Time quasi-periodic vortex patches for quasi-geostrophic shallow-water equations

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Abstract

In this paper, we shall implement KAM theory in order to construct a large class of time quasi-periodic solutions for an active scalar model arising in fluid dynamics. More precisely, the construction of invariant tori is performed for quasi-geostrophic shallow-water equations when the *Rossby deformation length* belongs to a massive Cantor set. As a consequence, we construct pulsating vortex patches whose boundary is localized in a thin annulus for any time.

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1 Introduction

We shall discuss in this introduction the quasi-geostrophic shallow-water equation which is a nonlinear and nonlocal transport equation generalizing 2D Euler equations and used to describe large scale motion for the atmosphere and the ocean circulation. A particular concern will be addressed to the long time dynamics behavior and especially to the emergence of long-lived structures in the vortex patch setting. The main novelty in this work is to explore quasi-periodic solutions around the Rankine vortices using KAM theory and Nash-Moser scheme in the spirit recent work of Baldi, Berti, Haus and Montalto [4].

1.1 Model, relative equilibria from periodic to quasi-periodic solutions

The current paper deals with the quasi-geostrophic shallow-water equations $(\mathrm{QGSW})_\lambda$, which is considered as one of the most common asymptotic models used to describe the large scale motion of the atmospheric and oceanic circulation and can be derived asymptotically from the rotating shallow-water equations when Rossby and Froude numbers are small enough, for more details we refer to [24, 52] and the references therein. This model is planar and the evolution of the potential vorticity $\boldsymbol{q}$ takes the form of a nonlinear and nonlocal transport equation,

$$(\mathrm{QGSW})_\lambda \begin{cases} \partial_t \boldsymbol{q} + \mathbf{v}\cdot\nabla\boldsymbol{q} = 0, & \text{in } \mathbb{R}_+\times\mathbb{R}^2 \\ \mathbf{v} = \nabla^\perp(\Delta-\lambda^2)^{-1}\boldsymbol{q}, & \\ \boldsymbol{q}(0,\cdot) = \boldsymbol{q}_0, & \end{cases} \qquad \text{where} \quad \nabla^\perp = \begin{pmatrix} -\partial_2 \\ \partial_1 \end{pmatrix}.$$

Here $\mathbf{v}$ denotes the velocity field which is solenoidal and $\boldsymbol{q}$ is a scalar function. Physically, the parameter λ is defined by

$$\lambda = \frac{\omega_c}{\sqrt{gH}},$$

where g is the gravity constant, H is the mean active layer depth and ω_c is the Coriolis frequency, assumed to be constant. In the literature, the number $\frac{1}{\lambda}$ is called the *Rossby deformation length* or *Rossby radius* and measures the length scale at which the rotation effects are balanced by the stratification. Notice that small values of λ corresponds to a free surface which is nearly rigid and when $\lambda = 0$ we get Euler equations written in the formulation velocity-vorticity.

The main purpose of this paper is to explore the emergence of time quasi-periodic solutions for $(\mathrm{QGSW})_\lambda$ when λ belongs to a massive Cantor like set. This goal will be accomplished in the special class of vortex patches. Before coming to the details we shall first discuss the existence of relative equilibria which are solutions that do not change their shape during the motion. This subject is well-explored for Euler equations and new results have been established recently for $(\mathrm{QGSW})_\lambda$ in [22]. Next we shall discuss the vortex patch problem and explore some of its specific feature.

► *Vortex patches/relative equilibria.* We consider an initial condition taking the form of a vortex patch, that is, $\boldsymbol{q}_0 = \mathbf{1}_{D_0}$ where D_0 is a bounded domain in $\mathbb{R}^2$. Since $\mathbf{1}_{D_0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then in a similar way to Euler equations Yudovich's theory applies and implies that this system admits a unique global in time weak solution. Then from the transport equation governing the potential vorticity we find that the patch structure is preserved during the motion and one gets

$$\boldsymbol{q}(t,\cdot) = \mathbf{1}_{D_t} \quad \text{where} \quad D_t := \boldsymbol{\Phi}_t(D_0),$$

where $\boldsymbol{\Phi}_t : \mathbb{C} \to \mathbb{C}$ stands for the flow map associated to the velocity field $\mathbf{v}$ and defined by

$$\partial_t \boldsymbol{\Phi}_t(z) = \mathbf{v}(t, \boldsymbol{\Phi}_t(z)) \quad \text{and} \quad \boldsymbol{\Phi}_0 = \mathrm{Id}_{\mathbb{C}}. \tag{1.1}$$

The boundary motion in the smooth case reduces to tackle the evolution of a curve in the complex plane surrounding a constant area domain and subject to the deformation induced by its own effect. Local/global in time persistence of the boundary regularity is a relevant subject in fluid dynamics and has attracted a lot of attention during the past decades, not only for Euler equations but also for similar active scalar equations such as generalized surface quasi-geostrophic equations, the aggregation equation. The literature is huge and we shall restrict the discussion to some suitable contributions that fit with our main task. Let us now briefly see how to write down the contour dynamics equations, more details can be found in [41, 42]. Given a smooth parametrization $z(t,\cdot) : \mathbb{T} \to \partial D_t$ of the boundary of the patch, then as particles located at the boundary move with the boundary then we get the evolution equation

$$[\partial_t z(t,\theta) - \mathbf{v}(t, z(t,\theta))] \cdot \mathbf{n}(t, z(t,\theta)) = 0, \tag{1.2}$$

where $\mathbf{n}(t,\cdot)$ is the outward normal vector to the boundary ∂D_t of D_t. This equation reflects the fact that the particle velocity and the boundary velocity admit the same normal components. As we shall see later in Section 2.1, one may deduce from the preceding equation that the boundary equation will evolve through a nonlinear integro-differential equation. Looking for particular solutions where the domain moves without any shape deformation is a traditional subject in fluid dynamics and important developments have been performed for long time ago. In the literature, these structures appear under different names: relative equilibria, V-states, long-lived structures, vortex crystals, etc... A particular case is given by rotating patches which are solutions rotating uniformly with constant angular velocity about their center of mass that can be fixed at the origin, namely,

$$\boldsymbol{q}(t,z) = \mathbf{1}_{D_t} \quad \text{with} \quad D_t = e^{it\Omega} D_0.$$

These solutions are periodic in time with period $\frac{2\pi}{\Omega}$ or equivalently with frequency Ω. For Euler equations, we know two explicit examples of rotating patches. The first one is the so-called Rankine vortices given by the discs which are stationary solutions not only for Euler but also for $(\mathrm{QGSW})_\lambda$. The second example is called Kirchhoff ellipses which rotate with the angular velocity $\Omega = \frac{ab}{(a+b)^2}$ where a and b are the semi-axes of the ellipse, we refer to [45] and [17, p. 304] for more explainations. Numerical experiments achieved by Deem and Zabusky in [25] show the existence of rotating solutions with $\mathbf{m}$-fold symmetry for $\mathbf{m} = 3, 4, 5$. An analytical proof based on the bifurcation theory and complex analysis tools was devised by Burbea in [18] who proved the existence of $\mathbf{m}$-fold (for any $\mathbf{m} \in \mathbb{N}^*$) symmetric V-states bifurcating from Rankine vortices with angular velocity $\Omega_{\mathbf{m}} := \frac{\mathbf{m}-1}{2\mathbf{m}}$. More investigations on the V-states in different settings like the doubly-connected patches, vortex pairs, boundary effects or for different models has been implemented during the past decade by several authors, for more details we refer to [19, 20, 27, 28, 29, 30, 31, 32, 33, 34, 36, 37, 38, 39, 40]. Concerning $(\mathrm{QGSW})_\lambda$ there are a few results dealing with relative equilibria. Interesting numerical simulations showing the complexity and the richness of the bifurcation diagram with respect to the parameter λ was studied in [23, 22]. In [22], using bifurcation tools the authors proved analogous results to those of Burbea. They show in particular the existence of branches of $\mathbf{m}$-fold symmetric V-states ($\mathbf{m} \geq 2$) bifurcating from Rankine vortex $\mathbf{1}_{\mathbb{D}}$ with the angular velocity

$$\Omega_{\mathbf{m}}(\lambda) = I_1(\lambda)K_1(\lambda) - I_{\mathbf{m}}(\lambda)K_{\mathbf{m}}(\lambda),$$

where I_m and K_m are the modified Bessel functions of first and second kind. For more details about these functions, we refer to the Appendix A. Notice that in the same paper the authors explored the two-fold branch when λ is small and proved first that it is located close to the ellipse branch of Euler equations and second it is not connected and from numerical simulations they put in evidence the fragmentation of this branch in multiple connected pieces. The second bifurcation from this branch was also analyzed leading to similar results as for Euler equations.

It is worthy to point out that the model under consideration is typically a reversible Hamiltonian equation with one degree of freedom given by the external the parameter λ. Therefore, it is a natural issue to explore whether or not quasi-periodic solutions constructed from the linearized operator can survive at the the nonlinear level when λ is selected in a suitable Cantor set. Studying the persistence of invariant tori is a relevant subject from KAM theory where a lot of important developments has been done in different directions. In the next paragraph we shall review and recall some basic notions and results in this subject.

► *Quasi-periodic solutions.* A function $f : \mathbb{R} \to \mathbb{R}$ is called quasi-periodic if there exists $F : \mathbb{T}^d \to \mathbb{R}$ such that

$$\forall\, t \in \mathbb{R}, \quad f(t) = F(\omega t)$$

for some frequency vector $\omega \in \mathbb{R}^d$ ($d \in \mathbb{N}^*$) which is non-resonant, that is

$$\forall\, l \in \mathbb{Z}^d \setminus \{0\}, \quad \omega \cdot l \neq 0. \tag{1.3}$$

Here and in the sequel, we denote $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$ the flat torus of dimension d. In the case $d = 1$, we recover from this definition periodic functions with frequency $\omega \in \mathbb{R}^*$.
The study of quasi-periodic solutions to Hamiltonian systems goes back to the pioneering works of Kolmogorov [46], Arnold [3] and Moser [48] where they proved, in finite dimension and under suitable non degeneracy and smoothness conditions, the persistence of invariant tori for small perturbation of integrable Hamiltonian systems. Namely, using action-angle variables (I, ϑ) (integrability in the Liouville sense) we may write

$$H(I, \vartheta) = h(I) + \varepsilon P(I, \vartheta).$$

The phase space of the integrable Hamiltonian system associated to h is foliated by Lagrangian invariant tori carrying a resonant or non-resonant quasi-periodic dynamics. Roughly speaking, Kolmogorov's theorem asserts that, for small ε and when the perturbation is analytic, many non-resonant Lagrangian invariant tori persist. Kolmogorov's proof is based on the reduction of the Hamiltonian to an integrable one using symplectic change of coordinates. This is done using a Newton scheme where at any level we may replace the remainder by an integrable contribution (depending only on the action) up to a new remainder which is smaller than the initial one. To do that we should solve a functional equation called the *homological equation* where we should avoid resonances and deal with small divisors problem. For a long time people like Poincaré thought that it was not possible to make the scheme convergent due to the uncontrolled loss of regularity. The key idea of Kolmogorov was to introduce Diophantine conditions to control the small divisors problem and get only an algebraic loss of regularity. Arnold made rigorous the idea of Kolmogorov and Moser extended it to a differentiable case using what is now called "Nash-Moser procedure". This is a modification of the standard Newton scheme making appeal to regularizing operators $(\Pi_N)_N$ in order to solve an equation $F(U) = 0$ in a Banach scale allowing some fixed loss of derivatives at each step. This strategy was first introduced by Nash in [49] to prove the isometric embedding theorem and improved later by Moser. The theory born from these works was named afterwards KAM theory in their honor.

Later on, this theory was explored and developed in partial differential equations by several authors leading to important contributions and opening new perspectives. The complexity of the problem depends on the space dimension and on the structure of the equations. For example in the semi-linear case the nonlinearity can be seen as a bounded perturbation of the linear problem and this simplifies a lot the problem of finding a right inverse of the linearized operator around a state close to the equilibrium. However in the quasi-linear case where the nonlinearity is unbounded and has the same order as the linear part the situation turns to be much more tricky. This is the case for instance in the water-wave equations where several results has been obtained in the past few years on the periodic or quasi-periodic, standing or traveling settings, see [1, 4, 12, 13, 14, 44, 50].
Next we shall give some insights on the general scheme performed to construct quasi-periodic solutions in the quasi-linear setting that was developed in particular by Berti and Bolle in [11]. This approach is robust and flexible and will be adapted to our framework with the suitable modifications. The first step is to write in a standard way the equations using the action-angle variables for the tangential part. When we linearize the nonlinear functional around a state near the equilibrium we end with an operator with variable coefficients that we should invert approximately up to small errors provided the external parameters belong to a suitable Cantor set defined through various Diophantine conditions. To do that we first look for an approximate inverse using an isotropic torus built around the initial one. It has the advantage to transform the linearized operator via symplectic change of coordinates into a triangular system up to errors that vanish when tested against an invariant torus. Notice that the outcome is that the Hamiltonian has a good normal form structure such that we can almost decouple the dynamics in the phase space in tangential and normal modes. On the tangential part the system can be solved in a triangular way provided we can invert the linearized operator on the normal part up to a small coupling error term. This is more or less a finite dimensional KAM theory appearing here. Then, the analysis reduces to invert the linearized operator on the normal part with is a small perturbation of a diagonal infinite dimensional matrix. This is done by conjugating the linearized operator to a diagonal one with constant coefficients. This step is long and technical and most of the non-resonance conditions in the Cantor set arise during this process. This allows the construction of an approximate inverse for the linearized operator with adequate tame estimates required along Nash-Moser scheme. We point out that this approach has been successfully implemented to generate quasi-periodic solutions to several quasi-linear and fully nonlinear autonomous PDE's, see for instance [4, 5, 6, 14]. Recent progress towards applying KAM theory for the vortex dynamics has been performed for gSQG equation [35] and Euler equation [15]. Finally, we point out that the use of suitable isotropic tori is a commodity but it is not essential to get the triangular structure up to small errors. This point will be discussed in Section 5, see also [35].

1.2 Main result and sketch of the proof

The contour dynamics equation stated in (1.2) can be written in a more tractable way using polar coordinates. This is meaningful at least for short time when the initial patch is close to the equilibrium state given by the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ where $\mathbb{D}$ is the unit disc of $\mathbb{R}^2$. Thus the boundary ∂D_t will be parametrized as follows

$$z(t,\theta)=R(t,\theta)e^{i\theta}\qquad\text{with}\qquad R(t,\theta)=\left(1+2r(t,\theta)\right)^{\frac{1}{2}}. \tag{1.4}$$

We shall prove in Section 2.1 that the function r satisfies a nonlinear and non-local transport equation taking the form

$$\partial_t r+F_\lambda[r]=0, \tag{1.5}$$

with

$$F_\lambda[r](t,\theta)=\int_{\mathbb{T}}K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big)d\eta$$

and

$$A_r(t,\theta,\eta)=\Big|R(t,\theta)e^{i\theta}-R(t,\eta)e^{i\eta}\Big|.$$

The function K_0 is a Bessel function of imaginary parts and it is defined in Appendix A. Next, we take a parameter $\Omega \neq 0$ and look for the solutions in the form

$$r(t,\theta) = \tilde{r}(t, \theta + \Omega t), \tag{1.6}$$

then the equation (1.5) is equivalent to (to alleviate the notation $\tilde{r}$ will be denoted by r)

$$\partial_t r + \Omega \partial_\theta r + F_\lambda[r] = 0. \tag{1.7}$$

We point out that the introduction of the parameter Ω seems at this level artificial but it will be used later to fix the degeneracy of the first eigenvalue associated with the linearized operator at the equilibrium state. As we shall see in Proposition 2.1, the equation (1.7) has an Hamiltonian structure

$$\partial_t r = \partial_\theta \nabla H(r), \tag{1.8}$$

where the Hamiltonian H is related to the kinetic energy and the angular momentum which are prime integrals of the system. In the quasi-periodic setting, we should find a frequency vector $\omega \in \mathbb{R}^d$, such that the equation (1.7) admits solutions in the form $r(t,\theta) = \widehat{r}(\omega t, \theta)$ with $\widehat{r}$ being a smooth $(2\pi)^{d+1}-$periodic function. Then $\widehat{r}$ satisfies (to alleviate the notation we keep the notation r for $\widehat{r}$)

$$\omega \cdot \partial_\varphi r + \Omega \partial_\theta r + F_\lambda[r] = 0. \tag{1.9}$$

To explore quasi-periodic solutions we should first check their existence at the linear level. Then according to Lemma 3.1 the linearized operator to (1.9) around a given small state r is given by the linear Hamiltonian equation,

$$\mathcal{L}_r \rho = 0 \quad \text{with} \quad \mathcal{L}_r = \partial_t + \partial_\theta \left[V_r \cdot - \mathbf{L}_r \right], \tag{1.10}$$

where V_r is a scalar function defined by

$$V_r(\lambda, t, \theta) = \Omega + \frac{1}{R(t,\theta)} \int_{\mathbb{T}} K_0\left(\lambda A_r(t,\theta,\eta)\right) \partial_\eta \left(R(t,\eta)\sin(\eta - \theta)\right) d\eta \tag{1.11}$$

and $\mathbf{L}_r$ is a non-local operator in the form

$$\mathbf{L}_r(\rho)(\lambda, t, \theta) = \int_{\mathbb{T}} K_0\left(\lambda A_r(t,\theta,\eta)\right) \rho(t,\eta) d\eta. \tag{1.12}$$

At the equilibrium state $r \equiv 0$, we find that the linearized operator is a Fourier multiplier, see Lemma 3.2,

$$\mathcal{L}_0 \rho = \partial_t \rho + V_0(\lambda) \partial_\theta \rho - \partial_\theta \mathcal{K}_\lambda * \rho \cdot \tag{1.13}$$

where $*$ denotes the convolution product in the variable θ and

$$V_0(\lambda) = \Omega + I_1(\lambda) K_1(\lambda) \quad \text{and} \quad \mathcal{K}_\lambda(\theta) = K_0\left(2\lambda \left|\sin\left(\tfrac{\theta}{2}\right)\right|\right).$$

Expanding into Fourier series

$$\rho(t,\theta) = \sum_{j \in \mathbb{Z}} \rho_j(t) e^{\mathrm{i} j \theta},$$

yields to

$$\rho \in \ker(\mathcal{L}_0) \quad \Longleftrightarrow \quad \rho(t,\theta) = \sum_{j\in\mathbb{Z}} \rho_j(0) e^{\mathrm{i}(j\theta - \Omega_j(\lambda) t)}, \tag{1.14}$$

where the eigenvalues Ω_j are defined by

$$\Omega_j(\lambda) = j\Big(\Omega + I_1(\lambda)K_1(\lambda) - I_j(\lambda)K_j(\lambda)\Big) \tag{1.15}$$

and the Bessel functions of imaginary argument I_n and K_n are given by (A.2). It is worthy to point out that the frequency associated to the mode $j = 0$ is vanishing and therefore it creates trivial resonance. This can be fixed by imposing a zero space average which can be maintained at the nonlinear level by virtue of the structure of (1.8). Hence we shall work with the phase space of real functions enjoying this property, namely,

$$H_0^s := H_0^s(\mathbb{T}, \mathbb{R}) = \Big\{ r(\theta) = \sum_{j \in \mathbb{Z}^*} r_j e^{\mathrm{i} j\theta} \quad \text{s.t.} \quad r_{-j} = \overline{r_j} \quad \text{and} \quad \|r\|_s^2 = \sum_{j \in \mathbb{Z}^*} |r_j|^2 |j|^{2s} < \infty \Big\}.$$

Another similar comment concerns the mode $j = 1$ which vanishes for any λ when $\Omega = 0$. This is why we have introduced Ω which should be strictly positive to remedy to this defect and avoid any resonance at higher frequencies. The reversibility of the system (1.8) can be also exploited to find the requested parity property of the solutions. Actually, we can check that if $(t, \theta) \mapsto r(t, \theta)$ is a solution then $(t, \theta) \mapsto r(-t, -\theta)$ is a solution too. Then the solutions to the linear problem with this symmetry are in the form

$$\rho(t, \theta) = \sum_{j \in \mathbb{Z}^*} \rho_j \cos\big(j\theta - \Omega_j(\lambda)t\big). \tag{1.16}$$

Now, in order to generate quasi-periodic solutions to the linear problem it suffices to excite a finite number of frequencies from the linear spectrum. We shall then consider the following frequency vector.

$$\omega_{\mathrm{Eq}}(\lambda) = (\Omega_j(\lambda))_{j \in \mathbb{S}} \quad \text{with} \quad \mathbb{S} = \{j_1, \ldots, j_d\} \subset \mathbb{N}^*.$$

Note that throughout the paper, we use the notation

$$\mathbb{N} = \{0, 1, 2, \ldots\} \quad \text{and} \quad \mathbb{N}^* = \{1, 2, \ldots\}.$$

Notice that the vector $\omega_{\mathrm{Eq}}(\lambda)$ gives periodic solutions provided that it satisfies the non-resonant condition (1.3). This property holds true for almost all the values of λ as it is proved in Proposition 3.1. Our main result concerns the persistence of quasi-periodic solutions for the nonlinear model (1.8) when the perturbation is small enough and the parameter λ is subject to be in a massive Cantor set.

Theorem 1.1. *Let $\lambda_1 > \lambda_0 > 0$, $d \in \mathbb{N}^*$ and $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. There exist $\varepsilon_0 \in (0, 1)$ small enough with the following properties : For every amplitudes $\mathtt{a} = (\mathtt{a}_j)_{j \in \mathbb{S}} \in (\mathbb{R}_+^*)^d$ satisfying*

$$|\mathtt{a}| \leqslant \varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty \subset (\lambda_0, \lambda_1)$ with asymptotically full Lebesgue measure as $\mathtt{a} \to 0$, i.e.

$$\lim_{\mathtt{a} \to 0} |\mathcal{C}_\infty| = \lambda_1 - \lambda_0,$$

such that for any $\lambda \in \mathcal{C}_\infty$, the equation (1.8) admits a time quasi-periodic solution with diophantine frequency vector $\omega_{\mathrm{pe}}(\lambda, \mathtt{a}) := (\omega_j(\lambda, \mathtt{a}))_{j \in \mathbb{S}} \in \mathbb{R}^d$ and taking the form

$$r(t, \theta) = \sum_{j \in \mathbb{S}} \mathtt{a}_j \cos\big(j\theta + \omega_j(\lambda, \mathtt{a})t\big) + \mathtt{p}(\omega_{\mathrm{pe}} t, \theta),$$

with

$$\omega_{\mathrm{pe}}(\lambda, \mathtt{a}) \underset{\mathtt{a} \to 0}{\longrightarrow} (-\Omega_j(\lambda))_{j \in \mathbb{S}},$$

where $\Omega_j(\lambda)$ are the equilibrium frequencies defined in (1.15) and the perturbation $\mathtt{p} : \mathbb{T}^{d+1} \to \mathbb{R}$ is an even function satisfying

$$\|\mathtt{p}\|_{H^s(\mathbb{T}^{d+1}, \mathbb{R})} \underset{\mathtt{a} \to 0}{=} o(|\mathtt{a}|)$$

for some large index of regularity s.

Before discussing the main steps of the proof, some remarks are in order.

Remark 1.1. • *Combining this theorem with* (1.6) *and* (1.4) *we find that the boundary shape of the patch can be parametrized in polar coordinates as follows*

$$z(t,\theta)=R(t,\theta+\Omega t)e^{\mathrm{i}\theta}\quad \textit{with}\quad R(t,\theta)=(1+2r(t,\theta))^{\frac{1}{2}}$$

and r *is described as in the theorem. The time evolution of the shape is given by small pulsation around the unit disc and the boundary is localized in an annulus around the unit circle.*

- *It is not at clear what could happen when* λ *is in the complement of the Cantor set. We expect the linear invariant tori to be destroyed by the nonlinearity and filamentation may take place generating fast increase of the curvature and the boundary length.*

We shall now outline the main steps of the proof which will be developed following standard scheme as in the preceding works [4, 6, 11, 14] with different variations connected to the models structure. We mainly use techniques from KAM theory combined with Nash Moser scheme. This will be implemented along several steps which are detailed below.

▶ **Step 1.** *Action-angle reformulation.* We first notice that the equation (1.8) can be seen as a perturbation of the integrable system given by the linear dynamics at the equilibrium state. Indeed, by combining (1.13),(1.15) and (1.8) we may write

$$\partial_t r=\partial_\theta \mathrm{L}(\lambda)(r)+X_P(r),$$

where $\mathrm{L}(\lambda)$ and the perturbed Hamiltonian vector field X_P are defined by

$$\mathrm{L}(\lambda)(r)=-\big(\Omega+(I_1K_1)(\lambda)\big)r+\mathcal{K}_\lambda * r\quad \text{and}\quad X_P(r)=I_1(\lambda)K_1(\lambda)\partial_\theta r-\partial_\theta\mathcal{K}_\lambda * r-F_\lambda[r].$$

Since we are looking for small solutions then we find it convenient to rescale the solution $r\rightsquigarrow \varepsilon r$ with ε a small positive number and consequently the new unknown still denoted by r satisfies

$$\partial_t r=\partial_\theta \mathrm{L}(\lambda)(r)+\varepsilon X_{P_\varepsilon}(r),$$

where X_{P_ε} is the Hamiltonian vector field defined by $X_{P_\varepsilon}(r)=\varepsilon^{-2}X_P(\varepsilon r)$. Then finding quasi-periodic solutions with frequency $\omega\in\mathbb{R}^d$ amounts to solve the equation

$$\omega\cdot\partial_\varphi r=\partial_\theta \mathrm{L}(\lambda)(r)+\varepsilon X_{P_\varepsilon}(r).$$

Here we still use the same notation r for the new profile which depends in the variables $(\varphi,\theta)\in\mathbb{T}^{d+1}$. The next step consists in splitting the phase space H_0^s into an orthogonal sum of tangential and normal subspaces as follows

$$H_0^s=H_{\overline{\mathbb{S}}}\overset{\perp}{\oplus}H_\perp^s,$$

where $H_{\overline{\mathbb{S}}}$ is the finite dimensional subspace of real functions generated by $\{e^{\mathrm{i}j\theta}, j\in\overline{\mathbb{S}}\}$ with $\overline{\mathbb{S}}=\mathbb{S}\cup(-\mathbb{S})$. For more details on this description we refer to Section 5.1. To track the dynamics it seems to be more suitable to use the action-angle variables (I,ϑ) seen as symplectic polar variables for the Fourier coefficients of the tangential part in $H_{\overline{\mathbb{S}}}$. This leads to reformulate the problem in terms of the embedded torus,

$$\begin{array}{rccl} i: & \mathbb{T}^d & \to & \mathbb{T}^d\times\mathbb{R}^d\times H_\perp^s\\ & \varphi & \mapsto & (\vartheta(\varphi),I(\varphi),z(\varphi)),\end{array}$$

with

$$r(\varphi,\theta)=\underbrace{v(\vartheta(\varphi),I(\varphi))(\theta)}_{\in H_{\overline{\mathbb{S}}}}+\underbrace{z(\varphi,\theta)}_{\in H_\perp^s}:=A(i(\varphi))(\theta)$$

and

$$v(\vartheta,I)=\sum_{j\in\overline{\mathbb{S}}}\sqrt{\mathtt{a}_j^2+\tfrac{|j|}{2}I_j}\,e^{\mathrm{i}\vartheta_j}e_j,\quad e_j(\theta)=e^{\mathrm{i}j\theta}\,.$$

Notice that the action and angle variables should satisfy the symmetry properties

$$\forall j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{R}.$$

Therefore we reduce the problem in the new variables to construct invariant tori with non-resonant frequency vector ω to the system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_\varepsilon}(i(\varphi)), \tag{1.17}$$

where X_{H_ε} is the Hamiltonian vector field associated to the Hamiltonian H_ε given by

$$H_\varepsilon = -\omega_{\mathrm{Eq}}(\lambda) \cdot I + \tfrac{1}{2}\langle \mathrm{L}(\lambda) z, z\rangle_{L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon,$$

with $\mathcal{P}_\varepsilon$ defined by $\mathcal{P}_\varepsilon = P_\varepsilon \circ A$. A useful trick used by Berti and Bolle in [11] consists to solve first the relaxed problem

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_\varepsilon^\alpha}(i(\varphi)),$$

where the vector field $X_{H_\varepsilon^\alpha}$ is associated to the modified Hamiltonian H_ε^α given by

$$H_\varepsilon^\alpha = \alpha \cdot I + \tfrac{1}{2}\langle \mathrm{L}(\lambda) z, z\rangle_{L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon.$$

The advantage of this procedure is to get one degree of freedom with the vector $\alpha \in \mathbb{R}^d$ that will be used to ensure some compatibility assumptions during the construction of an approximate inverse of the linearized operator. At the end of Nash-Moser scheme we shall adjust implicitly the frequency ω so that α coincides with the equilibrium frequency $-\omega_{\mathrm{Eq}}(\lambda)$, which enables to finally get solutions to the original Hamiltonian equation. The relaxed problem can be written in the following form

$$\mathcal{F}(i, \alpha, \lambda, \omega, \varepsilon) = 0,$$

with

$$\begin{aligned} \mathcal{F}(i, \alpha, \lambda, \omega, \varepsilon) &:= \omega \cdot \partial_\varphi i(\varphi) - X_{H_\varepsilon^\alpha}(i(\varphi)) \\ &= \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \alpha - \varepsilon \partial_I \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \partial_\vartheta \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \partial_\theta \left(\mathrm{L}(\lambda) z(\varphi) + \varepsilon \nabla_z \mathcal{P}_\varepsilon(i(\varphi))\right) \end{pmatrix}. \end{aligned} \tag{1.18}$$

We point out that the linear torus corresponding to the linear solution

$$r(\varphi, \theta) = \sum_{j \in \overline{\mathbb{S}}} \mathtt{a}_j \, e^{\mathrm{i}\varphi_j} e^{\mathrm{i}j\theta}$$

is given in the new coordinates system by $i_{\text{flat}}(\varphi) = (\varphi, 0, 0)$ and it is obvious that

$$\mathcal{F}\big(i_{\text{flat}}, -\omega_{\mathrm{Eq}}(\lambda), \lambda, -\omega_{\mathrm{Eq}}(\lambda), 0\big) = 0.$$

We emphasize that at this stage the classical implicit function theorem does not work because the linearized operator at the equilibrium state is not invertible due to resonances. One can avoid resonances by restricting the parameter λ to a suitable Cantor set according to some Diophantine conditions on the linear frequency $\omega_{\mathrm{Eq}}(\lambda)$ allowing in particular to control the small divisors problem. By this way we get an inverse at the equilibrium state but with algebraic loss of regularity. Unfortunately, this is not enough to apply Nash-Moser scheme which requires to construct a right inverse with tame estimates in a small neighborhood of the equilibrium and this is the challenging deal in this topic. Indeed, the linearized operator is no longer with constant coefficients as for the integrable case and its main part is not a Fourier multiplier. At this level we are dealing with a quasilinear problem where the perturbation is unbounded.

► **Step 2.** *Approximate inverse.* Let $\alpha_0 \in \mathbb{R}^d$ (actually α_0 is a function of the parameters ω and λ) and consider an embedded torus $i_0 = (\vartheta_0, I_0, z_0)$ near the flat one with the reversible structure,

$$\vartheta_0(-\varphi) = -\vartheta_0(\varphi), \quad I_0(-\varphi) = I_0(\varphi) \quad \text{and} \quad z_0(-\varphi, -\theta) = z_0(\varphi, \theta).$$

To deal with the linearized operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$, which exhibits complicated structure, and see whether we can construct an approximate inverse we should fix two important issues. One is related to the coupling structure in the new coordinates system and the second is that the linearized operator is with variable coefficients. For this aim we shall follow the approach conceived by Berti and Bolle in [11] with making suitable modifications. This approach consists in linearizing around an isotropic torus close enough to the original one and then use a symplectic change of coordinates leading to a triangular system up to small errors, essentially of "type Z" or highly decaying in frequency, that can be incorporated in Nash-Moser scheme. Therefore to invert this triangular system it suffices to get an approximate right inverse for the linearized operator in the normal direction, denoted in what follows by $\widehat{\mathcal{L}}_\omega$. We notice that in Section 5.3, and similarly to [35], we can bypass the use of isotropic torus by a slight modification of Berti-Bolle approach. Actually, according to Proposition 5.1, we can conjugate the linearized operator with the transformation described by (5.63) computed at the torus i_0 and get a triangular system with small errors mainly of "type Z". The computations are performed in a straightforward way using in a crucial way the Hamiltonian structure of the original system. The main advantage that simplifies some arguments is to require the invertibility for the linearized operator only at the torus itself and not necessary at a closer isotropic one. By this way, we can avoid the accumulation of different errors induced by the isotropic torus that one encounters for example in the estimates of the approximate inverse or in the multiple Cantor sets generated along the different reduction steps where the coefficients should be computed at the isotropic torus. The final outcome of this first step is to reduce the invertibility to finding an approximate inverse of $\widehat{\mathcal{L}}_\omega$ which takes, according to Proposition 6.1, the form

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^{\perp}\left(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\right)\Pi_{\mathbb{S}_0}^{\perp} \quad \text{with} \quad \mathcal{L}_{\varepsilon r} = \omega\cdot\partial_\varphi + \partial_\theta\left[V_{\varepsilon r}\cdot - \mathbf{L}_{\varepsilon r}\right],$$

where $\varepsilon\partial_\theta\mathcal{R}$ is a perturbation of finite rank, the function $V_{\varepsilon r}$ and the nonlocal operator $\mathbf{L}_{\varepsilon r}$ are defined in (1.11) and (1.12), respectively. At the equilibrium state (corresponding to $\varepsilon = 0$) $\widehat{\mathcal{L}}_\omega$ is diagonal and we shall see that the set of parameters (λ, ω) leading to the existence of a right inverse is almost full. Now remark that even for ε small, the perturbation affects the main part of the operator in a similar way to water waves [4, 14] or generalized SQG equation [35] and then we should construct the suitable change of coordinates in order to reduce the positive part of the operator to a diagonal operator. Later we should implement KAM scheme to diagonalize the zero-order part. This will be done in three steps.

ⓐ **Reduction of the transport part.** This procedure will be discussed in Proposition 6.2 and Proposition 6.3. We basically use KAM techniques as in [12, 26] in order to conjugate the operator $\mathcal{L}_{\varepsilon r}$, through a suitable quasi-periodic symplectic change of coordinates $\mathscr{B}$, to a transport operator with constant coefficients. Indeed, we may construct an invertible transformation

$$\mathscr{B}\rho(\varphi,\theta) = \big(1 + \partial_\theta\beta(\varphi,\theta)\big)\rho\big(\varphi, \theta + \beta(\varphi,\theta)\big)$$

and a constant $c_{i_0}(\lambda,\omega)$ such that for any given number $n \in \mathbb{N}$, if the parameter (λ,ω) belongs to the truncated set defined through the first order Melnikov condition

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \left\{(\lambda,\omega)\in\mathcal{O} \quad \text{s.t.} \quad |\omega\cdot l + j c_{i_0}(\lambda,\omega)| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\},$$

then we have

$$\mathfrak{L}_{\varepsilon r} := \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}(\lambda,\omega)\partial_\theta - \partial_\theta\mathcal{K}_\lambda * \cdot + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathrm{E}_n^0, \tag{1.19}$$

with $N_n = N_0^{(\frac{3}{2})^n}$, $N_0 \geqslant 2$, $\upsilon \in (0,1)$, $\mathcal{O} = (\lambda_0,\lambda_1)\times\mathscr{U}$, $0 < \lambda_0 < \lambda_1$ and $\mathscr{U} = B(0,R_0)$ being an open ball of $\mathbb{R}^d$ containing the curve of the linear vector frequency $\lambda \in (\lambda_0,\lambda_1) \mapsto -\omega_{\mathrm{Eq}}(\lambda)$. The operator $\mathfrak{R}_{\varepsilon r}$ is a self-adjoint Toeplitz integral operator satisfying the estimates

$$\forall s\in[s_0,S], \quad \max_{k\in\{0,1,2\}} \|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\right),$$

where the off-diagonal norm $\|\cdot\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}$ is defined in (4.10) and the loss of regularity σ is connected to τ_1 and d but it is independent of the index regularity s. Concerning the operator $\mathtt{E}_n^0$, we can show that it is a small fast decaying remainder with the following estimate for low regularity

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\, N_0^{\mu_2} N_n^{-\mu_2} \|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}, \tag{1.20}$$

where the weighted norms $\|\cdot\|_{q,s_0}^{\gamma,\mathcal{O}}$ are defined in (4.5). For the number μ_2, it is connected to the regularity of the torus i_0 and can be taken large enough allowing to identify the contributions of $\mathtt{E}_n^0$ as small errors in the construction of the approximate inverse. The next step will be discussed in Proposition 6.4 where we explore the effect of the transport reduction on the original operator $\widehat{\mathcal{L}}_\omega$ which is localized to the normal direction. We prove that with the localized transformation $\mathscr{B}_\perp$ defined by

$$\mathscr{B}_\perp = \Pi_{\mathbb{S}_0}^\perp \mathscr{B} \Pi_{\mathbb{S}_0}^\perp,$$

one obtains in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$,

$$\mathscr{B}_\perp^{-1} \widehat{\mathcal{L}}_\omega \mathscr{B}_\perp = \omega\cdot\partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_0 + \mathscr{R}_0 + \mathtt{E}_n^1, \tag{1.21}$$

where $\mathscr{D}_0$ is a diagonal operator whose spectrum $\{\mathrm{i}\mu_j^0, j\in\mathbb{S}_0^c\}$ satisfies

$$\mu_j^0(\lambda,\omega,i_0) = \Omega_j(\lambda) + j r^1(\lambda,\omega,i_0) \quad \text{with} \quad \|r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon$$

and $\mathscr{R}_0$ is a remainder term taking the form of an integral operator with Toeplitz and reversibility structures with the estimates the asymptotic

$$\forall s\in[s_0,S], \quad \max_{k\in\{0,1\}} \|\partial_\theta^k \mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\right),$$

We remark that the operator $\mathtt{E}_n^1$ satisfies similar estimates as for $\mathtt{E}_n^0$ seen before in (1.20). Finally, we want to emphasize that the derivation of the asymptotic structure of the operator $\mathfrak{L}_{\varepsilon r}$ seen before in (1.19) requires some refined analysis. The delicate point concerns the expansion of the operator $\mathbf{L}_r$ defined in (1.12) and for this part we use the kernel structure detailed in (A.6)

$$K_0(z) = -\log\left(\tfrac{z}{2}\right) I_0(z) + \sum_{m=0}^{\infty} \tfrac{\psi(m+1)}{(m!)^2}\left(\tfrac{z}{2}\right)^{2m}.$$

with I_0 being analytic. This is different from the cases discussed before as for the water waves in [4, 14] where the kernel is given by that of Euler equations (corresponding to $\lambda=0$), that is, $K(z) = -\log\left(\frac{z}{2}\right)$. In this latter case the deformed kernel enjoys the structure

$$-\log\left(2A_r(t,\theta,\eta)\right) = -\log\left|\sin\left(\tfrac{\theta-\eta}{2}\right)\right| + \text{smooth nonhomogeneous kernel}.$$

This means that the associated operator is given by a diagonal operator of order -1 up to a smoothing non diagonal pseudo-differential operator in $OPS^{-\infty}$. In our context, this decomposition fails for $\lambda>0$ and we get a similar one but with less smoothing operator. Actually we obtain from (5.38) the splitting

$$K_0\left(2\lambda A_r(t,\theta,\eta)\right) = K_0\left(2\lambda\sin\left(\tfrac{\theta-\eta}{2}\right)\right) + \mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta) + \mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta), \tag{1.22}$$

where the kernels $\mathscr{K}_{r,1}^1$ and $\mathscr{K}_{r,1}^2$ are smooth whereas $\mathscr{K}$ is slightly singular taking the form

$$\mathscr{K}(\theta) = \sin^2\left(\tfrac{\theta}{2}\right)\log\left(\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right).$$

ⓑ **KAM reduction of the remainder term.** This is the main target of Section 6.3.2 and the result is stated in Proposition 6.5. The goal is to conjugate the remainder $\mathscr{R}_0$ of (1.21) and transform it into a diagonal operator. This will be developed in a standard way by constructing successive transformations through the KAM reduction allowing to replace at each step the old remainder by a new one which is much smaller provided that we make the suitable parameters extraction. This scheme can be achieved unless we solve the associated *homological equation.* To avoid resonances we should at each step make an extraction from the parameters set through the second order Melnikov conditions and the final outcome is as follows,

$$\mathscr{L}_\infty := \Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty = \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty,$$

where $\mathscr{D}_\infty = \Big(\mathrm{i}\mu_j^\infty(\lambda,\omega,i_0)\Big)_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}$ is a diagonal operator with pure imaginary spectrum and Φ_∞ is a reversible invertible operator. This reduction is possible when the parameters (λ,ω) belong to the following Cantor-like set,

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) = \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n}} \Big\{(\lambda,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\quad \text{s.t.}\quad \big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)-\mu_{j_0}^\infty(\lambda,\omega,i_0)\big| > \tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}.$$

The eigenvalues admit the following asymptotic

$$\mu_j^\infty(\lambda,\omega,i_0) = \Omega_j(\lambda) + jr^1(\lambda,\omega,i_0) + r_j^\infty(\lambda,\omega,i_0),$$

where r^1 and r_j^∞ are real small coefficients with Lipschitz dependence with respect to the torus. Indeed, we have

$$\|r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}|j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1},$$
$$\|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}\|\Delta_{12}r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma}^{\gamma,\mathcal{O}},$$

for some index regularity $\overline{s}_h+\sigma$ and $\Delta_{12}r^1 = r^1(\lambda,\omega,i_1)-r^1(\lambda,\omega,i_2)$.
ⓒ **Construction of the approximate inverse.** The next step is to invert approximately the operator $\widehat{\mathcal{L}}_\omega$ detailed in Proposition 6.6. First we establish an approximate inverse for $\mathscr{L}_\infty$, on the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}} \Big\{(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad |\omega\cdot l+\mu_j^\infty(\lambda,\omega)| > \tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

Then, introducing the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0) = \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),$$

we are able to construct an approximate inverse of $\widehat{\mathcal{L}}_\omega$ in the following sense,

$$\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n} = \mathrm{Id}+\mathtt{E}_n \quad \text{in } \mathcal{G}_n,$$

where $\mathtt{E}_n$ is a fast frequency decaying operator as in (1.20) and $\mathtt{T}_{\omega,n}$ satisfies tame estimates uniformly in n. Therefore coming back to Section 5.3, more precisely to Theorem 5.1, this enables to construct an approximate right inverse T_0 for the full differential $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ enjoying suitable tame estimates. In what follows we want to make some comments. The first one concerns the Lipschitz dependence of the eigenvalues with respsect to the torus. This is required in studying the stability of Cantor sets in Nash-Moser scheme and allows to construct a final massive Cantor set. As to the second one, it concerns the KAM reduction which allows to diagonalize the operator when the parameters belong to a Cantor set like, even though all the involved transformations and operators can be extended in the whole set of parameters using standard cut-off functions for the Fourier coefficients. This extension with adequate estimates will be needed later during the implementation of Nash-Moser scheme. This is not the only way to produce suitable extensions and one expects Whitney extension to be also well adapted as in [4, 14]. In our case we privilege the first procedure which can be easily set up and manipulated using classical functional tools. The last comment is related to a technical point in KAM reduction, Contrary to the preceding papers such as [4, 14], we do not need to use pseudo-differential operators techniques in the description of the aforementioned asymptotic structures of $\mathfrak{L}_{\varepsilon r}$ and $\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp$. In fact, they can be avoided since all the involved operators can be described through their kernels and therefore instead of splitting the symbols we simply expand the kernels as in (1.22) which sounds to be more appropriate in our context.

▶ **Step 3.** *Nash-Moser scheme.* This is the main purpose of Section 7.1 where we construct zeros for the nonlinear function $\mathcal{F}$ defined in (1.18) for small ε following Nash-Moser scheme in the spirit of the papers [4, 14]. Let us quickly sketch this scheme. We build by induction a sequence of approximate solutions U_n

$$U_{n+1} = U_n + H_{n+1} \quad \text{with} \quad H_{n+1} = -\Pi_{N_n}\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n).$$

with T_n an approximate inverse of $d_{i,\alpha}\mathcal{F}(U_n)$ constructed in **Step 2**. Thus using Taylor Formula we may write

$$\mathcal{F}(U_{n+1}) = \Pi_{N_n}^{\perp}\mathcal{F}(U_n) - \Pi_{N_n}\big(L_n\mathrm{T}_n - \mathrm{Id}\big)\Pi_{N_n}\mathcal{F}(U_n) + \big(L_n\Pi_n^{\perp} - \Pi_{N_n}^{\perp}L_n\big)\mathtt{T}_n\Pi_{N_n}\mathcal{F}(U_n) + Q_n,$$

where Q_n is a quadratic functional. Consider the Cantor set

$$\mathcal{A}_n^{\gamma} = \bigcap_{k=0}^{n-1}\mathcal{G}_k(\gamma_{k+1}, \tau_1, \tau_2, i_k),$$

with $\gamma_n = \gamma(1 + 2^{-n})$, then we show by induction that

$$\|U_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}, \quad \|U_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_{n-1}^{\mu}, \quad \|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} \lesssim \varepsilon N_{n-1}^{-a_1} \tag{1.23}$$

for a suitable choice of the parameters $a_1, b_1, \overline{a}, \mu, \overline{\sigma} > 0$ and O_n^{γ} is an open enlargement of $\mathcal{A}_n^{\gamma}$ needed to construct classical extensions to the whole set of parameters $\mathcal{O}$. Actually, we get a precise statement in Proposition 7.1 allowing to deduce that the sequence $(U_n)_n$ converges in a strong topology towards a sufficient smooth profile $(\lambda, \omega) \in \mathcal{O} \mapsto U_\infty(\lambda, \omega) = \big(i_\infty(\lambda, \omega), \alpha_\infty(\lambda, \omega), (\lambda, \omega)\big)$ with

$$\forall (\lambda, \omega) \in \mathcal{G}_\infty^{\gamma}, \quad \mathcal{F}(U_\infty(\lambda, \omega)) = 0, \quad \mathcal{G}_\infty^{\gamma} = \bigcap_{n\in\mathbb{N}}\mathcal{A}_n^{\gamma}.$$

Moreover, we get in view of Corollary 7.1 a smooth function $\lambda \in (\lambda_0, \lambda_1) \mapsto (\lambda, \omega(\lambda, \varepsilon))$ with

$$\omega(\lambda, \varepsilon) = -\omega_{\mathrm{Eq}}(\lambda) + \bar{r}_\varepsilon(\lambda), \quad \|\bar{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \tag{1.24}$$

and

$$\forall \lambda \in \mathcal{C}_\infty^{\varepsilon}, \quad \mathcal{F}\big(U_\infty(\lambda, \omega(\lambda, \varepsilon))\big) = 0 \quad \text{with} \quad \alpha_\infty\big(\lambda, \omega(\lambda, \varepsilon)\big) = -\omega_{\mathrm{Eq}}(\lambda),$$

where the Cantor set $\mathcal{C}_\infty^{\varepsilon}$ is defined by

$$\mathcal{C}_\infty^{\varepsilon} = \Big\{\lambda \in (\lambda_0, \lambda_1) \quad \text{s.t.} \quad \big(\lambda, \omega(\lambda, \varepsilon)\big) \in \mathcal{G}_\infty^{\gamma}\Big\}. \tag{1.25}$$

This gives solutions to the original equation (1.17) provided that λ belongs to the final Cantor set $\mathcal{C}_\infty^{\varepsilon}$ and the last point to deal with aims to measure this set.

▶ **Step 4.** *Measure estimates.* The measure of the final Cantor set $\mathcal{C}_\infty^{\varepsilon}$ will be explored in Section 7.2. We show in Proposition 7.2 that by fixing $\gamma = \varepsilon^a$ for some small a we get

$$|\mathcal{C}_\infty^{\varepsilon}| \geqslant (\lambda_1 - \lambda_0) - C\varepsilon^{\delta},$$

with small δ connected to the geometry of the Cantor set and the non degeneracy of the equilibrium spectrum. There are two main ingredients to get this result. The first one is the stability of the intermediate Cantor sets $(\mathcal{A}_n^{\gamma})_n$ following from the fast convergence of Nash-Moser scheme. However the second one is the transversality property stated in Lemma 7.3 used in the spirit of [8] and [51]. This property will be first established for the linear frequencies in Proposition 3.5, using the analyticity of the eigenvalues and their asymptotic behavior. Then the extension of the transversality assumption

to the perturbed frequencies is done using perturbative arguments together with the asymptotic description of the approximate eigenvalues detailed in (7.67), (7.82) and (7.81).
We emphasize that the transversality is strongly related to the non-degeneracy of the eigenvalues in the sense of the Definition 3.1 . For instance, we show that the curve $\lambda \in [\lambda_0, \lambda_1] \mapsto \big(\Omega_{j_1}(\lambda), ..., \Omega_{j_d}(\lambda)\big)$ is not contained in any vectorial plane, that is, if there exists a constant vector $c = (c_1, .., c_d)$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad \sum_{j=1}^{d} c_k \Omega_{j_k}(\lambda) = 0,$$

then $c = 0$. This is proved in Lemma 3.4 and follows from the asymptotic of the eigenvalues for large values of λ according to the law (A.11) combined with the invertibility of Vandermonde matrices.

2 Hamiltonian formulation of the patch motion

In this section we shall set up the contour dynamics equation governing the patch motion. A particular attention will be focused on the vortex patch equation in the polar coordinates system. We shall see that the Hamiltonian structure still survives at the level of the patch dynamics, which is the starting point towards the construction of quasi-periodic solutions.

2.1 Contour dynamics equation in polar coordinates

Here and in the sequel, we identify $\mathbb{C}$ with $\mathbb{R}^2$ equipped with the canonical Euclidean structure through the standard inner product defined for all $z_1 = a_1 + ib_1,\ z_2 = a_2 + ib_2 \in \mathbb{C}$ by

$$z_1 \cdot z_2 := \langle z_1, z_2 \rangle_{\mathbb{R}^2} = \mathrm{Re}\,(z_1 \overline{z_2}) = a_1 a_2 + b_1 b_2. \tag{2.1}$$

The planar set $\mathbb{D}$ stands for the open unit disc of $\mathbb{R}^2$ and the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ (actually any radial function) is a stationary solution to $(\mathrm{QGSW})_\lambda$. To look for ordered structure like periodic or quasi-periodic vortex patches $t \mapsto \mathbf{1}_{D_t}$ around this equilibrium state, we find it convenient to consider a polar parametrization of the boundary

$$\begin{array}{rrcl} z: & \mathbb{R}_+ \times [0, 2\pi] & \mapsto & \mathbb{C} \\ & (t, \theta) & \mapsto & z(t, \theta) = (1 + 2r(t,\theta))^{\frac{1}{2}}\, e^{i\theta}. \end{array} \tag{2.2}$$

Here r is the radial deformation of the patch which is small, namely $|r(t, \theta)| \ll 1$. Taking $r = 0$ gives a parametrization of the unit circle $\mathbb{T}$. We shall introduce the new symplectic unknown

$$R(t, \theta) = (1 + 2r(t, \theta))^{\frac{1}{2}}\,. \tag{2.3}$$

which will be useful to write down the equations into the Hamiltonian form. In what follows we want to explicit the contour dynamics equation with the polar coordinates. It is a classical fact, see for instance [41, 42], that the particles on the boundary move with the flow and remain at the boundary and therefore in the smooth case one has

$$[\partial_t z(t, \theta) - \mathbf{v}(t, z(t, \theta))] \cdot \mathbf{n}(t, z(t, \theta)) = 0,$$

where $\mathbf{n}(t, z(t, \theta))$ is the outward normal vector to the boundary ∂D_t of D_t at the point $z(t, \theta)$. Since one has, up to a real constant of renormalization, $\mathbf{n}(t, z(t, \theta)) = -\mathrm{i}\partial_\theta z(t, \theta)$, then we find the complex form of the contour dynamics motion,

$$\mathrm{Im}\left([\partial_t z(t, \theta) - \mathbf{v}(t, z(t, \theta))]\, \overline{\partial_\theta z(t, \theta)} \right) = 0. \tag{2.4}$$

In order to transform it into a nonlinear PDE, we need to recover the velocity field $\mathbf{v}$ from the patch parametrization. To do so, recall that $\mathbf{v} = \nabla^\perp \boldsymbol{\Psi}$ where $\boldsymbol{\Psi}$ is the stream function associated to the vorticity and governed by Helmholtz equation,

$$(\Delta - \lambda^2)\boldsymbol{\Psi}(t, \cdot) = \mathbf{1}_{D_t}.$$

To invert this operator we shall make appeal to the Green function T_λ solution of the equation

$$(-\Delta+\lambda^2)T_\lambda=\delta_0 \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Using the Fourier transform yields

$$\forall \xi\in\mathbb{R}^2,\quad \widehat{T_\lambda}(\xi)=\frac{1}{|\xi|^2+\lambda^2}.$$

Thus by Fourier inversion theorem and using a scaling argument, we find

$$T_\lambda(z)=T_1(\lambda z)\quad \text{with}\quad T_1(z)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\frac{e^{iz\cdot\xi}}{1+|\xi|^2}d\xi.$$

Applying a polar change of variables gives

$$T_1(z)=\frac{1}{4\pi^2}\int_0^\infty\frac{r}{1+r^2}\int_0^{2\pi}\cos(|z|r\cos(\theta))d\theta dr.$$

Simple arguments based on the symmetry of trigonometric functions allow to get the identity

$$\int_0^{2\pi}\cos(|z|r\cos(\theta))d\theta=2\int_0^{\pi}\cos(|z|r\sin(\theta))d\theta.$$

Consequently, we get in view of (A.1)

$$T_1(z)=\frac{1}{2\pi}\int_0^\infty\frac{rJ_0(|z|r)}{1+r^2}dr,$$

where J_n denotes the Bessel function. Applying (A.7) with $\nu=\mu=0$, $a=1$ and $b=|z|$, we finally deduce the representation

$$T_1(z)=\frac{1}{2\pi}K_0(|z|).$$

Therefore one gets the formula,

$$\mathbf{\Psi}(t,z)=-\frac{1}{2\pi}\int_{\mathbb{R}^2}K_0(\lambda|z-\xi|)\mathbf{1}_{D_t}(\xi)dA(\xi), \tag{2.5}$$

where dA denotes the planar Lebesgue measure. To get get explicit form of the velocity in terms of the patch boundary we shall use the complex version of Stokes theorem

$$2\mathrm{i}\int_D\partial_{\overline{\xi}}f(\xi,\overline{\xi})dA(\xi)=\int_{\partial D}f(\xi,\overline{\xi})d\xi. \tag{2.6}$$

In view of $\mathbf{v}(t,z)=2\mathrm{i}\partial_{\overline{z}}\mathbf{\Psi}(t,z)$, one deduces that

$$\mathbf{v}(t,z)=\frac{1}{2\pi}\int_{\partial D_t}K_0(\lambda|z-\xi|)d\xi. \tag{2.7}$$

Next we intend to write down the boundary motion in terms of the contour dynamics. First, from the polar parametrization, it is easy to check from (2.2) that

$$\operatorname{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big)=-\partial_t r(t,\theta).$$

On the other hand, using (2.7) and (A.3), we infer

$$\operatorname{Im}\Big(\mathbf{v}(t,z(t,\theta))\overline{\partial_\theta z(t,\theta)}\Big)=\int_{\mathbb{T}}K_0\left(\lambda|z(t,\theta)-z(t,\eta)|\right)\operatorname{Im}\Big(\partial_\eta z(t,\eta)\overline{\partial_\theta z(t,\theta)}\Big)\,d\eta.$$

Here and throughout this paper, we shall work with the following notation

$$\int_{\mathbb{T}}f(\eta)d\eta:=\tfrac{1}{2\pi}\int_0^{2\pi}f(\eta)d\eta. \tag{2.8}$$

Next we observe that,

$$\begin{aligned}\operatorname{Im}\Big(\partial_\eta z(t,\eta)\overline{\partial_\theta z(t,\theta)}\Big) &= \partial^2_{\theta\eta}\operatorname{Im}\Big(z(t,\eta)\overline{z(t,\theta)}\Big)\\ &= \partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big).\end{aligned}$$

Thus, by setting

$$A_r(t,\theta,\eta) := |z(t,\theta)-z(t,\eta)| = |R(t,\theta)e^{i\theta}-R(t,\eta)e^{i\eta}| \tag{2.9}$$

and

$$F_\lambda[r](t,\theta) := \int_{\mathbb{T}} K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big)d\eta, \tag{2.10}$$

we get the vortex patch equation in the polar coordinates

$$\partial_t r(t,\theta) + F_\lambda[r](t,\theta) = 0. \tag{2.11}$$

Now, we fix a parameter Ω that will be used later to get rid of trivial resonances, and we shall look for solutions in the form

$$r(t,\theta) = \tilde{r}(t,\theta+\Omega t). \tag{2.12}$$

Then elementary change of variables applied with (2.10) show that

$$F_\lambda[\tilde{r}](t,\theta+\Omega t) = F_\lambda[r](t,\theta)\,. \tag{2.13}$$

Thus, the equation (2.11) becomes (to alleviate the notation we simply use r instead of $\tilde{r}$)

$$\partial_t r(t,\theta) + \Omega\partial_\theta r(t,\theta) + F_\lambda[r](t,\theta) = 0, \tag{2.14}$$

which is a nonlinear and nonlocal transport PDE. To fix the terminology, we mean by a time quasi-periodic solution of (2.14), a solution in the form

$$r(t,\theta) = \widehat{r}(\omega t,\theta),$$

where $\widehat{r}:\ (\varphi,\theta)\in\mathbb{T}^{d+1}\mapsto \widehat{r}(\varphi,\theta)\in\mathbb{R}$, $\omega\in\mathbb{R}^d$ and $d\in\mathbb{N}^*$. Hence in this setting, the equation (2.14) becomes

$$\omega\cdot\partial_\varphi\widehat{r}(\varphi,\theta) + \Omega\partial_\theta\widehat{r}(\varphi,\theta) + F_\lambda[\widehat{r}](\varphi,\theta) = 0.$$

In the sequel, we shall alleviate the notation and denote $\widehat{r}$ simply by r and the foregoing equation writes

$$\forall(\varphi,\theta)\in\mathbb{T}^{d+1},\quad \omega\cdot\partial_\varphi r(\varphi,\theta) + \Omega\partial_\theta r(\varphi,\theta) + F_\lambda[r](\varphi,\theta) = 0. \tag{2.15}$$

2.2 Hamiltonian structure

We now move to a new consideration related to the analysis of the Hamiltonian structure behind the transport equation (2.14). This structure sounds to be essential if one wants to explore quasi-periodic solutions near Rankine vortices. Notice that it is a classical fact that incompressible active scalar equations such as 2D Euler equations are Hamiltonian and as we shall see in this section, we can find a suitable interpretation of this property at the level of the contour dynamics equations which is a stronger reformulation.

