\text{with}\quad N_0\gg1.$$

Then, we project in the normal directions by considering the operator

$$\mathscr{B}_\perp=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.$$

Therefore, in view of Proposition 6.3, we obtain the following decomposition in $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\omega\cdot\partial_\varphi+\mathscr{D}_0+\mathscr{R}_0+\mathtt{E}_n^1:=\mathscr{L}_0+\mathtt{E}_n^1,$$

where $\mathscr{D}_0=(\mathrm{i}\mu_j^0(b,\omega))_{j\in\mathbb{S}_0^c}$ is a diagonal and reversible operator and $\mathscr{R}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ is a real and reversible Toeplitz in time remainder integral operator in $OPS^{-\infty}$ in space and satisfying nice smallness properties. The term $\mathtt{E}_n^1$ plays a similar role as the previous one $\mathtt{E}_n^0$. The next goal is to reduce the remainder term $\mathscr{R}_0$. For this aim, we implement a KAM reduction process in the Toeplitz topology as in [47, Prop. 6.5]. The result is stated in Proposition 6.4 and provides two operators Φ_∞ and $\mathscr{D}_\infty=(\mathrm{i}\mu_j^\infty(b,\omega))_{j\in\mathbb{S}_0^c}$, with $\mathscr{D}_\infty$ a diagonal and reversible operator whose spectrum is described by

$$\forall j\in\mathbb{S}_0^c,\quad\mu_j^\infty(b,\omega)=\Omega_j(b)+j\big(V_{i_0}^\infty(b,\omega)-\tfrac{1}{2}\big)+r_j^\infty(b,\omega),$$

such that in the Cantor set

$$\mathscr{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0) = \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ \langle l,j-j_0\rangle\leqslant N_n\\ (l,j)\neq(0,j_0)}} \Big\{(b,\omega)\in\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\quad\text{s.t.}\quad \big|\omega\cdot l+\mu_j^\infty(b,\omega)-\mu_{j_0}^\infty(b,\omega)\big|>\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}$$

the following decomposition holds

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty=\omega\cdot\partial_\varphi+\mathscr{D}_\infty+\mathtt{E}_n^2:=\mathscr{L}_\infty+\mathtt{E}_n^2.$$

Now, we can invert the operator $\mathscr{L}_\infty$ when the parameters are restricted to the Cantor set

$$\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}}\Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^\infty(b,\omega)\big|>\tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

Therefore, we are able to construct an approximate right inverse of $\widehat{\mathcal{L}}_\omega$ in the Cantor set

$$\mathcal{G}_n^\gamma(i_0)=\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathscr{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)\cap\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0).$$

We refer to Proposition 6.5 for more details. Now we can implement a Nash-Moser scheme in a similar way to [16, 37, 47] to find a solution $(b,\omega)\mapsto(i_\infty(b,\omega),\alpha_\infty(b,\omega))$ to the equation $\mathcal{F}(i,\alpha,b,\omega,\varepsilon)=0$ provided that the parameters (b,ω) are selected among a Cantor set $\mathcal{G}^\gamma_\infty$ which is constructed as the intersection of all the Cantor sets appearing in the scheme to invert at each step the linearized operator. To find a solution to the original problem we construct a frequency curve $b\mapsto\omega(b,\varepsilon)$ implicitly defined by solving the equation

$$\alpha_\infty(b,\omega(b,\varepsilon))=-\omega_{\mathrm{Eq}}(b).$$

Hence, we obtain the desired result for any value of b in the Cantor set

$$\mathcal{C}^\varepsilon_\infty=\Big\{b\in(b_0,b_1)\quad\text{s.t.}\quad(b,\omega(b,\varepsilon))\in\mathcal{G}^\gamma_\infty\Big\}.$$

Then, it remains to check that this set is non-trivial. This is done by estimating its measure using perturbed Rüssemann conditions from the equilibrium. In Proposition 7.2, we find a lower bound for the measure of $\mathcal{C}^\varepsilon_\infty$, namely

$$|\mathcal{C}^\varepsilon_\infty|\geqslant(b_1-b_0)-C\varepsilon^\delta\quad\text{for some }\delta=\delta(q_0,d,\tau_1,\tau_2)>0.$$

Notations. Along this paper we shall make use of the following parameters and sets.

- We denote by

$$\mathbb{N}:=\{0,1,\cdots\},\qquad\mathbb{Z}:=\{\cdots,-1,0,1,\cdots\}$$

  the set of natural numbers and the set of integers, respectively, and we set

$$\mathbb{N}^*:=\mathbb{N}\backslash\{0\},\qquad\mathbb{Z}^*:=\mathbb{Z}\backslash\{0\}.$$

- The integer d is the number of excited frequencies that will generate the quasi-periodic solutions. This is the dimension of the space where lies the frequency vector $\omega\in\mathbb{R}^d$, that will be a perturbation of the equilibrium frequency $\omega_{\mathrm{Eq}}(b)$, defined by (3.28).

- The real numbers b_0 and b_1 are fixed such that

$$0<b_0<b_1<1.$$

  The parameter b is the radius of the disc corresponding the the equilibrium state and lies in the interval (b_0,b_1). However at the end it will belong to a Cantor set for which invariant torus can be constructed.

- Since the application $b \mapsto \omega_{\rm Eq}(b)$ is continuous then $\omega_{\rm Eq}([b_0,b_1])$ is a compact subset of $\mathbb{R}^d$. In particular, there exists $R_0>0$ such that

$$\omega_{\rm Eq}((b_0,b_1)) \subset \mathscr{U} := B(0,R_0).$$

We also consider $\mathcal{O}$ the open bounded subset of $\mathbb{R}^{d+1}$ defined by

$$\mathcal{O} := (b_0,b_1)\times\mathscr{U}. \tag{1.9}$$

- The integer q is the index of regularity of our functions/operators with respect to the parameters b and ω. It is chosen as

$$q := q_0+1,$$

with q_0 being the non-degeneracy index provided by Lemma 3.5, which only depends on the linear unperturbed frequencies.

- The real parameters γ, τ_1 and τ_2 satisfy

$$0<\gamma<1, \quad \tau_2>\tau_1>d \tag{1.10}$$

and are linked to different Diophantine conditions, see for instance Propositions 6.2 and 6.4. The choice of τ_1 and τ_2 will be finally fixed in (7.14). We point out that the parameter γ appears in the weighted Sobolev spaces and will be fixed in Proposition 7.1 with respect to the rescaling parameter ε giving the smallness condition of the solutions around the equilibrium.

- The real number s is the Sobolev index regularity of the functions in the variables φ and θ. The index s will vary between s_0 and S,

$$S\geqslant s\geqslant s_0>\tfrac{d+1}{2}+q+2, \tag{1.11}$$

where S is a fixed large number.

- For a given continuous complex function $f:\mathbb{T}^n\to\mathbb{C}$, $n\geqslant 1$, $\mathbb{T}:=\mathbb{R}/2\pi\mathbb{Z}$, we denote by

$$\int_{\mathbb{T}^n} f(x)dx := \frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} f(x)dx. \tag{1.12}$$

- We denote by $(\mathbf{e}_{l,j})_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}$ the Hilbert basis of the $L^2(\mathbb{T}^{d+1},\mathbb{C})$,

$$\mathbf{e}_{l,j}(\varphi,\theta) := e^{\mathrm{i}(l\cdot\varphi+j\theta)},$$

and we endow this space with the Hermitian inner product

$$\big\langle\rho_1,\rho_2\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})} = \int_{\mathbb{T}^{d+1}}\rho_1(\varphi,\theta)\overline{\rho_2(\varphi,\theta)}d\varphi d\theta. \tag{1.13}$$

2 Hamiltonian reformulation

In this section, we shall write down the equation governing the boundary dynamics. For that purpose, we shall consider a polar parametrization of the boundary and see that the radial deformation in there is subject to a non-linear and non-local Hamiltonian equation of transport type.

2.1 Equation satisfied by the radial deformation of the patch

Given $b \in (0,1)$, consider a vortex patch $t \mapsto \mathbf{1}_{D_t}$, near the Rankine vortex $\mathbf{1}_{b\mathbb{D}}$ with a smooth boundary whose polar parametrization is given by

$$z(t,\theta) = \big(b^2 + 2r(t,\theta)\big)^{\frac{1}{2}} e^{\mathrm{i}\theta}, \tag{2.1}$$

where r is the radial deformation assumed to be small, namely $|r(t,\theta)| \ll 1$. In the sequel, we shall frequently use the following notations

$$R(b,t,\theta) := \big(b^2 + 2r(t,\theta)\big)^{\frac{1}{2}}, \tag{2.2}$$

$$A_r(b,t,\theta,\eta) := \Big|R(b,t,\theta)e^{\mathrm{i}\theta} - R(b,t,\eta)e^{\mathrm{i}\eta}\Big|, \tag{2.3}$$

$$B_r(b,t,\theta,\eta) := \Big|1 - R(b,t,\theta)R(b,t,\eta)e^{\mathrm{i}(\eta-\theta)}\Big|. \tag{2.4}$$

The equation satisfied by r is given by the following lemma.

Lemma 2.1. *For short time $T > 0$, the radial deformation r, defined through* (2.2)*, satisfies the following nonlinear and nonlocal transport PDE:*

$$\forall (t,\theta) \in [0,T] \times \mathbb{T}, \quad \partial_t r(t,\theta) + F_b[r](t,\theta) = 0, \tag{2.5}$$

where

$$F_b[r] := -F_b^0[r] - F_b^1[r] + F_b^2[r], \tag{2.6}$$

$$F_b^0[r] := \tfrac{1}{2}\partial_\theta r(t,\theta) \int_{\mathbb{T}} \tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)} d\eta, \tag{2.7}$$

$$F_b^1[r] := \int_{\mathbb{T}} \log\big(A_r(b,t,\theta,\eta)\big)\partial^2_{\theta\eta}\Big(R(b,t,\theta)R(b,t,\eta)\sin(\eta-\theta)\Big)d\eta, \tag{2.8}$$

$$F_b^2[r] := \int_{\mathbb{T}} \log\big(B_r(b,t,\theta,\eta)\big)\partial^2_{\theta\eta}\Big(\tfrac{R(b,t,\eta)}{R(b,t,\theta)}\sin(\eta-\theta)\Big)d\eta, \tag{2.9}$$

where $R(b,t,\theta)$, $A_r(b,t,\theta,\eta)$ and $B_r(b,t,\theta,\eta)$ are given by (2.2)*-*(2.4)*.*

Proof. We start with the vortex patch equation. Denoting $\mathbf{n}$ the outward normal vector to the boundary of the patch, the evolution equation of the boundary can be written as

$$\partial_t z(t,\theta)\cdot \mathbf{n}(t,z(t,\theta)) = -\partial_\theta \mathbf{\Psi}(t,z(t,\theta)).$$

For a detailed proof see for instance [45, p.174]. We shall identify $\mathbb{C}$ with $\mathbb{R}^2$. In particular, the Euclidean structure of $\mathbb{R}^2$ is seen in the complex sense through the usual inner product defined for all $z_1 = a_1 + \mathrm{i}\, b_1 \in \mathbb{C}$ and $z_2 = a_2 + \mathrm{i}\, b_2 \in \mathbb{C}$ by

$$z_1 \cdot z_2 := \langle z_1, z_2\rangle_{\mathbb{R}^2} = \mathrm{Re}\,(z_1\overline{z_2}) = a_1a_2 + b_1b_2. \tag{2.10}$$

Since $\mathbf{n}(t,z(t,\theta)) = -\mathrm{i}\,\partial_\theta z(t,\theta)$ (up to a real constant of renormalization) then the complex formulation of the vortex patch equation is given by

$$\mathrm{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big) = \partial_\theta \mathbf{\Psi}(t,z(t,\theta)).$$

Using the parametrization (2.1), one easily checks that

$$\mathrm{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big) = -\partial_t r(t,\theta).$$

Thus, the vortex patch equation writes in the following way

$$\partial_t r(t,\theta) + \partial_\theta \mathbf{\Psi}(t,z(t,\theta)) = 0. \tag{2.11}$$

Now we shall compute $\partial_\theta \mathbf{\Psi}(t,z(t,\theta))$. Using complex notations, we have

$$\partial_\theta \mathbf{\Psi}(t,z(t,\theta)) = \nabla\mathbf{\Psi}(t,z(t,\theta))\cdot\partial_\theta z(t,\theta) = 2\mathrm{Re}\Big(\partial_{\overline{w}}\mathbf{\Psi}(t,z(t,\theta))\overline{\partial_\theta z(t,\theta)}\Big). \tag{2.12}$$

Recall, from (1.3), that the stream function $\mathbf{\Psi}$ writes

$$\forall w\in\mathbb{D},\quad \mathbf{\Psi}(t,w) = \tfrac{1}{4\pi}\int_{D_t}\log\big(|w-\xi|^2\big)dA(\xi) - \tfrac{1}{4\pi}\int_{D_t}\log\big(|\overline{\xi}w-1|^2\big)dA(\xi).$$

Let $\epsilon>0$. We set

$$f_\epsilon(\xi,\overline{\xi}) := (\overline{\xi}-\overline{w})\Big[\log\big(|\xi-w|^2+\epsilon\big)-1\Big] - \big(\overline{\xi}-\tfrac{1}{w}\big)\Big[\log\big(|1-w\overline{\xi}|^2\big)-1\Big].$$

Then

$$\partial_{\overline{\xi}}f_\epsilon(\xi,\overline{\xi}) = \log\big(|w-\xi|^2+\epsilon\big) - \frac{\epsilon}{|w-\xi|^2+\epsilon} - \log\big(|\overline{\xi}w-1|^2\big).$$

Using the complex version of Stokes' Theorem,

$$2\mathrm{i}\int_D \partial_{\overline{\xi}}f_\epsilon(\xi,\overline{\xi})dA(\xi) = \int_{\partial D} f_\epsilon(\xi,\overline{\xi})d\xi,$$

then passing to the limit as ϵ goes to 0 we obtain

$$\mathbf{\Psi}(t,w) = \tfrac{1}{8\mathrm{i}\pi}\int_{\partial D_t}(\overline{\xi}-\overline{w})\Big[\log\big(|\xi-w|^2\big)-1\Big]d\xi - \tfrac{1}{8\mathrm{i}\pi}\int_{\partial D_t}\big(\overline{\xi}-\tfrac{1}{w}\big)\Big[\log\big(|1-w\overline{\xi}|^2\big)-1\Big]d\xi.$$

Performing the change of variables $\xi=z(t,\eta)$, given by (2.1), and using the notation (1.12) we can write

$$\begin{aligned}\mathbf{\Psi}(t,w) = {} & \tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}(\overline{z}(t,\eta)-\overline{w})\Big[\log\big(|z(t,\eta)-w|^2\big)-1\Big]\partial_\eta z(t,\eta)d\eta\\ & -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\big(\overline{z}(t,\eta)-\tfrac{1}{w}\big)\Big[\log\big(|1-w\overline{z}(t,\eta)|^2\big)-1\Big]\partial_\eta z(t,\eta)d\eta.\end{aligned}$$

It follows that

$$\begin{aligned}\partial_{\overline{w}}\mathbf{\Psi}(t,w) = {} & -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|z(t,\eta)-w|^2\big)\,\partial_\eta z(t,\eta)d\eta\\ & -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\Big(\overline{z}(t,\eta)-\frac{1}{w}\Big)\Big[\frac{\frac{1}{\overline{w}^2}}{z(t,\eta)-\frac{1}{\overline{w}}}+\frac{1}{\overline{w}}\Big]\partial_\eta z(t,\eta)d\eta.\end{aligned}\tag{2.13}$$

Direct computations lead to

$$\Big[\frac{\overline{z}(t,\eta)-\frac{1}{w}}{z(t,\eta)-\frac{1}{\overline{w}}}\Big]\partial_\eta z(t,\eta) = \partial_\eta\Big[\log\big(\big|z(t,\eta)-\frac{1}{\overline{w}}\big|^2\big)+\log\big(|w|^2\big)\Big]\Big(\overline{z}(t,\eta)-\frac{1}{w}\Big)-\partial_\eta\overline{z}(t,\eta).$$

Inserting this identity into (2.13), integrating by parts, using the morphism property of the logarithm and the periodicity imply

$$\begin{aligned}\partial_{\overline{w}}\mathbf{\Psi}(t,w) = {} & -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|z(t,\eta)-w|^2\big)\,\partial_\eta z(t,\eta)d\eta\\ & +\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|1-\overline{w}z(t,\eta)|^2\big)\,\partial_\eta\overline{z}(t,\eta)\frac{1}{\overline{w}^2}d\eta\\ & -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{1}{\overline{w}}d\eta.\end{aligned}\tag{2.14}$$

As a consequence, one gets

$$\begin{aligned}2\mathrm{Re}\Big(\partial_{\overline{w}}\mathbf{\Psi}(t,z(t,\theta))\partial_\theta\overline{z}(t,\theta)\Big) = {} & -\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|z(t,\eta)-z(t,\theta)|^2\big)\,\mathrm{Im}\big(\partial_\eta z(t,\eta)\partial_\theta\overline{z}(t,\theta)\big)d\eta\\ & +\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|1-\overline{z}(t,\theta)z(t,\eta)|^2\big)\,\mathrm{Im}\Big(\partial_\eta\overline{z}(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)^2}\Big)d\eta\\ & -\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\Big(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\Big)d\eta.\end{aligned}$$

That is, by (2.12),

$$
\begin{aligned}
\partial_\theta \mathbf{\Psi}(t,z(t,\theta)) = &-\tfrac{1}{2}\int_{\mathbb{T}} \log\left(|z(t,\eta)-z(t,\theta)|^2\right)\partial^2_{\theta\eta}\mathrm{Im}\big(z(t,\eta)\overline{z}(t,\theta)\big)d\eta\\
&+\tfrac{1}{2}\int_{\mathbb{T}}\log\left(|1-\overline{z}(t,\theta)z(t,\eta)|^2\right)\partial^2_{\theta\eta}\mathrm{Im}\left(\frac{z(t,\eta)}{z(t,\theta)}\right)d\eta\\
&-\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\left(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\right)d\eta.
\end{aligned}
$$

From (2.1) we immediately get

$$
\begin{aligned}
\mathrm{Im}\big(z(t,\eta)\overline{z}(t,\theta)\big) &= R(b,t,\theta)R(b,t,\eta)\sin(\eta-\theta),\\
\mathrm{Im}\left(\frac{z(t,\eta)}{z(t,\theta)}\right) &= \frac{R(b,t,\theta)}{R(b,t,\eta)}\sin(\eta-\theta),\\
\mathrm{Im}\left(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\right) &= \frac{R^2(b,t,\eta)}{R^2(b,t,\theta)}\partial_\theta r(t,\theta)-\partial_\eta r(t,\eta).
\end{aligned}
$$

Combining the last four identities with (2.11) and using the notations (2.1)-(2.4) we conclude the desired result.

□

We look for time quasi-periodic solutions of (2.5); that are functions in the form

$$\widehat{r}(t,\theta) = r(\omega t,\theta),$$

where $r = r(\varphi,\theta) : \mathbb{T}^{d+1}\to\mathbb{R}$, $\omega\in\mathbb{R}^d$, $d\in\mathbb{N}^*$. With this ansatz, the equation (2.5) becomes

$$\omega\cdot\partial_\varphi r(\varphi,\theta) + F_b[r](\varphi,\theta) = 0. \tag{2.15}$$

2.2 Hamiltonian structure

In this section, we show that the contour dynamics equation (2.5) has a Hamiltonian structure related to the kinetic energy

$$E(r)(t) := \frac{1}{2\pi}\int_{D_t}\mathbf{\Psi}(t,z)dA(z), \tag{2.16}$$

which is a conserved quantity for (1.1). It is well-known that the bidimensional Euler equations admits a Hamiltonian structure and we shall see here that such structure still persists at the level of the boundary equation, which is a stronger formulation.

Proposition 2.1. *The equation* (2.5) *is a Hamiltonian equation in the form*

$$\partial_t r = \partial_\theta\nabla H(r), \quad \text{where} \quad H(r) = -\tfrac{1}{2}E(r), \tag{2.17}$$

and ∇ *is the* $L^2_\theta(\mathbb{T})$*-gradient associated with the* $L^2_\theta(\mathbb{T})$ *normalized inner product*

$$\big\langle\rho_1,\rho_2\big\rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}}\rho_1(\theta)\rho_2(\theta)d\theta.$$

Proof. In polar coordinates, the stream function, given by (1.3), at some point $w\in\mathbb{D}$ writes

$$\mathbf{\Psi}(t,w) = \int_{\mathbb{T}}\int_0^{R(b,t,\eta)} G\left(w,\ell_2 e^{\mathrm{i}\eta}\right)\ell_2 d\ell_2 d\eta \quad \text{with} \quad G(w,\xi) := \log\left(\left|\frac{w-\xi}{1-w\overline{\xi}}\right|\right) \tag{2.18}$$

and kinetic energy E, in (2.16), reads

$$E(r)(t) = \int_{\mathbb{T}}\int_{\mathbb{T}}\int_0^{R(b,t,\theta)}\left(\int_0^{R(b,t,\eta)} G\left(\ell_1 e^{\mathrm{i}\theta},\ell_2 e^{\mathrm{i}\eta}\right)\ell_2 d\ell_2\right)\ell_1 d\ell_1 d\theta d\eta.$$

Differentiating with respect to r in the direction ρ and using the symmetry of the kernel $G(w,\xi) = G(\xi,w)$ yields

$$\begin{aligned}
d_rE(r)[\rho](t) &= 2\int_{\mathbb{T}} \rho(t,\theta)\left(\int_{\mathbb{T}}\int_0^{R(b,t,\eta)} G\left(R(b,t,\theta)e^{\mathrm{i}\theta}, \ell_2 e^{\mathrm{i}\eta}\right)\ell_2 d\ell_2 d\eta\right)d\theta\\
&= 2\int_{\mathbb{T}} \rho(t,\theta)\boldsymbol{\Psi}\big(t, R(b,t,\theta)e^{\mathrm{i}\theta}\big)d\theta.
\end{aligned}$$

Since $d_rE(r)[\rho] = \langle \nabla E, \rho\rangle_{L^2(\mathbb{T})}$ then

$$\nabla E(r)(t,\theta) = 2\boldsymbol{\Psi}\big(t, R(b,t,\theta)e^{\mathrm{i}\theta}\big). \tag{2.19}$$

Finally, using (2.19) and comparing (2.17) with (2.11) we conclude the desired result. This achieves the proof of Proposition 2.1. □

Now, we shall present the symplectic structure associated with the Hamiltonian equation (2.17). This will be relevant later in Section 5.1 when introducing the action-angle variables. We shall also explore some symmetry property for (2.17). Observe that this latter equation implies

$$\frac{d}{dt}\int_{\mathbb{T}} r(t,\theta)d\theta = 0.$$

Therefore, we will consider the phase space with zero average in the space variable, that is

$$L^2_0(\mathbb{T}) := \Big\{r = \sum_{j\in\mathbb{Z}^*} r_j e_j \quad \text{s.t.} \quad r_{-j} = \overline{r_j} \quad \text{and} \quad \|r\|^2_{L^2} := \sum_{j\in\mathbb{Z}^*} |r_j|^2 < +\infty\Big\}, \qquad e_j(\theta) := e^{\mathrm{i}j\theta}.$$

The equation (2.17) induces on the phase space $L^2_0(\mathbb{T})$ a symplectic structure induced by the symplectic 2-form

$$\mathcal{W}(r,h) = \int_{\mathbb{T}} \partial_\theta^{-1} r(\theta)h(\theta)d\theta \quad \text{where} \quad \partial_\theta^{-1} r(\theta) = \sum_{j\in\mathbb{Z}^*} \frac{r_j}{\mathrm{i}j} e^{\mathrm{i}j\theta}. \tag{2.20}$$

The Hamiltonian vector field is $X_H(r) = \partial_\theta \nabla H(r)$ associated to the Hamiltonian H is defined as the symplectic gradient of the Hamiltonian H with respect to the symplectic 2-form $\mathcal{W}$, namely

$$dH(r)[\cdot] = \mathcal{W}(X_H(r), \cdot).$$

Decomposing into Fourier series

$$r = \sum_{j\in\mathbb{Z}^*} r_j e_j \quad \text{with} \quad r_{-j} = \overline{r_j},$$

the symplectic form $\mathcal{W}$ becomes

$$\mathcal{W}(r,h) = \sum_{j\in\mathbb{Z}^*} \frac{1}{\mathrm{i}j} r_j h_{-j} = \sum_{j\in\mathbb{Z}^*} \frac{1}{\mathrm{i}j} r_j \overline{h_j},$$

that is

$$\mathcal{W} = \frac{1}{2}\sum_{j\in\mathbb{Z}^*} \frac{1}{\mathrm{i}j} dr_j \wedge dr_{-j} = \sum_{j\in\mathbb{N}^*} \frac{1}{\mathrm{i}j} dr_j \wedge dr_{-j}, \tag{2.21}$$

where for any $j \in \mathbb{Z}^*$ the exterior product $dr_j \wedge dr_{-j}$ is defined by

$$dr_j \wedge dr_{-j}(r,h) = r_j h_{-j} - r_{-j} h_j.$$

We shall now look at the reversibility property of the equation (2.17). We consider the involution $\mathscr{S}$ defined on the phase space $L^2_0(\mathbb{T})$ by

$$(\mathscr{S} r)(\theta) := r(-\theta), \tag{2.22}$$

which satisfies

$$\mathscr{S}^2 = \mathrm{Id} \quad \text{and} \quad \partial_\theta \circ \mathscr{S} = -\mathscr{S} \circ \partial_\theta. \tag{2.23}$$

Using the change of variables $\eta \mapsto -\eta$ and parity arguments, one gets

$$F_b \circ \mathscr{S} = -\mathscr{S} \circ F_b,$$

where F_b is given by (2.6). Then we conclude by Lemma 2.1, (2.17) and (2.23) that the Hamiltonian vector field X_H satisfies

$$X_H \circ \mathscr{S} = -\mathscr{S} \circ X_H.$$

Thus, we will look for quasi-periodic solutions satisfying the reversibility condition

$$r(-t,-\theta) = r(t,\theta).$$

3 Linearization and structure of the equilibrium frequencies

In the current section, we linearize the equation (2.5) at a given small state r close to the equilibrium. At this latter, we shall see that the linear operator is a Fourier multiplier with polynomial linear frequencies with respect to the radius of the Rankine patch $b\mathbb{D}$. At the end of this section, we also check the transversality conditions for the unperturbed frequency vector.

3.1 Linearized operator

We shall first prove that the linearized operator at a general small state r can be decomposed into the sum of a variable coefficients transport operator, a non-local operator of order 0 and a smoothing non-local operator in the variable θ. More precisely, we have the following lemma.

Lemma 3.1. *The linearized Hamiltonian equation of* (2.17) *at a state r is the time-dependent Hamiltonian system*

$$\partial_t \rho(t,\theta) = -\partial_\theta\Big(V_r(b,t,\theta)\rho(t,\theta) + \mathbf{L}_r(\rho)(b,t,\theta) - \mathbf{S}_r(\rho)(b,t,\theta)\Big),$$

where the function V_r is defined by

$$\begin{aligned} V_r(b,t,\theta) = & -\tfrac{1}{2}\int_{\mathbb{T}} \tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)}d\eta \\ & - \tfrac{1}{R(b,t,\theta)}\int_{\mathbb{T}} \log\big(A_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta \\ & - \tfrac{1}{R^3(b,t,\theta)}\int_{\mathbb{T}} \log\big(B_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta, \end{aligned} \tag{3.1}$$

$\mathbf{L}_r$ *is a non-local operator in the form*

$$\mathbf{L}_r(\rho)(b,t,\theta) = \int_{\mathbb{T}} \rho(t,\eta)\log\left(A_r(t,\theta,\eta)\right)d\eta \tag{3.2}$$

and $\mathbf{S}_r$ is a smoothing non-local operator in the form

$$\mathbf{S}_r(\rho)(b,t,\theta) = \int_{\mathbb{T}} \rho(t,\eta)\log\left(B_r(t,\theta,\eta)\right)d\eta. \tag{3.3}$$

We recall that A_r, B_r and R are defined by (2.3), (2.4) *and* (2.2), *respectively.*
Moreover, if $r(-t,-\theta) = r(t,\theta)$, then

$$V_r(b,-t,-\theta) = V_r(b,t,\theta). \tag{3.4}$$

Proof. In all the proof, we shall omit the dependence of our quantities with respect to the variables b and t. Notice that linearizing (2.11) amounts to compute the Gâteaux derivative of the stream function $\boldsymbol{\Psi}(r,z(\theta)) := \boldsymbol{\Psi}(z(\theta))$ given by (1.3) at point r in the direction ρ (real-valued). All the computations are done at a formal level, but can be rigorously justified in a classical way in the functional context introduced in Section 4. Applying the chain rule gives

$$d_r\big(\boldsymbol{\Psi}\big(r,z(\theta)\big)\big)[\rho] = \big(d_r\boldsymbol{\Psi}(r,w)[\rho]\big)_{|w=z(\theta)} + 2\mathrm{Re}\left(\big(\partial_{\overline{w}}\boldsymbol{\Psi}(r,w)\big)_{|w=z(\theta)} d_r\overline{z}(\theta)[\rho]\right). \tag{3.5}$$

Differentiating (2.18) gives

$$d_r\boldsymbol{\Psi}(r,w)[\rho] = \int_{\mathbb{T}} \log\left(\left|\frac{w-R(\eta)e^{\mathrm{i}\eta}}{1-R(\eta)e^{-\mathrm{i}\eta}w}\right|\right)\rho(\eta)d\eta. \tag{3.6}$$

On the other hand, from (2.14) and the identity

$$d_r z(\theta)[\rho](\theta) = \tfrac{\rho(\theta)}{R(\theta)}e^{\mathrm{i}\theta},$$

we obtain

$$\begin{aligned} 2\mathrm{Re}\left(\big(\partial_{\overline{w}}\boldsymbol{\Psi}(r,w)\big)_{|w=z(\theta)} d_r\overline{z}(\theta)[\rho]\right) &= -\tfrac{\rho(\theta)}{R(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\log\left(|z(\eta)-z(\theta)|^2\right)\partial_\eta \mathrm{Im}\left(z(\eta)e^{-\mathrm{i}\theta}\right)d\eta \\ &\quad -\tfrac{\rho(\theta)}{R^3(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\log\left(|1-\overline{z(\theta)}z(\eta)|^2\right)\partial_\eta \mathrm{Im}\left(\overline{z}(\eta)e^{\mathrm{i}\theta}\right)d\eta \\ &\quad +\tfrac{\rho(\theta)}{R^2(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\big(\partial_\eta\overline{z}(\eta)z(\eta)\big)d\eta. \end{aligned} \tag{3.7}$$

Putting together (3.6), (3.5), (3.7) and using the identities

$$\mathrm{Im}\left(z(\eta)e^{-\mathrm{i}\theta}\right) = R(\eta)\sin(\eta-\theta), \quad \mathrm{Im}\big(\partial_\eta\overline{z}(\eta)z(\eta)\big) = -R^2(\eta),$$

we conclude the desired result. The symmetry property (3.4) is an immediate consequence of (3.1) with the change of variables $\eta \mapsto -\eta$. This achieves the proof of Lemma 3.1. □

The following lemma shows that the linearized operator at the equilibrium state is a Fourier multiplier. This provides an integrable Hamiltonian equation from which we shall generate, in Proposition 3.1, quasi-periodic solutions.