2.2.1 Hamiltonian reformulation

We consider the kinetic energy and the angular impulse associated to the patch $\omega(t) = \mathbf{1}_{D_t}$ and defined by

$$E(t) = \frac{1}{2\pi}\int_{D_t}\mathbf{\Psi}(t,z)dA(z) \quad\text{ and }\quad J(t) = \frac{1}{2\pi}\int_{D_t}|z|^2 dA(z), \tag{2.16}$$

where the stream function $\mathbf{\Psi}$ is defined according to (2.5). The following result dealing with the time conservation of the preceding quantities is classical and can be proved in a similar way to Euler equations.

Lemma 2.1. *The kinetic energy E and the angular impulse J are conserved during the motion,*

$$\frac{dE(t)}{dt}=0=\frac{dJ(t)}{dt}\cdot$$

In what follows we shall state the main result of this section on the Hamiltonian structure governing the equation (2.14).

Proposition 2.1. *The equation* (2.14) *is Hamiltonian and takes the form*

$$\partial_t r=\partial_\theta\nabla H(r), \tag{2.17}$$

where ∇ is the $L^2(\mathbb{T}_\theta)$-gradient with respect to the $L^2(\mathbb{T}_\theta)$-normalized scalar product defined by

$$\langle\rho_1,\rho_2\rangle_{L^2(\mathbb{T})}=\int_{\mathbb{T}}\rho_1(\theta)\rho_2(\theta)d\theta$$

and the hamiltonian H is defined by

$$H(r)=\tfrac{1}{2}\left(E(r)-\Omega J(r)\right).$$

In particular, we get the conservation of the average, that is,

$$\frac{d}{dt}\int_{\mathbb{T}}r(t,\theta)d\theta=0. \tag{2.18}$$

Proof. ▶ Using Stokes formula (2.6), we may write

$$J(r)(t)=\tfrac{1}{8i\pi}\int_{\partial D_t}|\xi|^2\overline{\xi}d\xi.$$

Then from the parametrization detailed in (2.2) one gets easily

$$\begin{aligned}J(r)(t)&=\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}|z(t,\theta)|^2\overline{z(t,\theta)}\partial_\theta z(t,\theta)d\theta\\&=\tfrac{1}{16\mathrm{i}}\int_{\mathbb{T}}\partial_\theta\left(R^4(t,\theta)\right)d\theta+\tfrac{1}{4}\int_{\mathbb{T}}R^4(t,\theta)d\theta\\&=\tfrac{1}{4}\int_{\mathbb{T}}R^4(t,\theta)d\theta.\end{aligned}$$

Consequently,

$$J(r)(t)=\tfrac{1}{4}\int_{\mathbb{T}}\left(1+2r(t,\theta)\right)^2d\theta. \tag{2.19}$$

Differentiating in r one gets for $\rho\in L^2(\mathbb{T})$

$$\langle\nabla J(r),\rho\rangle_{L^2(\mathbb{T})}(t)=\int_{\mathbb{T}}(1+2r(t,\theta))\rho(\theta)d\theta,\qquad\text{i.e.}\qquad\nabla J(r)=1+2r.$$

It follows that

$$\tfrac{1}{2}\Omega\partial_\theta\nabla J(r)=\Omega\partial_\theta r. \tag{2.20}$$

▶ Next, we shall compute the Gâteaux derivative of E in a given direction $\rho\in L^2(\mathbb{T})$. We point out that the computations done below are formal but they can be justified rigorously in a classical way. The first step is to express the energy

$$E(t)=\tfrac{1}{2\pi}\int_{D_t}\mathbf{\Psi}(t,z)dA(z)$$

in terms of the boundary parametrization of ∂D_t. According to Stokes theorem (2.6) we have

$$\begin{aligned}\boldsymbol{\Psi}(t,z) &= -\tfrac{1}{2\pi}\int_{D_t} K_0(\lambda|\xi-z|)dA(\xi)\\ &= \tfrac{1}{\pi\lambda^2}\int_{D_t}\partial_{\overline{\xi}}\left(\tfrac{(\overline{\xi}-\overline{z})[\lambda|\xi-z|K_1(\lambda|\xi-z|)-1]}{|\xi-z|^2}\right)dA(\xi)\\ &= \tfrac{1}{2\mathrm{i}\pi\lambda^2}\int_{\partial D_t}\tfrac{(\overline{\xi}-\overline{z})[\lambda|\xi-z|K_1(\lambda|\xi-z|)-1]}{|\xi-z|^2}d\xi.\end{aligned}$$

To prove the second equality above, it suffices to find an anti-derivative of $K_0(\lambda|\xi-z|)$ with respect to $\overline{\xi}$. We shall search it in the form

$$(\overline{\xi}-\overline{z})f(\lambda|\xi-z|).$$

Then we should get

$$K_0(\lambda|\xi-z|) = \partial_{\overline{\xi}}\left((\overline{\xi}-\overline{z})f(\lambda|\xi-z|)\right) = f(\lambda|\xi-z|) + \tfrac{\lambda|\xi-z|}{2}f'(\lambda|\xi-z|).$$

Hence f is a solution on $\mathbb{R}_+^*$ of the ordinary differential equation

$$\tfrac{1}{2}xf'(x)+f(x)=K_0(x),\qquad \text{i.e.}\qquad (x^2f(x))'=2xK_0(x).$$

Using (A.4), we obtain

$$f(x) = -\tfrac{2xK_1(x)+C}{x^2},$$

where C is a constant to be fixed so that the integral converges. Using (A.5), one has on the real line

$$K_1(x) \underset{x\to 0}{=} \tfrac{1}{x}+\tfrac{x}{2}\log\left(\tfrac{x}{2}\right)+o\left(x\log\left(\tfrac{x}{2}\right)\right),$$

so that

$$xK_1(x) \underset{x\to 0}{=} 1+\tfrac{x^2}{2}\log\left(\tfrac{x}{2}\right)+o\left(x^2\log\left(\tfrac{x}{2}\right)\right).$$

Making the choice $C=-2$ we get

$$f(x) = -\tfrac{2(xK_1(x)-1)}{x^2}, \tag{2.21}$$

which behaves like a logarithm near 0 and thus it is integrable. Therefore using the parametrization (2.2) we find

$$\boldsymbol{\Psi}(t,z(t,\theta)) = \tfrac{1}{\mathrm{i}\lambda^2}\int_{\mathbb{T}}\frac{(\overline{z}(t,\eta)-\overline{z}(t,\theta))\left[\lambda|z(t,\theta)-z(t,\eta)|K_1\left(\lambda|z(t,\eta)-z(t,\theta)|\right)-1\right]}{|z(t,\eta)-z(t,\theta)|^2}\partial_\eta z(t,\eta)d\eta.$$

Making appeal to f and removing the time dependence, we get

$$\boldsymbol{\Psi}(z(\theta)) = \tfrac{\mathrm{i}}{2}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)d\eta. \tag{2.22}$$

At this stage we need to look for an anti-derivative with respect to $\overline{z}$ of $\frac{-1}{2}(\overline{\xi}-\overline{z})f(\lambda|\xi-z|)$ in the form

$$(\overline{\xi}-\overline{z})^2 g(\lambda|\xi-z|).$$

Therefore we deduce the constraint

$$\tfrac{-1}{2}(\overline{\xi}-\overline{z})f(\lambda|\xi-z|) = \partial_{\overline{z}}\left((\overline{\xi}-\overline{z})^2g(\lambda|\xi-z|)\right) = -(\overline{\xi}-\overline{z})\left(2g(\lambda|\xi-z|)+\tfrac{\lambda|\xi-z|}{2}g'(\lambda|\xi-z|)\right).$$

Hence, g should be a solution on $\mathbb{R}_+^*$ of the ordinary differential equation

$$\tfrac{x}{2}g'(x)+2g(x)=\tfrac{f(x)}{2},\qquad \text{i.e.}\qquad (x^4g(x))'=x^3f(x)=2x-2x^2K_1(x). \tag{2.23}$$

Using once again (A.4) yields

$$g(x)=\tfrac{x^2+2x^2K_2(x)+C}{x^4},$$

where C is again a constant used to cancel the violent singularity. From (A.5) and (A.2), one obtains the asymptotic

$$K_2(x) \underset{x\to 0}{=} \tfrac{2}{x^2} - \tfrac{1}{2} + O\left(x^2 \log(x)\right).$$

Thus

$$x^2 K_2(x) \underset{x\to 0}{=} 2 - \tfrac{x^2}{2} + O(x^4 \log(x)).$$

Then by choosing $C = -4$ we deduce that the function below

$$g(x) = \tfrac{x^2 + 2x^2 K_2(x) - 4}{x^4}$$

is integrable. Hence, applying once again Stokes theorem (2.6), we infer

$$\begin{aligned} E(r)(t) &= -\tfrac{1}{4\pi^2\lambda^4} \int_{\partial D_t}\int_{\partial D_t} \frac{(\overline{\xi} - \overline{z})^2\left[\lambda^2|\xi - z|^2\left(1 + 2K_2(\lambda|\xi - z|)\right) - 4\right]}{|\xi - z|^4} dz d\xi \\ &= \tfrac{-1}{\lambda^4}\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(t,\eta) - \overline{z}(t,\theta))^2[\lambda|z(t,\eta)-z(t,\theta)|(1+2K_2(\lambda|z(t,\eta)-z(t,\theta)|))-4]}{|z(t,\eta)-z(t,\theta)|^2} \partial_\theta z(t,\theta)\partial_\eta z(t,\eta) d\eta d\theta. \end{aligned}$$

Hence using g and removing the dependence in time, we find

$$E(r) = -\tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))^2 g(\lambda|z(\theta) - z(\eta)|)\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta. \tag{2.24}$$

The next goal is to compute the derivative of E with respect to r in the direction ρ, which is straightforward

$$\begin{aligned} \langle \nabla E(r), \rho\rangle_{L^2(\mathbb{T})} &= -\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))\, g\left(\lambda|z(\theta) - z(\eta)|\right)\left(\tfrac{\rho(\theta)e^{-\mathrm{i}\theta}}{R(\theta)} - \tfrac{\rho(\eta)e^{-\mathrm{i}\eta}}{R(\eta)}\right)\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &- \tfrac{\lambda}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta) - \overline{z}(\eta))^2}{|z(\theta) - z(\eta)|} g'\left(\lambda|z(\theta) - z(\eta)|\right)\tfrac{\rho(\theta)}{R(\theta)}\mathrm{Re}\left((z(\theta) - z(\eta))e^{-\mathrm{i}\theta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &- \tfrac{\lambda}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta) - \overline{z}(\eta))^2}{|z(\theta) - z(\eta)|} g'\left(\lambda|z(\theta) - z(\eta)|\right)\tfrac{\rho(\eta)}{R(\eta)}\mathrm{Re}\left((z(\eta) - z(\theta))e^{-\mathrm{i}\eta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &- \tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))^2 g\left(\lambda|z(\theta) - z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta) d\theta d\eta \\ &- \tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))^2 g\left(\lambda|z(\theta) - z(\eta)|\right)\partial_\theta z(\theta)\partial_\eta\left(\tfrac{\rho(\eta)e^{\mathrm{i}\eta}}{R(\eta)}\right) d\theta d\eta. \end{aligned}$$

By exchanging in the double integral θ and η, we deduce

$$\begin{aligned} \langle \nabla E(r), \rho\rangle_{L^2(\mathbb{T})} &= -2\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))\, g\left(\lambda|z(\theta) - z(\eta)|\right)\tfrac{\rho(\theta)e^{-\mathrm{i}\theta}}{R(\theta)}\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &- \lambda\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta) - \overline{z}(\eta))^2}{|z(\theta) - z(\eta)|} g'\left(\lambda|z(\theta) - z(\eta)|\right)\tfrac{\rho(\theta)}{R(\theta)}\mathrm{Re}\left((z(\theta) - z(\eta))e^{-\mathrm{i}\theta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &- \int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))^2 g\left(\lambda|z(\theta) - z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta) d\theta d\eta. \end{aligned}$$

An integration by parts in the last integral leads to

$$\begin{aligned} &-\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta))^2 g\left(\lambda|z(\theta) - z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta) d\theta d\eta \\ &= 2\int_{\mathbb{T}}\int_{\mathbb{T}} (\overline{z}(\theta) - \overline{z}(\eta)) g(\lambda|z(\theta) - z(\eta)|)\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\partial_\theta\overline{z}(\theta)\partial_\eta z(\eta) d\theta d\eta \\ &\quad + \lambda\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta) - \overline{z}(\eta))^2}{|z(\theta) - z(\eta)|} g'(\lambda|z(\theta) - z(\eta)|)\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\mathrm{Re}\left[(z(\theta) - z(\eta))\partial_\theta\overline{z}(\theta)\right]\partial_\eta z(\eta) d\theta d\eta. \end{aligned}$$

Using the identities

$$e^{\mathrm{i}\theta}\partial_\theta\overline{z}(\theta) - e^{-\mathrm{i}\theta}\partial_\theta z(\theta) = -2\mathrm{i}R(\theta)$$

and

$$\mathrm{Re}\left[(z(\theta)-z(\eta))\partial_\theta\overline{z}(\theta)\right]e^{\mathrm{i}\theta} - \partial_\theta z(\theta)\mathrm{Re}\left[(z(\theta)-z(\eta))e^{-\mathrm{i}\theta}\right] = -\mathrm{i}R(\theta)(z(\theta)-z(\eta)),$$

we infer

$$\begin{aligned}\langle\nabla E(r),\rho\rangle_{L^2(\mathbb{T})} &= \tfrac{4}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))g(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta\\ &\quad+\tfrac{\lambda}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))|z(\theta)-z(\eta)|g'(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta.\end{aligned}$$

Applying (2.23), we find

$$\langle\nabla E(r),\rho\rangle_{L^2(\mathbb{T})} = \tfrac{1}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta,$$

which implies by virtue of (2.22)

$$\nabla E(r) = \tfrac{1}{\mathrm{i}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)d\eta = -2\mathbf{\Psi}(z(\theta)).$$

Now, using the complex notation we deduce that

$$\begin{aligned}\partial_\theta\mathbf{\Psi}(z(\theta)) &= \nabla\mathbf{\Psi}(z(\theta))\cdot\partial_\theta z(\theta)\\ &= \mathrm{Im}\left(\mathbf{v}(z(\theta))\overline{\partial_\theta z(\theta)}\right)\\ &= F_\lambda[r](\theta),\end{aligned}$$

where we used (2.1) and the facts that $\nabla^\perp\mathbf{\Psi} = \mathbf{v}$ and $\mathbf{\Psi}$ is real-valued. Recall that the functional $F_\lambda[r]$ was introduced in (2.10). Hence

$$\partial_\theta\nabla E(r) = -2\partial_\theta\mathbf{\Psi}(z(\theta)) = -2F_\lambda[r](\theta).$$

Finally we get

$$\tfrac{1}{2}\partial_\theta\nabla E(r) = -F_\lambda[r](\theta). \tag{2.25}$$

The conservation of the average is easy to check from the Hamiltonian equation. Therefore the proof of Proposition 2.1 is achieved. □

2.2.2 Reversibility

The main concern is to investigate the reversibility of the Hamiltonian equation (2.17). This property will be used in a crucial way to fix the symmetry in the function spaces and by this way remove from the phase space the trivial resonances. For more details we refer to Section 4.1 and Section 5. To define the reversibility, we shall introduce the involution $\mathscr{S}$

$$(\mathscr{S}r)(\theta) := r(-\theta), \tag{2.26}$$

which satisfies the obvious properties

$$\mathscr{S}^2 = \mathrm{Id} \qquad \text{and} \qquad \partial_\theta\circ\mathscr{S} = -\mathscr{S}\circ\partial_\theta. \tag{2.27}$$

The following elementary result is useful and can be easily checked from the structure of the Hamiltonian. Actually, it suffices to make changes of variables.

Lemma 2.2. *The Hamiltonian H and its associated vector field $X := \partial_\theta\nabla H$ satisfy*

$$H\circ\mathscr{S} = H \quad \textit{and} \quad X\circ\mathscr{S} = -\mathscr{S}\circ X.$$

3 Linearization and frequencies structure

This section is devoted to some aspects of the linearized operator associated to the evolution equation (2.14) or its Hamiltonian version (2.17). We shall in particular compute it at any state close to the equilibrium and reveal some of its main general feature. As we shall see, the radial shape is very special and gives rise to a Fourier multiplier and thus the spectral properties follow immediately. This latter case serves as a toy model to check the emergence of quasi-periodic solutions at the linear level provided that the Rossby radius λ belongs to a Cantor set, see Proposition 3.1 . However, around this ideal state the situation is roughly uncontrolled and the operator is no longer diagonal and its spectral study is extremely delicate due to resonances that prevent to diagonalize the operator. To deal with this problem we will implement some important tools borrowed from KAM theory as we shall see in Section 6.

3.1 Linearized operator

The main goal of this section to compute the differential of the nonlinear operator in (2.14) for any small state r. The computations will be conducted at a formal level by simply computing Gateaux derivatives which are related to Frechet derivatives. This formal part can be justified rigorously in a classical way for the suitable functional setting fixed in Section 4.1.

3.1.1 The general form

In what follows we shall derive a formula for the linearized operator associated to the equation (2.17). We shall see that it can be split into a transport part with variable coefficients and a nonlocal operator of order zero. More precisiely, we shall establish the following lemma.

Lemma 3.1. *The linearized equation of* (2.17) *at a given small state r is given by the time-dependent linear Hamiltonian equation,*

$$\partial_t \rho(t,\theta) = \partial_\theta \Big(-V_r(\lambda,t,\theta)\rho(t,\theta) + \mathbf{L}_r \rho(\lambda,t,\theta) \Big),$$

where V_r is a scalar function defined by

$$V_r(\lambda,t,\theta) = \Omega + \tfrac{1}{R(t,\theta)} \int_{\mathbb{T}} K_0\left(\lambda A_r(t,\theta,\eta)\right) \partial_\eta \left(R(t,\eta)\sin(\eta-\theta)\right) d\eta \tag{3.1}$$

and $\mathbf{L}_r$ is a non-local operator given by

$$\mathbf{L}_r(\rho)(\lambda,t,\theta) = \int_{\mathbb{T}} K_0\big(\lambda A_r(t,\theta,\eta)\big)\rho(t,\eta) d\eta. \tag{3.2}$$

We recall that K_0, A_r and R are defined by (A.6), (2.9) *and* (2.3)*, respectively.*
Moreover, if $r(-t,-\theta) = r(t,\theta)$, then

$$V_r(\lambda,-t,-\theta) = V_r(\lambda,t,\theta). \tag{3.3}$$

Proof. Throughout the proof, we shall remove the time dependency of the involved quantities except when it is relevant to keep it. The computations of the Gâteaux derivative of F_λ defined by (2.10) at a point r in the direction ρ are straightforward and standard and we shall only sketch the main lines. Notice that the functional F_λ is smooth in a suitable functional setting and therefore its differential should be recovered from its Gâteaux derivative. First, we observe that the function A_r defined in (2.9) can be written in the form

$$\begin{aligned} A_r(\theta,\eta) &= \big(R^2(\theta) + R^2(\eta) - 2R(\theta)R(\eta)\cos(\eta-\theta)\big)^{\frac{1}{2}} \\ &= \Big((R(\theta)-R(\eta))^2 + 4R(\theta)R(\eta)\sin^2\left(\tfrac{\eta-\theta}{2}\right)\Big)^{\frac{1}{2}}. \end{aligned} \tag{3.4}$$

This identity (3.4) will be of constant use in the sequel. Second, after straightforward computations, we obtain from (2.10),

$$\begin{aligned} d_rF_\lambda[r](\rho) &= \partial_\tau F_\lambda[r+\tau\rho]_{|\tau=0} \\ &= \mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4, \end{aligned}$$

where

$$\begin{aligned} \mathcal{I}_1 &:= \lambda\rho(\theta)\int_{\mathbb{T}} B_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)d\eta, \\ \mathcal{I}_2 &:= \lambda\int_{\mathbb{T}}\rho(\eta)B_r(\eta,\theta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)d\eta, \\ \mathcal{I}_3 &:= \int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\rho(\theta)\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta, \\ \mathcal{I}_4 &:= \int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\rho(\eta)\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\right)d\eta, \end{aligned}$$

with

$$B_r(\theta,\eta) := \frac{R(\theta)-R(\eta)\cos(\eta-\theta)}{R(\theta)A_r(\theta,\eta)}. \tag{3.5}$$

Next, we shall compute $\mathcal{I}_1+\mathcal{I}_3$. To do that, we split $\mathcal{I}_3$ into two terms as follows,

$$\begin{aligned} \mathcal{I}_3 &= \partial_\theta\rho(\theta)\int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta \\ &\quad+\rho(\theta)\int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta \\ &:= \partial_\theta\rho(\theta)\overline{V}_r(\lambda,\theta)+\rho(\theta)\overline{\mathcal{I}}_3. \end{aligned}$$

An integration by parts in $\overline{\mathcal{I}}_3$ allows to get,

$$\overline{\mathcal{I}}_3 = -\lambda\int_{\mathbb{T}}\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)R(\eta)\partial_\theta\left(\tfrac{\sin(\eta-\theta)}{R(\theta)}\right)d\eta.$$

Putting together the preceding identities yields to

$$\mathcal{I}_1+\mathcal{I}_3 = \partial_\theta\rho(\theta)\overline{V_r}(\lambda,\theta)+\rho(\theta)V_1(\lambda,\theta) \tag{3.6}$$

with

$$\begin{aligned} V_1(\lambda,\theta) &:= \lambda\int_{\mathbb{T}} B_r(\theta,\eta)\partial^2_{\theta\eta}\big(R(\theta)R(\eta)\sin(\eta-\theta)\big)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta \\ &\quad-\lambda\int_{\mathbb{T}}\partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta. \end{aligned} \tag{3.7}$$

Differentiating term by term $\overline{V_r}$ with respect to θ gives

$$\begin{aligned} \partial_\theta\overline{V_r}(\lambda,\theta) &= \lambda\int_{\mathbb{T}}\partial_\theta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta \\ &\quad+\int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta \\ &:= \mathcal{J}_1+\mathcal{J}_2. \end{aligned}$$

Integrating by parts in $\mathcal{J}_2$ yields

$$\mathcal{J}_2 = -\lambda\int_{\mathbb{T}} R(\eta)\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\theta\left(\tfrac{\sin(\eta-\theta)}{R(\theta)}\right)d\eta.$$

Combining the preceding identities allows to we deduce that

$$\partial_\theta \overline{V_r}(\lambda,\theta) = \lambda \int_{\mathbb{T}} K_0'\left(\lambda A_r(\theta,\eta)\right)\left[\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right) - \partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)\right] d\eta.$$

Next we shall check the following identity

$$\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right) = B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right) - \partial_\eta A_r(\theta,\eta). \tag{3.8}$$

Indeed, by (3.4) and (3.5), one finds

$$\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right) = \partial_\theta R(\theta)B_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta)) - \tfrac{R(\eta)\sin(\eta-\theta)\partial_\eta(R(\eta)\sin(\eta-\theta))}{A_r(\theta,\eta)}$$

and

$$\begin{aligned}B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right) = \partial_\theta R(\theta)B_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta))\\ - \tfrac{(R(\theta)-R(\eta)\cos(\eta-\theta))\partial_\eta(R(\eta)\cos(\eta-\theta))}{A_r(\theta,\eta)}.\end{aligned}$$

Putting together the foregoing identities leads to

$$\frac{\partial_\theta A_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta))}{R(\theta)} = B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right) + g(\theta,\eta),$$

where

$$\begin{aligned}g(\theta,\eta) &:= \tfrac{1}{A_r(\theta,\eta)}\left[(R(\theta)-R(\eta)\cos(\eta-\theta))\partial_\eta(R(\eta)\cos(\eta-\theta)) - R(\eta)\sin(\eta-\theta)\partial_\eta(R(\eta)\sin(\eta-\theta))\right]\\ &= \tfrac{R(\theta)\partial_\eta(R(\eta)\cos(\eta-\theta))-R(\eta)\partial_\eta R(\eta)}{A_r(\theta,\eta)}\\ &= -\partial_\eta A_r(\theta,\eta).\end{aligned}$$

This achieves the proof of (3.8). From the periodicity we get

$$\int_{\mathbb{T}} \lambda\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta = \int_{\mathbb{T}} \partial_\eta\left[K_0\left(\lambda A_r(\theta,\eta)\right)\right]d\eta = 0$$

and thus we get the following important identity

$$\partial_\theta \overline{V_r}(\lambda,\theta) = V_1(\lambda,\theta).$$

Plugging this into (3.6) allows to get

$$\mathcal{I}_1 + \mathcal{I}_3 = \partial_\theta\left(\overline{V_r}(\lambda,\theta)\rho(\theta)\right).$$

Notice that it is easy to check that if $r(-t,-\theta) = r(t,\theta)$, then

$$\overline{V_r}(\lambda,-t,-\theta) = \overline{V_r}(\lambda,t,\theta). \tag{3.9}$$

The next task is to compute $\mathcal{I}_2 + \mathcal{I}_4$. Using integration by parts in $\mathcal{I}_4$ gives,

$$\mathcal{I}_4 = -\lambda\int_{\mathbb{T}} \rho(\eta)\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\theta\left(\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\right)d\eta.$$

From the symmetry property $A_r(\theta,\eta) = A_r(\eta,\theta)$ and by exchanging the roles of θ and η in (3.8), one deduces

$$B_r(\eta,\theta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right) - \partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\right) = -\partial_\theta A_r(\theta,\eta).$$

Therefore we obtain

$$\mathcal{I}_2 + \mathcal{I}_4 = -\partial_\theta\left(\int_{\mathbb{T}}\rho(\eta)K_0\left(\lambda A_r(\theta,\eta)\right)d\eta\right) := -\partial_\theta \mathbf{L}_r(\rho)(\lambda,\theta).$$

Finally, by setting

$$V_r(\lambda,t,\theta) = \Omega + \overline{V_r}(\lambda,t,\theta)$$

and combining the preceding identities, we end the proof of Lemma 3.1. □

3.1.2 The integrable case

The main purpose here is to explore the structure of the linearized operator at the equilibrium state. We shall see that the radial shape is reflected on the structure the linearized operator which is a Fourier multiplier (of a convolution type). More precisely, we have the following result.

Lemma 3.2. *1. The linearized equation of* (2.17) *at the equilibrium state* ($r = 0$) *writes,*

$$\partial_t \rho = \partial_\theta \mathrm{L}(\lambda)\rho = \partial_\theta \nabla H_{\mathrm{L}}(\rho), \tag{3.10}$$

where $\mathrm{L}(\lambda)$ *is the self-adjoint operator defined by* $\mathrm{L}(\lambda) = -V_0(\lambda) + \mathcal{K}_\lambda *_\theta$ *with*

$$V_0(\lambda) = \Omega + I_1(\lambda)K_1(\lambda) \tag{3.11}$$

and

$$\mathcal{K}_\lambda(\theta) = K_0\left(2\lambda\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right). \tag{3.12}$$

We refer to the Appendix A for the definitions of the modified Bessel functions I_1, K_1 *and* K_0. *Moreover, the Hamiltonian* H_{L} *is quadratic and takes the form*

$$H_{\mathrm{L}}(\rho) = \tfrac{1}{2}\langle \mathrm{L}(\lambda)\rho, \rho\rangle_{L^2(\mathbb{T})}.$$

2. The solutions to (3.10) *with zero space average are given by*

$$\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*} \rho_j(0) e^{\mathrm{i}(j\theta - \Omega_j(\lambda)t)}, \tag{3.13}$$

with

$$\Omega_j(\lambda) = j\big[\Omega + (I_1K_1)(\lambda) - (I_jK_j)(\lambda)\big]. \tag{3.14}$$

and for $\rho(\theta) = \sum_{j\in\mathbb{Z}^*} \rho_j e^{\mathrm{i}j\theta}$ *we have*

$$\mathrm{L}(\lambda)\rho(\theta) = -\sum_{j\in\mathbb{Z}^*} \tfrac{\Omega_j(\lambda)}{j}\rho_j e^{\mathrm{i}j\theta} \quad \text{and} \quad H_{\mathrm{L}}\rho = -\sum_{j\in\mathbb{Z}^*} \tfrac{\Omega_j(\lambda)}{2j}|\rho_j|^2, \tag{3.15}$$

Before proceeding with the proof we want to give some remarks.

Remark 3.1.
- *When* $\Omega = 0$ *the eigenvalue* $\Omega_1(\lambda)$ *vanishes for any* λ *due to the rotation invariance of the equation and the use of the free parameter* Ω *is to avoid this degeneracy. However the trivial resonance* $\Omega_0(\lambda) = 0$ *can be removed by imposing the zero space average which is preserved by the nonlinear dynamics from the Hamiltonian structure as we have seen before in* (2.18).
- *The solutions to the linear equation at the equilibrium are aperiodic and if we excite only a finite number of frequencies with non-resonances assumption we get quasi-periodic solutions. We will make a precise comment later on Proposition* 3.1.

Proof. **1**. First observe that from (3.4), one deduces for $r = 0$ that $A_0(\theta,\eta) = 2\left|\sin\left(\frac{\eta-\theta}{2}\right)\right|$. Then we obtain from (3.1) and (3.2),

$$\begin{aligned}\mathbf{L}_0\rho(\lambda,\theta) &= \int_{\mathbb{T}} \rho(\eta)K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)d\eta\\ &= \mathcal{K}_\lambda * \rho(\theta),\end{aligned}$$

with $\mathcal{K}_\lambda$ defined in (3.12) and using the change of variables $\eta \mapsto \eta + \theta$ we obtain

$$\begin{aligned}V_0(\lambda,\theta) &= \Omega + \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\cos(\eta-\theta)d\eta\\ &= \Omega + \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)\cos(\eta)d\eta\\ &:= V_0(\lambda).\end{aligned}$$

We remark that if we write $e_j(\theta)=e^{\mathrm{i}j\theta}$, then direct computations yield

$$\begin{aligned}(\mathcal{K}_\lambda * e_j)(\theta) &= \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)e^{\mathrm{i}j(\theta-\eta)}d\eta\\ &= e_j(\theta)\int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)e^{-\mathrm{i}j\eta}d\eta.\end{aligned}$$

Since the function $\eta\mapsto K_0\left(2\lambda\left|\sin\left(\frac{\eta}{2}\right)\right|\right)$ is even, we deduce using the change of variables $\eta=2\tau+\pi$ and the formula (A.8) that

$$\begin{aligned}\int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)e^{-\mathrm{i}j\eta}d\eta &= \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)\cos(j\eta)\,d\eta\\ &= \tfrac{(-1)^j}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_0(2\lambda\cos(\tau))\cos(2j\tau)d\tau\\ &= (I_jK_j)(\lambda).\end{aligned}$$

Hence, the Fourier coefficients of $\mathcal{K}_\lambda$ are

$$\left(\mathcal{K}_\lambda\right)_j = (I_jK_j)(\lambda). \tag{3.16}$$

Similar arguments as before with $j=1$ allow to get

$$V_0(\lambda)=\Omega+(I_1K_1)(\lambda).$$

Recall that $\mathcal{K}_\lambda$ is even and then we find that $\mathrm{L}(\lambda)$ is self-adjoint in $L^2(\mathbb{T})$.
2. Starting from the Fourier expansion $\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j(t)e^{\mathrm{i}j\theta}$, then we can easily ensure from direct computations using the previous results, that ρ solves the equation (3.10) if and only if

$$\dot{\rho}_j=-\mathrm{i}\Omega_j(\lambda)\rho_j \quad \text{with} \quad \Omega_j(\lambda)=j\left[\Omega+(I_1K_1)(\lambda)-(I_jK_j)(\lambda)\right],$$

and therefore

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(\lambda)t)}.$$

Concerning the identities (3.15) they can be obtained from straightforward computations. This ends the proof of Lemma 3.2. □

3.2 Structure of the linear frequencies

The main target in this section is to explore some interesting structures of the equilibrium frequencies. We shall in particular focus on their monotonicity and detail some asymptotic behavior for large modes. Another important discussion will be devoted to the non-degeneracy of these frequencies through the so-called Rüsseman conditions. This is the cornerstone step in measuring the final Cantor set giving rise to quasi-periodic solutions for the linear/nonlinear problems. Actually, in the nonlinear case the final Cantor appears as a perturbation of the Cantor set constructed from the equilibrium eigenvalues and therefore perturbative arguments based on their non-degeneracy are very useful and will be performed in Section 7.2.

3.2.1 Monotonicity and asymptotic behaviour

Our purpose is to establish some useful properties related to the monotonicity and the asymptotic behavior for large modes of the eigenvalues of the linearized operator at the equilibrium state. Notice that their explicit values are detailed in (3.14). Our result reads as follows.

Lemma 3.3. *Let $\Omega>0$ and $\lambda\in\mathbb{R}$, then the frequencies $(\Omega_j(\lambda))_{j\in\mathbb{Z}^*}$ satisfy the following properties.*

(i) For any $j \in \mathbb{Z}^*$, $\lambda > 0$ *we have* $\Omega_{-j}(\lambda) = -\Omega_j(\lambda)$.

(ii) For any $\lambda > 0$, *the sequences* $(\Omega_j(\lambda))_{j\in\mathbb{N}^*}$ *and* $\left(\frac{\Omega_j(\lambda)}{j}\right)_{j\in\mathbb{N}^*}$ *are strictly increasing.*

(iii) For any $\lambda > 0$, *the following expansion holds*

$$\Omega_j(\lambda) \underset{j\to\infty}{=} V_0(\lambda)j - \tfrac{1}{2} + \tfrac{\lambda^2}{4j^2} + O_\lambda\left(\tfrac{1}{j^4}\right), \tag{3.17}$$

where $V_0(\lambda)$ *is defined in* (3.11).

(iv) For any $j \in \mathbb{Z}^*$, $\lambda > 0$ *we have*

$$|\Omega_j(\lambda)| \geqslant \Omega|j|.$$

(v) Given $0 < \lambda_0 < \lambda_1$, *there exists* $C_0 > 0$ *such that*

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall j, j_0 \in \mathbb{Z}, \quad |\Omega_j(\lambda) \pm \Omega_{j_0}(\lambda)| \geqslant C_0|j \pm j_0|.$$

(vi) Given $0 < \lambda_0 < \lambda_1$ *and* $q_0 \in \mathbb{N}$, *there exists* $C_0 > 0$ *such that*

$$\forall\, j, j_0 \in \mathbb{Z}^*, \quad \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q \left(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\right)\right| \leqslant C_0|j - j_0|.$$

Proof. **(i)** It is an immediate consequence of (3.14) and (A.3).
(ii) The monotonicity of the sequence $\left(\frac{\Omega_j(\lambda)}{j}\right)_{j\in\mathbb{N}^*}$ is proved in [22, Prop. 5.9. (1)], see also the Appendix A. It follows that the sequence $(\Omega_j(\lambda))_{j\in\mathbb{N}^*}$ is strictly increasing as the product of two strictly increasing positive sequences.
(iii) It is an immediate consequence of (3.14) and the asymptotic expansion (A.14)
(iv) Recall that $j \mapsto \Omega_j(\lambda)$ is odd and vanishes at $j = 0$. Then it suffices to check the result for $j \in \mathbb{N}^*$. According to the Appendix A, the sequence $j \mapsto (I_jK_j)(\lambda)$ is decreasing and therefore

$$\forall \lambda > 0, \quad (I_1K_1)(\lambda) - (I_jK_j)(\lambda) \geqslant 0. \tag{3.18}$$

It follows that

$$\forall\, \lambda > 0, \quad |\Omega_j(\lambda)| \geqslant \Omega j.$$

(v) By the oddness of $j \mapsto \Omega_j(\lambda)$ it is enough to establish the estimate for $j, j_0 \in \mathbb{N}$. We shall first focus on the estimate of the difference $\Omega_j(\lambda) - \Omega_{j_0}(\lambda)$. Without loss of generality we can assume that $j > j_0 \geqslant 1$, (The case $j = j_0$ is obvious and the case $j_0 = 0$ brings us back to the previous point). One may write by (3.14) that for $\lambda > 0$,

$$\begin{aligned}\Omega_j(\lambda) - \Omega_{j_0}(\lambda) = (j - j_0)\Big(\Omega + I_1(\lambda)K_1(\lambda) - I_j(\lambda)K_j(\lambda)\Big)\\ + j_0\Big(I_{j_0}(\lambda)K_{j_0}(\lambda) - I_j(\lambda)K_j(\lambda)\Big).\end{aligned} \tag{3.19}$$

Combining this estimate with (3.18) yields

$$\Omega_j(\lambda) - \Omega_{j_0}(\lambda) \geqslant (j - j_0)\Omega + j_0\Big(I_{j_0}(\lambda)K_{j_0}(\lambda) - I_j(\lambda)K_j(\lambda)\Big). \tag{3.20}$$

We need to get refined estimate for the last term of the right hand side. For this goal we use the formulae (A.9) to write

$$(I_nK_n)(\lambda) = \tfrac{1}{2}\int_0^\infty J_0\big(2\lambda\sinh(\tfrac{t}{2})\big)e^{-nt}dt. \tag{3.21}$$

This allows to construct for a fixed λ a smooth extension $n \in (0,\infty) \mapsto (I_nK_n)(\lambda)$. Thus differentiating term by term using change of variable we get for any $m \in \mathbb{N}$

$$
\begin{aligned}
\sup_{\lambda\in\mathbb{R}} |\partial_n^m (I_nK_n)(\lambda)| &\leqslant \tfrac{1}{2}\int_0^\infty t^m e^{-nt}dt \\
&\leqslant \tfrac{m!}{2n^{m+1}},
\end{aligned}
\tag{3.22}
$$

where we have used the classical estimates for Bessel functions (applied with $n = q = 0$)

$$
\sup_{\substack{n,q\in\mathbb{N}\\ x\in\mathbb{R}}} |J_n^{(q)}(x)| \leqslant 1, \tag{3.23}
$$

which follows easily from the integral representation (A.1). In particular, for $m = 1$ we find that for for any $n \geqslant 1$

$$
\sup_{\lambda\in\mathbb{R}} \Big|\partial_n (I_nK_n)(\lambda)\Big| \leqslant \tfrac{1}{2n^2}\cdot
$$

Therefore applying Taylor Formula we infer for $j > j_0 \geqslant 1$

$$
\begin{aligned}
\sup_{\lambda\in\mathbb{R}} \Big|(I_jK_j)(\lambda) - (I_{j_0}K_{j_0})(\lambda)\Big| &\leqslant \tfrac{1}{2}\int_{j_0}^{j} \frac{dn}{n^2} \\
&\leqslant \tfrac{|j-j_0|}{2\,j\,j_0}\cdot
\end{aligned}
\tag{3.24}
$$

Inserting this estimate into (3.20) gives

$$
\Omega_j(\lambda) - \Omega_{j_0}(\lambda) \geqslant (j - j_0)\Big(\Omega - \tfrac{1}{2j}\Big).
$$

Therefore for $j > N = \big[\Omega^{-1}\big]$ and $j > j_0 \geqslant 1$ we get

$$
\Omega_j(\lambda) - \Omega_{j_0}(\lambda) \geqslant \tfrac{1}{2}\Omega\,(j - j_0). \tag{3.25}
$$

Now for $j \neq j_0 \in [\![1, N]\!]$ we get from the point (ii) that the map $\lambda \in [\lambda_0, \lambda_1] \mapsto \Omega_j(\lambda) - \Omega_{j_0}(\lambda)$ does not vanish and therefore we can find by a compactness argument a constant $C > 0$ such that

$$
\forall \lambda \in [\lambda_0, \lambda_1], \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C|j - j_0|.
$$

Taking $C_0 = \min(C, \frac{1}{2}\Omega)$ and combining the preceding inequality with (3.25) we obtain

$$
\forall \lambda \in [\lambda_0, \lambda_1], \forall j \geqslant j_0 \geqslant 1, \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C_0|j - j_0|.
$$

Finally we get

$$
\forall \lambda \in [\lambda_0, \lambda_1], \forall j, j_0 \in \mathbb{N}, \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C_0|j - j_0|.
$$

Let us now move to the estimate $\Omega_j(\lambda) + \Omega_{j_0}(\lambda)$ for $j, j_0 \in \mathbb{N}$. Since both quantities are positive then using the point (iv) yields

$$
\forall \lambda \in [\lambda_0, \lambda_1], \quad |\Omega_j(\lambda) + \Omega_{j_0}(\lambda)| = \Omega_j(\lambda) + \Omega_{j_0}(\lambda) \geqslant \Omega(j + j_0) \geqslant C_0(j + j_0).
$$

This completes the proof of the desired estimate.
(vi) Let $q_0 \in \mathbb{N}^*$. let $q \in [\![0, q_0]\!]$. Differentiating q times (3.19) in λ, one obtains

$$
\begin{aligned}
\partial_\lambda^q\big(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\big) = (j - j_0)&\Big(\partial_\lambda^q\Omega + \partial_\lambda^q\big(I_1(\lambda)K_1(\lambda)\big) - \partial_\lambda^\alpha\big(I_j(\lambda)K_j(\lambda)\big)\Big) \\
&+ j_0\partial_\lambda^q\Big(I_{j_0}(\lambda)K_{j_0}(\lambda) - I_j(\lambda)K_j(\lambda)\Big).
\end{aligned}
\tag{3.26}
$$

Similarly, we get by differentiating q times in λ the identity (3.21)

$$
\partial_\lambda^q(I_nK_n)(\lambda) = 2^{q-1}\int_0^\infty J_0^{(q)}\big(2\lambda\sinh(\tfrac{t}{2})\big)\sinh^q(\tfrac{t}{2})e^{-nt}dt. \tag{3.27}
$$

From (3.23) we deduce for any $\lambda \in [\lambda_0, \lambda_1]$,

$$\left|\partial_\lambda^q (I_n K_n)(\lambda)\right| \leqslant 2^{q-1} \int_0^\infty \sinh^q(\tfrac{t}{2}) e^{-nt} dt.$$

Then using the inequality $\sinh x \leqslant \frac{e^x}{2}$ for $x \geqslant 0$ we get for $n > \frac{q}{2}$

$$\begin{aligned} \left|\partial_\lambda^q (I_n K_n)(\lambda)\right| &\leqslant \tfrac{1}{2} \int_0^\infty e^{\left(\frac{q}{2}-n\right)t} dt \\ &\leqslant \tfrac{1}{2n-q}. \end{aligned} \tag{3.28}$$

By compactness argument, we deduce that

$$\sup_{j\in\mathbb{N}} \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q \big(I_j(\lambda) K_j(\lambda)\big)\right| \leqslant C. \tag{3.29}$$

Differentiating in n (3.27) yields

$$\partial_\lambda^q \partial_n (I_n K_n)(\lambda) = -2^{q-1} \int_0^\infty J_0^{(q)}\big(2\lambda \sinh(t/2)\big) \sinh^q(t/2) t e^{-nt} dt.$$

Therefore applying similar arguments used to show (3.28) gives for $2n > q$

$$\begin{aligned} \left|\partial_\lambda^q \partial_n (I_n K_n)(\lambda)\right| &\leqslant \tfrac{1}{2} \int_0^\infty t e^{-\left(n-\frac{q}{2}\right)t} dt \\ &\leqslant \frac{2}{(2n-q)^2}. \end{aligned} \tag{3.30}$$

Then Taylor Formula allows to get for $j, j_0 > \frac{q}{2}$

$$\sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_{j_0} K_{j_0})(\lambda)\right| \leqslant C \tfrac{|j-j_0|}{j j_0}. \tag{3.31}$$

Setting $N = \lfloor \frac{q_0}{2} \rfloor + 1$, one obtains for any $j, j_0 \geqslant N$

$$\max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_{j_0} K_{j_0})(\lambda)\right| \leqslant C|j-j_0|.$$

By compactness argument, one obtains for any $j, j_0 \in [\![1, N]\!]$

$$\max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_{j_0} K_{j_0})(\lambda)\right| \leqslant C|j-j_0|.$$

Now for the remaining case $j_0 \in [\![1, N]\!]$ and $j \geqslant N$ one has gathering the previous two estimates

$$\begin{aligned} \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_{j_0} K_{j_0})(\lambda)\right| &\leqslant \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_N K_N)(\lambda)\right| \\ &\quad + \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_N K_N - I_{j_0} K_{j_0})(\lambda)\right| \\ &\leqslant C|j-N| + C|N-j_0| \leqslant C|j-j_0|. \end{aligned}$$

Thus we can find $C > 0$ such that for any $j, j_0 \in \mathbb{N}^*$

$$\max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_j K_j - I_{j_0} K_{j_0})(\lambda)\right| \leqslant C|j-j_0|.$$

Putting together (3.26), (3.29) and (3.14) yields

$$\max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q \left(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\right)\right| \leqslant C|j-j_0|.$$

This ends the proof of Lemma 3.3. □

3.2.2 Non-degeneracy and transversality

Fix finitely many tangential sites

$$\mathbb{S} := \{j_1, \dots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad d \geqslant 1 \quad \text{and} \quad 1 \leqslant j_1 < \dots < j_d.$$

We consider the linear vector frequency at the equilibrium state

$$\omega_{\text{Eq}}(\lambda) := (\Omega_j(\lambda))_{j \in \mathbb{S}}, \tag{3.32}$$

where $\Omega_j(\lambda)$ is defined by (3.14). The main purpose is to study some Diophantine structure of the curve $\lambda \in (\lambda_0, \lambda_1) \mapsto \omega_{\text{Eq}}(\lambda)$ for fixed $0 < \lambda_0 < \lambda_1$. In particular, we shall focus on the non-degeneracy and the transversality conditions of these eigenvalues which are essential in getting non trivial Cantor set from which quasi-periodic solutions emerge at the linear and nonlinear levels. Notice that the approach that we shall implement here has been developed before in several papers such as [4, 8, 51]. Before exploring these properties we need to fix some definitions.

Definition 3.1. *Given two numbers $\lambda_0 < \lambda_1$ and $d \in \mathbb{N}^*$, a vector-valued function $f = (f_1, ..., f_d) : [\lambda_0, \lambda_1] \to \mathbb{R}^d$ is called non-degenerate if, for any vector $c = (c_1, ..., c_d) \in \mathbb{R}^d \setminus \{0\}$, the function $f \cdot c = f_1 c_1 + ... + f_d c_d$ is not identically zero on the whole interval $[\lambda_0, \lambda_1]$. This means that the curve of f is not contained in an hyperplane.*

Now we shall prove the following result on the non-degeneracy of the linear frequencies which is related to the asymptotic behavior of Bessel functions $(I_j K_j)(\lambda)$ for large values of λ. This property will be crucial to check a suitable transversality assumption.

Lemma 3.4. *Let $\Omega \in \mathbb{R}^*$ and $0 < \lambda_0 < \lambda_1$, then the frequency curve ω_{Eq} defined by* (3.32) *and the vector-valued function $\lambda \mapsto (\Omega + I_1 K_1, \omega_{\text{Eq}}) \in \mathbb{R}^{d+1}$ are non degenerate on $[\lambda_0, \lambda_1]$ in the sense of the Definition* 3.1.

Proof. ▶ Let us start with checking the non-degeneracy of ω_{Eq}. For this aim, we shall argue by contradiction and assume the existence of a fixed vector $c = (c_k)_{0 \leqslant k \leqslant d} \in \mathbb{R}^d$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad \sum_{k=1}^{d} c_k \Omega_{j_k}(\lambda) = 0. \tag{3.33}$$

Since for all $j \in \mathbb{N}^*$, the application $\lambda \mapsto (I_j K_j)(\lambda)$ admits a holomorphic extension in the open connected set $\{\lambda \in \mathbb{C}, \text{Re}(\lambda) > 0\}$ (see Appendix A) then by the continuation principle we obtain

$$\forall \lambda > 0, \quad \sum_{k=1}^{d} c_k j_k (I_{j_k} K_{j_k})(\lambda) = \left(\sum_{k=1}^{d} c_k j_k \right) \big((I_1 K_1)(\lambda) + \Omega \big). \tag{3.34}$$

Using the asymptotic expansion (A.11) obtained for $I_j K_j$ with large λ, we first get

$$\forall j \in \mathbb{N}^*, \quad \lim_{\lambda \to \infty} (I_j K_j)(\lambda) = 0.$$

Then taking the limit in (3.34) as $\lambda \to \infty$ implies

$$\Omega \sum_{k=1}^{d} c_k j_k = 0.$$

Since we assumed that $\Omega \neq 0$, then necessary we find that $\sum_{k=1}^{d} c_k j_k = 0$ which implies in turn according to (3.34)

$$\forall \lambda > 0, \quad \sum_{k=1}^{d} c_k j_k (I_{j_k} K_{j_k})(\lambda) = 0.$$

Applying once again the expansion (A.11) yields

$$\forall m \in [\![1,d]\!], \quad \sum_{k=1}^{d} c_k j_k \alpha_{j_k,m} = 0. \tag{3.35}$$

We consider the matrix $A_d = (A_{m,k})_{1\leqslant m,k\leqslant d} \in M_d(\mathbb{R})$ defined by

$$\forall (m,k) \in [\![1,d]\!]^2, \quad A_{m,k} = j_k \alpha_{j_k,m}.$$

Then the system (3.35) is equivalent to $A_d c = 0$ with $c = \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$. To get the desired result, $c = 0$, it suffices to check that det $A_d \neq 0$. Using the expression of the coefficients $\alpha_{j_k,m}$ in (A.12) one deduces that

$$\alpha_{j_k,m} = a_m(\mu_{j_k} - 1)Q_m\big(\mu_{j_k}\big), \quad a_m = (-1)^m \frac{(2m)!}{4^m\big(m!\big)^2}, \quad \mu_j = 4j^2, \tag{3.36}$$

with $Q_1(X) = 1$ and for $m \geq 2$

$$Q_m(X) = \prod_{\ell=2}^{m} \big(X - (2\ell-1)^2\big).$$

Remark that Q_m is a unitary polynomial of degree $m-1$. Using the homogeneity of the determinant with respect to each column and row we find

$$\det\, A_d = \prod_{m,k=1}^{d} a_m(\mu_{j_k} - 1) \det\, B_d,$$

with B_d the matrix given by

$$B_d = \begin{pmatrix} Q_1(\mu_{j_1}) & \cdots & Q_1(\mu_{j_d}) \\ \vdots & & \vdots \\ Q_d(\mu_{j_1}) & \cdots & Q_d(\mu_{j_d}) \end{pmatrix}.$$

Therefore we infer that A_d is nonsingular if $\det B_d \neq 0$. On the other hand, the computation of $\det B_d$ can be done in a similar way to Vandermonde determinant. Indeed, define the polynomial given by the determinant

$$P(X) = \begin{vmatrix} Q_1(\mu_{j_1}) & \cdots & Q_1(\mu_{j_{d-1}}) & Q_1(X) \\ \vdots & & \vdots & \vdots \\ Q_d(\mu_{j_1}) & \cdots & Q_d(\mu_{j_{d-1}}) & Q_d(X) \end{vmatrix}.$$

Then P is a polynomial of degree $d-1$ and vanishes at all the points $X = \mu_{j_k}$ for $k \in [\![1, d-1]\!]$. Consequently, we get

$$\det B_d = P(\mu_{j_d}) = \det B_{d-1} \prod_{k=1}^{d-1} \big(\mu_{j_d} - \mu_{j_k}\big).$$

Therefore, iterating this identity yields

$$\det B_d = \prod_{1\leqslant k<\ell\leqslant d-1} \big(\mu_{j_\ell} - \mu_{j_k}\big).$$

Since $\mu_{j_\ell} \neq \mu_{j_k}$ for $\ell \neq k$ we get $\det B_d \neq 0$ which achieves the proof of the first point.
► Next we move to the second point of the lemma and show that if

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad c_0\Big(\Omega + (I_1K_1)(\lambda)\Big) + \sum_{k=1}^{d} c_k j_k \Big(\Omega + (I_1K_1)(\lambda) - (I_{j_k}K_{j_k})(\lambda)\Big) = 0,$$

then necessary $c_0 = ... = c_d = 0$. As before we can extend by analyticity the preceding identity to $(0, \infty)$ By checking the terms in $\frac{1}{\lambda}$ in the preceding identity using (A.11) we find immediately that $c_0 = 0$. Therefore the system reduces to (3.33) and then we may apply the result of the first point in order to get $c_1 = ... = c_d = 0$. This completes the proof of Lemma 3.4. □

The next goal is to check that Rüssemann transversality conditions are satisfied for the linear frequencies of the equilibrium state. Namely, we shall prove the following result in the spirit of the papers [4, 8, 51].

Lemma 3.5. [Transversality] *Given $0 < \lambda_0 < \lambda_1$, there exist $q_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that the following results hold true. Recall that ω_{Eq} and Ω_j are defined in* (3.32) *and* (3.14) *respectively.*

(i) For any $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} \max_{q \in [\![0, q_0]\!]} |\partial_\lambda^q \omega_{\mathrm{Eq}}(\lambda) \cdot l| \geqslant \rho_0 \langle l \rangle.$$

(ii) For any $(l, j) \in (\mathbb{Z}^d \times \mathbb{N}) \setminus \{(0, 0)\}$

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} \max_{q \in [\![0, q_0]\!]} \left|\partial_\lambda^q \big(\omega_{\mathrm{Eq}}(\lambda) \cdot l \pm j (I_1 K_1)(\lambda)\big)\right| \geqslant \rho_0 \langle l \rangle.$$

(iii) For any $(l, j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} \max_{q \in [\![0, q_0]\!]} \left|\partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot l \pm \Omega_j(\lambda)\right)\right| \geqslant \rho_0 \langle l \rangle.$$

(iv) For any $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l, j) \neq (0, j')$, we have

$$\inf_{\lambda \in [\lambda_0, \lambda_1]} \max_{q \in [\![0, q_0]\!]} \left|\partial_\lambda^q \big(\omega_{\mathrm{Eq}}(\lambda) \cdot l + \Omega_j(\lambda) \pm \Omega_{j'}(\lambda)\big)\right| \geqslant \rho_0 \langle l \rangle.$$

Proof. **(i)** We argue by contradiction by assuming that for any $q_0 \in \mathbb{N}$ and $\rho_0 > 0$, there exist $l \in \mathbb{Z}^d \setminus \{0\}$ and $\lambda \in [\lambda_0, \lambda_1]$ such that

$$\max_{q \in [\![0, q_0]\!]} |\partial_\lambda^q (\omega_{\mathrm{Eq}}(\lambda) \cdot l)| < \rho_0 \langle l \rangle.$$

It follows that for any $m \in \mathbb{N}$, and by taking $q_0 = m$ and $\rho_0 = \frac{1}{m+1}$, there exist $l_m \in \mathbb{Z}^d \setminus \{0\}$ and $\lambda_m \in [\lambda_0, \lambda_1]$ such that

$$\max_{q \in [\![0, m]\!]} |\partial_\lambda^q \omega_{\mathrm{Eq}}(\lambda_m) \cdot l_m| < \tfrac{\langle l_m \rangle}{m+1}$$

and therefore

$$\forall q \in \mathbb{N}, \quad \forall m \geqslant q, \quad \left|\partial_\lambda^q \omega_{\mathrm{Eq}}(\lambda_m) \cdot \tfrac{l_m}{\langle l_m \rangle}\right| < \tfrac{1}{m+1}. \tag{3.37}$$

Since the sequences $\left(\frac{l_m}{\langle l_m \rangle}\right)_m$ and $(\lambda_m)_m$ are bounded, then by compactness and up to an extraction we can assume that

$$\lim_{m \to \infty} \tfrac{l_m}{\langle l_m \rangle} = \bar{c} \neq 0 \quad \text{and} \quad \lim_{m \to \infty} \lambda_m = \bar{\lambda}.$$

Hence, passing to the limit in (3.37) as $m \to \infty$ leads to

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \omega_{\mathrm{Eq}}(\bar{\lambda}) \cdot \bar{c} = 0.$$

Thus, we conclude that the real analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c}$ is identically zero which contradicts the non-degeneracy condition stated in Lemma 3.4.

(ii) We shall first check the result for the case $l = 0$ and $j \in \mathbb{N}^*$. Obviously, one has from the monotonicity of $\lambda \mapsto I_1(\lambda)K_1(\lambda)$ stated in Appendix A,

$$\begin{aligned}\inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{q\in[\![0,q_0]\!]}\left|\partial_\lambda^q\big(j(I_1K_1)(\lambda)\big)\right| &\geqslant (I_1K_1)(\lambda_1)\\ &\geqslant \rho_0\langle l\rangle,\end{aligned}$$

for some $\rho_0 > 0$. Now let us consider $l \in \mathbb{Z}^d\backslash\{0\}$ and $j \in \mathbb{N}$. Then we may write according to the triangle and Cauchy-Schwarz inequalities combined with the boundedness of ω_{Eq} and the monotonicity of $\lambda \mapsto I_1(\lambda)K_1(\lambda)$ stated in Appendix A,

$$|\omega_{\mathrm{Eq}}(\lambda)\cdot l \pm jI_1(\lambda)K_1(\lambda)| \geqslant jI_1(\lambda_1)K_1(\lambda_1) - |\omega_{\mathrm{Eq}}(\lambda)\cdot l| \geqslant c_0 j - C\langle l\rangle \geqslant \langle l\rangle$$

provided that $j \geqslant C_0\langle l\rangle$ for some $C_0 > 0$. Therefore we reduce the proof to indices j and l with

$$0 \leqslant j < C_0\langle l\rangle, \quad j\in\mathbb{N} \quad \text{and} \quad l\in\mathbb{Z}^d\backslash\{0\}. \tag{3.38}$$

Arguing by contradiction as in the previous case, we may assume the existence of sequences $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m \in \mathbb{N}$ satisfying (3.38) and $\lambda_m \in [\lambda_0, \lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]}\left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|} \pm j_m\tfrac{(I_1K_1)(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right| < \tfrac{1}{m+1}$$

and therefore

$$\forall q\in\mathbb{N},\quad \forall m\geqslant q,\qquad \left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|} \pm \tfrac{j_m}{|l_m|}(I_1K_1)(\lambda)\right)_{|\lambda=\lambda_m}\right| < \tfrac{1}{m+1}\cdot \tag{3.39}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$, $\left(\frac{j_m}{|l_m|}\right)_m$ and $(\lambda_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty}\tfrac{l_m}{|l_m|} = \bar{c} \neq 0, \qquad \lim_{m\to\infty}\tfrac{j_m}{|l_m|} = \bar{d} \quad \text{and} \quad \lim_{m\to\infty}\lambda_m = \bar{\lambda}.$$

Hence, by letting $m \to \infty$ in (3.39), using that $\lambda \mapsto (I_1K_1)(\lambda)$ is smooth, we find

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\big(\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c} \pm \bar{d}\,(I_1K_1)(\lambda)\big)_{|\lambda=\bar{\lambda}} = 0.$$

Thus, the real analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c} \pm \bar{d}\,I_1(\lambda)K_1(\lambda)$ with $(\bar{c},\bar{d}) \neq (0,0)$ is identically zero and this contradicts Lemma 3.4.

(iii) Consider $(l,j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$. Then applying the triangle inequality and Lemma 3.3-(iv), yields

$$\begin{aligned}|\omega_{\mathrm{Eq}}(\lambda)\cdot l \pm \Omega_j(\lambda)| &\geqslant |\Omega_j(\lambda)| - |\omega_{\mathrm{Eq}}(\lambda)\cdot l|\\ &\geqslant \Omega j - C|l| \geqslant \langle l\rangle\end{aligned}$$

provided $j \geqslant C_0\langle l\rangle$ for some $C_0 > 0$. Thus as before we shall restrict the proof to indices j and l with

$$0 \leqslant j < C_0\langle l\rangle, \quad j\in\mathbb{N}^*\setminus\mathbb{S} \quad \text{and} \quad l\in\mathbb{Z}^d\backslash\{0\}. \tag{3.40}$$

Proceeding by contradiction as in the previous case, we may assume the existence of sequences $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m \in \mathbb{N}\setminus\mathbb{S}$ satisfying (3.40) and $\lambda_m \in [\lambda_0, \lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]}\left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|} \pm \tfrac{\Omega_{j_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right| < \tfrac{1}{m+1}$$

and therefore

$$\forall q\in\mathbb{N},\quad \forall m\geqslant q,\qquad \left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|} \pm \tfrac{\Omega_{j_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right| < \tfrac{1}{m+1}\cdot \tag{3.41}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$ and $(\lambda_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty} \tfrac{l_m}{|l_m|} = \bar{c} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} \lambda_m = \bar{\lambda}.$$

Now we shall distinguish two cases.

► Case ❶ : $(l_m)_m$ is bounded. In this case, by (3.40) we find that $(j_m)_m$ is bounded too and thus up to to an extraction we may assume $\lim\limits_{m\to\infty} l_m = \bar{l}$ and $\lim\limits_{m\to\infty} j_m = \bar{j}$. Since $(j_m)_m$ and $(|l_m|)_m$ are sequences of integers, then they are necessary stationary. In particular, the condition (3.40) implies $\bar{l} \neq 0$. Hence, taking the limit $n \to \infty$ in (3.41), yields

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot \bar{l} \pm \Omega_{\bar{j}}(\lambda)\right)_{|\lambda=\bar{\lambda}} = 0.$$

Thus, the analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{l} \pm \Omega_{\bar{j}}(\lambda)$ with $(\bar{l}, 1) \neq (0,0)$ is identically zero which contradicts Lemma 3.4.

► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty} |l_m| = \infty$. We have two sub-cases.

• Sub-case ① : $(j_m)_m$ is bounded. In this case and up to an extraction we can assume that it converges. Then, taking the limit $m \to \infty$ in (3.41), we find

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \omega_{\mathrm{Eq}}(\bar{\lambda}) \cdot \bar{c} = 0.$$

As before we conclude that function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c}$ with $\bar{c} \neq 0$ is identically zero which contradicts Lemma 3.4.

• Sub-case ② : $(j_m)_m$ is unbounded. Then up to an extraction we can assume that $\lim\limits_{m\to\infty} j_m = \infty$. We write according to (3.14)

$$\frac{\Omega_{j_m}(\lambda)}{|l_m|} = \frac{j_m}{|l_m|} \left(\Omega + (I_1K_1)(\lambda) - (I_{j_m}K_{j_m})(\lambda)\right). \tag{3.42}$$

By (3.40), the sequence $\left(\frac{j_m}{|l_m|}\right)_n$ is bounded, thus up to an extraction we can assume that it converges to $\bar{d}$. Using the first inequality of (3.22) we deduce that

$$\forall\, m \in \mathbb{N}, \quad \sup_{\lambda\in\mathbb{R}} \left|(I_{j_m}K_{j_m})(\lambda)\right| \leq \tfrac{1}{2j_m},$$

which implies that

$$\lim_{m\to\infty} \sup_{\lambda\in\mathbb{R}} (I_{j_m}K_{j_m})(\lambda) = 0.$$

Moreover by (3.28), we have

$$\lim_{m\to\infty} \sup_{\lambda\in[\lambda_0,\lambda_1]} \left|\partial_\lambda^q (I_{j_m}K_{j_m})(\lambda)\right| = 0. \tag{3.43}$$

Taking the limit in (3.42) and using (3.43) yields

$$\lim_{m\to\infty} \frac{\partial_\lambda^q \Omega_{j_m}(\lambda_m)}{|l_m|} = \partial_\lambda^q \Big(\bar{d}\Big(\Omega + \big(I_1K_1\big)(\bar{\lambda})\Big)\Big).$$

Consequently, taking the limit $m \to \infty$ in (3.41), we have

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c} \pm \bar{d}\big(\Omega + \big(I_1K_1\big)(\lambda)\big)\right)_{|\lambda=\lambda_m} = 0.$$

By continuation the analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c} \pm \bar{d}(\Omega + I_1(\lambda)K_1(\lambda))$ with $(\bar{c}, \bar{d}) \neq 0$ is identically zero which contradicts Lemma 3.4.

(iv) Consider $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l, j) \neq (0, j')$. Then applying the triangle inequality combined with Lemma 3.3- (v), we infer

$$|\omega_{\mathrm{Eq}}(\lambda) \cdot l + \Omega_j(\lambda) \pm \Omega_{j'}(\lambda)| \geqslant |\Omega_j(\lambda) \pm \Omega_{j'}(\lambda)| - |\omega_{\mathrm{Eq}}(\lambda) \cdot l| \geqslant C_0|j \pm j'| - C|l| \geqslant \langle l \rangle$$

provided that $|j \pm j'| \geqslant c_0\langle l\rangle$ for some $c_0>0$. Then it remains to check the proof for indices satisfying

$$|j \pm j'| < c_0\langle l\rangle, \quad l\in\mathbb{Z}^d\backslash\{0\} \quad \text{and} \quad j,j'\in\mathbb{N}^*\setminus\mathbb{S}. \tag{3.44}$$

Reasoning by contradiction as in the previous cases, we get for all $m\in\mathbb{N}$, real numbers $l_m\in\mathbb{Z}^d\setminus\{0\}$, $j_m, j'_m\in\mathbb{N}^*\setminus\mathbb{S}$ satisfying (3.44) and $\lambda_m\in[\lambda_0,\lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]}\left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|}+\tfrac{\Omega_{j_m}(\lambda)\pm\Omega_{j'_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}$$

implying in turn that

$$\forall q\in\mathbb{N},\quad \forall m\geqslant q,\qquad \left|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|}+\tfrac{\Omega_{j_m}(\lambda)\pm\Omega_{j'_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}\cdot \tag{3.45}$$

Up to an extraction we can assume that $\lim\limits_{m\to\infty}\frac{l_m}{|l_m|}=\bar{c}\neq 0$ and $\lim\limits_{m\to\infty}\lambda_m=\bar{\lambda}$.
As before we shall distinguish two cases.
► Case ❶ : $(l_m)_m$ is bounded. We shall only focus on the most delicate case associated to the difference $\Omega_{j_m}-\Omega_{j'_m}$. Up to an extraction we may assume that $\lim\limits_{m\to\infty} l_m=\bar{l}\neq 0$. Now according to (3.44) we have two sub-cases to discuss depending whether the sequences $(j_m)_m$ and $(j'_m)_m$ are simultaneously bounded or unbounded.
• Sub-case ① : $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case, up to an extraction we may assume that these sequences are stationary $j_m=\bar{j}$ and $j'_m=\bar{j}'$ with $\bar{j},\bar{j}'\in\mathbb{N}^*\setminus\mathbb{S}$. Hence taking the limit as $m\to\infty$ in (3.45), we infer

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{l}+\Omega_{\bar{j}}(\bar{\lambda})-\Omega_{\bar{j}'}(\lambda)\right)_{\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda\mapsto\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{l}+\Omega_{\bar{j}}(\lambda)-\Omega_{\bar{j}'}(\lambda)$ is identically zero. If $\bar{j}=\bar{j}'$ then this contradicts Lemma 3.4 since $\bar{l}\neq 0$. However in the case $\bar{j}\neq\bar{j}'\in\mathbb{N}^*\setminus\mathbb{S}$ this still contradicts this lemma applied with the vector frequency $(\omega_{\mathrm{Eq}},\Omega_{\bar{j}},\Omega_{\bar{j}'})$ instead of ω_{Eq}.
• Sub-case ② : $(j_m)_m$ and $(j'_m)_m$ are both unbounded and without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m=\lim\limits_{m\to\infty} j'_m=\infty$. From (3.31) combined with (3.44) and the boundedness of $(l_m)_m$ we deduce that

$$\left|\partial_\lambda^q(I_{j_m}K_{j_m}-I_{j'_m}K_{j'_m})(\lambda_m)\right|\leqslant\tfrac{C}{j_m j'_m},$$

which implies in turn

$$\lim_{m\to\infty} j'_m\,\partial_\lambda^q(I_{j_m}K_{j_m}-I_{j'_m}K_{j'_m})(\lambda_m)=0. \tag{3.46}$$

Coming back to (3.14) we get the splitting

$$\begin{aligned}\Omega_{j_m}(\lambda)-\Omega_{j'_m}(\lambda)=&(j_m-j'_m)\big(\Omega+(I_1K_1)(\lambda)\big)-(j_m-j'_m)(I_{j_m}K_{j_m})(\lambda)\\&+j'_m\Big((I_{j'_m}K_{j'_m})(\lambda)-(I_{j_m}K_{j_m})(\lambda)\Big).\end{aligned} \tag{3.47}$$

Therefore by applying (3.43) and (3.46) we get for any $q\in\mathbb{N}$,

$$\lim_{m\to\infty}\partial_\lambda^q\Big(\Omega_{j_m}(\lambda)-\Omega_{j'_m}(\lambda)-(j_m-j'_m)\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{\lambda=\lambda_m}=0.$$

Using once again (3.44) and up to an extraction we have $\lim\limits_{m\to\infty}\frac{j_m-j'_m}{|l_m|}=\bar{d}$. Thus

$$\lim_{m\to\infty}|l_m|^{-1}\partial_\lambda^q\left(\Omega_{j_m}(\lambda)-\Omega_{j'_m}(\lambda)\right)_{|\lambda=\lambda_m}=\bar{d}\,\partial_\lambda^q\left(\Omega+(I_1K_1)(\lambda)\right)_{|\lambda=\bar{\lambda}}.$$

By taking the limit as $m\to\infty$ in (3.45), we find

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}+\bar{d}\big(\Omega+(I_1K_1)(\lambda)\big)\right)_{|\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda \mapsto \omega_{\rm Eq}(\lambda)\cdot\bar{c}+\bar{d}(\Omega+I_1(\lambda)K_1(\lambda))$ with $(\bar{c},\bar{d})\neq 0$ is vanishing which contradicts Lemma 3.4. Now we shall move to the second case.
► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty}|l_m|=\infty$.
We shall distinguish three sub-cases.
• Sub-case ①. The sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case and up to an extraction they will converge and then taking the limit in (3.45) yields,

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\omega_{\rm Eq}(\bar{\lambda})\cdot\bar{c}=0.$$

which leads to a contradiction as before.
• Sub-case ②. The sequences $(j_m)_m$ and $(j'_m)_m$ are both unbounded. This is similar to the sub-case ② of the case ❶.
• Sub-case ③. The sequence $(j_m)_m$ is unbounded and $(j'_m)_m$ is bounded (the symmetric case is similar). Without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m=\infty$ and $j'_m=\bar{j}$. By (3.44) and up to an extraction one gets $\lim\limits_{m\to\infty}\frac{j_m\pm j'_m}{|l_m|}=\bar{d}$. One may use (3.14) combined with (3.43) and (3.46) in order to get for any $q\in\mathbb{N}$,

$$\lim_{m\to\infty}|l_m|^{-1}\partial_\lambda^q\Big(\Omega_{j_m}(\lambda)\pm\Omega_{j'_m}(\lambda)-(j_m\pm j'_m)\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{|\lambda=\lambda_m}=$$
$$\lim_{m\to\infty}\partial_\lambda^q\left(\tfrac{(j_m\pm j'_m)}{|l_m|}(I_{j_m}K_{j_m})(\lambda)\pm\tfrac{j'_m}{|l_m|}\Big((I_{j_m}K_{j_m})(\lambda)-(I_{j'_m}K_{j'_m})(\lambda)\Big)_{|\lambda=\lambda_m}\right)=0.$$

Hence, taking the limit in (3.45) implies

$$\forall q\in\mathbb{N},\partial_\lambda^q\left(\omega_{\rm Eq}(\lambda)\cdot\bar{c}+\bar{d}\big(\Omega+(I_1K_1)(\lambda)\big)\right)_{\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda\mapsto\omega_{\rm Eq}(\lambda)\cdot\bar{c}+\bar{d}(\Omega+I_1(\lambda)K_1(\lambda))$ is identically zero with $(\bar{c},\bar{d})\neq 0$ which contradicts Lemma 3.4.This completes the proof of Lemma 3.5. □

3.2.3 Linear quasi-periodic solutions

Notice that all the solutions of (3.10) taking the form (3.13) are either periodic, quasi-periodic or almost periodic in time, with linear frequencies of oscillations $\Omega_j(\lambda)$ defined by (3.14) These different notions differs on the irrationality properties of the frequencies $\{\Omega_j(\lambda)\}$ and on the cardinality of the Fourier-space support (finite for quasi-periodic functions and possibly infinite for almost periodic ones). Remark that we have the implications

$$\text{Periodic}\quad\Rightarrow\quad\text{Quasi-periodic}\quad\Rightarrow\quad\text{Almost periodic.}$$

We shall prove here the existence of quasi-periodic solutions for the linear equation (3.10) when λ belongs to a massive Cantor set.

Proposition 3.1. *Let $\lambda_1>\lambda_0>0$, $d\in\mathbb{N}^*$ and $\mathbb{S}\subset\mathbb{N}^*$ with $|\mathbb{S}|=d$. Then, there exists a Cantor-like set $\mathcal{C}\subset[\lambda_0,\lambda_1]$ satisfying $|\mathcal{C}|=\lambda_1-\lambda_0$ and such that for all $\lambda\in\mathcal{C}$, every function in the form*

$$\rho(t,\theta)=\sum_{j\in\mathbb{S}}\rho_j\cos(j\theta-\Omega_j(\lambda)t),\quad \rho_j\in\mathbb{R}^* \tag{3.48}$$

is a time quasi-periodic reversible solution to the equation (3.10) *with the vector frequency*

$$\omega_{\rm Eq}(\lambda)=(\Omega_j(\lambda))_{j\in\mathbb{S}}.$$

Proof. It is easy to check that any function in the form (3.48) is a reversible solution to (3.10), that is a solution satisfying the property

$$r(-t,-\theta)=r(t,\theta).$$

Then, it remains to check the non-resonance condition (1.3) for the frequency vector $\omega_{\rm Eq}$ for almost every $\lambda \in [\lambda_0, \lambda_1]$. For that purpose, we consider $\tau_1 > 0, \gamma \in (0,1)$ and define the set $\mathcal{C}_\gamma$ by

$$\mathcal{C}_\gamma = \bigcap_{l \in \mathbb{Z}^d \setminus \{0\}} \Big\{ \lambda \in [\lambda_0, \lambda_1] \quad \text{s.t.} \quad |\omega_{\rm Eq}(\lambda) \cdot l| > \tfrac{\gamma}{\langle l \rangle^{\tau_1}} \Big\}.$$

Therefore its complement set takes the form

$$[\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma = \bigcup_{l \in \mathbb{Z}^d \setminus \{0\}} \mathcal{R}_l \quad \text{where} \quad \mathcal{R}_l = \Big\{ \lambda \in [\lambda_0, \lambda_1] \quad \text{s.t.} \quad |\omega_{\rm Eq}(\lambda) \cdot l| \leqslant \tfrac{\gamma}{\langle l \rangle^{\tau_1}} \Big\}.$$

It follows that

$$\Big| [\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma \Big| \leqslant \sum_{l \in \mathbb{Z}^d \setminus \{0\}} |\mathcal{R}_l|.$$

Now applying Lemma 3.6 together with Lemma 3.5-(i), one obtains

$$|\mathcal{R}_l| \lesssim \gamma^{\frac{1}{q_0}} \langle l \rangle^{-1 - \frac{\tau_1 + 1}{q_0}}.$$

Then by imposing

$$\tau_1 > (d-1) q_0 - 1,$$

one gets a convergent series with

$$\big| [\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma \big| \leqslant C \gamma^{\frac{1}{q_0}}.$$

Now, we define the Cantor set

$$\mathcal{C} = \bigcup_{\gamma > 0} \mathcal{C}_\gamma.$$

Then one gets easily for any $\gamma > 0$

$$\lambda_1 - \lambda_0 - C \gamma^{\frac{1}{q_0}} \leqslant |\mathcal{C}_\gamma| \leqslant |\mathcal{C}| \leqslant \lambda_1 - \lambda_0.$$

Passing to the limit as $\gamma \to 0$ yields

$$|\mathcal{C}| = \lambda_1 - \lambda_0,$$

which achieves the proof of Proposition 3.1. $\square$

In the previous proof, we used the following Lemma whose proof can be found in [51, Thm. 17.1]. Notice that in all the paper, we use the notation $|A|$ as the Lebesgue measure of a given measurable set A.

Lemma 3.6. *Let $q_0 \in \mathbb{N}^*$ and $\alpha, \beta \in \mathbb{R}_+$. Let $f \in C^{q_0}([a,b], \mathbb{R})$ such that*

$$\inf_{x \in [a,b]} \max_{k \in [\![0, q_0]\!]} |f^{(k)}(x)| \geqslant \beta.$$

Then, there exists $C = C(a, b, q_0, \|f\|_{C^{q_0}([a,b],\mathbb{R})}) > 0$ such that

$$\Big| \{ x \in [a,b] \quad \text{s.t.} \quad |f(x)| \leqslant \alpha \} \Big| \leqslant C \frac{\alpha^{\frac{1}{q_0}}}{\beta^{1 + \frac{1}{q_0}}}.$$

4 Functional setting and technical Lemmas

In this section, we set up the general topological framework for both the functions and the operators classes. We also provide some classical results on the law products, composition rule, Toeplitz operators, etc...

Next, we intend to introduce some parameters with some restrictions that will be used later.

$$\gamma \in (0,1), \quad q,d \in \mathbb{N}^*, \quad S \geqslant s \geqslant s_0 > \tfrac{d+1}{2} + q + 2, \tag{4.1}$$

where S is a fixed large number.

$$\tau_2 > \tau_1 > d. \tag{4.2}$$

Let

$$0 < \lambda_0 < \lambda_1.$$

Since the mapping ω_{Eq}, defined by (3.32), is continuous, then we can find a radius $R_0 > 0$ such that

$$\omega_{\mathrm{Eq}}\left((\lambda_0,\lambda_1)\right) \subset \mathscr{U} := B(0,R_0).$$

We consider $\mathcal{O}$ the open bounded subset of $\mathbb{R}^{d+1}$ defined by

$$\mathcal{O} = (\lambda_0,\lambda_1) \times \mathscr{U}. \tag{4.3}$$

Now, we explain the role played by the foregoing parameters throughout this paper.

Remark 4.1.
- *The parameter λ comes from the model $(QGSW)_\lambda$ and it is free in a fixed interval (λ_0,λ_1). However at the end it will belong to a Cantor set for which invariant torus can be constructed.*
- *The integer d is the number of excited frequencies that will generate the quasi-periodic solutions. This is the dimension of the space where lies the frequency vector $\omega \in \mathscr{U} \subset \mathbb{R}^d$, that will be a perturbation of the equilibrium frequency $\omega_{\mathrm{Eq}}(\lambda)$.*
- *The real number s is the Sobolev index regularity of the functions in the variables φ and θ. The index s will vary between s_0 and a large enough parameter S and at this end of Nash-Moser scheme it will be fixed as a large number related to the geometry of the intermediate Cantor sets.*
- *The integer q is the index of regularity of our functions/operators with respect to the parameters λ and ω. We have to consider such regularity in order to perform measure estimates in Section 7.2 by checking the Rüssemann conditions. Its value will be fixed equal to q_0+1, where q_0 is the non degeneracy index of the tangential frequencies given in Lemma 3.5.*
- *All the remaining parameters γ, τ_1 and τ_2 are linked to different Diophantine conditions, see for instance Lemma 5.4 and Propositions 6.2 and 6.5. The choice of τ_1 and τ_2 will be finally fixed in (7.64). We point out that the parameter γ appears in the weighted Sobolev spaces and will be fixed in Proposition 7.1 with respect to the rescaling parameter ε giving the smallness condition of the solutions around the equilibrium.*

4.1 Function spaces

We shall introduce the function spaces that will be frequently used along the paper. They are given by weighted Sobolev spaces with respect to a parameter $\gamma \in (0,1)$ used in defining the Cantor sets to track the regularity with respect to the external parameters λ and ω of the solutions to the nonlinear equation. We denote by $(\mathbf{e}_{l,j})_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}$ the Hilbert basis of the complex Hilbert space $L^2(\mathbb{T}^{d+1},\mathbb{C})$ defined by

$$\mathbf{e}_{l,j}(\varphi,\theta) = e^{\mathrm{i}(l\cdot\varphi+j\theta)}.$$

We endow this space with the Hermitian inner product

$$\begin{aligned}\big\langle \rho_1,\rho_2\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})}&=\int_{\mathbb{T}^{d+1}}\rho_1(\varphi,\theta)\overline{\rho_2(\varphi,\theta)}d\varphi d\theta\\&=\frac{1}{(2\pi)^{d+1}}\int_{[0,2\pi]^{d+1}}\rho_1(\varphi,\theta)\overline{\rho_2(\varphi,\theta)}d\varphi d\theta.\end{aligned}$$

To get the last line of the preceding identity we use the notation (2.8). Given $\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})$, we may decompose it in Fourier expansion as

$$\rho=\sum_{(l,j)\in\mathbb{Z}^{d+1}}\rho_{l,j}\,\mathbf{e}_{l,j}\qquad\text{where}\qquad\rho_{l,j}=\big\langle\rho,\mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})}.$$

Next, we introduce for $s\in\mathbb{R}$ the complex Sobolev space $H^s(\mathbb{T}^{d+1},\mathbb{C})$ by

$$H^s(\mathbb{T}^{d+1},\mathbb{C})=\Big\{\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\|\rho\|_{H^s}^2:=\sum_{(l,j)\in\mathbb{Z}^{d+1}}\langle l,j\rangle^{2s}|\rho_{l,j}|^2<\infty\Big\},$$

where $\langle l,j\rangle:=\max(1,|l|,|j|)$ with $|\cdot|$ denoting either the ℓ^1 norm in $\mathbb{R}^d$ or the absolute value in $\mathbb{R}$. The real Sobolev spaces can be viewed as closed sub-spaces of the preceding one,

$$\begin{aligned}H^s=H^s(\mathbb{T}^{d+1},\mathbb{R})&=\Big\{\rho\in H^s(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\ \rho(\varphi,\theta)=\overline{\rho(\varphi,\theta)}\Big\}\\&=\Big\{\rho\in H^s(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\ \rho_{-l,-j}=\overline{\rho_{l,j}}\Big\}.\end{aligned}$$

We shall also make use of the following subspaces of H^s taking into account of some particular symmetries on odd and even functions,

$$\begin{aligned}H^s_{\text{even}}&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\ \rho(-\varphi,-\theta)=\rho(\varphi,\theta)\Big\}\\&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\ \rho_{-l,-j}=\rho_{l,j}\Big\}\end{aligned}$$

and

$$\begin{aligned}H^s_{\text{odd}}&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\ \rho(-\varphi,-\theta)=-\rho(\varphi,\theta)\Big\}\\&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\ \rho_{-l,-j}=-\rho_{l,j}\Big\}.\end{aligned}$$

For $N\in\mathbb{N}^*$, we define the cut-off frequency projectors on $H^s(\mathbb{T}^{d+1},\mathbb{C})$ as follows

$$\Pi_N\rho=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\\langle l,j\rangle\leqslant N}}\rho_{l,j}\mathbf{e}_{l,j}\qquad\text{and}\qquad\Pi_N^\perp=\mathrm{Id}-\Pi_N.\tag{4.4}$$

We shall also make use of the following mixed weighted Sobolev spaces.

$$\begin{aligned}W^{q,\infty,\gamma}(\mathcal{O},H^s)&=\Big\{\rho:\mathcal{O}\to H^s\quad\text{s.t.}\quad\|\rho\|_{q,s}^{\gamma,\mathcal{O}}<\infty\Big\},\\W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})&=\Big\{\rho:\mathcal{O}\to\mathbb{C}\quad\text{s.t.}\quad\|\rho\|_{q}^{\gamma,\mathcal{O}}<\infty\Big\},\end{aligned}$$

where $\mu\in\mathcal{O}\mapsto\rho(\mu)\in H^s$ and

$$\begin{aligned}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}&=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{|\alpha|}\sup_{\mu\in\mathcal{O}}\|\partial_\mu^\alpha\rho(\mu,\cdot)\|_{H^{s-|\alpha|}},\\\|\rho\|_{q}^{\gamma,\mathcal{O}}&=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{|\alpha|}\sup_{\mu\in\mathcal{O}}|\partial_\mu^\alpha\rho(\mu)|.\end{aligned}\tag{4.5}$$

Note that a function $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ can be written in the form

$$\rho(\mu,\varphi,\theta)=\sum_{(l,j)\in\mathbb{Z}^{d+1}}\rho_{l,j}(\mu)\mathbf{e}_{l,j}(\varphi,\theta).$$

Remark 4.2. • *From Sobolev embeddings, we obtain*

$$W^{q,\infty,\gamma}(\mathcal{O},H^s)\hookrightarrow C^{q-1}(\mathcal{O},H^s)\quad \textit{and}\quad W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C}).$$

• *The spaces* $\left(W^{q,\infty,\gamma}(\mathcal{O},H^s),\|\cdot\|_{q,s}^{\gamma,\mathcal{O}}\right)$ *and* $\left(W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C}),\|\cdot\|_{q}^{\gamma,\mathcal{O}}\right)$ *are complete.*

In the next lemma we collect some useful classical results dealing with various operations in weighted Sobolev spaces. The proofs are very close to those in [12, 13, 14], so we omit them.

Lemma 4.1. *Let* (γ,q,d,s_0,s) *satisfying* (4.1), *then the following assertions hold true.*

(i) Space translation invariance: Let $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$, *then for all* $\eta\in\mathbb{T}$, *the function* $(\varphi,\theta)\mapsto\rho(\varphi,\eta+\theta)$ *belongs to* $W^{q,\infty,\gamma}(\mathcal{O},H^s)$, *and satisfies*

$$\|\rho(\cdot,\eta+\cdot)\|_{q,s}^{\gamma,\mathcal{O}}=\|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

(ii) Projectors properties: Let $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$, *then for all* $N\in\mathbb{N}^*$ *and for all* $t\in\mathbb{R}_+^*$,

$$\|\Pi_N\rho\|_{q,s+t}^{\gamma,\mathcal{O}}\leqslant N^t\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\quad \textit{and}\quad \|\Pi_N^\perp\rho\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N^{-t}\|\rho\|_{q,s+t}^{\gamma,\mathcal{O}},$$

where the projectors are defined in (4.4).

(iii) Interpolation inequality: Let $q<s_1\leqslant s_3\leqslant s_2$ *and* $\overline{\theta}\in[0,1]$, *with* $s_3=\overline{\theta}s_1+(1-\overline{\theta})s_2$. *If* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_2})$, *then* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_3})$ *and*

$$\|\rho\|_{q,s_3}^{\gamma,\mathcal{O}}\lesssim\left(\|\rho\|_{q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|\rho\|_{q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iv) Law products:

(a) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$. *Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

(b) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$. *Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ *and*

$$\|\rho_1\rho_2\|_{q}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q}^{\gamma,\mathcal{O}}.$$

(c) Let $(\rho_1,\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\times W^{q,\infty,\gamma}(\mathcal{O},H^s)$. *Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(v) Composition law: Let $f\in C^\infty(\mathcal{O}\times\mathbb{R},\mathbb{R})$ *and* $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *such that*

$$\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}},\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C_0$$

for an arbitrary constant $C_0>0$ *and define the pointwise composition*

$$\forall(\mu,\varphi,\theta)\in\mathcal{O}\times\mathbb{T}^{d+1},\quad f(\rho)(\mu,\varphi,\theta):=f(\mu,\rho(\mu,\varphi,\theta)).$$

Then $f(\rho_1)-f(\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *with*

$$\|f(\rho_1)-f(\rho_2)\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C(s,d,q,f,C_0)\|\rho_1-\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(vi) Composition law 2: Let $f\in C^\infty(\mathbb{R},\mathbb{R})$ *with bounded derivatives. Let* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$. *Then*

$$\|f(\rho)-f(0)\|_{q}^{\gamma,\mathcal{O}}\leqslant C(q,d,f)\|\rho\|_{q}^{\gamma,\mathcal{O}}\left(1+\|\rho\|_{L^\infty(\mathcal{O})}^{q-1}\right).$$

The following technical lemma turns out to be very useful in the study of the linearized operator.

Lemma 4.2. *Let* (γ, q, d, s_0, s) *satisfy* (4.1) *and* $f \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$.
We consider the function $g : \mathcal{O} \times \mathbb{T}^d_\varphi \times \mathbb{T}_\theta \times \mathbb{T}_\eta \to \mathbb{C}$ *defined by*

$$g(\mu, \varphi, \theta, \eta) = \begin{cases} \frac{f(\mu,\varphi,\eta) - f(\mu,\varphi,\theta)}{\sin\left(\frac{\eta-\theta}{2}\right)} & \text{if } \theta \neq \eta \\ 2\partial_\theta f(\mu, \varphi, \theta) & \text{if } \theta = \eta. \end{cases}$$

Then

$$\forall k \in \mathbb{N}, \quad \|(\partial^k_\theta g)(*, \cdot, \centerdot, \eta + \centerdot)\|^{\gamma,\mathcal{O}}_{q,s} \lesssim \|\partial_\theta f\|^{\gamma,\mathcal{O}}_{q,s+k} \lesssim \|f\|^{\gamma,\mathcal{O}}_{q,s+k+1}.$$

Proof. Since the differentiation with respect to μ can be transported from g to f, then it is enough to check the result for $q = 0$ and therefore we shall remove the dependence in μ. We start with expanding f into its Fourier series,

$$f(\varphi, \theta) = \sum_{(l,j)\in\mathbb{Z}^{d+1}} f_{l,j} \mathbf{e}_{l,j}(\varphi, \theta).$$

Thus, one can write

$$\begin{aligned} g(\varphi, \theta, \eta) &= \sum_{(l,j)\in\mathbb{Z}^{d+1}} f_{l,j} \frac{e^{\mathrm{i}j\eta} - e^{\mathrm{i}j\theta}}{\sin\left(\frac{\eta-\theta}{2}\right)} e^{il\cdot\varphi} \\ &= 2\mathrm{i} \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1} \\ j\neq 0}} f_{l,j} e^{\mathrm{i}j\frac{\theta+\eta}{2}} \frac{\sin\left(j\frac{\eta-\theta}{2}\right)}{\sin\left(\frac{\eta-\theta}{2}\right)} e^{il\cdot\varphi}. \end{aligned}$$

We shall introduce the Chebychev polynomials of second kind $(U_n)_{n\in\mathbb{N}}$. They are defined for all $n \in \mathbb{N}$ by the following relation

$$\forall \theta \in \mathbb{R}, \quad \sin(\theta) U_n(\cos(\theta)) = \sin((n+1)\theta).$$

Using these polynomials, we obtain a new formulation for g, namely

$$g(\varphi, \theta, \eta) = 2\mathrm{i} \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1} \\ j\neq 0}} \frac{j f_{l,j}}{|j|} e^{\mathrm{i}j\frac{\theta+\eta}{2}} U_{|j|-1}\left(\cos\left(\frac{\theta-\eta}{2}\right)\right) e^{il\cdot\varphi}.$$

Differentiating in θ yields by Leibniz rule

$$\partial^k_\theta g(\varphi, \theta, \eta) = 2\mathrm{i} \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1} \\ j\neq 0}} \sum_{m=0}^{k} \binom{k}{m} \frac{j^{k+1-m} f_{l,j} \mathrm{i}^{k-m}}{|j| 2^{k-m}} e^{\mathrm{i}j\frac{\theta+\eta}{2}} \partial^m_\theta \left(U_{|j|-1}\left(\cos\left(\frac{\theta-\eta}{2}\right)\right)\right). \tag{4.6}$$

For all $j \in \mathbb{N}^*$, we consider the function f_j defined by

$$f_j(\theta) = U_{j-1}(\cos(\theta)) = \frac{\sin(j\theta)}{\sin(\theta)}.$$

Notice that f_j is even and 2π-periodic. Thus, we restrict its study to the interval $[0, \pi]$. Also remark that

$$f_j(\pi - \theta) = (-1)^j f_j(\theta).$$

Hence, we restrict the study to the interval $[0, \frac{\pi}{2}]$. We first consider the function f_j on the interval $[\frac{\pi}{6}, \frac{\pi}{2}]$. There, the function f_j writes as the quotient of two smooth functions with non vanishing denominator. Therefore, differentiating in θ leads to

$$\forall k \in \mathbb{N}, \quad \sup_{\theta\in[\frac{\pi}{6},\frac{\pi}{2}]} \left|\partial^k_\theta f_j(\theta)\right| \lesssim |j|^k.$$

Now we look at the behaviour close to 0 by looking at the function f_j restricted to $[0,\frac{\pi}{4}]$. Using Taylor Formula, we can write

$$\begin{aligned} f_j(\theta) &= \tfrac{\sin(j\theta)}{\theta}\times\tfrac{\theta}{\sin(\theta)} \\ &= j\int_0^1 \cos(tj\theta)dt\times\tfrac{\theta}{\sin(\theta)}. \end{aligned}$$

The function $\theta\mapsto\frac{\theta}{\sin(\theta)}$ being smooth on $[0,\frac{\pi}{4}]$, then differentiating in θ leads to

$$\forall k\in\mathbb{N},\qquad \sup_{\theta\in[0,\frac{\pi}{4}]}\left|\partial_\theta^k f_j(\theta)\right|\lesssim |j|^{k+1}.$$

Combining the previous estimates, one gets

$$\forall j\in\mathbb{N}^*,\quad \forall k\in\mathbb{N},\quad \sup_{\theta\in\mathbb{R}}\left|\partial_\theta^k\Big(U_{j-1}\left(\cos\left(\theta\right)\right)\Big)\right|\lesssim |j|^{k+1}. \tag{4.7}$$

Gathering (4.6) and (4.7), we deduce that

$$(\partial_\theta^k g)(\varphi,\theta,\eta+\theta)=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ j\neq 0}} c_{l,j,k}(\eta)\mathbf{e}_{l,j}(\varphi,\theta),$$

with

$$\sup_{\eta\in\mathbb{T}}|c_{l,j,k}(\eta)|\lesssim |j|^{k+1}|f_{l,j}|.$$

Therefore,

$$\begin{aligned} \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k g)(\cdot,\bullet,\eta+\bullet)\|_{H^s_{\varphi,\theta}}^2 &= \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ j\neq 0}}\langle l,j\rangle^{2s}\sup_{\eta\in\mathbb{T}}|c_{l,j,k}(\eta)|^2 \\ &\lesssim \sum_{(l,j)\in\mathbb{Z}^{d+1}}\langle l,j\rangle^{2s}|j|^{2k+2}|f_{l,j}|^2 \\ &\lesssim \|\partial_\theta f\|_{H^{s+k}}^2. \end{aligned}$$

This concludes the proof of Lemma 4.2. □

4.2 Operators

We shall focus in this section on some useful norms related to suitable operators class. These notions were used before in [4, 12, 13, 14]. We consider a smooth family of bounded operators on Sobolev spaces $H^s(\mathbb{T}^{d+1},\mathbb{C})$, that is a smooth map $T:\mu=(\lambda,\omega)\in\mathcal{O}\mapsto T(\mu)\in\mathcal{L}(H^s(\mathbb{T}^{d+1},\mathbb{C}))$ of linear continuous operators on Sobolev space $H^s(\mathbb{T}^{d+1},\mathbb{C})$, with $\mathcal{O}$ being an open bounded set of $\mathbb{R}^{d+1}$. Then we find it convenient to encode $T(\mu)$ in terms of the infinite dimensional matrix $\Big(T_{l_0,j_0}^{l,j}(\mu)\Big)_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2\\ (j,j_0)\in\mathbb{Z}^2}}$ with

$$T(\mu)\mathbf{e}_{l_0,j_0}=\sum_{(l,j)\in\mathbb{Z}^{d+1}}T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j}$$

and

$$T_{l_0,j_0}^{l,j}(\mu)=\big\langle T(\mu)\mathbf{e}_{l_0,j_0},\mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}. \tag{4.8}$$

Next, we need to fix a notation that we are implicitly using along the paper. For a given family of multi-parameter operators $T(\mu)$, it acts on $W^{q,\infty,\gamma}(\mathcal{O},H^s(\mathbb{T}^{d+1},\mathbb{C}))$ in the following sense,

$$\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^s(\mathbb{T}^{d+1},\mathbb{C})),\qquad (T\rho)(\mu,\varphi,\theta):=T(\mu)\rho(\mu,\varphi,\theta).$$

4.2.1 Toeplitz in time operators

In this short section we shall introduce a suitable class of Toeplitz operators.

Definition 4.1. *We say that an operator $T(\mu)$ is Toeplitz in time (actually in the variable φ) if its Fourier coefficients defined by* (4.8)*, satisfy*

$$\forall\, l_o, l, j, j_0 \in \mathbb{Z}, \quad T_{l_0,j_0}^{l,j}(\mu) = T_{0,j_0}^{l-l_0,j}(\mu).$$

Or equivalently

$$T_{l_0,j_0}^{l,j}(\mu) = T_{j_0}^{j}(\mu, l-l_0),$$

with $T_{j_0}^{j}(\mu,l) := T_{0,j_0}^{l,j}(\mu)$.

The action of a Toeplitz operator $T(\mu)$ on a function $\rho = \displaystyle\sum_{(l_0,j_0)\in\mathbb{Z}^{d+1}} \rho_{l_0,j_0}\mathbf{e}_{l_0,j_0}$ is then given by

$$T(\mu)\rho = \sum_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2\\ (j,j_0)\in\mathbb{Z}^2}} T_{j_0}^{j}(\mu, l-l_0)\rho_{l_0,j_0}\mathbf{e}_{l,j}. \tag{4.9}$$

In this paper, we will encounter several operators acting only on the variable θ and that can be considered as φ-dependent operators $T(\mu,\varphi)$ taking the form

$$T(\mu,\varphi)\rho(\varphi,\theta) = \int_{\mathbb{T}} K(\mu,\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta.$$

One can easily check that those operators are Toeplitz and therefore they satisfy (4.9).
For $q\in\mathbb{N}$ and $s\in\mathbb{R}$, we can equip Toeplitz operators with the off-diagonal norm given by,

$$\|T\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} = \sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\ |\alpha|\leqslant q}} \gamma^{|\alpha|} \sup_{\mu\in\mathcal{O}} \|\partial_\mu^\alpha(T)(\mu)\|_{\textnormal{O-d},s-|\alpha|}, \tag{4.10}$$

where

$$\|T\|_{\textnormal{O-d},s}^2 = \sum_{(l,m)\in\mathbb{Z}^{d+1}} \langle l,m\rangle^{2s} \sup_{j-k=m} |T_j^k(l)|^2.$$

This norm will be of important use later during the KAM reduction of the remainder. The cut-off projectors $(P_N)_{N\in\mathbb{N}^*}$ are defined as follows:

$$(P_N T(\mu))\,\mathbf{e}_{l_0,j_0} = \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ |l-l_0|,|j-j_0|\leqslant N}} T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j} \quad \text{and} \quad P_N^\perp T = T - P_N T. \tag{4.11}$$

In the next lemma we shall gather classical results whose proofs are very close to those in [14] concerning pseudo-differential operators. We recall that the weighted norms on functions that will be used below are defined in (4.5).

Lemma 4.3. *Let (γ,q,d,s_0,s) satisfying* (4.1)*. Let T, T_1 and T_2 be Toeplitz in time operators.*

(i) Projectors properties: Let $N\in\mathbb{N}^*$. Let $t\in\mathbb{R}_+$. Then

$$\|P_N T\rho\|_{\textnormal{O-d},q,s+t}^{\gamma,\mathcal{O}} \leqslant N^t \|T\rho\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \quad \text{and} \quad \|P_N^\perp T\rho\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant N^{-t}\|T\rho\|_{\textnormal{O-d},q,s+t}^{\gamma,\mathcal{O}}.$$

(ii) Interpolation inequality: Let $q<s_1\leqslant s_3\leqslant s_2$, $\overline{\theta}\in[0,1]$ with $s_3=\overline{\theta}s_1+(1-\overline{\theta})s_2$. Then

$$\|T\|_{\textnormal{O-d},q,s_3}^{\gamma,\mathcal{O}} \lesssim \left(\|T\|_{\textnormal{O-d},q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}} \left(\|T\|_{\textnormal{O-d},q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iii) Composition law:

$$\|T_1T_2\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \|T_1\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|T_2\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|T_1\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|T_2\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}.$$

(iv) Link between operators and off-diagonal norms:

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|T\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|T\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In particular

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|T\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

4.2.2 Reversible and reversibility preserving operators

In this section we intend to collect some definitions and properties related to different reversibility notions for operators and give practical characterizations. We shall also come back to Toeplitz operators defined before in Section 4.2.1 and discuss two important examples frequently encountered during this paper and given by multiplications and integral operators.
First, we give the following definitions following [5, Def. 2.2].

Definition 4.2. *Introduce the following involution*

$$(\mathscr{S}_2\rho)(\varphi,\theta)=\rho(-\varphi,-\theta). \tag{4.12}$$

We say that an operator $T(\mu)$ is

- *real if for all $\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})$, we have*
$$\overline{\rho}=\rho \quad\Longrightarrow\quad \overline{T\rho}=T\rho.$$
- *reversible if*
$$T(\mu)\circ\mathscr{S}_2=-\mathscr{S}_2\circ T(\mu).$$
- *reversibility preserving if*
$$T(\mu)\circ\mathscr{S}_2=\mathscr{S}_2\circ T(\mu).$$

We now detail the following characterizations needed at several places in this paper and the proofs are quite easy and follow from Fourier expansion. One can find a similar result in [5, Lem. 2.6].

Proposition 4.1. *Let T be an operator. Then T is*

- *real if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=\overline{T_{l_0,j_0}^{l,j}}.$$
- *reversible if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=-T_{l_0,j_0}^{l,j}.$$
- *reversibility-preserving if and only if*
$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=T_{l_0,j_0}^{l,j}.$$

In what follows, we shall focus on two particular cases of operators which will be of constant use throughout this paper. Namely, multiplication and integral operators.

Definition 4.3. *Let T be an operator as in Section 4.2. We say that*

- *T is a multiplication operator if there exists a function $M:(\mu,\varphi,\theta)\mapsto M(\mu,\varphi,\theta)$ such that*
$$(T\rho)(\mu,\varphi,\theta)=M(\mu,\varphi,\theta)\rho(\mu,\varphi,\theta).$$
- *T is an integral operator if there exists a function (called the kernel) $K:(\mu,\varphi,\theta,\eta)\mapsto K(\mu,\varphi,\theta,\eta)$ such that*
$$(T\rho)(\mu,\varphi,\theta)=\int_{\mathbb{T}}\rho(\mu,\varphi,\eta)K(\mu,\varphi,\theta,\eta)d\eta.$$

We intend to prove the following lemma.

Lemma 4.4. *Let (γ,q,d,s_0,s) satisfy (4.1), then the following assertions hold true.*

(i) Let T be a multiplication operator by a real-valued function M, then

- *If* $M(\mu,-\varphi,-\theta)=M(\mu,\varphi,\theta)$*, then* T *is real and reversibility preserving Toeplitz in time and space operator.*
- *If* $M(\mu,-\varphi,-\theta)=-M(\mu,\varphi,\theta)$*, then* T *is real and reversible Toeplitz in time and space operator.*

Moreover,

$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|M\|_{q,s+s_0}^{\gamma,\mathcal{O}}.$$

(ii) Let T *be an integral operator with a real-valued kernel* K*.*

- *If* $K(\mu,-\varphi,-\theta,-\eta)=K(\mu,\varphi,\theta,\eta)$*, then* T *is a real and reversibility preserving Toeplitz in time operator.*
- *If* $K(\mu,-\varphi,-\theta,-\eta)=-K(\mu,\varphi,\theta,\eta)$*, then* T *is a real and reversible Toeplitz in time operator.*

In addition,

$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta$$

and

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}d\eta+\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}d\eta,$$

where the notation $*,\cdot,\bullet$ *denote* μ,φ,θ*, respectively.*

Proof. We point out that the proofs will be implemented for the particular case $q=0$ and the general case can be done similarly by differentiating with respect to μ and using Leibniz rule.
(i) Since M is a real-valued function, then we get by the definition

$$\begin{aligned}\overline{T_{-l_0,-j_0}^{-l,-j}}&=\int_{\mathbb{T}^{d+1}}M(\varphi,\theta)\overline{\mathbf{e}_{-l_0,-j_0}(\varphi,\theta)\mathbf{e}_{l,j}(\varphi,\theta)}d\varphi d\theta\\&=\int_{\mathbb{T}^{d+1}}M(\varphi,\theta)\mathbf{e}_{l_0,j_0}(\varphi,\theta)\mathbf{e}_{-l,-j}(\varphi,\theta)d\varphi d\theta=T_{l_0,j_0}^{l,j}.\end{aligned}$$

This shows in view of Proposition 4.1 that the operator T is a real. It remains to check the reversibility preserving property. We write from the definition

$$\begin{aligned}T(\mathscr{S}_2\rho)(\varphi,\theta)&=M(\varphi,\theta)\rho(-\varphi,-\theta)\\&=M(-\varphi,-\theta)\rho(-\varphi,-\theta)\\&=\mathscr{S}_2\left(T\rho\right)(\varphi,\theta).\end{aligned}$$

This gives the desired result. As to the reversible Toepliz structure, it can be checked in a similar way. To achieve the proof of the first point it remains to establish the suitable estimate. Using a duality argument $H^{s+s_0}-H^{-s-s_0}$, we may write,

$$|T_j^{j'}(l)|=\left|\int_{\mathbb{T}^{d+1}}M(\varphi,\theta)\mathbf{e}_{l,j-j'}(\varphi,\theta)d\varphi d\theta\right|\lesssim\langle l,j-j'\rangle^{-s-s_0}\|M\|_{H^{s+s_0}}.$$

It follows that

$$\begin{aligned}\|T\|_{\text{O-d},s}^2&=\sum_{(l,m)\in\mathbb{Z}^{d+1}}\langle l,m\rangle^{2s}\sup_{j-j'=m}|T_j^{j'}(l)|^2\\&\lesssim\|M\|_{H^{s+s_0}}^2\sum_{(l,m)\in\mathbb{Z}^{d+1}}\langle l,m\rangle^{2s}\langle l,m\rangle^{-2s-2s_0}\\&\lesssim\|M\|_{H^{s+s_0}}^2.\end{aligned}$$

Therefore we find

$$\|T\|_{\text{O-d},s}\lesssim\|M\|_{H^{s+s_0}}.$$

(ii) By assumption, K is real and thus

$$\begin{aligned}\overline{T_{-l_0,-j_0}^{-l,-j}} &= \int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)\overline{\mathbf{e}_{-l_0,-j_0}(\varphi,\eta)\mathbf{e}_{l,j}(\varphi,\theta)}d\varphi d\theta \\ &= \int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)\mathbf{e}_{l_0,j_0}(\varphi,\eta)\mathbf{e}_{-l,-j}(\varphi,\theta)d\varphi d\theta d\eta = T_{l_0,j_0}^{l,j}.\end{aligned}$$

This implies, according to Proposition 4.1, that T is a real operator. Now we shall check the reversibility preserving. The reversibility can be checked in a similar way. By the change of variables $\eta \mapsto -\eta$, we may write,

$$\begin{aligned}T(\mathscr{S}_2\rho)(\varphi,\theta) &= \int_{\mathbb{T}} K(\varphi,\theta,\eta)\rho(-\varphi,-\eta)d\eta \\ &= \int_{\mathbb{T}} K(-\varphi,-\theta,-\eta)\rho(-\varphi,-\eta)d\eta \\ &= \int_{\mathbb{T}} K(-\varphi,-\theta,\eta)\rho(-\varphi,\eta)d\eta = \mathscr{S}_2\left(T\rho\right)(\varphi,\theta).\end{aligned}$$

From Fubini's theorem and the duality $H^{s+s_0}_{\varphi,\theta} - H^{-s-s_0}_{\varphi,\theta}$, we infer,

$$\begin{aligned}|T_j^{j'}(l)| &= \left|\int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)e^{\mathrm{i}(l\cdot\varphi+j\theta-j'\eta)}d\varphi d\theta d\eta\right| \\ &= \left|\int_{\mathbb{T}^{d+1}} e^{\mathrm{i}(l\cdot\varphi+(j-j')\theta)}\left(\int_{\mathbb{T}} K(\varphi,\theta,\eta+\theta)e^{-\mathrm{i}j'\eta}d\eta\right)d\varphi d\theta\right| \\ &\lesssim \langle l, j-j'\rangle^{-s-s_0}\int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{H^{s+s_0}_{\varphi,\theta}}d\eta.\end{aligned}$$

Hence, we deduce that

$$\|T\|_{\text{O-d},s} \lesssim \int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{H^{s+s_0}_{\varphi,\theta}}d\eta.$$

The last estimate in Lemma 4.4 can be obtained from the expression

$$(T\rho)(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\theta+\eta)K(\varphi,\theta,\theta+\eta)d\eta,$$

combined with the law products and the translation invariance in Lemma 4.1-(i)-(iv).
This concludes the proof of Lemma 4.4. □

5 Hamiltonian toolkit and approximate inverse

In this section, we shall reformulate the problem into the form of searching for zeros of a functional $\mathcal{F}$. We first rescale the equation by introducing a small parameter ε. This allows us to see the Hamiltonian equation (2.17) as a perturbation of the equilibirum one (3.10). The latter being integrable and admitting quasi-periodic solutions in view of Lemma 3.2-2 and Lemma 3.1, we can hope using KAM technics to find quasi-periodic solutions to the first one. This approach has been intensively used before in [4, 6, 12, 13, 14]. According to Proposition 2.1, it seems more convenient to work with the phase space

$$H^s_0 = H^s_0(\mathbb{T},\mathbb{R}) = \left\{ r(\theta) = \sum_{j\in\mathbb{Z}^*} r_j e^{ij\theta} \quad \text{s.t.} \quad r_{-j} = \overline{r_j} \quad \text{and} \quad \|r\|_s^2 = \sum_{j\in\mathbb{Z}^*} |r_j|^2|j|^{2s} < \infty \right\}.$$

Therefore, we select finitely-many tangential sites $\mathbb{S}$ and decompose the phase space into tangential and normal subspaces described by the selection of Fourier modes belonging to $\mathbb{S}$ or not. On the tangential part, containing the main part of the quasi-periodic solutions, we introduce action-angle variables allowing to reformulate the problem in terms of embedded invariant tori. We shall also be

concerned with some regularity aspects for the perturbed Hamiltonian vector field appearing in $\mathcal{F}$ and needed during the Nash-Moser scheme. Finally, we construct an approximate right inverse for the linearized operator associated to $\mathcal{F}$.
The symplectic structure on $L^2_0(\mathbb{T})$ (corresponding to the subspace of $L^2(\mathbb{T})$ of real functions with zero average) induced by (2.17) is given by the symplectic 2-form

$$\mathcal{W}(r,h)=\int_{\mathbb{T}}\partial_\theta^{-1}r(\theta)h(\theta)d\theta \quad \text{with} \quad \partial_\theta^{-1}r(\theta)=\sum_{j\in\mathbb{Z}^*}\frac{r_j}{\mathrm{i}j}e^{\mathrm{i}j\theta}. \tag{5.1}$$

Then for a given function H, its symplectic gradient X_H is defined through the identity

$$dH(r)[\cdot]=\mathcal{W}(X_H(r),\cdot).$$

Using the Fourier expansion

$$r(\theta)=\sum_{j\in\mathbb{Z}^*}r_je^{\mathrm{i}j\theta} \quad \text{with} \quad r_{-j}=\overline{r_j},$$

we easily find that the symplectic form $\mathcal{W}$ writes

$$\mathcal{W}(r,h)=\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}r_jh_{-j}=\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}r_j\overline{h_j},$$

that is

$$\mathcal{W}=\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}dr_j\wedge dr_{-j}=2\sum_{j\in\mathbb{N}^*}\frac{1}{\mathrm{i}j}dr_j\wedge dr_{-j}. \tag{5.2}$$

Next, with the result of Lemma 3.2 we can easily check that the equation (2.17) can be written in the form

$$\partial_t r=\partial_\theta\mathrm{L}(\lambda)(r)+X_P(r),$$

where X_P is the Hamiltonian vector field defined by

$$X_P(r)=I_1(\lambda)K_1(\lambda)\partial_\theta r-\partial_\theta\mathcal{K}_\lambda * r-F_\lambda[r]. \tag{5.3}$$

Remind that $F_\lambda[r]$ is introduced in (2.10) and the convolution kernel is stated in (3.12). To measure the smallness condition it seems to be more convenient to introduce a small parameter ε and rescale the Hamiltonian as done for instance in the papers [4, 14]. To do that we rescale the solution as follows $r\mapsto\varepsilon r$ with r bounded. Therefore the Hamiltonian equation takes the form

$$\partial_t r=\partial_\theta\mathrm{L}(\lambda)(r)+\varepsilon X_{P_\varepsilon}(r), \tag{5.4}$$

where X_{P_ε} is the rescaled Hamiltonian vector field defined by $X_{P_\varepsilon}(r):=\varepsilon^{-2}X_P(\varepsilon r)$. Notice that (5.4) can be recast in the Hamiltonian form

$$\partial_t r=\partial_\theta\nabla\mathcal{H}_\varepsilon(r), \tag{5.5}$$

where the rescaled Hamiltonian $\mathcal{H}_\varepsilon(r)$ is given by

$$\begin{aligned}\mathcal{H}_\varepsilon(r)&=\varepsilon^{-2}H(\varepsilon r)\\&:=H_{\mathrm{L}}(r)+\varepsilon P_\varepsilon(r),\end{aligned} \tag{5.6}$$

with H_{L} being the quadratic Hamiltonian defined in Lemma 3.2 and $\varepsilon P_\varepsilon(r)$ is composed with terms of higher order more than cubic.