Lemma 3.2. *1. The linearized equation of (2.17) at the equilibrium state ($r=0$) writes*

$$\partial_t\rho = \partial_\theta \mathrm{L}(b)\rho = \partial_\theta\nabla H_{\mathrm{L}}(\rho), \tag{3.8}$$

where $\mathrm{L}(b)$ is the self-adjoint operator on $L^2_0(\mathbb{T})$ defined by

$$\mathrm{L}(b) := -\frac{1}{2} - \mathcal{K}_b * \cdot \tag{3.9}$$

with

$$\mathcal{K}_b := \mathcal{K}_{1,b} - \mathcal{K}_{2,b}, \tag{3.10}$$

$$\mathcal{K}_{1,b}(\theta) := \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right), \tag{3.11}$$

$$\mathcal{K}_{2,b}(\theta) := \log\left(|1-b^2e^{\mathrm{i}\theta}|\right). \tag{3.12}$$

It is generated by the quadratic Hamiltonian

$$H_{\mathrm{L}}(\rho) := \frac{1}{2}\langle \mathrm{L}(b)\rho,\rho\rangle_{L^2(\mathbb{T})}. \tag{3.13}$$

2. From Fourier point of view, if we write $\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*} \rho_j(t)e^{\mathrm{i}j\theta}$ *with* $\rho_{-j}(t) = \overline{\rho_j}(t)$*, then the self-adjoint operator* $\mathrm{L}(b)$ *and the Hamiltonian* H_{L} *write*

$$\mathrm{L}(b)\rho(\theta) = -\sum_{j\in\mathbb{Z}^*} \tfrac{\Omega_j(b)}{j}\rho_j e^{\mathrm{i}j\theta} \quad \textit{and} \quad H_{\mathrm{L}}(\rho) = -\sum_{j\in\mathbb{Z}^*} \tfrac{\Omega_j(b)}{2j}|\rho_j|^2, \tag{3.14}$$

where $\big(\Omega_j(b)\big)_{j\in\mathbb{Z}^*}$ *is defined by*

$$\forall j\in\mathbb{N}^*, \quad \Omega_j(b) = \frac{j-1+b^{2j}}{2} \quad \text{and} \quad \Omega_{-j}(b) = -\Omega_j(b). \tag{3.15}$$

Moreover, the reversible solutions of the equation (3.8) *take the form*

$$\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*} \rho_j \cos\left(j\theta - \Omega_j(b)t\right), \quad \rho_j\in\mathbb{R}. \tag{3.16}$$

Proof. **1.** Notice that we can rewrite A_r and B_r, defined in (2.3), (2.4), as

$$\begin{aligned} A_r(b,t,\theta,\eta) &= \Big(R^2(b,t,\theta) + R^2(b,t,\eta) - 2R(b,t,\theta)R(b,t,\eta)\cos(\eta-\theta)\Big)^{\frac12} \\ &= \Big(\big(R(b,t,\theta) - R(b,t,\eta)\big)^2 + 4R(b,t,\theta)R(b,t,\eta)\sin^2(\eta-\theta)\Big)^{\frac12} \end{aligned} \tag{3.17}$$

and

$$B_r(b,t,\theta,\eta) = \Big(R^2(b,t,\theta)R^2(b,t,\eta) - 2R(b,t,\theta)R(b,t,\eta)\cos(\eta-\theta) + 1\Big)^{\frac12}. \tag{3.18}$$

Taking $r=0$ in (3.17), (2.4) and (2.2) gives

$$A_0(b,t,\theta,\eta) = 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|, \quad B_0(b,t,\theta,\eta) = |1-b^2e^{\mathrm{i}(\eta-\theta)}| \quad \text{and} \quad R(b,t,\theta) = b. \tag{3.19}$$

According to (3.1), (3.2) and (3.3) we obtain, after straightforward simplifications using (3.19),

$$\begin{aligned} V_0(b,t,\theta) &= -\tfrac12 - \tfrac12\int_{\mathbb{T}} \log\left(4b^2\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(\eta)d\eta - \frac{1}{b^2}\int_{\mathbb{T}}\log\left(|1-b^2e^{\mathrm{i}\eta}|\right)\cos(\eta)d\eta, \\ \mathbf{L}_0(\rho)(b,t,\theta) &= \int_{\mathbb{T}} \log\left(2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\rho(t,\eta)d\eta, \\ \mathbf{S}_0(\rho)(b,t,\theta) &= \int_{\mathbb{T}} \log\left(\left|1-b^2e^{\mathrm{i}(\eta-\theta)}\right|\right)\rho(t,\eta)d\eta. \end{aligned}$$

We then see that $\mathbf{L}_0$ and $\mathbf{S}_0$ are convolution operators given by

$$\begin{aligned} \mathbf{L}_0 &= \mathcal{K}_{1,b} * \cdot \quad \text{with} \quad \mathcal{K}_{1,b}(\theta) := \tfrac12\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right), \\ \mathbf{S}_0 &= \mathcal{K}_{2,b} * \cdot \quad \text{with} \quad \mathcal{K}_{2,b}(\theta) := \log\left(|1-b^2e^{\mathrm{i}\theta}|\right). \end{aligned}$$

2. To describe the operators above, it suffices to look for their actions on the Fourier basis $(e_j)_{j\in\mathbb{Z}^*}$ of $L^2_0(\mathbb{T})$. We first study the operator $\mathbf{L}_0$. Recall the following formula which can be found in [21, Lem. A.3]

$$\forall j\in\mathbb{Z}^*, \quad \int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(j\eta)d\eta = -\frac{1}{|j|}. \tag{3.20}$$

Using (3.20) together with symmetry arguments, one obtains

$$\begin{aligned} \forall j\in\mathbb{Z}^*, \quad \mathcal{K}_{1,b} * e_j(\theta) &= \tfrac12\int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)e^{\mathrm{i}j(\theta-\eta)}d\eta \\ &= \frac{e_j(\theta)}{2}\int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(j\eta)d\eta \qquad (3.21) \\ &= -\frac{e_j(\theta)}{2|j|}. \qquad (3.22) \end{aligned}$$

We now turn to the study of the operator $\mathbf{S}_0$. Using the following identity proved in [61, Lem. 3.2]

$$\forall j \in \mathbb{Z}^*, \quad \int_{\mathbb{T}} \log\left(|1-b^2 e^{\mathrm{i}\eta}|\right)\cos(j\eta)d\eta = -\frac{b^{2|j|}}{2|j|}, \tag{3.23}$$

we obtain

$$\begin{aligned}\forall j \in \mathbb{Z}^*, \quad \mathcal{K}_{2,b} * e_j(\theta) &= e_j(\theta)\int_{\mathbb{T}} \log\left(|1-b^2 e^{\mathrm{i}\eta}|\right)\cos(j\eta)d\eta \\ &= -\frac{b^{2|j|}e_j(\theta)}{2|j|}.\end{aligned} \tag{3.24}$$

In view of the expression of V_0 and using formulae (3.20) and (3.23) we find

$$V_0(b,t,\theta) = \frac{1}{2}. \tag{3.25}$$

Notice that, the kernels $\mathcal{K}_{1,b}$ and $\mathcal{K}_{2,b}$ being even, the operator $\mathrm{L}(b)$ is self-adjoint. The identities in (3.14) follows immediately from (3.9), (3.22), (3.24) and (3.25). Then, according to (3.14), a real function ρ with Fourier representation $\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*}\rho(t)e^{\mathrm{i}j\theta}$ is a solution to (3.8) if and only if

$$\forall j \in \mathbb{Z}^*, \quad \dot{\rho_j} = -\mathrm{i}\,\Omega_j(b)\rho_j,$$

where $\Omega_j(b)$ is defined by (3.15). Solving the previous ODE gives

$$\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(b)t)}.$$

Therefore, every real-valued reversible solution to (3.8) has the form (3.16). This ends the proof of Lemma 3.2. □

3.2 Properties of the equilibrium frequencies

The goal of this section is to explore some important properties of the equilibrium frequencies. We shall first show some bounds on these frequencies then discuss their non-degeneracy through the transversality conditions. Such conditions are crucial in the measure estimates of the final Cantor set giving rise to quasi-periodic solutions for the linear and the nonlinear problems.

Lemma 3.3. *(i) For all $b \in (0,1)$, the sequence $\left(\frac{\Omega_j(b)}{j}\right)_{j\in\mathbb{N}^*}$ is strictly increasing.*

(ii) For all $j \in \mathbb{Z}^*$, we have

$$\forall\, 0 < b_0 \leqslant b < 1, \quad |\Omega_j(b)| \geqslant \frac{b_0^2}{2}|j|.$$

(iii) For all $j, j' \in \mathbb{Z}^*$, we have

$$\forall\, 0 < b_0 \leqslant b < 1, \quad |\Omega_j(b) \pm \Omega_{j'}(b)| \geqslant \frac{b_0^2}{6}|j \pm j'|.$$

(iv) Given $0 < b_0 < b_1 < 1$ and $q_0 \in \mathbb{N}$, there exists $C_0 > 0$ such that

$$\forall j, j' \in \mathbb{Z}^*, \quad \max_{q\in[\![0,q_0]\!]}\sup_{b\in[b_0,b_1]}\left|\partial_b^q\big(\Omega_j(b)-\Omega_{j'}(b)\big)\right| \leqslant C_0|j-j'|.$$

Proof. **(i)** This point was proved in the proof of [36, Prop. 2].
(ii) By symmetry (3.15), it suffices to show the inequality for $j \in \mathbb{N}^*$. From (i) we have

$$\frac{\Omega_j(b)}{j} \geqslant \Omega_1(b) = \frac{b^2}{2} \geqslant \frac{b_0^2}{2}.$$

(iii) In view of the symmetry (3.15), it suffices to check the property for $j, j' \in \mathbb{N}^*$. By symmetry in j, j' we may assume that $j \geqslant j'$. For $j = j' = 1$ one has

$$\Omega_1(b) + \Omega_1(b) = b^2 \geqslant b_0^2.$$

In the case where $j \geqslant 2$ and $j' \geqslant 1$ we get

$$\Omega_j(b) + \Omega_{j'}(b) = \frac{j + j' - 2}{2} + \frac{b^{2j} + b^{2j'}}{2} \geqslant (j + j')\frac{j + j' - 2}{2(j + j')} \geqslant \frac{j + j'}{6}.$$

Now we shall move to the difference. Using Taylor formula we obtain, for all $j > j' \geqslant 1$,

$$\begin{aligned}\Omega_j(b) - \Omega_{j'}(b) &= \frac{j - j'}{2} + \frac{b^{2j} - b^{2j'}}{2} \\ &= \frac{j - j'}{2} + \log(b) \int_{j'}^{j} b^{2x} dx \\ &\geqslant \frac{j - j'}{2}(1 + 2\log(b)b^{2j}) \geqslant \frac{j - j'}{4}.\end{aligned}$$

(iv) The case $j = j'$ is trivial, then from the symmetry (3.15) and without loss of generality we shall assume that $j > j' \geqslant 1$. First, remark that

$$\forall b \in (0, 1), \quad |\Omega_j(b) \pm \Omega_{j'}(b)| \leqslant \frac{(j - 1) \pm (j' - 1)}{2} + \frac{b^{2j'} \pm b^{2j}}{2} \leqslant j \pm j'.$$

Now, for all $q \in \mathbb{N}^*$, one has

$$\partial_b^q \Big(\Omega_j(b) \pm \Omega_{j'}(b)\Big) = \frac{1}{2}\partial_b^q \Big(b^{2j} \pm b^{2j'}\Big).$$

Moreover, for all $q \in [\![1, q_0]\!]$ and $n \in \mathbb{N}^*$,

$$0 \leqslant \partial_b^q(b^n) \leqslant q!\binom{n}{q} b^{n-q} \leqslant \frac{n^{q_0} b_1^n}{b_0^{q_0}}.$$

Since $b_1 \in (0, 1)$ then the sequence $(n^{q_0} b_1^n)_{n\in\mathbb{N}}$ is bounded. Therefore, there exists $C_0 := C_0(q_0, b_0, b_1) > 0$ such that

$$\forall n \in \mathbb{N}, \quad 0 \leqslant \partial_b^q(b^n) \leqslant C_0. \tag{3.26}$$

We deduce that for all $q \in [\![1, q_0]\!]$,

$$\left|\partial_b^q \Big(\Omega_j(b) \pm \Omega_{j'}(b)\Big)\right| \leqslant C_0 \leqslant C_0(j \pm j').$$

This concludes the proof of Lemma 3.3. □

Let us consider finitely many Fourier modes, called tangential sites, gathered in the tangential set $\mathbb{S}$ defined by

$$\mathbb{S} := \{j_1, \ldots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < j_2 < \ldots < j_d. \tag{3.27}$$

Now, we define the equilibrium frequency vector by

$$\omega_{\mathrm{Eq}}(b) = (\Omega_j(b))_{j\in\mathbb{S}}, \tag{3.28}$$

where $\Omega_j(b)$ is defined by (3.15). We shall now investigate the non-degeneracy and the transversality properties satisfied by ω_{Eq}. Let us first start with the non-degeneracy, for which we recall the definition.

Definition 3.1. *Given two numbers $b_0 < b_1$ and $d \in \mathbb{N}^*$. A vector-valued function $f = (f_1, ..., f_d) : [b_0, b_1] \to \mathbb{R}^d$ is called non-degenerate if, for any vector $c = (c_1, ..., c_d) \in \mathbb{R}^d \setminus \{0\}$, the function $f \cdot c = f_1 c_1 + ... + f_d c_d$ is not identically zero on the whole interval $[b_0, b_1]$. In other words, the curve of f is not contained in an hyperplane.*

We have the following result.

Lemma 3.4. *The equilibrium frequency vector ω_{Eq} and the vector-valued function $(\omega_{\mathrm{Eq}}, 1)$ are non-degenerate on $[b_0, b_1]$ in the sense of Definition 3.1.*

Proof. ▶ We shall first prove that the equilibrium frequency vector ω_{Eq} is non-degenerate on $[b_0, b_1]$. Arguing by contradiction, suppose that there exists $c := (c_1, \ldots, c_d) \in \mathbb{R}^d \backslash \{0\}$ such that

$$\forall b \in [b_0, b_1], \quad \sum_{k=1}^{d} c_k \Omega_{j_k}(b) = 0. \tag{3.29}$$

Since $\Omega_j(b)$ is polynomial in b then, from (3.15), one has

$$\forall b \in \mathbb{R}, \quad \sum_{k=1}^{d} c_k (j_k - 1 + b^{2j_k}) = 0. \tag{3.30}$$

Taking the limit $b \to 0$ in (3.30) gives the relation $\sum_{k=1}^{d} c_k (j_k - 1) = 0$, which, inserted into (3.30), implies

$$\forall b \in \mathbb{R}, \quad \sum_{k=1}^{d} c_k b^{2j_k} = 0.$$

Since $j_1 < j_2 < \ldots < j_d$, then

$$\forall k \in [\![1, d]\!], \quad c_k = 0,$$

which contradicts the assumption.

▶ Next, we shall check that the function $(\omega_{\mathrm{Eq}}, 1)$ is non-degenerate on $[b_0, b_1]$. Suppose, by contradiction, that there exists $c := (c_1, \ldots, c_d, c_{d+1}) \in \mathbb{R}^{d+1} \backslash \{0\}$ such that

$$\forall b \in [b_0, b_1], \quad c_{d+1} + \sum_{k=1}^{d} c_k \Omega_{j_k}(b) = 0. \tag{3.31}$$

Since $\Omega_j(b)$ is polynomial in b then, from (3.15), one may writes

$$\forall b \in \mathbb{R}, \quad c_{d+1} + \tfrac{1}{2} \sum_{k=1}^{d} c_k (j_k - 1 + b^{2j_k}) = 0. \tag{3.32}$$

Taking the limit $b \to 0$ in (3.32) yields

$$c_{d+1} + \tfrac{1}{2} \sum_{k=1}^{d} c_k (j_k - 1) = 0.$$

Inserting this relation into (3.32) gives

$$\forall b \in \mathbb{R}, \quad \sum_{k=1}^{d} c_k b^{2j_k} = 0.$$

Reasoning as in the previous point, we obtain

$$\forall k \in [\![1, d]\!], \, c_k = 0$$

and then $c_{d+1} = 0$, by coming back to (3.32), contradicting the assumption. □

We shall now state the transversality conditions satisfied by the unperturbed frequencies.

Lemma 3.5. [Transversality] *Let $0 < b_0 < b_1 < 1$. Set $q_0 = 2j_d+2$. Then, there exists $\rho_0 > 0$ such that the following results hold true. Recall that $\omega_{\rm Eq}$ and Ω_j are defined in* (3.28) *and* (3.15)*, respectively.*

(i) For all $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{b\in[b_0,b_1]} \max_{q\in[\![0,q_0]\!]} |\partial_b^q \omega_{\rm Eq}(b)\cdot l| \geqslant \rho_0\langle l\rangle.$$

(ii) For all $(l,j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$

$$\inf_{b\in[b_0,b_1]} \max_{q\in[\![0,q_0]\!]} \big|\partial_b^q\big(\omega_{\rm Eq}(b)\cdot l \pm \tfrac{j}{2}\big)\big| \geqslant \rho_0\langle l\rangle.$$

(iii) For all $(l,j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$

$$\inf_{b\in[b_0,b_1]} \max_{q\in[\![0,q_0]\!]} \big|\partial_b^q\big(\omega_{\rm Eq}(b)\cdot l \pm \Omega_j(b)\big)\big| \geqslant \rho_0\langle l\rangle.$$

(iv) For all $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l,j) \neq (0,j')$, we have

$$\inf_{b\in[b_0,b_1]} \max_{q\in[\![0,q_0]\!]} \big|\partial_b^q\big(\omega_{\rm Eq}(b)\cdot l + \Omega_j(b) \pm \Omega_{j'}(b)\big)\big| \geqslant \rho_0\langle l\rangle.$$

Proof. **(i)** Assume by contradiction that for all $\rho_0 > 0$, there exist $l \in \mathbb{Z}^d \setminus \{0\}$ and $b \in [b_0, b_1]$ such that

$$\max_{q\in[\![0,q_0]\!]} |\partial_b^q \omega_{\rm Eq}(b)\cdot l| < \rho_0\langle l\rangle.$$

In particular, for the choice $\rho_0 = \frac{1}{m+1}$, we can construct sequences $l_m \in \mathbb{Z}^d \setminus \{0\}$ and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0,q_0]\!], \quad \big|\partial_b^q \omega_{\rm Eq}(b_m)\cdot \tfrac{l_m}{\langle l_m\rangle}\big| < \tfrac{1}{m+1}. \tag{3.33}$$

Since the sequences $\left(\frac{l_m}{\langle l_m\rangle}\right)_m$ and $(b_m)_m$ are bounded, then by compactness arguments and, up to an extraction, we can assume that

$$\lim_{m\to\infty} \tfrac{l_m}{\langle l_m\rangle} = \bar{c} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} b_m = \bar{b}.$$

Therefore, denoting

$$P_0 := \omega_{\rm Eq}(X)\cdot \bar{c} \in \mathbb{R}_{2j_d}[X]$$

then passing to the limit in (3.33) as $m \to \infty$ leads to

$$\forall q \in [\![0,q_0]\!], \quad P_0^{(q)}(\bar{b}) = 0.$$

Hence, using the particular choice of q_0, we conclude that the polynomial $(X-\bar{b})^{2j_d+3}$ divides P_0,

$$(X-\bar{b})^{2j_d+3} | P_0.$$

Since $\deg(P_0) \leqslant 2j_d$, we conclude that P_0 is identically zero. This contradicts the non-degeneracy of the equilibrium frequency vector $\omega_{\rm Eq}$ stated in Lemma 3.4.
(ii) The case $l = 0, j \in \mathbb{N}^*$ is trivially satisfied. Thus, we shall consider the case $j \in \mathbb{N}$, $l \in \mathbb{Z}^d \setminus \{0\}$. By the triangle inequality combined with the boundedness of $\omega_{\rm Eq}$ we find

$$\big|\omega_{\rm Eq}(b)\cdot l + \tfrac{j}{2}\big| \geqslant \tfrac{1}{2}|j| - |\omega_{\rm Eq}(b)\cdot l| \geqslant \tfrac{1}{2}|j| - C|l| \geqslant |l|$$

provided that $|j| \geqslant C_0|l|$ for some $C_0 > 0$. Thus, we shall restrict the proof to indices j and l with

$$|j| \leqslant C_0|l|, \quad j \in \mathbb{N}, \quad l \in \mathbb{Z}^d \setminus \{0\}. \tag{3.34}$$

Arguing by contradiction as in the previous case, we may assume the existence of sequences $l_m \in \mathbb{Z}^d\backslash\{0\}$, $j_m \in \mathbb{N}$ satisfying (3.34) and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0, q_0]\!], \quad \left|\partial_b^q \left(\omega_{\mathrm{Eq}}(b_m) \cdot \tfrac{l_m}{\langle l_m\rangle} + \tfrac{j_m}{2\langle l_m\rangle}\right)\right| < \tfrac{1}{1+m}. \tag{3.35}$$

Since the sequences $(b_m)_m$, $\left(\frac{j_m}{2\langle l_m\rangle}\right)$ and $\left(\frac{l_m}{\langle l_m\rangle}\right)$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty} b_m = \bar{b}, \quad \lim_{m\to\infty} \tfrac{j_m}{2\langle l_m\rangle} = \bar{d} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} \tfrac{l_m}{\langle l_m\rangle} = \bar{c} \neq 0.$$

Denoting

$$Q_0 := \omega_{\mathrm{Eq}}(X) \cdot \bar{c} + \bar{d} \in \mathbb{R}_{2j_d}[X]$$

and letting $m \to \infty$ in (3.35) we obtain

$$\forall q \in [\![0, q_0]\!], \quad Q_0^{(q)}(\bar{b}) = 0.$$

Consequently, using the particular choice of q_0, we get

$$(X - \bar{b})^{2j_d+3} | Q_0.$$

Since $\deg(Q_0) \leqslant 2j_d$, we conclude that Q_0 is identically zero. This contradicts Lemma 3.4.
(iii) Consider $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$. Then applying the triangle inequality and Lemma 3.3-(ii), yields

$$\begin{aligned}|\omega_{\mathrm{Eq}}(b) \cdot l \pm \Omega_j(b)| &\geqslant |\Omega_j(b)| - |\omega_{\mathrm{Eq}}(b) \cdot l| \\ &\geqslant \tfrac{b_0^2}{2} j - C|l| \geqslant \langle l\rangle\end{aligned}$$

provided $j \geqslant C_0\langle l\rangle$ for some $C_0 > 0$. Thus as before we shall restrict the proof to indices j and l with

$$0 \leqslant j < C_0\langle l\rangle, \quad j \in \mathbb{N}^* \setminus \mathbb{S} \quad \text{and} \quad l \in \mathbb{Z}^d\backslash\{0\}. \tag{3.36}$$

Proceeding by contradiction, we may assume the existence of sequences $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m \in \mathbb{N} \setminus \mathbb{S}$ satisfying (3.36) and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0, q_0]\!], \quad \left|\partial_b^q \left(\omega_{\mathrm{Eq}}(b) \cdot \tfrac{l_m}{|l_m|} \pm \tfrac{\Omega_{j_m}(b)}{|l_m|}\right)_{|b=b_m}\right| < \tfrac{1}{m+1}. \tag{3.37}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$ and $(b_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty} \tfrac{l_m}{|l_m|} = \bar{c} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} b_m = \bar{b}.$$

Now we shall distinguish two cases.
► Case ❶ : $(l_m)_m$ is bounded. In this case, by (3.36) we find that $(j_m)_m$ is bounded too and thus up to to an extraction we may assume $\lim\limits_{m\to\infty} l_m = \bar{l}$ and $\lim\limits_{m\to\infty} j_m = \bar{\jmath}$. Since $(j_m)_m$ and $(|l_m|)_m$ are sequences of integers, then they are necessary stationary. In particular, the condition (3.36) implies $\bar{l} \neq 0$ and $\bar{\jmath} \in \mathbb{N} \setminus \mathbb{S}$. Hence, denoting

$$P_{0,\bar{\jmath}} := \omega_{\mathrm{Eq}}(X) \cdot \bar{l} \pm \Omega_{\bar{\jmath}}(X) \in \mathbb{R}_{\max(2j_d, 2\bar{\jmath})}[X],$$

then taking the limit $m \to \infty$ in (3.37), yields

$$\forall q \in [\![0, q_0]\!], \quad P_{0,\bar{\jmath}}^{(q)}(\bar{b}) = 0.$$

If $\bar{\jmath} < j_d$, then in a similar way to the point (i), we find that $P_{0,\bar{\jmath}} = 0$ which contradicts Lemma 3.4, applied with $(\omega_{\mathrm{Eq}}, \Omega_{\bar{\jmath}})$ in place of ω_{Eq}. Hence, we shall restrict the discussion to the case $\bar{\jmath} > j_d$. Since $\omega_{\mathrm{Eq}}(X) \cdot \bar{l}$ is of degree $2j_d$, then we obtain in view of our choice of q_0 that

$$\tfrac{1}{2} q! \binom{2\bar{\jmath}}{q} b^{2\bar{\jmath}-2j_d-1} = \partial_b^{2j_d+1} \Omega_{\bar{\jmath}}(\bar{b}) = 0.$$

This implies that $\bar{b}=0$ which contradicts the fact that $\bar{b}\in[b_0,b_1]\subset(0,1)$.
► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty}|l_m|=\infty$. We have two sub-cases.
• Sub-case ① : $(j_m)_m$ is bounded. In this case and up to an extraction we can assume that it converges. Then, taking the limit $m\to\infty$ in (3.37), we find

$$\forall q\in[\![0,q_0]\!],\quad \partial_b^q\omega_{\mathrm{Eq}}(\bar{b})\cdot\bar{c}=0.$$

Therefore, we obtain a contradiction in a similar way to the point (i).
• Sub-case ② : $(j_m)_m$ is unbounded. Then up to an extraction we can assume that $\lim\limits_{m\to\infty}j_m=\infty$. We write according to (3.15)

$$\frac{\Omega_{j_m}(b)}{|l_m|}=\frac{j_m}{2|l_m|}-\frac{1}{2|l_m|}+\frac{b^{2j_m}}{2|l_m|}. \tag{3.38}$$

By (3.36), the sequence $\left(\frac{j_m}{2|l_m|}\right)_n$ is bounded, thus up to an extraction we can assume that it converges to $\bar{d}$. Moreover, since $\lim\limits_{m\to\infty}j_m=\lim\limits_{m\to\infty}|l_m|=\infty$ and $b_m\in(b_0,b_1)$, then taking the limit in (3.38), one obtains from (3.26),

$$\lim_{m\to\infty}\frac{\partial_b^q\Omega_{j_m}(b)_{|b=b_m}}{|l_m|}=\begin{cases}\bar{d} & \text{if } q=0\\ 0 & \text{else.}\end{cases}$$

Consequently, taking the limit $m\to\infty$ in (3.37), we have

$$\forall q\in[\![0,q_0]\!],\quad \partial_b^q\left(\omega_{\mathrm{Eq}}(b)\cdot\bar{c}\pm\bar{d}\right)_{|b=\bar{b}}=0.$$

Then, in a similar way the the point (ii), we deduce that the polynomial $\omega_{\mathrm{Eq}}(X)\cdot\bar{c}+\bar{d}$ is identically zero, which is in contradiction with Lemma 3.4.
(iv) Consider $l\in\mathbb{Z}^d, j,j'\in\mathbb{N}^*\setminus\mathbb{S}$ with $(l,j)\neq(0,j')$. Then applying the triangle inequality combined with Lemma 3.3-(iii), we infer that

$$|\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)\pm\Omega_{j'}(b)|\geqslant|\Omega_j(b)\pm\Omega_{j'}(b)|-|\omega_{\mathrm{Eq}}(b)\cdot l|\geqslant\tfrac{b_0^2}{6}|j\pm j'|-C|l|\geqslant\langle l\rangle$$

provided that $|j\pm j'|\geqslant c_0\langle l\rangle$ for some $c_0>0$. Then it remains to check the proof for indices satisfying

$$|j\pm j'|<c_0\langle l\rangle,\quad l\in\mathbb{Z}^d\backslash\{0\},\quad j,j'\in\mathbb{N}^*\setminus\mathbb{S}. \tag{3.39}$$

Reasoning by contradiction as in the previous cases, we get for all $m\in\mathbb{N}$, real numbers $l_m\in\mathbb{Z}^d\setminus\{0\}$, $j_m,j'_m\in\mathbb{N}^*\setminus\mathbb{S}$ satisfying (3.39) and $b_m\in[b_0,b_1]$ such that

$$\forall q\in[\![0,q_0]\!],\quad \left|\partial_b^q\left(\omega_{\mathrm{Eq}}(b)\cdot\tfrac{l_m}{|l_m|}+\tfrac{\Omega_{j_m}(b)\pm\Omega_{j'_m}(b)}{|l_m|}\right)_{|b=b_m}\right|<\tfrac{1}{m+1}\cdot \tag{3.40}$$

Up to an extraction we can assume that $\lim\limits_{m\to\infty}\frac{l_m}{|l_m|}=\bar{c}\neq0$ and $\lim\limits_{m\to\infty}b_m=\bar{b}$.
As before we shall distinguish two cases.
► Case ❶ : $(l_m)_m$ is bounded. Up to an extraction we may assume that $\lim\limits_{m\to\infty}l_m=\bar{l}\neq0$. Now according to (3.39) we have two sub-cases to discuss depending whether the sequences $(j_m)_m$ and $(j'_m)_m$ are simultaneously bounded or unbounded.
• Sub-case ① : $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case, up to a extraction we may assume that these sequences are stationary $j_m=\bar{\jmath}$ and $j'_m=\bar{\jmath}'$ with $\bar{\jmath},\bar{\jmath}'\in\mathbb{N}^*\setminus\mathbb{S}$. Hence, denoting

$$P_{0,\bar{\jmath},\bar{\jmath}'}:=\omega_{\mathrm{Eq}}(X)\cdot\bar{l}+\Omega_{\bar{\jmath}}(X)\pm\Omega_{\bar{\jmath}'}(X)\in\mathbb{R}_{\max(2j_d,2\bar{\jmath},2\bar{\jmath}')}[X],$$

then, taking the limit $m\to\infty$ in (3.37), we have

$$\forall q\in[\![0,q_0]\!],\quad P^{(q)}_{0,\bar{\jmath},\bar{\jmath}'}(\bar{b})=0.$$

If $\max(\bar{\jmath},\bar{\jmath}') < j_d$, then, we deduce that $P_{0,\bar{\jmath},\bar{\jmath}'} = 0$ which gives a contradiction as the previous cases, up to replacing ω_{Eq} by $\big(\omega_{\mathrm{Eq}}, \Omega_{\bar{\jmath}}, \Omega_{\bar{\jmath}'}\big)$. Therefore, we are left to study the case $\max(\bar{\jmath},\bar{\jmath}') > j_d$. Notice that the cases $\bar{\jmath} = \bar{\jmath}'$ and $\min(\bar{\jmath},\bar{\jmath}') > j_d$ are byproducts of point (i) and (iii). Without loss of generality, we may assume that $\bar{\jmath} > \bar{\jmath}' \geqslant j_d + 1$. In particular, since $\omega_{\mathrm{Eq}}(X)\cdot\bar{l}$ is of degree $2j_d$, then, according to our choice of q_0, we obtain

$$\begin{cases} C_1 \bar{b}^{\alpha} \pm C_2 \bar{b}^{\beta} = 0 \\ C_1 \alpha \bar{b}^{\alpha} \pm C_2 \beta \bar{b}^{\beta} = 0, \end{cases} \tag{3.41}$$

with

$$\alpha := 2\bar{\jmath} - 2j_d - 1, \quad \beta := 2\bar{\jmath}' - 2j_d - 1, \quad C_1 := q_0! \binom{2\bar{\jmath}}{q_0} \quad \text{and} \quad C_2 := q_0! \binom{2\bar{\jmath}'}{q_0}.$$

Since C_1 and C_2 are positive, we immediately get from the first equation in (3.41) that

$$C_1 \bar{b}^{\alpha} + C_2 \bar{b}^{\beta} = 0 \quad \Rightarrow \quad \bar{b} = 0.$$

This contradicts the fact that $\bar{b} \in [b_0, b_1] \subset (0,1)$. In the case where we have the difference, the system (3.41) gives

$$\frac{C_2}{C_1} = \frac{C_2 \beta}{C_1 \alpha},$$

which implies in turn that $\alpha = \beta$, that is $\bar{\jmath} = \bar{\jmath}'$ which is excluded by hypothesis.

• Sub-case ② : $(j_m)_m$ and $(j'_m)_m$ are both unbounded and without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m = \lim\limits_{m\to\infty} j'_m = \infty$. Coming back to (3.15) we get the splitting

$$\frac{\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)}{|l_m|} = \frac{j_m \pm j'_m}{2|l_m|} + \frac{b^{2j_m} \pm b^{2j'_m}}{2|l_m|}.$$

Using once again (3.39) and up to an extraction we have $\lim\limits_{m\to\infty} \frac{j_m \pm j'_m}{|l_m|} = \bar{d}$. Thus

$$\lim_{m\to\infty} |l_m|^{-1} \partial_b^q \left(\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)\right)_{|b=b_m} = \begin{cases} \bar{d} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

By taking the limit as $m \to \infty$ in (3.40), we find

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \left(\omega_{\mathrm{Eq}}(b)\cdot\bar{c} + \bar{d}\right)_{|b=\bar{b}} = 0.$$

This leads to a contradiction as in the point (ii).

► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty} |l_m| = \infty$.

We shall distinguish three sub-cases.

• Sub-case ①. The sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case and up to an extraction they will converge and then taking the limit in (3.40) yields,

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \omega_{\mathrm{Eq}}(\bar{b})\cdot\bar{c} = 0.$$

which leads to a contradiction as before.

• Sub-case ②. The sequences $(j_m)_m$ and $(j'_m)_m$ are both unbounded. This is similar to the sub-case ② of the case ❶.

• Sub-case ③. The sequence $(j_m)_m$ is unbounded and $(j'_m)_m$ is bounded (the symmetric case is similar). Without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m = \infty$ and $j'_m = \bar{\jmath}'$. By (3.39) and up to an extraction one gets $\lim\limits_{m\to\infty} \frac{j_m \pm j'_m}{|l_m|} = \bar{d}$. Once again, we have

$$\lim_{m\to\infty} |l_m|^{-1} \partial_b^q \left(\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)\right)_{|b=b_m} = \begin{cases} \bar{d} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

Hence, taking the limit in (3.40) implies

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \left(\omega_{\mathrm{Eq}}(b)\cdot\bar{c} + \bar{d}\right)_{b=\bar{b}} = 0,$$

which also gives a contradiction as the previous cases. This completes the proof of Lemma 3.5. □

Notice that by selecting only a finite number of frequencies, the sum in (3.16) give rise to quasi-periodic solutions of the linearized equation (3.8), up to selecting the parameter b in a Cantor-like set of full measure. We have the following result.

Proposition 3.1. *Let $0 < b_0 < b_1 < 1$, $d \in \mathbb{N}^*$ and $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. Then, there exists a Cantor-like set $\mathcal{C} \subset [b_0, b_1]$ satisfying $|\mathcal{C}| = b_1 - b_0$ and such that for all $\lambda \in \mathcal{C}$, every function in the form*

$$\rho(t,\theta) = \sum_{j\in\mathbb{S}} \rho_j \cos(j\theta - \Omega_j(b)t), \quad \rho_j \in \mathbb{R}^* \tag{3.42}$$

is a time quasi-periodic reversible solution to the equation (2.17) *with the vector frequency*

$$\omega_{\mathrm{Eq}}(b) = (\Omega_j(b))_{j\in\mathbb{S}}.$$

The proof of this proposition follows in a similar way to [37, Lem 3.3] or [47, Prop. 3.1] where the main ingredient is the following Rüssmann Lemma [62, Thm. 17.1]. This latter is also needed to prove Proposition 7.2 .

Lemma 3.6. *Let $q_0 \in \mathbb{N}^*$ and $\alpha, \beta \in \mathbb{R}_+$. Let $f \in C^{q_0}([a,b],\mathbb{R})$ such that*

$$\inf_{x\in[a,b]} \max_{k\in[\![0,q_0]\!]} |f^{(k)}(x)| \geqslant \beta.$$

Then, there exists $C = C(a,b,q_0,\|f\|_{C^{q_0}([a,b],\mathbb{R})}) > 0$ such that

$$\Big|\{x \in [a,b] \quad \text{s.t.} \quad |f(x)| \leqslant \alpha\}\Big| \leqslant C\frac{\alpha^{\frac{1}{q_0}}}{\beta^{1+\frac{1}{q_0}}},$$

where the notation $|A|$ corresponds to the Lebesgue measure of a given measurable set A.

4 Topological and algebraic aspects for functions and operators

In this section we present the general topological framework for both functions and operators classes. We also recall basic definitions and gather some technical results that will be used along the paper.

4.1 Function spaces

Recall the classical complex Sobolev space $H^s(\mathbb{T}^{d+1},\mathbb{C})$ which is the set of all complex periodic functions with the Fourier expansion

$$\rho = \sum_{(l,j)\in\mathbb{Z}^{d+1}} \rho_{l,j}\,\mathbf{e}_{l,j} \quad \text{where} \quad \rho_{l,j} = \big\langle \rho, \mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})}$$

such that

$$\|\rho\|_{H^s}^2 := \sum_{(l,j)\in\mathbb{Z}^{d+1}} \langle l,j\rangle^{2s}|\rho_{l,j}|^2 < \infty \quad \text{where} \quad \langle l,j\rangle := \max(1,|l|,|j|),$$

with $|\cdot|$ denoting either the ℓ^1 norm in $\mathbb{R}^d$ or the absolute value in $\mathbb{R}$. The real Sobolev spaces can be viewed as closed sub-spaces of the preceding one,

$$H^s = H^s(\mathbb{T}^{d+1},\mathbb{R}) = \Big\{\rho \in H^s(\mathbb{T}^{d+1},\mathbb{C}) \quad \text{s.t.} \quad \forall\,(l,j) \in \mathbb{Z}^{d+1},\ \rho_{-l,-j} = \overline{\rho_{l,j}}\Big\}.$$

For $N \in \mathbb{N}^*$, we define the cut-off frequency projectors on $H^s(\mathbb{T}^{d+1},\mathbb{C})$ as follows

$$\Pi_N\rho = \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ \langle l,j\rangle\leqslant N}} \rho_{l,j}\mathbf{e}_{l,j} \quad \text{and} \quad \Pi_N^{\perp} = \mathrm{Id} - \Pi_N. \tag{4.1}$$

We shall also make use of the following mixed weighted Sobolev spaces with respect to the parameter γ. Let $\mathcal{O}$ be an open bounded set of $\mathbb{R}^{d+1}$ and define

$$W^{q,\infty,\gamma}(\mathcal{O},H^s)=\Big\{\rho:\mathcal{O}\to H^s\quad\text{s.t.}\quad\|\rho\|_{q,s}^{\gamma,\mathcal{O}}<\infty\Big\},$$
$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})=\Big\{\rho:\mathcal{O}\to \mathbb{C}\quad\text{s.t.}\quad\|\rho\|_{q}^{\gamma,\mathcal{O}}<\infty\Big\},$$

where for $\mu\in\mathcal{O}\mapsto\rho(\mu)\in H^s$ or $\mu\in\mathcal{O}\mapsto\rho(\mu)\in\mathbb{C}$ the norm is defined by

$$\begin{aligned}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}&=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{|\alpha|}\sup_{\mu\in\mathcal{O}}\|\partial_\mu^\alpha\rho(\mu,\cdot)\|_{H^{s-|\alpha|}},\\ \|\rho\|_{q}^{\gamma,\mathcal{O}}&=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{|\alpha|}\sup_{\mu\in\mathcal{O}}|\partial_\mu^\alpha\rho(\mu)|.\end{aligned}\tag{4.2}$$

Remark 4.1.

- *From Sobolev embeddings we obtain,*

$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C}).$$

- *The spaces* $\Big(W^{q,\infty,\gamma}(\mathcal{O},H^s),\|\cdot\|_{q,s}^{\gamma,\mathcal{O}}\Big)$ *and* $\Big(W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C}),\|\cdot\|_{q}^{\gamma,\mathcal{O}}\Big)$ *are complete.*
- *For needs related to the use of the kernels of integral operators, we will have to duplicate the variable* θ*. Thus we may define the weighted Sobolev space* $W^{q,\infty,\gamma}(\mathcal{O},H^s_{\varphi,\theta,\eta})$ *similarly as above and denote the corresponding norm by* $\|\cdot\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}$.

The next lemma gathers some useful classical results related to various operations in weighted Sobolev spaces. The proofs are standard and can be found for instance in [13, 14, 16].

Lemma 4.1. *Let* (γ,q,d,s_0,s) *satisfy* (1.11)*, then the following assertions hold true.*

(i) Space translation invariance: Let $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*, then for all* $\eta\in\mathbb{T}$*, the function* $(\varphi,\theta)\mapsto\rho(\varphi,\eta+\theta)$ *belongs to* $W^{q,\infty,\gamma}(\mathcal{O},H^s)$*, and satisfies*

$$\|\rho(\cdot,\eta+\cdot)\|_{q,s}^{\gamma,\mathcal{O}}=\|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

(ii) Projectors properties: Let $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*, then for all* $N\in\mathbb{N}^*$ *and for all* $t\in\mathbb{R}_+^*$,

$$\|\Pi_N\rho\|_{q,s+t}^{\gamma,\mathcal{O}}\leqslant N^t\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\qquad\text{and}\qquad\|\Pi_N^\perp\rho\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N^{-t}\|\rho\|_{q,s+t}^{\gamma,\mathcal{O}},$$

where the projectors are defined in (4.1).

(iii) Interpolation inequality: Let $q<s_1\leqslant s_3\leqslant s_2$ *and* $\overline{\theta}\in[0,1]$*, with* $s_3=\overline{\theta}s_1+(1-\overline{\theta})s_2$.
If $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_2})$*, then* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_3})$ *and*

$$\|\rho\|_{q,s_3}^{\gamma,\mathcal{O}}\lesssim\left(\|\rho\|_{q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|\rho\|_{q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iv) Law products:

(a) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

(b) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ *and*

$$\|\rho_1\rho_2\|_{q}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q}^{\gamma,\mathcal{O}}.$$

(c) Let $(\rho_1,\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\times W^{q,\infty,\gamma}(\mathcal{O},H^s)$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(v) Composition law 1: Let $f\in C^\infty(\mathcal{O}\times\mathbb{R},\mathbb{R})$ *and* $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *such that*

$$\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}},\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C_0$$

for an arbitrary constant $C_0>0$ *and define the pointwise composition*

$$\forall(\mu,\varphi,\theta)\in\mathcal{O}\times\mathbb{T}^{d+1},\quad f(\rho)(\mu,\varphi,\theta):=f(\mu,\rho(\mu,\varphi,\theta)).$$

Then $f(\rho_1)-f(\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *with*

$$\|f(\rho_1)-f(\rho_2)\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C(s,d,q,f,C_0)\|\rho_1-\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(vi) Composition law 2: Let $f\in C^\infty(\mathbb{R},\mathbb{R})$ *with bounded derivatives. Let* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$*. Then*

$$\|f(\rho)-f(0)\|_{q}^{\gamma,\mathcal{O}}\leqslant C(q,d,f)\|\rho\|_{q}^{\gamma,\mathcal{O}}\left(1+\|\rho\|_{L^\infty(\mathcal{O})}^{q-1}\right).$$

This estimate is also true for $\gamma=1$*, corresponding to the classical Sobolev space* $W^{q,\infty}(\mathcal{O},\mathbb{C})$*.*

The next result will be useful later in the study of the linearized operator.

Lemma 4.2. *Let* (γ,q,d,s_0,s) *satifying* (1.11) *and* $f\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*.*
We consider the function $g:\mathcal{O}\times\mathbb{T}^d_\varphi\times\mathbb{T}_\theta\times\mathbb{T}_\eta\to\mathbb{C}$ *defined by*

$$g(\mu,\varphi,\theta,\eta)=\begin{cases}\frac{f(\mu,\varphi,\eta)-f(\mu,\varphi,\theta)}{\sin\left(\frac{\eta-\theta}{2}\right)} & \text{if } \theta\neq\eta\\ 2\partial_\theta f(\mu,\varphi,\theta) & \text{if } \theta=\eta.\end{cases}$$

Then

(i) $\forall k\in\mathbb{N},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k g)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\partial_\theta f\|_{q,s+k}^{\gamma,\mathcal{O}}\lesssim\|f\|_{q,s+k+1}^{\gamma,\mathcal{O}}.$

(ii) $\|g\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\|f\|_{q,s+1}^{\gamma,\mathcal{O}}.$

Proof. **(i)** This result has been proved in [47, Lem. 4.2].
(ii) It suffices to prove the case $q=0$. Recall the following classical norm estimate

$$\|g\|_{H^s_{\varphi,\theta,\eta}}\lesssim\|g\|_{H^s_{\varphi,\theta}L^2_\eta}+\|g\|_{L^2_\theta H^s_{\varphi,\eta}}. \tag{4.3}$$

By the translation invariance property

$$\begin{aligned}\|g\|^2_{L^2_\theta H^s_{\varphi,\eta}}&=\int_0^{2\pi}\|g(\cdot,\theta+\bullet,\bullet)\|^2_{H^s_{\varphi,\eta}}d\theta\\&\lesssim\sup_{\theta\in\mathbb{T}}\|g(*,\cdot,\theta+\bullet,\bullet)\|_{H^s_{\varphi,\eta}}.\end{aligned}$$

Using the first point and the symmetry g in (η,θ) we obtain

$$\|g\|_{L^\infty_\theta H^s_{\varphi,\eta}}\lesssim\|f\|_{s+1}.$$

Introducing the Bessel potential J^s defined in Fourier by

$$\forall j\in\mathbb{Z}^d,\quad (J^s u)_j=\max(1,|j|)^s u_j, \tag{4.4}$$

a use of Fubini's Theorem implies

$$\|g\|_{H^s_{\varphi,\theta}L^2_\eta}=\|J^s_{\varphi,\theta}\,g\|_{L^2_\varphi L^2_\theta L^2_\eta}=\|J^s_{\varphi,\theta}\,g\|_{L^2_\eta L^2_\varphi L^2_\theta}=\|g\|_{L^2_\eta H^s_{\varphi,\theta}}.$$

Since g is symmetric in the variables θ and η, we get

$$\|g\|_{L^2_\eta H^s_{\varphi,\theta}} = \|g\|_{L^2_\theta H^s_{\varphi,\eta}}.$$

Combining the foregoing estimates leads to

$$\|g\|_{H^s_{\varphi,\theta,\eta}} \lesssim \|f\|_{s+1}.$$

This ends the proof of Lemma 4.2. □

We now turn to the presentation of quasi-periodic symplectic change of variables needed for the reduction of the transport part of the linearized operator in the construction of the approximate inverse in the normal directions. Let $\beta : \mathcal{O} \times \mathbb{T}^{d+1} \to \mathbb{T}$ be a smooth function such that $\sup_{\mu\in\mathcal{O}} \|\beta(\mu,\cdot,\centerdot)\|_{\textnormal{Lip}} < 1$ then the map

$$(\varphi,\theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \beta(\mu,\varphi,\theta)) \in \mathbb{T}^{d+1}$$

is a diffeomorphism with inverse having the form

$$(\varphi,\theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \widehat{\beta}(\mu,\varphi,\theta)) \in \mathbb{T}^{d+1}.$$

Moreover, one has the relation

$$y = \theta + \beta(\mu,\varphi,\theta) \Longleftrightarrow \theta = y + \widehat{\beta}(\mu,\varphi,y). \tag{4.5}$$

Define the operators

$$\mathscr{B} = (1 + \partial_\theta\beta)\mathcal{B}, \tag{4.6}$$

with

$$\mathcal{B}\rho(\mu,\varphi,\theta) = \rho\big(\mu,\varphi,\theta + \beta(\mu,\varphi,\theta)\big). \tag{4.7}$$

By straightforward computations we obtain

$$\mathscr{B}^{-1}\rho(\mu,\varphi,y) = \Big(1 + \partial_y\widehat{\beta}(\mu,\varphi,y)\Big)\rho\big(\mu,\varphi,y + \widehat{\beta}(\mu,\varphi,y)\big) \tag{4.8}$$

and

$$\mathcal{B}^{-1}\rho(\mu,\varphi,y) = \rho\big(\mu,\varphi,y + \widehat{\beta}(\mu,\varphi,y)\big).$$

The following lemma gives some elementary algebraic properties for $\mathcal{B}^{\pm1}$ and $\mathscr{B}^{\pm1}$.

Lemma 4.3. *The following assertions hold true.*

(i) The action of $\mathscr{B}^{-1}$ on the derivative is given by

$$\mathscr{B}^{-1}\partial_\theta = \partial_\theta\mathcal{B}^{-1}.$$

(ii) Denote by $\mathscr{B}^\star$ the $L^2_\theta(\mathbb{T})$-adjoint of $\mathscr{B}$, then

$$\mathscr{B}^\star = \mathcal{B}^{-1} \quad \textit{and} \quad \mathcal{B}^\star = \mathscr{B}^{-1}.$$

Now we shall state the following result proved in [47, Lem. 6.2].

Lemma 4.4. *Let $(q, d\gamma, s_0)$ as in (1.11). Let $\beta \in W^{q,\infty,\gamma}\big(\mathcal{O}, H^\infty(\mathbb{T}^{d+1})\big)$ such that*

$$\|\beta\|^{\gamma,\mathcal{O}}_{q,2s_0} \leqslant \varepsilon_0, \tag{4.9}$$

with ε_0 small enough. Then the following assertions hold true.

(i) The linear operators $\mathcal{B}, \mathscr{B} : W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big) \to W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big)$ are continuous and invertible, with

$$\forall s \geqslant s_0, \quad \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \left(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\right) + C\|\beta\|_{q,s}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \tag{4.10}$$

and

$$\forall s \geqslant s_0, \quad \|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \left(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\right) + C\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{4.11}$$

(ii) The functions β and $\widehat{\beta}$ are linked through

$$\forall s \geqslant s_0, \quad \|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\|\beta\|_{q,s}^{\gamma,\mathcal{O}}. \tag{4.12}$$

(iii) Let $\beta_1, \beta_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^\infty(\mathbb{T}^{d+1}))$ satisfying (4.9). If we denote

$$\Delta_{12}\beta = \beta_1 - \beta_2 \quad \text{and} \quad \Delta_{12}\widehat{\beta} = \widehat{\beta}_1 - \widehat{\beta}_2,$$

then they are linked through

$$\forall s \geqslant s_0, \quad \|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}} \max_{j\in\{1,2\}} \|\beta_j\|_{q,s+1}^{\gamma,\mathcal{O}}\right). \tag{4.13}$$

4.2 Operators

In this section we shall collect some basic definitions and properties related to suitable operators class. Consider a smooth family of bounded operators on Sobolev spaces $H^s(\mathbb{T}^{d+1})$,

$$T : \mu = (b,\omega) \in \mathcal{O} \mapsto T(\mu) \in \mathcal{L}(H^s(\mathbb{T}^{d+1}, \mathbb{C})).$$

The linear operator $T(\mu)$ can be represented by the infinite dimensional matrix $\Big(T_{l_0,j_0}^{l,j}(\mu)\Big)_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2\\(j,j_0)\in\mathbb{Z}^2}}$ with

$$T(\mu)\mathbf{e}_{l_0,j_0} = \sum_{(l,j)\in\mathbb{Z}^{d+1}} T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j} \quad \text{where} \quad T_{l_0,j_0}^{l,j}(\mu) = \big\langle T(\mu)\mathbf{e}_{l_0,j_0}, \mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Along this paper the operators and the test functions may depend on the same parameter μ and thus the action of the operator $T(\mu)$ on a scalar function $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C}))$ is by convention defined through

$$(T\rho)(\mu,\varphi,\theta) := T(\mu)\rho(\mu,\varphi,\theta).$$

4.2.1 Toeplitz in time operators

In this paper we always consider Toeplitz in time operators, whose definition is the following.

Definition 4.1. *An operator $T(\mu)$ is said Toeplitz in time (actually in the variable φ) if its Fourier coefficients satisfy,*

$$T_{l_0,j_0}^{l,j}(\mu) = T_{0,j_0}^{l-l_0,j}(\mu).$$

Or equivalently,

$$T_{l_0,j_0}^{l,j}(\mu) = T_{j_0}^{j}(\mu, l-l_0) \quad \text{with} \quad T_{j_0}^{j}(\mu,l) := T_{0,j_0}^{l,j}(\mu).$$

Notice that the action of a Toeplitz operator $T(\mu)$ on a function $\rho = \sum_{(l_0,j_0)\in\mathbb{Z}^{d+1}} \rho_{l_0,j_0}\mathbf{e}_{l_0,j_0}$ is given by

$$T(\mu)\rho = \sum_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2\\(j,j_0)\in\mathbb{Z}^2}} T_{j_0}^{j}(\mu, l-l_0)\rho_{l_0,j_0}\mathbf{e}_{l,j}. \tag{4.14}$$

For Toeplitz operators, we can define the off-diagonal norm as

$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{|\alpha|}\sup_{(b,\omega)\in\mathcal{O}}\|\partial_\mu^\alpha(T)(\mu)\|_{\text{O-d},s-|\alpha|},\tag{4.15}$$

where

$$\|T\|_{\text{O-d},s}^2=\sum_{(l,m)\in\mathbb{Z}^{d+1}}\langle l,m\rangle^{2s}\sup_{j-k=m}|T_j^k(l)|^2.$$

The cut-off projectors $(P_N)_{N\in\mathbb{N}^*}$ are defined as follows

$$(P_N T(\mu))\,\mathbf{e}_{l_0,j_0}=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\|l-l_0|,|j-j_0|\leqslant N}}T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j}\quad\text{and}\quad P_N^\perp T=T-P_NT.\tag{4.16}$$

The next lemma lists some classical results. The proofs are very similar to those in [16].

Lemma 4.5. *Let (γ,q,d,s_0,s) satisfying (1.11). Let T, T_1 and T_2 be Toeplitz in time operators.*

(i) Projectors properties : Let $N\in\mathbb{N}^*$. Let $t\in\mathbb{R}_+$. Then

$$\|P_NT\rho\|_{\text{O-d},q,s+t}^{\gamma,\mathcal{O}}\leqslant N^t\|T\rho\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\quad\text{and}\quad\|P_N^\perp T\rho\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant N^{-t}\|T\rho\|_{\text{O-d},q,s+t}^{\gamma,\mathcal{O}}.$$

(ii) Interpolation inequality : Let $q<s_1\leqslant s_3\leqslant s_2$, $\overline{\theta}\in[0,1]$ with $s_3=\overline{\theta}s_1+(1-\overline{\theta})s_2$. Then

$$\|T\|_{\text{O-d},q,s_3}^{\gamma,\mathcal{O}}\lesssim\left(\|T\|_{\text{O-d},q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|T\|_{\text{O-d},q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iii) Composition law :

$$\|T_1T_2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|T_1\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|T_2\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|T_1\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|T_2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.$$

(iv) Link between operators and off-diagonal norms :

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|T\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In particular

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

4.2.2 Reversible and reversibility preserving operators

We recall the following definitions of real reversible and reversibility preserving operators, see for instance [4, Def. 2.2].

Definition 4.2. *We define the following involution*

$$(\mathscr{S}_2\rho)(\varphi,\theta)=\rho(-\varphi,-\theta).\tag{4.17}$$

We say that an operator $T(\mu)$ is

- *real if for all $\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})$, we have*

$$\overline{\rho}=\rho\quad\Longrightarrow\quad\overline{T\rho}=T\rho.$$

- *reversible if*

$$T(\mu)\circ\mathscr{S}_2=-\mathscr{S}_2\circ T(\mu).$$

- *reversibility preserving if*

$$T(\mu)\circ\mathscr{S}_2=\mathscr{S}_2\circ T(\mu).$$

The following lemma gives characterizations of the reversibility notion in terms of Fourier coefficients. A similar result is stated in [4, Lem. 2.6].

Lemma 4.6. *Let T be an operator. Then T is*

- *real if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=\overline{T_{l_0,j_0}^{l,j}}.$$
- *reversible if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=-T_{l_0,j_0}^{l,j}.$$
- *reversibility-preserving if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=T_{l_0,j_0}^{l,j}.$$

Throughout this paper we shall constantly make use of two kinds of operators : multiplication and integral operators.

Definition 4.3. *Let T be an operator as in Section 4.2. We say that*

- *T is a multiplication operator if there exists a function $M:(\mu,\varphi,\theta)\mapsto M(\mu,\varphi,\theta)$ such that*
$$(T\rho)(\mu,\varphi,\theta)=M(\mu,\varphi,\theta)\rho(\mu,\varphi,\theta).$$
- *T is an integral operator if there exists a function (called the kernel) $K:(\mu,\varphi,\theta,\eta)\mapsto K(\mu,\varphi,\theta,\eta)$ such that*
$$(T\rho)(\mu,\varphi,\theta)=\int_{\mathbb{T}}\rho(\mu,\varphi,\eta)K(\mu,\varphi,\theta,\eta)d\eta.$$

We shall need the following lemma whose proof can be found in [47, Lem. 4.4].

Lemma 4.7. *Let (γ,q,d,s_0,s) satisfy (1.11), then the following assertions hold true.*

(i) Let T be a multiplication operator by a real-valued function M, then the following holds true.

- *If $M(\mu,-\varphi,-\theta)=M(\mu,\varphi,\theta)$, then T is real and reversibility preserving Toeplitz in time and space operator.*
- *If $M(\mu,-\varphi,-\theta)=-M(\mu,\varphi,\theta)$, then T is real and reversible Toeplitz in time and space operator.*

Moreover,
$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|M\|_{q,s+s_0}^{\gamma,\mathcal{O}}.$$

(ii) Let T be an integral operator with a real-valued kernel K.

- *If $K(\mu,-\varphi,-\theta,-\eta)=K(\mu,\varphi,\theta,\eta)$, then T is a real and reversibility preserving Toeplitz in time operator.*
- *If $K(\mu,-\varphi,-\theta,-\eta)=-K(\mu,\varphi,\theta,\eta)$, then T is a real and reversible Toeplitz in time operator.*

In addition,
$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\int_{\mathbb{T}}\|K(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\lesssim\|K\|_{q,H^{s+s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}$$

and
$$\begin{aligned}\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\int_{\mathbb{T}}\|K(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}d\eta+\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\int_{\mathbb{T}}\|K(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\|K\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\|K\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\end{aligned}$$

where the notation $*,\cdot,\centerdot$ denote μ,φ,θ, respectively.

In the following lemma we shall study the action of a change of variables as in (4.7) on an integral operator. More precisely, we shall need two partial change of coordinates $\mathcal{B}^1$ and $\mathcal{B}^2$ acting respectively on the variables θ and η and defined through

$$(\mathcal{B}^1\rho)(\mu,\varphi,\theta,\eta)=\rho\big(\mu,\varphi,\theta+\beta_1(\mu,\varphi,\theta),\eta\big),\qquad (4.18)$$
$$(\mathcal{B}^2\rho)(\mu,\varphi,\theta,\eta)=\rho\big(\mu,\varphi,\theta,\eta+\beta_2(\mu,\varphi,\eta)\big),$$

with β_1,β_2 two smooth functions satisfying (4.9). A similar result is proved in [16, Lem. 2.34] for pseudo-differential integral operators, so we omit the proof here. We also include the difference estimate which is useful to study the stability of the Cantor sets in Section 7.2. The proof of the difference estimate is standard and we shall also skip it here.

Lemma 4.8. *Let (γ,q,d,s_0,s) satisfy (1.11). Given $r\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$, we consider a C^∞ function in the form*

$$K:(\mu,\varphi,\theta,\eta)\mapsto K(\mu,\varphi,\theta,\eta).$$

We consider the integral operator associated to K, namely

$$T\rho(\mu,\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)K(\mu,\varphi,\theta,\eta)d\eta.$$

Then the following assertions hold true.