5.1 Action-angle reformulation

Let us consider finitely many Fourier-frenquencies, called tangential sites, gathered in the tangantial set $\mathbb{S}$ defined by

$$\mathbb{S}=\{j_1,\ldots,j_d\}\subset\mathbb{N}^*\quad\text{with}\quad 1\leqslant j_1<j_2<\ldots<j_d.$$

We now define the symmetrized tangential sets $\overline{\mathbb{S}}$ and $\mathbb{S}_0$ by

$$\overline{\mathbb{S}}=\mathbb{S}\cup(-\mathbb{S})=\{\pm j,\ j\in\mathbb{S}\}\quad\text{and}\quad\mathbb{S}_0=\overline{\mathbb{S}}\cup\{0\}.\tag{5.7}$$

Recall from (3.32) that we denote the unperturbed tangential frequency vector by

$$\omega_{\mathrm{Eq}}(\lambda)=(\Omega_j(\lambda))_{j\in\mathbb{S}},\tag{5.8}$$

where $\Omega_j(\lambda)$ are given by (3.14). For $s\in\mathbb{R}$, we decompose the phase space of $H_0^s(\mathbb{T})$ as the direct sum

$$\begin{aligned}H_0^s(\mathbb{T})=&H_{\overline{\mathbb{S}}}\overset{\perp}{\oplus}H_\perp^s,\\ H_{\overline{\mathbb{S}}}=\Big\{v=\sum_{j\in\overline{\mathbb{S}}}r_je^{\mathrm{i}j\theta},\ \overline{r_j}=r_{-j}\Big\},\quad &H_\perp^s=\Big\{z=\sum_{j\in\mathbb{Z}\setminus\mathbb{S}_0}z_je_j\in H^s,\ \overline{z_j}=z_{-j}\Big\},\end{aligned}\tag{5.9}$$

where $e_j(\theta)=e^{\mathrm{i}j\theta}$. We denote by $\Pi_{\overline{\mathbb{S}}},\Pi_{\mathbb{S}_0}^{\perp}$ the corresponding orthogonal projectors defined by

$$r=v+z,\quad v:=\Pi_{\overline{\mathbb{S}}}r:=\sum_{j\in\overline{\mathbb{S}}}r_je_j,\quad z:=\Pi_{\mathbb{S}_0}^{\perp}r:=\sum_{j\in\mathbb{Z}\setminus\mathbb{S}_0}r_je_j\,,\tag{5.10}$$

where v and z are called the tangential and normal variables, respectively. Fix some small amplitudes $(\mathtt{a}_j)_{j\in\mathbb{S}}\in(\mathbb{R}_+^*)^d$ and set $\mathtt{a}_{-j}=\mathtt{a}_j$. We shall now introduce the action-angle variables on the tangential set $H_{\overline{\mathbb{S}}}$ by making the following symplectic polar change of coordinates

$$\forall\, j\in\overline{\mathbb{S}},\quad r_j=\sqrt{\mathtt{a}_j^2+\tfrac{|j|}{2}I_j}\,e^{\mathrm{i}\vartheta_j},\tag{5.11}$$

where

$$\forall\, j\in\overline{\mathbb{S}},\quad I_{-j}=I_j\in\mathbb{R}\quad\text{and}\quad\vartheta_{-j}=-\vartheta_j\in\mathbb{R}.\tag{5.12}$$

Thus, any function of the phase space H_0^s decomposes as

$$r=A(\vartheta,I,z):=v(\vartheta,I)+z\quad\text{where}\quad v(\vartheta,I):=\sum_{j\in\overline{\mathbb{S}}}\sqrt{\mathtt{a}_j^2+\tfrac{|j|}{2}I_j}\,e^{\mathrm{i}\vartheta_j}e_j\,.\tag{5.13}$$

In these coordinates the solutions (3.48) of the linear system (3.10) simply read as $v(-\omega_{\mathrm{Eq}}(\lambda)t,I)$ where ω_{Eq} is defined in (5.8) and $I\in\mathbb{R}^d$ such that the quantity under the square root is positive. The involution $\mathscr{S}$ defined in (2.26) now reads in the new variables

$$\mathfrak{S}:(\vartheta,I,z)\mapsto(-\vartheta,I,\mathscr{S}z)\tag{5.14}$$

and the symplectic 2-form in (5.2) becomes after straightforward computations using (5.11) and (5.12)

$$\mathcal{W}=\sum_{j\in\mathbb{S}}d\vartheta_j\wedge dI_j+\sum_{j\in\mathbb{Z}\setminus\mathbb{S}_0}\frac{1}{\mathrm{i}j}dr_j\wedge dr_{-j}=\Big(\sum_{j\in\mathbb{S}}d\vartheta_j\wedge dI_j\Big)\oplus\mathcal{W}_{|H_\perp^s},\tag{5.15}$$

where $\mathcal{W}_{|H_\perp^s}$ denotes the restriction of $\mathcal{W}$ to $H_\perp^s$. Note that $\mathcal{W}$ is an exact 2-form as

$$\mathcal{W}=d\Lambda,$$

where Λ is the Liouville 1-form defined by

$$\Lambda_{(\vartheta,I,z)}[\widehat{\vartheta},\widehat{I},\widehat{z}]=-\sum_{j\in\mathbb{S}}I_j\widehat{\vartheta}_j+\tfrac{1}{2}\langle\partial_\theta^{-1}z,\widehat{z}\rangle_{L^2(\mathbb{T})}\,.\tag{5.16}$$

The next goal is to study the Hamiltonian system generated by the Hamiltonian $\mathcal{H}_\varepsilon$ in (5.6), in the action-angle and normal coordinates $(\vartheta, I, z) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times H^s_\perp$. We consider the Hamiltonian H_ε defined by

$$H_\varepsilon = \mathcal{H}_\varepsilon \circ A, \tag{5.17}$$

where A is the map described before in (5.13). Since $\mathrm{L}(\lambda)$ in (3.15) is a Fourier multiplier keeping invariant the subspaces $H_{\mathbb{S}_0}$ and $H^s_\perp$, then the quadratic Hamiltonian H_{L} in (3.15) in the variables (ϑ, I, z) reads, up to an additive constant which can be removed since it does not change the dynamics in view of (2.17),

$$H_{\mathrm{L}} \circ A = -\sum_{j\in\mathbb{S}} \Omega_j(\lambda) I_j + \tfrac{1}{2}\langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})} = -\omega_{\mathrm{Eq}}(\lambda)\cdot I + \tfrac{1}{2}\langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})}, \tag{5.18}$$

where $\omega_{\mathrm{Eq}} \in \mathbb{R}^d$ is the unperturbed tangential frequency vector defined by (3.14). According to (5.6) and (5.18), one deduces that the Hamiltonian H_ε in (5.17) has the form

$$H_\varepsilon = \mathcal{N} + \varepsilon \mathcal{P}_\varepsilon \quad \text{with} \quad \mathcal{N} := -\omega_{\mathrm{Eq}}(\lambda)\cdot I + \tfrac{1}{2}\langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})} \quad \text{and} \quad \mathcal{P}_\varepsilon := P_\varepsilon \circ A. \tag{5.19}$$

We look for an embedded invariant torus

$$\begin{aligned} i:\ \mathbb{T}^d &\to \mathbb{R}^d \times \mathbb{R}^d \times H^s_\perp \\ \varphi &\mapsto i(\varphi) := (\vartheta(\varphi), I(\varphi), z(\varphi)) \end{aligned} \tag{5.20}$$

of the Hamiltonian vector field

$$X_{H_\varepsilon} := (\partial_I H_\varepsilon, -\partial_\vartheta H_\varepsilon, \Pi^\perp_{\mathbb{S}_0}\partial_\theta \nabla_z H_\varepsilon) \tag{5.21}$$

filled by quasi-periodic solutions with Diophantine frequency vector ω. Remark that for the value $\varepsilon = 0$, the Hamiltonian system reduces to the linear equation

$$\omega\cdot\partial_\varphi i(\varphi) = X_{H_0}(i(\varphi))$$

which admits the trivial solution given by the flat torus $i_{\mathrm{flat}}(\varphi) = (\varphi, 0, 0)$ provided that $\omega = -\omega_{\mathrm{Eq}}(\lambda)$. In what follows we shall consider the modified Hamiltonian equation indexed with a parameter $\alpha \in \mathbb{R}^d$,

$$H^\alpha_\varepsilon := \mathcal{N}_\alpha + \varepsilon\mathcal{P}_\varepsilon \quad \text{where} \quad \mathcal{N}_\alpha := \alpha\cdot I + \frac{1}{2}\langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})}. \tag{5.22}$$

For the value $\alpha = -\omega_{\mathrm{Eq}}(\lambda)$ we have $H^\alpha_\varepsilon = H_\varepsilon$. The parameter α will play the role of a Lagrangian multiplier in order to satisfy a compatibility condition during the approximate inverse process. Notice that the initial problem reduces after this multiple transformations to find zeros of the nonlinear operator

$$\begin{aligned} \mathcal{F}(i,\alpha,\mu,\varepsilon) &:= \omega\cdot\partial_\varphi i(\varphi) - X_{H^\alpha_\varepsilon}(i(\varphi)) \\ &= \begin{pmatrix} \omega\cdot\partial_\varphi\vartheta(\varphi) - \alpha - \varepsilon\partial_I\mathcal{P}_\varepsilon(i(\varphi)) \\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\theta\mathcal{P}_\varepsilon(i(\varphi)) \\ \omega\cdot\partial_\varphi z(\varphi) - \partial_\theta\big[\mathrm{L}(\lambda) z(\varphi) + \varepsilon\nabla_z\mathcal{P}_\varepsilon\big(i(\varphi)\big)\big] \end{pmatrix}, \ \mu = (\lambda,\omega), \end{aligned} \tag{5.23}$$

where $\mathcal{P}_\varepsilon$ is defined in (5.6). We point out that we can easily check that the Hamiltonian H^α_ε is reversible in the sense of the Definition 4.2, that is,

$$H^\alpha_\varepsilon \circ \mathfrak{S} = H^\alpha_\varepsilon, \tag{5.24}$$

where the involution $\mathfrak{S}$ is defined in (5.14). Thus, we shall look for reversible solutions of

$$\mathcal{F}(i,\alpha,\mu,\varepsilon) = 0,$$

that is, solutions satisfying

$$\mathfrak{S} i(\varphi) = i(-\varphi),$$

or equivalently,

$$\vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathscr{S}z)(\varphi). \tag{5.25}$$

We define the periodic component $\mathfrak{I}$ of the torus i by

$$\mathfrak{I}(\varphi) := i(\varphi) - (\varphi, 0, 0) = (\Theta(\varphi), I(\varphi), z(\varphi)) \quad \text{with} \quad \Theta(\varphi) = \vartheta(\varphi) - \varphi.$$

We define the weighted Sobolev norm of $\mathfrak{I}$ as

$$\|\mathfrak{I}\|^{\gamma,\mathcal{O}}_{q,s} := \|\Theta\|^{\gamma,\mathcal{O}}_{q,s} + \|I\|^{\gamma,\mathcal{O}}_{q,s} + \|z\|^{\gamma,\mathcal{O}}_{q,s}.$$

5.2 Hamiltonian regularity

This section is devoted to some regularity aspects of the Hamiltonian vector field introduced in (5.3), together with the rescaled one associated to the Hamiltonian described in (5.19). The first main result reads as follows.

Lemma 5.1. *Let (γ, q, s_0, s) satisfying (4.1). There exists $\varepsilon_0 \in (0,1)$ such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0,$$

then the vector field X_P defined in (5.3) satisfies the following estimates

(i) $\|X_P(r)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}$.

(ii) $\|d_r X_P(r)[\rho]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

Proof. **(i)** According to (3.16), the Fourier coefficients of $\partial_\theta \mathcal{K}_\lambda$ are $(\mathrm{i}jI_j(\lambda)K_j(\lambda))_{j\in\mathbb{Z}}$. Hence

$$\|\partial_\theta \mathcal{K}_\lambda * r\|_{H^s}^2 = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}} \langle l,j\rangle^{2s} j^2 I_{|j|}^2(\lambda)K_{|j|}^2(\lambda)|r_{l,j}|^2 \leqslant \tfrac{1}{4}\|r\|_{H^s}^2.$$

Notice that the last inequality is obtained by the decay property of the product I_jK_j on $\mathbb{R}_+^*$, (A.3) and (A.10). Thus we deduce that

$$\|\partial_\theta \mathcal{K}_\lambda * r\|_{H^s} \leqslant \tfrac{1}{2}\|r\|_{H^s} \leqslant \|r\|_{H^s}.$$

Now we claim that

$$\|\partial_\theta \mathcal{K}_\lambda * r\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s}^{\gamma,\mathcal{O}}. \tag{5.26}$$

Indeed, from (3.16), we infer that

$$\partial_\theta \mathcal{K}_\lambda * r = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}} \mathrm{i}jI_j(\lambda)K_j(\lambda)r_{l,j}(\lambda,\omega)\mathbf{e}_{l,j}. \tag{5.27}$$

At this stage we need to explore the regularity of the multiplier with respect to λ. By using (A.8), we write

$$I_j(\lambda)K_j(\lambda) = \tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} K_0\left(2\lambda\cos(\tau)\right)\cos(2j\tau)d\tau.$$

From (A.6), we have the decomposition

$$K_0(z) = -\log(z/2)I_0(z) + f(z), \tag{5.28}$$

with I_0 being the modified Bessel function of the first kind and f an analytic function. By the morphism property of the logarithm, we get

$$\begin{aligned}
I_j(\lambda)K_j(\lambda) = & -\log(\lambda)\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} I_0(2\lambda\cos(\tau))\cos(2j\tau)d\tau \\
& -\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} \log(\cos(\tau))\cos(2j\tau)d\tau \\
& -\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} \log(\cos(\tau))\left(I_0(2\lambda\cos(\tau))-1\right)\cos(2j\tau)d\tau \\
& +\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} f(2\lambda\cos(\tau))\cos(2j\tau)d\tau \\
:= & \ \mathcal{I}_{1,j}(\lambda) + \mathcal{I}_{2,j} + \mathcal{I}_{3,j}(\lambda) + \mathcal{I}_{4,j}(\lambda).
\end{aligned}$$

Since I_0 and f are analytic, then the above expressions are smooth with respect to the parameter $\lambda\in(\lambda_0,\lambda_1)\subset\mathbb{R}_+^*$. An integration by parts in $\mathcal{I}_{1,j}(\lambda)$ and $\mathcal{I}_{4,j}(\lambda)$ yields

$$\forall\, i\in\{1,4\},\quad \sup_{j\in\mathbb{Z}}\left(|j|\max_{n\in[\![0,q]\!]}\|\partial_\lambda^{(n)}\mathcal{I}_{i,j}\|_{L^\infty([\lambda_0,\lambda_1])}\right)\lesssim 1.$$

Looking at the definition of I_0 in (A.2), we see that we have uniformly in $\lambda\in[\lambda_0,\lambda_1]$,

$$\forall\, n\in[\![0,q]\!],\quad \partial_\lambda^{(n)}\left(I_0\left(2\lambda\cos(\tau)\right)-1\right)=O\left(\cos(\tau)\right).$$

Hence, an integration by parts in $\mathcal{I}_{3,j}(\lambda)$ yields

$$\sup_{j\in\mathbb{Z}}\left(|j|\max_{n\in[\![0,q]\!]}\|\partial_\lambda^{(n)}\mathcal{I}_{3,j}\|_{L^\infty([\lambda_0,\lambda_1])}\right)\lesssim 1.$$

It remains to study the integral $\mathcal{I}_{2,j}$. One can easily check from the above decomposition that

$$\mathcal{I}_{2,j}=\lim_{\lambda\to 0^+}I_j(\lambda)K_j(\lambda).$$

Using (A.10), we then find

$$\mathcal{I}_{2,j}=\frac{1}{2j}.$$

Putting together the preceding estimates, we obtain

$$\sup_{j\in\mathbb{Z}}\left(|j|\max_{n\in[\![0,q]\!]}\|\partial_\lambda^{(n)}\left(I_jK_j\right)\|_{L^\infty([\lambda_0,\lambda_1])}\right)\lesssim 1.$$

Then coming back to (5.27) and using Leibniz formula, we obtain (5.26). On the other hand, applying Lemma 4.1-(iv)-(c) we get

$$\|(I_1K_1)\,\partial_\theta r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.29}$$

Next we shall move to the estimate of $F_\lambda[r]$ defined in (2.10). According to (3.4) we may write

$$\begin{aligned}A_r(\varphi,\theta,\eta)&=2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\left(\left(\frac{R(\varphi,\eta)-R(\varphi,\theta)}{2\sin\left(\frac{\eta-\theta}{2}\right)}\right)^2+R(\varphi,\eta)R(\varphi,\theta)\right)^{\frac12}\\&:=2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|v_{r,1}(\varphi,\theta,\eta).\end{aligned}\tag{5.30}$$

Notice that $v_{r,1}$ is smooth when r is smooth and small enough, and $v_{0,1}=1$. More precisely, Lemma 4.1-(v) combined with Lemma 4.2 allow to get

$$\begin{aligned}&\sup_{\eta\in\mathbb{T}}\|v_{r,1}(*,\cdot,\centerdot,\eta+\centerdot)-1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1}^{\gamma,\mathcal{O}},\\\forall k\in\mathbb{N}^*,\ &\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k v_{r,1})(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}.\end{aligned}\tag{5.31}$$

Here and in the sequel, the symbols $*,\cdot,\centerdot$ denote the variables $\mu=(\lambda,\omega),\varphi,\theta$, respectively. Then from the identity (5.28) we infer

$$\begin{aligned}K_0(\lambda A_r(\varphi,\theta,\eta))=K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\log\left(\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\left[I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)-I_0(\lambda A_r(\varphi,\theta,\eta))\right]\\-\log\left(v_{r,1}(\varphi,\theta,\eta)\right)I_0(\lambda A_r(\varphi,\theta,\eta))+f(\lambda A_r(\varphi,\theta,\eta))-f\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right).\end{aligned}\tag{5.32}$$

By virtue of the expansion (A.2), we can write

$$I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)-I_0(\lambda A_r(\varphi,\theta,\eta))=\sin^2\left(\tfrac{\eta-\theta}{2}\right)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta),$$

with $\mathscr{K}^1_{r,1}$ being smooth and vanishing at $r=0$. More precisely, we have the expansion

$$\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta)=\sum_{m=1}^{\infty}\tfrac{(2\lambda)^{2m}}{(m!)^2}\sin^{2m-2}\left(\tfrac{\eta-\theta}{2}\right)\left(1-v_{r,1}^{2m}(\varphi,\theta,\eta)\right). \tag{5.33}$$

Now our aim is to establish the following estimate.

$$\forall\, k\in\mathbb{N},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathscr{K}^1_{r,1})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{5.34}$$

For this goal we apply Taylor Formula at the order 2,

$$\begin{aligned}I_0\big(\lambda A_r(\varphi,\theta,\eta)\big)-I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)&=2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)I_0'\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\\&+4\lambda^2\sin^2\left(\tfrac{\eta-\theta}{2}\right)\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)^2\int_0^1(1-t)I_0''\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\big(1-t+tv_{r,1}(\varphi,\theta,\eta)\big)\right)dt.\end{aligned}$$

Consequently, the kernel $\mathscr{K}^1_{r,1}$ can be rewritten into the form

$$\begin{aligned}\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta)&=2\lambda\Big(1-v_{r,1}(\varphi,\theta,\eta)\Big)\frac{I_0'\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)}{\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|}\\&-4\lambda^2\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)^2\int_0^1(1-t)I_0''\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\big(1-t+tv_{r,1}(\varphi,\theta,\eta)\big)\right)dt.\end{aligned} \tag{5.35}$$

Using the structure (A.2) and Lemma 4.1-(iv)-(v) combined with (5.31) we deduce the estimate (5.34). Coming back to (5.32) and set

$$\begin{aligned}\mathscr{K}^2_{r,1}(\lambda,\varphi,\theta,\eta)&=\log(\lambda)\sin^2\left(\tfrac{\eta-\theta}{2}\right)\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta)-\log(v_{r,1}(\varphi,\theta,\eta))I_0(\lambda A_r(\varphi,\theta,\eta))\\&+f(\lambda A_r(\varphi,\theta,\eta))-f\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right).\end{aligned} \tag{5.36}$$

Then, by virtue of the law products and the composition laws of Lemma 4.1 combined with (5.31), (5.34) and the fact that f is analytic and even, we get

$$\forall k\in\mathbb{N},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathscr{K}^2_{r,1})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{5.37}$$

Consequently we obtain the decomposition

$$K_0(\lambda A_r(\varphi,\theta,\eta))=K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\mathscr{K}(\eta-\theta)\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta)+\mathscr{K}^2_{r,1}(\lambda,\varphi,\theta,\eta), \tag{5.38}$$

where $\mathscr{K}$ is defined by

$$\mathscr{K}(\theta)=\sin^2\left(\tfrac{\theta}{2}\right)\log\left(\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right) \tag{5.39}$$

and the functions $\mathscr{K}^1_{r,1}$ and $\mathscr{K}^2_{r,1}$ satisfy the estimates (5.34) and (5.37). We can obviously check that $\mathscr{K}$ is an even function satisfying

$$\mathscr{K},\,\partial_\theta\mathscr{K}\in L^\infty(\mathbb{T},\mathbb{R})\subset L^1(\mathbb{T},\mathbb{R})\quad\text{and}\quad\partial_\theta^2\mathscr{K}\in L^1(\mathbb{T},\mathbb{R})\setminus L^\infty(\mathbb{T},\mathbb{R}). \tag{5.40}$$

Introduce

$$\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\eta)=\mathscr{K}(\eta-\theta)\mathscr{K}^1_{r,1}(\lambda,\varphi,\theta,\eta)+\mathscr{K}^2_{r,1}(\lambda,\varphi,\theta,\eta). \tag{5.41}$$

Hence, putting together (5.34), (5.37) and (5.40), we obtain

$$\forall k\in\{0,1\},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathbb{K}_{r,1})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{5.42}$$

In addition, if $r(-\varphi,-\theta)=r(\varphi,\theta)$, then the kernel $\mathbb{K}_{r,1}$ satisfies the following symmetry property

$$\mathbb{K}_{r,1}(\lambda,-\varphi,-\theta,-\eta)=\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\eta). \tag{5.43}$$

Plugging (5.41) and (5.38) into $F_\lambda[r]$ defined in (2.10) yields

$$F_\lambda[r](\varphi,\theta)=\int_{\mathbb{T}}\Big(K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\eta)\Big)\,\partial^2_{\theta\eta}\big[R(\varphi,\theta)R(\varphi,\eta)\sin(\eta-\theta)\big]d\eta.$$

We denote

$$f(\varphi,\theta,\eta)=\partial^2_{\theta\eta}\Big(R(\varphi,\theta)R(\varphi,\eta)\sin(\eta-\theta)\Big),$$

then straightforward computations yield

$$f(\varphi,\theta,\eta)=\Big(\partial_\theta R(\theta)\partial_\eta R(\eta)+R(\theta)R(\eta)\Big)\sin(\eta-\theta)+\Big(\partial_\theta R(\theta)R(\eta)-\partial_\eta R(\eta)R(\theta)\Big)\cos(\eta-\theta).$$

We immediately deduce by law products, translation invariance property and composition laws in Lemma 4.1 that

$$\|f(*,\cdot,\centerdot,\eta+\centerdot)-\sin(\eta)\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\|r\|^{\gamma,\mathcal{O}}_{q,s+1}.\tag{5.44}$$

Using a change of variables, we obtain

$$\begin{aligned}F_\lambda[r](\varphi,\theta)=&\int_{\mathbb{T}}\Big(K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)+\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\theta+\eta)\Big)\Big(f(\varphi,\theta,\eta+\theta)-\sin\eta\Big)d\eta\\&+\int_{\mathbb{T}}\Big(K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)+\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\theta+\eta)\Big)\sin\eta\,d\eta.\end{aligned}$$

By symmetry, we find

$$\int_{\mathbb{T}}K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)\sin\eta\,d\eta=0,$$

which allows to get

$$\begin{aligned}F_\lambda[r](\varphi,\theta)=&\int_{\mathbb{T}}\Big(K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)+\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\theta+\eta)\Big)\Big(f(\varphi,\theta,\eta+\theta)-\sin\eta\Big)d\eta\\&+\int_{\mathbb{T}}\mathbb{K}_{r,1}\big(\lambda,\varphi,\theta,\theta+\eta\big)\sin\eta\,d\eta.\end{aligned}$$

Recall that K_0 admits a logarithmic behavior around 0, hence it is integrable at 0. Therefore, using the law products of Lemma 4.1, (5.42), (5.44) and the smallness property on r, we infer

$$\|F_\lambda[r]\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\|r\|^{\gamma,\mathcal{O}}_{q,s+1}.\tag{5.45}$$

Combining (5.45) with (5.26) and (5.29) achieves the proof of the first point.

(ii) From (3.2), (3.12), (5.38) and (5.41) we deduce that the operator $\mathbf{L}_r$ writes

$$\mathbf{L}_r=\mathcal{K}_\lambda*\cdot+\mathbf{L}_{r,1},\tag{5.46}$$

where $\mathbf{L}_{r,1}$ is the integral operator of kernel $\mathbb{K}_{r,1}$ introduced in (5.41). From Lemma 3.1 and its proof we find

$$d_rF_\lambda[r]\rho=\partial_\theta\left((V_r-\Omega)\rho\right)-\partial_\theta\mathcal{K}_\lambda*\rho-\partial_\theta\mathbf{L}_{r,1}\rho.$$

Thus, we get according to the definition (5.3)

$$d_rX_P(r)\rho=\partial_\theta\mathbf{L}_{r,1}\rho-\partial_\theta\left((V_r-V_0)\rho\right).\tag{5.47}$$

Coming back to (3.1) and using the kernel decomposition (5.38) together with the law products, the composition laws in Lemma 4.1 and the smallness condition, we deduce for any $s\geqslant s_0$,

$$\|V_r-V_0\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\|r\|^{\gamma,\mathcal{O}}_{q,s+1}.\tag{5.48}$$

Therefore, we obtain from the law products, (5.48) and the smallness property on r,

$$\begin{aligned}\|\partial_\theta\left((V_r-V_0)\right)\rho)\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|V_r-V_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|V_r-V_0\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Now, by using the last point in Lemma 4.4 with (5.42) and the smallness property on r, we obtain

$$\|\partial_\theta\mathbf{L}_{r,1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Putting together (5.47) and the last two estimates gives

$$\|d_rX_P(r)\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$$

(iii) Differentiating in r the identity (5.47) yields,

$$d_r^2X_P(r)[\rho_1,\rho_2]=\partial_\theta\left(d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\right)-\partial_\theta\left(\big(d_rV_r(r)[\rho_2]\big)\rho_1\right). \tag{5.49}$$

For the first member of the right-hand side we first recall from (5.41) that

$$\mathbf{L}_{r,1}\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\big[\mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta)+\mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta)\big]d\eta.$$

Hence, by differentiation and change of variables, we obtain

$$\begin{aligned}d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1(\varphi,\theta)&=\int_{\mathbb{T}}\rho_1(\varphi,\eta)\big[\mathscr{K}(\eta-\theta)\big(d_r\mathscr{K}_{r,1}^1\big)[\rho_2](\varphi,\theta,\eta)+d_r\mathscr{K}_{r,1}^2\big)[\rho_2](\varphi,\theta,\eta)\big]d\eta \qquad (5.50)\\ &=\int_{\mathbb{T}}\rho_1(\varphi,\theta+\eta)\big[\mathscr{K}(\eta)\big(d_r\mathscr{K}_{r,1}^1\big)[\rho_2](\varphi,\theta,\theta+\eta)+d_r\mathscr{K}_{r,1}^2\big)[\rho_2](\varphi,\theta,\theta+\eta)\big]d\eta.\end{aligned}$$

Coming back to (5.35), we emphasize that the dependence in r of the functional $\mathscr{K}_{r,1}^1$ is smooth since the function $v_{r,1}$, introduced in (5.30), depends smoothly in r. In addition $d_r\mathscr{K}_{r,1}^1$ can be easily related to $d_rv_{r,1}$. From straightforward calculus we see that, for the sake of simple notation we remove the dependence in the parameters and φ,

$$d_rv_{r,1}(r)[\rho](\theta,\eta)=\frac{1}{v_{r,1}(\theta,\eta)}\left(\frac{R(\theta)-R(\eta)}{\sin^2\left(\frac{\eta-\theta}{2}\right)}\left(\frac{\rho(\theta)}{R(\theta)}-\frac{\rho(\eta)}{R(\eta)}\right)+\frac{\rho(\theta)R^2(\eta)+\rho(\eta)R^2(\theta)}{2R(\theta)R(\eta)}\right). \tag{5.51}$$

Therefore using (5.31) combined with the law products stated in Lemma 4.1, Lemma 4.2 and the smallness condition of Lemma 5.1, we find that

$$\sup_{\eta\in\mathbb{T}}\|d_rv_{r,1}(r)[\rho](*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.52}$$

Similarly to (5.52), one gets from (5.35) and (5.36),

$$\forall\, i\in\{1,2\},\quad \sup_{\eta\in\mathbb{T}}\|d_r\mathscr{K}_{r,1}^i[\rho](*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.53}$$

Inserting (5.53) into (5.50) and using once again the law products and the smallness condition we obtain,

$$\begin{aligned}\|\partial_\theta d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{5.54}$$

Next we shall move to the estimate of the last member of (5.49). Differentiating the definition of V_r in the proof of Lemma 3.1, we infer

$$
\begin{aligned}
d_r V_r(r)[\rho_2](\theta) = & \int_{\mathbb{T}} K_0\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{\rho_2(\eta)R^2(\theta)-\rho_2(\theta)R^2(\eta)}{R^3(\theta)R(\eta)}\sin(\eta-\theta)\right)d\eta\\
& + \tfrac{\lambda}{R(\theta)}\int_{\mathbb{T}} \tfrac{(R(\theta)-R(\eta))\left(\frac{\rho_2(\theta)}{R(\theta)}-\frac{\rho_2(\eta)}{R(\eta)}\right)}{A_r(\theta,\eta)} K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta\\
& + 2\lambda\int_{\mathbb{T}} \tfrac{\rho_2(\theta)R^2(\eta)+\rho_2(\eta)R^2(\theta)}{R^2(\theta)R(\eta)A_r(\theta,\eta)}\sin^2\left(\tfrac{\eta-\theta}{2}\right)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta\\
:= & \mathcal{I}_1(\theta)+\mathcal{I}_2(\theta)+\mathcal{I}_3(\theta).
\end{aligned}
$$

The estimate of $\mathcal{I}_1$ is similar to (5.45), we use the same tools and one finds

$$
\|\mathcal{I}_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.55}
$$

For the terms $\mathcal{I}_2$ and $\mathcal{I}_3$ the computations are straightforward and we shall only extract their main parts and give the suitable estimates. For this goal we differentiate (A.6), leading to

$$
K_0'(z) = \frac{-1}{z} + \log(z)F(z) + G(z),
$$

with F and G being entire functions. Hence, applying (5.30), we deduce that $\mathcal{I}_2$ takes the form

$$
\mathcal{I}_2(\theta) = -\frac{1}{4}\int_{\mathbb{T}} \tfrac{(R(\theta)-R(\eta))\left(\frac{\rho_2(\theta)}{R(\theta)}-\frac{\rho_2(\eta)}{R(\eta)}\right)}{R^2(\theta)v_{r,1}^2(\theta,\eta)\sin^2\left(\frac{\eta-\theta}{2}\right)}\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta + \text{l.o.t.}
$$

Hence we proceed as for (5.53) and one finds

$$
\|\mathcal{I}_2\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.56}
$$

As to the last term $\mathcal{I}_3$, we write

$$
\mathcal{I}_3(\theta) = -\frac{1}{2}\int_{\mathbb{T}} \tfrac{\rho_2(\theta)R^2(\eta)+\rho_2(\eta)R^2(\theta)}{R^2(\theta)R(\eta)v_{r,1}^2(\theta,\eta)}\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta + \text{l.o.t.}
$$

Then, we get

$$
\|\mathcal{I}_3\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s}^{\gamma,\mathcal{O}} + \|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.57}
$$

Putting together (5.55), (5.56) and (5.57) yields

$$
\|d_r V_r(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{5.58}
$$

Therefore we obtain according to the law products in Lemma 4.1, (5.58) and the smallness condition,

$$
\begin{aligned}
\big\|\partial_\theta\big(d_r V_r(r)[\rho_2]\rho_1\big)\big\|_{q,s}^{\gamma,\mathcal{O}} & \lesssim \big\|d_r V_r(r)[\rho_2]\big\|_{q,s+1}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s_0}^{\gamma,\mathcal{O}} + \big\|d_r V_r(r)[\rho_2]\big\|_{q,s_0}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s+1}^{\gamma,\mathcal{O}}\\
& \lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.
\end{aligned}
$$

Combining the latter estimate with (5.49) and (5.54) allows to get

$$
\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.
$$

Using Sobolev embeddings we get the desired result. This achieves the proof of Lemma 5.1. □

As an application of Lemma 5.1, we shall establish tame estimates for the Hamiltonian vector field

$$
X_{\mathcal{P}_\varepsilon} = (\partial_I\mathcal{P}_\varepsilon, -\partial_\vartheta\mathcal{P}_\varepsilon, \Pi_{\mathbb{S}}^{\perp}\partial_\theta\nabla_z\mathcal{P}_\varepsilon)
$$

defined through (5.19) and (5.21).

Lemma 5.2. *Let (γ, q, s_0, s) satisfy (4.1). There exists $\varepsilon_0 \in (0,1)$ such that if*

$$\varepsilon \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant 1,$$

then the perturbed Hamiltonian vector field $X_{\mathcal{P}_\varepsilon}$ satisfies the following estimates,

(i) $\|X_{\mathcal{P}_\varepsilon}(i)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}$.

(ii) $\big\|d_i X_{\mathcal{P}_\varepsilon}(i)[\widehat{i}\,]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\,\widehat{i}\,\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\,\widehat{i}\,\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\big\|d_i^2 X_{\mathcal{P}_\varepsilon}(i)[\widehat{i},\widehat{i}\,]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\,\widehat{i}\,\|_{q,s+2}^{\gamma,\mathcal{O}}\|\,\widehat{i}\,\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\left(\|\,\widehat{i}\,\|_{q,s_0+2}^{\gamma,\mathcal{O}}\right)^2$.

Proof. These estimates can be recovered from Lemma 5.1 combined with the following estimate on the action-angle change of variables introduced in (5.13)

$$\forall \alpha,\beta \in \mathbb{N}^d, \quad \|\partial_\vartheta^\alpha \partial_I^\beta v(\vartheta, I)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}\|_{q,s}^{\gamma,\mathcal{O}}. \tag{5.59}$$

This estimate follows from Lemma 4.1-(iv)-(v) provided that $\|\vartheta\|_{q,s_0}^{\gamma,\mathcal{O}}, \|I\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant 1$. This latter condition is satisfied due to the smallness condition in the Lemma. For more details, we refer to [14, Lem. 5.1]. □

5.3 Berti-Bolle approach for the approximate inverse

In this section, we shall follow the remarkable procedure developed by Berti and Bolle in [11] to construct an approximate right inverse for the linearized operator

$$d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)[\widehat{i},\widehat{\alpha}] = \omega\cdot\partial_\varphi \widehat{i} - d_i X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))[\widehat{i}] - (\widehat{\alpha},0,0), \tag{5.60}$$

where $\mathcal{F}$ is the nonlinear functional defined in (5.23). This construction is crucial for the Nash-Moser scheme that we shall perform later in Section 7. From (5.20), we denote by i_0 an embedded torus with

$$i_0(\varphi) = (\vartheta_0(\varphi), I_0(\varphi), z_0(\varphi)) \quad \text{and} \quad \mathfrak{I}_0(\varphi) = i_0(\varphi) - (\varphi, 0, 0).$$

Throughout this section, we shall assume the following smallness condition : the application $(\lambda,\omega)\mapsto \mathfrak{I}_0(\lambda,\omega)$ is q-times differentiable on $\mathcal{O}$ and there exists $\varepsilon_0 \in (0,1)$ (small enough) such that

$$\|\mathfrak{I}_0\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\alpha_0 - \omega\|_q^{\gamma,\mathcal{O}} \leqslant \varepsilon_0. \tag{5.61}$$

We mainly follow the same approach as in [11] which reduces the search of an approximate right inverse of (5.60) to the search of an approximate right inverse in the normal directions. The main difference with [11] is to be able to bypass the use of the isotropic torus in a similar way to the recent paper [35].

5.3.1 Triangularization up to error terms

Given a linear operator $A \in \mathcal{L}(\mathbb{R}^d, H_\perp^s)$, we define the transposed operator $A^\top : H_\perp^s \to \mathbb{R}^d$ by the duality relation

$$\forall (u,v) \in H_\perp^s \times \mathbb{R}^d, \quad \langle A^\top u, v\rangle_{\mathbb{R}^d} = \langle u, Av\rangle_{L^2(\mathbb{T})}. \tag{5.62}$$

We introduce the following change of coordinates $G_0 : (\phi, y, w) \to (\vartheta, I, z)$ of the phase space $\mathbb{T}^d \times \mathbb{R}^d \times H_\perp^s$ defined by

$$\begin{pmatrix}\vartheta\\ I\\ z\end{pmatrix} := G_0\begin{pmatrix}\phi\\ y\\ w\end{pmatrix} := \begin{pmatrix}\vartheta_0(\phi)\\ I_0(\phi) + L_1(\varphi)y + L_2(\varphi)w\\ z_0(\phi) + w\end{pmatrix}, \tag{5.63}$$

where

$$L_1(\varphi) := [\partial_\varphi \vartheta_0(\varphi)]^{-\top}, \tag{5.64}$$

$$L_2(\varphi) := -[(\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\varphi))]^\top \partial_\theta^{-1}, \tag{5.65}$$

$$\widetilde{z}_0(\vartheta) := z_0(\vartheta_0^{-1}(\vartheta)), \tag{5.66}$$

provided that $\vartheta_0 : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism. Notice that one recovers the torus i_0 by taking in the new coordinates, the flat torus $i_{\text{flat}}(\varphi) = (\varphi, 0, 0)$ namely

$$G_0(i_{\text{flat}}(\varphi)) = i_0(\varphi).$$

Next, we shall adopt the notation $\mathtt{u} = (\phi, y, w)$ to denote the new coordinates induced by G_0 in (5.63) and we simply set $\mathtt{u}_0(\varphi) = i_{\text{flat}}(\varphi)$. Now, to measure to which extent an embedded torus $i_0(\mathbb{T})$ is close to be invariant for the Hamiltonian vector field $X_{H_\varepsilon^{\alpha_0}}$, we shall make appeal to the error function

$$Z(\varphi) := (Z_1, Z_2, Z_3)(\varphi) := \mathcal{F}(i_0, \alpha_0)(\varphi) = \omega \cdot \partial_\varphi i_0(\varphi) - X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi)) . \tag{5.67}$$

We say that a quantity is of "type Z" is it is $O(Z)$, and particular it is vanishing at an exact solution. In the next Proposition, we study the conjugation of the linear operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ by the linear change of variables induced by G_0 defined in (5.63),

$$DG_0(\varphi, 0, 0) \begin{pmatrix} \widehat{\phi} \\ \widehat{y} \\ \widehat{w} \end{pmatrix} := \begin{pmatrix} \partial_\varphi \vartheta_0(\varphi) & 0 & 0 \\ \partial_\varphi I_0(\varphi) & L_1(\varphi) & L_2(\varphi) \\ \partial_\varphi z_0(\varphi) & 0 & I \end{pmatrix} \begin{pmatrix} \widehat{\phi} \\ \widehat{y} \\ \widehat{w} \end{pmatrix}. \tag{5.68}$$

The following result is proved in [35].

Proposition 5.1. *The conjugation of the linearized operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ by the linear change of variables $DG_0(\mathtt{u}_0)$ writes as follows*

$$[DG_0(\mathtt{u}_0)]^{-1} d_{i,\alpha}\mathcal{F}(i_0, \alpha_0) D\widetilde{G}_0(\mathtt{u}_0)[\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}] = \mathbb{D}[\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}] + \mathbb{E}[\widehat{\phi}, \widehat{y}, \widehat{w}], \tag{5.69}$$

where $\widetilde{G}_0$ is defined by

$$\widehat{G}_0(\mathtt{u}, \alpha) := (G_0(\mathtt{u}), \alpha)$$

and where

(i) the operator $\mathbb{D}$ admits a triangular structure in the variables $(\widehat{\phi}, \widehat{y}, \widehat{w})$ in the form

$$\mathbb{D}[\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}] := \begin{pmatrix} \omega \cdot \partial_\varphi \widehat{\phi} - \big[\mathsf{K}_{20}(\varphi)\widehat{y} + \mathsf{K}_{11}^\top(\varphi)\widehat{w} + L_1^\top(\varphi)\widehat{\alpha}\big] \\ \omega \cdot \partial_\varphi \widehat{y} + \mathsf{B}(\varphi)\widehat{\alpha} \\ \omega \cdot \partial_\varphi \widehat{w} - \partial_\theta\big[\mathsf{K}_{11}(\varphi)\widehat{y} + \mathsf{K}_{02}(\varphi)\widehat{w} - L_2^\top(\varphi)\widehat{\alpha}\big] \end{pmatrix},$$

$\mathsf{B}(\varphi)$ and $\mathsf{K}_{20}(\varphi)$ are $d \times d$ real matrices given by

$$\mathsf{B}(\varphi) := [\partial_\varphi \vartheta_0(\varphi)]^\top \partial_\varphi I_0(\varphi) L_1^\top(\varphi) + [\partial_\varphi z_0(\varphi)]^\top L_2^\top(\varphi), \tag{5.70}$$

$$\mathsf{K}_{20}(\varphi) := \varepsilon L_1^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi), \tag{5.71}$$

$\mathsf{K}_{02}(\varphi)$ is a linear self-adjoint operator of $H_\perp^s$ in the form

$$\begin{aligned} \mathsf{K}_{02}(\varphi) := (\partial_z \nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) + \varepsilon L_2^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_2(\varphi) \\ + \varepsilon L_2^\top(\varphi)(\partial_{zI}\mathcal{P}_\varepsilon)(i_0(\varphi)) + \varepsilon(\partial_I \nabla_z \mathcal{P}_\varepsilon)(i_0(\varphi))L_2(\varphi) \end{aligned} \tag{5.72}$$

and $\mathsf{K}_{11}(\varphi) \in \mathcal{L}(\mathbb{R}^d, H_\perp^s)$ is given by

$$\mathsf{K}_{11}(\varphi) := \varepsilon L_2^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi) + \varepsilon(\partial_I \nabla_z \mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi). \tag{5.73}$$

(ii) the operator $\mathbb{E}$ is an error term in the form

$$\mathbb{E}[\widehat{\phi},\widehat{y},\widehat{w}] := [DG_0(\mathtt{u}_0)]^{-1}\partial_\varphi Z(\varphi)\widehat{\phi} + \begin{pmatrix} 0 \\ \mathtt{A}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y} + \mathsf{K}_{11}^\top(\varphi)\widehat{w}\big] - R_{10}(\varphi)\widehat{y} - R_{01}(\varphi)\widehat{w} \\ 0 \end{pmatrix},$$

where $\mathtt{A}(\varphi)$ and $R_{10}(\varphi)$ are $d\times d$ matrices defined by

$$\mathtt{A}(\varphi) := [\partial_\varphi\vartheta_0(\varphi)]^\top\partial_\varphi I_0(\varphi) - [\partial_\varphi I_0(\varphi)]^\top\partial_\varphi\vartheta_0(\varphi) + [\partial_\varphi z_0(\varphi)]^\top\partial_\theta^{-1}\partial_\varphi z_0(\varphi), \tag{5.74}$$

$$R_{10}(\varphi) := [\partial_\varphi Z_1(\varphi)]^\top L_1(\varphi) \tag{5.75}$$

and $R_{01}(\varphi)\in\mathcal{L}(H^s_\perp,\mathbb{R}^d)$ with

$$R_{01}(\varphi) := -[\partial_\varphi Z_1(\varphi)]^\top L_2(\varphi) + [\partial_\varphi Z_3(\varphi)]^\top\partial_\theta^{-1}. \tag{5.76}$$

Now we recall the following result, for the proof we refer to [35, Lemma 6.1] and Lemmata 5.6-5.7 in [14],

Lemma 5.3. *The following assertions hold true.*

(i) The operator $DG_0(\mathtt{u}_0)$ and $[DG_0(\mathtt{u}_0)]^{-1}$ satisfy for all $\widehat{\mathtt{u}} = (\widehat{\phi},\widehat{y},\widehat{w})$,

$$\forall s\in[s_0,S],\quad \|[DG_0(\mathtt{u}_0)]^{\pm1}[\widehat{\mathtt{u}}]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{\mathtt{u}}\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{\mathtt{u}}\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

(ii) The operators R_{10} and R_{01}, defined in (5.75) and (5.76), satisfy the estimates

$$\forall s\in[s_0,S],\quad \|R_{10}\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|Z\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|Z\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+1}^{\gamma,\mathcal{O}},$$
$$\forall s\in[s_0,S],\quad \|R_{01}\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|Z\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|Z\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$$

(iii) The operators K_{20} and K_{11}, defined in (5.71) and (5.73), satisfy the estimates

$$\forall s\in[s_0,S],\quad \|\mathsf{K}_{20}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\big(1+\|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\big),$$
$$\forall s\in[s_0,S],\quad \|\mathsf{K}_{11}\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\big(\|\widehat{y}\|_{q,s+3}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+3}^{\gamma,\mathcal{O}}\big),$$
$$\forall s\in[s_0,S],\quad \|\mathsf{K}_{11}^\top\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\big(\|\widehat{w}\|_{q,s+3}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+3}^{\gamma,\mathcal{O}}\big).$$

(iv) The matrices $\mathtt{A}$ and $\mathtt{B}$ defined in (5.74) and (5.70) satisfy

$$\forall s\in[s_0,S],\quad \|\mathtt{A}\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathtt{B}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}.$$

Notice that the matrix $\mathtt{A}(\varphi)$ measures the defect of the symplectic structure. In the following, we shall see that it is of order $O(Z)$. Notice that according to (5.74) and [11, Lem. 5], the coefficients $\mathtt{A}_{jk}$ of the matrix $\mathtt{A}$ can be written

$$\mathtt{A}_{jk}(\varphi) = \partial_{\varphi_k}I_0(\varphi)\cdot\partial_{\varphi_j}\vartheta_0(\varphi) - \partial_{\varphi_k}\vartheta_0(\varphi)\cdot\partial_{\varphi_j}I_0(\varphi) + \langle\partial_\theta^{-1}\partial_{\varphi_k}z_0(\varphi),\partial_{\varphi_j}z_0(\varphi)\rangle_{L^2(\mathbb{T})}, \tag{5.77}$$

and satisfy

$$\omega\cdot\partial_\varphi\mathtt{A}_{jk}(\varphi) = \mathcal{W}\big(\partial_\varphi Z(\varphi)\underline{e}_k,\partial_\varphi i_0(\varphi)\underline{e}_j\big) + \mathcal{W}\big(\partial_\varphi i_0(\varphi)\underline{e}_k,\partial_\varphi Z(\varphi)\underline{e}_j\big), \tag{5.78}$$

where $\mathcal{W}$ is the symplectic form defined in (5.1) and $(\underline{e}_1,\ldots,\underline{e}_d)$ denotes the canonical basis of $\mathbb{R}^d$. In order to estimate $\mathtt{A}_{jk}(\varphi)$, we shall discuss the invertibility of the operator $\omega\cdot\partial_\varphi$. This task was accomplished in several paper [4, 11, 14, 35]. and we shall outline here the main lines.
Let $\gamma\in(0,1]$ and $\tau_1>0$ be defined as in (4.1). We introduce the Diophantine Cantor set

$$\mathtt{DC}(\gamma,\tau_1) := \bigcap_{l\in\mathbb{Z}^d\setminus\{0\}}\Big\{\omega\in\mathbb{R}^d\quad\text{s.t.}\quad|\omega\cdot l| > \frac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}$$

and for $N\in\mathbb{N}^*$ we define the truncated Diophantine Cantor set

$$\mathtt{DC}_N(\gamma,\tau_1):=\bigcap_{\substack{l\in\mathbb{Z}^d\setminus\{0\}\\ |l|\leqslant N}}\Big\{\omega\in\mathbb{R}^d\quad\text{s.t.}\quad|\omega\cdot l|>\frac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}. \tag{5.79}$$

Given $f:\mathcal{O}\times\mathbb{T}^{d+1}\to\mathbb{R}$ a smooth function with zero φ-average, that can be expanded in Fourier series as follows

$$f=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ l\neq 0}}f_{l,j}(\lambda,\omega)\mathbf{e}_{l,j},\quad \mathbf{e}_{l,j}(\varphi,\theta):=e^{i(l\cdot\varphi+j\theta)}.$$

If $\omega\in\mathtt{DC}(\gamma,\tau_1)$ then the equation $\omega\cdot\partial_\varphi u=f$ has a periodic solution $u:\mathbb{T}^{d+1}\to\mathbb{R}$ given by

$$u(\lambda,\varphi,\theta)=-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ l\neq 0}}\frac{f_{l,j}(\lambda)}{\omega\cdot l}\mathbf{e}_{l,j}(\varphi,\theta).$$

For all $\omega\in\mathcal{O}$, we define the smooth extension of u by

$$(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}f:=-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ l\neq 0}}\frac{\chi\big(\gamma^{-1}\langle l\rangle^{\tau_1}\,\omega\cdot l\big)f_{l,j}(\lambda)}{\omega\cdot l}\mathbf{e}_{l,j}. \tag{5.80}$$

where $\chi\in\mathscr{C}^\infty(\mathbb{R},[0,1])$ is an even positive cut-off function such that

$$\chi(\xi)=\begin{cases}0 & \text{if}\quad |\xi|\leqslant\frac{1}{3}\\ 1 & \text{if}\quad |\xi|\geqslant\frac{1}{2}.\end{cases} \tag{5.81}$$

Notice that this operator is well-defned in the whole set of parameters $\mathcal{O}$ and coincides with the formal inverse of $(\omega\cdot\partial_\varphi)^{-1}$ when the frequency ω belongs to $\mathtt{DC}(\gamma,\tau_1)$. The next result is the fundamental theorem of calculus in the quasi-periodic setting. It is proved in [4, Lem. 2.5] and [35, Lem. 5.4].

Lemma 5.4. *Let $\gamma\in(0,1],q\in\mathbb{N}^*$. Then for any $s\geqslant q$ we have*

$$\big\|(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}f\big\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\gamma^{-1}\|f\|^{\gamma,\mathcal{O}}_{q,s+\tau_1 q+\tau_1}.$$

In addition, for any $N\in\mathbb{N}^*$ and for any $\omega\in\mathtt{DC}_N(\gamma,\tau_1)$ we have

$$(\omega\cdot\partial_\varphi)(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}\Pi_N=\Pi_N,$$

where Π_N is the orthogonal projection defined by

$$\Pi_N\sum_{(l,j)\in\mathbb{Z}^{d+1}}f_{l,j}\mathbf{e}_{l,j}=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ |l|\leqslant N}}f_{l,j}\mathbf{e}_{l,j}.$$

For later purposes we need to fix some notation that will be adopted in the sequel. Take $N_0\geq 2$ and define the sequence

$$N_{-1}=1,\quad\forall n\in\mathbb{N},\quad N_n=N_0^{\left(\frac{3}{2}\right)^n}. \tag{5.82}$$

Next, we shall split the coefficients of the matrix $\mathtt{A}=\mathtt{A}(\varphi)$ defined in (5.74) as

$$\mathtt{A}_{kj}=\mathtt{A}^{(n)}_{kj}+\mathtt{A}^{(n),\perp}_{kj},\qquad \mathtt{A}^{(n)}_{kj}:=\Pi_{N_n}\mathtt{A}_{kj},\qquad \mathtt{A}^{(n),\perp}_{kj}:=\Pi^\perp_{N_n}\mathtt{A}_{kj}. \tag{5.83}$$

The proof of the following lemma is quite similar to Lemma 5.3. in [7] with the a minor difference in the weighted norms. See also [35, Lem. 6.3].

Lemma 5.5. *Let $n\in\mathbb{N}$, then the following results hold true.*

(i) The function $\mathtt{A}^{(n),\perp}_{kj}$ satisfies

$$\forall b\geqslant 0,\quad\forall s\in[s_0,S],\quad\|\mathtt{A}^{(n),\perp}_{kj}\|^{\gamma,\mathcal{O}}_{q,s}\lesssim N_n^{-b}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+2+b}.$$

(ii) There exist functions $\mathtt{A}^{(n),\text{ext}}_{kj}$ defined for any $(\lambda,\omega)\in\mathcal{O}$, q-times differentiable with respect to λ and satisfying the estimate

$$\forall s\in[s_0,S],\quad\|\mathtt{A}^{(n),\text{ext}}_{kj}\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\gamma^{-1}\big(\|Z\|^{\gamma,\mathcal{O}}_{s+\tau_1 q+\tau_1+1}+\|Z\|^{\gamma,\mathcal{O}}_{q,s_0+1}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\tau_1 q+\tau_1+1}\big)\,.$$

Moreover, $\mathtt{A}^{(n),\text{ext}}_{kj}$ coincides with $\mathtt{A}^{(n)}_{kj}$ in the Cantor set $\mathtt{DC}_{N_n}(\gamma,\tau_1)$.

5.3.2 Construction of the approximate inverse

This section is devoted to the construction of an approximate right inverse of the operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ that will be discussed in Theorem 5.1. One first may observe according to Proposition 5.1-(ii) and Lemmas 5.3 and 5.5, that the operator $\mathbb{E}$ vanishes at an exact solution up to fast decaying remainder terms. As a consequence, getting an approximate inverse for the full operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ amounts simply to invert the operator $\mathbb{D}$ up to small errors of type "Z" mixed with fast frequency decaying error. Let us consider the triangular system given by

$$\mathbb{D}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]=\begin{pmatrix}g_1\\ g_2\\ g_3\end{pmatrix}, \tag{5.84}$$

where $\mathbb{D}$ is defined in Proposition 5.1-(i). The system (5.84) writes more explicitly in the following way

$$\begin{cases}\omega\cdot\partial_\varphi\widehat{\phi}=g_1+[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}+L_1^\top(\varphi)\widehat{\alpha}]\\ \omega\cdot\partial_\varphi\widehat{y}=g_2-\mathsf{B}(\varphi)\widehat{\alpha}\\ \big(\omega\cdot\partial_\varphi-\partial_\theta\mathsf{K}_{02}(\varphi)\big)\widehat{w}=g_3+\partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y}-L_2^\top(\varphi)\widehat{\alpha}].\end{cases} \tag{5.85}$$

The strategy to solve the above system in the variables $(\widehat{\phi},\widehat{y},\widehat{w})$ is first to solve the second action-component equation, then to solve the third normal-component equation and finally to solve the first angle-component equation.
Due to the fact that the Cantor set should be truncated then we need to solve approximately the system (5.85) and for this aim we need the following statement proved in [35, Lem. 6.4] and [4].

Lemma 5.6. *The following results hold true.*

(i) There exists a function $\mathsf{g}:\mathbb{Z}^d\setminus\{0\}\to\{-1,1\}$ *such that*

$$\forall l\in\mathbb{Z}^d\setminus\{0\},\quad \mathsf{g}(-l)=-\mathsf{g}(l).$$

(ii) For all $(\lambda,\omega)\in\mathcal{O}$ *the operator* $\omega\cdot\partial_\varphi$ *can be split as follows*

$$\omega\cdot\partial_\varphi=\mathcal{D}_{(n)}+\mathcal{D}_{(n)}^\perp,$$

with

$$\begin{aligned}\mathcal{D}_{(n)}&:=\omega\cdot\partial_\varphi\,\Pi_{N_n}+\Pi_{N_n,\mathsf{g}}^\perp\\ \mathcal{D}_{(n)}^\perp&:=\omega\cdot\partial_\varphi\,\Pi_{N_n}^\perp-\Pi_{N_n,\mathsf{g}}^\perp,\end{aligned}$$

where

$$\Pi_{N_n,\mathsf{g}}^\perp\sum_{(l,j)\in\mathbb{Z}^{d+1}}f_{l,j}\mathbf{e}_{l,j}=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ |l|>N_n}}\mathsf{g}(l)f_{l,j}\mathbf{e}_{l,j}.$$

(iii) The operator $\mathcal{D}_{(n)}^\perp$ *satisfies*

$$\forall b\geqslant 0,\quad \forall s\in[s_0,S],\quad \|\mathcal{D}_{(n)}^\perp h\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N_n^{-b}\|h\|_{q,s+b+1}^{\gamma,\mathcal{O}}.$$

(iv) There exists a family of linear operators $\big([\mathcal{D}_{(n)}]_{\textrm{ext}}^{-1}\big)_n$ *satisfying, for any* $h\in W^{q,\infty,\gamma}(\mathcal{O},H_0^s(\mathbb{T}^{d+1}))$,

$$\forall s\in[s_0,S],\quad \sup_{n\in\mathbb{N}}\|[\mathcal{D}_{(n)}]_{\textrm{ext}}^{-1}h\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|h\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

Moreover, for all $\omega\in\mathtt{DC}_{N_n}(\gamma,\tau_1)$ *one has the identity*

$$\mathcal{D}_{(n)}[\mathcal{D}_{(n)}]_{\textrm{ext}}^{-1}=\textrm{Id}. \tag{5.86}$$

Consider the linearized operator restricted to the normal directions $\widehat{\mathcal{L}}_\omega$ and defined by

$$\widehat{\mathcal{L}}_\omega := \Pi_{\mathbb{S}_0}^{\perp}\big(\omega\cdot\partial_\varphi - \partial_\theta \mathsf{K}_{02}(\varphi)\big)\Pi_{\mathbb{S}_0}^{\perp}, \tag{5.87}$$

which appears in the last equation of (5.85). The construction of an approximate right inverse of this operator is the heart part of this paper and will be discussed in Proposition 6.6. Here we give only a partial statement.

Proposition 5.2. *Let $(\gamma, q, d, \tau_1, \tau_2, s_0, s_h, \mu_2, S)$ satisfy (4.1) (4.2) and (6.244). There exist $\varepsilon_0 > 0$ and $\sigma = \sigma(\tau_1, \tau_2, q, d) > 0$ such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}} \leqslant 1, \tag{5.88}$$

then there exist a family of linear operator $(\mathtt{T}_{\omega,n})_n$ satisfying

$$\forall\, s\in\,[s_0,S],\quad \sup_{n\in\mathbb{N}}\|\mathtt{T}_{\omega,n}h\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|h\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|h\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right) \tag{5.89}$$

and a family of Cantor sets $\{\mathcal{G}_n = \mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)\}_n$, satisfying the inclusion

$$\mathcal{G}_n \subset (\lambda_0,\lambda_1)\times \mathtt{DC}_{N_n}(\gamma,\tau_1)$$

such that in each set $\mathcal{G}_n$ we have the splitting

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n,$$

with

$$\widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n} = \mathrm{Id}, \tag{5.90}$$

where the operators $\widehat{\mathtt{L}}_{\omega,n}$ and $\widehat{\mathtt{R}}_n$ are defined in the whole set $\mathcal{O}$ with the estimates

$$\begin{aligned}
\forall\, s\in[s_0,S],\quad & \|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}},\\
\forall\, b\in[0,S],\quad & \|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\gamma^{-1}\left(\|\rho\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right)\\
&\qquad + \varepsilon\overline{\gamma}^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.
\end{aligned}$$

For the splitting below which follows from the foregoing results we refer to (6.45) in [35]. Consider the linear operator $\mathbb{L}_{\mathrm{ext}}$ defined by

$$\mathbb{L}_{\mathrm{ext}} = \mathbb{D}_n + \mathbb{E}_n^{\mathrm{ext}} + \mathscr{P}_n + \mathscr{Q}_n, \tag{5.91}$$

where, for any $(\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha})\in\mathbb{T}^d\times\mathbb{R}^d\times H_\perp^s\times\mathbb{R}^d$

$$\mathbb{D}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}] := \begin{pmatrix} \mathcal{D}_{(n)}\widehat{\phi} - \mathsf{K}_{20}(\varphi)\widehat{y} - \mathsf{K}_{11}^{\top}(\varphi)\widehat{w} - L_1^{\top}(\varphi)\widehat{\alpha}\\ \mathcal{D}_{(n)}\widehat{y} + \mathtt{B}(\varphi)\widehat{\alpha}\\ \widehat{\mathtt{L}}_{\omega,n}\widehat{w} - \partial_\theta\big[\mathsf{K}_{11}(\varphi)\widehat{y} - L_2^{\top}(\varphi)\widehat{\alpha}\big]\end{pmatrix}, \tag{5.92}$$

$$\begin{aligned}\mathbb{E}_n^{\mathrm{ext}}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}] &:= [DG_0(\mathtt{u}_0(\varphi))]^{-1}\partial_\varphi Z(\varphi)\widehat{\phi} - \begin{pmatrix}0\\ R_{10}(\varphi)\widehat{y} + R_{01}(\varphi)\widehat{w}\\ 0\end{pmatrix}\\
&\quad + \begin{pmatrix}0\\ \mathtt{A}^{(n),\mathrm{ext}}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y} + \mathsf{K}_{11}^{\top}(\varphi)\widehat{w}\big]\\ 0\end{pmatrix},\end{aligned} \tag{5.93}$$

$$\mathscr{P}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}] := \begin{pmatrix}\mathcal{D}_{(n)}^{\perp}\widehat{\phi}\\ \mathcal{D}_{(n)}^{\perp}\widehat{y} + \mathtt{A}^{(n),\perp}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y} + \mathsf{K}_{11}^{\top}(\varphi)\widehat{w}\big]\\ 0\end{pmatrix}, \tag{5.94}$$

$$\mathscr{Q}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}] := \begin{pmatrix}0\\0\\ \widehat{\mathtt{R}}_n[\widehat{w}]\end{pmatrix}. \tag{5.95}$$

Then, the operator $\mathbb{L}_{\rm ext}$ is defined on the whole set $\mathcal{O}$ and when it is restricted to the Cantor set $\mathcal{G}_n$ it coincides with the conjugated linearized operator obtained in (5.69), that is,

$$\mathbb{L}_{\rm ext} = [DG_0(\mathtt{u}_0)]^{-1} d_{i,\alpha}\mathcal{F}(i_0,\alpha_0) D\widetilde{G}_0(\mathtt{u}_0) \quad \text{in } \mathcal{G}_n. \tag{5.96}$$

In the next result, we give some useful estimates for the different terms appearing in $\mathbb{L}_{\rm ext}$ needed to obtain good tame estimates for the approximate inverse.