(i) Let $\mathcal{B}^1$ and $\mathcal{B}^2$ as in (4.18) associated to β_1 and β_2, respectively and enjoying the smallness condition (4.9). Then,

$$\|\mathcal{B}^1\mathcal{B}^2K\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\|K\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\Big(\max_{i\in\{1,2\}}\|\beta_i\|_{q,s}^{\gamma,\mathcal{O}}\Big)\|K\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}.\qquad (4.19)$$

Now, assume that $\beta_1=\beta_2=\beta$ satisfies the following symmetry conditions

$$\beta(\mu,-\varphi,-\theta)=-\beta(\mu,\varphi,\theta).\qquad (4.20)$$

Consider $\mathscr{B}$, $\mathcal{B}$ be quasi-periodic changes of variables as in (4.6)-(4.7), then

- *if $K(\mu,-\varphi,-\theta,-\eta)=K(\mu,\varphi,\theta,\eta)$, then $\mathcal{B}^{-1}T\mathscr{B}$ is a real and reversibility preserving Toeplitz in time integral operator.*
- *if $K(\mu,-\varphi,-\theta,-\eta)=-K(\mu,\varphi,\theta,\eta)$, then $\mathcal{B}^{-1}T\mathscr{B}$ is a real and reversible Toeplitz in time inegral operator.*

In this case, for any $k\in\mathbb{N}$,

$$\|\partial_\theta^k\mathcal{B}^{-1}T\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|K\|_{q,H^{s+s_0+k}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\beta\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\|K\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}.\qquad (4.21)$$

(ii) Introduce $\mathcal{B}_r$ a quasi-periodic change of variables as in (4.7) associated to β_r (linked to r) Consider r_1, $r_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$. Denote

$$\Delta_{12}r=r_1-r_2,\quad \Delta_{12}f_r=f_{r_1}-f_{r_2}$$

for any quantity f_r depending on r and assume that there exist $\varepsilon_0>0$ small enough such that

$$\forall i\in\{1,2\},\quad \|\beta_{r_i}\|_{q,2s_0}^{\gamma,\mathcal{O}}+\|K_{r_i}\|_{q,H^{s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.\qquad (4.22)$$

Then, for any $k\in\mathbb{N}$, the following estimate holds

$$\begin{aligned}\|\Delta_{12}\partial_\theta^k\mathcal{B}_r^{-1}T_r\mathscr{B}_r\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}K_r\|_{q,H^{s+s_0+k}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_r\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\qquad (4.23)\\&+\Big(\max_{i\in\{1,2\}}\|\beta_{r_i}\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big)\|\Delta_{12}K_r\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&+\Big(\max_{i\in\{1,2\}}\|K_{r_i}\|_{q,H^{s+s_0+k+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\max_{i\in\{1,2\}}\|\beta_{r_i}\|_{q,s+s_0+k+1}^{\gamma,\mathcal{O}}\Big)\|\Delta_{12}\beta_r\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

5 Functional of interest and regularity aspects

The main goal of this section is to reformulate the problem in a dynamical system language more adapted to KAM techniques. More precisely, we shall write the equation (2.17) as a Hamiltonian perturbation of an integrable system, given by the linear dynamics at the equilibrium state. Then, by selecting finitely-many tangential sites and decomposing the phase space into tangential and normal subspaces we can introduce action-angle variables on the tangential part allowing to reformulate the problem in terms of embedded tori. This reduces the problem into the search for zeros of a functional $\mathcal{F}$ to which the Nash-Moser implicit function theorem will be applied. We shall also study in this section some regularity aspects for the perturbed Hamiltonian vector field appearing in $\mathcal{F}$ and needed during the Nash-Moser scheme. This approach has been intensively used before, for instance in [3, 5, 13, 14, 16].

Notice that, according to Lemmata 2.1 and 3.2, the equation (2.17), that is also (2.5), can be written in the form

$$\partial_t r = \partial_\theta \mathrm{L}(b)(r) + X_P(r) \quad \text{with} \quad X_P(r) := \tfrac{1}{2}\partial_\theta r + \partial_\theta \mathcal{K}_b * r - F_b[r], \tag{5.1}$$

where the nonlinear functional $F_b[r]$ is introduced in (2.6) and the convolution kernel is given by (3.10). Since we shall look for small amplitude quasi-periodic solutions then it is more convenient to rescale the solution as follows $r \mapsto \varepsilon r$ with r bounded. Hence, the Hamiltonian equation (2.17) takes the form

$$\partial_t r = \partial_\theta \mathrm{L}(b)(r) + \varepsilon X_{P_\varepsilon}(r), \tag{5.2}$$

where X_{P_ε} is the Hamiltonian vector field defined by $X_{P_\varepsilon}(r) := \varepsilon^{-2} X_P(\varepsilon r)$. Notice that (5.2) is the Hamiltonian system generated by the rescaled Hamiltonian

$$\begin{aligned}\mathcal{H}_\varepsilon(r) &= \varepsilon^{-2} H(\varepsilon r)\\ &:= H_{\mathrm{L}}(r) + \varepsilon P_\varepsilon(r),\end{aligned} \tag{5.3}$$

with H_{L} the quadratic Hamiltonian defined in Lemma 3.2 and $\varepsilon P_\varepsilon(r)$ containing terms of higher order more than cubic.

5.1 Reformulation with the action-angle and normal variables

Recall from (3.27) that the tangential sites are defined by

$$\mathbb{S} := \{j_1, \ldots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < j_2 < \ldots < j_d.$$

We now define the symmetrized tangential sets $\overline{\mathbb{S}}$ and $\mathbb{S}_0$ by

$$\overline{\mathbb{S}} := \mathbb{S} \cup (-\mathbb{S}) = \{\pm j,\ j \in \mathbb{S}\} \quad \text{and} \quad \mathbb{S}_0 = \overline{\mathbb{S}} \cup \{0\}. \tag{5.4}$$

Then we decompose the phase space $L^2_0(\mathbb{T})$ into the following $L^2(\mathbb{T})$-orthogonal direct sum

$$L^2_0(\mathbb{T}) = L_{\overline{\mathbb{S}}} \overset{\perp}{\oplus} L^2_\perp, \quad L_{\overline{\mathbb{S}}} := \Big\{ \sum_{j \in \overline{\mathbb{S}}} r_j e_j,\ \overline{r_j} = r_{-j} \Big\}, \quad L^2_\perp := \Big\{ z = \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} z_j e_j \in L^2_0(\mathbb{T}) \Big\}, \tag{5.5}$$

where we denote $e_j(\theta) = e^{\mathrm{i} j \theta}$. The associated orthogonal projectors $\Pi_{\overline{\mathbb{S}}}, \Pi^\perp_{\mathbb{S}_0}$ are defined by

$$r = \sum_{j \in \mathbb{Z}^*} r_j e_j = v + z, \quad v := \Pi_{\overline{\mathbb{S}}} r := \sum_{j \in \overline{\mathbb{S}}} r_j e_j, \quad z := \Pi^\perp_{\mathbb{S}_0} r := \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} r_j e_j, \tag{5.6}$$

where v and z are respectively called the tangential and normal variables. For fixed small amplitudes $(\mathtt{a}_j)_{j \in \overline{\mathbb{S}}} \in (\mathbb{R}^*_+)^d$ satisfying $\mathtt{a}_{-j} = \mathtt{a}_j$, we introduce the action-angle variables on the tangential set $L_{\overline{\mathbb{S}}}$ by making the following symplectic polar change of coordinates

$$\forall j \in \overline{\mathbb{S}}, \quad r_j = \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j}, \tag{5.7}$$

where

$$\forall j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{T}. \tag{5.8}$$

Thus, any function r of the phase space L_0^2 can be represented as

$$r = A(\vartheta, I, z) := v(\vartheta, I) + z \quad \text{where} \quad v(\vartheta, I) := \sum_{j \in \overline{\mathbb{S}}} \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j} e_j\,. \tag{5.9}$$

Observe that the function $v(-\omega_{\rm Eq}(b)t, 0)$, where $\omega_{\rm Eq}$ is defined in (3.28), corresponds to the solution of the linear system (3.8) described by (3.42). In these new coordinates, the involution $\mathscr{S}$ defined in (2.22) reads

$$\mathfrak{S} : (\vartheta, I, z) \mapsto (-\vartheta, I, \mathscr{S} z) \tag{5.10}$$

and the symplectic 2-form in (2.20) becomes, after straightforward computations using (5.7) and (5.8),

$$\mathcal{W} = \sum_{j \in \mathbb{S}} d\vartheta_j \wedge dI_j + \frac{1}{2} \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} \frac{1}{\mathrm{i}j} dr_j \wedge dr_{-j} = \Big(\sum_{j \in \mathbb{S}} d\vartheta_j \wedge dI_j \Big) \oplus \mathcal{W}_{|L_\perp^2}\,, \tag{5.11}$$

where $\mathcal{W}_{|L_\perp^2}$ denotes the restriction of $\mathcal{W}$ to $L_\perp^2$. This proves that the transformation A defined in (5.9) is symplectic and in the action-angle and normal coordinates $(\vartheta, I, z) \in \mathbb{T}^d \times \mathbb{R}^d \times L_\perp^2$, the Hamiltonian system generated by $\mathcal{H}_\varepsilon$ in (5.3) transforms into the one generated by the Hamiltonian

$$H_\varepsilon = \mathcal{H}_\varepsilon \circ A. \tag{5.12}$$

Since $\mathrm{L}(b)$ in Lemma 3.2 preserves the subspaces $L_{\overline{\mathbb{S}}}$ and $L_\perp^2$ then the quadratic Hamiltonian $H_{\rm L}$ in (3.13) (see (3.14)) in the variables (ϑ, I, z) reads, up to an additive constant,

$$H_{\rm L} \circ A = -\sum_{j \in \mathbb{S}} \Omega_j(b) I_j + \frac{1}{2} \langle \mathrm{L}(b)\, z, z \rangle_{L^2(\mathbb{T})} = -\omega_{\rm Eq}(b) \cdot I + \frac{1}{2} \langle \mathrm{L}(b)\, z, z \rangle_{L^2(\mathbb{T})}, \tag{5.13}$$

where $\omega_{\rm Eq} \in \mathbb{R}^d$ is the unperturbed tangential frequency vector defined by (3.28). By (5.3) and (5.13), the Hamiltonian H_ε in (5.12) reads

$$H_\varepsilon = \mathcal{N} + \varepsilon \mathcal{P}_\varepsilon \quad \text{with} \quad \mathcal{N} := -\omega_{\rm Eq}(b) \cdot I + \frac{1}{2} \langle \mathrm{L}(b)\, z, z \rangle_{L^2(\mathbb{T})} \quad \text{and} \quad \mathcal{P}_\varepsilon := P_\varepsilon \circ A. \tag{5.14}$$

We look for an embedded invariant torus

$$\begin{array}{rccl} i: & \mathbb{T}^d & \rightarrow & \mathbb{R}^d \times \mathbb{R}^d \times L_\perp^2 \\ & \varphi & \mapsto & i(\varphi) := (\vartheta(\varphi), I(\varphi), z(\varphi)) \end{array} \tag{5.15}$$

of the Hamiltonian vector field

$$X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\vartheta H_\varepsilon, \Pi_{\mathbb{S}_0}^\perp \partial_\theta \nabla_z H_\varepsilon) \tag{5.16}$$

filled by quasi-periodic solutions with Diophantine frequency vector ω. We point out that for the value $\varepsilon = 0$ the Hamiltonian system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_0}(i(\varphi))$$

possesses, for any value of the parameter $b \in (b_0, b_1)$, the invariant torus

$$i_{\rm flat}(\varphi) := (\varphi, 0, 0). \tag{5.17}$$

Now we consider the family of Hamiltonians,

$$H_\varepsilon^\alpha := \mathcal{N}_\alpha + \varepsilon \mathcal{P}_\varepsilon \quad \text{where} \quad \mathcal{N}_\alpha := \alpha \cdot I + \frac{1}{2} \langle \mathrm{L}(b)\, z, z \rangle_{L^2(\mathbb{T})}, \tag{5.18}$$

which depends on the constant vector $\alpha \in \mathbb{R}^d$. For the value $\alpha = -\omega_{\rm Eq}(b)$ we have $H_\varepsilon^\alpha = H_\varepsilon$. The parameter α is introduced in order to ensure the validity of some compatibility conditions during the approximate inverse process. We look for zeros of the nonlinear operator

$$\begin{aligned}\mathcal{F}(i,\alpha,(b,\omega),\varepsilon) &:= \omega\cdot\partial_\varphi i(\varphi) - X_{H_\varepsilon^\alpha}(i(\varphi))\\ &= \begin{pmatrix} \omega\cdot\partial_\varphi\vartheta(\varphi) - \alpha - \varepsilon\partial_I\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\vartheta\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi z(\varphi) - \partial_\theta\big[\mathrm{L}(b)z(\varphi) + \varepsilon\nabla_z\mathcal{P}_\varepsilon\big(i(\varphi)\big)\big]\end{pmatrix},\end{aligned} \tag{5.19}$$

where $\mathcal{P}_\varepsilon$ is defined in (5.3). For any $\alpha \in \mathbb{R}^d$, the Hamiltonian H_ε^α is invariant under the involution $\mathfrak{S}$ defined in (5.10),

$$H_\varepsilon^\alpha \circ \mathfrak{S} = H_\varepsilon^\alpha.$$

Thus, we look for reversible solutions of $\mathcal{F}(i,\alpha,(b,\omega),\varepsilon) = 0$, namely satisfying

$$\vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathscr{S}z)(\varphi). \tag{5.20}$$

We define the periodic component $\mathfrak{I}$ of the torus i by

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi,0,0) = (\Theta(\varphi), I(\varphi), z(\varphi)) \quad \text{with} \quad \Theta(\varphi) = \vartheta(\varphi) - \varphi,$$

and the weighted Sobolev norm of $\mathfrak{I}$ as

$$\|\mathfrak{I}\|_{q,s}^{\gamma,\mathcal{O}} := \|\Theta\|_{q,s}^{\gamma,\mathcal{O}} + \|I\|_{q,s}^{\gamma,\mathcal{O}} + \|z\|_{q,s}^{\gamma,\mathcal{O}}.$$

5.2 Regularity of the perturbed Hamiltonian vector field

This section is devoted to some regularity aspects of the Hamiltonian involved in the equation (2.17). We shall need the following lemma.

Lemma 5.1. *Let (γ,q,s_0,s) satisfy* (1.11). *There exists $\varepsilon_0 \in (0,1]$ such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0,$$

then the operators $\partial_\theta\mathbf{L}_r$ and $\partial_\theta\mathbf{S}_r$, defined in (3.2) *and* (3.3) *write*

$$\partial_\theta\mathbf{L}_r = \partial_\theta\mathcal{K}_{1,b}*\cdot + \partial_\theta\mathbf{L}_{r,1} \quad \text{with} \quad \mathbf{L}_{r,1}(\rho)(b,\varphi,\theta) := \int_{\mathbb{T}}\rho(\varphi,\eta)\mathbb{K}_{r,1}(b,\varphi,\theta,\eta)d\eta, \tag{5.21}$$

$$\partial_\theta\mathbf{S}_r = \partial_\theta\mathcal{K}_{2,b}*\cdot + \partial_\theta\mathbf{S}_{r,1} \quad \text{with} \quad \mathbf{S}_{r,1}(\rho)(b,\varphi,\theta) = \int_{\mathbb{T}}\rho(\varphi,\eta)\mathscr{K}_{r,1}(b,\varphi,\theta,\eta)d\eta \tag{5.22}$$

where $\mathcal{K}_{1,b}$, $\mathcal{K}_{2,b}$ are given by (3.11)-(3.12) *and the kernels $\mathbb{K}_{r,1}(b,\varphi,\theta,\eta), \mathscr{K}_{r,1}(b,\varphi,\theta,\eta) \in \mathbb{R}$ satisfy the following symmetry property: if $r(-\varphi,-\theta) = r(\varphi,\theta)$ then*

$$\mathbb{K}_{r,1}(b,-\varphi,-\theta,-\eta) = \mathbb{K}_{r,1}(b,\varphi,\theta,\eta), \tag{5.23}$$

$$\mathscr{K}_{r,1}(b,-\varphi,-\theta,-\eta) = \mathscr{K}_{r,1}(b,\varphi,\theta,\eta) \tag{5.24}$$

and the following estimates

$$\|\mathbb{K}_{r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}, \tag{5.25}$$

$$\|\mathscr{K}_{r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s}^{\gamma,\mathcal{O}}. \tag{5.26}$$

Moreover,

$$\|\partial_\theta\mathcal{K}_{1,b}*\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}, \tag{5.27}$$

$$\|\partial_\theta\mathcal{K}_{2,b}*\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}, \tag{5.28}$$

$$\|\partial_\theta\mathbf{L}_{r,1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s_0+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}, \tag{5.29}$$

$$\|\partial_\theta\mathbf{S}_{r,1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{5.30}$$

Proof. According to (3.17) we may write

$$A_r(\varphi,\theta,\eta) = 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\left(\left(\frac{R(b,\varphi,\eta)-R(b,\varphi,\theta)}{2b\sin\left(\frac{\eta-\theta}{2}\right)}\right)^2 + \frac{1}{b^2}R(b,\varphi,\eta)R(b,\varphi,\theta)\right)^{\frac{1}{2}}$$
$$:= 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right| v_{r,1}(b,\varphi,\theta,\eta). \tag{5.31}$$

Notice that $v_{r,1}$ is smooth when r is smooth and small enough, and $v_{0,1}=1$. More precisely, by using Lemma 4.1-(iv)-(v) combined with Lemma 4.2-(ii) and the smallness condition on r, we get

$$\|v_{r,1}-1\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.32}$$

Using the morphism property of the logarithm, we can write

$$\log(A_r(b,\varphi,\theta,\eta)) = \log\big(2b\big) + \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\eta-\theta}{2}\right)\right) + \log\left(v_{r,1}(b,\varphi,\theta,\eta)\right)$$
$$:= \log\big(2b\big) + \mathcal{K}_{1,b}(\eta-\theta) + \mathbb{K}_{r,1}(b,\varphi,\theta,\eta) \tag{5.33}$$

and (5.23) immediately follows. Moreover, (3.2) and (5.33) give (5.21). Applying Lemma 4.1-(v) together with (5.32) and the smallness condition on r, we obtain (5.25). Using (5.25), Lemma 4.7-(ii) and the smallness property on r, we get (5.29). Similarly, from (3.18) we can link B_r^2 to B_0^2 by

$$B_r^2(b,\varphi,\theta,\eta) = B_0^2(b,\varphi,\theta,\eta) + \Big(R^2(b,\varphi,\theta)R^2(b,\varphi,\eta)-b^4\Big) - 2\Big(R(b,\varphi,\theta)R(b,\varphi,\eta)-b^2\Big)\cos(\eta-\theta)$$
$$= B_0^2(b,\varphi,\theta,\eta)\big(1+P_r(b,\varphi,\theta,\eta)\big)$$

with

$$P_r(b,\varphi,\theta,\eta) := \tfrac{\left(R^2(b,\varphi,\theta)R^2(b,\varphi,\eta)-b^4\right)-2\left(R(b,\varphi,\theta)R(b,\varphi,\eta)-b^2\right)\cos(\eta-\theta)}{1+b^4-2b^2\cos(\eta-\theta)}.$$

so that we can write

$$\log\big(B_r(b,\varphi,\theta,\eta)\big) = \log\big(B_0(b,\varphi,\theta,\eta)\big) + \tfrac{1}{2}\log\big(1+P_r(b,\varphi,\theta,\eta)\big)$$
$$:= \mathcal{K}_{2,b}(\eta-\theta) + \mathscr{K}_{r,1}(b,\varphi,\theta,\eta) \tag{5.34}$$

and (5.24) immediately follows. Moreover, (3.3) and (5.34) give (5.22). Notice that that P_r is smooth with respect to each variable and with respect to r with $P_0=0$. We conclude by Lemma 4.1-(iv)-(v) and the smallness property on r that

$$\|P_r\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s}^{\gamma,\mathcal{O}}.$$

As a consequence, composition laws in Lemma 4.1 together with the smallness property on r imply (5.26). Then, using (5.26), Lemma 4.7-(ii) and the smallness property on r, we get (5.30). The estimates (5.27)-(5.28) can be obtained using (3.22), (3.24) and Leibniz rule combined with the following estimate

$$\sup_{n\in\mathbb{N}}\|b\mapsto b^n\|_q^{\gamma,\mathcal{O}} \lesssim 1.$$

This ends the proof of Lemma 5.1. □

We now provide tame estimates for the vector field X_P defined in (5.1).

Lemma 5.2. *Let* (γ,q,s_0,s) *satisfy* (1.11). *There exists* $\varepsilon_0\in(0,1]$ *such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0,$$

then the vector field X_P*, defined in* (5.1) *satisfies the following estimates*

(i) $\|X_P(r)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(ii) $\|d_rX_P(r)[\rho]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \big(\|\rho_1\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$

Proof. We shall follow the proof developed in [15, Lem 10.2]. We first prove the estimate (iii) and the estimates (ii) and (i) then follow by Taylor formula since $d_r X_P(0)=0$ and $X_P(0)=0$. Recall from Lemma 3.1, (5.21) and (5.22) that

$$d_r X_H(r)[\rho] = -d_r F_b(r)[\rho] = -\partial_\theta\left(V_r\rho\right) - \partial_\theta\mathcal{K}_b * \rho - \partial_\theta\mathbf{L}_{r,1}\rho + \partial_\theta\mathbf{S}_{r,1}\rho.$$

According to (5.1), P is is the Hamiltonian generated by higher order more than cubic terms $H_{\geqslant 3}$. Then differentiating with respect to r the last expression we obtain

$$d_r^2 X_P(r)[\rho_1,\rho_2] = -\partial_\theta\left(\big(d_r V_r[\rho_2]\big)\rho_1\right) - \partial_\theta\left(d_r\mathbf{L}_{r,1}[\rho_2]\rho_1\right) + \partial_\theta\left(d_r\mathbf{S}_{r,1}[\rho_2]\rho_1\right). \tag{5.35}$$

Recall, from (5.21) and (5.33), that

$$\mathbf{L}_{r,1}(\rho)(b,\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\log(v_{r,1}(b,\varphi,\theta,\eta))d\eta. \tag{5.36}$$

Hence by differentiation we obtain

$$d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta) = \frac{1}{2}\int_{\mathbb{T}}\rho_1(\varphi,\eta)\frac{\big(d_r v_{r,1}^2\big)[\rho_2](b,\varphi,\theta,\eta)}{v_{r,1}^2(b,\varphi,\theta,\eta)}d\eta. \tag{5.37}$$

Coming back to (5.31) it is obvious that the dependance in r of the functional $v_{r,1}^2$ is smooth. Straightforward calculus leads to

$$\frac{1}{2}d_r v_{r,1}^2(r)[\rho_2](b,\varphi,\theta,\eta) = \frac{R(b,\varphi,\theta)-R(b,\varphi,\eta)}{\sin^2\left(\frac{\eta-\theta}{2}\right)}\left(\frac{\rho_2(\varphi,\theta)}{R(b,\varphi,\theta)} - \frac{\rho_2(\varphi,\eta)}{R(b,\varphi,\eta)}\right) + \frac{\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)+\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)}{R(b,\varphi,\theta)R(b,\varphi,\eta)}.$$

Using (5.32) combined with the law products stated in Lemma 4.1, Lemma 4.2-(ii) and the smallness condition of Lemma 5.2 we find that

$$\|d_r v_{r,1}^2(r)[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{5.38}$$

According to (5.38), (5.37) and using Lemma 4.1-(iv)-(v), Lemma 4.7-(ii) and the smallness condition we obtain,

$$\begin{aligned}\|\partial_\theta d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim& \|d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ \lesssim& \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\big(\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big).\end{aligned} \tag{5.39}$$

Now we shall move to the estimate of $d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta)$. By differentiating with respect to r in (5.22) and (5.34), we obtain

$$d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta) = \frac{1}{2}\int_{\mathbb{T}}\rho_1(\varphi,\eta)\frac{\big(d_r B_r^2\big)[\rho_2](b,\varphi,\theta,\eta)}{B_r^2(b,\varphi,\theta,\eta)}d\eta.$$

In view of (3.18), direct computations yield

$$\begin{aligned}\frac{1}{2}d_r B_r^2(r)[\rho_2](b,\varphi,\theta,\eta) =& \rho_2(\varphi,\theta)R^2(b,\varphi,\eta) + \rho_2(\varphi,\eta)R^2(b,\varphi,\theta)\\ &- \Big(\rho_2(\varphi,\theta)\tfrac{R(b,\varphi,\eta)}{R(b,\varphi,\theta)} + \rho_2(\varphi,\eta)\tfrac{R(b,\varphi,\theta)}{R(b,\varphi,\eta)}\Big)\cos(\eta-\theta).\end{aligned}$$

Then, Lemma 4.1-(iv)-(v) and the smallness condition on r imply

$$\|d_r B_r^2(r)[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

It follows from Lemma 4.7-(ii), that

$$\begin{aligned}\|\partial_\theta d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim& \|d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ \lesssim& \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\big(\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big).\end{aligned} \tag{5.40}$$

Next we shall move to the estimate of $d_rV_r[\rho_2]$. From Lemma 3.1, we can write

$$V_r = V_r^0 + V_r^1 + V_r^2, \quad \text{with} \quad V_r^0(b,\varphi,\theta) := -\tfrac{1}{2}\int_{\mathbb{T}} \tfrac{R^2(b,\varphi,\eta)}{R^2(b,\varphi,\theta)} d\eta,$$
$$V_r^1(b,\varphi,\theta) := -\tfrac{1}{R(b,\varphi,\theta)}\int_{\mathbb{T}} \log\big(A_r(b,\varphi,\theta,\eta)\big)\partial_\eta\big(R(b,\varphi,\eta)\sin(\eta-\theta)\big)d\eta,$$
$$V_r^2(b,\varphi,\theta) := -\tfrac{1}{R^3(b,\varphi,\theta)}\int_{\mathbb{T}} \log\big(B_r(b,\varphi,\theta,\eta)\big)\partial_\eta\big(R(b,\varphi,\eta)\sin(\eta-\theta)\big)d\eta.$$

Differentiating V_r^0 with respect to r in the direction ρ_2 yields

$$d_rV_r^0(r)[\rho_2](\theta) = -\int_{\mathbb{T}} \tfrac{\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)-\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)}{R^4(b,\varphi,\theta)} d\eta.$$

Law products in Lemma 4.1 and the smallness condition in r then imply

$$\|d_rV_r^0(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{5.41}$$

Differentiating V_r^1 with respect to r in the direction ρ_2 gives

$$\begin{aligned} d_rV_r^1(r)[\rho_2](\theta) &= -\int_{\mathbb{T}} \log\left(A_r(b,\varphi,\theta,\eta)\right)\partial_\eta d_rf_r[\rho_2](b,\varphi,\theta,\eta)d\eta \\ &\quad - \frac{1}{2}\int_{\mathbb{T}} \frac{\big(d_rv_{r,1}^2\big)[\rho_2](b,\varphi,\theta,\eta)}{v_{r,1}^2(b,\varphi,\theta,\eta)}\partial_\eta f_r(b,\varphi,\theta,\eta)d\eta \\ &:= \mathcal{I}_1(\theta) + \mathcal{I}_2(\theta), \end{aligned}$$

with

$$f_r(b,\varphi,\theta,\eta) := \tfrac{R(b,\varphi,\eta)}{R(b,\varphi,\theta)}\sin(\eta-\theta).$$

Straightforward computations give

$$d_rf_r[\rho_2](b,\varphi,\theta) = \tfrac{\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)-\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)}{R^3(b,\varphi,\theta)R(b,\varphi,\eta)}\sin(\eta-\theta).$$

Then, by law products and composition laws in Lemma 4.1 we immediately deduce that

$$\|\partial_\eta f_r\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim 1 + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}, \tag{5.42}$$
$$\|\partial_\eta d_rf_r[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \big(1+\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \big(1+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{5.43}$$

The following estimate on $\mathcal{I}_2$ can be obtained combining (5.38), (5.32) and (5.42) together with Lemma 4.1-(iv)-(v) and the smallness property on r.

$$\|\mathcal{I}_2\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{5.44}$$

As for $\mathcal{I}_1$ we argue in a similar way to Lemma 5.1 to get

$$\|\mathcal{I}_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{5.45}$$

Putting together (5.44) and (5.45) yields

$$\|d_rV_r^1(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{5.46}$$

Differentiating V_r^2 with respect to r in the direction ρ_2 yields

$$\begin{aligned} d_rV_r^2(r)[\rho_2](b,\varphi,\theta) &= -\int_{\mathbb{T}} \log\big(B_r(b,\varphi,\theta,\eta)\big)\partial_\eta\left(\tfrac{\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)-3\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)}{R^5(b,\varphi,\theta)R(b,\varphi,\eta)}\sin(\eta-\theta)\right)d\eta \\ &\quad - \frac{1}{2}\int_{\mathbb{T}} \frac{\big(d_rB_r^2\big)[\rho_2](b,\varphi,\theta,\eta)}{B_r^2(b,\varphi,\theta,\eta)}\partial_\eta\Big(\tfrac{R(b,\varphi,\eta)}{R^3(b,\varphi,\theta)}\sin(\eta-\theta)\Big)d\eta. \end{aligned}$$

Arguing in a similar way as above we find

$$\|d_r V_r^2(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{5.47}$$

Putting together (5.41), (5.46) and (5.47) gives

$$\|d_r V_r(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{5.48}$$

Therefore, according to the law products in Lemma 4.1, (5.48) and the smallness condition we obtain

$$\begin{aligned}\big\|\partial_\theta\big(d_r V_r(r)[\rho_2]\rho_1\big)\big\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \|d_r V_r(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \big\|d_r V_r(r)[\rho_2]\big\|_{q,s+1}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s_0}^{\gamma,\mathcal{O}} + \big\|d_r V_r(r)[\rho_2]\big\|_{q,s_0}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combining the latter estimate with (5.35), (5.39) and (5.40) allows to get

$$\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$$

Using Sobolev embeddings we get the desired result. This concludes the proof of Lemma 5.2. □

Notice in particular that Lemma 5.2-(i) implies that there is no singlarity in ε for the rescaled vector field X_{P_ε} defined in (5.2). Based on the previous lemma we obtain tame estimates for the Hamiltonian vector field

$$X_{\mathcal{P}_\varepsilon} = (\partial_I\mathcal{P}_\varepsilon, -\partial_\vartheta\mathcal{P}_\varepsilon, \Pi_{\mathbb{S}}^\perp\partial_\theta\nabla_z\mathcal{P}_\varepsilon)$$

defined by (5.14) and (5.16). The proof can be done in a similar way to [16, Lem. 5.1].

Lemma 5.3. *Let (γ, q, s_0, s) satisfy (1.11). There exists $\varepsilon_0 \in (0,1)$ such that if*

$$\varepsilon \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant 1,$$

then the perturbed Hamiltonian vector field $X_{\mathcal{P}_\varepsilon}$ satisfies the following tame estimates,

(i) $\|X_{\mathcal{P}_\varepsilon}(i)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}$.

(ii) $\big\|d_i X_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath}]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{\imath}\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\widehat{\imath}\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\big\|d_i^2 X_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath},\widehat{\imath}]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{\imath}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\widehat{\imath}\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\big(\|\widehat{\imath}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)^2$.