Proposition 5.3. *Let* $(\gamma, q, d, \tau_1, s_0, \mu_2)$ *satisfy* (4.1) *and* (6.244) *and assume the conditions* (5.61) *and* (5.88). *Then, denoting* $\widehat{\mathtt{v}} = (\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha})$, *the following assertions hold true.*

(i) The operator $\mathbb{E}_n^{\rm ext}$ *satisfies the estimate*

$$\|\mathbb{E}_n^{\rm ext}[\widehat{\mathtt{v}}]\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \|Z\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}} \|\widehat{\mathtt{v}}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.$$

(ii) The operator $\mathscr{P}^{(n)}$ *satisfies the estimate*

$$\forall b \geqslant 0, \quad \|\mathscr{P}_n[\widehat{\mathtt{v}}]\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\big(\|\widehat{\mathtt{v}}\|_{q,s_0+\sigma+b}^{\gamma,\mathcal{O}} + \varepsilon\|\mathfrak{I}_0\|_{q,s_0+\sigma+b}^{\gamma,\mathcal{O}}\|\widehat{\mathtt{v}}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\big).$$

(iii) The operator $\mathscr{Q}_n$ *satisfies the estimate*

$$\begin{aligned}\forall\, b \in [0,S], \quad \|\mathscr{Q}_n\widehat{\mathtt{v}}\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\gamma^{-1}\Big(\|\widehat{w}\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big) \\ + \varepsilon\overline{\gamma}^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2}\|\widehat{w}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

(iv) There exists a family of operators $\big([\mathbb{D}_n]_{\rm ext}^{-1}\big)_n$ *such that for all* $g := (g_1, g_2, g_3)$ *satisfying the reversibility property*

$$g_1(\varphi) = g_1(-\varphi)\,, \quad g_2(\varphi) = -g_2(-\varphi)\,, \quad g_3(\varphi) = -(\mathscr{S}g_3)(-\varphi)\,,$$

the function $[\mathbb{D}_n]_{\rm ext}^{-1} g$ *satisfies the estimate*

$$\forall\, s \in [s_0, S], \quad \|[\mathbb{D}_n]_{\rm ext}^{-1} g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\big(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\big)$$

and for all $(\lambda,\omega) \in \mathcal{G}_n$ *one has*

$$\mathbb{D}_n\,[\mathbb{D}_n]_{\rm ext}^{-1} = \mathrm{Id}.$$

Proof. **(i)** The estimate of $\mathbb{E}_n^{\rm ext}$ is obtained from (5.93), Lemma 5.3, Lemma 4.1-(iv) and Lemma 5.5-(ii).
(ii) From (5.94), Lemma 5.6-(iii), Lemma 4.1-(iv), Lemma 5.5-(i), Lemma 5.3-(ii) we obtain the estimate on $\mathscr{P}_n$.
(iii) It is a consequence of (5.95) and Proposition 5.2.
(iv) The proof can be found in [35, Prop. 6.3] and for the sake of completeness we shall sketch the main ideas. We intend to look for an exact inverse of $\mathbb{D}_n$ by solving the system

$$\mathbb{D}_n[\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}] = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}, \tag{5.97}$$

where (g_1, g_2, g_3) satisfy the reversibility property

$$g_1(\varphi) = g_1(-\varphi)\,, \quad g_2(\varphi) = -g_2(-\varphi)\,, \quad g_3(\varphi) = -(\mathscr{S}g_3)(-\varphi), \tag{5.98}$$

with $\mathscr{S}$ being the involution defined in (2.26). Note that in view of (5.92), the system (5.97) writes

$$\begin{cases} \mathcal{D}_{(n)}\widehat{\phi} = g_1 + [\mathsf{K}_{20}(\varphi)\widehat{y} + \mathsf{K}_{11}^\top(\varphi)\widehat{w} + L_1^\top(\varphi)\widehat{\alpha}] \\ \mathcal{D}_{(n)}\widehat{y} = g_2 - \mathsf{B}(\varphi)\widehat{\alpha} \\ \widehat{\mathsf{L}}_{\omega,n}\widehat{w} = g_3 + \partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y} - L_2^\top(\varphi)\widehat{\alpha}]. \end{cases} \tag{5.99}$$

We first consider the second action-component equation in (5.99), namely

$$\mathcal{D}_{(n)}\widehat{y} = g_2 - \mathtt{B}(\varphi)\widehat{\alpha}.$$

In view of (5.98), (5.70) and (5.77), g_2 and $\mathtt{B}$ are odd in the variable φ. Thus, the φ-average of the right hand side of this equation is zero. Then, by Lemma 5.6-(iv) its solution in the Cantor set $\mathtt{DC}_{N_n}(\gamma, \tau_1)$ is given by

$$\widehat{y} := [\mathcal{D}_{(n)}]_{\rm ext}^{-1}\big(g_2 - \mathtt{B}(\varphi)\widehat{\alpha}\big)\,. \tag{5.100}$$

Then we turn to the third normal-component equation in (5.99), namely

$$\widehat{\mathtt{L}}_{\omega,n}\widehat{w} = g_3 + \partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y} - L_2^\top(\varphi)\widehat{\alpha}].$$

By Proposition 5.2, this equation admits as a solution

$$\widehat{w} := \mathtt{T}_{\omega,n}\big(g_3 + \partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y} - L_2^\top(\varphi)\widehat{\alpha}]\big)\,. \tag{5.101}$$

Finally, we solve the first angle-equation in (5.99), which, substituting (5.100), (5.101), becomes

$$\mathcal{D}_{(n)}\widehat{\phi} = g_1 + M_1(\varphi)\widehat{\alpha} + M_2(\varphi)g_2 + M_3(\varphi)g_3\,, \tag{5.102}$$

where

$$M_1(\varphi) := L_1^\top(\varphi) - M_2(\varphi)\mathtt{B}(\varphi) - M_3(\varphi)\partial_\theta L_2^\top(\varphi)\,, \tag{5.103}$$

$$M_2(\varphi) := \mathsf{K}_{20}(\varphi)[\mathcal{D}_{(n)}]_{\rm ext}^{-1} + \mathsf{K}_{11}^\top(\varphi)\mathtt{T}_{\omega,n}\partial_\vartheta\mathsf{K}_{11}(\varphi)[\mathcal{D}_{(n)}]_{\rm ext}^{-1}\,, \tag{5.104}$$

$$M_3(\varphi) := \mathsf{K}_{11}^\top(\varphi)\mathtt{T}_{\omega,n}\,. \tag{5.105}$$

To solve the equation (5.102) we choose $\widehat{\alpha}$ such that the right hand side has zero φ-average. Notice that Lemma 5.3, (5.61), (5.89) and Lemma 5.6-(ii) imply

$$\forall s \in [s_0, S], \quad \|M_2 g_2\|_{q,s}^{\gamma,\mathcal{O}} + \|M_3 g_3\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right). \tag{5.106}$$

By Lemma 5.3-(iii), (5.61), the ϕ-averaged matrix is $\langle M_1\rangle = \mathrm{Id} + O(\varepsilon\gamma^{-1})$. Therefore, for $\varepsilon\gamma^{-1}$ small enough, $\langle M_1\rangle$ is invertible and $\langle M_1\rangle^{-1} = \mathrm{Id} + O(\varepsilon\gamma^{-1})$. We thus define

$$\widehat{\alpha} := -\langle M_1\rangle^{-1}(\langle g_1\rangle + \langle M_2 g_2\rangle + \langle M_3 g_3\rangle)\,. \tag{5.107}$$

Remark that $\widehat{\alpha}$ satisfies

$$\|\widehat{\alpha}\|_q^{\gamma,\mathcal{O}} \lesssim \|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}. \tag{5.108}$$

Coming back to (5.100) and using (5.108), (5.61) together with Lemma 5.6-(iv) and Lemma 5.3-(iv), we obtain

$$\forall s \in [s_0, S], \quad \|\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right). \tag{5.109}$$

Putting together (5.101), (5.89), Lemma 5.3-(iii), (5.108), (5.109) and (5.61), one should get, up to redefine the value of σ,

$$\forall s \in [s_0, S], \quad \|\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right). \tag{5.110}$$

With the choice (5.107) of $\widehat{\alpha}$, the equation (5.102) admits as a solution

$$\widehat{\phi} := [\mathcal{D}_{(n)}]_{\rm ext}^{-1}\big(g_1 + M_1(\varphi)\widehat{\alpha} + M_2(\varphi)g_2 + M_3(\varphi)g_3\big)\,. \tag{5.111}$$

Putting together (5.111), Lemma 5.6-(ii), (5.108) and (5.106), one obtains

$$\forall s \in [s_0, S], \quad \|\widehat{\phi}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right). \tag{5.112}$$

In conclusion, we have obtained a solution $(\widehat{\phi}, \widehat{y}, \widehat{w}, \widehat{\alpha}) := [\mathbb{D}_n]_{\rm ext}^{-1} g$ of the linear system (5.97) satisfying in virtue of (5.108), (5.112), (5.110) and (5.109),

$$\forall s \in [s_0, S], \quad \|[\mathbb{D}_n]_{\rm ext}^{-1} g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).$$

Notice that the relation

$$\mathbb{D}_n[\mathbb{D}_n]_{\rm ext}^{-1} = \mathrm{Id} \quad \text{in } \mathcal{G}_n$$

is a direct consequence of (5.86) and (5.90). □

The last point is to prove that the operator

$$\mathrm{T}_0 := \mathrm{T}_0(i_0) := (D\widetilde{G}_0)(\mathrm{u}_0) \circ [\mathbb{D}_n]_{\mathrm{ext}}^{-1} \circ (DG_0)(\mathrm{u}_0)^{-1} \tag{5.113}$$

is an approximate right inverse for $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$.

Theorem 5.1 (Approximate inverse)**.** *Let* $(\gamma, q, d, \tau_1, \tau_2, s_0, s_h, \mu_2, S)$ *satisfy* (4.1), (4.2), (6.244) *and* (6.3). *Then there exists* $\overline{\sigma} = \overline{\sigma}(\tau_1, \tau_2, d, q) > 0$ *such that if the smallness conditions* (5.61) *and* (5.88) *hold, then the operator* T_0 *defined in* (5.113) *is reversible and satisfies for all* $g = (g_1, g_2, g_3)$, *with* (5.98),

$$\forall s \in [s_0, S], \quad \|\mathrm{T}_0 g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right). \tag{5.114}$$

Moreover T_0 *is an almost-approximate right inverse of* $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ *in the Cantor set* $\mathcal{G}_n$. *More precisely, for all* $(\lambda, \omega) \in \mathcal{G}_n$ *one has*

$$d_{i,\alpha}\mathcal{F}(i_0) \circ \mathrm{T}_0 - \mathrm{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)}, \tag{5.115}$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ *and* $\mathcal{E}_3^{(n)}$ *are defined in the set* $\mathcal{O}$ *with the estimates*

$$\|\mathcal{E}_1^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\|\mathcal{F}(i_0, \alpha_0)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}, \tag{5.116}$$

$$\forall b \geqslant 0, \quad \|\mathcal{E}_2^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}N_n^{-b}\left(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right), \tag{5.117}$$

$$\begin{aligned}\forall b \in [0, S], \quad \|\mathcal{E}_3^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim N_n^{-b}\gamma^{-2}\left(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)\\ &\quad + \varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.\end{aligned} \tag{5.118}$$

Proof. The estimate (5.114) is a consequence of (5.113), Proposition 5.3-(iv) and Lemma 5.3-(i). Then, according to (5.91) and (5.96), in the Cantor set $\mathcal{G}_n$ we have the decomposition

$$\begin{aligned}d_{i,\alpha}\mathcal{F}(i_0, \alpha_0) &= DG_0(\mathrm{u}_0) \circ \mathbb{L}_{\mathrm{ext}} \circ D[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\\ &= DG_0(\mathrm{u}_0) \circ \mathbb{D}_n \circ D[\widetilde{G}_0(\mathrm{u}_0)]^{-1} + DG_0(\mathrm{u}_0) \circ \mathbb{E}_n^{\mathrm{ext}} \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1}\\ &\quad + DG_0(\mathrm{u}_0) \circ \mathscr{P}_n \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1} + DG_0(\mathrm{u}_0) \circ \mathscr{Q}_n \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1}.\end{aligned}$$

By applying T_0, defined in (5.113), to the last identity we get for all $(\lambda, \omega) \in \mathcal{G}_n$

$$d_{i,\alpha}\mathcal{F}(i_0, \alpha_0) \circ \mathrm{T}_0 - \mathrm{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)},$$

with

$$\begin{aligned}\mathcal{E}_1^{(n)} &:= DG_0(\mathrm{u}_0) \circ \mathbb{E}_n^{\mathrm{ext}} \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1} \circ \mathrm{T}_0,\\ \mathcal{E}_2^{(n)} &:= DG_0(\mathrm{u}_0) \circ \mathscr{P}_n \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1} \circ \mathrm{T}_0,\\ \mathcal{E}_3^{(n)} &:= DG_0(\mathrm{u}_0) \circ \mathscr{Q}_n \circ [\widetilde{G}_0(\mathrm{u}_0)]^{-1} \circ \mathrm{T}_0.\end{aligned}$$

The estimates on $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ come from (5.114), Proposition 5.3 and Lemma 5.3-(i). □

6 Reduction of the linearized operator in the normal directions

In this section, we fix a torus $i_0 = (\vartheta_0, I_0, z_0)$ close to the flat one and satisfying the reversibility condition (5.25), that is

$$\vartheta_0(-\varphi) = -\vartheta_0(\varphi), \quad I_0(-\varphi) = I_0(\varphi), \quad z_0(-\varphi) = (\mathscr{S} z_0)(\varphi). \tag{6.1}$$

As in the previous section, we denote $\mathfrak{I}_0(\varphi) = i_0(\varphi) - (\varphi, 0, 0)$. Our main goal here is to explore the invertibility of the operator

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathcal{L}}_\omega(i_0) = \Pi_{\mathbb{S}_0}^{\perp}\left(\omega \cdot \partial_\varphi - \partial_\theta \mathsf{K}_{02}(\varphi)\right)\Pi_{\mathbb{S}_0}^{\perp} \tag{6.2}$$

defined through (5.87) and (5.72) with the suitable tame estimates for the inverse. For a precise statement we refer to Proposition 6.6. Notice that this operator will be described as a quasilinear perturbation of the diagonal operator stated in Lemma 3.1 and we expect that suitable standard reductions can be performed to conjugate it to a diagonal one provided that the exterior parameters are subject to live in a Cantor set allowing to prevent resonances. For this aim, we shall implement with suitable adaptions the strategy developed in the works [4, 14]. We distinguish two long reduction steps. First, we perform a quasi-periodic change of variables such that in the new coordinates system the transport part is straightened to a constant coefficient operator. The construction of this transformation is based on a KAM reducibility procedure as in [26]. The outcome of this first step is a new operator whose positive part is diagonal with a small nonlocal perturbation of order -1. Then the second step consists in applying KAM scheme in order to reduce the remainder and conjugate the resulting operator from step 1 into a diagonal one up to small errors. The proof follows basically a common procedure that can be found for instance in [10]. We point out that our results differ slightly from the preceding ones in [4, 14], especially at the level of Cantor sets which are constructed over the final targets.
We shall use throughout the proofs some frequency cut-offs with respect to the sequence defined in (5.82), with N_0 a constant needed to be large enough. In the current section, the numbers $N_0 \geqslant 2$ and $\gamma \in (0,1)$ are a priori free parameters, but during the Nash-Moser scheme, see Proposition 7.1, they will be adjusted with respect to ε according to the relations

$$N_0 = \gamma^{-1} \quad \text{and} \quad \gamma = \varepsilon^a \quad \text{for some } a > 0.$$

We shall set the following parameters required along the different reductions that we intend to perform,

$$\begin{array}{ll} s_l := s_0 + \tau_1 q + \tau_1 + 2, & \overline{\mu}_2 := 4\tau_1 q + 6\tau_1 + 3, \\ \overline{s}_l := s_l + \tau_2 q + \tau_2, & \overline{s}_h := \frac{3}{2}\overline{\mu}_2 + s_l + 1, \end{array} \tag{6.3}$$

supplemented with the assumptions (4.1) and (4.2).

6.1 Localization on the normal directions

According to Theorem 5.1, the construction of an approximate inverse for $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ is based on Proposition 5.2 dealing with finding an approximate right inverse for the operator $\widehat{\mathcal{L}}_\omega$. This program will be achieved along several steps and in the first one we shall describe its asymptotic structure around the linearized operator at the equilibrium state described in Lemma 3.1. More precisely, we shall prove the following result.

Proposition 6.1. *Let (γ, q, d, s_0) satisfy (4.1). Then the operator $\widehat{\mathcal{L}}_\omega$ defined in (6.2) takes the form*

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^{\perp} \left(\mathcal{L}_{\varepsilon r} - \varepsilon \partial_\theta \mathcal{R}\right) \Pi_{\mathbb{S}_0}^{\perp}, \quad \mathcal{L}_{\varepsilon r} = \omega \cdot \partial_\varphi + \partial_\theta \left(V_{\varepsilon r} \cdot\right) - \partial_\theta \mathbf{L}_{\varepsilon r},$$

where $V_{\varepsilon r}$ and $\mathbf{L}_{\varepsilon r}$ are defined in Lemma 3.1, and from (5.13) we have

$$\begin{aligned} r(\varphi) &= A\big(\vartheta_0(\varphi), I_0(\varphi), z_0(\varphi)\big) \\ &= v\big(\vartheta_0(\varphi), I_0(\varphi)\big) + z_0(\varphi), \end{aligned}$$

supplemented with the reversibility assumption

$$r(\lambda, \omega, -\varphi, -\theta) = r(\lambda, \omega, \varphi, \theta). \tag{6.4}$$

Moreover, $\mathcal{R}$ is an integral operator in the sense of the Definition 4.3, whose kernel J satisfies the symmetry property

$$J(\lambda, \omega, -\varphi, -\theta, -\eta) = J(\lambda, \omega, \varphi, \theta, \eta). \tag{6.5}$$

and under the assumption

$$\|\mathfrak{I}_0\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.6}$$

we have for all $s \geqslant s_0$,

(i) The function r satisfies the estimates,

$$\|r\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1+\|\mathfrak{I}_0\|_{q,s}^{\gamma,\mathcal{O}} \tag{6.7}$$

and

$$\|\Delta_{12} r\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\Delta_{12} i\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12} i\|_{q,s_0}^{\gamma,\mathcal{O}} \max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}}. \tag{6.8}$$

(ii) The kernel J satisfies the following estimates for all $\ell\in\mathbb{N}$,

$$\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^\ell J)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1+\|\mathfrak{I}_0\|_{q,s+3+\ell}^{\gamma,\mathcal{O}} \tag{6.9}$$

and

$$\sup_{\eta\in\mathbb{T}}\|\Delta_{12}(\partial_\theta^\ell J)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\Delta_{12} i\|_{q,s+3+\ell}^{\gamma,\mathcal{O}} + \|\Delta_{12} i\|_{q,s_0+3}^{\gamma,\mathcal{O}} \max_{j=1,2}\|\mathfrak{I}_j\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}. \tag{6.10}$$

Here $*,\cdot,\centerdot$ stand for $(\lambda,\omega),\varphi,\theta$, respectively and $\mathfrak{I}_\ell(\varphi)=i_\ell(\varphi)-(\varphi,0,0)$. In addition, for any function f, $\Delta_{12}f:=f(i_1)-f(i_2)$ refers for the difference of f taken at two different states i_1 and i_2 satisfying (6.6).

Proof. To alleviate the notation we shall at several stages of the proof remove the dependence of the involved functions/operators with respect to (λ,ω) and keep it when we deem it relevant. Recall that the operator $\widehat{\mathcal{L}}_\omega$ is defined in (6.2). To describe $\mathsf{K}_{02}(\varphi)$ we follow [4, 14]. First, we observe from (5.72) and (5.22) that

$$\mathsf{K}_{02}(\varphi)=\mathrm{L}(\lambda)+\varepsilon\partial_w\nabla_w(\mathcal{P}_\varepsilon(i_0(\varphi))+\varepsilon\mathcal{R}(\varphi),$$

with

$$\mathcal{R}(\varphi)=\mathcal{R}_1(\varphi)+\mathcal{R}_2(\varphi)+\mathcal{R}_3(\varphi),$$

where

$$\begin{aligned}\mathcal{R}_1(\varphi)&:=L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi),\\ \mathcal{R}_2(\varphi)&:=L_2^\top(\varphi)\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)&:=\partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi).\end{aligned}$$

As we shall see, all the operators $\mathcal{R}_1(\varphi)$, $\mathcal{R}_2(\varphi)$ and $\mathcal{R}_3(\varphi)$ have a finite-dimensional rank. This property is obvious for the operator $L_2(\varphi)$ defined in (5.65), which sends in view of (5.62) the space $H_\perp^s$ to $\mathbb{R}^d$ and therefore for any $\rho\in H_\perp^s$ we write

$$L_2(\varphi)[\rho]=\sum_{k=1}^d\big\langle L_2(\varphi)[\rho],\underline{e}_k\big\rangle_{\mathbb{R}^d}\,\underline{e}_k=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}\,\underline{e}_k,$$

with $(\underline{e}_k)_{k=1}^d$ being the canonical basis of $\mathbb{R}^d$. Hence

$$\begin{aligned}\mathcal{R}_1(\varphi)[\rho]&=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_1(\varphi)[\underline{e}_k]\qquad\text{with}\qquad A_1(\varphi)=L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)[\rho]&=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_3(\varphi)[\underline{e}_k]\qquad\text{with}\qquad A_3(\varphi)=\partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi)).\end{aligned}$$

In a similar way, by setting $A_2(\varphi):=\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)):H_\perp^s\to\mathbb{R}^d$, then we may write

$$\mathcal{R}_2(\varphi)[\rho]=\sum_{k=1}^d\big\langle\rho,A_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}L_2^\top(\varphi)[\underline{e}_k].$$

Define

$$g_{k,1}(\varphi,\theta)=g_{k,3}(\varphi,\theta)=\chi_{k,2}(\varphi,\theta):=L_2^\top(\varphi)[\underline{e}_k](\theta),\quad g_{k,2}(\varphi,\theta):=A_2^\top(\varphi)[\underline{e}_k](\theta)$$

and

$$\chi_{k,1}(\varphi,\theta):=A_1(\varphi)[\underline{e}_k](\theta),\quad \chi_{k,3}(\varphi,\theta):=A_3(\varphi)[\underline{e}_k](\theta),$$

then we can see that the operator $\mathcal{R}$ takes the integral form

$$\begin{aligned}\mathcal{R}\rho(\varphi,\theta)&=\sum_{k'=1}^{3}\sum_{k=1}^{d}\langle\rho(\varphi,\cdot),g_{k,k'}(\varphi,\cdot)\rangle_{L^2(\mathbb{T})}\chi_{k,k'}(\varphi,\theta)\\&=\int_{\mathbb{T}}\rho(\varphi,\eta)J(\varphi,\theta,\eta)d\eta,\end{aligned}$$

with

$$J(\varphi,\theta,\eta):=\sum_{k'=1}^{3}\sum_{k=1}^{d}g_{k,k'}(\varphi,\eta)\chi_{k,k'}(\varphi,\theta).$$

Now we remark that by construction $g_{k,k'},\chi_{k,k'}\in H_\perp^s$ with

$$\|g_{k,k'}\|_{q,s}^{\gamma,\mathcal{O}}+\|\chi_{k,k'}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\tag{6.11}$$

and straightforward computations yield

$$\|d_i g_{k,k'}[\widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}}+\|d_i\chi_{k,k'}[\widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{i}\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+4}^{\gamma,\mathcal{O}}\|\widehat{i}\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\tag{6.12}$$

On the other hand, one has from direct computations that

$$\forall\ell\in\mathbb{N},\quad(\partial_\theta^\ell J)(\varphi,\theta,\eta+\theta)=\sum_{k'=1}^{3}\sum_{k=1}^{d}g_{k,k'}(\varphi,\eta+\theta)(\partial_\theta^\ell\chi_{k,k'})(\varphi,\theta).$$

Hence, we may combine (6.11) with Lemma 4.1-(iv) and (6.6) allowing to get

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^\ell J)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\centerdot)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\centerdot)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\centerdot)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\&\lesssim 1+\|\mathfrak{I}_0\|_{q,s+3+\ell}^{\gamma,\mathcal{O}},\end{aligned}$$

where we have used the interpolation inequality: for $s\geqslant s_0$

$$\begin{aligned}\|g_{k,k'}(*,\cdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\centerdot)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}&\lesssim\|g_{k,k'}(*,\cdot,\eta+\centerdot)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\centerdot)\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\quad+\|g_{k,k'}(*,\cdot,\eta+\centerdot)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\centerdot)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\end{aligned}$$

In addition, to estimate the difference we simply write

$$\begin{aligned}\forall\ell\in\mathbb{N},\quad\Delta_{12}(\partial_\theta^\ell J)(\varphi,\theta,\eta+\theta)&=\sum_{k'=1}^{3}\sum_{k=1}^{d}\Delta_{12}g_{k,k'}(\varphi,\eta+\theta)(\partial_\theta^\ell(\chi_{k,k'})_{r_1})(\varphi,\theta)\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}(g_{k,k'})_{r_2}(\varphi,\eta+\theta)(\Delta_{12}\partial_\theta^\ell\chi_{k,k'})(\varphi,\theta).\end{aligned}$$

By applying the mean value theorem combined with (6.12) and (6.6) combined with interpolation inequalities

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|\Delta_{12}(\partial_\theta^\ell J)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\sum_{k'=1}^{3}\sum_{k=1}^{d}\|\Delta_{12}g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\&+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|\Delta_{12}g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\&+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\&+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\&\lesssim\|\Delta_{12}i\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0+3}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}.\end{aligned}$$

The symmetry property detailed in (6.5) is a consequence of the definition of r and the reversibility condition (6.1) imposed on the torus i_0. Consequently, putting together (5.6) and (5.13) gives

$$\begin{aligned}\mathsf{K}_{02}(\varphi)&=\mathrm{L}(\lambda)\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\partial_z\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))+\varepsilon\mathcal{R}(\varphi)\\&=\mathrm{L}(\lambda)\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_rP_\varepsilon(A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi)\\&=\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r\mathcal{H}_\varepsilon(A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi)\\&=\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_rH(\varepsilon A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi).\end{aligned}$$

Recall from (5.13) that

$$r(\varphi,\cdot)=A(i_0(\varphi)),\tag{6.13}$$

then according to the general form of the linearized operator stated in Lemma 3.1 one has

$$-\partial_\theta\partial_r\nabla_rH(\varepsilon r(\varphi,\cdot))=\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r},$$

which implies in turn

$$-\mathsf{K}_{02}(\varphi)=\Pi_{\mathbb{S}_0}^{\perp}\big(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r})-\varepsilon\mathcal{R}(\varphi)\big)\Pi_{\mathbb{S}_0}^{\perp}.$$

Plugging this identity into (6.2) gives the desired result. Next, using (6.13), (5.13) and (5.59), we obtain

$$\begin{aligned}\|r\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|v(\vartheta_0,I_0)\|_{q,s}^{\gamma,\mathcal{O}}+\|z_0\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim1+\|\mathfrak{I}_0\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

We shall now move to the proof of the bound (6.8). First, we observe from (5.13) that

$$\|\Delta_{12}r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}v(\vartheta,I)\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}z\|_{q,s}^{\gamma,\mathcal{O}}.$$

Therefore, Taylor Formula with (5.59) and law products allow to get

$$\|\Delta_{12}v(\vartheta,I)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}(I,\vartheta)\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}(I,\vartheta)\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}},$$

which implies that

$$\|\Delta_{12}r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}i\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}}.$$

This achieves the proof of Proposition 6.1. $\square$

6.2 Reduction of order 1

In this section, we perform the reduction of the transport part of the linearized operator $\mathcal{L}_{\varepsilon r}$ described in Proposition 6.1. More precisely, we conjugate the operator $\mathcal{L}_{\varepsilon r}$ by a quasi-periodic symplectic change of variables $\mathscr{B}$ leading to a transport part with constant coefficients depending only on the torus i_0 and the parameters ε, λ and ω. To get a precise information on the remainder, which is of order -1 in θ, we need to describe the action of this conjugation on the nonlocal term using the kernel structure rather than pseudo-differential theory. The reduction to a constant coefficient operator is based on KAM scheme through the construction of successive quasi-periodic symplectic change of coordinates. This will be implemented in the same spirit of [13, 26]. Here we need to extend their construction to the framework of of symplectic change of coordinates with C^q regularity. We point out that similar results with slight variations have been established in [7, 13] in a non-symplectic framework.

6.2.1 Reduction of the transport part

Before stating our result, we need to introduce some transformations. Let $\beta : \mathcal{O} \times \mathbb{T}^{d+1} \to \mathbb{T}$ be a smooth function such that $\sup_{\mu \in \mathcal{O}} \|\beta(\mu, \cdot, \cdot)\|_{\mathrm{Lip}} < 1$ then the map

$$(\varphi, \theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \beta(\mu, \varphi, \theta)) \in \mathbb{T}^{d+1}$$

is a diffeomorphism and its inverse takes the form

$$(\varphi, \theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \widehat{\beta}(\mu, \varphi, \theta)) \in \mathbb{T}^{d+1}.$$

The relation between β and $\widehat{\beta}$ is described through,

$$y = \theta + \beta(\mu, \varphi, \theta) \Longleftrightarrow \theta = y + \widehat{\beta}(\mu, \varphi, y). \tag{6.14}$$

Now we define the operators

$$\mathscr{B} = (1 + \partial_\theta \beta)\mathcal{B}, \tag{6.15}$$

with

$$\mathcal{B}\rho(\mu, \varphi, \theta) = \rho\big(\mu, \varphi, \theta + \beta(\mu, \varphi, \theta)\big).$$

Direct computations show that the inverse $\mathscr{B}^{-1}$ keeps the same form, that is,

$$\mathscr{B}^{-1}\rho(\mu, \varphi, y) = \Big(1 + \partial_y \widehat{\beta}(\mu, \varphi, y)\Big)\rho\big(\mu, \varphi, y + \widehat{\beta}(\mu, \varphi, y)\big) \tag{6.16}$$

and

$$\mathcal{B}^{-1}\rho(\mu, \varphi, y) = \rho\big(\mu, \varphi, y + \widehat{\beta}(\mu, \varphi, y)\big).$$

We shall now give some elementary algebraic properties for $\mathcal{B}^{\pm 1}$ and $\mathscr{B}^{\pm 1}$ which can be checked by straightforward computations.

Lemma 6.1. *The following assertions hold true.*

(i) The action of $\mathscr{B}^{-1}$ on the derivative is given by

$$\mathscr{B}^{-1}\partial_\theta = \partial_\theta \mathcal{B}^{-1}.$$

(ii) The conjugation of the transport operator by $\mathscr{B}$ keeps the same structure

$$\mathscr{B}^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\big(V(\varphi, \theta) \cdot\big)\Big)\mathscr{B} = \omega \cdot \partial_\varphi + \partial_y(\mathscr{V}(\varphi, y) \cdot),$$

with

$$\mathscr{V}(\varphi, y) = \mathcal{B}^{-1}\Big(\omega \cdot \partial_\varphi \beta(\varphi, \theta) + V(\varphi, \theta)\big(1 + \partial_\theta \beta(\varphi, \theta)\big)\Big).$$

(iii) Denote by $\mathscr{B}^\star$ the $L^2_\theta(\mathbb{T})$-adjoint of $\mathscr{B}$, then

$$\mathscr{B}^\star = \mathcal{B}^{-1} \quad \textit{and} \quad \mathcal{B}^\star = \mathscr{B}^{-1}.$$

Now we shall state the following result proved in [26] for $q = 1$ and which can be obtained by induction for a general $q \in \mathbb{N}^*$ up to slight modifications. We also refer to [13, (A.2)].

Lemma 6.2. *Let $(q, d\gamma, s_0)$ as in (4.1). Let $\beta \in W^{q,\infty,\gamma}\big(\mathcal{O}, H^\infty(\mathbb{T}^{d+1})\big)$ such that*

$$\|\beta\|_{q,2s_0}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0, \tag{6.17}$$

with ε_0 small enough. Then the following assertions hold true.

(i) The linear operators $\mathcal{B}, \mathscr{B} : W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big) \to W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big)$ are continuous and invertible, with

$$\forall s \geqslant s_0, \quad \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \big(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\big) + C\|\beta\|_{q,s}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \tag{6.18}$$

and

$$\forall s \geqslant s_0, \quad \|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \big(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\big) + C\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}} \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.19}$$

(ii) The functions β and $\widehat{\beta}$ are linked through

$$\forall s \geqslant s_0, \quad \|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\|\beta\|_{q,s}^{\gamma,\mathcal{O}}. \tag{6.20}$$

(iii) Let $\beta_1, \beta_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^\infty(\mathbb{T}^{d+1}))$ satisfying (6.17). If we denote

$$\Delta_{12}\beta = \beta_1 - \beta_2 \quad \text{and} \quad \Delta_{12}\widehat{\beta} = \widehat{\beta}_1 - \widehat{\beta}_2,$$

then they are linked through

$$\forall s \geqslant s_0, \quad \|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C \left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}} \max_{j\in\{1,2\}} \|\beta_j\|_{q,s+1}^{\gamma,\mathcal{O}} \right). \tag{6.21}$$

Proof. **(i)-(ii)** For (6.18) and (6.20), we refer to [13, (A.2)] and [26, Lem. A.3.]. The estimate (6.19) is obtained from (6.18) and law product in Lemma 4.1.
(iii) One has by Taylor Formula

$$\begin{aligned}
\Delta_{12}\widehat{\beta}(y) &= \widehat{\beta}_1(y) - \widehat{\beta}_2(y) \\
&= \beta_2(y + \widehat{\beta}_2(y)) - \beta_1(y + \widehat{\beta}_1(y)) \\
&= -\Delta_{12}\beta(y + \widehat{\beta}_2(y)) - \Delta_{12}\widehat{\beta}(y) \int_0^1 \partial_\theta \beta_1(y + \widehat{\beta}_1(y) - t\Delta_{12}\widehat{\beta}(y))dt.
\end{aligned}$$

Hence

$$\Delta_{12}\widehat{\beta}(y) = \tfrac{-\mathcal{B}_2^{-1}\Delta_{12}\beta(y)}{1+\mathscr{I}(y)} \quad \text{with} \quad \mathscr{I}(y) := \int_0^1 \partial_\theta \beta_1(y + \widehat{\beta}_1(y) - t\Delta_{12}\widehat{\beta}(y))dt.$$

By composition estimate in Lemma 4.1, one has

$$\left\| \tfrac{1}{1+\mathscr{I}} \right\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}}.$$

Thus, applying the law product in Lemma 4.1 implies

$$\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \big(1 + \|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}}\big) \|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}} + \big(1 + \|\mathscr{I}\|_{q,s_0}^{\gamma,\mathcal{O}}\big) \|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}.$$

Using (6.18), (6.20) and (6.17) yields

$$\begin{aligned}\|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}\left(1+\|\widehat{\beta}_2\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\|\widehat{\beta}_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\end{aligned}$$

and

$$\begin{aligned}\|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}\left(1+\|\widehat{\beta}_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\left(\|\widehat{\beta}_1\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\right)\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the foregoing estimates gives

$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\left(1+\|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\right)\left(1+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad+C\left(1+\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\right)\left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.22}$$

From the triangle inequality, (6.20) and (6.17), one has

$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant \|\widehat{\beta}_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant \|\beta_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant 2\varepsilon_0.\end{aligned}$$

From Sobolev embeddings we infer that

$$\max_{j\in\{1,2\}}\|\beta_j\|_{q,s_0+1}^{\gamma,\mathcal{O}}\leqslant \max_{j\in\{1,2\}}\|\beta_j\|_{q,2s_0}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.$$

Thus, by choosing ε_0 small enough, we can ensure

$$C\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(1+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\frac{1}{2}\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}.$$

Inserting this term into the left hand side in (6.22) and using Sobolev embeddings, we find

$$\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C\left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j\in\{1,2\}}\|\beta_j\|_{q,s+1}^{\gamma,\mathcal{O}}\right).$$

This ends the proof of Lemma 6.2. □

Now we shall state the main result of this section concerning the reduction of the transport part of the linearized operator $\mathcal{L}_{\varepsilon r}$.

Proposition 6.2. *Let $(\gamma,q,d,\tau_1,s_0,S,s_l,\overline{s}_h,\overline{\mu}_2)$ satisfy (4.1), (4.2) and (6.3). Let $\upsilon\in\left(0,\frac{1}{q+2}\right]$. We set*

$$\sigma_1=s_0+\tau_1 q+2\tau_1+4.\tag{6.23}$$

For any $(\mu_2,\mathtt{p},s_h)$ satisfying

$$\mu_2\geqslant\overline{\mu}_2:=4\tau_1 q+6\tau_1+3,\quad \mathtt{p}\geqslant 0,\quad s_h\geqslant\max\left(\frac{3}{2}\mu_2+s_l+1,\overline{s}_h+\mathtt{p}\right),\tag{6.24}$$

there exists $\varepsilon_0>0$ such that if

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_1}^{\gamma,\mathcal{O}}\leqslant 1,\tag{6.25}$$

there exist

$$c_{i_0}\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{R})\quad\text{and}\quad\beta\in\bigcap_{s\in[s_0,S]}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\text{odd}})$$

such that with $\mathscr{B}$ defined in (6.15) one gets the following results.

(i) The function c_{i_0} satisfies the following estimate,

$$\|c_{i_0} - V_0\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon, \tag{6.26}$$

where V_0 is defined in Lemma 3.2.

(ii) The transformations $\mathscr{B}^{\pm 1}, \mathcal{B}^{\pm 1}, \beta$ and $\widehat{\beta}$ satisfy the following estimates for all $s \in [s_0, S]$

$$\|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \tag{6.27}$$

and

$$\|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\beta\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right). \tag{6.28}$$

(iii) Let $n \in \mathbb{N}$, then in the truncated Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \left\{(\lambda,\omega) \in \mathcal{O} \quad \text{s.t.} \quad \big|\omega\cdot l + jc_{i_0}(\lambda,\omega)\big| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\},$$

we have

$$\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi + \partial_\theta\big(V_{\varepsilon r}\,\cdot\,\big)\big)\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}\partial_\theta + \mathtt{E}_n^0,$$

with $\mathtt{E}_n^0 = \mathtt{E}_n^0(\lambda,\omega,i_0)$ a linear operator satisfying

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon N_0^{\mu_2} N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{6.29}$$

(iv) Given two tori i_1 and i_2 both satisfying (6.25), *we have*

$$\|\Delta_{12}c_i\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} \tag{6.30}$$

and

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} + \|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}. \tag{6.31}$$

Before giving the proof, some remarks are in order.

Remark 6.1.

- *The final Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is constructed over the limit coefficient c_{i_0} but it is still truncated in the time frequency, that is $|l| \leqslant N_n$, leading to a residual remainder with enough decay. This induces a suitable stability property that is crucial during the Nash-Moser scheme achieved with the nonlinear functional.*
- *Notice that, since $4\gamma^{\upsilon} \geqslant \gamma$, then looking at $j = 0$ we find that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is contained in the Diophantine Cantor set $(\lambda_0,\lambda_1)\times\mathtt{DC}_{N_n}(\gamma,\tau_1)$ introduced in* (5.79).
- *The parameter υ is introduced for technical reasons appearing later in the measure estimates of the final Cantor set and it will be fixed in* (7.64).
- *The constant 4 used in the definition of the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is useful to ensure the inclusion of this set in all the Cantor sets built in the KAM procedure (see* (6.90) *in the proof below) and also to establish some inclusions related to the final Cantor set (see the proof of Lemma* 7.2*).*
- *We emphasize here that the functions β and $\widehat{\beta}$ are odd in the sense*

$$\beta(\lambda,\omega,-\varphi,-\theta) = -\beta(\lambda,\omega,\varphi,\theta) \quad \text{and} \quad \widehat{\beta}(\lambda,\omega,-\varphi,-\theta) = -\widehat{\beta}(\lambda,\omega,\varphi,\theta) \tag{6.32}$$

which will be crucial later to get the Toeplitz structure of the new remainder term emerging after this reduction.

Proof. Since we are looking at a state near the disc, we can split $V_{\varepsilon r}$ defined by (3.1) according to

$$V_{\varepsilon r}(\lambda,\varphi,\theta)=V_0(\lambda)+f_0(\lambda,\varphi,\theta), \tag{6.33}$$

with f_0 being a perturbation term of small size. We refer to (6.56) for a more precise quantification of this smallness. The proof is an iteration process introducing at each step a linear quasi-periodic symplectic change of coordinates. This transformation is linked to the remainder term of the previous step. Roughly speaking, if the latter is of size ε, then we choose the change of coordinates in such a way that we extract the main diagonal part of the previous remainder and keep a new perturbation term of size ε^2. The choice of the transformation is done through the resolution of an homological equation requiring non-resonance conditions capted by a suitable selection of the parameters of the system. Thus, by iteration, we can construct a final Cantor set gathering all the parameters restrictions of all steps in which we completely reduced the transport operator into a constant coefficient one. We shall now explain a typical step of the procedure Later, we shall implement the scheme.

(i)-(ii) ▶ KAM step. Let us consider a transport operator in the form,

$$\omega\cdot\partial_\varphi+\partial_\theta\Big(V+f\Big)$$

for suitable parameters (λ,ω) that belong to a subset $\mathcal{O}_-^\gamma\subset\mathcal{O}$, where $\mathcal{O}$ is the ambient set and

$$V=V(\lambda,\omega)\quad\text{and}\quad f=f(\lambda,\omega,\varphi,\theta),$$

where f enjoys the following symmetry condition

$$f(\lambda,\omega,-\varphi,-\theta)=f(\lambda,\omega,\varphi,\theta). \tag{6.34}$$

To alleviate the notations we shall use during the proof the variable $\mu:=(\lambda,\omega)$. We consider a symplectic quasi-periodic change of coordinates close to the identity taking the form

$$\begin{aligned}\mathscr{G}\rho(\mu,\varphi,\theta)&:=\big(1+\partial_\theta g(\mu,\varphi,\theta)\big)\mathcal{G}\rho(\mu,\varphi,\theta)\\&:=\big(1+\partial_\theta g(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+g(\mu,\varphi,\theta)\big),\end{aligned} \tag{6.35}$$

where $g:\mathcal{O}\times\mathbb{T}^{d+1}\to\mathbb{R}$ is a small which will be later linked to f. Then, by using Lemma 6.1, we can write for any $N\geqslant 2$

$$\mathscr{G}^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta\left(V+f\right)\Big)\mathscr{G}=\omega\cdot\partial_\varphi+\partial_\theta\mathcal{G}^{-1}\Big(V+\omega\cdot\partial_\varphi g+V\partial_\theta g+\Pi_N f+\Pi_N^\perp f+f\partial_\theta g\Big). \tag{6.36}$$

Recall that the projections Π_N are defined in (4.4). The basic idea is to obtain after this transformation a new transport operator in the form

$$\mathscr{G}^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta\left(V+f\right)\Big)\mathscr{G}=\omega\cdot\partial_\varphi+\partial_\theta\big(V_++f_+\big), \tag{6.37}$$

where

$$V_+=V_+(\mu)\quad\text{and}\quad f_+=f_+(\mu,\varphi,\theta),$$

with f_+ quadratically smaller than f. In order to get rid of the terms wich are not small of quadratic in f, then, in view of (6.36), we shall select g solving the following *homological equation*

$$\omega\cdot\partial_\varphi g+V\partial_\theta g+\Pi_N f=\langle f\rangle_{\varphi,\theta}, \tag{6.38}$$

where

$$\langle f\rangle_{\varphi,\theta}(\mu):=\int_{\mathbb{T}^{d+1}}f(\mu,\varphi,\theta)d\varphi d\theta.$$

To find a solution to the *homological equation* (6.38), we use Fourier decomposition and look for g in the form

$$g(\mu,\varphi,\theta):=\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\backslash\{0\}\\ \langle l,j\rangle\leqslant N}}\tfrac{f_{l,j}(\mu)}{\omega\cdot l+jV(\mu)}e^{\mathrm{i}(l\cdot\varphi+j\theta)}. \tag{6.39}$$

The denominators appearing in the Fourier decomposition of g may be small and generate problems in the convergence of the series in (6.39) for large N. This is a well-knonw phenomenon in KAM theory called "small divisors problem". To overcome this difficulty, one has to avoid the resonances and, following the ideas of Kolmogorov, we introduce Diophantine conditions gathered in the following Cantor set

$$\mathcal{O}_{+}^{\gamma}:=\bigcap_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N}}\Big\{\mu:=(\lambda,\omega)\in\mathcal{O}_{-}^{\gamma}\quad\text{s.t.}\quad\big|\omega\cdot l+jV(\mu)\big|>\tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.\tag{6.40}$$

Such a selection of the external parameters allows us to control the size of the denominators in (6.39). As we shall see in (6.48), the quantification of this control, linked to the parameters γ and τ_1, allows to get suitable estimates for g with some loss of regularity uniform with respect to N. Before performing this estmate, we shall first construct an extension of g to the whole set $\mathcal{O}$. In what follows, we still denote g this extension. This is done by extending the Fourier coefficients of g using the cut-off function χ defined in (5.81). More precisely, we define

$$\begin{aligned}g_{l,j}(\mu)&:=\mathrm{i}\tfrac{\chi\left((\omega\cdot l+jV(\mu))(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}\right)}{\omega\cdot l+jV(\mu)}f_{l,j}(\mu)\\ &:=\widetilde{g_{l,j}}(\mu)f_{l,j}(\mu).\end{aligned}\tag{6.41}$$

Notice that the extension g is a solution to (6.38) only when the parameters are restricted to the Cantor set $\mathcal{O}_{+}^{\gamma}$. Then, we define

$$V_{+}=V+\langle f\rangle_{\varphi,\theta}\quad\text{and}\quad f_{+}=\mathcal{G}^{-1}\big(\Pi_N^{\perp}f+f\partial_\theta g\big),$$

so that in restriction to the Cantor set $\mathcal{O}_{+}^{\gamma}$, the identity (6.37) holds. Remark that V_{+} and f_{+} are well-defined in the whole set of parameters $\mathcal{O}$ and the function g is smooth since it is generated by a finite number of frequencies. According to (6.34), we obtain that g is odd. As a consequence,

$$g\in\bigcap_{s\geqslant 0}W^{q,\infty,\gamma}(\mathcal{O},H^{s}_{\mathrm{odd}}).\tag{6.42}$$

Our next task is to estimate the Fourier coefficients $\widetilde{g_{l,j}}$ defined by (6.41). Notice that we can write them in the following form

$$\begin{aligned}\widetilde{g_{l,j}}(\mu)&=\mathrm{i}\,a_{l,j}\widehat{\chi}(a_{l,j}A_{l,j}(\mu)),\quad\widehat{\chi}(x):=\tfrac{\chi(x)}{x}\\ A_{l,j}(\mu)&:=\omega\cdot l+jV(\mu),\quad a_{l,j}:=(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}.\end{aligned}\tag{6.43}$$

Since $\widehat{\chi}$ is C^∞ with bounded derivatives and $\widehat{\chi}(0)=0$, then applying Lemma 4.1-(vi), we obtain

$$\forall q'\in[\![0,q]\!],\quad\|\widetilde{g_{l,j}}\|_{q'}^{\gamma,\mathcal{O}}\lesssim a_{l,j}^{2}\|A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\Big(1+a_{l,j}^{q'-1}\|A_{l,j}\|_{L^\infty(\mathcal{O})}^{q'-1}\Big).$$

Direct computations lead to

$$\begin{aligned}\forall(l,j)\in\mathbb{Z}^{d+1},\,\forall\alpha\in\mathbb{N}^{d+1},\quad|\alpha|\leqslant q,\quad\sup_{\mu\in\mathcal{O}}\big|\partial_\mu^\alpha A(\mu)\big|&\lesssim\langle l,j\rangle\max\left(1,\sup_{\mu\in\mathcal{O}}\big|\partial_\mu^\alpha V(\mu)\big|\right)\\ &\lesssim\gamma^{-|\alpha|}\langle l,j\rangle\max\big(1,\|V\|_q^{\gamma,\mathcal{O}}\big).\end{aligned}$$

Assuming

$$\|V\|_q^{\gamma,\mathcal{O}}\leqslant C,\tag{6.44}$$

we then obtain

$$\forall q'\in[\![0,q]\!],\quad\forall(l,j)\in\mathbb{Z}^{d+1},\quad\|A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\langle l,j\rangle.\tag{6.45}$$

Added to the fact that $0\leqslant a_{l,j}\leqslant\gamma^{-\upsilon}\langle l\rangle^{\tau_1}$, we then find that

$$\forall q'\in[\![0,q]\!],\quad\|\widetilde{g_{l,j}}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\gamma^{-\upsilon(q'+1)}\langle l,j\rangle^{\tau_1 q'+\tau_1+q'}.\tag{6.46}$$

Our choice of υ in Proposition 6.2 implies in particular that

$$\upsilon \leqslant \tfrac{1}{q+1}. \tag{6.47}$$

Therefore, we deduce from (6.41) and Leibniz rule that for all $\alpha \in \mathbb{N}^{d+1}$ with $|\alpha| \leqslant q$

$$\begin{aligned}\gamma^{2|\alpha|}\|\partial_\mu^\alpha g(\mu,\cdot,\centerdot)\|_{H^{s-|\alpha|}}^2 &\lesssim \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\ \langle l,j\rangle\leqslant N}} \sum_{\substack{\beta\in\mathbb{N}^{d+1}\\ \beta\leqslant\alpha}} \gamma^{2|\alpha|-2|\beta|}\big|\partial_\mu^{\alpha-\beta}\widetilde{g_{l,j}}(\mu)\big|^2\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2s-2|\alpha|}\\ &\lesssim \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\ \langle l,j\rangle\leqslant N}} \sum_{\substack{\beta\in\mathbb{N}^{d+1}\\ \beta\leqslant\alpha}} \left(\|\widetilde{g_{l,j}}\|_{|\alpha|-|\beta|}^{\gamma,\mathcal{O}}\right)^2\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2s-2|\alpha|}\\ &\lesssim \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\ \langle l,j\rangle\leqslant N}} \sum_{\substack{\beta\in\mathbb{N}^{d+1}\\ \beta\leqslant\alpha}} \gamma^{-2}\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2(s+\tau_1 q+\tau_1-|\beta|)}.\end{aligned}$$

As a consequence, by interverting the summation symbols, we find

$$\|g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\|\Pi_N f\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}. \tag{6.48}$$

Assume now that

$$\gamma^{-1}N^{\tau_1 q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0. \tag{6.49}$$

Then added to (6.48) and Lemma 4.1-(ii), we get

$$\|g\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant C\gamma^{-1}N^{\tau_1 q+\tau_1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant C\varepsilon_0.$$

On the other hand if we assume

$$\|f\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right),$$

then (6.48) gives

$$\begin{aligned}\|g\|_{q,2s_0+1}^{\gamma,\mathcal{O}} &\lesssim \gamma^{-1}\|f\|_{q,2s_0+\tau_1 q+\tau_1+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,2s_0+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_h+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Notice that to obtain the last inequality we used the fact that (6.24) and (6.23) imply

$$2s_0+\tau_1 q+\tau_1+2 \leqslant s_h+\sigma_1.$$

Using interpolation inequality and (6.25), one gets for some $\overline{\theta}\in(0,1)$.

$$\begin{aligned}\|g\|_{q,2s_0}^{\gamma,\mathcal{O}} &\lesssim \left(\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|g\|_{q,2s_0+1}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}\\ &\lesssim \varepsilon_0.\end{aligned} \tag{6.50}$$

Thus, taking ε_0 small enough, we can ensure the smallness condition in Lemma 6.2 and get that the linear operator $\mathscr{G}$ is invertible. Now, we introduce

$$u = \Pi_N^\perp f + f\partial_\theta g.$$

By the triangle inequality, Lemma 4.1-(ii) and (6.48), we obtain for all $s\in[s_0,S]$

$$\begin{aligned}\|u\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Pi_N^\perp f\|_{q,s}^{\gamma,\mathcal{O}} + C\Big(\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta g\|_{q,s}^{\gamma,\mathcal{O}} + \|f\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta g\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\ &\leqslant \|\Pi_N^\perp f\|_{q,s}^{\gamma,\mathcal{O}} + C\Big(\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|g\|_{q,s+1}^{\gamma,\mathcal{O}} + \|f\|_{q,s}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+1}^{\gamma,\mathcal{O}}\Big)\\ &\leqslant \|\Pi_N^\perp f\|_{q,s}^{\gamma,\mathcal{O}} + C\gamma^{-1}N^{\tau_1 q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combined with Lemma 6.2, Lemma 4.1-(ii) and (6.49), we get for all $s\in[s_0,S]$

$$\begin{aligned}\|f_+\|_{q,s}^{\gamma,\mathcal{O}}&=\|\mathcal{G}^{-1}(u)\|_{q,s}^{\gamma,\mathcal{O}}\\&\leqslant\|u\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|u\|_{q,s}^{\gamma,\mathcal{O}}\|\widehat{g}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\widehat{g}\|_{q,s}^{\gamma,\mathcal{O}}\|u\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|u\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|u\|_{q,s}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}+\|g\|_{q,s}^{\gamma,\mathcal{O}}\|u\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|\Pi_N^{\perp}f\|_{q,s}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Using once again Lemma 4.1-(ii), we find for $S\geqslant\overline{s}\geqslant s\geqslant s_0$

$$\|f_+\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N^{s-\overline{s}}\|f\|_{q,\overline{s}}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\tag{6.51}$$

► **KAM scheme.** Let us now assume that we have constructed V_m and f_m, well-defined in the whole set of parameters $\mathcal{O}$ and satisfying the assumptions (6.44) and (6.49). We shall now construct the corresponding quantity at the next order, namely V_{m+1} and f_{m+1}, still satisfying (6.44) and (6.49). For this aim, we shall implement the KAM step with (V,f,V_+,f_+,N) replaced by $(V_m,f_m,V_{m+1},f_{m+1},N_m)$. More precisely, we will shall prove by induction the existence of a sequence $\{V_m,f_m\}_{m\in\mathbb{N}}$ such that

$$\delta_m(s_l)\leqslant\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\quad\text{ and }\quad\delta_m(s_h)\leqslant\left(2-\tfrac{1}{m+1}\right)\delta_0(s_h)\tag{6.52}$$

and

$$\|V_m\|_q^{\gamma,\mathcal{O}}\leqslant C\quad\text{and}\quad N_m^{\tau_1q+\tau_1+1}\delta_m(s_0)\leqslant\varepsilon_0,\tag{6.53}$$

with f_m satisfying the following symmetry condition

$$f_m(\mu,-\varphi,-\theta)=f_m(\mu,\varphi,\theta)\tag{6.54}$$

and where we denote

$$\delta_m(s):=\gamma^{-1}\|f_m\|_{q,s}^{\gamma,\mathcal{O}}.$$

Recall that the parameters s_l and s_h were intruduced in (6.3) and (6.24).

➢ *Initialization.* We shall first check that the estimates (6.52) and (6.53) are satisfied for $m=0$. In which case the functions V_0 and f_0 are defined by (3.11) and (6.33). By (5.48) and (6.7) we infer

$$\begin{aligned}\delta_0(s)&=\gamma^{-1}\|V_{\varepsilon r}-V_0\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.55}$$

Thus, the notation (6.24) and the smallness condition (6.25) imply that

$$N_0^{\mu_2}\delta_0(s_h)\leqslant C\varepsilon_0.\tag{6.56}$$

In addition, by (6.4) and (3.3), we deduce that f_0 satisfies the following symmetry condition

$$f_0(\lambda,-\varphi,-\theta)=f_0(\lambda,\varphi,\theta).\tag{6.57}$$

We set $\mathcal{O}_0^{\gamma}=\mathcal{O}$ and consider $N_0\geqslant2$. Our next task is to check that the assumptions (6.44) and (6.49) are satisfied by V_0 and f_0. First recall that V_0 is defined by

$$V_0(\lambda)=\Omega+I_1(\lambda)K_1(\lambda).$$

Using the smooth regularity of (A.9), we obtain

$$\|V_0\|_q^{\gamma,\mathcal{O}}\leqslant C.\tag{6.58}$$

Therefore, the required boundedness property (6.44) is satisfied with $V=V_0$. Now by (6.24), we have

$$\mu_2\geqslant\tau_1q+\tau_1+2.\tag{6.59}$$

Hence, using (6.56), we obtain

$$\begin{aligned}\gamma^{-1}N_0^{\tau_1 q+\tau_1+1}\|f_0\|_{q,s_0}^{\gamma,\mathcal{O}} &= N_0^{\tau_1 q+\tau_1+1}\delta_0(s_0)\\ &\leqslant N_0^{\tau_1 q+\tau_1+1-\mu_2}N_0^{\mu_2}\delta_0(s_h)\\ &\leqslant C\varepsilon_0 N_0^{-1}.\end{aligned}$$

By taking N_0 large enough we get

$$CN_0^{-1}\leqslant 1, \tag{6.60}$$

so that

$$\gamma^{-1}N_0^{\tau_1 q+\tau_1+1}\|f_0\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant \varepsilon_0.$$

Hence, the assumption (6.49) is satisfied for $f=f_0$. This ends the initialization step.

➢ *Iteration.* let us now assume that we have constructed V_m and f_m enjoying the properties (6.52), (6.53) and (6.54). We shall see how to construct V_{m+1} and f_{m+1}. According to the KAM step, we consider a symplectic quasi-periodic change of variables $\mathscr{G}_m$ taking the form

$$\begin{aligned}\mathscr{G}_m\rho(\mu,\varphi,\theta) &:= \big(1+\partial_\theta g_m(\mu,\varphi,\theta)\big)\mathcal{G}_m\rho(\mu,\varphi,\theta)\\ &= \big(1+\partial_\theta g_m(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+g_m(\mu,\varphi,\theta)\big),\end{aligned}$$

with

$$g_m(\mu,\varphi,\theta):=\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}}\tfrac{\chi\left((\omega\cdot l+jV_m(\mu))(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}\right)}{\omega\cdot l+jV_m(\mu)}(f_m)_{l,j}(\mu)\,e^{i(l\cdot\varphi+j\theta)}, \tag{6.61}$$

where χ is the cut-off function introduced in (5.81) and N_m is defined in (5.82). As explained in the KAM step, g_m is well-defined on the whole set of parameters $\mathcal{O}$ and solves the *homological equation*

$$\omega\cdot\partial_\varphi g_m+V_m\partial_\theta g_m+\Pi_{N_m}f_m=\langle f_m\rangle_{\varphi,\theta}$$

when restricted to the Cantor set

$$\mathcal{O}_{m+1}^{\gamma}:=\bigcap_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}}\Big\{\mu=(\lambda,\omega)\in\mathcal{O}_m^{\gamma}\quad\text{s.t.}\quad\big|\omega\cdot l+jV_m(\mu)\big|>\tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}. \tag{6.62}$$

Hence, in the Cantor set $\mathcal{O}_{m+1}^{\gamma}$, the following reduction holds

$$\mathscr{G}_m^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta(V_m+f_m)\Big)\mathscr{G}_m=\omega\cdot\partial_\varphi+\partial_\theta(V_{m+1}+f_{m+1}),$$

with V_{m+1} and f_{m+1} defined by

$$\begin{cases} V_{m+1}=V_m+\langle f_m\rangle_{\varphi,\theta}\\ f_{m+1}=\mathcal{G}_m^{-1}\Big(\Pi_{N_m}^{\perp}f_m+f_m\partial_\theta g_m\Big).\end{cases} \tag{6.63}$$

In view of (6.54), the function f_m is even and therefore g_m is odd. Consequently, we deduce through elementary manipulations that f_{m+1} is also even. This allows us to follow the symmetry persistence along the scheme. Besides, in a similar way to (6.42), one obtains

$$g_m\in\bigcap_{s\geqslant 0}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\mathrm{odd}}). \tag{6.64}$$

Now, we set

$$\mathscr{B}_{-1}=\mathscr{G}_{-1}=\mathrm{Id}\quad\text{ and }\quad\forall m\in\mathbb{N},\ \mathscr{B}_m=\mathscr{G}_0\circ\mathscr{G}_1\circ...\circ\mathscr{G}_m.$$

One easily finds that

$$\begin{aligned}\mathscr{B}_m\rho(\mu,\varphi,\theta) &= \big(1+\partial_\theta\beta_m(\mu,\varphi,\theta)\big)\mathcal{B}_m\rho(\mu,\varphi,\theta)\\ &= \big(1+\partial_\theta\beta_m(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+\beta_m(\mu,\varphi,\theta)\big),\end{aligned}$$

where the sequence $(\beta_m)_{m\in\mathbb{N}}$ is defined by $\beta_{-1}=g_{-1}=0$ and

$$\beta_0=g_0 \quad \text{ and } \quad \beta_m(\mu,\varphi,\theta)=\beta_{m-1}(\mu,\varphi,\theta)+g_m\big(\mu,\varphi,\theta+\beta_{m-1}(\mu,\varphi,\theta)\big). \tag{6.65}$$

A trivial induction based on (6.64) yields

$$\beta_m\in\bigcap_{s\geqslant 0}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\text{odd}}). \tag{6.66}$$

According to Sobolev embeddings, (6.63) and the induction assumption (6.52), we infer

$$\begin{aligned}\|V_m-V_{m-1}\|_q^{\gamma,\mathcal{O}}&=\|\langle f_{m-1}\rangle_{\varphi,\theta}\|_q^{\gamma,\mathcal{O}}\\ &\leqslant \|f_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &=\gamma\delta_{m-1}(s_0)\\ &\leqslant\gamma\delta_0(s_h)N_0^{\mu_2}N_{m-1}^{-\mu_2}.\end{aligned} \tag{6.67}$$

As a consequence, by using the triangle inequality, (6.56) and choosing ε_0 small enough we deduce

$$\begin{aligned}\|V_m\|_q^{\gamma,\mathcal{O}}&\leqslant\|V_{m-1}\|_q^{\gamma,\mathcal{O}}+\gamma\delta_0(s_h)N_0^{\mu_2}N_{m-1}^{-\mu_2}\\ &\leqslant\|V_0\|_q^{\gamma,\mathcal{O}}+\gamma\delta_0(s_h)N_0^{\mu_2}\left(\sum_{k=0}^{m-1}N_k^{-\mu_2}\right)\\ &\leqslant\|V_0\|_q^{\gamma,\mathcal{O}}+\sum_{k=0}^{\infty}N_k^{-\mu_2}.\end{aligned}$$

Now, remark that (6.24) implies in particular

$$\mu_2\geqslant\tau_1q+\tau_1+2.$$

Hence, by the induction hypothesis (6.52), (6.56), (6.59) and (6.60), we have

$$\begin{aligned}\delta_m(s_0)N_m^{\tau_1q+\tau_1+1}&\leqslant\delta_0(s_h)N_0^{\mu_2}N_m^{\tau_1q+\tau_1+1-\mu_2}\\ &\leqslant\varepsilon_0N_0^{-1}\\ &\leqslant\varepsilon_0.\end{aligned} \tag{6.68}$$

Using (6.58) and the previous estimate, we deduce that

$$\sup_{m\in\mathbb{N}}\|V_m\|_q^{\gamma,\mathcal{O}}\leqslant C \quad \text{ and } \quad \delta_m(s_0)N_m^{\tau_1q+\tau_1+1}\leqslant\varepsilon_0. \tag{6.69}$$

Thus, the KAM step applies and, in particular, the estimate (6.51) becomes

$$\delta_{m+1}(s)\leqslant N_m^{s-\overline{s}}\delta_m(\overline{s})+CN_m^{\tau_1q+\tau_1+1}\delta_m(s)\delta_m(s_0). \tag{6.70}$$

If we apply (6.70) with $s=s_l$ and $\overline{s}=s_h$, we obtain

$$\delta_{m+1}(s_l)\leqslant N_m^{s_l-s_h}\delta_m(s_h)+CN_m^{\tau_1q+\tau_1+1}\delta_m(s_l)\delta_m(s_0).$$

Using the induction assumption (6.52) and the fact that $s_l\geqslant s_0$ yields

$$\begin{aligned}\delta_{m+1}(s_l)&\leqslant N_m^{s_l-s_h}\delta_m(s_h)+CN_m^{\tau_1q+\tau_1+1}(\delta_m(s_l))^2\\ &\leqslant\left(2-\tfrac{1}{m+1}\right)N_m^{s_l-s_h}\delta_0(s_h)+CN_0^{2\mu_2}N_m^{\tau_1q+\tau_1+1-2\mu_2}(\delta_0(s_h))^2\\ &\leqslant 2N_m^{s_l-s_h}\delta_0(s_h)+CN_0^{2\mu_2}N_m^{\tau_1q+\tau_1+1-2\mu_2}(\delta_0(s_h))^2.\end{aligned}$$

The conditions (6.24) imply

$$s_h\geqslant\tfrac{3}{2}\mu_2+s_l+1, \quad \text{ and } \quad \mu_2\geqslant 2(\tau_1q+\tau_1+1)+1.$$

Also, using the fact that $N_0 \geqslant 2$ and choosing ε_0 small enough, we get in view of (6.56),

$$4N_0^{-\mu_2} \leqslant 1 \quad \text{and} \quad 2C\delta_0(s_h)N_0^{\mu_2} \leqslant 1.$$

As a consequence, one has

$$N_m^{s_l - s_h} \leqslant \tfrac{1}{4}N_0^{\mu_2}N_{m+1}^{-\mu_2} \quad \text{and} \quad CN_0^{2\mu_2}N_m^{\tau_1 q+\tau_1+1-2\mu_2}\delta_0(s_h) \leqslant \tfrac{1}{2}N_0^{\mu_2}N_{m+1}^{-\mu_2}, \tag{6.71}$$

which implies in turn

$$\delta_{m+1}(s_l) \leqslant \delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.$$

This proves the first statement of the induction in (6.52) and we now turn to the proof of the second statement. Applying (6.70) with $s = \overline{s} = s_h$ and and using the induction (6.52), we get

$$\begin{aligned}\delta_{m+1}(s_h) &\leqslant \delta_m(s_h)\left(1 + CN_m^{\tau_1 q+\tau_1+1}\delta_m(s_0)\right)\\ &\leqslant \left(2 - \tfrac{1}{m+1}\right)\delta_0(s_h)\left(1 + CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h)\right).\end{aligned}$$

Notice that if the condition

$$\left(2 - \tfrac{1}{m+1}\right)\left(1 + CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h)\right) \leqslant 2 - \tfrac{1}{m+2} \tag{6.72}$$

holds true, then

$$\delta_{m+1}(s_h) \leqslant \left(2 - \tfrac{1}{m+2}\right)\delta_0(s_h),$$

which achieves the induction argument of (6.52). Notice that (6.72) is equivalent to

$$\left(2 - \tfrac{1}{m+1}\right)CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h) \leqslant \tfrac{1}{(m+1)(m+2)}.$$

Using (6.59), the preceding condition holds true if

$$CN_0^{\mu_2}N_m^{-1}\delta_0(s_h) \leqslant \tfrac{1}{(m+1)(m+2)}. \tag{6.73}$$

Since $N_0 \geqslant 2$, then in view of (5.82) there exists a small enough constant $c_0 > 0$ such that

$$\forall m \in \mathbb{N}, \quad c_0 N_m^{-1} \leqslant \tfrac{1}{(m+1)(m+2)}.$$

Consequently, (6.73) is ensured provided that

$$CN_0^{\mu_2}\delta_0(s_h) \leqslant c_0. \tag{6.74}$$

Choosing ε_0 small enough and making use of (6.56), we obtain

$$\begin{aligned}CN_0^{\mu_2}\delta_0(s_h) &\leqslant C\varepsilon_0\\ &\leqslant c_0.\end{aligned}$$

Hence, the condition (6.74) is satisfied and the proof of (6.52) is now achieved.
➢ *Persistence of the regularity.* Putting together (6.70), applied with $\overline{s} = s \in [s_0, S]$, (6.52) and (6.59), we infer

$$\begin{aligned}\delta_{m+1}(s) &\leqslant \delta_m(s)\left(1 + CN_m^{\tau_1 q+\tau_1+1}\delta_m(s_0)\right)\\ &\leqslant \delta_m(s)\left(1 + C\delta_0(s_h)N_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\right)\\ &\leqslant \delta_m(s)\left(1 + CN_m^{-1}\right).\end{aligned}$$

Gathering this estimate with (6.55), implies, up to a trivial induction,

$$\begin{aligned}\delta_m(s) &\leqslant \delta_0(s)\prod_{k=0}^{\infty}\left(1 + CN_k^{-1}\right)\\ &\leqslant C\delta_0(s)\\ &\leqslant C\varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{6.75}$$

Then, (6.48), interpolation inequality in Lemma 4.1 and (6.52) give

$$\begin{aligned}\|g_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\delta_m(s+\tau_1 q+\tau_1)\\ &\leqslant C\left(\delta_m(s_0)\right)^{\overline{\theta}(s)}\left(\delta_m(s+\tau_1 q+\tau_1+1)\right)^{1-\overline{\theta}(s)}\\ &\leqslant C\delta_0^{\overline{\theta}(s)}(s_h)\delta_0^{1-\overline{\theta}(s)}(s+\tau_1 q+\tau_1+1)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2},\end{aligned}$$

with $\overline{\theta}(s):=\frac{1}{s+\tau_1 q+\tau_1+1-s_0}$. From (6.75), (6.25) and (6.55), we deduce

$$\begin{aligned}\|g_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_h+1}^{\gamma,\mathcal{O}}\right)\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2}\\ &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2}.\end{aligned}\tag{6.76}$$

Using (6.65) and (6.18), we get for all $s\in[s_0,S]$

$$\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\left(1+\|\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\|g_m\|_{q,s}^{\gamma,\mathcal{O}}.\tag{6.77}$$

If we apply this estimate with $s=s_0$ and use Sobolev embeddings, we deduce

$$\|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\|\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\left(1+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

The previous two expressions make appear recurrent relation for the weighted norms of the sequence $(\beta_m)_m$. To get good estimate for β_m, we shall make use the following result which is quite easy to prove by induction : Given three positive sequences $(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ satisfying

$$\forall\, n\in\mathbb{N},\quad a_{n+1}\leqslant b_n a_n+c_n,$$

we have

$$\begin{aligned}\forall\, n\geqslant 2,\quad a_n &\leqslant a_0\prod_{i=0}^{n-1}b_i+\sum_{k=0}^{n-2}c_k\prod_{i=k+1}^{n-1}b_i+c_{n-1}\\ &\leqslant\Big(a_0+\sum_{k=0}^{n-1}c_k\Big)\prod_{i=0}^{n-1}b_i.\end{aligned}\tag{6.78}$$

In particular, if $\displaystyle\prod_{n=0}^{\infty}b_n$ and $\displaystyle\sum_{n=0}^{\infty}c_n$ converge then

$$\sup_{n\in\mathbb{N}}a_n\leqslant\Big(a_0+\sum_{n=0}^{\infty}c_n\Big)\prod_{n=0}^{\infty}b_i.\tag{6.79}$$

Since the conditions (6.24) and (6.23) imply

$$s_0+\tau_1 q+\tau_1+2\leqslant s_h+\sigma_1\quad\text{and}\quad\overline{\theta}(s_0)\mu_2\geqslant 1,\tag{6.80}$$

then, from (6.76) and (6.25), we deduce

$$\begin{aligned}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}\left(1+\|\mathfrak{I}_0\|_{q,s_0+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)N_m^{-\overline{\theta}(s_0)\mu_2}\\ &\leqslant C\varepsilon_0 N_m^{-1}.\end{aligned}$$

Choosing ε_0 small enough to ensure $C\varepsilon_0\leqslant 1$, N_0 sufficiently large to ensure $\displaystyle\sum_{m=0}^{\infty}N_m^{-1}<\infty$ and we can apply (6.79) together with the fact that $\beta_0=g_0$ to obtain

$$\begin{aligned}\sup_{m\in\mathbb{N}}\|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant\left(\|\beta_0\|_{q,s_0}^{\gamma,\mathcal{O}}+C\sum_{k=0}^{\infty}\|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\prod_{k=0}^{\infty}\left(1+C\|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\leqslant\left(1+C\sum_{k=0}^{\infty}N_k^{-1}\right)\prod_{k=0}^{\infty}\left(1+N_k^{-1}\right)\\ &\leqslant C.\end{aligned}\tag{6.81}$$

Hence the sequence $\left(\|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)_{m\in\mathbb{N}}$ is bounded and inserting this information in (6.77) gives for all $s\in[s_0,S]$

$$\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|g_m\|_{q,s}^{\gamma,\mathcal{O}}.$$

Similarly to what preceeds, if we apply (6.79) and (6.76), we infer

$$\begin{aligned}\sup_{m\in\mathbb{N}}\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \left(\|\beta_0\|_{q,s}^{\gamma,\mathcal{O}}+C\sum_{k=0}^{\infty}\|g_k\|_{q,s}^{\gamma,\mathcal{O}}\right)\prod_{k=0}^{\infty}\left(1+C\|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\left(1+N_0^{\overline{\theta}(s)\mu_2}\sum_{k=0}^{\infty}N_k^{-\overline{\theta}(s)\mu_2}\right).\end{aligned}$$

From Lemma A.1 we get

$$\forall s\in[s_0,S],\quad N_0^{\overline{\theta}(s)\mu_2}\sum_{k=0}^{\infty}N_k^{-\overline{\theta}(s)\mu_2}\lesssim 1$$

which implies in turn

$$\forall s\in[s_0,S],\quad \sup_{m\in\mathbb{N}}\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right). \tag{6.82}$$

From the condition (6.3) we have $s_l=s_0+\tau_1 q+\tau_1+2$, and consequently we deduce from (6.18), (6.81), (6.48) and (6.52),

$$\begin{aligned}\|\beta_m-\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}} &\leqslant C\|g_m\|_{q,s_0+2}^{\gamma,\mathcal{O}}\left(1+\|\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\|g_m\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant C\delta_m(s_l)\\ &\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h).\end{aligned} \tag{6.83}$$

Applying once again Lemma A.1, we deduce that

$$\sum_{m=0}^{\infty}\|\beta_m-\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant C\delta_0(s_h).$$

Hence there exists $\beta\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_0+2})$ such that

$$\beta_m\underset{m\to\infty}{\longrightarrow}\beta\quad\text{(strongly) in}\quad W^{q,\infty,\gamma}(\mathcal{O},H^{s_0+2}).$$

By (6.82) the sequence $(\beta_m)_{m\in\mathbb{N}}$ is bounded in $W^{q,\infty,\gamma}(\mathcal{O},H^s)$, then by a weak-compactness argument we find that $\beta\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$. Using (6.82), we obtain

$$\begin{aligned}\forall s\in[s_0,S],\quad \|\beta\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \liminf_{m\to\infty}\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{6.84}$$

We then can consider the quasi-periodic symplectic change of variables $\mathscr{B}$ associated with β and defined by

$$\begin{aligned}\mathscr{B}\rho(\lambda,\omega,\varphi,\theta) &= \left(1+\partial_\theta\beta(\lambda,\omega,\varphi,\theta)\right)\mathcal{B}\rho(\lambda,\omega,\varphi,\theta)\\ &= \left(1+\partial_\theta\beta(\lambda,\omega,\varphi,\theta)\right)\rho(\lambda,\omega,\varphi,\theta+\beta(\lambda,\omega,\varphi,\theta)).\end{aligned}$$

By (6.84), (6.80) and (6.25), we have

$$\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_0+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\lesssim\varepsilon_0. \tag{6.85}$$

Proceeding as for (6.50), using interpolation (6.84), (6.85), (6.25) and the fact that $2s_0+\tau_1 q+\tau_1+3\leqslant s_h+\sigma_1$, one obtains

$$\|\beta\|_{q,2s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon_0.$$

Therefore, choosing ε_0 small enough, we deduce in view of Lemma 6.2 that $\mathscr{B}$ is an invertible operator. Moreover, by (6.19) and (6.84), we get

$$\|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.86}$$

In addition, by (6.66), and Sobolev embeddings (to get pointwise convergence), we find

$$\beta \in \bigcap_{s\in[s_0,S]} W^{q,\infty,\gamma}(\mathcal{O}, H^s_{\mathrm{odd}}).$$

We also have an estimate of the rate of convergence for the sequence $(\beta_m)_m$ towards β,

$$\begin{aligned}\|\beta-\beta_m\|_{q,s_0+2}^{\gamma,\mathcal{O}} &\leqslant \sum_{k=m}^{\infty}\|\beta_{k+1}-\beta_k\|_{q,s_0+2}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}\sum_{k=m+1}^{\infty}N_k^{-\mu_2}.\end{aligned} \tag{6.87}$$

From Lemma A.1, one obtains

$$\sum_{k=m}^{\infty}N_k^{-\mu_2} \underset{m\to\infty}{=} O\left(N_m^{-\mu_2}\right). \tag{6.88}$$

Gathering (6.88), (6.87) and (6.55), we get

$$\begin{aligned}\|\beta-\beta_m\|_{q,s_0+2}^{\gamma,\mathcal{O}} &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{m+1}^{-\mu_2}.\end{aligned} \tag{6.89}$$

▶ **KAM conclusion**

By (6.67), we have

$$\begin{aligned}\sum_{m=0}^{\infty}\|V_{m+1}-V_m\|_q^{\gamma,\mathcal{O}} &\leqslant \gamma\delta_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\ &\lesssim \gamma\delta_0(s_h).\end{aligned}$$

We deduce that the sequence $(V_m)_{m\in\mathbb{N}}$ is convergent in $W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ and let us denote by c_{i_0} its limit. Moreover, we have by (6.88), (6.55) and (6.25)

$$\begin{aligned}\|c_{i_0}-V_0\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\|V_{m+1}-V_m\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h) \lesssim \varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s_h+1}^{\gamma,\mathcal{O}}\right)\\ &\lesssim \varepsilon.\end{aligned}$$

Now, we introduce the truncated Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0\}\\ |l|\leqslant N_n}}\Big\{\mu:=(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+jc_{i_0}(\mu)\big|>\tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

In what follows, we shall prove that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ satisfies the inclusion

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\bigcap_{m=0}^{n+1}\mathcal{O}_m^{\gamma}=\mathcal{O}_{n+1}^{\gamma},$$

where the intermediate Cantor sets are defined in (6.62). For this aim, we shall argue by induction. We first remark that by construction $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}=:\mathcal{O}_0^{\gamma}$. Now assume that $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}_m^{\gamma}$ for $m\leqslant n$ and let us check that

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}_{m+1}^{\gamma}. \tag{6.90}$$

Putting together (6.67) and (6.88) we infer

$$\begin{aligned}\|V_m - c_{i_0}\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{l=m}^{\infty} \|V_{l+1} - V_l\|_q^{\gamma,\mathcal{O}}\\ &\leqslant \gamma\delta_0(s_h)N_0^{\mu_2}\sum_{l=m}^{\infty} N_l^{-\mu_2}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \end{aligned}\tag{6.91}$$

Given $\mu \in \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ and $(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}$ such that $0\leqslant |l|\leqslant N_m$, we have then $|l|\leqslant N_n$ and by triangle inequality,

$$\begin{aligned}|\omega\cdot l + jV_m(\mu)| &\geqslant |\omega\cdot l + jc_{i_0}(\mu)| - |j||V_m(\mu) - c_{i_0}(\mu)|\\ &\geqslant \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\langle j\rangle\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\ &\geqslant \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\langle j\rangle\gamma^{\upsilon}\varepsilon_0\langle l\rangle^{-\mu_2}.\end{aligned}$$

Since (6.24) implies $\mu_2\geqslant\tau_1$, then taking $\varepsilon_0\leqslant\frac{1}{C}$, we deduce from the previous estimate

$$|\omega\cdot l + jV_m(\mu)| > \tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Consequently, $\mu\in\mathcal{O}_{m+1}^{\gamma}$ and the inclusion (6.90) holds.
(iii) We can write for all $n\in\mathbb{N}$,

$$\begin{aligned}\mathscr{B}^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B} &= \left(\mathscr{B}^{-1}-\mathscr{B}_n^{-1}\right)\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B}\\ &\quad+\mathscr{B}_n^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\left(\mathscr{B}-\mathscr{B}_n\right)\\ &\quad+\mathscr{B}_n^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B}_n.\end{aligned}$$

In view of (6.90) and the definition of $\mathscr{B}_n$, we have in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathscr{B}_n^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B}_n = \omega\cdot\partial_\varphi + \partial_\theta\left(V_{n+1}+f_{n+1}\right).$$

Therefore, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, the following decomposition holds

$$\mathscr{B}^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}\partial_\theta + \mathtt{E}_n^0(i_0),$$

where

$$\begin{aligned}\mathtt{E}_n^0(i_0) &:= \left(V_{n+1}-c_{i_0}\right)\partial_\theta + \partial_\theta\left(f_{n+1}\cdot\right) + \left(\mathscr{B}^{-1}-\mathscr{B}_n^{-1}\right)\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\mathscr{B}\\ &\quad+\mathscr{B}^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\left(V_0+f_0\right)\Big)\left(\mathscr{B}-\mathscr{B}_n\right)\\ &:= \mathtt{E}_{n,1}^0 + \mathtt{E}_{n,2}^0 + \mathtt{E}_{n,3}^0 + \mathtt{E}_{n,4}^0.\end{aligned}$$

By the law products in Lemma 4.1, (6.91) and (6.55) we have

$$\begin{aligned}\|\mathtt{E}_{n,1}^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \|V_{n+1}-c_{i_0}\|_q^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.92}$$

From (6.52) and since (6.3) implies in particular $s_l\geqslant s_0+1$, we obtain

$$\begin{aligned}\|\mathtt{E}_{n,2}^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \gamma\delta_{n+1}(s_0+1)\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.93}$$

We now turn to the estimate of $\mathtt{E}^0_{n,4}$. First remark that by the law products in Lemma 4.1, we have

$$\begin{aligned}\|\omega\cdot\partial_\varphi\rho+\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0}&\leqslant\|\omega\cdot\partial_\varphi\rho\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0}\\&\lesssim\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\left(1+\|V_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).\end{aligned}$$

But combining (6.33), (6.58), (6.75) and (6.25), we obtain

$$\begin{aligned}\|V_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{q,s_0}&\leqslant\|V_0\|^{\gamma,\mathcal{O}}_{q}+\|f_0\|^{\gamma,\mathcal{O}}_{q,s_0}\\&\leqslant C+C\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s_0+1}\\&\leqslant C.\end{aligned}$$

Therefore, we get

$$\|\omega\cdot\partial_\varphi\rho+\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}. \tag{6.94}$$

Putting together (6.94), (6.86) and (6.25), gives

$$\|\mathtt{E}^0_{n,4}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\|(\mathscr{B}-\mathscr{B}_n)\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}. \tag{6.95}$$

Applying Taylor Formula, we may write

$$\begin{aligned}(\mathscr{B}-\mathscr{B}_n)\rho(\theta)&=(1+\partial_\theta\beta(\theta))\rho(\theta+\beta(\theta))-(1+\partial_\theta\beta_n(\theta))\rho(\theta+h_n(\theta))\\&=(1+\partial_\theta\beta(\theta))\left[\rho(\theta+\beta(\theta))-\rho(\theta+\beta_n(\theta))\right]+\partial_\theta\left(\beta-\beta_n\right)(\theta)\rho(\theta+\beta_n(\theta))\\&:=(1+\partial_\theta\beta(\theta))(\beta-\beta_n)(\theta)\mathscr{I}_n(\theta)+\partial_\theta(\beta-\beta_n)(\theta)\mathcal{B}_n\rho(\theta),\end{aligned}$$

where

$$\mathscr{I}_n\rho(\theta):=\int_0^1\left(\partial_\theta\rho\right)\left(\theta+\beta_n(\theta)+t(\beta(\theta)-\beta_n(\theta))\right)dt.$$

Hence, we get by the law products, 6.18 and (6.89)

$$\begin{aligned}\|\partial_\theta\left(\beta-\beta_n\right)\mathcal{B}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}&\lesssim\|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+2}\|\rho\|_{q,s_0+1}\left(1+\|\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\\&\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}.\end{aligned}$$

Using the law products together with (6.89), (6.81) and (6.85) we find

$$\begin{aligned}\left\|(1+\partial_\theta\beta)\left(\beta-\beta_n\right)\mathscr{I}_n\rho\right\|^{\gamma,\mathcal{O}}_{q,s_0+1}&\lesssim\left(1+\|\beta\|^{\gamma,\mathcal{O}}_{q,s_0+2}\right)\|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}\\&\quad\times\left(1+\|\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}+\|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\\&\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}.\end{aligned}$$

Gathering the foregoing estimates leads to

$$\|(\mathscr{B}-\mathscr{B}_n)\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{6.96}$$

Plugging (6.96) into (6.95) gives

$$\|\mathtt{E}^0_{n,4}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{6.97}$$

Proceeding in a similar way as before using in particular the identity (6.16) and (6.20) we find

$$\|\mathtt{E}^0_{n,3}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{6.98}$$

Putting together (6.92), (6.93), (6.97), (6.98) allows to get

$$\|\mathtt{E}^0_{n}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}.$$

(iv) ▶ Estimate of $\Delta_{12}\beta$. First notice that, since $\beta_{-1}=0$, then

$$\Delta_{12}\beta=\sum_{m=0}^{\infty}\Delta_{12}(\beta_m-\beta_{m-1}). \tag{6.99}$$

The triangle inequality allows us to write

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\leqslant\sum_{m=0}^{\infty}\|\Delta_{12}(\beta_m-\beta_{m-1})\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}. \tag{6.100}$$

According to Taylor Formula and (6.65), we infer

$$\begin{aligned}\Delta_{12}\beta_m(\theta)=\Delta_{12}\beta_{m-1}(\theta)&+(\mathcal{B}_{m-1})_{r_1}(\Delta_{12}g_m)(\theta)\\&+\Delta_{12}\beta_{m-1}(\theta)\int_0^1(\partial_\theta(g_m)_{r_2})\big(\theta+(\beta_{m-1})_{r_2}(\theta)+t\Delta_{12}\beta_{m-1}(\theta)\big)dt.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_{12}(\beta_m-\beta_{m-1})(\theta)&=(\mathcal{B}_{m-1})_{r_1}(\Delta_{12}g_m)(\theta)\\&\quad+\Delta_{12}\beta_{m-1}(\theta)\int_0^1(\partial_\theta(g_m)_{r_2})\big(\theta+(\beta_{m-1})_{r_2}(\theta)+t\Delta_{12}\beta_{m-1}(\theta)\big)dt.\end{aligned}$$

Consequently, using the law product in Lemma 4.1, Lemma 6.2 and Sobolev embeddings we obtain

$$\begin{aligned}\|\Delta_{12}(\beta_m-\beta_{m-1})\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\left(1+C\|(\beta_{m-1})_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|(\beta_{m-1})_{r_1}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&+C\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(g_m)_{r_2}\|_{q,\overline{s}_h+\mathtt{p}+1}^{\gamma,\mathcal{O}}\left(1+\|(\beta_{m-1})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&+C\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(g_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(\|(\beta_{m-1})_{r_2}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\right)\\&+C\|\Delta_{12}\beta_{m-1}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\|(g_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(1+\|(\beta_{m-1})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\end{aligned}$$

and for all $s\in[s_0,\overline{s}_h+\mathtt{p}]$

$$\begin{aligned}\|\Delta_{12}\beta_m\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(\beta_{m-1})_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|(\beta_{m-1})_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(g_m)_{r_2}\|_{q,s+1}^{\gamma,\mathcal{O}}\left(1+\|(\beta_{m-1})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&+C\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(g_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(\|(\beta_{m-1})_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}}\right)\\&+\|\Delta_{12}\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(g_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(1+\|(\beta_{m-1})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\right).\end{aligned}$$

Notice that (6.24) implies in particular $\overline{s}_h+\mathtt{p}+\tau_1q+\tau_1+3\leqslant s_h+\sigma_1$. Therefore, using (6.76) and (6.25), we get

$$\begin{aligned}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|_{q,\overline{s}_h+\mathtt{p}+1}^{\gamma,\mathcal{O}}&\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|_{q,\overline{s}_h+\mathtt{p}+\tau_1q+\tau_1+3}^{\gamma,\mathcal{O}}\right)\\&\leqslant C.\end{aligned} \tag{6.101}$$

Notice that the previous estimate is sufficient to easily get rid of most of terms in the estimates of $\Delta_{12}\beta_m$ and $\Delta_{12}(\beta_m-\beta_{m-1})$, but not enough to make the series (6.99) convergent. For this purpose, we shall refine the estimates. By (6.76), (6.24) and (6.25), we have

$$\begin{aligned}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|_{q,\overline{s}_h+\mathtt{p}+1}&\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_k\|_{q,\overline{s}_h+\mathtt{p}+\tau_1q+\tau_1+3}^{\gamma,\mathcal{O}}\right)N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}.\end{aligned} \tag{6.102}$$

Combining (6.82) and (6.25)

$$\begin{aligned}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(\beta_m)_{r_k}\|_{q,\overline{s}_h+\mathtt{p}}&\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\tau_1q+\tau_1+3}\right)\\&\leqslant C.\end{aligned}\tag{6.103}$$

Hence, using (6.101), (6.103) and Sobolev embeddings, the previous two estimates can be reduced to

$$\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant C\left(\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+1}\right),\tag{6.104}$$

$$\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant C\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\left(1+C\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\tag{6.105}$$

and

$$\begin{aligned}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}&\leqslant C\left(\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|_{q,\overline{s}_h+\mathtt{p}+1}\right)\\&\quad+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\left(1+C\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).\end{aligned}\tag{6.106}$$

From (6.105), using (6.79) and the fact that $\beta_0=g_0$, we deduce that

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant\left(\|\Delta_{12}g_0\|^{\gamma,\mathcal{O}}_{q,s_0}+C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\prod_{k=0}^{\infty}\left(1+\|(g_k)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).$$

Adding (6.102), we obtain

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,s_0}.$$

Similarly, (6.106), (6.79), (6.102) and the previous estimate allow to get

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}.$$

Putting together the previous bounds, (6.104) and (6.102) gives

$$\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\lesssim\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\varepsilon\gamma^{-1}N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}.\tag{6.107}$$

Thus, the main delicate point is to estimate $\Delta_{12}g_m$. First remark that according to (6.43) and (6.61), we can make the splitting

$$\begin{aligned}g_m(\mu,\varphi,\theta)&=\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\\langle l,j\rangle\leqslant N_m}}a_{l,j}\widehat{\chi}\big(a_{l,j}(A_{l,j})_{r_2}(\mu)\big)(\Delta_{12}f_m)_{l,j}(\mu)\mathbf{e}_{l,j}\\&\quad+\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\\langle l,j\rangle\leqslant N_m}}a_{l,j}\Delta_{12}\widehat{\chi}\big(a_{l,j}A_{l,j}(\mu)\big)((f_m)_{r_1})_{l,j}(\mu)\mathbf{e}_{l,j}\\&:=\mathbf{I}_1+\mathbf{I}_2.\end{aligned}$$

Similarly to (6.48), one obtains

$$\|\mathbf{I}_1\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\gamma^{-1}\|\Pi_{N_m}\Delta_{12}f_m\|^{\gamma,\mathcal{O}}_{q,s+\tau_1q+\tau_1}.\tag{6.108}$$

We shall now estimate the second term. Applying Taylor Formula, we get

$$\begin{aligned}\mathbf{I}_2=\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} a_{l,j}^2(\Delta_{12}A_{l,j})\int_0^1\widehat{\chi}'\Big(a_{l,j}\Big[\tau(A_{l,j})_{r_1}(\mu)+(1-\tau)(A_{l,j})_{r_2}(\mu)\Big]\Big)d\tau((f_m)_{r_1})_{l,j}\mathbf{e}_{l,j}\\ :=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}}\widetilde{h}_{l,j}(\mu)((f_m)_{r_1})_{l,j}(\mu)\mathbf{e}_{l,j}.\end{aligned}\tag{6.109}$$

Remark that direct computations yield

$$\forall q'\in[\![0,q]\!],\quad \|\Delta_{12}A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\langle l,j\rangle\|\Delta_{12}V_m\|_{q'}^{\gamma,\mathcal{O}}.\tag{6.110}$$

Since that $\widehat{\chi}'\in C^\infty$ with $\widehat{\chi}'(0)=0$, then applying Lemma 4.1-(iv)-(vi) together with (6.45) and (6.110), we get

$$\begin{aligned}\forall q'\in[\![0,q]\!],\quad \|\widetilde{h_{l,j}}\|_{q'}^{\gamma,\mathcal{O}}&\lesssim a_{l,j}^3\|\Delta_{12}A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\left(\|(A_{l,j})_{r_1}\|_{q'}^{\gamma,\mathcal{O}}+\|(A_{l,j})_{r_2}\|_{q'}^{\gamma,\mathcal{O}}\right)\\ &\quad\times\left(1+a_{l,j}^{q'-1}\left(\|(A_{l,j})_{r_1}\|_{L^\infty(\mathcal{O})}+\|(A_{l,j})_{r_2}\|_{L^\infty(\mathcal{O})}\right)^{q'-1}\right)\\ &\lesssim\gamma^{-\upsilon(q'+2)}\langle l,j\rangle^{\tau_1q'+2\tau_1+q'+1}\|\Delta_{12}V_m\|_{q'}^{\gamma,\mathcal{O}}.\end{aligned}$$

By assumption in Proposition 6.2, we have

$$\upsilon\leqslant\tfrac{1}{q+2}\tag{6.111}$$

and using Leibniz rule, we deduce that

$$\|\mathbf{I}_2\|_q^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|\Pi_{N_m}(f_m)_{r_1}\|_{q,s+\tau_1q+2\tau_1+1}^{\gamma,\mathcal{O}}.\tag{6.112}$$

Putting together (6.108) and (6.112), we obtain for all $s\geqslant s_0$

$$\begin{aligned}\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\gamma^{-1}\|\Pi_{N_m}\Delta_{12}f_m\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}+\gamma^{-1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|\Pi_{N_m}(f_m)_{r_1}\|_{q,s+\tau_1q+2\tau_1+1}^{\gamma,\mathcal{O}}\\ &\lesssim\gamma^{-1}N_m^{\tau_1q+\tau_1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}+\gamma^{-2}N_m^{\tau_1q+2\tau_1+1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|(f_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.113}$$

Therefore, estimating $\Delta_{12}g_m$ can be done through the estimate of $\Delta_{12}f_m$. To do so, we shall argue by induction. For that purpose, we shall consider a parameter $\widetilde{\mathtt{p}}$ (which can depend on the parameter $\mathtt{p}$, see for instance (6.127)) satisfying the following constraint

$$\overline{s}_h+\widetilde{\mathtt{p}}+3\leqslant s_h+\sigma_1.\tag{6.114}$$

We denote

$$u_m:=\Pi_{N_m}^\perp f_m+f_m\partial_\theta g_m.$$

Then, we can write

$$\Delta_{12}f_{m+1}=(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m+\left(\Delta_{12}\mathcal{G}_m^{-1}\right)(u_m)_{r_2},$$

with

$$\Delta_{12}u_m=\Pi_{N_m}^\perp\Delta_{12}f_m+\Delta_{12}f_m\partial_\theta(g_m)_{r_1}+(f_m)_{r_2}\partial_\theta\Delta_{12}g_m.$$

By the triangle inequality, we have for all $s\geqslant s_0$

$$\|\Delta_{12}f_{m+1}\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}+\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}.\tag{6.115}$$

Therefore, combining (6.18), (6.20), (6.48) and Lemma 4.1-(ii), we get for all $s\geqslant s_0$

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(\widehat{g_m})_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|(\widehat{g_m})_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(g_m)_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|(g_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\gamma^{-1}N_m^{\tau_1q+\tau_1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\quad+C\gamma^{-1}N_m^{\tau_1q+\tau_1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Using (6.75),(6.114) and (6.25), one gets

$$\gamma^{-1}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+1}^{\gamma,\mathcal{O}}\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+2}^{\gamma,\mathcal{O}}\right)\\ \leqslant C. \tag{6.116}$$

Therefore, from (6.52) and (6.116), we get for all $s\in[s_0,\overline{s}_h]$

$$\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1-\overline{\mu}_2}\right)+CN_m^{\tau_1q+\tau_1}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

At this level we need to give a suitable estimate for $\Delta_{12}u_m$. For this aim, we apply the law products in Lemma 4.1, ensuring that for all $s\geqslant s_0$

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant&\|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}+C\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta(g_m)_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}+C\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta(g_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\|(f_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+C\|(f_m)_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence we deduce by (6.48) and Lemma 4.1-(ii),

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant&\|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+C\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Added to (6.113), we finally obtain for all $s\geqslant s_0$

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant&\|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\gamma^{-2}N_m^{\tau_1q+2\tau_1+1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}$$

Consequently, we find from (6.52), Lemma 4.1-(ii) and (6.116),

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant&N_m^{s_0-\overline{s}_h-\widetilde{\mathtt{p}}}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+CN_0^{2\overline{\mu}_2}N_m^{\tau_1q+2\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}\end{aligned}$$

and

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\leqslant&\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\left(1+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right)+CN_m^{\tau_1q+\tau_1+1}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+2\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}},\end{aligned}$$

where we use the notation

$$\delta_0^{1,2}(s):=\gamma^{-1}\max_{k\in\{1,2\}}\|(f_0)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}.$$

It follows from the preceding estimates that,

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant&CN_m^{\tau_1q+\tau_1}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\\leqslant&CN_m^{s_0+\tau_1q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}+CN_0^{\overline{\mu}_2}N_m^{2(\tau_1q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.117}$$

In a similar way, direct computations yield

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\leqslant&\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\left(1+N_m^{s_0+\tau_1q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right)\\&+C\left(N_0^{\overline{\mu}_2}N_m^{2(\tau_1q+\tau_1)+1-\overline{\mu}_2}+N_m^{\tau_1q+\tau_1+1}\right)\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+CN_0^{\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.118}$$

By a new use of Taylor Formula, we can write

$$(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}(\theta)=\Delta_{12}\widehat{g_m}(\theta)\int_0^1\partial_\theta(u_m)_{r_2}\Big(\theta+(\widehat{g_m})_{r_2}(\theta)+t\Delta_{12}\widehat{g_m}(\theta)\Big)dt.$$

Applying Lemma 4.1 and (6.48), we deduce for all $s\geqslant s_0$

$$\begin{aligned}\|u_m\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Pi_{N_m}^{\perp}f_m\|_{q,s}^{\gamma,\mathcal{O}}+C\|f_m\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+C\|f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta g_m\|_{q,s}^{\gamma,\mathcal{O}}\\&\leqslant\|f_m\|_{q,s}\left(1+CN_m^{\tau_1q+\tau_1+1}\|f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&\leqslant C\|f_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.119}$$

Using once again the law products in Lemma 4.1 combined with (6.18) yield for all $s\geqslant s_0$

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant C\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s_0+1}\left(1+\|(\widehat{g_m})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&\quad+C\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s+1}^{\gamma,\mathcal{O}}\left(1+\|(\widehat{g_m})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&\quad+C\|\Delta_{12}\widehat{g_m}\|_{q,s_0}\|(u_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\left(\|(\widehat{g_m})_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

In view of (6.21), (6.102) and Sobolev embeddings, one gets for all $s\in[s_0,\overline{s}_h+\mathtt{p}]$

$$\begin{aligned}\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant C\left(\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}\right)\\&\leqslant C\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the previous estimates, (6.101) and (6.20) gives for all $s\in[s_0,\overline{s}_h]$

$$\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s_0+1}+C\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s+1}^{\gamma,\mathcal{O}}.$$

Thus, by virtue of (6.113), (6.119), we get for all $s\in[s_0,\overline{s}_h]$

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant C\gamma^{-1}N_m^{\tau_1q+\tau_1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\quad+C\gamma^{-1}N_m^{\tau_1q+\tau_1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\quad+C\gamma^{-2}N_m^{\tau_1q+2\tau_1+1}\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence, (6.52), (6.75) and (6.116) allow to get (since $s_l\geqslant s_0+1$)

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}&\leqslant CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\quad+CN_0^{2\overline{\mu}_2}N_m^{\tau_1q+2\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}\end{aligned}\tag{6.120}$$

and

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}&\leqslant CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\\&\quad+CN_m^{\tau_1q+\tau_1}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\quad+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+2\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.121}$$

Gathering (6.115), (6.117) and (6.120) implies (since $N_m^{-\widetilde{\mathtt{p}}}\leqslant 1$)

$$\begin{aligned}\|\Delta_{12}f_{m+1}\|_{q,s_0}^{\gamma,\mathcal{O}}&\leqslant N_m^{s_0+\tau_1q+\tau_1-\overline{s}_h}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\\&\quad+CN_0^{\overline{\mu}_2}N_m^{2(\tau_1q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\quad+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.122}$$

In a similar war, we get in view of (6.115), (6.118) and (6.121)

$$\begin{aligned}\|\Delta_{12}f_{m+1}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} \leqslant \|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}&\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right)\\ &+C\left(N_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}+N_m^{\tau_1 q+\tau_1+1}\right)\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &+CN_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.123}$$

In the sequel, we shall use the following notations

$$\overline{\delta}_m(s)=\gamma^{-1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\quad\text{and}\quad\varkappa_m=\gamma^{-1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}.$$

Notice that

$$\Delta_{12}V_{m+1}=\Delta_{12}V_m+\langle\Delta_{12}f_m\rangle_{\varphi,\theta}\quad\text{and}\Delta_{12}V_0=0.$$

Then, by using Sobolev embeddings, we obtain

$$\varkappa_m\leqslant\sum_{k=0}^{m-1}\overline{\delta}_k(s_0).\tag{6.124}$$

We shall now prove by induction that, for all $\widetilde{\mathtt{p}}$ satisfying the condition (6.114), we have

$$\forall k\leqslant m,\quad\overline{\delta}_k(s_0)\leqslant N_0^{\overline{\mu}_2}N_k^{-\overline{\mu}_2}\nu(\overline{s}_h+\widetilde{\mathtt{p}})\quad\text{and}\quad\overline{\delta}_k(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\left(2-\tfrac{1}{k+1}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}}),\tag{6.125}$$

with

$$\nu(s):=\overline{\delta}_0(s)+\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{s_0+2}.$$

First remark that the property (6.125) is trivially satisfied for $m=0$ according to Sobolev embeddings. We now assume that (6.125) is true at the order m and let us check it at the next order. By the induction assumption (6.125) and (6.124), one obtains the following estimate

$$\sup_{m\in\mathbb{N}}\varkappa_m\leqslant C\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\tag{6.126}$$

Using (6.122), (6.126) and hypothesis of induction (6.125), we find

$$\begin{aligned}\overline{\delta}_{m+1}(s_0)&\leqslant N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}\overline{\delta}_m(\overline{s}_h+\widetilde{\mathtt{p}})+CN_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\overline{\delta}_m(s_0)\\ &\quad+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\varkappa_m\\ &\leqslant\left[2N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right]\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\end{aligned}$$

Then, in view of (6.24), we infer

$$\begin{aligned}2N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}&=2N_m^{-\frac{3}{2}\overline{\mu}_2-3}=2N_m^{-3}N_{m+1}^{-\overline{\mu}_2}\\ &\leqslant 2N_0^{-3}N_{m+1}^{-\overline{\mu}_2}\\ &\leqslant\frac{1}{2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.\end{aligned}$$

To prove the last inequality, we remark that since $N_0\geqslant 2$ and $\overline{\mu}_2\geqslant 0$ (according to (6.3)), then

$$4\leqslant N_0^{\overline{\mu}_2+3}.$$

Similarly, from the expression of $\overline{\mu}_2$ in (6.24) and using (5.82) one obtains

$$\begin{aligned}CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)&\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\frac{1}{2}\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\\ &\leqslant C\varepsilon\gamma^{-1}N_0^{2\tau_1 q+3\tau_1+1+\frac{1}{2}\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\\ &\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.\end{aligned}$$

Hence, choosing ε_0 small enough and using (6.25) we deduce that

$$CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\leqslant\frac{1}{2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.$$

Gathering the preceding estimates gives

$$\overline{\delta}_{m+1}(s_0)\leqslant N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\nu(\overline{s}_h+\widetilde{\mathtt{p}}).$$

This ends the proof of the first statement in (6.125). As to the second one, we shall first write in view of (6.123),

$$\begin{aligned}\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\overline{\delta}_m(\overline{s}_h+\widetilde{\mathtt{p}})\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right)\\+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}\right)\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\overline{\delta}_m(s_0)\\+CN_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\varkappa_m.\end{aligned}$$

Notice that since $\overline{s}_h+\widetilde{\mathtt{p}}+2\leqslant s_h+\sigma_1$, then by (6.55) and (6.25), one has

$$\begin{aligned}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)&\lesssim\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+2}^{\gamma,\mathcal{O}}\right)\\&\lesssim\varepsilon\gamma^{-1}.\end{aligned}$$

It follows from (6.125) and (6.126),

$$\begin{aligned}\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\left(2-\tfrac{1}{m+1}\right)\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}})\\+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\right)N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\varepsilon\gamma^{-1}\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\end{aligned}$$

Proceeding as for (6.72), taking ε_0 small enough and thanks to (6.24), we obtain

$$\begin{aligned}&\left(2-\tfrac{1}{m+1}\right)\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\right)\\&+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\right)N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\varepsilon\gamma^{-1}\\&\leqslant 2-\tfrac{1}{m+2},\end{aligned}$$

so that

$$\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\left(2-\tfrac{1}{m+2}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}}).$$

This completes the proof of the second statement in (6.125).
➢ *Conclusion.* From (6.113), we get for $s=s_0$.

$$\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\overline{\delta}_m(s_0+\tau_1 q+\tau_1)+\varkappa_m\delta_m(s_0+\tau_1 q+2\tau_1+1).$$

By interpolation inequality in Lemma 4.1, (6.52) applied with $\mu_2=\overline{\mu}_2$, (6.125) applied with $\widetilde{\mathtt{p}}=0$ and Sobolev embeddings, we have for some $\overline{\theta}\in(0,1)$

$$\begin{aligned}\overline{\delta}_m(s_0+\tau_1 q+\tau_1)&\leqslant\overline{\delta}_m(s_0+\tau_1 q+2\tau_1+1)\\&\lesssim\overline{\delta}_m(s_0)^{\overline{\theta}}\overline{\delta}_m(\overline{s}_h)^{1-\overline{\theta}}\\&\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h)\end{aligned}$$

and

$$\begin{aligned}\delta_m(s_0+\tau_1 q+2\tau_1+1)&\lesssim\delta_m(s_0)^{\overline{\theta}}\delta_m(\overline{s}_h)^{1-\overline{\theta}}\\&\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\delta_0(\overline{s}_h)\\&\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}.\end{aligned}$$

Therefore
$$\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h).$$
Now from (6.113), we have
$$\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+1}^{\gamma,\mathcal{O}} \lesssim \overline{\delta}_m(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)+\varkappa_m\delta_m(\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+2).$$
Applying (6.125) with
$$\widetilde{\mathtt{p}}=\mathtt{p}+\tau_1 q+\tau_1+1, \tag{6.127}$$
which is possible since from (6.3), (6.24) and (6.23), one has $\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+4\leqslant s_h+\sigma_1$, we find
$$\begin{aligned}\overline{\delta}_m(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)&\leqslant 2\nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)\\&\leqslant 2\overline{\delta}_0(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)+2\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\end{aligned}$$
Implementing a similar proof to (5.48) based on the kernel decomposition (5.38), the composition laws and (6.8), we find
$$\begin{aligned}\forall s\geqslant s_0,\quad \overline{\delta}_0(s)&=\gamma^{-1}\|\Delta_{12}V_{\varepsilon r}\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0+1}^{\gamma,\mathcal{O}}\max_{\ell=1,2}\|r_\ell\|_{q,s+1}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$
On the other hand, since
$$\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+3\leqslant s_h+\sigma_1,$$
one may obtain through combining (6.75) and (6.25)
$$\begin{aligned}\delta_m(\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+2)&\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+3}^{\gamma,\mathcal{O}}\right)\\&\leqslant C\varepsilon\gamma^{-1}.\end{aligned}$$
Thus, by interpolation inequality in Lemma 4.1, we finally obtain for some $\overline{\theta}\in(0,1)$
$$\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1). \tag{6.128}$$
Choosing N_0 sufficiently large, then the composition law in Lemma 4.1 allows to get
$$\begin{aligned}\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)N_0^{\overline{\theta}\overline{\mu}_2}\sum_{k=0}^{\infty}N_m^{-\overline{\theta}\overline{\mu}_2}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.129}$$
Finally, gathering (6.100), (6.107), (6.128) and (6.129), we get
$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}.$$
Putting together this estimate, (6.21) and (6.103) yields
$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}.\end{aligned}$$
► **Estimate on** $\Delta_{12}c_i$. Since $V_0=\Omega+I_1K_1$ is independent of r, then
$$\Delta_{12}c_i=\sum_{m=0}^{\infty}\Delta_{12}(V_{m+1}-V_m).$$
Therefore we obtain in view of (6.63), Sobolev embeddings and (6.125) applied with $\widetilde{\mathtt{p}}=0$,
$$\begin{aligned}\|\Delta_{12}(V_{m+1}-V_m)\|_q^{\gamma,\mathcal{O}}&=\|\langle\Delta_{12}f_m\rangle_{\varphi,\theta}\|_q^{\gamma,\mathcal{O}}\\&\leqslant C\gamma\overline{\delta}_m(s_0)\\&\leqslant C\gamma N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\nu(\overline{s}_h).\end{aligned}$$

Hence by the composition law in Lemma 4.1, Lemma A.1 and (6.8) one may find

$$\begin{aligned}\|\Delta_{12}c_i\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\|\Delta_{12}(V_{m+1}-V_m)\|_q^{\gamma,\mathcal{O}}\\ &\leqslant C\gamma\nu(\overline{s}_h)N_0^{\overline{\mu}_2}\sum_{m=0}^{\infty}N_m^{-\overline{\mu}_2}\\ &\leqslant C\varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.\end{aligned}$$

This achieves the proof of Proposition 6.2. □

6.2.2 Action on the non-local term

In this section, we shall analyze the conjugation action by $\mathscr{B}$ on the nonlocal term appearing in the linearized operator $\mathcal{L}_{\varepsilon r}$ described in Proposition 6.1. The main result reads as follows.