6 Construction of an approximate right inverse

In order to apply a modified Nash-Moser scheme, we need to construct an approximate right inverse of the linearized operator associated to the functional $\mathcal{F}$, that is

$$d_{(i,\alpha)}\mathcal{F}(i_0,\alpha_0)[\widehat{\imath},\widehat{\alpha}] = \omega\cdot\partial_\varphi\widehat{\imath} - d_i X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))[\widehat{\imath}] - (\widehat{\alpha},0,0). \tag{6.1}$$

where $\mathcal{F}$ is defined in (5.19), $\alpha_0 : \mathcal{O} \to \mathbb{R}^d$ is a vector-valued function and $i_0 = (\vartheta_0, I_0, z_0)$ is an arbitrary torus close to the flat one and satisfying the reversibility condition

$$\vartheta_0(-\varphi) = -\vartheta_0(\varphi), \quad I_0(-\varphi) = I_0(\varphi) \quad \text{and} \quad z_0(-\varphi) = (\mathscr{S}z_0)(\varphi). \tag{6.2}$$

For this aim, we may follow the procedure introduced in [11] and slightly simplified in [37, Sec. 6]. The main idea consists in conjugating (6.1) by a linear diffeomorphism of the toroidal phase space $\mathbb{T}^d\times\mathbb{R}^d\times L^2_\perp$ to a triangular system in the action-angles-normal variables up to small fast decaying error terms and terms vanishing at an exact solution. Then, to solve the triangular system we are led to almost invert the linearized operator in the normal directions, given by

$$\widehat{\mathcal{L}}_\omega := \Pi_{\mathbb{S}_0}^\perp\Big(\omega\cdot\partial_\varphi - \partial_\theta(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) - \varepsilon\partial_\theta\mathcal{R}(\varphi)\Big)\Pi_{\mathbb{S}_0}^\perp, \tag{6.3}$$

where $H_\varepsilon^{\alpha_0}$ is given by (5.18),

$$\mathcal{R}(\varphi) := L_2^\top(\varphi)\partial_I \nabla_I \mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi) + L_2^\top(\varphi)\partial_z \nabla_I \mathcal{P}_\varepsilon(i_0(\varphi)) + \partial_I \nabla_z \mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi), \tag{6.4}$$

$\mathcal{P}_\varepsilon$ is defined by (5.14) and

$$L_2(\phi) := -[(\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\phi))]^\top \partial_\theta^{-1}, \quad \widetilde{z}_0(\vartheta) \quad := z_0(\vartheta_0^{-1}(\vartheta)). \tag{6.5}$$

Here, for any linear operator $A \in \mathcal{L}(\mathbb{R}^d, L^2_\perp)$ the transposed operator $A^\top : L^2_\perp \to \mathbb{R}^d$ is defined through the duality relation

$$\forall\, u \in L^2_\perp\,, \quad \forall\, v \in \mathbb{R}^d, \quad \langle A^\top u, v\rangle_{\mathbb{R}^d} = \langle u, Av\rangle_{L^2(\mathbb{T}^d)}. \tag{6.6}$$

We point out the presence of the remainder term due to the linear change of variables performed to decouple the dynamics of the action-angle components from the normal ones. For more details we refer the reader to [37, Sec. 6].

6.1 Linearized operator in the normal direction

Our main goal here is to explore the structure of the linear operator $\widehat{\mathcal{L}}_\omega$, introduced in (6.3). We have the following result. The following lemma describes the asymptotic structure of $\widehat{\mathcal{L}}_\omega$ around the equilibrium state, described in Lemma 3.1.

Proposition 6.1. *Let* (γ, q, d, s_0) *satisfy* (1.11). *Then, the operator* $\widehat{\mathcal{L}}_\omega$ *defined in* (6.3) *takes the form*

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^\perp \Big(\mathcal{L}_{\varepsilon r} - \varepsilon \partial_\theta \mathcal{R}\Big)\Pi_{\mathbb{S}_0}^\perp, \tag{6.7}$$

where

(i) the operator $\mathcal{L}_{\varepsilon r}$ *is given by*

$$\mathcal{L}_{\varepsilon r} := \omega \cdot \partial_\varphi + \partial_\theta\big(V_{\varepsilon r}\, \cdot\,\big) + \partial_\theta \mathbf{L}_{\varepsilon r} - \partial_\theta \mathbf{S}_{\varepsilon r}, \tag{6.8}$$

with $V_{\varepsilon r}$, $\mathbf{L}_{\varepsilon r}$ *and* $\mathbf{S}_{\varepsilon r}$ *defined by* (3.1), (3.2) *and* (3.3).

(ii) the function r *is given by*

$$r(\varphi, \cdot) = A(i_0(\varphi)), \tag{6.9}$$

satisfies the following symmetry property

$$r(-\varphi, -\theta) = r(\varphi, \theta) \tag{6.10}$$

and the following estimates

$$\|r\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}_0\|_{q,s}^{\gamma,\mathcal{O}}, \tag{6.11}$$

$$\|\Delta_{12} r\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\Delta_{12} i\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12} i\|_{q,s_0}^{\gamma,\mathcal{O}} \max_{j\in\{1,2\}} \|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}}. \tag{6.12}$$

(iii) the operator $\mathcal{R}$, *defined in* (6.4), *is an integral operator with kernel* J *satisfying the symmetry property*

$$J(-\varphi, -\theta, -\eta) = J(\varphi, \theta, \eta) \tag{6.13}$$

and the following estimates: for all $\ell \in \mathbb{N}$,

$$\sup_{\eta\in\mathbb{T}} \|(\partial_\theta^\ell J)(*, \cdot, \centerdot, \eta + \centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}_0\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}, \tag{6.14}$$

$$\sup_{\eta\in\mathbb{T}} \|\Delta_{12}(\partial_\theta^\ell J)(*, \cdot, \centerdot, \eta + \centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\Delta_{12} i\|_{q,s+3+\ell}^{\gamma,\mathcal{O}} + \|\Delta_{12} i\|_{q,s_0+3}^{\gamma,\mathcal{O}} \max_{j\in\{1,2\}} \|\mathfrak{I}_j\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}. \tag{6.15}$$

where $*, \cdot, \centerdot,$ *denote successively the variables* b, φ, θ *and* $\mathfrak{I}_j(\varphi) = i_j(\varphi) - (\varphi, 0, 0)$.

Proof. From (5.18), (5.12), (5.9) and (5.3) we obtain

$$\begin{aligned}(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) &= \mathrm{L}(b)\Pi_{\mathbb{S}_0}^\perp + \varepsilon\partial_z\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))\\ &= \mathrm{L}(b)\Pi_{\mathbb{S}_0}^\perp + \varepsilon\Pi_{\mathbb{S}_0}^\perp\partial_r\nabla_r P_\varepsilon(A(i_0(\varphi)))\\ &= \Pi_{\mathbb{S}_0}^\perp\partial_r\nabla_r\mathcal{H}_\varepsilon(A(i_0(\varphi)))\\ &= \Pi_{\mathbb{S}_0}^\perp\partial_r\nabla_r H(\varepsilon A(i_0(\varphi))).\end{aligned}$$

According to the general form of the linearized operator stated in Lemma 3.1 one has

$$-\partial_\theta(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) = \Pi_{\mathbb{S}_0}^\perp\big(\partial_\theta\big(V_{\varepsilon r}(b,\varphi,\cdot)\,\cdot\,\big) + \partial_\theta\mathbf{L}_{\varepsilon r} - \partial_\theta\mathbf{S}_{\varepsilon r}\big)\Pi_{\mathbb{S}_0}^\perp.$$

Inserting this identity into (6.3) gives (6.7). The operator $\mathcal{R}(\varphi)$ in (6.4) may be written as

$$\mathcal{R}(\varphi) = \mathcal{R}_1(\varphi) + \mathcal{R}_2(\varphi) + \mathcal{R}_3(\varphi), \quad \text{with} \quad \begin{aligned}\mathcal{R}_1(\varphi) &:= L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi),\\ \mathcal{R}_2(\varphi) &:= L_2^\top(\varphi)\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi) &:= \partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi).\end{aligned}$$

Notice that $\mathcal{R}_1(\varphi)$, $\mathcal{R}_2(\varphi)$ and $\mathcal{R}_3(\varphi)$ have a finite-dimensional rank. In fact, from (6.5) and (6.6) one may write

$$L_2(\varphi)[\rho] = \sum_{k=1}^d \big\langle L_2(\varphi)[\rho], \underline{e}_k\big\rangle_{\mathbb{R}^d}\,\underline{e}_k = \sum_{k=1}^d \big\langle \rho, L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}\,\underline{e}_k,$$

with $(\underline{e}_k)_{k=1}^d$ being the canonical basis of $\mathbb{R}^d$. Hence

$$\begin{aligned}\mathcal{R}_1(\varphi)[\rho] &= \sum_{k=1}^d \big\langle \rho, L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})} A_1(\varphi)[\underline{e}_k] \qquad \text{with} \quad A_1(\varphi) = L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)[\rho] &= \sum_{k=1}^d \big\langle \rho, L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})} A_3(\varphi)[\underline{e}_k] \qquad \text{with} \quad A_3(\varphi) = \partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi)).\end{aligned}$$

Analogously, since $A_2(\varphi) := \partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)) : L^2_\perp \to \mathbb{R}^d$, then we may write

$$\mathcal{R}_2(\varphi)[\rho] = \sum_{k=1}^d \big\langle \rho, A_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})} L_2^\top(\varphi)[\underline{e}_k].$$

By setting

$$\begin{aligned}g_{k,1}(\varphi,\theta) = g_{k,3}(\varphi,\theta) = \chi_{k,2}(\varphi,\theta) &:= L_2^\top(\varphi)[\underline{e}_k](\theta), \qquad & g_{k,2}(\varphi,\theta) &:= A_2^\top(\varphi)[\underline{e}_k](\theta),\\ \chi_{k,1}(\varphi,\theta) &:= A_1(\varphi)[\underline{e}_k](\theta), & \chi_{k,3}(\varphi,\theta) &:= A_3(\varphi)[\underline{e}_k](\theta),\end{aligned}$$

we can see that the operator $\mathcal{R}$ takes the integral form

$$\mathcal{R}\rho(\varphi,\theta) = \int_{\mathbb{T}}\rho(\varphi,\eta)J(\varphi,\theta,\eta)d\eta, \quad \text{with} \quad J(\varphi,\theta,\eta) := \sum_{k'=1}^3\sum_{k=1}^d g_{k,k'}(\varphi,\eta)\chi_{k,k'}(\varphi,\theta).$$

The symmetry property (6.13) is a consequence of the definition of r and the reversibility condition (6.2) imposed on the torus i_0. The estimates (6.15), (6.14), (6.11) and (6.12) are straightforward and follow in a similar way to Proposition 6.1 in [47]. □

6.2 Diagonalization of the linearized operator in the normal directions

This section is devoted to the reduction of the linearized operator $\widehat{\mathcal{L}}_\omega$, defined in (6.7), to constant coefficients. This procedure is done in three steps. First, we reduce the operator $\mathcal{L}_{\varepsilon r}$ introduced in (6.8) up to smoothing reminders. Then we study the action of the localization in the normal directions. Finally, we almost eliminate the remainders by using a KAM reduction procedure. We fix the following parameters.

$$\begin{aligned} s_l &:= s_0 + \tau_1 q + \tau_1 + 2, \quad & \overline{\mu}_2 &:= 4\tau_1 q + 6\tau_1 + 3,\\ \overline{s}_l &:= s_l + \tau_2 q + \tau_2, & \overline{s}_h &:= \tfrac{3}{2}\overline{\mu}_2 + s_l + 1.\end{aligned} \tag{6.16}$$

6.2.1 Leading orders reduction

In this section, we shall straighten the transport part by using a suitable quasi-periodic symplectic change of variables and look at its conjugation action on the non-local terms. The reduction of the transport part is done by a KAM iterative scheme. Such technique is now well-developed in [3, 7, 14, 28, 37, 47]. The result reads as follows.

Proposition 6.2. *Let* $(\gamma, q, d, \tau_1, s_0, \overline{\mu}_2, s_l, \overline{s}_h, S)$ *satisfy* (1.11), (1.10) *and* (6.16). *Let* $\upsilon \in \left(0, \frac{1}{q+2}\right]$. *We set*

$$\sigma_1 = s_0 + \tau_1 q + 2\tau_1 + 4 \quad \text{and} \quad \sigma_2 = s_0 + \sigma_1 + 3. \tag{6.17}$$

For any $(\mu_2, \mathtt{p}, s_h)$ *satisfying*

$$\mu_2 \geqslant \overline{\mu}_2, \quad \mathtt{p} \geqslant 0, \quad s_h \geqslant \max\left(\frac{3}{2}\mu_2 + s_l + 1, \overline{s}_h + \mathtt{p}\right), \tag{6.18}$$

there exists $\varepsilon_0 > 0$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_2}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.19}$$

then following assertions hold true.

1. *There exist*

$$V_{i_0}^{\infty} \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) \quad \textit{and} \quad \beta \in W^{q,\infty,\gamma}(\mathcal{O}, H^S)$$

such that with $\mathscr{B}$ *defined in* (4.6) *one gets the following results.*

(i) The function $V_{i_0}^{\infty}$ *satisfies the estimate:*

$$\|V_{i_0}^{\infty} - \tfrac{1}{2}\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon. \tag{6.20}$$

(ii) The transformations $\mathscr{B}^{\pm 1}, \mathcal{B}^{\pm 1}, \beta$ *and* $\widehat{\beta}$ *satisfy the following estimates: for all* $s \in [s_0, S]$,

$$\|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}, \tag{6.21}$$

$$\|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\beta\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right). \tag{6.22}$$

Moreover, β *and* $\widehat{\beta}$ *satisfy the following symmetry condition:*

$$\beta(\mu, -\varphi, -\theta) = -\beta(\mu, \varphi, \theta) \quad \text{and} \quad \widehat{\beta}(\mu, -\varphi, -\theta) = -\widehat{\beta}(\mu, \varphi, \theta). \tag{6.23}$$

(iii) Let $n \in \mathbb{N}$, *then in the truncated Cantor set*

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \left\{(b,\omega) \in \mathcal{O} \quad \text{s.t.} \quad \left|\omega\cdot l + jV_{i_0}^{\infty}(b,\omega)\right| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\}, \tag{6.24}$$

we have the decomposition

$$\mathfrak{L}_{\varepsilon r} := \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} = \omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathtt{E}_n^0,$$

where $\mathcal{L}_{\varepsilon r}$ *is given by* (6.8), $\mathcal{K}_b$ *is defined in Lemma 3.2 and* $\mathtt{E}_n^0 = \mathtt{E}_n^0(b,\omega,i_0)$ *is a linear operator satisfying*

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{6.25}$$

The operator $\mathfrak{R}_{\varepsilon r}$ *is a real and reversibility preserving integral operator satisfying*

$$\forall s \in [s_0, S], \quad \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right). \tag{6.26}$$

2. Given two tori i_1 and i_2 both satisfying (6.19), we have

$$\|\Delta_{12}V_i^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}, \tag{6.27}$$

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} + \|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}. \tag{6.28}$$

In addition, we have

$$\max_{k\in\{0,1\}}\|\Delta_{12}(\partial_\theta^k\mathfrak{R}_{\varepsilon r})\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_2}^{\gamma,\mathcal{O}}. \tag{6.29}$$

Proof. Notice that along the proof, to simplify the notation, we shall omit the dependence with respect to the parameters b,ω kipping in mind that the functions appearing actually depend on them. We begin by setting

$$V_0 = \tfrac{1}{2} \quad\text{and}\quad f_0(\varphi,\theta) := V_{\varepsilon r}(\varphi,\theta) - \tfrac{1}{2}, \tag{6.30}$$

with $V_{\varepsilon r}$ defined by (3.1). According to (6.10) and (3.4), one gets

$$f_0(-\varphi,-\theta) = f_0(\varphi,\theta). \tag{6.31}$$

Notice that according to (3.1), (5.48) and Taylor formula, one has

$$\|f_0\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\left(1 + \|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right). \tag{6.32}$$

These properties allow to apply [47, Prop. 6.2], whose proof is based on a KAM iterative scheme reduction of the perturbation term f_0 and construct β and $V_{i_0}^\infty$. In particular, for any $n\in\mathbb{N}$, we are able to construct a Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ in the form (6.24) in which the following reduction holds

$$\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi + \partial_\theta\big(V_{\varepsilon r}\,\cdot\,\big)\big)\mathscr{B} = \omega\cdot\partial_\varphi + V_{i_0}^\infty\partial_\theta + \mathtt{E}_n^0, \tag{6.33}$$

where $\mathtt{E}_n^0$ is an operator enjoying the decay property stated in (6.25). Using (6.33), (5.21), (5.22) and Lemma 4.3-(i), one obtains in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ the following decomposition

$$\begin{aligned}\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} &= \mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_{\varepsilon r}\cdot\right)\big)\mathscr{B} + \mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r}\mathscr{B} - \mathscr{B}^{-1}\partial_\theta\mathbf{S}_{\varepsilon r}\mathscr{B}\\ &= \omega\cdot\partial_\varphi + V_{i_0}^\infty\partial_\theta + \mathscr{B}^{-1}\partial_\theta\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B} + \mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r,1}\mathscr{B}\\ &\quad - \mathscr{B}^{-1}\partial_\theta\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B} - \mathscr{B}^{-1}\partial_\theta\mathbf{S}_{\varepsilon r,1}\mathscr{B} + \mathtt{E}_n^0\\ &= \omega\cdot\partial_\varphi + V_{i_0}^\infty\partial_\theta + \partial_\theta\mathcal{K}_{1,b}*\cdot - \partial_\theta\mathcal{K}_{2,b}*\cdot + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathtt{E}_n^0,\end{aligned}$$

wehre

$$\mathfrak{R}_{\varepsilon r} := \Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B} - \mathcal{K}_{1,b}*\cdot\Big] - \Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B} - \mathcal{K}_{2,b}*\cdot\Big] + \mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B} - \mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}. \tag{6.34}$$

Direct computations using (3.11) lead to

$$\mathcal{B}^{-1}(\mathcal{K}_{1,b}*(\mathscr{B}\rho))(\varphi,\theta) = \int_{\mathbb{T}}\rho(\varphi,\eta)\log(\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta))d\eta,$$

where

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta) := \left|\sin\left(\tfrac{\eta-\theta}{2} + \widehat{h}(\varphi,\theta,\eta)\right)\right| \quad\text{with}\quad \widehat{h}(\varphi,\theta,\eta) := \tfrac{\widehat{\beta}(\varphi,\theta)-\widehat{\beta}(\varphi,\eta)}{2}.$$

Using elementary trigonometric identities, we can write

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta) = \left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right| v_{\widehat{\beta}}(\varphi,\theta,\eta) \quad\text{with}\quad v_{\widehat{\beta}}(\varphi,\theta,\eta) := \cos\left(\widehat{h}(\varphi,\theta,\eta)\right) + \frac{\sin\big(\widehat{h}(\varphi,\theta,\eta)\big)}{\tan\left(\frac{\eta-\theta}{2}\right)}.$$

In view of (6.23), one finds that $v_{\widehat{\beta}}$ enjoys the following symmetry property,

$$v_{\widehat{\beta}}(-\varphi,-\theta,-\eta) = v_{\widehat{\beta}}(\varphi,\theta,\eta). \tag{6.35}$$

Using the morphism property of the logarithm, one gets

$$\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\rho\big)\mathscr{B}-\mathcal{K}_{1,b}*\rho\Big](\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)d\eta$$

where

$$\mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta):=\log\big(v_{\widehat{\beta}}(\varphi,\theta,\eta)\big).\tag{6.36}$$

Notice that (6.36) and (6.35) imply

$$\mathbb{K}_{\widehat{\beta},2}(-\varphi,-\theta,-\eta)=\mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)\in\mathbb{R}.\tag{6.37}$$

Hence, we deduce from Lemma 4.7 that $\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{1,b}*\cdot$ is a real and reversibility preserving Toeplitz in time operator. Writting

$$v_{\widehat{\beta}}(\varphi,\theta,\eta)=1+\Big(\cos\big(\widehat{h}(\varphi,\theta,\eta)\big)-1\Big)+\frac{\sin\big(\widehat{h}(\varphi,\theta,\eta)\big)}{\tan\left(\frac{\eta-\theta}{2}\right)},$$

one finds, by Lemma 4.1-(v), Lemma 4.2 and (6.22),

$$\begin{aligned}\|v_{\widehat{\beta}}-1\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Moreover, by (6.28) and the Mean Value Theorem (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0+2$), we find

$$\begin{aligned}\|\Delta_{12}v_{\widehat{\beta}}\|_{q,H^{\overline{s}_h+\mathtt{p}+s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}+s_0+2}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}.\end{aligned}$$

In a similar way, we deduce that

$$\|\mathbb{K}_{\widehat{\beta},2}\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1+\sigma_1}^{\gamma,\mathcal{O}}\right),\tag{6.38}$$

$$\|\Delta_{12}\mathbb{K}_{\widehat{\beta},2}\|_{q,H^{\overline{s}_h+\mathtt{p}+s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}.\tag{6.39}$$

In view of Lemma 4.7 we get, from (6.38) and (6.39),

$$\max_{k\in\{0,1,2\}}\left\|\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{1,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+3+\sigma_1}^{\gamma,\mathcal{O}}\right),\tag{6.40}$$

$$\max_{k\in\{0,1\}}\left\|\Delta_{12}\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{1,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}.\tag{6.41}$$

According to (3.12), one finds

$$\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*(\mathscr{B}\rho)\big)(\varphi,\theta)-\mathcal{K}_{2,b}*\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)d\eta,$$

with

$$\begin{aligned}\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)&:=\tfrac{1}{2}\Big[\log\big(1+b^4-2b^2\cos(\eta-\theta+\widehat{h}(\varphi,\theta,\eta))\big)-\log\big(1+b^4-2b^2\cos(\eta-\theta)\big)\Big]\\&=\tfrac{1}{2}\log\left(\frac{1+b^4-2b^2\cos\big(\eta-\theta+\widehat{h}(\varphi,\theta,\eta)\big)}{1+b^4-2b^2\cos(\eta-\theta)}\right).\end{aligned}$$

From (6.23), we deduce that

$$\mathscr{K}_{\widehat{\beta},2}(-\varphi,-\theta,-\eta)=\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)\in\mathbb{R}.\tag{6.42}$$

It follows from Lemma 4.7 that $\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot$ is a real and reversibility preserving Toeplitz in time operator. Arguing as for (5.26) and using (6.22), we obtain

$$\begin{aligned}\|\mathscr{K}_{\widehat{\beta},2}\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+\sigma_{1}}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.43}$$

Using Mean Value theorem, applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0+1$, one also gets by (6.28)

$$\begin{aligned}\|\Delta_{12}\mathscr{K}_{\widehat{\beta},2}\|_{q,H^{\overline{s}_{h}+\mathtt{p}+s_{0}+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_{h}+\mathtt{p}+s_{0}+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_{h}+\mathtt{p}+s_{0}+1+\sigma_{1}}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.44}$$

Consequently, in view of Lemma 4.7, we get from (6.43)

$$\max_{k\in\{0,1,2\}}\left\|\partial_{\theta}^{k}\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+s_{0}+2+\sigma_{1}}^{\gamma,\mathcal{O}}\right)\tag{6.45}$$

and from (6.44)

$$\max_{k\in\{0,1\}}\left\|\Delta_{12}\partial_{\theta}^{k}\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,\overline{s}_{h}+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_{h}+\mathtt{p}+s_{0}+1+\sigma_{1}}^{\gamma,\mathcal{O}}.\tag{6.46}$$

Next, putting together (5.23), (6.23) and Lemma 4.8, we infer that $\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}$ is a real and reversibility preserving Toeplitz in time operator. Moreover, we obtain from (4.21) in Lemma 4.8, (6.22), (5.25), (6.11) and the smallness condition (6.19),

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_{\theta}^{k}\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathbb{K}_{\varepsilon r,1}\|_{q,H^{s+s_{0}+2}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\beta\|_{q,s+s_{0}+2}^{\gamma,\mathcal{O}}\|\mathbb{K}_{\varepsilon r,1}\|_{q,H^{s_{0}}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+s_{0}+2+\sigma_{1}}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.47}$$

Applying Lemma 4.1, we get from (5.31), Lemma 4.2 and (6.12),

$$\begin{aligned}\|\Delta_{12}\mathbb{K}_{\varepsilon r,1}\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}v_{\varepsilon r,1}\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_{0}}^{\gamma,\mathcal{O}}\max_{j\in\{1,2\}}\|\mathfrak{I}_{j}\|_{q,s+1}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Added to Lemma 4.8-(ii), (6.28), (6.22), (5.25) and (6.19), we infer

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_{\theta}^{k}\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\|_{\textnormal{O-d},q,\overline{s}_{h}+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_{h}+\mathtt{p}+s_{0}+1+\sigma_{1}}^{\gamma,\mathcal{O}}.\tag{6.48}$$

The next task is to estimate the term $\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}$ in (6.34). Note that (5.24), (6.23) and Lemma 4.8 imply that $\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}$ is a real and reversibility preserving Toeplitz in time operator. In addition, Lemma 4.8 together with the estimates (5.26), (6.11) and (6.22) give

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_{\theta}^{k}\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathscr{K}_{\varepsilon r,1}\|_{q,H^{s+2}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\beta\|_{q,s+2}^{\gamma,\mathcal{O}}\|\mathscr{K}_{\varepsilon r,1}\|_{q,H^{s_{0}}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+s_{0}+2+\sigma_{1}}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.49}$$

Applying Lemma 4.1, we get from (5.34) and (6.12),

$$\|\Delta_{12}\mathscr{K}_{\varepsilon r,1}\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_{0}}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\mathfrak{I}_{i}\|_{q,s}^{\gamma,\mathcal{O}}\Big).$$

Then, combining Lemma 4.8-(ii), (6.28), (6.22), (5.26) and (6.19), we get

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_{\theta}^{k}\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}\|_{\textnormal{O-d},q,\overline{s}_{h}+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_{h}+\mathtt{p}+s_{0}+1+\sigma_{1}}^{\gamma,\mathcal{O}}.\tag{6.50}$$

In view of (6.34), Lemma 4.8 and the previous computations, we conclude that $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving toeplitz in time integral operator which satisfies, by (6.40), (6.45), (6.47) and (6.49),

$$\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+3+\sigma_1}^{\gamma,\mathcal{O}}\right).$$

In addition, combining (6.41), (6.46), (6.48) and (6.50) yields

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}.$$

This ends the proof of Proposition 6.2. □

6.2.2 Projection in the normal directions

In this section, we study the effects of the localization in the normal directions for the reduction of the transport part. For that purpose, we consider the localized quasi-periodic symplectic change of coordinates defined by

$$\mathscr{B}_\perp=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.$$

Then, the main result of this section reads as follows.

Proposition 6.3. *Let* $(\gamma,q,d,\tau_1,s_0,s_h,\overline{s}_h,\mathtt{p},S)$ *satisfy the assumptions* (1.11), (1.10) *and* (6.18). *There exist* $\varepsilon_0>0$ *and* $\sigma_3=\sigma_3(\tau_1,q,d,s_0)\geqslant\sigma_2$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_3}^{\gamma,\mathcal{O}}\leqslant1,\tag{6.51}$$

then the following assertions hold true.

(i) The operators $\mathscr{B}_\perp^{\pm1}$ *satisfy the following estimate*

$$\|\mathscr{B}_\perp^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{6.52}$$

(ii) For any $n\in\mathbb{N}^*$, *in the Cantor set* $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *introduced in Proposition 6.2, we have*

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp&=\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{R}_0+\mathtt{E}_n^1\\&:=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_0+\mathscr{R}_0+\mathtt{E}_n^1\\&:=\mathscr{L}_0+\mathtt{E}_n^1,\end{aligned}$$

where $\mathscr{R}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ *is reversible and* $\mathscr{D}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{D}_0\Pi_{\mathbb{S}_0}^\perp$ *is a reversible Fourier multiplier operator given by*

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad\mathscr{D}_0\mathbf{e}_{l,j}=\mathrm{i}\,\mu_j^0\,\mathbf{e}_{l,j},$$

with

$$\mu_j^0(b,\omega,i_0)=\Omega_j(b)+jr^1(b,\omega,i_0)\quad\textit{and}\quad r^1(b,\omega,i_0)=V_{i_0}^\infty(b,\omega)-\tfrac12\tag{6.53}$$

and such that

$$\|r^1\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\quad\text{and}\quad\|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.\tag{6.54}$$

(iii) The operator $\mathtt{E}_n^1$ *satisfies the following estimate*

$$\|\mathtt{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\tag{6.55}$$

(iv) The operator $\mathscr{R}_0$ *is a real and reversible Toeplitz in time operator satisfying*

$$\forall s\in[s_0,S],\quad\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\tag{6.56}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{6.57}$$

(v) Furthermore the operator $\mathscr{L}_0$ satisfies

$$\forall s \in [s_0, S], \quad \|\mathscr{L}_0 \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.58}$$

Proof. **(i)** Follows from (6.21) and Lemma 4.1-(ii).
(ii) From (6.7) and the decomposition $\mathrm{Id} = \Pi_{\mathbb{S}_0} + \Pi_{\mathbb{S}_0}^{\perp}$ we write

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp &= \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R})\mathscr{B}_\perp \\ &= \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathcal{L}_{\varepsilon r}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} - \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathcal{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} - \varepsilon\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathcal{R}\mathscr{B}_\perp.\end{aligned}$$

According to the definitions of $\mathfrak{L}_{\varepsilon r}$ and $\mathcal{L}_{\varepsilon r}$ seen in Proposition 6.2 and in Lemma 3.1 and using (5.21), (5.22) and (3.10), one has in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathcal{L}_{\varepsilon r}\mathscr{B} = \mathscr{B}\mathfrak{L}_{\varepsilon r} \quad \text{and} \quad \mathcal{L}_{\varepsilon r} = \omega\cdot\partial_\varphi + \partial_\theta\left(V_{\varepsilon r}\cdot\right) + \partial_\theta\mathcal{K}_b * \cdot + \partial_\theta\mathbf{L}_{\varepsilon r,1} - \partial_\theta\mathbf{S}_{\varepsilon r,1}$$

and therefore

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp = \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} - \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right) + \partial_\theta\mathbf{L}_{\varepsilon r,1} - \partial_\theta\mathbf{S}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} - \varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp,$$

where we have used the identities

$$\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp} = \mathscr{B}_\perp^{-1} \quad \text{and} \quad [\Pi_{\mathbb{S}_0}^{\perp}, T] = 0 = [\Pi_{\mathbb{S}_0}, T],$$

for any Fourier multiplier T. The structure of $\mathfrak{L}_{\varepsilon r}$ is detailed in Proposition 6.2, and from this we deduce that

$$\begin{aligned}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} &= \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathtt{E}_n^0\big)\Pi_{\mathbb{S}_0}^{\perp} \\ &= \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big) + \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} + \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp} \\ &= \mathscr{B}_\perp\big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big) + \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} + \Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp}.\end{aligned}$$

It follows that

$$\begin{aligned}\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} &= \big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big)\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp} \\ &= \big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big)\Pi_{\mathbb{S}_0}^{\perp} + \Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} \\ &\quad + \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp}.\end{aligned}$$

Consequently, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, one has the following reduction

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp =& \big(\omega\cdot\partial_\varphi + V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big)\Pi_{\mathbb{S}_0}^{\perp} + \Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp} \\ & - \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right) + \partial_\theta\mathbf{L}_{\varepsilon r,1} - \partial_\theta\mathbf{S}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} - \varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp + \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp} \\ :=& \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{D}_0 + \mathscr{R}_0 + \mathtt{E}_n^1,\end{aligned} \tag{6.59}$$

where we set

$$\mathscr{D}_0 := \big(V_{i_0}^{\infty}\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big)\Pi_{\mathbb{S}_0}^{\perp}$$

and

$$\mathtt{E}_n^1 := \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^{\perp}. \tag{6.60}$$

(iii) Results from (6.60), (6.52), (6.21), (6.25) and Lemma 4.1-(ii).
(iv) For the estimates (6.56) and (6.57), we refer to Lemma [47, Prop 6.3 and Lem. 6.3]. They are based on suitable duality representations of $\mathscr{B}_\perp^{\pm 1}$ linked to $\mathscr{B}^{\pm 1}$.
(v) It is obtained by (5.27), (5.28), (6.20), (6.56) and Lemma 4.5-(iv). ☐

6.2.3 Elimination of the remainder term

We perform here the KAM reduction of the remainder $\mathscr{R}_0$ of Proposition 6.3. This procedure allows to diagonalise the linearized operator in the normal directions, namely to conjugate it to a constant coefficients operator $\mathscr{L}_\infty$, up to fast decaying terms.