Proposition 6.3. *Let $(\gamma,q,d,\tau_1,s_0,\overline{s}_h,\sigma_1,S)$ satisfy* (4.1), (4.2), (6.3) *and* (6.23). *We set*

$$\sigma_2 = s_0+\sigma_1+3. \tag{6.130}$$

For any $(\mu_2,\mathtt{p},s_h)$ satisfying the condition (6.24), *there exists $\varepsilon_0>0$ such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0 \quad\text{and}\quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_2}^{\gamma,\mathcal{O}}\leqslant 1, \tag{6.131}$$

then in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, we have

$$\mathfrak{L}_{\varepsilon r} := \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}\partial_\theta - \partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0,$$

where $\mathcal{K}_\lambda$ is defined in (3.12), *$\mathtt{E}_n^0$ is introduced in Proposition* 6.2 *and $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving self-adjoint integral operator satisfying*

$$\forall s\in[s_0,S],\qquad \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textrm{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right). \tag{6.132}$$

In addition, if i_1 and i_2 are two tori satisfying the smallness property (6.131), *then*

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_2}^{\gamma,\mathcal{O}}. \tag{6.133}$$

Proof. We recall from Proposition 6.1 and Lemma 3.1. that

$$\mathcal{L}_{\varepsilon r}=\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r},$$

where $\mathbf{L}_{\varepsilon r}$ is a non-local operator defined by

$$\mathbf{L}_{\varepsilon r}(\rho)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)K_0(\lambda A_{\varepsilon r}(\varphi,\theta,\eta))d\eta,$$

with

$$A_{\varepsilon r}(\varphi,\theta,\eta)=\Big(\big(R(\varphi,\eta)-R(\varphi,\theta)\big)^2+4R(\varphi,\eta)R(\varphi,\theta)\sin^2\left(\tfrac{\eta-\theta}{2}\right)\Big)^{\frac12}$$

and

$$R(\varphi,\theta)=\big(1+2\varepsilon r(\varphi,\theta)\big)^{\frac12}.$$

Notice that we have removed the dependance in (λ,ω) from the functions in order to alleviate the notation. Hence by Proposition 6.2, Lemma 6.1-(i) and (5.46), we have in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\begin{aligned}\mathfrak{L}_{\varepsilon r} &:= \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}\\ &=\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)\big)\mathscr{B}-\mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r}\mathscr{B}\\ &=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r}\mathscr{B}+\mathtt{E}_n^0\\ &=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\Big(\mathcal{B}^{-1}\left(\mathcal{K}_\lambda*\cdot\right)\mathscr{B}+\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\Big)+\mathtt{E}_n^0.\end{aligned} \tag{6.134}$$

From a direct computation using (3.12) combined with (6.16) and (6.15), we find

$$\mathcal{B}^{-1}\big(\mathcal{K}_\lambda * \mathscr{B}\rho\big)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)K_0\big(\lambda\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta)\big)d\eta,$$

where

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta):=2\left|\sin\left(\tfrac{\eta-\theta}{2}+\widehat{h}(\varphi,\theta,\eta)\right)\right|,$$

with

$$\widehat{h}(\varphi,\theta,\eta):=\tfrac{\widehat{\beta}(\varphi,\eta)-\widehat{\beta}(\varphi,\theta)}{2}.$$

Using elementary trigonometric identities, we can write

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta)=2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|v_{\widehat{\beta},2}(\varphi,\theta,\eta),\tag{6.135}$$

with

$$v_{\widehat{\beta},2}(\varphi,\theta,\eta):=\cos\left(\widehat{h}(\varphi,\theta,\eta)\right)+\tfrac{\sin\left(\widehat{h}(\varphi,\theta,\eta)\right)}{\tan\left(\frac{\eta-\theta}{2}\right)}.$$

Notice that $v_{0,2}=1$ and one may write

$$v_{\widehat{\beta},2}(\theta,\eta)=1+\big(\cos\big(\widehat{h}(\theta,\eta)\big)-1\big)+\tfrac{\widehat{h}(\theta,\eta)}{\tan\left(\frac{\eta-\theta}{2}\right)}+\left(\tfrac{\sin(\widehat{h}(\theta,\eta))}{\widehat{h}(\theta,\eta)}-1\right)\tfrac{\widehat{h}(\theta,\eta)}{\tan\left(\frac{\eta-\theta}{2}\right)}$$

and then using Lemma 4.1-(iv)-(v), Lemma 4.2 and (6.28), we obtain

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\big\|v_{\widehat{\beta},2}(*,\cdot,\centerdot,\eta+\centerdot)-1\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+1}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right),\\ \forall k\in\mathbb{N}^*,\ \sup_{\eta\in\mathbb{T}}\big\|(\partial_\theta^k v_{\widehat{\beta},2})(*,\cdot,\centerdot,\eta+\centerdot)\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+k+1}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1+k}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.136}$$

Proceeding as for (5.38), one obtains the decomposition

$$K_0(\lambda\mathscr{A}_{\widehat{\beta}}(\lambda,\omega,\varphi,\theta,\eta))=K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\mathscr{K}(\eta-\theta)\mathscr{K}^1_{\widehat{\beta},2}(\varphi,\theta,\eta)+\mathscr{K}^2_{\widehat{\beta},2}(\varphi,\theta,\eta)$$

with similar estimates to (5.34) and (5.37), that is, for all $k\in\mathbb{N}$,

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\left(\big\|(\partial_\theta^k\mathscr{K}^1_{\widehat{\beta},2})(*,\cdot,\centerdot,\eta+\centerdot)\big\|_{q,s}^{\gamma,\mathcal{O}}+\big\|(\partial_\theta^k\mathscr{K}^2_{\widehat{\beta},2})(*,\cdot,\centerdot,\eta+\centerdot)\big\|_{q,s}^{\gamma,\mathcal{O}}\right)&\lesssim\|\widehat{\beta}\|_{q,s+1+k}^{\gamma,\mathcal{O}}\\ &\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1+k}^{\gamma,\mathcal{O}}\right),\end{aligned}\tag{6.137}$$

where the symbols $*,\cdot,\centerdot$ stand for $(\lambda,\omega),\varphi,\theta$, respectively. Now we shall denote by $\mathbf{L}_{\varepsilon r,2}$ the integral operator with the kernel $\mathbb{K}_{\varepsilon r,2}$ defined by

$$\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta):=\mathscr{K}(\eta-\theta)\mathscr{K}^1_{\widehat{\beta},2}(\varphi,\theta,\eta)+\mathscr{K}^2_{\widehat{\beta},2}(\varphi,\theta,\eta).\tag{6.138}$$

Then we find the decomposition

$$\mathcal{B}^{-1}\left(\mathcal{K}_\lambda*\cdot\right)\mathscr{B}=\mathcal{K}_\lambda*\cdot+\mathbf{L}_{\varepsilon r,2}.$$

Inserting this identity into (6.134) allows to get

$$\mathfrak{L}_{\varepsilon r}=\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}^0_n,$$

with

$$\mathfrak{R}_{\varepsilon r}=-\mathbf{L}_{\varepsilon r,2}-\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}.\tag{6.139}$$

Observe that by (6.4) and (6.32) we can easily check that the kernel $\mathbb{K}_{\varepsilon r,2}$ satisfies the following symmetry property

$$\mathbb{K}_{\varepsilon r,2}(-\varphi,-\theta,-\eta)=\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)\in\mathbb{R},\tag{6.140}$$

which implies in turn, according to Lemma 4.4, that $\mathbf{L}_{\varepsilon r,2}$ is a real and reversibility preserving operator. Moreover, one obtains from (6.137)

$$\max_{k\in\{0,1,2\}}\left\|(\partial_\theta^k\mathbb{K}_{\varepsilon r,2})(*,\cdot,\centerdot,\eta+\centerdot)\right\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right). \tag{6.141}$$

Our next purpose is to highlight some properties of the operator $\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}$ which takes the integral form

$$\big(\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\big)\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)d\eta, \tag{6.142}$$

where the kernel $\widehat{\mathbb{K}}_{\varepsilon r,1}$ is related to the kernel $\mathbb{K}_{\varepsilon r,1}$ defined in (5.41) through the formula,

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)=\mathbb{K}_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big). \tag{6.143}$$

It is quite easy to check from (5.43) and (6.32), that

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(-\varphi,-\theta,-\eta)=\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.144}$$

According to (5.41), one gets the decomposition

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)=\widehat{\mathscr{K}}(\varphi,\theta,\eta)\widehat{\mathscr{K}_{\varepsilon r,1}^1}(\varphi,\theta,\eta)+\widehat{\mathscr{K}_{\varepsilon r,1}^2}(\varphi,\theta,\eta), \tag{6.145}$$

with

$$\begin{aligned}\widehat{\mathscr{K}}(\varphi,\theta,\eta)&:=\mathscr{K}\big(\eta-\theta+\widehat{\beta}(\varphi,\eta)-\widehat{\beta}(\varphi,\theta)\big),\\ \widehat{\mathscr{K}_{\varepsilon r,1}^1}(\varphi,\theta,\eta)&:=\mathscr{K}_{\varepsilon r,1}^1\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big),\\ \widehat{\mathscr{K}_{\varepsilon r,1}^2}(\varphi,\theta,\eta)&:=\mathscr{K}_{\varepsilon r,1}^2\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\end{aligned}$$

Coming back to (5.39) and using the morphism property of the logarithm, combined with (6.135) we deduce that

$$\begin{aligned}\widehat{\mathscr{K}}(\varphi,\theta,\eta)&=\sin^2\left(\tfrac{\eta-\theta}{2}\right)v_{\widehat{\beta},2}^2(\varphi,\theta,\eta)\left(\log\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|+\log\left|v_{\widehat{\beta},2}(\varphi,\theta,\eta)\right|\right)\\&=\mathscr{K}\left(\eta-\theta\right)+\mathscr{K}\left(\eta-\theta\right)\left(v_{\widehat{\beta},2}^2(\varphi,\theta,\eta)-1\right)\\&\quad+\sin^2\left(\tfrac{\eta-\theta}{2}\right)v_{\widehat{\beta},2}^2(\varphi,\theta,\eta)\log\left|v_{\widehat{\beta},2}(\varphi,\theta,\eta)\right|.\end{aligned}$$

Combining Lemma 4.1-(iv)-(v), (6.136), (6.28) gives for any $\eta\in\mathbb{T}$

$$\begin{aligned}\max_{k\in\{0,1,2\}}\left\|(\partial_\theta^k\widehat{\mathscr{K}})(*,\cdot,\centerdot,\eta+\centerdot)\right\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+3}^{\gamma,\mathcal{O}}\left(1-\log\left|\sin\left(\frac{\eta}{2}\right)\right|\right)-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|+1\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\frac{\eta}{2}\right)\right|\right)-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|+1.\end{aligned} \tag{6.146}$$

The next goal is to prove that

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\left\|(\partial_\theta^k\widehat{\mathscr{K}_{\varepsilon r,1}^1})(*,\cdot,\centerdot,\eta+\centerdot)\right\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right). \tag{6.147}$$

For this aim we first write from (5.35) and (A.2)

$$\begin{aligned}\mathscr{K}_{\varepsilon r,1}^1(\varphi,\theta,\eta)&=4\lambda^2\big(1-v_{\varepsilon r,1}(\varphi,\theta,\eta)\big)\tilde{I}_\lambda(\eta-\theta)\\&\quad-4\lambda^2(v_{\varepsilon r,1}(\varphi,\theta)-1)^2\int_0^1(1-t)I_0''\left(2\lambda\sin\left(\tfrac{\eta-\theta}{2}\right)(1-t+tv_{\varepsilon r,1}(\varphi,\theta,\eta))\right)dt\\&:=4\lambda^2\big(1-v_{\varepsilon r,1}(\varphi,\theta,\eta)\big)\tilde{I}_\lambda(\eta-\theta)+G(\varphi,\theta,\eta),\end{aligned} \tag{6.148}$$

with

$$\tilde{I}_\lambda(\eta) := \frac{I_0'\left(2\lambda\left|\sin\left(\frac{\eta}{2}\right)\right|\right)}{2\lambda\left|\sin\left(\frac{\eta}{2}\right)\right|} = \frac{1}{2}\sum_{m=0}^{\infty}\frac{\lambda^{2m}\sin^{2m}\left(\frac{\eta}{2}\right)}{m!(m+1)!}.$$

Then we get the decomposition

$$\widehat{\mathscr{K}_{\varepsilon r,1}^1}(\varphi,\theta,\eta) = 4\lambda^2\big[1-\widehat{v}_{\varepsilon r,1}(\varphi,\theta,\eta)\big]\widehat{I}_\lambda(\varphi,\theta,\eta) + \widehat{G}(\varphi,\theta,\eta),$$

with

$$\begin{aligned}\widehat{v}_{\varepsilon r,1}(\varphi,\theta,\eta) &:= v_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big),\\ \widehat{I}_\lambda(\varphi,\theta,\eta) &:= \tilde{I}_\lambda\big(\eta+\widehat{\beta}(\varphi,\eta)-\theta-\widehat{\beta}(\varphi,\theta)\big),\\ \widehat{G}(\varphi,\theta,\eta) &:= G\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{\mathscr{K}_{\varepsilon r,1}^1}(\varphi,\theta,\theta+\eta) =& 4\lambda^2\big[1-v_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\theta+\eta+\widehat{\beta}(\varphi,\theta+\eta)\big)\big]\tilde{I}_\lambda\big(\eta+\widehat{\beta}(\varphi,\theta+\eta)-\widehat{\beta}(\varphi,\theta)\big)\\ &+ G\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\theta+\widehat{\beta}(\varphi,\eta+\theta)\big).\end{aligned} \tag{6.149}$$

Notice that $(\lambda,\eta)\mapsto\tilde{I}_\lambda(\eta)$ is $\mathcal{C}^\infty$, then using Lemma 4.1-(v) and (6.28) yields for any $k\in\mathbb{N}$

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\widehat{I}_\lambda)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim 1+\|\widehat{\beta}\|_{q,s+k}^{\gamma,\mathcal{O}}\\ &\lesssim 1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+k}^{\gamma,\mathcal{O}}.\end{aligned}$$

Now using (5.30), Lemma 4.1-(v), (6.27), (6.7), (6.136) and proceeding as in (5.31) we obtain

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|\widehat{v}_{\varepsilon r,1}\big(*,\cdot,\centerdot,\eta+\centerdot\big)-1\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\|r\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon^2\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)\end{aligned}$$

and

$$\begin{aligned}\max_{k\in\{1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\widehat{v}_{\varepsilon r,1})\big(*,\cdot,\centerdot,\eta+\centerdot\big)\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\|r\|_{q,s+3}^{\gamma,\mathcal{O}}+\varepsilon^2\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+3}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Arguing as above using the structure of G detailed in (6.148) allows to get

$$\sup_{\eta\in\mathbb{T}}\|\widehat{G}\big(*,\cdot,\centerdot,\eta+\centerdot\big)-1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)$$

and

$$\max_{k\in\{1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\widehat{G}\big(*,\cdot,\centerdot,\eta+\centerdot\big)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).$$

Thus applying the law products in Lemma 4.1 and using the preceding estimates combined with (6.149) imply

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|\big(\partial_\theta^k\widehat{\mathscr{K}_{\varepsilon r,1}^1}\big)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right), \tag{6.150}$$

which gives in particular (6.147). The estimate of the last term $\mathscr{K}_{\varepsilon r,1}^2$ in (6.145), which is connected to (5.36), can be treated in a similar way to the estimate (6.150) and one finds

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\big\|(\partial_\theta^k\widehat{\mathscr{K}_{\varepsilon r,1}^2})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).\tag{6.151}$$

Consequently, putting together (6.145),(6.146), (6.147) and (6.151) yields

$$\max_{k\in\{0,1,2\}}\big\|(\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right).\tag{6.152}$$

By (6.139) we infer that $\mathfrak{R}_{\varepsilon r}$ is an integral operator of kernel $\mathbb{K}_{\varepsilon r}$ given by

$$\mathbb{K}_{\varepsilon r}=-\widehat{\mathbb{K}}_{\varepsilon r,1}-\mathbb{K}_{\varepsilon r,2}.$$

Therefore, by virtue of Lemma 4.4 combined with (6.141) and (6.152) we find, taking $\sigma_2=s_0+\sigma_1+3$,

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\max_{k\in\{0,1,2\}}\int_{\mathbb{T}}\left(\|(\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}}+\|(\partial_\theta^k\mathbb{K}_{\varepsilon r,2})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}}\right)d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\int_{\mathbb{T}}\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Notice that by (6.144), (6.140), the kernel $\mathbb{K}_{\varepsilon r}$ satisfies the following symmetry property

$$\mathbb{K}_{\varepsilon r}(-\varphi,-\theta,-\eta)=\mathbb{K}_{\varepsilon r}(\varphi,\theta,\eta)\in\mathbb{R},\tag{6.153}$$

which implies in view of Lemma 4.4 that $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving Toeplitz in time integral operator. It remains to estimate the quantity $\max\limits_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}$. This is will be implemented as before and we shall here sketch the main ideas. First we observe that for $k\in\{0,1\}$ the kernel of $\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}$ is given by

$$\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}=-\Delta_{12}\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1}-\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}.$$

To estimate $\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}$ we shall use (6.138) leading to

$$\Delta_{12}\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)=\mathscr{K}(\theta-\eta)\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)+\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^2(\varphi,\theta,\eta)\tag{6.154}$$

and

$$\begin{aligned}\Delta_{12}\partial_\theta\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)=\mathscr{K}(\theta-\eta)\Delta_{12}\partial_\theta\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)+\mathscr{K}'(\theta-\eta)\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)\\+\Delta_{12}\partial_\theta\mathscr{K}_{\widehat{\beta},2}^2(\varphi,\theta,\eta).\end{aligned}\tag{6.155}$$

Observe from (6.135) that the preceding kernels can be expressed with respect to $\widehat{\beta}$. Then proceeding in a similar way to (5.53) we obtain

$$\forall\, i\in\{1,2\},\quad\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\|d_{\widehat{\beta}}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i[\rho](*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\widehat{\beta}\|_{q,s+2}^{\gamma,\mathcal{O}}.\tag{6.156}$$

Applying Taylor Formula yields for all $i\in\{1,2\}$ and for all $k\in\{0,1\}$,

$$\Delta_{12}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i(\varphi,\theta,\theta+\eta)=\int_0^1 d_{\widehat{\beta}}\partial_\theta^k\mathscr{K}_{(1-\tau)\widehat{\beta}_2+\tau\widehat{\beta}_1,2}^i[\widehat{\beta}_1-\widehat{\beta}_2](\varphi,\theta,\theta+\eta)\,d\tau.$$

It follows from (6.156) that for all $i\in\{1,2\}$ and for all $k\in\{0,1\}$

$$\big\|\Delta_{12}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{\beta}_2-\widehat{\beta}_1\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2-\widehat{\beta}_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\int_0^1\|(1-\tau)\widehat{\beta}_2+\tau\widehat{\beta}_1\|_{q,s+2}^{\gamma,\mathcal{O}}\,d\tau.$$

Therefore, by our previous choice of σ_2, we obtain in view of (6.28), (6.31) (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0$) and the smallness condition (6.131),

$$\forall i\in\{1,2\},\quad \max_{k\in\{0,1\}}\big\|\Delta_{12}\partial_\theta^k\mathscr{K}^i_{\widehat{\beta},2}(*,\cdot,\bullet,\eta+\bullet)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}\Big(1+\varepsilon\gamma^{-1}\big(1+\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}\big)\Big)$$
$$\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.$$

Inserting this estimate into (6.154) and (6.155) yields

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}(*,\cdot,\bullet,\eta+\bullet)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\tag{6.157}$$

Using similar techniques based on Taylor Formula, one can estimate $\Delta_{12}\partial_\theta\widehat{\mathbb{K}}_{\varepsilon r,1}$. We use in particular the identity (6.145) combined with (5.53), (6.28), (6.31) and the smallness condition (6.131) allowing to get

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1}(*,\cdot,\bullet,\eta+\bullet)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\tag{6.158}$$

Putting together (6.157) and (6.158) gives

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}(*,\cdot,\bullet,\eta+\bullet)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.$$

Comibining this estimate with Lemma 4.4 yields

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,\overline{s}_h+\mathtt{p}}\lesssim\max_{k\in\{0,1\}}\int_{\mathbb{T}}\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}(*,\cdot,\bullet,\eta+\bullet)\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}d\eta$$
$$\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.$$

This completes the proof of the Proposition 6.3. □

6.3 Diagonalization up to small errors

The main goal of this section is to diagonalize, up to small errors, the operator $\widehat{\mathcal{L}}_\omega$ discussed in Proposition 6.1 and given by

$$\widehat{\mathcal{L}}_\omega=\Pi_{\mathbb{S}_0}^\perp(\mathcal{L}_{\varepsilon r}-\varepsilon\partial_\theta\mathcal{R})\Pi_{\mathbb{S}_0}^\perp.$$

This will be performed in two main steps. First, we shall explore the effect of the frequency localization in the normal direction on the transport reduction discussed in Section 6.2. We essentially get the same structure up to a small perturbation of finite-dimensional rank. Then, in the second step we shall implement a KAM reducibility scheme in order to reduce the remainder to a diagonal one modulo small fast decaying operators. This will be performed through the use of a suitable strong topology on continuous operators given by (4.10). With this topology one has tame estimates and the Toeplitz structure of the remainder is very important in this part. The reduction will be conducted by assuming non resonance conditions stemming from the second order Melnikov conditions needed in the resolution of adequate homological equations during the scheme.

6.3.1 Projection in the normal directions

In this section, we study the effects of the reduction of the transport part when the linearized operator is localized in the normal directions. Notice that the change of coordinates does not stabilize the normal subspace and as we shall see the defect of the commutation can be modeled by projectors of finite ranks. Let us define

$$\mathscr{B}_\perp:=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp,$$

where the transformation $\mathscr{B}$ is introduced in (6.15) and constructed in Proposition 6.2. Recall that the projection $\Pi_{\mathbb{S}_0}^\perp$ and the Soboloev space $H^s_\perp$ were respectively defined in (5.10) and (5.9). We also recall the following notations

$$\mathbf{e}_{l,j}(\varphi,\theta)=e^{\mathrm{i}(l\cdot\varphi+j\theta)}\quad\text{and}\quad e_m(\theta)=e^{\mathrm{i}m\theta}.$$

The first main result of this section reads as follows.

Lemma 6.3. *Let $\mathscr{B}$ the transformation constructed in Proposition* 6.2*, then under the condition* (6.131) *and* (6.24)*, the following assertions hold.*

(i) For all $s \in [s_0, S]$, the operator $\mathscr{B}_\perp : W^{q,\infty,\gamma}(\mathcal{O}, H^s_\perp) \to W^{q,\infty,\gamma}(\mathcal{O}, H^s_\perp)$ is continuous and invertible, with

$$\|\mathscr{B}_\perp^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.159}$$

In addition, we have the representations

$$\mathscr{B}_\perp\rho = \mathscr{B}\rho - \sum_{m\in\mathbb{S}_0}\big\langle\rho, \big(\mathcal{B}^{-1} - \mathrm{Id}\big)e_m\big\rangle_{L^2_\theta(\mathbb{T})}e_m$$

and

$$\mathscr{B}_\perp^{-1}\rho = \mathscr{B}^{-1}\rho - \sum_{m\in\mathbb{S}_0}\big\langle\rho, \big(\mathcal{B} - \mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,$$

where

$$\mathbf{A}(\varphi) = \Big(\langle e_m, \mathcal{B}e_k\rangle_{L^2_\theta(\mathbb{T})}\Big)_{\substack{m\in\mathbb{S}_0\\ k\in\mathbb{S}_0}}, \quad \mathbf{A}^{-1}(\varphi) = \Big(\alpha_{k,m}\Big)_{\substack{m\in\mathbb{S}_0\\ k\in\mathbb{S}_0}}, \quad g_m(\varphi,\theta) = \sum_{k\in\mathbb{S}_0}\alpha_{k,m}(\varphi)e_k(\theta),$$

with the estimate

$$\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m} - \delta_{km}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\Big(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\Big).$$

(ii) Given two tori i_1 and i_2 satisfying the smallness condition (6.131)*, one has*

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}. \tag{6.160}$$

Proof. **(i)** The first estimate concerning $\mathscr{B}_\perp$ follows easily from the continuity of the orthogonal projector $\Pi_{\mathbb{S}_0}^\perp$ on Sobolev spaces $H^s_\perp$, combined with (6.27). For the representation of $\mathscr{B}_\perp$, take $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s_\perp)$ and set

$$\mathscr{B}_\perp\rho = \Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\rho = \Pi_{\mathbb{S}_0}^\perp\mathscr{B}\rho := g.$$

Next, we write the following splitting

$$\mathscr{B}\rho = g + h \quad \text{with} \quad \Pi_{\mathbb{S}_0}h = h. \tag{6.161}$$

Notice that the projector $\Pi_{\mathbb{S}_0}$ is defined by

$$\Pi_{\mathbb{S}_0}\rho = \sum_{j\in\mathbb{S}_0}\rho_j e_j = \Pi_{\overline{\mathbb{S}}}\rho + \langle\rho\rangle_\theta,$$

where $\Pi_{\overline{\mathbb{S}}}$ is defined in (5.10) and $\langle\cdot\rangle_\theta$ denotes the avarage in the variable θ. Therefore

$$h(\varphi,\theta) = \sum_{m\in\mathbb{S}_0}h_m(\varphi)e_m(\theta),$$

supplemented with the orthogonal conditions

$$\forall\, k \in \mathbb{S}_0, \quad \langle\mathscr{B}\rho - h, e_k\rangle_{L^2_\theta(\mathbb{T})} = 0.$$

This implies

$$h(\varphi,\theta) = \sum_{m\in\mathbb{S}_0}\langle\mathscr{B}\rho, e_m\rangle_{L^2_\theta(\mathbb{T})}e_m(\theta).$$

Using Lemma 6.1-(iii) leads to

$$h(\varphi,\theta) = \sum_{m\in\mathbb{S}_0}\langle\rho, \mathcal{B}^{-1}e_m\rangle_{L^2_\theta(\mathbb{T})}e_m(\theta).$$

Inserting this identity into (6.161) yields

$$\mathscr{B}_{\perp}\rho = g = \mathscr{B}\rho - \sum_{m\in\mathbb{S}_0} \big\langle \rho, \mathcal{B}^{-1}e_m\big\rangle_{L^2_\theta(\mathbb{T})} e_m.$$

Since $\forall m \in \mathbb{S}_0$, $\big\langle \rho, e_m\big\rangle_{L^2_\theta(\mathbb{T})} = 0$, then

$$\mathscr{B}_{\perp}\rho = g = \mathscr{B}\rho - \sum_{m\in\mathbb{S}_0} \big\langle \rho, \big(\mathcal{B}^{-1} - \mathrm{Id}\big)e_m\big\rangle_{L^2_\theta(\mathbb{T})} e_m.$$

This ensures the desired representation of $\mathscr{B}_{\perp}$.
Next, we intend to establish similar representation for $\mathscr{B}_{\perp}^{-1}$. Let $g \in W^{q,\infty,\gamma}(\mathcal{O}, H^s_{\perp})$ and we need to solve the equation

$$f \in W^{q,\infty,\gamma}(\mathcal{O}, H^s_{\perp}), \quad \mathscr{B}_{\perp} f = \Pi^{\perp}_{\mathbb{S}_0}\mathscr{B} f = g.$$

This is equivalent to

$$\mathscr{B} f = g + h, \quad \text{with} \quad \Pi_{\mathbb{S}_0} h = h \quad \text{and} \quad \Pi_{\mathbb{S}_0} f = 0.$$

Then we get

$$f = \mathscr{B}^{-1}\big(g + h\big), \quad \text{with} \quad \Pi_{\mathbb{S}_0} h = h \quad \text{and} \quad \Pi_{\mathbb{S}_0} f = 0. \tag{6.162}$$

The condition $\Pi_{\mathbb{S}_0} f = 0$ is equivalent to,

$$\forall k \in \mathbb{S}_0, \quad \big\langle \mathscr{B}^{-1}\big(g + h\big), e_k\big\rangle_{L^2_\theta(\mathbb{T})} = 0.$$

Therefore using Lemma 6.1-(iii) the latter equation reads

$$\forall k \in \mathbb{S}_0, \quad \big\langle g + h, \widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})} = 0 \quad \text{with} \quad \widehat{e}_k(\varphi,\theta) := \mathcal{B}e_k(\varphi,\theta) = e^{\mathrm{i}k(\theta+\beta(\varphi,\theta))},$$

which will fix h. Indeed, by expanding $h(\varphi,\theta) = \sum_{m\in\mathbb{S}_0} a_m(\varphi)e_m(\theta)$, we can transform the preceding system into

$$\forall k \in \mathbb{S}_0, \quad \sum_{m\in\mathbb{S}_0} a_m(\varphi)\big\langle e_m, \widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})} = -\big\langle g, \widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})}. \tag{6.163}$$

Define the matrix

$$\mathbf{A}(\varphi) = \big(c_{m,k}(\varphi)\big)_{(m,k)\in\mathbb{S}_0^2}, \quad c_{m,k}(\varphi) = \langle e_m, \widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})} = \int_{\mathbb{T}} e^{\mathrm{i}((m-k)\theta - k\beta(\varphi,\theta))}d\theta. \tag{6.164}$$

Notice that according to (6.32) and the change of variables $\theta \mapsto -\theta$, one obtains

$$\forall (m,k) \in \mathbb{S}_0^2, \quad \forall \varphi \in \mathbb{T}^d, \quad c_{m,k}(-\varphi) = c_{-m,-k}(\varphi) = \overline{c_{m,k}(\varphi,\theta)}. \tag{6.165}$$

One can check by slight adaptation of the composition law in Lemma 4.1 and using the smallness condition (6.131) and (6.28)

$$\begin{aligned}
\|c_{m,m} - 1\|^{\gamma,\mathcal{O}}_{q,s} &\leqslant \int_{\mathbb{T}} \|e^{-\mathrm{i}m\beta(\cdot,\theta))} - 1\|^{\gamma,\mathcal{O}}_{q,H^s_\varphi} d\theta \\
&\lesssim \|\beta\|^{\gamma,\mathcal{O}}_{q,s} \\
&\lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma_1}\right).
\end{aligned} \tag{6.166}$$

For $k \neq m \in \mathbb{S}_0$ we use integration by parts,

$$c_{m,k}(\varphi) = \tfrac{k}{\mathrm{i}(m-k)} \int_{\mathbb{T}} e^{\mathrm{i}((m-k)\theta - k\beta(\varphi,\theta))}\partial_\theta\beta(\varphi,\theta)d\theta.$$

Then using law products and composition laws in Lemma 4.1 combined with (6.28) yield

$$\begin{aligned}\sup_{\substack{(m,k)\in\mathbb{S}_0^2\\ m\neq k}}\|c_{m,k}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Finally, we get that

$$\mathbf{A}(\varphi)=\mathrm{Id}+\mathbf{R}(\varphi)\quad\text{with}\quad\|\mathbf{R}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}.\tag{6.167}$$

Hence under the smallness condition $\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\ll 1$ following from (6.131), combined with the law products in Lemma 4.1 we get that $\mathbf{A}$ is invertible with

$$\begin{aligned}\|\mathbf{A}^{-1}-\mathrm{Id}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.168}$$

Therefore the system (6.163) is invertible and one gets a unique solution given by

$$a_m(\varphi)=-\sum_{k\in\mathbb{S}_0}\alpha_{m,k}(\varphi)\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\quad\text{with}\quad\mathbf{A}^{-1}(\varphi):=\Big(\alpha_{m,k}(\varphi)\Big)_{(m,k)\in\mathbb{S}_0^2}.\tag{6.169}$$

We claim that the coefficients of $\mathbf{A}^{-1}$ admit the same symmetry conditions as (6.165), that is

$$\forall(m,k)\in\mathbb{S}_0^2,\quad\forall\varphi\in\mathbb{T}^d,\quad\alpha_{m,k}(-\varphi)=\alpha_{-m,-k}(\varphi)=\overline{\alpha_{m,k}(\varphi)}.\tag{6.170}$$

This can be done through the series expansion $A^{-1}=\sum_{n\in\mathbb{N}}(-1)^n(A-\mathrm{Id})^n$ together with the fact that the entries of the monomials $(A-\mathrm{Id})^n$ satisfy in turn (6.165). Next, using the law products yields

$$\sup_{m\in\mathbb{S}_0}\|a_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\sup_{k\in\mathbb{S}_0}\Big(\|\mathbf{A}^{-1}\|_{q,s}^{\gamma,\mathcal{O}}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|\mathbf{A}^{-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\Big).\tag{6.171}$$

Notice that one gets from (6.168)

$$\sup_{(m,k)\in\mathbb{S}_0^2}\|\alpha_{k,m}-\delta_{km}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right),$$

where δ_{km} denotes the Kronecker symbol. Let us now move to the estimate of the partial scalar product containing g in (6.171). Using the law products in Lemma 4.1 with Cauchy-Schwarz inequality gives

$$\begin{aligned}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\Big(\|g(\cdot,\theta)\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\|e^{i\beta(\cdot,\theta)}\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|g(\cdot,\theta)\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}\|e^{i\beta(\cdot,\theta)}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\Big)d\theta\\&\lesssim\|g\|_{q,L^2_\theta H^s_\varphi}^{\gamma,\mathcal{O}}\|e^{i\beta}\|_{q,L^2_\theta H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|g\|_{q,L^2_\theta H^{s_0}_\varphi}^{\gamma,\mathcal{O}}\|e^{i\beta}\|_{q,L^2_\theta H^s_\varphi}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}\|e^{i\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\|e^{i\beta}\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Then applying the composition law as in (6.166) combined with with (6.28) and the smallness condition (6.131) gives

$$\begin{aligned}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\|\beta\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right)\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Plugging this estimate into (6.171) and using (6.28), (6.168) combined with the smallness condition (6.131) and Sobolev embeddings implies

$$\begin{aligned}\sup_{m\in\mathbb{S}_0}\|a_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|h\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \sum_{m\in\mathbb{S}_0}\|a_m\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\\ &\lesssim \|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Coming back to (6.162) and using (6.27), we get

$$\begin{aligned}\|f\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|g+h\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g+h\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

It follows that

$$\|\mathscr{B}_\perp^{-1}g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In addition from (6.169) and (6.162) we deduce the formula

$$\begin{aligned}\mathscr{B}_\perp^{-1}g(\varphi,\theta) &= \mathscr{B}^{-1}g(\varphi,\theta)-\sum_{\substack{m\in\mathbb{S}_0\\ k\in\mathbb{S}_0}}\alpha_{m,k}(\varphi)\big\langle g,\mathcal{B}e_k\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m(\theta)\\ &= \mathscr{B}^{-1}g(\varphi,\theta)-\sum_{m\in\mathbb{S}_0}\big\langle g,\mathcal{B}g_m\big\rangle_{L^2_\theta(\mathbb{T})}\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta),\end{aligned}\tag{6.172}$$

with

$$g_m(\varphi,\theta):=\sum_{k\in\mathbb{S}_0}\alpha_{m,k}(\varphi)e_k(\theta).\tag{6.173}$$

From (6.170) and the symmetry of $\mathbb{S}_0$, we infer

$$\forall m\in\mathbb{S}_0,\quad \forall(\varphi,\theta)\in\mathbb{T}^{d+1},\quad g_m(-\varphi,-\theta)=g_{-m}(\varphi,\theta)=\overline{g_m(\varphi,\theta)}.\tag{6.174}$$

Since $\Pi_{\mathbb{S}_0}^\perp g=g$ and $\Pi_{\mathbb{S}_0}^\perp g_m=0$ then $\big\langle g,g_m\big\rangle_{L^2_\theta(\mathbb{T})}=0$ and therefore

$$\big\langle g,\mathcal{B}g_m\big\rangle_{L^2_\theta(\mathbb{T})}=\big\langle g,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}.$$

Plugging this identity into (6.172) yields

$$\mathscr{B}_\perp^{-1}g=\mathscr{B}^{-1}g-\sum_{m\in\mathbb{S}_0}\big\langle g,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m.$$

(ii) Coming back to the definition of $c_{m,k}$ in (6.164), one can write

$$\forall(m,k)\in\mathbb{S}_0^2,\quad \Delta_{12}c_{m,k}=\langle e_m,(\Delta_{12}\mathcal{B})e_k\rangle_{L^2_\theta(\mathbb{T})}.$$

Hence, using Taylor Formula and (6.31), we have

$$\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}c_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}.$$

From (6.173), one has

$$\Delta_{12}g_m=\sum_{k\in\mathbb{S}_0}\Delta_{12}\alpha_{m,k}\,e_k.$$

Thus

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}\alpha_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}.\tag{6.175}$$

Using Neumann series, we can write

$$\mathbf{A}^{-1}(\varphi)=\mathrm{Id}+\sum_{n=1}^{\infty}(-1)^n\mathbf{R}^n(\varphi).$$

Therefore, the law products in Lemma 4.1 combined with (6.167) and the smallness condition (6.131) lead to

$$\begin{aligned}\|\Delta_{12}\mathbf{A}^{-1}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \sum_{n=1}^{\infty}\|\Delta_{12}\mathbf{R}^n\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\ &\lesssim \|\Delta_{12}\mathbf{R}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

As a consequence,

$$\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}\alpha_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}. \tag{6.176}$$

Gathering (6.176) and (6.175) finally gives

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}.$$

This achieves the proof of Lemma 6.3. □

In Lemma 6.3, the parameter $\mathtt{p}$ is subject to the constraint (6.24) and from now on, we shall fix it to the value

$$\mathtt{p} = 4\tau_2 q + 4\tau_2. \tag{6.177}$$

This particular choice is determined through some constraints in the proof of the remainder reduction. More precisely, it appears in (6.373). Next we shall establish the second main result of this section.

Proposition 6.4. *Let* $(\gamma, q, d, \tau_1, s_0, s_h, \overline{s}_h, \mu_2, \mathtt{p}, \sigma_2, S)$ *satisfy the assumptions* (4.1), (4.2), (6.3), (6.24), (6.130) *and* (6.177). *Consider the operator* $\widehat{\mathcal{L}}_\omega$ *defined in Proposition* 6.1.
There exists $\varepsilon_0 > 0$ *and* $\sigma_3 = \sigma_3(\tau_1, q, d, s_0) \geqslant \sigma_2$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_3}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.178}$$

then the following assertions hold true.

(i) For any $n \in \mathbb{N}^*$, *in the Cantor set* $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *introduced in Proposition* 6.2, *we have*

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp &= \big(\omega\cdot\partial_\varphi + c_{i_0}\partial_\theta - \partial_\theta\mathcal{K}_\lambda * \cdot\big)\Pi_{\mathbb{S}_0}^\perp + \mathscr{R}_0 + \mathtt{E}_n^1\\ &:= \big(\omega\cdot\partial_\varphi + \mathscr{D}_0\big)\Pi_{\mathbb{S}_0}^\perp + \mathscr{R}_0 + \mathtt{E}_n^1\\ &:= \mathscr{L}_0 + \mathtt{E}_n^1,\end{aligned}$$

where $\mathscr{D}_0$ *is a reversible Fourier multiplier given by*

$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c, \quad \mathscr{D}_0\mathbf{e}_{l,j} = \mathrm{i}\mu_j^0\,\mathbf{e}_{l,j},$$

with

$$\mu_j^0(\lambda,\omega,i_0) = \Omega_j(\lambda) + jr^1(\lambda,\omega,i_0) \quad \text{and} \quad r^1(\lambda,\omega,i_0) = c_{i_0}(\lambda,\omega) - V_0(\lambda)$$

and such that

$$\|r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon \quad \text{and} \quad \|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}. \tag{6.179}$$

(ii) The operator $\mathtt{E}_n^1$ *satisfies the following estimate*

$$\|\mathtt{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{6.180}$$

(iii) $\mathscr{R}_0$ *is a real and reversible Toeplitz in time operator satisfying* $\mathscr{R}_0 = \Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ *with*

$$\forall s\in[s_0,S], \quad \max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\textrm{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right) \tag{6.181}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.182}$$

(iv) The operator $\mathscr{L}_0$ satisfies

$$\forall s\in[s_0,S],\quad \|\mathscr{L}_0\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{6.183}$$

Proof. **(i)** We shall first start with finding a suitable expansion for $\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp$. Using the expression of $\widehat{\mathcal{L}}_\omega$ given in Proposition 6.1 and the decomposition $\mathrm{Id}=\Pi_{\mathbb{S}_0}+\Pi_{\mathbb{S}_0}^\perp$ we write

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp&=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp(\mathcal{L}_{\varepsilon r}-\varepsilon\partial_\theta\mathcal{R})\mathscr{B}_\perp\\&=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathcal{R}\mathscr{B}_\perp.\end{aligned}$$

According to the definitions of $\mathfrak{L}_{\varepsilon r}$ and $\mathcal{L}_{\varepsilon r}$ seen in Proposition 6.3 and in Lemma 3.1 and using (5.46), one has in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathcal{L}_{\varepsilon r}\mathscr{B}=\mathscr{B}\mathfrak{L}_{\varepsilon r}\quad\text{and}\quad\mathcal{L}_{\varepsilon r}=\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}-\partial_\theta\mathcal{K}_\lambda*\cdot$$

and therefore

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp,$$

where we have used the identities

$$\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp=\mathscr{B}_\perp^{-1}\quad\text{and}\quad[\Pi_{\mathbb{S}_0}^\perp,T]=0=[\Pi_{\mathbb{S}_0},T],$$

for any Fourier multiplier T. The structure of $\mathfrak{L}_{\varepsilon r}$ is detailed in Proposition 6.3, and from this we deduce that

$$\begin{aligned}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0\big)\Pi_{\mathbb{S}_0}^\perp\\&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\mathscr{B}_\perp\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

It follows that

$$\begin{aligned}\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&\quad+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

Consequently, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, one has the following reduction

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=&\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\:=&\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{R}_0+\mathtt{E}_n^1,\end{aligned}\tag{6.184}$$

where we set

$$\mathtt{E}_n^1:=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.$$

Notice that the estimates (6.179) are simple reformulations of (6.26) and (6.30) since $\Delta_{12}r^1=\Delta_{12}c_i$.
(ii) By using (6.159), (6.27), the continuity of the projectors, (6.29) and (6.178), one obtains

$$\begin{aligned}\|\mathtt{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\|\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\end{aligned}$$

(iii) Now, we shall prove the following estimates,

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\tag{6.185}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|^{\gamma,\mathcal{O}}_{\text{O-d},q,\overline{s}_h+\mathtt{p}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}. \tag{6.186}$$

To do that, we shall study separately the different terms appearing in (6.184) in the definition of $\mathscr{R}_0$. Notice that in the various estimates below, we use the notation σ_3 to denote some loss of regularity. This index depends only on τ_1, q, d, s_0 and may change increasingly from one line to another and it is always taken greater than the σ_2 introduced in Proposition 6.3.

► *Study of the term* $\Pi^{\perp}_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}$. One gets easily according to (6.132) and (6.133)

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^k\Pi^{\perp}_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} &\lesssim \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma_3}\right)\end{aligned} \tag{6.187}$$

and

$$\begin{aligned}\|\Delta_{12}\big(\Pi^{\perp}_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}\big)\|^{\gamma,\mathcal{O}}_{\text{O-d},q,\overline{s}_h+\mathtt{p}} &\lesssim \|\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,\overline{s}_h+\mathtt{p}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\end{aligned} \tag{6.188}$$

► *Study of the term* $\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}$. Using the first point of Proposition 6.4 yields

$$\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0} = \Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0} - \mathcal{T}_0\mathcal{S}_1, \tag{6.189}$$

where

$$\mathcal{T}_0\rho = \sum_{m\in\mathbb{S}_0}\big\langle\rho,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m \quad\text{and}\quad \mathcal{S}_1 := \mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}. \tag{6.190}$$

To estimate the first term, we use Proposition 6.3

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^k\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi^{\perp}_{\mathbb{S}_0}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} &\lesssim \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma_3}\right).\end{aligned} \tag{6.191}$$

As to the second term, we write

$$\begin{aligned}\mathcal{T}_0\mathcal{S}_1\rho &= \sum_{m\in\mathbb{S}_0}\big\langle\mathcal{S}_1\rho,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\ &= \sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

where $\mathcal{S}_1^\star$ is the $L^2_\theta(\mathbb{T})$-adjoint of $\mathcal{S}_1$. This is an integral operator taking the form

$$\begin{aligned}\big(\mathcal{T}_0\mathcal{S}_1\rho\big)(\varphi,\theta) &= \int_{\mathbb{T}}\mathcal{K}_1(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\ \mathcal{K}_1(\varphi,\theta,\eta) &:= \sum_{m\in\mathbb{S}_0}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

Recall from Proposition 6.3 that $\mathfrak{R}_{\varepsilon r}$ is self-adjoint and using Lemma 6.1 we have the identities $\mathscr{B}^\star = \mathcal{B}^{-1}$ and $\mathcal{B}^\star = \mathscr{B}^{-1}$, then

$$\mathcal{S}_1^\star = -\Pi^{\perp}_{\mathbb{S}_0}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\mathcal{B}^{-1}. \tag{6.192}$$

Therefore, combining (6.174), (6.32) and (6.153) imply

$$\mathcal{K}_1(-\varphi,-\theta,-\eta) = -\mathcal{K}_1(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.193}$$

Applying Lemma 4.4 combined with the law products yield for any $k\in\mathbb{N}$

$$\begin{aligned}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_1\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} &\lesssim \int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_1)(\ast,\cdot,\centerdot,\eta+\centerdot)\|^{\gamma,\mathcal{O}}_{q,s+s_0}d\eta \\ &\lesssim \sum_{m\in\mathbb{S}_0}\Big(\big\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\|^{\gamma,\mathcal{O}}_{q,s+s_0}\big\|\mathscr{B}^{-1}e_m\big\|^{\gamma,\mathcal{O}}_{q,s_0+k} + \big\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\|^{\gamma,\mathcal{O}}_{q,s_0}\big\|\mathscr{B}^{-1}e_m\big\|^{\gamma,\mathcal{O}}_{q,s+s_0+k}\Big).\end{aligned} \tag{6.194}$$

Remark that (6.192) implies

$$\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m=-\Pi_{\mathbb{S}_0}^\perp\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m.$$

Hence according to Lemma 4.3 combined with Proposition 6.3 we find

$$\begin{aligned}\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big)\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\quad+\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s_0+\sigma_3}^{\gamma,\mathcal{O}}\Big)\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s+1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.195}$$

Using (6.27) together with Lemma 6.3 and the smallness condition (6.178) leads to

$$\begin{aligned}\big\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g_m\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m}\|_{q,H_\varphi^s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m}\|_{q,H_\varphi^{s_0}}^{\gamma,\mathcal{O}}\\&\lesssim1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.196}$$

Therefore, inserting this estimate into (6.195) and using (6.178) allow to get

$$\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big).$$

Plugging this estimate into (6.194) and using (6.27) ensure

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_1\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big).\tag{6.197}$$

Consequently, by combining (6.189), (6.191) and (6.197), we find

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big).\tag{6.198}$$

We now turn to the difference estimate. From (6.189), it is obvious that

$$\Delta_{12}\big(\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\big)=\Pi_{\mathbb{S}_0}\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp-\Delta_{12}(\mathcal{T}_0\mathcal{S}_1).\tag{6.199}$$

To estimate the first term, we use (6.133)

$$\begin{aligned}\|\Pi_{\mathbb{S}_0}\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.200}$$

As to the second term, we notice that $\Delta_{12}\big(\mathcal{T}_0\mathcal{S}_1\big)$ is an integral operator whose kernel $\Delta_{12}\mathcal{K}_1$ is

$$\begin{aligned}\Delta_{12}\mathcal{K}_1(\varphi,\theta,\eta)=\sum_{m\in\mathbb{S}_0}&\Delta_{12}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}_{r_1}e_m\big)(\varphi,\theta)\\&+\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)_{r_2}(\varphi,\eta)\big(\Delta_{12}\mathscr{B}e_m\big)(\varphi,\theta).\end{aligned}$$

Hence, using Lemma 4.4-(ii) together with the law products we deduce that

$$\begin{aligned}\|\Delta_{12}\mathcal{T}_0\mathcal{S}_1\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}\mathcal{K}_1(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\sum_{m\in\mathbb{S}_0}\|\Delta_{12}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}_{r_1}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\\&\quad+\|\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)_{r_2}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}\mathscr{B}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Notice that by Taylor Formula and (6.31) (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0$), one has

$$\sup_{m\in\mathbb{S}_0}\|\Delta_{12}\mathscr{B}^{-1}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}. \tag{6.201}$$

On the other hand, we have

$$\Delta_{12}\mathcal{S}_1^{\star}=-\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\mathcal{B}_{r_1}^{-1}-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1},$$

leading to

$$\begin{aligned}\Delta_{12}\big(\mathcal{S}_1^{\star}(\mathcal{B}-\mathrm{Id})g_m\big)=&-\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1}\big(\mathcal{B}_{r_1}-\mathrm{Id}\big)g_{m,r_1}\\&+\mathcal{S}_{1,r_2}^{\star}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}+\mathcal{S}_{1,r_2}^{\star}\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m.\end{aligned}$$

AAccording to Lemma 4.3, we obtain

$$\|\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}\mathfrak{R}_{\varepsilon r}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\big(\mathrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.$$

From (6.196), one has

$$\|\big(\mathrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\varepsilon\gamma^{-1}\|\mathfrak{I}_1\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Thus, from (6.133) and(6.178), we infer

$$\|\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.202}$$

Applying Lemma 4.3, (6.132) and (6.178) we deduce that

$$\begin{aligned}\|\mathcal{S}_{1,r_2}^{\star}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}&\lesssim\|\mathfrak{R}_{\varepsilon r_2}\|_{\textrm{O-d},q,\overline{s}_h+s_0}\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

To estimate the right hand side member, it suffices to use (6.27) and (6.178), leading to

$$\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\lesssim\|\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.$$

By Taylor Formula, we may write

$$\Delta_{12}\mathcal{B}\rho(\theta)=\Delta_{12}\beta(\theta)\int_0^1\partial_\theta\rho\big(\theta+\beta_2(\theta)+t\Delta_{12}\beta(\theta)\big)dt.$$

It follows from the law products in Lemma 4.1, (6.31) and (6.178) that

$$\begin{aligned}\|\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+1}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\|g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+2}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Thus

$$\|\mathcal{S}_{1,r_2}^{\star}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.203}$$

In the same way, using Taylor Formula together with (6.31), we get

$$\|\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1}\big(\mathcal{B}_{r_1}-\mathrm{Id}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.204}$$

By Lemma 4.3,(6.132) and (6.178), one finds

$$\begin{aligned}\|\mathcal{S}_{1,r_2}^{\star}\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}&\lesssim\|\mathfrak{R}_{\varepsilon r_2}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Applying (6.27) and (6.178), we obtain

$$\|\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.$$

Using (6.160) (applied with $\mathtt{p}=s_0+1$), we finally get

$$\|\mathcal{S}_{1,r_2}^{\star}\big(\mathcal{B}_{r_2}-\mathrm{Id}\big)\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{6.205}$$

Gathering (6.201), (6.202), (6.203), (6.204) and (6.205) implies

$$\|\Delta_{12}\big(\mathcal{T}_0\mathcal{S}_1\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{6.206}$$

Putting together (6.199), (6.200) and (6.206), one obtains

$$\|\Delta_{12}\big(\mathscr{B}_{\perp}^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{6.207}$$

► *Study of the term* $\mathscr{B}_{\perp}^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}$. We first write,

$$\mathscr{B}_{\perp}^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}:=\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp},$$

with

$$\mathcal{S}_2=\big(\big(V_{\varepsilon r}-c_{i_0}\big)\cdot-\mathbf{L}_{\varepsilon r,1}\big)\,\Pi_{\mathbb{S}_0}.$$

Notice that to get the above identity we have used the identity

$$\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta(c_{i_0}\cdot)\Pi_{\mathbb{S}_0}=0.$$

Recall from (5.46) and (5.41) that

$$\mathbf{L}_{\varepsilon r,1}\rho(\varphi,\theta)=\int_{\mathbb{T}}\mathbb{K}_{\varepsilon r,1}(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta.$$

Then from elementary computations we find

$$\mathcal{S}_2\rho(\varphi,\theta)=\int_{\mathbb{T}}\mathcal{K}_2(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,$$

with

$$\begin{aligned}\mathcal{K}_2(\varphi,\theta,\eta)&:=\big(V_{\varepsilon r}(\varphi,\theta)-c_{i_0}\big)D_{\mathbb{S}_0}(\theta-\eta)-\int_{\mathbb{T}}\mathbb{K}_{\varepsilon r,1}(\varphi,\theta,\eta')D_{\mathbb{S}_0}(\eta'-\eta)d\eta',\\ D_{\mathbb{S}_0}(\theta)&:=\sum_{n\in\mathbb{S}_0}e^{in\theta}.\end{aligned}$$

Combining (5.43), (6.4), (3.3) and the change of variables $\eta'\mapsto\eta'$, one gets

$$\mathcal{K}_2(-\varphi,-\theta,-\eta)=\mathcal{K}_2(\varphi,\theta,\eta)\in\mathbb{R}.\tag{6.208}$$

Proceeding as in (6.189) we obtain

$$\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}=\mathscr{B}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}-\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}.\tag{6.209}$$

It follows that

$$\big\|\partial_\theta^k\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\big\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\big\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.\tag{6.210}$$

The expression of the first term is similar to that of (6.142), namely, one has

$$\big(\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\widehat{\mathcal{K}}_2(\varphi,\theta,\eta)d\eta,$$

with

$$\widehat{\mathcal{K}}_2(\varphi,\theta,\eta):=\mathcal{K}_2\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).$$

Combining (6.208) and (6.32), one gets

$$\widehat{\mathcal{K}}_2(-\varphi,-\theta,-\eta)=\widehat{\mathcal{K}}_2(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.211}$$

Then coming bacl to (6.143) and arguing as for (6.152), we find

$$\sup_{k\in\{0,1,2\}}\big\|(\partial_\theta^k\widehat{\mathcal{K}}_2)\big(*,\cdot,\centerdot,\eta+\centerdot\big)\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\Big(1-\log\big|\sin\left(\eta/2\right)\big|\Big). \tag{6.212}$$

By virtue of Lemma 4.4 and (6.212) we obtain

$$\begin{aligned}\sup_{k\in\{0,1\}}\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\sup_{k\in\{0,1\}}\int_{\mathbb{T}}\big\|(\partial_\theta^{k+1}\widehat{\mathcal{K}}_2)\big(*,\cdot,\centerdot,\eta+\centerdot\big)\big\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{6.213}$$

Notice that from (6.190), we can write

$$\begin{aligned}\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\rho&=\sum_{m\in\mathbb{S}_0}\big\langle\partial_\theta\mathcal{S}_2\mathscr{B}\rho,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\&=-\sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

where $\mathcal{S}_2^\star$ is the adjoint of $\mathcal{S}_2$ and is given by

$$\mathcal{S}_2^\star=\Pi_{\mathbb{S}_0}\big(\big(V_{\varepsilon r}-c_{i_0}\big)\cdot-\mathbf{L}_{\varepsilon r,1}\big). \tag{6.214}$$

This is an integral operator taking the form

$$\begin{aligned}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\rho\big)(\varphi,\theta)&=\int_{\mathbb{T}}\mathcal{K}_3(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\\mathcal{K}_3(\varphi,\theta,\eta)&:=\sum_{m\in\mathbb{S}_0}\big(\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

According to (6.174), (6.32), (6.214), (6.4), (3.3) and (5.43), one gets

$$\mathcal{K}_3(-\varphi,-\theta,-\eta)=-\mathcal{K}_3(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.215}$$

On the other hand, applying Lemma 4.4 combined with the law products yield for any $k\in\mathbb{N}$

$$\begin{aligned}&\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_3)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\sum_{m\in\mathbb{S}_0}\Big(\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s_0+k}^{\gamma,\mathcal{O}}+\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Applying (6.27) we find

$$\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Now, from (6.214), the law products and Lemma 4.3, we find

$$\begin{aligned}\|\mathcal{S}_2^\star\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim&\|V_{\varepsilon r}-c_{i_0}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|V_{\varepsilon r}-c_{i_0}\|_{q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

From the composition law and (6.26), one has

$$\begin{aligned}\|V_{\varepsilon r}-c_{i_0}\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|V_{\varepsilon r}-V_0\|_{q,s}^{\gamma,\mathcal{O}}+\|V_0-c_{i_0}\|_{q}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

According to Lemma 4.4 and (5.42), we deduce that

$$\begin{aligned}\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}} \|\mathbb{K}_{\varepsilon r,1}(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta \\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Using (6.178), one gets

$$\|\mathcal{S}_2^\star\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Combining this with (6.27) allow to get

$$\begin{aligned}\|\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|g_m\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}} \\ &\lesssim 1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore,

$$\max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right). \tag{6.216}$$

Plugging the estimates (6.213) and (6.216) into (6.210) we find

$$\begin{aligned}\max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \max_{k\in\{0,1\}}\left\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{6.217}$$

We now turn to the estimate of the difference. Coming back to (6.209), one can write

$$\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big) = \Delta_{12}\big(\mathscr{B}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big) - \Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big).$$

It follows that

$$\left\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \left\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} + \left\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}. \tag{6.218}$$

Arguing as for (6.158), one obtains

$$\|\Delta_{12}(\partial_\theta\widehat{\mathcal{K}}_2)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,\overline{s}_h+\mathtt{p}+s_0} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right).$$

Then, using Lemma 4.4 implies

$$\begin{aligned}\left\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widehat{\mathcal{K}}_2)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,\overline{s}_h+\mathtt{p}+s_0}d\eta \\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.219}$$

On the other hand, proceeding as for (6.206), and using in particular (6.30),

$$\left\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.220}$$

Putting together (6.219), (6.220) and (6.218), ensures that

$$\left\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.221}$$

► *Study of the term* $\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp$. Using the relation $\mathrm{Id} = \Pi_{\mathbb{S}_0}+\Pi_{\mathbb{S}_0}^\perp$, we can write

$$\begin{aligned}\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp &= \partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}_\perp - \partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}_\perp \\ &= \partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp - \partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp \\ &= \partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp - \partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp - \partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp + \partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.\end{aligned} \tag{6.222}$$

Hence

$$\|\partial_\theta^k \mathscr{B}_\perp^{-1}\partial_\theta \mathcal{R}\mathscr{B}_\perp\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant \|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}$$
$$+ \|\partial_\theta^k \mathcal{T}_0\partial_\theta \mathcal{R}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \|\partial_\theta^k \mathcal{T}_0\partial_\theta \mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}. \tag{6.223}$$

Recall that from Proposition 6.1 that $\mathcal{R}$ is an integral operator of kernel J and therefore direct computations give

$$\big(\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\rho\big)(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\widehat{J}(\varphi,\theta,\eta)d\eta, \tag{6.224}$$

with

$$\widehat{J}(\varphi,\theta,\eta) := J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big). \tag{6.225}$$

Combining (6.32) and (6.5), one gets

$$\widehat{J}(-\varphi,-\theta,-\eta) = \widehat{J}(\varphi,\theta,\eta) \in \mathbb{R}. \tag{6.226}$$

Using the composition law and (6.9), we obtain

$$\max_{k\in\{0,1,2\}} \sup_{\eta\in\mathbb{T}} \|(\partial_\theta^k \widehat{J})\big(*,\cdot,\centerdot,\eta+\centerdot\big)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Thus, applying Lemma 4.4-(ii) implies

$$\max_{k\in\{0,1\}} \|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \max_{k\in\{0,1,2\}} \int_{\mathbb{T}} \|(\partial_\theta^k \widehat{J})\big(*,\cdot,\centerdot,\eta+\centerdot\big)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta$$
$$\lesssim 1 + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.227}$$

On the other hand we notice from (6.224) that we get the structure

$$\big(\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\rho\big)(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\widetilde{J}(\varphi,\theta,\eta)d\eta, \tag{6.228}$$

with

$$\widetilde{J}(\varphi,\theta,\eta) := \int_{\mathbb{T}} J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta'\big)D_{\mathbb{S}_0}(\eta'-\eta)d\eta'. \tag{6.229}$$

Combining (6.32), (6.5) and the change of variables $\eta'\mapsto -\eta'$, one finds

$$\widetilde{J}(-\varphi,-\theta,-\eta) = \widetilde{J}(\varphi,\theta,\eta) \in \mathbb{R}. \tag{6.230}$$

Using the change of variables $\eta'\mapsto \eta'+\theta$ yields

$$\widetilde{J}(\varphi,\theta,\eta+\theta) := \int_{\mathbb{T}} J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta'+\theta\big)D_{\mathbb{S}_0}(\eta'-\eta)d\eta'.$$

Then by the composition law, we infer

$$\max_{k\in\{0,1,2\}} \sup_{\eta\in\mathbb{T}} \|(\partial_\theta^k \widetilde{J})(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Consequently, we find in view of Lemma 4.4

$$\max_{k\in\{0,1\}} \|\partial_\theta^{k}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \max_{k\in\{0,1,2\}} \int_{\mathbb{T}} \|(\partial_\theta^k \widetilde{J})\big(*,\cdot,\centerdot,\eta+\centerdot\big)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta$$
$$\lesssim 1 + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.231}$$

If wet set

$$\mathcal{S}_3 = \partial_\theta \mathcal{R}\mathscr{B} \quad \text{or} \quad \partial_\theta \mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B},$$

then using (6.190), we deduce that

$$\begin{aligned}\mathcal{T}_0\mathcal{S}_3\rho &= \sum_{m\in\mathbb{S}_0}\big\langle \mathcal{S}_3\rho, \big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\ &= \sum_{m\in\mathbb{S}_0}\big\langle \rho, \mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

with $\mathcal{S}_3^\star$ is the adjoint of $\mathcal{S}_3$ given by

$$\mathcal{S}_3^\star = -\mathcal{B}^{-1}\mathcal{R}^\star\partial_\theta \quad \text{or} \quad -\mathcal{B}^{-1}\Pi_{\mathbb{S}_0}\mathcal{R}^\star\partial_\theta, \tag{6.232}$$

and $\mathcal{R}^\star$ the adjoint of $\mathcal{R}$ which is an integral operator with kernel

$$J^\star(\varphi,\theta,\eta) := \sum_{k'=1}^{3}\sum_{k=1}^{d} g_{k,k'}(\varphi,\theta)\chi_{k,k'}(\varphi,\eta), \tag{6.233}$$

where we use the notations of the proof of Proposition 6.1. Notice that similarly to (6.9) and (6.5), the kernel $J^\star$ satisfies

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k J^\star)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}} \tag{6.234}$$

and

$$J^\star(-\varphi,-\theta,-\eta) = J^\star(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.235}$$

Now, we have the integral representation

$$\begin{aligned}\big(\mathcal{T}_0\mathcal{S}_3\rho\big)(\varphi,\theta) &= \int_{\mathbb{T}}\mathcal{K}_4(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\ \mathcal{K}_4(\varphi,\theta,\eta) &:= \sum_{m\in\mathbb{S}_0}\big(\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

Then by virtue of (6.174), (6.32), (6.214) and (6.235) we obtain

$$\mathcal{K}_4(-\varphi,-\theta,-\eta) = -\mathcal{K}_4(\varphi,\theta,\eta)\in\mathbb{R}. \tag{6.236}$$

Applying Lemma 4.4 combined with the law products, we get for all $k\in\{0,1\}$

$$\begin{aligned}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_3\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_4)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\ &\lesssim \sum_{m\in\mathbb{S}_0}\Big(\|\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s_0+k}^{\gamma,\mathcal{O}} + \|\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Consequently, using Lemma 4.4 and (6.234), we get

$$\begin{aligned}\|\mathcal{R}^\star\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}}\|J^\star(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s+s_0}^{\gamma,\mathcal{O}}\\ &\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Applying (6.27), Lemma 4.3 and the previous estimate implies

$$\begin{aligned}\|\mathcal{S}_3^\star\rho\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\mathcal{R}^\star\partial_\theta\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\mathcal{R}^\star\partial_\theta\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \Big(\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big)+\|\mathcal{R}^\star\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\Big)\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\mathcal{R}^\star\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|g_m\|_{q,s+1}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_3\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}. \tag{6.237}$$

Putting together (6.223), (6.227), (6.231) and (6.237) allows to get

$$\max_{k\in\{0,1\}}\varepsilon\big\|\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right). \tag{6.238}$$

We now move to the estimate of the difference. From (6.222), one has

$$\begin{aligned}\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\quad+\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.239}$$

Combining Lemma 4.4 with Taylor Formula, (6.224), (6.225), (6.10) and (6.31) one obtains

$$\begin{aligned}\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widehat{J})(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.240}$$

In the same spirit, (6.228) and (6.229) give

$$\begin{aligned}\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widetilde{J})(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.241}$$

According to the structure of $J^\star$ detailed in (6.233) one can check that $J^\star$ satisfies similar estimates as (6.10). Then using (6.31), one finds in a similar way to (6.206),

$$\begin{aligned}\|\Delta_{12}(\mathcal{T}_0\mathcal{S}_3)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}\mathcal{K}_4(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.242}$$

Hence, putting together, (6.240), (6.241), (6.242) and (6.239) gives

$$\varepsilon\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big)\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}. \tag{6.243}$$

On the other hand, gathering (6.184), (6.153), (6.193), (6.211), (6.215), (6.236), (6.226) and (6.230) together with Lemma 4.4, we find that $\mathscr{R}_0$ is a real and reversible Toeplitz in time integral operator. In addition, (6.184), (6.187), (6.198), (6.217) and (6.238) give (6.185).
Furthermore, (6.184), (6.188), (6.207), (6.221) and (6.243) imply (6.186).
(iv) Using Lemma 4.3 together with (6.181), (5.26), (6.26) and (6.178), one obtains for all $s\in[s_0,S]$

$$\begin{aligned}\|\mathscr{L}_0\rho\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta+\partial_\theta\mathcal{K}_\lambda*\cdot)\big)\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\rho\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

This ends the proof of Proposition 6.4. □

6.3.2 KAM reduction of the remainder term

The goal of this section is to conjugate $\mathscr{L}_0$ defined in Proposition 6.4 to a diagonal operator, up to a fast decaying small remainder. This will be achieved through a standard KAM reducibility techniques in the spirit of Proposition 6.2 but well-adapted to the operators setting. This will be implemented by taking advantage of the exterior parameters which are restricted to a suitable Cantor set that prevents the resonances in the second Melnikov assumption. Notice that one gets from this study some estimates on the distribution of the eigenvalues and their stability with respect to the torus parametrization. This is considered as the key step not only to get an approximate inverse but also to achieve Nash-Moser scheme with a final massive Cantor set. The main result of this section reads as follows.