Proposition 6.4. *Let* $(\gamma, q, d, \tau_1, \tau_2, s_0, s_l, \overline{s}_l, \overline{s}_h, \overline{\mu}_2, S)$ *satisfy* (1.11), (1.10), (6.16). *For any* (μ_2, s_h) *satisfying*

$$\mu_2 \geqslant \overline{\mu}_2 + 2\tau_2 q + 2\tau_2, \quad \text{and} \quad s_h \geqslant \frac{3}{2}\mu_2 + \overline{s}_l + 1, \tag{6.61}$$

there exist $\varepsilon_0 \in (0,1)$ *and* $\sigma_4 = \sigma_4(\tau_1, \tau_2, q, d) \geqslant \sigma_3$ *such that if*

$$\varepsilon\gamma^{-2-q} N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.62}$$

then the following assertions hold true.

(i) There exists a family of invertible linear operator $\Phi_\infty : \mathcal{O} \to \mathcal{L}\big(H^s \cap L^2_\perp\big)$ *satisfying the estimates*

$$\forall s \in [s_0, S], \qquad \|\Phi_\infty^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.63}$$

There exists a diagonal operator $\mathscr{L}_\infty = \mathscr{L}_\infty(b, \omega, i_0)$ *taking the form*

$$\mathscr{L}_\infty = \omega \cdot \partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty$$

where $\mathscr{D}_\infty = \mathscr{D}_\infty(b, \omega, i_0) = \Pi_{\mathbb{S}_0}^\perp \mathscr{D}_\infty \Pi_{\mathbb{S}_0}^\perp$ *is a reversible Fourier multiplier operator given by,*

$$\forall (l, j) \in \mathbb{Z}^d \times \mathbb{S}_0^c, \quad \mathscr{D}_\infty \mathbf{e}_{l,j} = \mathrm{i}\, \mu_j^\infty\, \mathbf{e}_{l,j},$$

with

$$\forall j \in \mathbb{S}_0^c, \quad \mu_j^\infty(b, \omega, i_0) = \mu_j^0(b, \omega, i_0) + r_j^\infty(b, \omega, i_0) \tag{6.64}$$

and

$$\sup_{j \in \mathbb{S}_0^c} |j| \|r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} \tag{6.65}$$

such that in the Cantor set

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) := \bigcap_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |l| \leqslant N_n \\ (l,j) \neq (0,j_0)}} \Big\{(b, \omega) \in \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0), \big|\omega \cdot l + \mu_j^\infty(b, \omega, i_0) - \mu_{j_0}^\infty(b, \omega, i_0)\big| > \tfrac{2\gamma\langle j - j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}$$

we have

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty = \mathscr{L}_\infty + \mathtt{E}_n^2,$$

and the linear operator $\mathtt{E}_n^2$ *satisfies the estimate*

$$\|\mathtt{E}_n^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-2} N_0^{\mu_2} N_{n+1}^{-\mu_2} \|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{6.66}$$

Notice that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *was introduced in Proposition 6.2, the operator* $\mathscr{L}_0$ *and the frequencies* $\big(\mu_j^0(b, \omega, i_0)\big)_{j \in \mathbb{S}_0^c}$ *were stated in Proposition 6.3.*

(ii) Given two tori i_1 *and* i_2 *both satisfying* (6.62), *then*

$$\forall j \in \mathbb{S}_0^c, \quad \|\Delta_{12} r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} \tag{6.67}$$

$$\forall j \in \mathbb{S}_0^c, \quad \|\Delta_{12}\mu_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}|j|\|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}. \tag{6.68}$$

Proof. The proof consists in a KAM reduction algorithm. We may refer for instance to [4, 8, 29, 37, 47] to see some implementations of this method. Here, the main reference which fits with our purpose is [47, Prop 6.5]. Hence, we only explain here the main lines of this procedure and refer to this work for the technical details. In Proposition 6.3, we managed to find the following reduction valid in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$,

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\mathscr{L}_0+\mathtt{E}_n^1,$$

where $\mathscr{L}_0$ is an operator admitting the following structure

$$\mathscr{L}_0=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_0+\mathscr{R}_0,\tag{6.69}$$

with $\mathscr{D}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{D}_0\Pi_{\mathbb{S}_0}^\perp=\big(\mathrm{i}\mu_j^0(b,\omega)\big)_{j\in\mathbb{S}_0^c}$ a diagonal operator of pure imaginary spectrum and $\mathscr{R}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ a real and reversible Toeplitz in time operator of zero order. The proof consists in a recursive elimination of the remainder term $\mathscr{R}_0$. First notice that if we denote

$$\delta_0(s):=\gamma^{-1}\|\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}},$$

then, in view of (6.56), we get

$$\delta_0(s)\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\tag{6.70}$$

Hence, according to (6.61), (6.62) and the fact that $\sigma_4\geqslant\sigma_3$, we infer

$$\begin{aligned}N_0^{\mu_2}\delta_0(s_h)&\leqslant CN_0^{\mu_2}\varepsilon\gamma^{-2}\\&\leqslant C\varepsilon_0.\end{aligned}\tag{6.71}$$

We construct recursively a sequence $(\mathscr{L}_m)_{m\in\mathbb{N}}$ of operators in the form

$$\mathscr{L}_m:=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_m+\mathscr{R}_m,\tag{6.72}$$

with $\mathscr{D}_m=\Pi_{\mathbb{S}_0}^\perp\mathscr{D}_m\Pi_{\mathbb{S}_0}^\perp=\big(\mathrm{i}\mu_j^m(b,\omega)\big)_{j\in\mathbb{S}_0^c}$ a diagonal real reversible operator, that is,

$$\mathscr{D}_m\mathbf{e}_{l,j}=\mathrm{i}\,\mu_j^m(b,\omega)\,\mathbf{e}_{l,j}\quad\text{and}\quad\mu_{-j}^m(b,\omega)=-\mu_j^m(b,\omega)\tag{6.73}$$

and $\mathscr{R}_m=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_m\Pi_{\mathbb{S}_0}^\perp$ a real and reversible Toeplitz in time operator of zero order quadratically smaller than the previous one in the Toeplitz topology introduced in Section 4.2. To construct $\mathscr{D}_{m+1}$ and $\mathscr{R}_{m+1}$, we consider a linear invertible transformation close to the identity

$$\Phi_m=\Pi_{\mathbb{S}_0}^\perp+\Psi_m:\mathcal{O}\to\mathcal{L}(H^s\cap L_\perp^2),$$

where Ψ_m is small and depends on $\mathscr{R}_m$. Straightforward computations give

$$\Phi_m^{-1}\mathscr{L}_m\Phi_m=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_m+\Phi_m^{-1}\Big(\big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_m,\Psi_m\big]+P_{N_m}\mathscr{R}_m+P_{N_m}^\perp\mathscr{R}_m+\mathscr{R}_m\Psi_m\Big),$$

where the projector P_{N_m} was defined in (4.16). Therefore, we shall define Ψ_m so that it satisfies the following *homological equation*

$$\big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_m,\Psi_m\big]+P_{N_m}\mathscr{R}_m=\lfloor P_{N_m}\mathscr{R}_m\rfloor,\tag{6.74}$$

where $\lfloor P_{N_m}\mathscr{R}_m\rfloor$ is the diagonal part of the operator $P_{N_m}\mathscr{R}_m$. We emphasize that the notation $\lfloor\mathcal{R}\rfloor$ with a general operator $\mathcal{R}$ is defined as follows, for all $(l_0,j_0)\in\mathbb{Z}^d\times\mathbb{S}_0^c$,

$$\mathcal{R}\mathbf{e}_{l_0,j_0}=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\mathcal{R}_{l_0,j_0}^{l,j}\mathbf{e}_{l,j}\Longrightarrow\lfloor\mathcal{R}\rfloor\mathbf{e}_{l_0,j_0}=\mathcal{R}_{l_0,j_0}^{l_0,j_0}\mathbf{e}_{l_0,j_0}=\big\langle\mathcal{R}\mathbf{e}_{l_0,j_0},\mathbf{e}_{l_0,j_0}\big\rangle_{L^2(\mathbb{T}^{d+1})}\mathbf{e}_{l_0,j_0}.\tag{6.75}$$

Recall that we denote $\mathbf{e}_{l_0,j_0}(\varphi,\theta)=e^{\mathrm{i}(l_0\cdot\varphi+j_0\theta)}$. The equation (6.74) can be solved using the Fourier decomposition of operators. Indeed, since $\mathscr{R}_m$ is a real and reversible Toeplitz in time operator, then by virtue of Lemma 4.6, we can write

$$(\mathscr{R}_m)_{l_0,j_0}^{l,j}:=\mathrm{i}\,r_{j_0,m}^j(b,\omega,l_0-l)\in\mathrm{i}\,\mathbb{R}\quad\text{and}\quad(\mathscr{R}_m)_{-l_0,-j_0}^{-l,-j}=-(\mathscr{R}_m)_{l_0,j_0}^{l,j}.\tag{6.76}$$

Therefore, we define the operator Ψ_m by

$$(\Psi_m)_{j_0}^j(b,\omega,l) = \begin{cases} -\varrho_{j_0,m}^j(b,\omega,l)\ r_{j_0,m}^j(b,\omega,l), & \text{if} \quad (l,j)\neq(0,j_0) \\ 0, & \text{if} \quad (l,j)=(0,j_0), \end{cases} \tag{6.77}$$

with

$$\varrho_{j_0,m}^j(b,\omega,l) := \frac{\chi\Big((\omega\cdot l+\mu_j^m(b,\omega)-\mu_{j_0}^m(b,\omega))(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}\Big)}{\omega\cdot l+\mu_j^m(b,\omega)-\mu_{j_0}^m(b,\omega)}, \tag{6.78}$$

where the cut-off function $\chi\in C^\infty(\mathbb{R},[0,1])$ is even and defined by

$$\chi(\xi) = \begin{cases} 0 & \text{if } |\xi|\leqslant\frac{1}{3} \\ 1 & \text{if } |\xi|\geqslant\frac{1}{2}. \end{cases} \tag{6.79}$$

supplemented by the normal conditions

$$\forall\, l\in\mathbb{Z}^d,\ \forall\, j \text{ or } j_0\in\mathbb{S}_0, \quad (\Psi_m)_{j_0}^j(b,\omega,l)=0.$$

Using Lemma 4.1-(v), we can prove that

$$\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}, \tag{6.80}$$

provided that

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2, \qquad \max_{|\alpha|\in[\![0,q]\!]}\ \sup_{(b,\omega)\in\mathcal{O}}\left|\partial_{b,\omega}^\alpha\left(\mu_j^m(b,\omega)-\mu_{j_0}^m(b,\omega)\right)\right|\leqslant C\,|j-j_0|. \tag{6.81}$$

Remark that (6.81) is satisfied for $m=0$ in view of Lemma 3.3-(iv) and (6.20). By construction, $\Psi_m=\Pi_{\mathbb{S}_0}^\perp\Psi_m\Pi_{\mathbb{S}_0}^\perp$ is a real and reversibility preserving Toeplitz in time operator well-defined in the whole set of parameters $\mathcal{O}$ and solves the equation (6.74) when restricted to the Cantor set $\mathscr{O}_{m+1}^\gamma$ defined recursively by $\mathscr{O}_0^\gamma=\mathcal{O}$ and

$$\mathscr{O}_{m+1}^\gamma = \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_m\\ (l,j)\neq(0,j_0)}}\Big\{(b,\omega)\in\mathscr{O}_m^\gamma \quad \text{s.t.} \quad |\omega\cdot l+\mu_j^m(b,\omega)-\mu_{j_0}^m(b,\omega)|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}. \tag{6.82}$$

Consequently, in the Cantor set $\mathscr{O}_{m+1}^\gamma$, one has

$$\mathscr{L}_{m+1} := \Phi_m^{-1}\mathscr{L}_m\Phi_m = \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_{m+1}+\mathscr{R}_{m+1}, \tag{6.83}$$

with

$$\mathscr{D}_{m+1}=\mathscr{D}_m+\lfloor P_{N_m}\mathscr{R}_m\rfloor \quad \text{and} \quad \mathscr{R}_{m+1}=\Phi_m^{-1}\big(-\Psi_m\lfloor P_{N_m}\mathscr{R}_m\rfloor+P_{N_m}^\perp\mathscr{R}_m+\mathscr{R}_m\Psi_m\big). \tag{6.84}$$

Remark that $\mathscr{D}_m$ and $\lfloor P_{N_m}\mathscr{R}_m\rfloor$ are Fourier multiplier Toeplitz operators that can be identified to their spectra $(\mathrm{i}\mu_j^m)_{j\in\mathbb{S}_0^c}$ and $(\mathrm{i}r_j^m)_{j\in\mathbb{S}_0^c}$, namely

$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c, \quad \mathscr{D}_m\mathbf{e}_{l,j}=\mathrm{i}\mu_j^m\,\mathbf{e}_{l,j} \quad \text{and} \quad \lfloor P_{N_m}\mathscr{R}_m\rfloor\mathbf{e}_{l,j}=\mathrm{i}r_j^m\,\mathbf{e}_{l,j}. \tag{6.85}$$

By construction, we find

$$\mu_j^{m+1}=\mu_j^m+r_j^m. \tag{6.86}$$

We set

$$\delta_m(s) := \gamma^{-1}\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \quad \text{and} \quad \widehat{\delta}_m(s):=\max\Big(\gamma^{-1}\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}},\delta_m(s)\Big).$$

By Lemma 4.5 and (6.80), we get for all $S \geqslant \overline{s} \geqslant s \geqslant s_0$,

$$\delta_{m+1}(s) \leqslant N_m^{s-\overline{s}}\delta_m(\overline{s}) + CN_m^{\tau_2 q+\tau_2}\delta_m(s_0)\delta_m(s), \tag{6.87}$$
$$\widehat{\delta}_{m+1}(s) \leqslant N_m^{s-\overline{s}}\widehat{\delta}_m(\overline{s}) + CN_m^{\tau_2 q+\tau_2+1}\widehat{\delta}_m(s_0)\widehat{\delta}_m(s). \tag{6.88}$$

These recursive relations allow us to prove by induction on m that

$$\delta_m(\overline{s}_l) \leqslant \delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2} \quad \text{and} \quad \delta_m(s_h) \leqslant \left(2-\tfrac{1}{m+1}\right)\delta_0(s_h), \tag{6.89}$$

$$\widehat{\delta}_m(s_0) \leqslant \widehat{\delta}_0(s_h)N_0^{\mu_2}N_m^{-\mu_2} \quad \text{and} \quad \widehat{\delta}_m(s_h) \leqslant \left(2-\tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h), \tag{6.90}$$

with $\overline{s}_l$ and s_h fixed by (6.16) and (6.61). Using the definition of the Fourier coefficients for an operator and the Toeplitz structure of $\mathscr{R}_m$, we find

$$\begin{aligned}\mu_j^{m+1} - \mu_j^m &= r_j^m \\ &= -\mathrm{i}\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{l,j}, \mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})} \\ &= -\mathrm{i}\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}.\end{aligned}$$

By a duality argument combined with Lemma 4.5, (6.89), (6.70) and (6.62), we infer

$$\|\mu_j^{m+1} - \mu_j^m\|_q^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_m^{-\mu_2}. \tag{6.91}$$

As the assumption (6.81) is satisfied with $\mathscr{D}_m$, then we obtain by (6.91)

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\quad \max_{|\alpha|\in[\![0,q]\!]}\ \sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^\alpha\left(\mu_j^{m+1}(\lambda,\omega) - \mu_{j_0}^{m+1}(\lambda,\omega)\right)\right| \leqslant C\big(1+\varepsilon\gamma^{-1-q}N_0^{\mu_2}N_m^{-\mu_2}\big)\,|j-j_0|.$$

Therefore, the sequence $\big(\mu_j^m\big)_{m\in\mathbb{N}}$ converges in $W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$. We denote μ_j^∞ its limit and consider the associated diagonal operator $\mathscr{D}_\infty = \big(\mathrm{i}\mu_j^\infty\big)_{j\in\mathbb{S}_0^c}$. We introduce the sequence of operators $\left(\widehat{\Phi}_m\right)_{m\in\mathbb{N}}$ by

$$\widehat{\Phi}_0 := \Phi_0 \quad \text{and} \qquad \forall m\geqslant 1,\ \widehat{\Phi}_m := \Phi_0\circ\Phi_1\circ\ldots\circ\Phi_m. \tag{6.92}$$

It is obvious from the identity $\Phi_m = \mathrm{Id} + \Psi_m$ that

$$\widehat{\Phi}_{m+1} = \widehat{\Phi}_m + \widehat{\Phi}_m\Psi_{m+1}. \tag{6.93}$$

From (6.93), (6.80) and (6.62), we can make the sequence $(\widehat{\Phi}_m)_{m\in\mathbb{N}}$ converge to an element Φ_∞ in the Toeplitz operator topology introduced in Section 4.2. Moreover, arguing in a similar way to [47] we may prove that

$$\|\widehat{\Phi}_m^{-1} - \Phi_\infty^{-1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|\widehat{\Phi}_m - \Phi_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant C\,\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}. \tag{6.94}$$

In addition, for any $m\in\mathbb{N}$, we find in view of (6.91)

$$\begin{aligned}\|\mu_j^\infty - \mu_j^m\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{n=m}^{\infty}\|\mu_j^{n+1} - \mu_j^n\|_q^{\gamma,\mathcal{O}} \\ &\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}.\end{aligned} \tag{6.95}$$

Therefore, we deduce

$$\begin{aligned}\mu_j^\infty &= \mu_j^0 + \sum_{m=0}^{\infty}\big(\mu_j^{m+1} - \mu_j^m\big) \\ &:= \mu_j^0 + r_j^\infty,\end{aligned} \tag{6.96}$$

where $(\mu_j^0)_{j\in\mathbb{S}_0^c}$ is described in Proposition 6.3 and takes the form

$$\mu_j^0(b,\omega,i_0) = \Omega_j(b) + j\big(V_{i_0}^\infty(b,\omega) - \tfrac{1}{2}\big).$$

Define the diagonal operator $\mathscr{D}_\infty$ by

$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad \mathscr{D}_\infty \mathbf{e}_{l,j} = \mathrm{i}\mu_j^\infty \mathbf{e}_{l,j}. \tag{6.97}$$

By the norm definition we obtain

$$\|\mathscr{D}_m - \mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} = \sup_{j\in\mathbb{S}_0^c}\|\mu_j^m - \mu_j^\infty\|_q^{\gamma,\mathcal{O}},$$

which gives by virtue of (6.95)

$$\|\mathscr{D}_m - \mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \tag{6.98}$$

Let us consider the following Cantor set for a given $n\in\mathbb{N}$,

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) := \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ \langle l,j-j_0\rangle\leqslant N_n\\ (l,j)\neq(0,j_0)}} \Big\{(b,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\quad\text{s.t.}\quad \big|\omega\cdot l + \mu_j^\infty(b,\omega) - \mu_{j_0}^\infty(b,\omega)\big| > \tfrac{2\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\},$$

where the intermediate Cantor sets are defined in (6.82). A finite induction, in a similar way to [47], allows to prove that

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\subset\mathscr{O}_{n+1}^{\gamma},$$

Now define the operator

$$\mathscr{L}_\infty := \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty.$$

Writing

$$\mathscr{L}_m - \mathscr{L}_\infty = \mathscr{D}_m - \mathscr{D}_\infty + \mathscr{R}_m,$$

then using (6.98) and (6.89), we can deduce the convergence of the sequence $(\mathscr{L}_m)_{m\in\mathbb{N}}$ to the operator $\mathscr{L}_\infty$ in the Toeplitz operator topology. In view of (6.92) and (6.83) one obtains

$$\begin{aligned}\forall(\lambda,\omega)\in\mathscr{O}_{n+1}^\gamma,\quad \widehat{\Phi}_n^{-1}\mathscr{L}_0\widehat{\Phi}_n &= \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_{n+1} + \mathscr{R}_{n+1}\\ &= \mathscr{L}_\infty + \mathscr{D}_{n+1} - \mathscr{D}_\infty + \mathscr{R}_{n+1}.\end{aligned}$$

It follows that for any $(\lambda,\omega)\in\mathscr{O}_{n+1}^\gamma$

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty &= \mathscr{L}_\infty + \mathscr{D}_{n+1} - \mathscr{D}_\infty + \mathscr{R}_{n+1}\\ &\quad + \Phi_\infty^{-1}\mathscr{L}_0\left(\Phi_\infty - \widehat{\Phi}_n\right) + \left(\Phi_\infty^{-1} - \widehat{\Phi}_n^{-1}\right)\mathscr{L}_0\widehat{\Phi}_n\\ &:= \mathscr{L}_\infty + \mathtt{E}_n^2.\end{aligned}$$

According to Lemma 4.5, (6.58), (6.95), (6.62) and (6.94) we get (6.66). Next let us see how to get the estimate (6.65). Recall that

$$r_j^\infty = \sum_{m=0}^{\infty} r_j^m \qquad\text{with}\qquad r_j^m = -\mathrm{i}\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j},\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

An integration by parts gives

$$\begin{aligned}\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j},\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})} &= \tfrac{\mathrm{i}}{j}\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j},\partial_\theta\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}\\ &= \tfrac{-\mathrm{i}}{j}\big\langle P_{N_m}\partial_\theta\mathscr{R}_m\mathbf{e}_{0,j},\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.\end{aligned}$$

Using a duality argument $H^{s_0}-H^{-s_0}$ combined with Lemma 4.5, (6.90), (6.56) and (6.62), we obtain

$$\|r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim |j|^{-1}\varepsilon\gamma^{-1}.$$

The difference estimates (6.67) and (6.68) can be obtained by fixing $\mathtt{p} = 4\tau_2 q + 4\tau_2$ as in [47]. The proof of Proposition 6.4 is now complete. □

6.3 Construction and tame estimates for the approximate inverse

At this step, we can construct an almost approximate right inverse for $\widehat{\mathcal{L}}_\omega$ defined in (6.7). This enables to find in turn an almost approximate right inverse for the whole operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ given by (6.1).

Proposition 6.5. *Let $(\gamma,q,d,\tau_1,s_0,s_h,\mu_2,S)$ satisfying* (1.11)*,* (1.10) *and* (6.61)*. There exists $\sigma:=\sigma(\tau_1,\tau_2,q,d)\geqslant\sigma_4$ such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}}\leqslant 1,\tag{6.99}$$

then, the following assertions hold true.

(i) Consider the operator $\mathscr{L}_\infty$ defined in Proposition 6.4*, then there exists a family of linear reversible operators $\big(\mathtt{T}_n\big)_{n\in\mathbb{N}}$ defined in $\mathcal{O}$ satisfying the estimate*

$$\forall s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}$$

and such that for any $n\in\mathbb{N}$, in the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^\infty(b,\omega,i_0)\big|>\tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},$$

we have

$$\mathscr{L}_\infty\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3,$$

with

$$\forall s_0\leqslant s\leqslant\overline{s}\leqslant S,\quad\|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

(ii) There exists a family of linear reversible operators $\big(\mathtt{T}_{\omega,n}\big)_{n\in\mathbb{N}}$ satisfying

$$\forall\,s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|\mathtt{T}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\tag{6.100}$$

and such that in the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0):=\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\tag{6.101}$$

we have

$$\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n}=\mathrm{Id}+\mathtt{E}_n,$$

where $\mathtt{E}_n$ satisfies the following estimate

$$\begin{aligned}\forall\,s\in[s_0,S],\quad\|\mathtt{E}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.102}$$

Recall that $\widehat{\mathcal{L}}_\omega$, $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ and $\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ are given by (6.7) *and Propositions 6.2 and 6.4, respectively.*

(iii) In the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$, we have the following splitting

$$\widehat{\mathcal{L}}_\omega=\widehat{\mathtt{L}}_{\omega,n}+\widehat{\mathtt{R}}_n\quad\text{with}\quad\widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n}=\mathrm{Id}\quad\text{and}\quad\widehat{\mathtt{R}}_n=\mathtt{E}_n\widehat{\mathtt{L}}_{\omega,n},$$

where $\widehat{\mathtt{L}}_{\omega,n}$ and $\widehat{\mathtt{R}}_n$ are reversible operators defined in $\mathcal{O}$ and satisfy the following estimates

$$\forall s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}},\tag{6.103}$$

$$\begin{aligned}\forall s\in[s_0,S],\quad\|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.104}$$

Proof. **(i)** First recall from Proposition 6.4 that

$$\mathscr{L}_\infty = \omega\cdot\partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty.$$

Using the projectors defined in (4.1), we can split this operator as follows

$$\begin{aligned}\mathscr{L}_\infty &= \Pi_{N_n}\omega\cdot\partial_\varphi \Pi_{N_n}\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty - \Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp\\ &:= \mathtt{L}_n - \mathtt{R}_n,\end{aligned}\tag{6.105}$$

where

$$\mathtt{R}_n := \Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp.$$

According to the structure of $\mathscr{D}_\infty$ in Proposition 6.4, we obtain from (6.105),

$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad \mathbf{e}_{-l,-j}\mathtt{L}_n\mathbf{e}_{l,j} = \begin{cases} \mathrm{i}\big(\omega\cdot l+\mu_j^\infty\big) & \text{if } |l|\leqslant N_n\\ \mathrm{i}\,\mu_j^\infty & \text{if } |l|>N_n.\end{cases}$$

Let us now consider the diagonal operator $\mathtt{T}_n$ defined by

$$\begin{aligned}\mathtt{T}_n\rho(b,\omega,\varphi,\theta) := &-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}}\tfrac{\chi\big((\omega\cdot l+\mu_j^\infty(b,\omega,i_0))\gamma^{-1}\langle l\rangle^{\tau_1}\big)}{\omega\cdot l+\mu_j^\infty(b,\omega,i_0)}\rho_{l,j}(b,\omega)\,e^{\mathrm{i}(l\cdot\varphi+j\theta)}\\ &-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|> N_n}}\tfrac{\rho_{l,j}(b,\omega)}{\mu_j^\infty(b,\omega,i_0)}\,e^{\mathrm{i}(l\cdot\varphi+j\theta)},\end{aligned}$$

where χ is the cut-off function introduced in (6.79) and $\big(\rho_{l,j}(b,\omega)\big)_{l,j}$ are the Fourier coefficients of ρ. Now recall the expansion of the perturbed eigenvalues given by Proposition 6.4, namely

$$\mu_j^\infty(b,\omega,i_0) = \Omega_j(b) + jr^1(b,\omega) + r_j^\infty(b,\omega)\quad\text{with}\quad r^1(b,\omega) = V_{i_0}^\infty(b,\omega)-\tfrac{1}{2}.$$

In view of Lemma 3.3-(iv), (6.54) and (6.65), they satisfy the following estimates

$$\forall j\in\mathbb{S}_0^c,\quad \|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim |j|.$$

According to Lemma 3.3-(ii), (6.54), (6.65) and the smallness condition (6.99) we infer

$$|j|\lesssim \|\mu_j^\infty\|_0^{\gamma,\mathcal{O}}\leqslant \|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}.$$

Computations based on Lemma 4.1-(vi) give

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \gamma^{-1}\|\rho\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}.\tag{6.106}$$

In addition, by construction

$$\mathtt{L}_n\mathtt{T}_n = \mathrm{Id}\quad\text{in }\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)\tag{6.107}$$

since $\chi(\cdot)=1$ in this set. Gathering (6.107) and (6.105) yields

$$\begin{aligned}\forall\,(b,\omega)\in\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\quad \mathscr{L}_\infty\mathtt{T}_n &= \mathrm{Id}-\mathtt{R}_n\mathtt{T}_n\\ &:= \mathrm{Id}+\mathtt{E}_n^3.\end{aligned}\tag{6.108}$$

Remark that by Lemma 4.1-(ii),

$$\forall\, s_0\leqslant s\leqslant\overline{s}\leqslant S,\quad \|\mathtt{R}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\|\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}.$$

Putting this estimate with (6.106) implies

$$\forall\, s_0\leqslant s\leqslant\overline{s}\leqslant S,\quad \|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}.\tag{6.109}$$

(ii) We set

$$\mathtt{T}_{\omega,n} := \mathscr{B}_\perp \Phi_\infty \mathtt{T}_n \Phi_\infty^{-1} \mathscr{B}_\perp^{-1}, \tag{6.110}$$

where the operators $\mathscr{B}_\perp$ and Φ_∞ are defined in Propositions 6.3 and 6.4 respectively. Notice that $\mathtt{T}_{\omega,n}$ is defined in the whole range of parameters $\mathcal{O}$. Since the condition (6.99) is satisfied, then, both Propositions 6.2 and 6.4 apply and the estimate (6.100) is obtained combining (6.52), (6.63), (6.106) and (6.99). Now combining Propositions 6.3 and 6.4, we find that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \cap \mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ the following decomposition holds

$$\begin{aligned}\Phi_\infty^{-1} \mathscr{B}_\perp^{-1} \widehat{\mathcal{L}}_\omega \mathscr{B}_\perp \Phi_\infty &= \Phi_\infty^{-1} \mathscr{L}_0 \Phi_\infty + \Phi_\infty^{-1} \mathtt{E}_n^1 \Phi_\infty \\ &= \mathscr{L}_\infty + \mathtt{E}_n^2 + \Phi_\infty^{-1} \mathtt{E}_n^1 \Phi_\infty.\end{aligned}$$

According to (6.108), one finds that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \cap \mathscr{O}_{\infty,n}^{\gamma,\tau_2}(i_0) \cap \Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)$ the following identity holds

$$\Phi_\infty^{-1} \mathscr{B}_\perp^{-1} \widehat{\mathcal{L}}_\omega \mathscr{B}_\perp \Phi_\infty \mathtt{T}_n = \mathrm{Id} + \mathtt{E}_n^3 + \mathtt{E}_n^2 \mathtt{T}_n + \Phi_\infty^{-1} \mathtt{E}_n^1 \Phi_\infty \mathtt{T}_n,$$

which implies in turn, in view of (6.110), the following identity in $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$

$$\begin{aligned}\widehat{\mathcal{L}}_\omega \mathtt{T}_{\omega,n} &= \mathrm{Id} + \mathscr{B}_\perp \Phi_\infty \big(\mathtt{E}_n^3 + \mathtt{E}_n^2 \mathtt{T}_n + \Phi_\infty^{-1} \mathtt{E}_n^1 \Phi_\infty \mathtt{T}_n\big) \Phi_\infty^{-1} \mathscr{B}_\perp^{-1} \\ &:= \mathrm{Id} + \mathtt{E}_n.\end{aligned} \tag{6.111}$$

Combining (6.111), (6.55), (6.66), (6.109), (6.106), (6.52), (6.63) and (6.99), we get (6.102), up to take σ large enough.
(iii) By virtue of (6.111), one can write in the Cantor set $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$

$$\widehat{\mathcal{L}}_\omega = \mathtt{T}_{\omega,n}^{-1} + \mathtt{E}_n \mathtt{T}_{\omega,n}^{-1}. \tag{6.112}$$

Putting together (6.110) and (6.107), one finds in the Cantor set $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$

$$\widehat{\mathtt{L}}_{\omega,n} := \mathtt{T}_{\omega,n}^{-1} = \mathscr{B}_\perp \Phi_\infty \mathtt{L}_n \Phi_\infty^{-1} \mathscr{B}_\perp^{-1}.$$

Therefore, (6.112) can be rewritten

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n \quad \text{with} \quad \widehat{\mathtt{R}}_n := \mathtt{E}_n \widehat{\mathtt{L}}_{\omega,n}.$$

The estimate (6.103) is obtained gathering (6.105), (6.52), (6.63) and (6.99). Finally, (6.103) together with (6.102) implies (6.104). This ends the proof of Proposition 6.5. □

The following theorem, which can be found in [11, 16, 37], states that the linearized operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ in (6.1) admits an approximate right inverse on a suitable Cantor set.

Theorem 6.1. (Approximate inverse)
Let $(\gamma, q, d, \tau_1, \tau_2, s_0, s_h, \mu_2)$ satisfy (1.11), (1.10), (6.16) and (6.61). Then there exists $\overline{\sigma} = \overline{\sigma}(\tau_1, \tau_2, d, q) > 0$ and a family of reversible operators $\mathrm{T}_0 := \mathrm{T}_{0,n}(i_0)$ such that if the smallness condition (6.99) holds, then for all $g = (g_1, g_2, g_3)$, satisfying

$$g_1(\varphi) = g_1(\varphi), \quad g_2(-\varphi) = -g_2(\varphi) \quad \text{and} \quad g_3(-\varphi) = (\mathscr{S} g_3)(\varphi),$$

the function $\mathrm{T}_0 g$ satisfies the following estimate

$$\forall s \in [s_0, S], \quad \|\mathrm{T}_0 g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} \left(\|g\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \right).$$

Moreover T_0 is an almost-approximate right inverse of $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ in the Cantor set $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$ defined by (6.101). More precisely,

$$\forall (b, \omega) \in \mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0), \quad d_{i,\alpha}\mathcal{F}(i_0) \circ \mathrm{T}_0 - \mathrm{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)},$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ *and* $\mathcal{E}_3^{(n)}$ *are defined in the whole set* $\mathcal{O}$ *with the estimates*

$$\begin{aligned}
&\|\mathcal{E}_1^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\|\mathcal{F}(i_0,\alpha_0)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}},\\
\forall b\geqslant 0,\quad &\|\mathcal{E}_2^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}N_n^{-b}\big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}+\varepsilon\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\big),\\
\forall b\in[0,S],\quad &\|\mathcal{E}_3^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\gamma^{-2}\Big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\\
&\qquad\qquad\qquad + \varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.
\end{aligned}$$

7 Nash-Moser iteration and measure of the final Cantor set

In this last section, we shall find a non-trivial solution $(b,\omega)\mapsto(i_\infty(b,\omega),\alpha_\infty(b,\omega))$ to the equation

$$\mathcal{F}(i,\alpha,b,\omega,\varepsilon)=0,$$

where $\mathcal{F}$ is the functional defined in (5.19). This done by using a Nash-Moser scheme in a similar way to the series of papers [3, 16, 37, 47]. The solutions are constructed for parameters (b,ω) belonging to the intersection of all the Cantor sets $\mathcal{G}_\infty^\gamma$ on which we are able to invert the linearized operator at the different steps. In order to find a solution to the original problem, we must rigidify the frequencies ω so that it coincides with the equilibrium frequencies. This amounts to consider a frequency curve $b\mapsto\omega(b,\varepsilon)$ implicitly defined by the equation

$$\alpha_\infty\big(b,\omega(b,\varepsilon)\big)=-\omega_{\mathrm{Eq}}(b).$$

Considering the associated rigidified Cantor set

$$\mathcal{C}_\infty^\varepsilon=\Big\{b\in(b_0,b_1)\quad\text{s.t.}\quad\big(b,\omega(b,\varepsilon)\big)\in\mathcal{G}_\infty^\gamma\Big\},$$

we have a solution to the original problem provided that the measure of $\mathcal{C}_\infty^\varepsilon$ is non-zero. This will be checked, in Section 7.2, by perturbative arguments in the spirit of the previous works [3, 4, 16, 37, 47]. This proves in particular Theorem 1.1.

7.1 Nash-Moser iteration

In this section we implement the Nash-Moser scheme, which is a modified Newton method consisting in a recursive construction of approximate solutions of the equation $\mathcal{F}\big(i,\alpha,b,\omega\big):=\mathcal{F}\big(i,\alpha,b,\omega,\varepsilon\big)=0$ where the functional $\mathcal{F}$ is defined in (5.19). At each step of this procedure, we need to construct an approximate inverse of the linearized operator at a state near the equilibrium by applying the reduction procedure developed in Section 6. This allows to get Theorem 6.1 with the suitable tame estimates associated to the final loss of regularity $\overline{\sigma}$ that could be arranged to be large enough. We point out that $\overline{\sigma}$ depends only on the shape of the Cantor set through the parameters τ_1,τ_2,d and q but it is independent of the regularity of the solutions that we want to construct. Now, we shall fix the following parameters needed to implement the Nash-Moser scheme and related to the loss of regularity $\overline{\sigma}$.

$$\begin{cases}\overline{a} &= \tau_2+2\\ \mu_1 &= 3q(\tau_2+2)+6\overline{\sigma}+6\\ a_1 &= 6q(\tau_2+2)+12\overline{\sigma}+15\\ a_2 &= 3q(\tau_2+2)+6\overline{\sigma}+9\\ \mu_2 &= 2q(\tau_2+2)+5\overline{\sigma}+7\\ s_h &= s_0+4q(\tau_2+2)+9\overline{\sigma}+11\\ b_1 &= 2s_h-s_0.\end{cases}\tag{7.1}$$

We shall now impose the constraint relating γ and N_0 to ε

$$0<a<\tfrac{1}{\mu_2+q+2},\quad \gamma:=\varepsilon^a,\quad N_0:=\gamma^{-1}.\tag{7.2}$$

We consider the finite dimensional subspaces

$$E_n := \Big\{ \mathfrak{I} = (\Theta, I, z) \quad \text{s.t.} \quad \Theta = \Pi_n \Theta, \quad I = \Pi_n I \quad \text{and} \quad z = \Pi_n z \Big\},$$

where Π_n is the projector defined by

$$f(\varphi,\theta) = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}} f_{l,j} e^{\mathrm{i}(l\cdot\varphi+j\theta)} \quad \Rightarrow \quad \Pi_n f(\varphi,\theta) = \sum_{\langle l,j\rangle\leqslant N_n} f_{l,j} e^{\mathrm{i}(l\cdot\varphi+j\theta)}.$$

The main result of this section can be stated as follows. The proof is now standard and to avoid the technical details we refer to the works [16, 37, 47]. In our context, we may follow closely the version detailed in [47, Prop. 7.1]. In particular the roles of the parameters in (7.1) are explained there.

Proposition 7.1. ***(Nash-Moser)***
Let $(\tau_1, \tau_2, q, d, s_0)$ satisfy (1.11) *and* (1.10). *Consider the parameters fixed by* (7.1) *and* (7.2). *There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ we get for all $n \in \mathbb{N}$ the following properties.*

$(\mathcal{P}1)_n$ *There exists a q-times differentiable function*

$$\begin{array}{rccl} W_n : & \mathcal{O} & \rightarrow & E_{n-1}\times\mathbb{R}^d\times\mathbb{R}^{d+1} \\ & (b,\omega) & \mapsto & \big(\mathfrak{I}_n, \alpha_n - \omega, 0\big) \end{array}$$

satisfying

$$W_0 = 0 \quad \textit{and} \quad \textit{for} \quad n\in\mathbb{N}^*, \quad \|W_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}.$$

By setting

$$U_0 = \Big((\varphi,0,0),\omega,(b,\omega) \Big) \quad \text{and} \quad \textit{for} \quad n\in\mathbb{N}^*, \quad U_n = U_0 + W_n \quad \text{and} \quad H_n = U_n - U_{n-1}, \tag{7.3}$$

then

$$\forall s\in[s_0,S],\ \|H_1\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \frac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} \quad \textit{and} \quad \forall\, 2\leqslant k\leqslant n,\ \|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_{k-1}^{-a_2}. \tag{7.4}$$

We also have for $n \geqslant 2$,

$$\|H_n\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}. \tag{7.5}$$

$(\mathcal{P}2)_n$ *Define*

$$i_n = (\varphi,0,0) + \mathfrak{I}_n, \quad \gamma_n = \gamma(1+2^{-n}), \tag{7.6}$$

then i_n satisfies the following reversibility condition

$$\mathfrak{S} i_n(\varphi) = i_n(-\varphi), \tag{7.7}$$

where $\mathfrak{S}$ is defined by (5.10). *Define also*

$$\mathcal{A}_0^\gamma = \mathcal{O} \quad \textit{and} \quad \mathcal{A}_{n+1}^\gamma = \mathcal{A}_n^\gamma \cap \mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$$

where $\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$ is defined in Proposition 6.5. *Consider the open sets*

$$\forall \mathtt{r}>0, \quad \mathrm{O}_n^{\mathtt{r}} := \Big\{ (b,\omega)\in\mathcal{O} \quad \text{s.t.} \quad \mathtt{dist}\big((b,\omega),\mathcal{A}_n^\gamma\big) < \mathtt{r}N_n^{-\overline{a}} \Big\}$$

where $\mathtt{dist}(x,A) = \inf_{y\in A}\|x-y\|$. Then we have the following estimate

$$\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^\gamma} \leqslant C_*\varepsilon N_{n-1}^{-a_1}.$$

$(\mathcal{P}3)_n$ $\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}.$

A non trivial reversible quasi-periodic solution of our problem is obtained as the limit of the sequence $(U_n)_{n\in\mathbb{N}}$ according to the fast convergence stated in Proposition 7.1. This is explained in the following corollary.

Corollary 7.1. *There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the following assertions hold true. We consider the Cantor set $\mathcal{G}_\infty^\gamma$, related to ε through γ, and defined by*

$$\mathcal{G}_\infty^\gamma := \bigcap_{n\in\mathbb{N}} \mathcal{A}_n^\gamma.$$

There exists a function

$$\begin{array}{rccl} U_\infty : & \mathcal{O} & \to & \big(\mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp \cap H^{s_0}\big) \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ & (b,\omega) & \mapsto & \big(i_\infty(b,\omega), \alpha_\infty(b,\omega), (b,\omega)\big) \end{array}$$

such that

$$\forall (b,\omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}(U_\infty(b,\omega)) = 0.$$

In addition, i_∞ is reversible and $\alpha_\infty \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{R}^d)$ with

$$\alpha_\infty(b,\omega) = \omega + \mathrm{r}_\varepsilon(b,\omega) \quad \text{and} \quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} N_0^{q\overline{a}}. \tag{7.8}$$

Moreover, there exists a q-times differentiable function $b \in (b_0, b_1) \mapsto \omega(b,\varepsilon)$ with

$$\omega(b,\varepsilon) = -\omega_{\mathrm{Eq}}(b) + \bar{r}_\varepsilon(b), \quad \|\bar{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} N_0^{q\overline{a}}, \tag{7.9}$$

and

$$\forall b \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\big(U_\infty(b,\omega(b,\varepsilon))\big) = 0 \quad \text{and} \quad \alpha_\infty\big(b,\omega(b,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(b),$$

where the Cantor set $\mathcal{C}_\infty^\varepsilon$ is defined by

$$\mathcal{C}_\infty^\varepsilon = \Big\{ b \in (b_0, b_1) \quad \text{s.t.} \quad \big(b, \omega(b,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma \Big\}. \tag{7.10}$$

Proof. In view of (7.3) and (7.4), we obtain

$$\|W_{n+1} - W_n\|_{q,s_0}^{\gamma,\mathcal{O}} = \|H_{n+1}\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant \|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1} N_n^{-a_2}.$$

This implies the convergence of the sequence $(W_n)_{n\in\mathbb{N}}$. Its limit is denoted by

$$W_\infty := \lim_{n\to\infty} W_n := (\mathfrak{I}_\infty, \alpha_\infty - \omega, 0, 0)$$

and we set

$$U_\infty := \big(i_\infty, \alpha_\infty, (b,\omega)\big) = U_0 + W_\infty.$$

Taking $n \to \infty$ in (7.7) gives

$$\mathfrak{S} i_\infty(\varphi) = i_\infty(-\varphi).$$

According to Proposition 7.1-$(\mathcal{P}2)_n$, we get for small values of ε

$$\forall (b,\omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}\Big(i_\infty(b,\omega), \alpha_\infty(b,\omega), (b,\omega), \varepsilon\Big) = 0, \tag{7.11}$$

where $\mathcal{F}$ is the functional defined in (5.19). We emphasize that the Cantor set $\mathcal{G}_\infty^\gamma$ depends on ε through γ fixed in (7.2). Now, from Proposition 7.1-$(\mathcal{P}1)_n$, we deduce that

$$\alpha_\infty(b,\omega) = \omega + \mathrm{r}_\varepsilon(b,\omega) \quad \text{with} \quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} N_0^{q\overline{a}}.$$

Next we shall prove the second result and check the existence of solutions to the original Hamiltonian equation. First recall that the open set $\mathcal{O}$ is defined in (1.9) by

$$\mathcal{O} = (b_0, b_1) \times \mathscr{U} \quad \text{with} \quad \mathscr{U} = B(0, R_0) \quad \text{for some large } R_0 > 0,$$

where the ball $\mathscr{U}$ is taken to contain the equilibrium frequency vector $b \mapsto \omega_{\rm Eq}(b)$. In view of (7.8), we obtain that for any $b \in (b_0, b_1)$, the mapping $\omega \mapsto \alpha_\infty(b,\omega)$ is invertible from $\mathscr{U}$ into its image $\alpha_\infty(b,\mathscr{U})$ and we have

$$\widehat{\omega} = \alpha_\infty(b,\omega) = \omega + \mathrm{r}_\varepsilon(b,\omega) \Leftrightarrow \omega = \alpha_\infty^{-1}(b,\widehat{\omega}) = \widehat{\omega} + \widehat{\mathrm{r}}_\varepsilon(b,\widehat{\omega}).$$

In particular,

$$\widehat{\mathrm{r}}_\varepsilon(b,\widehat{\omega}) = -\mathrm{r}_\varepsilon(b,\omega).$$

Differentiating the previous relation and using (7.8), we find

$$\|\widehat{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{7.12}$$

Now, we set

$$\omega(b,\varepsilon) := \alpha_\infty^{-1}(b,-\omega_{\rm Eq}(b)) = -\omega_{\rm Eq}(b) + \overline{\mathrm{r}}_\varepsilon(b) \quad \text{with} \quad \overline{\mathrm{r}}_\varepsilon(b) := \widehat{\mathrm{r}}_\varepsilon\big(b,-\omega_{\rm Eq}(b)\big)$$

and consider the following Cantor set

$$\mathcal{C}_\infty^\varepsilon := \Big\{ b \in (b_0,b_1) \quad \text{s.t.} \quad \big(b,\omega(b,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma \Big\}.$$

Then, according to (7.11), we get

$$\forall b \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\Big(U_\infty\big(b,\omega(b,\varepsilon)\big)\Big) = 0.$$

This gives a nontrivial reversible solution for the original Hamiltonian equation provided that $b \in \mathcal{C}_\infty^\varepsilon$. From Lemma 3.3, we obtain that all the derivatives up to order q of $\omega_{\rm Eq}$ are uniformly bounded on $[b_0,b_1]$. As a consequence, the chain rule and (7.12) imply

$$\|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \quad \text{and} \quad \|\omega(\cdot,\varepsilon)\|_q^{\gamma,\mathcal{O}} \lesssim 1 + \varepsilon\gamma^{-1}N_0^{q\overline{a}} \lesssim 1. \tag{7.13}$$

This achieves the proof of Corollary 7.1. □

7.2 Measure estimates

In this last section, we check that the Cantor set $\mathcal{C}_\infty^\varepsilon$, defined in (7.10), of parameters generating non-trivial quasi-periodic solutions is non trivial. More precisely, we have the following proposition giving a lower bound measure for $\mathcal{C}_\infty^\varepsilon$.

Proposition 7.2. *Let q_0 be defined as in Lemma 3.5 and impose* (7.1) *and* (7.2) *with $q = q_0 + 1$. Assume the additional conditions*

$$\begin{cases} \tau_1 > dq_0 \\ \tau_2 > \tau_1 + dq_0 \\ \upsilon = \frac{1}{q_0+3}. \end{cases} \tag{7.14}$$

Then there exists $C > 0$ such that

$$\big|\mathcal{C}_\infty^\varepsilon\big| \geqslant (b_1 - b_0) - C\varepsilon^{\frac{a\upsilon}{q_0}}.$$

In particular,

$$\lim_{\varepsilon\to 0} \big|\mathcal{C}_\infty^\varepsilon\big| = b_1 - b_0.$$

Remark 7.1. *The constraints listed in* (7.14) *appear naturally in the proof, see* (7.22) *and* (7.27), *for the convergence of series and for smallness conditions. Notice that these conditions agree with* (1.10) *and Proposition 6.2.*

Proof. According to Corollary 7.1, we can decompose the Cantor set $\mathcal{C}_\infty^\varepsilon$ in the following intersection

$$\mathcal{C}_\infty^\varepsilon := \bigcap_{n\in\mathbb{N}} \mathcal{C}_n^\varepsilon \quad \text{where} \quad \mathcal{C}_n^\varepsilon := \Big\{ b \in (b_0, b_1) \quad \text{s.t} \quad \big(b, \omega(b,\varepsilon)\big) \in \mathcal{A}_n^\gamma \Big\}. \tag{7.15}$$

Recall that the intermediate sets $\mathcal{A}_n^\gamma$ and the perturbed frequency vector $\omega(b,\varepsilon)$ are respectively defined in Proposition 7.1 and in (7.8). Instead of measuring directly $\mathcal{C}_\infty^\varepsilon$, we rather estimate the measure of its complementary set in (b_0, b_1). Thus, we write

$$(b_0, b_1) \setminus \mathcal{C}_\infty^\varepsilon = \big((b_0, b_1) \setminus \mathcal{C}_0^\varepsilon\big) \sqcup \bigsqcup_{n=0}^{\infty} \big(\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon\big). \tag{7.16}$$

Then, we have to measure all the sets appearing in the decomposition (7.16). This can be done by using Lemma 3.6 together with some trivial inclusions allowing to link the time and space Fourier modes in order to make the series converge. For more details, we refer to Lemmata 7.1, 7.2 and 7.3.

From (7.2) and (7.9), one obtains

$$\sup_{b\in(b_0,b_1)} |\omega(b,\varepsilon) + \omega_{\mathrm{Eq}}(b)| \leqslant \|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1} N_0^{q\overline{a}} = C\varepsilon^{1-a(1+q\overline{a})}.$$

Notice that the conditions (7.1) and (7.2) imply in particular

$$0 < a < \frac{1}{1+q\overline{a}}.$$

Therefore, taking ε small enough yields

$$\sup_{b\in(b_0,b_1)} |\omega(b,\varepsilon) + \omega_{\mathrm{Eq}}(b)| \leqslant \|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}} \leqslant 1.$$

Recall that $\mathscr{U} = B(0, R_0)$, then, up to take R_0 large enough, we get

$$\forall b \in (b_0, b_1),\ \forall \varepsilon \in [0, \varepsilon_0), \quad \omega(b,\varepsilon) \in \mathscr{U} = B(0, R_0).$$

Recall that $\mathcal{A}_0^\gamma = \mathcal{O} = (b_0, b_1) \times \mathscr{U}$ then, from (7.15),

$$\mathcal{C}_0^\varepsilon = (b_0, b_1)$$

and coming back to (7.16), we find

$$\begin{aligned} \Big|(b_0, b_1) \setminus \mathcal{C}_\infty^\varepsilon\Big| &\leqslant \sum_{n=0}^{\infty} \Big|\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon\Big| \\ &:= \sum_{n=0}^{\infty} \mathcal{S}_n. \end{aligned} \tag{7.17}$$

In accordance with the notations used in Propositions 6.3 and 6.4, we denote the perturbed frequencies associated with the reduced linearized operator at state i_n in the following way

$$\begin{aligned} \mu_j^{\infty,n}(b,\varepsilon) &:= \mu_j^\infty\big(b, \omega(b,\varepsilon), i_n\big) \\ &= \Omega_j(b) + j r^{1,n}(b,\varepsilon) + r_j^{\infty,n}(b,\varepsilon), \end{aligned} \tag{7.18}$$

where

$$\begin{aligned} r^{1,n}(b,\varepsilon) &:= V_n^\infty(b,\varepsilon) - \tfrac{1}{2}, \\ V_n^\infty(b,\varepsilon) &:= V_{i_n}^\infty(b, \omega(b,\varepsilon)), \\ r_j^{\infty,n}(b,\varepsilon) &:= r_j^\infty\big(b, \omega(b,\varepsilon), i_n\big). \end{aligned}$$

Now, according to (7.15), Propositions 6.4, 6.5 and 6.2 one can write for any $n\in\mathbb{N}$,

$$\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n),\tag{7.19}$$

where we denote

$$\begin{aligned}\mathcal{R}_{l,j}^{(0)}(i_n)&:=\Big\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon)\Big|\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},\\ \mathcal{R}_{l,j,j_0}(i_n)&:=\Big\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)\Big|\leqslant\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\},\\ \mathcal{R}_{l,j}^{(1)}(i_n)&:=\Big\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\Big|\leqslant\tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.\end{aligned}$$

In view of the inclusion

$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C})$$

and the fact that $q=q_0+1$, one obtains that for any $n\in\mathbb{N}$ the curves

$$\begin{aligned}&b\mapsto\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon),\quad(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\&b\mapsto\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon),\quad(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\&b\mapsto\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon),\quad(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\end{aligned}$$

are of regularity C^{q_0}. Therefore, applying Lemma 3.6 together with Lemma 7.3 yields

$$\begin{aligned}\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big|&\lesssim\gamma^{\frac{\upsilon}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\ \Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big|&\lesssim\gamma^{\frac{1}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\ \Big|\mathcal{R}_{l,j,j_0}(i_n)\Big|&\lesssim\gamma^{\frac{1}{q_0}}\langle j-j_0\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}.\end{aligned}\tag{7.20}$$

We first estimate the measure of $\mathcal{S}_0$ and $\mathcal{S}_1$ defined in (7.17). From Lemma 7.2, we have some trivial inclusions allowing us to write for $n\in\{0,1\}$,

$$\mathcal{S}_n\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|j|\leqslant C_0\langle l\rangle,|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big|+\sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|j-j_0|\leqslant C_0\langle l\rangle,|l|\leqslant N_n\\\min(|j|,|j_0|)\leqslant c_2\gamma_1^{-\upsilon}\langle l\rangle^{\tau_1}}}\Big|\mathcal{R}_{l,j,j_0}(i_n)\Big|+\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|j|\leqslant C_0\langle l\rangle,|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big|.\tag{7.21}$$

Inserting (7.20) into (7.21) implies that for $n\in\{0,1\}$,

$$\begin{aligned}\mathcal{S}_n\lesssim\gamma^{\frac{1}{q_0}}&\Bigg(\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}+\sum_{\substack{|j-j_0|\leqslant C_0\langle l\rangle\\\min(|j|,|j_0|)\leqslant c_2\gamma^{-\upsilon}\langle l\rangle^{\tau_1}}}|j-j_0|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}\Bigg)\\&+\gamma^{\frac{\upsilon}{q_0}}\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}.\end{aligned}$$

The first two conditions listed in (7.14) write

$$\tau_1>d\,q_0\quad\text{and}\quad\tau_2>\tau_1+d\,q_0.\tag{7.22}$$

Hence, we can make the series appearing in the following expression converge and write

$$\begin{aligned}\max_{n\in\{0,1\}}\mathcal{S}_n&\lesssim\gamma^{\frac{1}{q_0}}\Bigg(\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg)+\gamma^{\frac{\upsilon}{q_0}}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}\\&\lesssim\gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.\end{aligned}\tag{7.23}$$

Let us now move to the estimate of $\mathcal{S}_n$ for $n\geqslant 2$ defined by (7.17). Using Lemma 7.1 and Lemma 7.2, we infer

$$\mathcal{S}_n\leqslant\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|j|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n}}\left|\mathcal{R}_{l,j}^{(0)}(i_n)\right|+\sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|j-j_0|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n\\\min(|j|,|j_0|)\leqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}}}\left|\mathcal{R}_{l,j,j_0}(i_n)\right|+\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|j|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n}}\left|\mathcal{R}_{l,j}^{(1)}(i_n)\right|.$$

Notice that if $|j-j_0|\leqslant C_0\langle l\rangle$ and $\min(|j|,|j_0|)\leqslant\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\max(|j|,|j_0|)=\min(|j|,|j_0|)+|j-j_0|\leqslant\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}+C_0\langle l\rangle\lesssim\gamma^{-\upsilon}\langle l\rangle^{\tau_1}.$$

Hence, we deduce from (7.20) that

$$\mathcal{S}_n\lesssim\gamma^{\frac{1}{q_0}}\left(\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{|l|>N_{n-1}}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\right)+\gamma^{\frac{\upsilon}{q_0}}\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}.$$

Now according to (7.22), we obtain

$$\sum_{n=2}^{\infty}\mathcal{S}_n\lesssim\gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.\tag{7.24}$$

Inserting (7.24) and (7.23) into (7.17) yields

$$\left|(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon\right|\lesssim\gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.$$

Remark also that (7.14) implies

$$\min\left(\tfrac{\upsilon}{q_0},\tfrac{1}{q_0}-\upsilon\right)=\tfrac{\upsilon}{q_0}.$$

Consequently, using the fact that $\gamma=\varepsilon^a$ due to (7.2), we finally get

$$\left|(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon\right|\lesssim\varepsilon^{\frac{a\upsilon}{q_0}}.$$

This ends the proof of Proposition 7.2. □

We shall now prove Lemmata 7.1, 7.2 and 7.3 used in the proof of Proposition 7.2.

Lemma 7.1. *Let $n\in\mathbb{N}\setminus\{0,1\}$ and $l\in\mathbb{Z}^d$ such that $|l|\leqslant N_{n-1}$. Then the following assertions hold true.*

(i) For $j\in\mathbb{Z}$ with $(l,j)\neq(0,0)$, we get $\mathcal{R}_{l,j}^{(0)}(i_n)=\varnothing$.

(ii) For $(j,j_0)\in(\mathbb{S}_0^c)^2$ with $(l,j)\neq(0,j_0)$, we get $\mathcal{R}_{l,j,j_0}(i_n)=\varnothing$.

(iii) For $j\in\mathbb{S}_0^c$, we get $\mathcal{R}_{l,j}^{(1)}(i_n)=\varnothing$.

(iv) For any $n\in\mathbb{N}\setminus\{0,1\}$,

$$\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\cup\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\cup\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n).$$

Proof. The following estimate, obtained from (7.5), turns to be very useful in the sequel. For any $n\geqslant 2$, we have

$$\begin{aligned}\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}&\leqslant\|U_n-U_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\&\leqslant\|H_n\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\\&\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}.\end{aligned}\tag{7.25}$$

Since (7.25) is only true for $n \geqslant 2$, we had to estimate the measures of $\mathcal{S}_0$ and $\mathcal{S}_1$ differently in the proof of Proposition 7.2.

(i) Assume that $|l| \leqslant N_{n-1}$ and $(l,j) \neq (0,0)$. Let us prove that

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{R}_{l,j}^{(0)}(i_{n-1}). \tag{7.26}$$

Take $b \in \mathcal{R}_{l,j}^{(0)}(i_n)$. In view of (7.19), we have in particular that $b \in \mathcal{C}_n^{\varepsilon} \subset \mathcal{C}_{n-1}^{\varepsilon}$. In addition, the triangle inequality gives

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^{\infty}(b,\varepsilon)\big| &\leqslant \big|\omega(b,\varepsilon)\cdot l + jV_{n}^{\infty}(b,\varepsilon)\big| + |j|\big|V_{n}^{\infty}(b,\varepsilon) - V_{n-1}^{\infty}(b,\varepsilon)\big| \\ &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C|j|\|V_{i_n}^{\infty} - V_{i_{n-1}}^{\infty}\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}$$

Thus, putting together (6.27), (7.25), (7.2) and the fact that $\sigma_4 \geqslant 2$, we obtain

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^{\infty}(b,\varepsilon)\big| &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon\langle j\rangle\|i_n - i_{n-1}\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} \\ &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon^{2-a}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

According the definition of γ_n in Proposition 7.1-$(\mathcal{P}2)_n$, we infer

$$\exists c_0 > 0, \quad \forall n \in \mathbb{N}, \quad \gamma_{n+1}^{\upsilon} - \gamma_n^{\upsilon} \leqslant -c_0\,\gamma^{\upsilon}2^{-n}.$$

Notice that (7.14), (7.1) and (7.2) give

$$2 - a - a\upsilon > 1 \quad \text{and} \quad a_2 > \tau_1, \tag{7.27}$$

which implies in turn

$$\sup_{n\in\mathbb{N}} 2^n N_{n-1}^{-a_2+\tau_1} < \infty.$$

Consequently, for ε small enough and $|l| \leqslant N_{n-1}$,

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^{\infty}(b,\varepsilon)\big| &\leqslant \tfrac{4\gamma_{n}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\tfrac{\langle j\rangle\gamma^{\upsilon}}{2^n\langle l\rangle^{\tau_1}}\Big(-4c_0 + C\varepsilon 2^n N_{n-1}^{-a_2+\tau_1}\Big) \\ &\leqslant \tfrac{4\gamma_{n}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.\end{aligned}$$

It follows that $b \in \mathcal{R}_{l,j}^{(0)}(i_{n-1})$ and this proves (7.26). Now, from (7.19) we deduce

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{R}_{l,j}^{(0)}(i_{n-1}) \subset \mathcal{C}_{n-1}^{\varepsilon} \setminus \mathcal{C}_n^{\varepsilon}.$$

In view of (7.26) and (7.19), we get $\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{C}_n^{\varepsilon} \setminus \mathcal{C}_{n+1}^{\varepsilon}$ and thus we conclude

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \big(\mathcal{C}_n^{\varepsilon} \setminus \mathcal{C}_{n+1}^{\varepsilon}\big) \cap \big(\mathcal{C}_{n-1}^{\varepsilon} \setminus \mathcal{C}_n^{\varepsilon}\big) = \varnothing.$$

This proves the first point.

(ii) Let $(j,j_0) \in (\mathbb{S}_0^c)^2$ and $(l,j) \neq (0,j_0)$. If $j = j_0$ then by construction $\mathcal{R}_{l,j_0,j_0}(i_n) = \mathcal{R}_{l,0}^{(0)}(i_n)$ and then the result is an immediate consequence of the first point. Then, we restrict the discussion to the case $j \neq j_0$. In a similar way to the point (i), we only have to check that

$$\mathcal{R}_{l,j,j_0}(i_n) \subset \mathcal{R}_{l,j,j_0}(i_{n-1}).$$

Take $b \in \mathcal{R}_{l,j,j_0}(i_n)$. Then coming back to (7.19), we deduce from the triangle inequality that $b \in \mathcal{C}_n^{\varepsilon} \subset \mathcal{C}_{n-1}^{\varepsilon}$ and

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(b,\varepsilon) - \mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} + \varrho_{j,j_0}^{n}(b,\varepsilon), \tag{7.28}$$

where

$$\varrho_{j,j_0}^{n}(b,\varepsilon) := \big|\mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon) - \mu_j^{\infty,n-1}(b,\varepsilon) + \mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big|.$$

According to (7.18), one obtains

$$\varrho^n_{j,j_0}(b,\varepsilon)\leqslant|j-j_0|\big|r^{1,n}(b,\varepsilon)-r^{1,n-1}(b,\varepsilon)\big|+\big|r_j^{\infty,n}(b,\varepsilon)-r_j^{\infty,n-1}(b,\varepsilon)\big|\\ +\big|r_{j_0}^{\infty,n}(b,\varepsilon)-r_{j_0}^{\infty,n-1}(b,\varepsilon)\big|. \tag{7.29}$$

From (6.54), (7.25), (7.2) and the fact that $\sigma_4\geqslant\sigma_3$, we deduce that

$$\begin{aligned}\big|r^{1,n}(b,\varepsilon)-r^{1,n-1}(b,\varepsilon)\big|&\lesssim\varepsilon\|i_n-i_{n-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma_3}\\&\lesssim\varepsilon^2\gamma^{-1}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2-a}N_{n-1}^{-a_2}.\end{aligned}$$

Similarly, (6.67), (7.25) and (7.2) imply

$$\begin{aligned}\big|r_j^{\infty,n}(b,\varepsilon)-r_j^{\infty,n-1}(b,\varepsilon)\big|&\lesssim\varepsilon\gamma^{-1}\|i_n-i_{n-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma_4}\\&\lesssim\varepsilon^2\gamma^{-2}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Plugging the preceding two estimates into (7.29) yields

$$\varrho^n_{j,j_0}(b,\varepsilon)\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}. \tag{7.30}$$

Gathering (7.30) and (7.28) and using $\gamma_{n+1}=\gamma_n-\varepsilon^a2^{-n-1}$, we obtain

$$\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(b,\varepsilon)-\mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big|\leqslant\tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-\varepsilon^a\langle j-j_0\rangle2^{-n}\langle l\rangle^{-\tau_2}\\+C\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.$$

Using the fact that $|l|\leqslant N_{n-1}$, we deduce

$$-\varepsilon^a2^{-n}\langle l\rangle^{-\tau_2}+C\varepsilon^{2(1-a)}N_{n-1}^{-a_2}\leqslant\varepsilon^a2^{-n}\langle l\rangle^{-\tau_2}\Big(-1+C\varepsilon^{2-3a}2^nN_{n-1}^{-a_2+\tau_2}\Big).$$

Notice that (7.1) and (7.2) imply in particular

$$a_2>\tau_2\quad\text{and}\quad a<\tfrac{2}{3}. \tag{7.31}$$

Therefore, for ε small enough, we get

$$\forall\, n\in\mathbb{N},\quad -1+C\varepsilon^{2-3a}2^nN_{n-1}^{-a_2+\tau_2}\leqslant0,$$

which implies in turn

$$\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(b,\varepsilon)-\mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big|\leqslant\tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Finally, $b\in\mathcal{R}_{l,j,j_0}(i_{n-1})$. This achieves the proof of the second point.
(iii) Let $j\in\mathbb{S}_0^c$. In particular, one has $(l,j)\neq(0,0)$. In a similar line to the first point, we shall prove that if $|l|\leqslant N_{n-1}$ and then

$$\mathcal{R}^{(1)}_{l,j}(i_n)\subset\mathcal{R}^{(1)}_{l,j}(i_{n-1}),$$

where the set $\mathcal{R}^{(1)}_{l,j}(i_n)$ is defined below (7.19). Take $b\in\mathcal{R}^{(1)}_{l,j}(i_n)$. Then, by construction, $b\in\mathcal{C}_n^\varepsilon\subset\mathcal{C}_{n-1}^\varepsilon$. By using the triangle inequality, (6.68), (7.25) and the choice $\gamma=\varepsilon^a$, we obtain

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(b,\varepsilon)\big|&\leqslant\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\big|+|\mu_j^{\infty,n}(b,\varepsilon)-\mu_j^{\infty,n-1}(b,\varepsilon)|\\&\leqslant\tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon\gamma^{-1}|j|\|i_n-i_{n-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma_4}\\&\leqslant\tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon^{2(1-a)}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Now recalling that $\gamma_{n+1}=\gamma_n-\varepsilon^a 2^{-n-1}$ and $|l|\leqslant N_{n-1}$, we get

$$\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(b,\varepsilon)\big|\leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}+\tfrac{\langle j\rangle\varepsilon^a}{2^{n+1}\langle l\rangle^{\tau_1}}\Big(-1+\varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\Big).$$

As a byproduct of (7.31), we infer

$$a_2>\tau_1\quad\text{and}\quad a<\tfrac{2}{3}.\tag{7.32}$$

Therefore, up to take ε small enough, we deduce

$$\forall\, n\in\mathbb{N},\quad -1+\varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\leqslant 0,$$

which implies in turn that

$$\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(b,\varepsilon)\big|\leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Finally, $b\in\mathcal{R}_{l,j}^{(1)}(i_{n-1})$ and the proof of the third point is now complete.
(iv) Follows immediately from (7.19) and the points (i)-(ii)-(iii). □

The following lemma provides necessary constraints on the time and space Fourier modes so that the sets in (7.19) are not void.

Lemma 7.2. *There exists ε_0 such that for any $\varepsilon\in[0,\varepsilon_0]$ and $n\in\mathbb{N}$ the following assertions hold true.*

(i) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}$. If $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(ii) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. If $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$, then $|j-j_0|\leqslant C_0\langle l\rangle$.

(iii) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$. If $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(iv) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. There exists $c_2>0$ such that if $\min(|j|,|j_0|)\geqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\mathcal{R}_{l,j,j_0}(i_n)\subset\mathcal{R}_{l,j-j_0}^{(0)}(i_n).$$

Proof. **(i)** Let us assume that $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$. Then, there exists $b\in(b_0,b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon)|\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

From triangle and Cauchy-Schwarz inequalities, (7.6) and (7.2), we deduce

$$\begin{aligned}|V_n^\infty(b,\varepsilon)||j|&\leqslant 4|j|\gamma_{n+1}^{\upsilon}\langle l\rangle^{-\tau_1}+|\omega(b,\varepsilon)\cdot l|\\&\leqslant 4|j|\gamma_{n+1}^{\upsilon}+C\langle l\rangle\\&\leqslant 8\varepsilon^{a\upsilon}|j|+C\langle l\rangle.\end{aligned}$$

Remark that we used the fact that $(b,\varepsilon)\mapsto\omega(b,\varepsilon)$ is bounded. Also notice that the identity

$$V_n^\infty(b,\varepsilon)=\tfrac12+r^{1,n}(b,\varepsilon)$$

together with (6.20), (6.65) and Proposition 7.1-$(\mathcal{P}1)_n$ imply

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad \sup_{n\in\mathbb{N}}\sup_{b\in(b_0,b_1)}|\partial_b^k r^{1,n}(b,\varepsilon)|&\leqslant\gamma^{-k}\sup_{n\in\mathbb{N}}\|r^{1,n}\|_q^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-k}\\&\lesssim\varepsilon^{1-ak}.\end{aligned}\tag{7.33}$$

Hence, taking ε small enough, we infer

$$\inf_{n\in\mathbb{N}}\inf_{b\in(b_0,b_1)}|V_n^\infty(b,\varepsilon)|\geqslant\tfrac14.$$

Therefore, up to choose ε small enough we can ensure $|j| \leqslant C_0\langle l\rangle$ for some $C_0 > 0$.
(ii) In the case $j = j_0$ we get by definition $\mathcal{R}_{l,j_0,j_0}(i_n) = \mathcal{R}^{(0)}_{l,0}(i_n)$, so this case can be treated by the first point. Then, we shall restrict the discussion to the case $j \neq j_0$. Let us assume that $\mathcal{R}_{l,j,j_0}(i_n) \neq \varnothing$. Then, there exists $b \in (b_0, b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon)| \leqslant \tfrac{2\gamma_{n+1}|j-j_0|}{\langle l\rangle^{\tau_2}}.$$

By using triangle and Cauchy-Schwarz inequalities, (7.6) and (7.2), we get

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon)| &\leqslant 2\gamma_{n+1}|j-j_0|\langle l\rangle^{-\tau_2} + |\omega(b,\varepsilon)\cdot l| \\ &\leqslant 2\gamma_{n+1}|j-j_0| + C\langle l\rangle \\ &\leqslant 4\varepsilon^a|j-j_0| + C\langle l\rangle.\end{aligned}$$

In a similar way to (7.33), we may obtain

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad \sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}\sup_{b\in(b_0,b_1)}|j||\partial_b^k r_j^{\infty,n}(b,\varepsilon)| &\leqslant \gamma^{-k}\sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^{\infty,n}\|_q^{\gamma,\mathcal{O}} \\ &\lesssim \varepsilon\gamma^{-1-k} \\ &\lesssim \varepsilon^{1-a(1+k)}.\end{aligned} \tag{7.34}$$

From the triangle inequality, Lemma 3.3-(iii), (7.33) and (7.34) we infer for $j \neq j_0$,

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon)| &\geqslant |\Omega_j(b) - \Omega_{j_0}(b)| - |r^{1,n}(b,\varepsilon)||j-j_0| - |r_j^{\infty,n}(b,\varepsilon)| - |r_{j_0}^{\infty,n}(b,\varepsilon)| \\ &\geqslant \big(\tfrac{b_0^2}{6} - C\varepsilon^{1-a}\big)|j-j_0| \\ &\geqslant \tfrac{b_0^2}{12}|j-j_0|.\end{aligned}$$

Notice that the last inequality is obtained for ε sufficiently small. Gathering the previous inequalities implies that, up to choose ε small enough, we can ensure $|j-j_0| \leqslant C_0\langle l\rangle$, for some $C_0 > 0$.
(iii) First notice that the case $j = 0$ is obvious. Now for $j \neq 0$ we assume that $\mathcal{R}^{(1)}_{l,j}(i_n) \neq \varnothing$. Then, there exists $b \in (b_0, b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon)| \leqslant \tfrac{\gamma_{n+1}|j|}{\langle l\rangle^{\tau_1}}.$$

Thus, triangle and Cauchy-Schwarz inequalities, (7.6) and (7.2) imply

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)| &\leqslant \gamma_{n+1}|j|\langle l\rangle^{-\tau_1} + |\omega(b,\varepsilon)\cdot l| \\ &\leqslant 2\varepsilon^a|j| + C\langle l\rangle.\end{aligned}$$

According to the definition (7.18) together with the triangle inequality, Lemma 3.3-(ii), (7.33) and (7.34), we obtain

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)| &\geqslant \tfrac{b_0^2}{2}|j| - |j||r^{1,n}(b,\varepsilon)| - |r_j^{\infty,n}(b,\varepsilon)| \\ &\geqslant \tfrac{b_0^2}{2}|j| - C\varepsilon^{1-a}|j|.\end{aligned}$$

Putting together the previous two inequalities and the second condition in (7.32) yields

$$\big(\tfrac{b_0^2}{2} - C\varepsilon^{1-a} - 2\varepsilon^a\big)|j| \leqslant C\langle l\rangle.$$

Finally, by choosing ε small enough we get $|j| \leqslant C_0\langle l\rangle$, for some $C_0 > 0$.
(iv) First remark that the case $j = j_0$ is obvious as a direct consequence of the definition (7.19). Let $j \neq j_0$. In view of the symmetry property $\mu_{-j}^{\infty,n} = -\mu_j^{\infty,n}$ of the perturbed eigenvalues, we can always assume that $0 < j < j_0$. Take $b \in \mathcal{R}_{l,j,j_0}(i_n)$. Then by construction

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) \pm \mu_{j_0}^{\infty,n}(b,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Putting together (7.18), (3.15) and the triangle inequality, we find

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l+(j\pm j_0)V_n^\infty(b,\varepsilon)\big|\leqslant&\big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\pm\mu_{j_0}^{\infty,n}(b,\varepsilon)\big|+\tfrac{1}{2}|b^{2j}\pm b^{2j_0}|\\&+\tfrac12\big|(j-1)\pm(j_0-1)-(j\pm j_0)\big|+\big|r_j^{\infty,n}(b,\varepsilon)\pm r_{j_0}^{\infty,n}(b,\varepsilon)\big|.\end{aligned}$$

Hence, we deduce

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l+(j\pm j_0)V_n^\infty(b,\varepsilon)\big|\leqslant&\tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}}+\tfrac12|b^{2j}\pm b^{2j_0}|\\&+\tfrac12\big|(j-1)\pm(j_0-1)-(j\pm j_0)\big|+\big|r_j^{\infty,n}(b,\varepsilon)\pm r_{j_0}^{\infty,n}(b,\varepsilon)\big|.\end{aligned}\tag{7.35}$$

Notice that

$$b^{2j}+b^{2j_0}\leqslant C\tfrac{\langle j+j_0\rangle}{j}.$$

In addition, Taylor formula implies

$$b^{2j}-b^{2j_0}\leqslant-2\ln(b)\int_j^{j_0}b^{2x}dx\leqslant\tfrac{c_1\langle j-j_0\rangle}{j},$$

where $c_1=\sup\limits_{j\in\mathbb{N},b\in(0,1)}\big(-2\ln(b)jb^{2j}\big)>0$. On the other hand, one has

$$\big|(j-1)\pm(j_0-1)-(j\pm j_0)\big|=1\pm1\leqslant\tfrac{\langle j+j_0\rangle}{j}.$$

Applying (6.65), we find for $j\neq j_0$,

$$\begin{aligned}\big|r_j^{\infty,n}(b,\varepsilon)\pm r_{j_0}^{\infty,n}(b,\varepsilon)\big|\leqslant&C\varepsilon^{1-a}\big(|j|^{-1}+|j_0|^{-1}\big)\\\leqslant&C\varepsilon^{1-a}\tfrac{\langle j\pm j_0\rangle}{j}.\end{aligned}$$

Plugging the preceding estimates into (7.35) yields

$$\big|\omega(b,\varepsilon)\cdot l+(j\pm j_0)V_n^\infty(b,\varepsilon)\big|\leqslant\tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}}+C\tfrac{\langle j\pm j_0\rangle}{j}.$$

Therefore, if we assume $j\geqslant\frac12C\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$ and $\tau_2>\tau_1$, then we deduce

$$\big|\omega(b,\varepsilon)\cdot l+(j\pm j_0)V_n^\infty(b,\varepsilon)\big|\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_1}}.$$

This achieves the proof of Lemma 7.2, taking $c_2=\frac{C}{2}$. □

We shall now establish that the perturbed frequencies $\omega(b,\varepsilon)$ satisfy the Rüssemann conditions. This is done by a perturbation argument on the transversality conditions of the equilibrium linear frequencies $\omega_{\mathrm{Eq}}(b)$ stated in Lemma 3.5.

Lemma 7.3. *Let q_0, C_0 and ρ_0 as in Lemma 3.5. There exist $\varepsilon_0>0$ small enough such that for any $\varepsilon\in[0,\varepsilon_0]$ the following assertions hold true.*

(i) For all $l\in\mathbb{Z}^d\setminus\{0\}$, we have

$$\inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_b^k\left(\omega(b,\varepsilon)\cdot l\right)\big|\geqslant\tfrac{\rho_0\langle l\rangle}{2}.$$

(ii) For all $(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}$ such that $|j|\leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad\inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon)\big)\big|\geqslant\tfrac{\rho_0\langle l\rangle}{2}.$$

(iii) For all $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$ such that $|j|\leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad\inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\big)\big|\geqslant\tfrac{\rho_0\langle l\rangle}{2}.$$

(iv) For all $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ *such that* $|j-j_0|\leqslant C_0\langle l\rangle$*, we have*

$$\forall n\in\mathbb{N},\qquad \inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)\big)\right|\geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

Proof. **(i)** From the triangle and Cauchy-Schwarz inequalities together with (7.13), (7.2) and Lemma 3.5-(i), we deduce

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}|\partial_b^k\left(\omega(b,\varepsilon)\cdot l\right)| &\geqslant \max_{k\in[\![0,q_0]\!]}|\partial_b^k\left(\omega_{\mathrm{Eq}}(b)\cdot l\right)|-\max_{k\in[\![0,q]\!]}|\partial_b^k\left(\bar{\mathrm{r}}_\varepsilon(b)\cdot l\right)|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon\gamma^{-1-q}N_0^{q\overline{a}}\langle l\rangle\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

provided that ε is small enough and

$$1-a(1+q+q\overline{a})>0. \tag{7.36}$$

Notice that the condition (7.36) is automatically satisfied by (7.2) and (7.1).
(ii) As before, using the triangle and Cauchy-Schwarz inequalities combined with (7.13), (7.33), Lemma 3.5-(ii) and the fact that $|j|\leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}|\partial_b^k\left(\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon)\right)| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega_{\mathrm{Eq}}(b)\cdot l+\tfrac{j}{2}\big)\right|-\max_{k\in[\![0,q]\!]}|\partial_b^k\left(\bar{\mathrm{r}}_\varepsilon(b)\cdot l+jr^{1,n}(b,\varepsilon)\right)|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle - C\varepsilon^{1-aq}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough and with the condition (7.36).
(iii) As before, using triangle and Cauchy-Schwarz inequalities combined with (7.13), (7.33), (7.34), Lemma 3.5-(iii) and the fact that $|j|\leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}|\partial_b^k\left(\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)\right)|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_b^k\big(\bar{\mathrm{r}}_\varepsilon(b)\cdot l+jr^{1,n}(b,\varepsilon)+r_j^{\infty,n}(b,\varepsilon)\big)\right|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle - C\varepsilon^{1-a(1+q)}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough with the condition (7.36).
(iv) Arguing as in the preceding point, using (7.33), (7.34), Lemma 3.5-(iv) and the fact that $0<|j-j_0|\leqslant C_0\langle l\rangle$ (notice that the case $j=j_0$ is trivial), we have

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)-\Omega_{j_0}(b)\big)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_b^k\big(\bar{\mathrm{r}}_\varepsilon(b)\cdot l+(j-j_0)r^{1,n}(b,\varepsilon)+r_j^{\infty,n}(b,\varepsilon)-r_{j_0}^{\infty,n}(b,\varepsilon)\big)\right|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle - C\varepsilon^{1-a(1+q)}|j-j_0|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough. This ends the proof of Lemma 7.3. □

Acknowledgements : The work of Z. Hassainia is supported by Tamkeen under the NYU Abu Dhabi Research Institute grant of the center SITE.
The authors would like to thank Taoufik Hmidi for introducing the problem and for his precious advices when writing this document.

References

[1] T. Alazard, P. Baldi, *Gravity capillary standing water waves*, Arch. Ration. Mech. Anal. 217 (2015), no. 3, 741–830.

[2] V. I. Arnold, *Small denominators and problems of stability of motion in classical mechanics and celestial mechanics,* Uspekhi Mat. Nauk 18 (1963), 91–192.

[3] P. Baldi, M. Berti, E. Haus, R. Montalto, *Time quasi-periodic gravity water waves in finite depth,* Invent. Math. 214 (2018), no. 2, 739–911.

[4] P. Baldi, M. Berti, R. Montalto, *KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation,* Math. Ann. 359 (2014), no. 1-2, 471–536.

[5] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of KdV,* Ann. Inst. H. Poincaré Analyse Non. Lin. 33 (2016), no. 6, 1589–1638.

[6] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of mKdV,* Boll. Unione Mat. Ital. 9 (2016), 143–188.

[7] P. Baldi, R. Montalto, *Quasi-periodic incompressible Euler flows in 3D,* Advances in Mathematics 384 (2021), 107730.

[8] M. Berti, *KAM theory for partial differential equations*, Analysis in Theory and Applications, 35 (2019), no. 3, 235–267.

[9] M. Berti, L. Biasco, M. Procesi, *KAM theory for the Hamiltonian derivative wave equation,* Ann. Sci. Éc. Norm. Supér. Volume 4, 46 (2013), no. 2, 301–373.

[10] M. Berti, L. Biasco, M. Procesi, *KAM for Reversible Derivative Wave Equations,* Arch. Ration. Mech. Anal. 212 (2014), no. 3, 905–955.

[11] M. Berti, P. Bolle, *A Nash-Moser approach to KAM theory*, Fields Institute Communications, special volume "Hamiltonian PDEs and Applications", (2015), 255–284.

[12] M. Berti, T. Kappeler, R. Montalto, *Large KAM tori for quasi-linear perturbations of KdV,* Archive for Rational Mechanics, 239 (2021), 1395–1500.

[13] M. Berti, L. Franzoi, A. Maspero, *Traveling quasi-periodic water waves with constant vorticity*, Archive for Rational Mechanics and Analysis, 240 (2021), 99–202.

[14] M. Berti, L. Franzoi, A. Maspero, *Pure gravity traveling quasi-periodic water waves with constant vorticity*, arXiv:2101.12006.

[15] M. Berti, Z. Hassainia, N. Masmoudi, *Time quasi-periodic vortex patches*, arXiv:2202.062015.

[16] M. Berti , R. Montalto, *Quasi-periodic standing wave solutions of gravity-capillary water waves*, MEMO, Volume 263, 1273, Memoires AMS, ISSN 0065-9266, (2020).

[17] A. L. Bertozzi, P. Constantin, *Global regularity for vortex patches,* Comm. Math. Phys., 152 (1993), no. 1, 9–28.

[18] J. Bourgain, *Green's function estimates for lattice Schrödinger operators and applications,* Annals of Mathematics Studies 158 (2005), Princeton University Press, Princeton.

[19] J. Bourgain, *Construction of quasi-periodic solutions for Hamiltonian perturbations of linear equations and applications to nonlinear PDE,* Internat. Math. Res. Notices (1994), no. 11, 475ff., approx. 21 pp

[20] J. Burbea, *Motions of vortex patches,* Lett. Math. Phys. 6 (1982), no. 1, 1–16.

[21] A. Castro, D. Córdoba, J. Gómez-Serrano, *Uniformly rotating analytic global patch solutions for active scalars,* Ann. PDE, 2 (2016), no. 1, 1–34.

[22] J. Y. Chemin, *Fluides parfaits incompressibles,* Astérisque 230, Société Mathématique de France, (1995).

[23] L. Chierchia, J. You, *KAM tori for 1D nonlinear wave equations with periodic boundary conditions,* Comm. Math. Phys. 211 (2000), 497–525.

[24] W. Craig, C. E. Wayne, *Newton's method and periodic solutions of nonlinear wave equation,* Comm. Pure Appl. Math. 46 (1993), 1409–1498.

[25] G. S. Deem, N. J. Zabusky, *Vortex waves : Stationary "V-states", Interactions, Recurrence, and Breaking,* Phys. Rev. Lett. 40 (1978), no. 13, 859–862.

[26] N. Depauw, *Poche de tourbillon pour Euler 2D dans un ouvert à bord,* J. Math. Pures Appl. 78 (1999), no. 3, 313–351.

[27] L. H. Eliasson, S. B. Kuksin, *KAM for the nonlinear Schrödinger equation,* Ann. Math. 172 (2010), 371–435.

[28] R. Feola, F. Giuliani, R. Montalto, M. Procesi, *Reducibility of first order linear operators on tori via Moser's theorem*, J. Funct. Anal. 276 (2019), no. 3, 932–970.

[29] R. Feola, M. Procesi, *Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations,* Journal of Differential Equations 259 (2015), no. 7, 3389–3447.

[30] E. Fraenkel, *An introduction to maximum principles and symmetry in elliptic problems,* Cambridge University Press 128 (2000).

[31] M. Gao, X. Yuan, J. Zhang, *KAM tori for reversible partial differential equations,* Nonlinearity 24 (2011), 1189–1228.

[32] C. García, *Kármán vortex street in incompressible fluid models*, Nonlinearity, 33 (2020), no. 4, 1625–1676.

[33] C. García, *Vortex patches choreography for active scalar equations*, Journal of Nonlinear Science, 31 (2021), no. 75, 1432–1467.

[34] J. Gómez-Serrano, J. Park, J. Shi, Y. Yao, *Symmetry in stationary and uniformly-rotating solutions of active scalar equations,* Duke Math. J. 170 (2021), no. 13, 2957–3038.

[35] Z. Hassainia, T. Hmidi, *Steady asymmetric vortex pairs for Euler equations*, American Institut of Mathematical Science, 41 (2021), no. 4, 1939–1969.

[36] Z. Hassainia, T. Hmidi, F. de la Hoz, J. Mateu, *An analytical and numerical study of steady patches in the disc*, Analysis and PDE, 9 (2015), no. 10.

[37] Z. Hassainia, T. Hmidi, N. Masmoudi, *KAM theory for active scalar equations,* arXiv:2110.08615.

[38] Z. Hassainia, N. Masmoudi, M. H. Wheeler, *Global bifurcation of rotating vortex patches*, Comm. Pure Appl. Math. Vol. LXXIII (2020), 1933–1980

[39] Z. Hassainia, M. Wheeler, *Multipole vortex patch equilibria for active scalar equations*, arXiv:2103.06839.

[40] T. Hmidi, *On the trivial solutions for the rotating patch model,* J. Evol. Equ., 15 (2015), no. 4, 801–816.

[41] T. Hmidi, F. de la Hoz, J. Mateu, J. Verdera, *Doubly connected V-states for the planar Euler equations,* SIAM J. Math. Anal. 48 (2016), no. 3, 1892–1928.

[42] T. Hmidi, J. Mateu, *Bifurcation of rotating patches from Kirchhoff vortices,* Discrete Contin. Dyn. Syst. 36 (2016), no. 10, 5401–5422.

[43] T. Hmidi, J. Mateu, *Degenerate bifurcation of the rotating patches,* Adv. Math. 302 (2016), 799–850.

[44] T. Hmidi, J. Mateu, *Existence of corotating and counter-rotating vortex pairs for active scalar equations.* Comm. Math. Phys. 350 (2017), no. 2, 699–747.

[45] T. Hmidi, J. Mateu, J. Verdera, *Boundary Regularity of Rotating Vortex Patches.* Arch. Ration. Mech. Anal. 209 (2013), no. 1, 171–208.

[46] T. Hmidi, J. Mateu, J. Verdera, *On rotating doubly connected vortices.* J. Differential Equations 258 (2015), no. 4, 1395–1429.

[47] T. Hmidi, E. Roulley, *Time quasi-periodic vortex patches for quasi-geostrophic shallow-water equations*, arXiv:2110.13751.

[48] G. Iooss, P. Plotnikov, J. Toland, *Standing waves on an infinitely deep perfect fluid under gravity*, Arch. Ration. Mech. Anal., 177 (2005), no. 3, 367–478.

[49] T. Kappeler, J. Pöschel, *KdV and KAM,* Springer, Berlin, (2003).

[50] G. Kirchhoff, *Vorlesungen uber mathematische Physik*, Leipzig, (1874).

[51] S. Kuksin, *Hamiltonian perturbations of in finite-dimensional linear systems with imaginary spectrum,* Funktsional. Anal. i Prilozhen. 21 (1987), no. 3, 22–37, 95.

[52] S. Kuksin, *A KAM theorem for equations of the Korteweg-de Vries type,* Rev. Math. Phys. 10 (1998), no. 3, 1–64.

[53] S. Kuksin, *Analysis of Hamiltonian PDEs,* Oxford Lecture Series in Mathematics and its Applications, vol. 19. Oxford University Press, Oxford, (2000).

[54] A. N. Kolmogorov, *On the persistence of conditionally periodic motions under a small change of the hamiltonian function,* Doklady Akad. Nauk SSSR 98 (1954), 527–530.

[55] J. Liu, X. Yuan, *A KAM theorem for Hamiltonian partial differential equations with unbounded perturbations,* Commun. Math. Phys. 307 (2011), 629–673.

[56] J. Moser, *On invariant curves of area-preserving mappings of an annulus*, Nachr. Akad. Wiss., Göttingen, Math. Phys. Kl. (1962), 1–20.

[57] J. Nash, *C1-isometric imbeddings*, Annals of Mathematics 60 (1954), no. 3, 383–396.

[58] J. Pöschel, *A KAM-Theorem for some nonlinear PDEs,* Ann. Sc. Norm. Pisa 23 (1996), 119–148.

[59] J. Pöschel, *Quasi-periodic solutions for a nonlinear wave equation*, Comment. Math. Helv. 71 (1996), no. 2, 269–296.

[60] P. Plotnikov, J. Toland, *Nash-Moser theory for standing water waves*, Arch. Ration. Mech. Anal. 159 (2001), no. 1, 1–83.

[61] E. Roulley, *Vortex rigid motion in quasi-geostrophic shallow-water equations,* arXiv:2202.00404.

[62] H. Rüssmann, *Invariant tori in non-degenerate nearly integrable Hamiltonian systems,* Regul. Chaotic Dyn. 6 (2001), no. 2, 119–204.

[63] C. E. Wayne, *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory,* Communications in Mathematical Physics 127 (1990), no. 3, 479–528.

[64] Y. Yudovich, *Nonstationary flow of an ideal incompressible liquid,* Zh. Vych. Mat. 3 (1963), 1032–1066.