Proposition 6.5. *Let $(\gamma,q,d,\tau_1,\tau_2,s_0,\overline{s}_l,\overline{\mu}_2,S)$ satisfy (4.1), (4.2) and (6.3). For any (μ_2,s_h) satisfying*

$$\mu_2\geqslant\overline{\mu}_2+2\tau_2q+2\tau_2\quad\text{and}\quad s_h\geqslant\frac{3}{2}\mu_2+\overline{s}_l+1,\tag{6.244}$$

there exist $\varepsilon_0\in(0,1)$ and $\sigma_4=\sigma_4(\tau_1,\tau_2,q,d)\geqslant\sigma_3$, with σ_3 defined in Proposition 6.4, such that if

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\leqslant 1,\tag{6.245}$$

then the following assertions hold true.

(i) There exists a family of invertible linear operator $\Phi_\infty:\mathcal{O}\to\mathcal{L}\big(H^s_\perp\big)$ satisfying the estimates

$$\forall s\in[s_0,S],\qquad\|\Phi_\infty^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{6.246}$$

There exists a diagonal operator $\mathscr{L}_\infty=\mathscr{L}_\infty(\lambda,\omega,i_0)$ taking the form

$$\mathscr{L}_\infty=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp}+\mathscr{D}_\infty$$

where $\mathscr{D}_\infty=\mathscr{D}_\infty(\lambda,\omega,i_0)$ is a reversible Fourier multiplier operator given by,

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad\mathscr{D}_\infty\mathbf{e}_{l,j}=\mathrm{i}\mu_j^\infty\,\mathbf{e}_{l,j},$$

with

$$\forall j\in\mathbb{S}_0^c,\quad\mu_j^\infty(\lambda,\omega,i_0)=\mu_j^0(\lambda,\omega,i_0)+r_j^\infty(\lambda,\omega,i_0),\quad\|r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\tag{6.247}$$

and

$$\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1},\tag{6.248}$$

such that in the Cantor set

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0):=\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N_n\\(l,j)\neq(0,j_0)}}\Big\{(\lambda,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0),\big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)-\mu_{j_0}^\infty(\lambda,\omega,i_0)\big|>\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}$$

we have

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty=\mathscr{L}_\infty+\mathtt{E}_n^2,$$

and the linear operator $\mathtt{E}_n^2$ satisfies the estimate

$$\|\mathtt{E}_n^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-2}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\tag{6.249}$$

Notice that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ was introduced in Proposition 6.2, the operator $\mathscr{L}_0$ and the frequencies $\big(\mu_j^0(\lambda,\omega,i_0)\big)_{j\in\mathbb{S}_0^c}$ were stated in Proposition 6.4.

(ii) Given two tori i_1 and i_2 both satisfying (6.245), then

$$\forall j\in\mathbb{S}_0^c,\quad\|\Delta_{12}r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\tag{6.250}$$

and

$$\forall j\in\mathbb{S}_0^c,\quad\|\Delta_{12}\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}|j|\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.\tag{6.251}$$

Proof. **(i)** We shall introduce the quantity

$$\delta_0(s) := \gamma^{-1}\|\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}},$$

where $\mathscr{R}_0$ is the remainder seen in Proposition 6.4. By applying (6.181), we deduce that

$$\delta_0(s) \leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right). \tag{6.252}$$

Therefore with the notation of (6.244), (6.245) and the fact that $\sigma_4 \geqslant \sigma_3$ we obtain

$$\begin{aligned} N_0^{\mu_2}\delta_0(s_h) &\leqslant CN_0^{\mu_2}\varepsilon\gamma^{-2} \\ &\leqslant C\varepsilon_0. \end{aligned} \tag{6.253}$$

▶ **KAM step.** Recall from Proposition 6.4 that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ one has

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp = \mathscr{L}_0 + \mathtt{E}_n^1,$$

where the operator $\mathscr{L}_0$ has the following structure

$$\mathscr{L}_0 = \big(\omega\cdot\partial_\varphi + \mathscr{D}_0\big)\Pi_{\mathbb{S}_0}^\perp + \mathscr{R}_0, \tag{6.254}$$

with $\mathscr{D}_0$ a diagonal operator of pure imaginary spectrum and $\mathscr{R}_0$ a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp = \mathscr{R}_0$. Similarly to the reduction of the transport part, we shall first expose a typical step of the iteration process of the KAM scheme whose goal is to reduce to a diagonal part $\mathscr{R}_0$. Notice that the scheme is flexible and has been used in the literature to deal with various equations. Assume that we have a linear operator $\mathscr{L}$ taking the following form in restriction to some Cantor set $\mathscr{O}$ one has

$$\mathscr{L} = \big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^\perp + \mathscr{R},$$

where $\mathscr{D}$ is real and reversible diagonal Toeplitz in time operator, that is,

$$\mathscr{D}\mathbf{e}_{l,j} = i\mu_j(\lambda,\omega)\,\mathbf{e}_{l,j} \quad \text{and} \quad \mu_{-j}(\lambda,\omega) = -\mu_j(\lambda,\omega). \tag{6.255}$$

The operator $\mathscr{R}$ is assumed to be a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^\perp\mathscr{R}\Pi_{\mathbb{S}_0}^\perp = \mathscr{R}$. Consider a linear invertible transformation close to the identity

$$\Phi = \Pi_{\mathbb{S}_0}^\perp + \Psi : \mathcal{O} \to \mathcal{L}(H_\perp^s),$$

where Ψ is small and depends on $\mathscr{R}$. Then straightforward calculus show that in $\mathscr{O}$

$$\begin{aligned} \Phi^{-1}\mathscr{L}\Phi &= \Phi^{-1}\Big(\Phi\left(\omega\cdot\partial_\varphi + \mathscr{D}\right)\Pi_{\mathbb{S}_0}^\perp + \Big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D},\Psi\Big] + \mathscr{R} + \mathscr{R}\Psi\Big) \\ &= \big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^\perp + \Phi^{-1}\Big(\big[\big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^\perp,\Psi\big] + P_N\mathscr{R} + P_N^\perp\mathscr{R} + \mathscr{R}\Psi\Big), \end{aligned}$$

where the projector P_N was defined in (4.11). The main idea consists in replacing the remainder $\mathscr{R}$ with another quadratic one up to a diagonal part and provided that the parameters (λ,ω) belongs to a Cantor set connected to non-resonance conditions associated to the *homological equation*. Iterating this scheme will generate new remainders which become smaller and smaller up to new contributions on the diagonal part and with more extraction on the parameters. Then by passing to the limit we expect to diagonalize completely the operators provided that the parameters belong to a limit Cantor set. Notice that the Cantor set should be truncated in the time mode in order to get a stability form required later in Nash-Moser scheme and during the measure of the final Cantor set. This will induce a diagonalization up to small fast decaying remainders modeled by the operators $\mathtt{E}_n^2$ in Proposition 6.5. Now the first step is to impose the following homological equation,

$$\big[\big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^\perp,\Psi\big] + P_N\mathscr{R} = \lfloor P_N\mathscr{R}\rfloor, \tag{6.256}$$

where $\lfloor P_N\mathscr{R}\rfloor$ is the diagonal part of the operator $\mathscr{R}$. We emphasize that the notation $\lfloor\mathcal{R}\rfloor$ with a general operator $\mathcal{R}$ is defined as follows, for all $(l_0,j_0)\in\mathbb{Z}^d\times\mathbb{S}_0^c$,

$$\mathcal{R}\mathbf{e}_{l_0,j_0}=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\mathcal{R}_{l_0,j_0}^{l,j}\mathbf{e}_{l,j}\Longrightarrow\lfloor\mathcal{R}\rfloor\mathbf{e}_{l_0,j_0}=\mathcal{R}_{l_0,j_0}^{l_0,j_0}\mathbf{e}_{l_0,j_0}=\big\langle\mathcal{R}\mathbf{e}_{l_0,j_0},\mathbf{e}_{l_0,j_0}\big\rangle_{L^2(\mathbb{T}^{d+1})}\mathbf{e}_{l_0,j_0}. \tag{6.257}$$

Remind the notation $\mathbf{e}_{l_0,j_0}(\varphi,\theta)=e^{\mathrm{i}(l_0\cdot\varphi+j_0\theta)}$. The Fourier coefficients of Ψ are defined through

$$\Psi\mathbf{e}_{l_0,j_0}=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\mathbf{e}_{l,j},\quad\Psi_{l_0,j_0}^{l,j}\in\mathbb{C}.$$

From direct computations based on the above Fourier decomposition, we infer

$$\big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp},\Psi\big]\mathbf{e}_{l_0,j_0}=\mathrm{i}\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\;\omega\cdot(l-l_0)\;\mathbf{e}_{l,j}$$

and using the diagonal structure of $\mathscr{D}$,

$$[\mathscr{D}_0,\Psi]\mathbf{e}_{l_0,j_0}=\mathrm{i}\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\left(\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\right)\mathbf{e}_{l,j}.$$

By hypothesis, $\mathscr{R}$ is a real and reversible Toeplitz in time operator. Hence its Fourier coefficients write in view of Proposition 4.1,

$$\mathscr{R}_{l_0,j_0}^{l,j}:=\mathrm{i}\,r_{j_0}^{j}(\lambda,\omega,l_0-l)\in\mathrm{i}\,\mathbb{R}\quad\text{ and }\quad\mathscr{R}_{-l_0,-j_0}^{-l,-j}=-\mathscr{R}_{l_0,j_0}^{l,j}. \tag{6.258}$$

Consequently Ψ is a solution of (6.256) if and only if

$$\Psi\mathbf{e}_{l_0,j_0}=\sum_{\substack{|l-l_0|\leqslant N\\|j-j_0|\leqslant N}}\Psi_{l_0,j_0}^{l,j}\mathbf{e}_{l,j}$$

and

$$\Psi_{l_0,j_0}^{l,j}\Big(\omega\cdot(l-l_0)+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\Big)=\begin{cases}-r_{j_0}^{j}(\lambda,\omega,l_0-l) & \text{if } (l,j)\neq(l_0,j_0)\\ 0 & \text{if } (l,j)=(l_0,j_0).\end{cases}$$

In particular, we get that Ψ is a Toeplitz in time operator with $\Psi_{j_0}^{j}(l_0-l):=\Psi_{l_0,j_0}^{l,j}$. Moreover, for $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ with $|l|,|j-j_0|\leqslant N$, one obtains

$$\Psi_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}\frac{-r_{j_0}^{j}(\lambda,\omega,l)}{\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)} & \text{if } (l,j)\neq(0,j_0)\\ 0 & \text{if } (l,j)=(0,j_0),\end{cases} \tag{6.259}$$

provided that the denominator is non zero. In addition, from $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}\Pi_{\mathbb{S}_0}^{\perp}=\mathscr{R}$, one easily gets

$$\forall\,l\in\mathbb{Z}^d,\ \forall\,j\text{ or }\ j_0\in\mathbb{S}_0,\quad r_{j_0}^{j}(\lambda,\omega,l)=0.$$

Therfore, we should impose the compatibility condition

$$\forall\,l\in\mathbb{Z}^d,\ \forall\,j\text{ or }\ j_0\in\mathbb{S}_0,\quad\Psi_{j_0}^{j}(\lambda,\omega,l)=0.$$

This implies that $\Pi_{\mathbb{S}_0}^{\perp}\Psi\Pi_{\mathbb{S}_0}^{\perp}=\Psi$. To justify the formula given by (6.259) we need to avoid resonances and restrict the parameters to the following open set according to the so-called second Melnikov condition,

$$\mathscr{O}_+^{\gamma}=\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N\\(l,j)\neq(0,j_0)}}\Big\{(\lambda,\omega)\in\mathscr{O}\quad\text{s.t.}\quad|\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}.$$

In view of this restriction, the identity (6.259) is well defined and to extend Ψ to the whole set $\mathcal{O}$ we shall use the cut-off function χ of (5.81). We set

$$\Psi_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}-\varrho_{j_0}^{j}(\lambda,\omega,l)\,r_{j_0}^{j}(\lambda,\omega,l), & \text{if}\quad (l,j)\neq(0,j_0)\\ 0, & \text{if}\quad (l,j)=(0,j_0),\end{cases}\tag{6.260}$$

with

$$\varrho_{j_0}^{j}(\lambda,\omega,l):=\frac{\chi\big((\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega))(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}\big)}{\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)}.\tag{6.261}$$

To simplify the notation, in the sequel we shall still write Ψ to denote this extension. Note that the extension (6.260) is smooth and in restriction to the Cantor set $\mathscr{O}_{+}^{\gamma}$ coincides with Ψ. On the other hand, (6.258) and (6.260) imply that $\Psi_{j_0}^{j}(l)\in\mathbb{R}$. In addition, (6.261) combined with (6.255) give

$$\Psi_{-j_0}^{-j}(-l)=\Psi_{j_0}^{j}(l).$$

Consequently, in view of Proposition 4.1, we deduce that Ψ is a real and reversibility preserving operator. Now consider,

$$\mathscr{D}_{+}=\mathscr{D}+\lfloor P_N\mathscr{R}\rfloor,\qquad \mathscr{R}_{+}=\Phi^{-1}\big(-\Psi\lfloor P_N\mathscr{R}\rfloor+P_N^{\perp}\mathscr{R}+\mathscr{R}\Psi\big)\tag{6.262}$$

and

$$\mathscr{L}_{+}:=\big(\omega\cdot\partial_\varphi+\mathscr{D}_{+}+\mathscr{R}_{+}\big)\Pi_{\mathbb{S}_0}^{\perp}.$$

Therefore, in restriction to the Cantor set $\mathscr{O}_{+}^{\gamma}$, we can write

$$\mathscr{L}_{+}=\Phi^{-1}\mathscr{L}\Phi.$$

Our next task is to estimate $\varrho_{j_0}^{j}$ defined by (6.261). Notice that this quantity can be written in the following form

$$\begin{aligned}&\varrho_{j_0}^{j}(\lambda,\omega,l)=a_{l,j,j_0}\widehat{\chi}\big(a_{l,j,j_0}A_{l,j,j_0}(\lambda,\omega)\big),\quad \widehat{\chi}(x)=\frac{\chi(x)}{x},\\ &A_{l,j,j_0}(\lambda,\omega)=\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega),\quad a_{l,j,j_0}=(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2},\end{aligned}\tag{6.263}$$

where $\widehat{\chi}(x):=\frac{\chi(x)}{x}$ is $\mathcal{C}^\infty$ with bounded derivatives. Assume now the following estimate

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\ \sup_{(\lambda,\omega)\in\mathcal{O}}\big|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\right)\big|\leqslant C\,|j-j_0|.\tag{6.264}$$

Then, we find

$$\forall\,(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2,\qquad \max_{q'\in[\![0,q]\!]}\|A_{l,j,j_0}\|_{q'}^{\gamma,\mathcal{O}}\leqslant C\,\langle l,j-j_0\rangle.\tag{6.265}$$

In a similar way to (6.46), using Lemma 4.1-(vi) and (6.265), we obtain

$$\forall q'\in[\![0,q]\!],\quad \|\varrho_{j_0}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-(q'+1)}\langle l,j-j_0\rangle^{\tau_2 q'+\tau_2+q'}.\tag{6.266}$$

Similarly to (6.48), using Leibniz rule, we get

$$\|\Psi\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-1}\|P_N\mathscr{R}\|_{\textnormal{O-d},q,s+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}}.\tag{6.267}$$

We also assume that the following smallness condition holds

$$\gamma^{-1}\|\mathscr{R}\|_{\textnormal{O-d},q,s_0+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}}\leqslant C\varepsilon_0.\tag{6.268}$$

Hence, by virtue of (6.267), we get

$$\begin{aligned}\|\Psi\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant C\gamma^{-1}\|\mathscr{R}\|_{\textnormal{O-d},q,s_0+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}}\\&\leqslant C\varepsilon_0.\end{aligned}\tag{6.269}$$

As a consequence, up to take ε_0 small enough, the operator Φ is invertible and

$$\Phi^{-1}=\sum_{n=0}^{\infty}(-1)^n\Psi^n:=\mathrm{Id}+\Sigma.$$

According to the law products in Lemma 4.1, Lemma 4.3, (6.267) and (6.269) one gets

$$\begin{aligned}\|\Sigma\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Psi\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+\sum_{n=1}^{\infty}\left(C\|\Psi\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)^n\right)\\&\leqslant C\,\gamma^{-1}N^{\tau_2q+\tau_2}\|\mathscr{R}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.270}$$

Therefore, we conclude with the assumption (6.268) that Φ^{-1} is satisfies the following estimate

$$\|\Phi^{-1}-\mathrm{Id}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-1}N^{\tau_2q+\tau_2}\|\mathscr{R}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}.\tag{6.271}$$

From (6.262), we can write

$$\mathscr{R}_+=P_N^{\perp}\mathscr{R}+\Phi^{-1}\mathscr{R}\Psi-\Psi\lfloor P_N\mathscr{R}\rfloor+\Sigma\big(P_N^{\perp}\mathscr{R}-\Psi\lfloor P_N\mathscr{R}\rfloor\big).$$

Thus, by virtue of Lemma 4.3 and (6.271), we infer

$$\begin{aligned}\|\mathscr{R}_+\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|P_N^{\perp}\mathscr{R}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}+C\|\Sigma\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\left(\|P_N^{\perp}\mathscr{R}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\Psi\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)\\&\quad+C\left(1+\|\Sigma\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)\left(\|\Psi\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\Psi\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.272}$$

By Lemma 4.3 , (6.267),(6.269) and (6.271), we get for all $S\geqslant\overline{s}\geqslant s\geqslant s_0$,

$$\|\mathscr{R}_+\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant N^{s-\overline{s}}\|\mathscr{R}\|_{\textnormal{O-d},q,\overline{s}}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_2q+\tau_2}\|\mathscr{R}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}.\tag{6.273}$$

► **Initialization** We shall verify that the assumptions (6.264) and (6.268) required along the KAM step to get the final form (6.273) are satisfied for $\mathscr{L}=\mathscr{L}_0$ in (6.254). Indeed, (6.264) is an immediate consequence of Lemma 3.3-(vi), that is

$$\exists C>0,\ \forall(j,j_0)\in\mathbb{Z}^2,\qquad\max_{|\alpha|\in[\![0,q]\!]}\sup_{\lambda\in(\lambda_0,\lambda_1)}\left|\partial_\lambda^\alpha\left(\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\right)\right|\leqslant C\,|j-j_0|.\tag{6.274}$$

Thus, applying (6.179) we obtain

$$\exists C>0,\ \forall(j,j_0)\in\mathbb{Z}^2,\qquad\max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^\alpha\left(\mu_j^0(\lambda,\omega)-\mu_{j_0}^0(\lambda,\omega)\right)\right|\leqslant C\,|j-j_0|.$$

Concerning the second assumption (6.268), we may combine (6.181) and (6.245) to find

$$\begin{aligned}\gamma^{-1}\|\mathscr{R}_0\|_{\textnormal{O-d},q,s_0+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}&\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\right)\\&\leqslant C\varepsilon_0.\end{aligned}$$

► **KAM iteration**. Let $m\in\mathbb{N}$ and consider a linear operator

$$\mathscr{L}_m:=\big(\omega\cdot\partial_\varphi+\mathscr{D}_m+\mathscr{R}_m\big)\Pi_{\mathbb{S}_0}^{\perp}\tag{6.275}$$

with $\mathscr{D}_m$ a diagonal real reversible operator and $\mathscr{R}_m$ a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}_m\Pi_{\mathbb{S}_0}^{\perp}=\mathscr{R}_m$. We assume that both assumptions (6.264) and (6.268) are satisfied for $\mathscr{D}_m$ and $\mathscr{R}_m$. Remark that for $m=0$ we take the operator $\mathscr{L}_0$ defined in (6.254). Let $\Phi_m=\mathrm{Id}+\Psi_m$ be a linear invertible operator such that

$$\Phi_m^{-1}\mathscr{L}_m\Phi_m:=\big(\omega\cdot\partial_\varphi+\mathscr{D}_{m+1}+\mathscr{R}_{m+1}\big)\Pi_{\mathbb{S}_0}^{\perp},\tag{6.276}$$

with Ψ_m satisfying the homological equation

$$\big[(\omega\cdot\partial_\varphi+\mathscr{D}_m)\Pi_{\mathbb{S}_0}^{\perp},\Psi_m\big]+P_{N_m}\mathscr{R}_m=\lfloor P_{N_m}\mathscr{R}_m\rfloor.$$

Recall that N_m was defined in (5.82). The diagonal parts $(\mathscr{D}_m)_{m\in\mathbb{N}}$ and the remainders $(\mathscr{R}_m)_{m\in\mathbb{N}}$ are defined similarly to (6.262) by the recursive formulas,

$$\mathscr{D}_{m+1}=\mathscr{D}_m+\lfloor P_{N_m}\mathscr{R}_m\rfloor\quad\text{ and }\quad\mathscr{R}_{m+1}=\Phi_m^{-1}\Big(-\Psi_m\lfloor P_{N_m}\mathscr{R}_m\rfloor+P_{N_m}^{\perp}\mathscr{R}_m+\mathscr{R}_m\Psi_m\Big).\tag{6.277}$$

Remark that $\mathscr{D}_m$ and $\lfloor P_{N_m}\mathscr{R}_m\rfloor$ are Fourier multiplier Toeplitz operators that can be identified to their spectra $(\mathrm{i}\mu_j^m)_{j\in\mathbb{S}_0^c}$ and $(\mathrm{i}r_j^m)_{j\in\mathbb{S}_0^c}$, namely

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad\mathscr{D}_m\mathbf{e}_{l,j}=\mathrm{i}\mu_j^m\,\mathbf{e}_{l,j}\quad\text{ and }\quad\lfloor P_{N_m}\mathscr{R}_m\rfloor\mathbf{e}_{l,j}=\mathrm{i}r_j^m\,\mathbf{e}_{l,j}.\tag{6.278}$$

By construction, we find

$$\mu_j^{m+1}=\mu_j^m+r_j^m.\tag{6.279}$$

In a similar way to (6.259) we obtain

$$(\Psi_m)_{j_0}^j(\lambda,\omega,l)=\begin{cases}\frac{-r_{j_0,m}^j(\lambda,\omega,l)}{\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)} & \text{if } (l,j)\neq(0,j_0)\\ 0 & \text{if } (l,j)=(0,j_0),\end{cases}\tag{6.280}$$

where the collection $\{r_{j_0,m}^j(\lambda,\omega,l)\}$ describes the Fourier coefficients of $\mathscr{R}_m$, that is,

$$\mathscr{R}_m\mathbf{e}_{l_0,j_0}=\mathrm{i}\sum_{(l,j)\in\mathbb{Z}^{d+1}}r_{j_0,m}^j(\lambda,\omega,l_0-l)\mathbf{e}_{l,j}.$$

Now we shall define the open Cantor set where the preceding formula is meaningful,

$$\mathscr{O}_{m+1}^{\gamma}=\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N_m\\(l,j)\neq(0,j_0)}}\Big\{(\lambda,\omega)\in\mathscr{O}_m^{\gamma}\quad\text{s.t.}\quad|\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}.\tag{6.281}$$

Similarly to (6.260) and (6.261) we can extend (6.280) as follows

$$(\Psi_m)_{j_0}^j(\lambda,\omega,l)=\begin{cases}-\frac{\chi\big((\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega))(\gamma|j-j_0|)^{-1}\langle l\rangle^{\tau_2}\big)r_{j_0,m}^j(\lambda,\omega,l)}{\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)} & \text{if } (l,j)\neq(0,j_0)\\ 0 & \text{if } (l,j)=(0,j_0).\end{cases}\tag{6.282}$$

We point out that working with this extension for Ψ_m allows to extend both $\mathscr{D}_{m+1}$ and the remainder $\mathscr{R}_{m+1}$ provided that the operators $\mathscr{D}_m$ and $\mathscr{R}_m$ are defined in the whole range of parameters. Thus the operator defined by the right-hand side in (6.276) can be extended to the whole set $\mathcal{O}$ and we denote this extension by $\mathscr{L}_{m+1}$, that is,

$$\big(\omega\cdot\partial_\varphi+\mathscr{D}_{m+1}+\mathscr{R}_{m+1}\big)\Pi_{\mathbb{S}_0}^{\perp}:=\mathscr{L}_{m+1}.\tag{6.283}$$

This enables to construct by induction the sequence of operators $(\mathscr{L}_{m+1})$ in the full set $\mathcal{O}$. Similarly the operator $\Phi_m^{-1}\mathscr{L}_m\Phi_m$ admits an extension in $\mathcal{O}$ induced by the extension of $\Phi_m^{\pm1}$. However, by construction the identity $\mathscr{L}_{m+1}=\Phi_m^{-1}\mathscr{L}_m\Phi_m$ in (6.276) occurs in the Cantor set $\mathscr{O}_{m+1}^{\gamma}$ and may fail outside this set. We define

$$\delta_m(s):=\gamma^{-1}\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\tag{6.284}$$

and we want to prove by induction in $m\in\mathbb{N}$ that

$$\forall\,m\in\mathbb{N},\quad\forall s\in[s_0,\overline{s}_l],\;\delta_m(s)\leqslant\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\quad\text{ and }\quad\delta_m(s_h)\leqslant\Big(2-\tfrac{1}{m+1}\Big)\,\delta_0(s_h),\tag{6.285}$$

with $\overline{s}_l$ and s_h fixed by (6.3) and (6.244). Moreover, we should check the validity of the assumptions (6.264) and (6.268) for $\mathscr{D}_{m+1}$ and $\mathscr{R}_{m+1}$. Notice that by Sobolev embeddings, it is sufficient to prove the first inequality with $s = \overline{s}_l$. The property is obvious for $m = 0$. Now, assume that the property (6.285) is true for $m \in \mathbb{N}$ and let us check it at the next order. We write

$$\Phi_m^{-1} = \mathrm{Id} + \Sigma_m \quad \text{with} \quad \Sigma_m = \sum_{n=1}^{\infty} (-1)^n \Psi_m^n. \tag{6.286}$$

Thus similarly to (6.270), using in particular (6.267) and (6.269) we deduce successively

$$\begin{aligned}\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} &\leqslant \|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \left(1 + \sum_{n=0}^{\infty} \left(C\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)^n\right)\\ &\leqslant \delta_m(s_0 + \tau_2 q + \tau_2) \left(1 + \sum_{n=0}^{\infty} \left(C\delta_m(s_0 + \tau_2 q + \tau_2)\right)^n\right)\end{aligned}$$

and for any $s \in [s_0, S]$,

$$\begin{aligned}\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \left(1 + \sum_{n=0}^{\infty} \left(C\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)^n\right)\\ &\leqslant N_m^{\tau_2 q+\tau_2}\delta_m(s) \left(1 + \sum_{n=0}^{\infty} \left(C\delta_m(s_0 + \tau_2 q + \tau_2)\right)^n\right).\end{aligned}$$

Hence, from the induction assumption, the fact that $N_m \geqslant N_0$ and since (6.3) implies in particular $s_0 + \tau_2 q + \tau_2 \leqslant \overline{s}_l$, we obtain

$$\begin{aligned}\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} &\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h) \left(1 + \sum_{n=0}^{\infty} \left(CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\right)^n\right)\\ &\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h) \left(1 + \sum_{n=0}^{\infty} \left(C\delta_0(s_h)\right)^n\right)\end{aligned}$$

and for any $s \in [s_0, S]$,

$$\begin{aligned}\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant N_m^{\tau_2 q+\tau_2}\delta_m(s) \left(1 + \sum_{n=0}^{\infty} \left(CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\right)^n\right)\\ &\leqslant N_m^{\tau_2 q+\tau_2}\delta_m(s) \left(1 + \sum_{n=0}^{\infty} \left(C\delta_0(s_h)\right)^n\right).\end{aligned}$$

It follows from the condition (6.253) that

$$\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h) \quad \text{and} \quad \|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant CN_m^{\tau_2 q+\tau_2}\delta_m(s). \tag{6.287}$$

One also gets

$$\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant C\delta_m(s + \tau_2 q + \tau_2). \tag{6.288}$$

From KAM step (6.273) and Sobolev embeddings, we infer

$$\delta_{m+1}(\overline{s}_l) \leqslant N_m^{\overline{s}_l - s_h}\delta_m(s_h) + CN_m^{\tau_2 q+\tau_2}\left(\delta_m(\overline{s}_l)\right)^2.$$

Using the induction assumption (6.285) yields

$$\begin{aligned}\delta_{m+1}(\overline{s}_l) &\leqslant N_m^{\overline{s}_l - s_h}\left(2 - \tfrac{1}{m+1}\right)\delta_0(s_h) + CN_m^{\tau_2 q+\tau_2}\delta_0^2(s_h)N_0^{2\mu_2}N_m^{-2\mu_2}\\ &\leqslant 2N_m^{\overline{s}_l - s_h}\delta_0(s_h) + CN_m^{\tau_2 q+\tau_2}\delta_0^2(s_h)N_0^{2\mu_2}N_m^{-2\mu_2}.\end{aligned}$$

At this level we need to select the parameters $\overline{s}_l, s_h$ and μ_2 in such a way

$$N_m^{\overline{s}_l - s_h} \leqslant \frac{1}{4} N_0^{\mu_2} N_{m+1}^{-\mu_2} \qquad \text{and} \qquad C N_m^{\tau_2 q + \tau_2} \delta_0(s_h) N_0^{2\mu_2} N_m^{-2\mu_2} \leqslant \frac{1}{2} N_0^{\mu_2} N_{m+1}^{-\mu_2} \tag{6.289}$$

leading to

$$\delta_{m+1}(\overline{s}_l) \leqslant \delta_0(s_h) N_0^{\mu_2} N_{m+1}^{-\mu_2}.$$

The conditions (6.244) imply in particular

$$s_h \geqslant \frac{3}{2}\mu_2 + \overline{s}_l + 1 \quad \text{and} \quad \mu_2 \geqslant 2(\tau_2 q + \tau_2) + 1.$$

Then, using (5.82), we conclude that the assumptions of (6.289) hold true provided that

$$4 N_0^{-\mu_2} \leqslant 1 \qquad \text{and} \qquad 2 C N_0^{\mu_2} \delta_0(s_h) \leqslant 1, \tag{6.290}$$

which follow from (6.253), since the first condition $4N_0^{-\mu_2} \leqslant 1$ is automatically satisfied because $N_0 \geqslant 2$ and $\mu_2 \geqslant 2$, according to (6.244). Therefore, under the assumptions (6.244) we get the first statement of the induction in (6.285). The next goal is to establish the second estimate in (6.285). By KAM step (6.273) combined with the induction assumptions (6.285) we deduce that

$$\begin{aligned}
\delta_{m+1}(s_h) &\leqslant \delta_m(s_h) + C N_m^{\tau_2 q + \tau_2} \delta_m(s_0) \delta_m(s_h) \\
&\leqslant \Big(2 - \tfrac{1}{m+1}\Big) \delta_0(s_h) \Big(1 + C N_0^{\mu_2} N_m^{\tau_2 q + \tau_2 - \mu_2} \delta_0(s_h)\Big).
\end{aligned}$$

Thus if one has

$$\Big(2 - \tfrac{1}{m+1}\Big)\Big(1 + C N_0^{\mu_2} N_m^{\tau_2 q + \tau_2 - \mu_2} \delta_0(s_h)\Big) \leqslant 2 - \tfrac{1}{m+2}, \tag{6.291}$$

then we get

$$\delta_{m+1}(s_h) \leqslant \Big(2 - \tfrac{1}{m+2}\Big) \delta_0(s_h),$$

which ends the induction argument of (6.285). Remark that with the choice $\mu_2 \geqslant 2(\tau_2 q + \tau_2)$ fixed in (6.244), the condition (6.291) is staisfied if

$$C N_0^{\mu_2} N_m^{-\tau_2 q - \tau_2} \delta_0(s_h) \leqslant \tfrac{1}{(2m+1)(m+2)}. \tag{6.292}$$

Since $N_0 \geqslant 2$ we may find a constant $c_0 > 0$ small enough such that

$$\forall\, m \in \mathbb{N}, \quad c_0 N_m^{-1} \leq \tfrac{1}{(2m+1)(m+2)}.$$

Consequently, (6.292) is satisfied provided that

$$C N_0^{\mu_2} N_m^{-\tau_2 q - \tau_2 + 1} \delta_0(s_h) \leqslant c_0. \tag{6.293}$$

By virtue of the assumption (4.2) we get in particular

$$\tau_2 q + \tau_2 - 1 \geqslant 0. \tag{6.294}$$

Thus (6.293) is satisfied in view of (6.253). To conclude the induction proof of (6.285) it remains to check that the assumptions (6.264) and (6.268) are satisfied for $\mathscr{D}_{m+1}$ and $\mathscr{R}_{m+1}$. First, the assumption (6.268) is a consequence of the first inequality of (6.285) applied at the order $m+1$ with the regularity index $s = s_0 + \tau_2 q + \tau_2 \leqslant \overline{s}_l$ supplemented with (6.253). Concerning the validity of (6.264) for $\mathscr{D}_{m+1}$, we combine (6.278) and (6.279) and (6.257), in order to find

$$\|\mu_j^{m+1} - \mu_j^m\|_q^{\gamma,\mathcal{O}} = \big\| \langle P_{N_m} \mathscr{R}_m \mathbf{e}_{l,j}, \mathbf{e}_{l,j} \rangle_{L^2(\mathbb{T}^{d+1})} \big\|_q^{\gamma,\mathcal{O}}.$$

From the Topeplitz structure of $\mathscr{R}_m$ we may write

$$\|\mu_j^{m+1} - \mu_j^m\|_q^{\gamma,\mathcal{O}} = \big\| \langle P_{N_m} \mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j} \rangle_{L^2(\mathbb{T}^{d+1})} \big\|_q^{\gamma,\mathcal{O}}.$$

By a duality argument combined with Lemma 4.3 and (6.284) we infer

$$\begin{aligned}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}} &\lesssim \big\|\mathscr{R}_m\mathbf{e}_{0,j}\|_{q,s_0}^{\gamma,\mathcal{O}}\ \langle j\rangle^{-s_0}\\ &\lesssim \big\|\mathscr{R}_m\big\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathbf{e}_{0,j}\|_{H^{s_0}}\ \langle j\rangle^{-s_0}\\ &\lesssim \big\|\mathscr{R}_m\big\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}=\gamma\delta_m(s_0).\end{aligned}\tag{6.295}$$

Hence we deduce from (6.285), (6.252) and (6.245)

$$\begin{aligned}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\ &\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_m^{-\mu_2}.\end{aligned}\tag{6.296}$$

As the assumption (6.264) is satisfied with $\mathscr{D}_m$, that is,

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)\right)\right|\leqslant C\,|j-j_0|,\tag{6.297}$$

then we obtain by (6.296)

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j^{m+1}(\lambda,\omega)-\mu_{j_0}^{m+1}(\lambda,\omega)\right)\right|\leqslant C\big(1+\varepsilon\gamma^{-1-q}N_0^{\mu_2}N_m^{-\mu_2}\big)\,|j-j_0|.$$

Consequently, the convergence of the series $\sum N_m^{-\mu_2}$ gives the required assumption with the same constant C independently of m. This completes the induction principle. In what follows, we shall provide some estimates for Ψ_m that will be used later to study the string convergence. Using (6.267) combined with Lemma 4.3 and $s_0+\tau_2 q+\tau_2+1\leqslant\overline{s}_l$ we find

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s_0+\tau_2q+\tau_2+1}^{\gamma,\mathcal{O}}\\ &\leqslant\ C\,\delta_m(\overline{s}_l).\end{aligned}\tag{6.298}$$

Thus (6.285) and (6.253) yield

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant\ C\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\ &\leqslant\ C\,\varepsilon\gamma^{-2}N_0^{\mu_2}\,N_m^{-\mu_2}.\end{aligned}\tag{6.299}$$

Next, we discuss the persistence of higher regularity. Let $s\in[s_0,S]$, then from (6.273), (6.285) and (6.253) and (6.294)

$$\begin{aligned}\delta_{m+1}(s) &\leqslant \delta_m(s)\Big(1+CN_m^{\tau_2q+\tau_2}\delta_m(s_0)\Big)\\ &\leqslant \delta_m(s)\Big(1+CN_0^{\mu_2}N_m^{\tau_2q+\tau_2-\mu_2}\delta_0(s_h)\Big)\\ &\leqslant \delta_m(s)\big(1+CN_m^{-1}\big).\end{aligned}$$

Combining this estimate with (5.82) and (6.252) yields

$$\begin{aligned}\forall\,s\geqslant s_0,\,\forall\,m\in\mathbb{N},\quad \delta_m(s) &\leqslant \delta_0(s)\prod_{n=0}^{\infty}\big(1+CN_n^{-1}\big)\\ &\leqslant C\delta_0(s)\\ &\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.300}$$

Using (6.267) combined with Lemma 4.3, applying in particular interpolation inequalities, leads to

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}\\ &\leqslant\ C\,\delta_m(s+\tau_2q+\tau_2)\\ &\leqslant\ C\,\delta_m^{\overline{\theta}}(s_0)\delta_m^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1).\end{aligned}\tag{6.301}$$

with $\overline{\theta}=\frac{1}{s-s_0+\tau_2 q+\tau_2+1}$. Inserting (6.285) and (6.300) into (6.301) and using (6.253) give

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\,\delta_0^{\overline{\theta}}(s_2)\delta_0^{1-\overline{\theta}}(s+\tau_2 q+\tau_2+1)N_0^{\mu_2\overline{\theta}}N_m^{-\mu_2\overline{\theta}}\\ &\leqslant C\,\varepsilon_0{}^{\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2 q+\tau_2+1)N_m^{-\mu_2\overline{\theta}}.\end{aligned}\tag{6.302}$$

We point out that one also finds from (6.288), the second inequality of (6.301) and (6.300) that

$$\forall\, s\in[s_0,S],\quad \sup_{m\in\mathbb{N}}\Big(\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big)\leqslant C\varepsilon\gamma^{-2}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\Big).\tag{6.303}$$

► **KAM conclusion**. Let us examine the sequence of operators $\big(\widehat{\Phi}_m\big)_{m\in\mathbb{N}}$ defined by

$$\widehat{\Phi}_0:=\Phi_0\quad\text{and}\qquad\forall m\geqslant 1,\ \widehat{\Phi}_m:=\Phi_0\circ\Phi_1\circ\ldots\circ\Phi_m.\tag{6.304}$$

It is obvious from the identity $\Phi_m=\mathrm{Id}+\Psi_m$ that $\widehat{\Phi}_{m+1}=\widehat{\Phi}_m+\widehat{\Phi}_m\Psi_{m+1}$. Applying the law products yields

$$\|\widehat{\Phi}_{m+1}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\leqslant\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\Big(1+C\|\Psi_{m+1}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\Big).$$

By iterating this inequality and using (6.299) we infer

$$\begin{aligned}\|\widehat{\Phi}_{m+1}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant\|\Phi_0\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\prod_{n=1}^{m+1}\Big(1+C\|\Psi_n\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\Big)\\ &\leqslant\prod_{n=0}^{\infty}\Big(1+C\,\varepsilon_0 N_n^{-\mu_2}\Big).\end{aligned}$$

Using the first condition of (6.253) and (5.82) imply

$$\|\widehat{\Phi}_{m+1}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\leqslant\prod_{n=0}^{\infty}\Big(1+C\,\varepsilon_0 4^{-(\frac{3}{2})^n}\Big)$$

and since the infinite product converges, we obtain for ε_0 small enough

$$\sup_{m\in\mathbb{N}}\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\leqslant 2.\tag{6.305}$$

Now we shall estimate the difference $\widehat{\Phi}_{m+1}-\widehat{\Phi}_m$ and for this aim we use the law products combined with (6.299) and (6.305)

$$\begin{aligned}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant C\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\\ &\leqslant C\,\delta_0(s_h)N_0^{\mu_2}\,N_{m+1}^{-\mu_2}.\end{aligned}\tag{6.306}$$

Applying Lemma A.1 gives

$$\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\leqslant C\,\delta_0(s_h).\tag{6.307}$$

Therefore, by a completeness argument we deduce that the series $\sum_{m\in\mathbb{N}}(\widehat{\Phi}_{m+1}-\widehat{\Phi}_m)$ converges to an element Φ_∞. In addition, we get in view of (6.306) and Lemma A.1

$$\begin{aligned}\|\widehat{\Phi}_m-\Phi_\infty\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant\sum_{j=m}^{\infty}\|\widehat{\Phi}_{j+1}-\widehat{\Phi}_j\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\\ &\leqslant C\,\delta_0(s_h)N_0^{\mu_2}\sum_{j=m}^{\infty}N_{j+1}^{-\mu_2}\\ &\leqslant C\,\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.\end{aligned}\tag{6.308}$$

Remark that one also finds from (6.305)

$$\|\Phi_\infty\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} \leqslant 2. \tag{6.309}$$

Using (6.307) combined with (6.301) for $m=0$ and (6.244)

$$\begin{aligned}\|\Phi_\infty - \mathrm{Id}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty} \|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} + \|\Psi_0\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}} \\ &\leqslant C\,\delta_0(s_h). \end{aligned} \tag{6.310}$$

Let us now check the convergence with higher order norms. Take $s \in [s_0, S]$, then using the law products, (6.299), (6.302) and (6.305) we infer

$$\begin{aligned}\|\widehat{\Phi}_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big(1 + C\|\Psi_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big) + C\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \\ &\leqslant \|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big(1 + C\,\varepsilon_0\,N_{m+1}^{-\mu_2}\Big) + C\,\delta_0^{\overline{\theta}}(s_h) N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2 q+\tau_2+1) N_m^{-\mu_2\overline{\theta}}. \end{aligned} \tag{6.311}$$

According to the first condition of (6.253) and (5.82) one finds

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1 + C\,\varepsilon_0\,N_n^{-\mu_2}\right) &\leqslant \prod_{n=0}^{\infty}\left(1 + C\,\varepsilon_0 4^{-(\frac{3}{2})^n}\right) \\ &\leqslant 2,\end{aligned}$$

where the last inequality holds if ε_0 is chosen small enough. Applying (6.79) together with (6.311) and Lemma A.1 and using (6.267) yield

$$\begin{aligned}\sup_{m\in\mathbb{N}}\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\Big(\|\Phi_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2 q+\tau_2+1)\Big) \\ &\leqslant C\Big(1 + \delta_0(s+\tau_2 q+\tau_2) + \delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}\big(s+\tau_2 q+\tau_2+1\big)\Big).\end{aligned}$$

Interpolation inequalities and (6.253) allow to get

$$\sup_{m\in\mathbb{N}}\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant C\Big(1 + \delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big). \tag{6.312}$$

The next task is to estimate the difference $\|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|_{q,s}^{\gamma,\mathcal{O}}$. By the law products combined with the first inequality in (6.299), (6.302), (6.305) and (6.312) we obtain

$$\begin{aligned}\|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\Big(\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|\widehat{\Phi}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big) \\ &\leqslant C\,\delta_0(s_h)N_0^{\mu_2}\,N_{m+1}^{-\mu_2}\Big(1 + \delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big) \\ &\quad + C\,\delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\,\delta_0^{1-\overline{\theta}}\big(s+\tau_2 q+\tau_2+1\big)N_{m+1}^{-\mu_2\overline{\theta}}.\end{aligned}$$

Thus, we obtain in view of Lemma A.1

$$\begin{aligned}\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\,\delta_0(s_h)\Big(1 + \delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big) \\ &\quad + C\,\delta_0^{\overline{\theta}}(s_h)\delta_0^{1-\overline{\theta}}\big(s+\tau_2 q+\tau_2+1\big).\end{aligned}$$

Combining the interpolation inequalities with the second condition in (6.253) gives

$$\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant C\Big(\delta_0(s_h) + \delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big). \tag{6.313}$$

From this latter inequality combined with (6.253) and (6.312) we infer

$$\begin{aligned}\|\Phi_\infty\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\widehat{\Phi}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\&\leqslant C\Big(1+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big).\end{aligned}\tag{6.314}$$

On the other hand, using (6.313) and the second inequality in (6.301) with $m=0$, one can check that

$$\begin{aligned}\|\Phi_\infty-\mathrm{Id}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\Psi_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\&\leqslant C\Big(\delta_0(s_h)+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big).\end{aligned}\tag{6.315}$$

Therefore, Lemma 4.3 together with (6.309), (6.314) and Sobolev embeddings give

$$\begin{aligned}\|\Phi_\infty\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\Phi_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\Phi_\infty\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\Big(1+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\delta_0\big(s+\tau_2q+\tau_2+1\big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.316}$$

Applying (6.181) and (6.284) we obtain

$$\begin{aligned}\delta_0\big(s+\tau_2q+\tau_2+1\big)&=\gamma^{-1}\|\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{6.317}$$

Plugging (6.317) into (6.316) and using (6.245) combined with Sobolev embeddings and (6.244) yield

$$\begin{aligned}\|\Phi_\infty\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.318}$$

In a similar way to (6.316) we get by Lemma 4.3 combined with (6.315) and (6.310)

$$\begin{aligned}\big\|\big(\Phi_\infty-\mathrm{Id}\big)\rho\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\Phi_\infty-\mathrm{Id}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\Phi_\infty-\mathrm{Id}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\delta_0(s_h)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\Big(\delta_0(s_h)+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\delta_0(s_h)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\delta_0\big(s+\tau_2q+\tau_2+1\big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence we find from (6.317) and (6.245) combined with Sobolev embeddings and (6.252)

$$\begin{aligned}\big\|\big(\Phi_\infty-\mathrm{Id}\big)\rho\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\big(\varepsilon\gamma^{-2}+\delta_0(s_h)\big)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-2}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.319}$$

The estimates Φ_∞^{-1} and $\Phi_\infty^{-1}-\widehat{\Phi}_n^{-1}$ follow from the same type pf arguments.
➢ In what follows we plan to study the asymptotic of the eigenvalues. Summing up in m the estimates (6.296) and using Lemma A.1, we find

$$\begin{aligned}\sum_{m=0}^{\infty}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}}&\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\&\leqslant C\gamma\delta_0(s_h).\end{aligned}\tag{6.320}$$

Thus for each $j\in\mathbb{S}_0^c$ the sequence $(\mu_j^m)_{m\in\mathbb{N}}$ converges in the space $W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ to an element denoted by $\mu_j^\infty\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$. Moreover, for any $m\in\mathbb{N}$, we find in view of (6.296)

$$\begin{aligned}\|\mu_j^\infty-\mu_j^m\|_q^{\gamma,\mathcal{O}}&\leqslant\sum_{n=m}^{\infty}\|\mu_j^{n+1}-\mu_j^n\|_q^{\gamma,\mathcal{O}}\\&\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}\sum_{n=m}^{\infty}N_n^{-\mu_2}.\end{aligned}$$

Applying Lemma A.1

$$\sup_{j\in\mathbb{S}_0^c}\|\mu_j^\infty-\mu_j^m\|_q^{\gamma,\mathcal{O}}\leqslant C\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \tag{6.321}$$

Therefore, we deduce

$$\begin{aligned}\mu_j^\infty&=\mu_j^0+\sum_{m=0}^{\infty}\left(\mu_j^{m+1}-\mu_j^m\right)\\ &:=\mu_j^0+r_j^\infty,\end{aligned} \tag{6.322}$$

where (μ_j^0) is described in Proposition 6.4 and takes the form

$$\mu_j^0(\lambda,\omega,i_0)=\Omega_j(\lambda)+j\big(c_{i_0}(\lambda,\omega)-I_1(\lambda)K_1(\lambda)\big).$$

Hence (6.320), (6.252) and (6.245) yield

$$\begin{aligned}\|r_j^\infty\|_q^{\gamma,\mathcal{O}}&\leqslant C\,\gamma\,\delta_0(s_h)\\ &\leqslant C\,\varepsilon\gamma^{-1}\end{aligned}$$

and this gives the first result in (6.248). Define the diagonal operator $\mathscr{D}_\infty$ defined on the normal modes by

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad \mathscr{D}_\infty\mathbf{e}_{l,j}=\mathrm{i}\mu_j^\infty\mathbf{e}_{l,j}. \tag{6.323}$$

By the norm definition we obtain

$$\|\mathscr{D}_m-\mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}=\sup_{j\in\mathbb{S}_0^c}\|\mu_j^m-\mu_j^\infty\|_q^{\gamma,\mathcal{O}},$$

which gives by virtue of (6.321)

$$\|\mathscr{D}_m-\mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \tag{6.324}$$

➢ The next goal is to prove that the Cantor set $\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ defined in Proposition 6.5 satisfies

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\subset\bigcap_{m=0}^{n+1}\mathscr{O}_m^\gamma=\mathscr{O}_{n+1}^\gamma.$$

where the intermediate Cantor sets are defined in (6.281). For this aim we shall proceed by finite induction on m with n fixed. First, we get by construction $\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\subset\mathcal{O}=:\mathscr{O}_0^\gamma$. Now assume that $\mathscr{O}_{\infty,n}^{\gamma,\tau_2}(i_0)\subset\mathscr{O}_m^\gamma$ for $m\leqslant n$ and let us check that

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\subset\mathscr{O}_{m+1}^\gamma. \tag{6.325}$$

Let $(\lambda,\omega)\in\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ and $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ such that $0\leqslant|l|\leqslant N_m$ and $(l,j)\neq(0,j_0)$. Then, the triangle inequality, (6.321), (6.244) and (6.253) imply

$$\begin{aligned}|\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)|&\geqslant|\omega\cdot l+\mu_j^\infty(\lambda,\omega)-\mu_{j_0}^\infty(\lambda,\omega)|-2\sup_{j\in\mathbb{S}_0^c}\|\mu_j^m-\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\\ &\geqslant\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-2\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\ &\geqslant\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-2\gamma\varepsilon_0\langle l\rangle^{-\mu_2}\langle j-j_0\rangle.\end{aligned}$$

Thus for ε_0 small enough and by (6.244)(implying that $\mu_2\geqslant\tau_2$) we get

$$\big|\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)\big|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}$$

which shows that $(\lambda,\omega)\in\mathscr{O}_{m+1}^{\gamma}$ and therefore the inclusion (6.325) is satisfied.
➢ Next we shall discuss the convergence of the sequence $(\mathscr{L}_m)_{m\in\mathbb{N}}$ introduced in (6.275) towards the diagonal operator $\mathscr{L}_\infty:=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp}+\mathscr{D}_\infty$, where $\mathscr{D}_\infty$ is detailed in (6.323). Applying (6.324) and (6.285)

$$\begin{aligned}\|\mathscr{L}_m-\mathscr{L}_\infty\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant\|\mathscr{D}_m-\mathscr{D}_\infty\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\\&\leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2},\end{aligned}\tag{6.326}$$

which gives in particular that

$$\lim_{m\to\infty}\|\mathscr{L}_m-\mathscr{L}_\infty\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}=0.\tag{6.327}$$

By virtue of (6.304) and (6.276) one gets

$$\begin{aligned}\forall(\lambda,\omega)\in\mathscr{O}_{n+1}^{\gamma},\quad\widehat{\Phi}_n^{-1}\mathscr{L}_0\widehat{\Phi}_n&=\big(\omega\cdot\partial_\varphi+\mathscr{D}_{n+1}+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^{\perp}\\&=\mathscr{L}_\infty+\big(\mathscr{D}_{n+1}-\mathscr{D}_\infty+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^{\perp},\end{aligned}$$

It follows that any $(\lambda,\omega)\in\mathscr{O}_{n+1}^{\gamma}$

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty&=\mathscr{L}_\infty+\big(\mathscr{D}_{n+1}-\mathscr{D}_\infty+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^{\perp}\\&\quad+\Phi_\infty^{-1}\mathscr{L}_0\left(\Phi_\infty-\widehat{\Phi}_n\right)+\left(\Phi_\infty^{-1}-\widehat{\Phi}_n^{-1}\right)\mathscr{L}_0\widehat{\Phi}_n\\&:=\mathscr{L}_\infty+\mathtt{E}_{n,1}^2+\mathtt{E}_{n,2}^2+\mathtt{E}_{n,3}^2:=\mathscr{L}_\infty+\mathtt{E}_n^2.\end{aligned}$$

For the estimate $\mathtt{E}_{n,1}^2$ we use (6.324) combined with (6.284), (6.285), (6.252) and (6.245)

$$\begin{aligned}\|\mathtt{E}_{n,1}^2\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}.\end{aligned}\tag{6.328}$$

According to Lemma 4.11 with (6.328) we obtain

$$\|\mathtt{E}_{n,1}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Now let us move to the estimates of $\mathtt{E}_{n,2}^2$ and $\mathtt{E}_{n,3}^2$. They can be treated in a similar way. Therefore we shall restrict the discussion to the term $\mathtt{E}_{n,2}^2$. Using (6.246) yields

$$\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\left\|\mathscr{L}_0\big(\Phi_\infty-\widehat{\Phi}_n\big)\rho\right\|_{q,s_0}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+\sigma_4}^{\gamma,\mathcal{O}}\left\|\mathscr{L}_0\big(\Phi_\infty-\widehat{\Phi}_n\big)\rho\right\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{6.329}$$

Therefore we get from (6.183) combined with (6.245)

$$\begin{aligned}\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\left\|\mathscr{L}_0\big(\Phi_\infty-\widehat{\Phi}_n\big)\rho\right\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\left\|\big(\Phi_\infty-\widehat{\Phi}_n\big)\rho\right\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Applying (6.308) with Lemma 4.11, (6.245) and (6.252) allow to get

$$\begin{aligned}\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\left\|\Phi_\infty-\widehat{\Phi}_n\right\|_{\textnormal{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\leqslant C\,\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\leqslant C\,\varepsilon\gamma^{-2}N_0^{\mu_2}N_{m+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Notice that for $\mathtt{E}_{n,3}^2$ we get the same estimate as the preceding one. Consequently, putting together the foregoing estimates yields (6.249).
➢ The goal now is to prove (6.248). We set

$$\widehat{\delta}_m(s):=\max\left(\gamma^{-1}\|\partial_\theta\mathscr{R}_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}},\gamma^{-1}\|\mathscr{R}_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\right).$$

Then we shall prove by induction on $m \in \mathbb{N}$ that

$$\widehat{\delta}_m(s_0) \leqslant \widehat{\delta}_0(s_h) N_0^{\mu_2} N_m^{-\mu_2} \quad \text{and} \quad \widehat{\delta}_m(s_h) \leqslant \left(2 - \tfrac{1}{m+1}\right) \widehat{\delta}_0(s_h). \tag{6.330}$$

According to Sobolev embeddings, the property is trivially satisfied for $m = 0$. Notice that from (6.181) and (6.245) one gets

$$\begin{aligned} \widehat{\delta}_0(s_h) &\lesssim \varepsilon\gamma^{-2}\left(1 + \|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\right) \\ &\lesssim \varepsilon\gamma^{-2}. \end{aligned} \tag{6.331}$$

We ssume that (6.330) is satisfied at the order m and let us check it at the order $m+1$. Applying ∂_θ to the second identity in (6.277) and using (6.286) we obtain the expression

$$\begin{aligned} \partial_\theta \mathscr{R}_{m+1} &= \Phi_m^{-1}\Big(P_{N_m}^{\perp}\partial_\theta\mathscr{R}_m + \partial_\theta\mathscr{R}_m\Psi_m - \Psi_m\partial_\theta\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor - \big[\partial_\theta, \Psi_m\big]\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big) \\ &\quad + \big[\partial_\theta, \Sigma_m\big]\Big(P_{N_m}^{\perp}\mathscr{R}_m + \mathscr{R}_m\Psi_m - \Psi_m\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big). \\ &:= \mathscr{U}_m^1 + \mathscr{U}_m^2 \end{aligned}$$

with

$$\mathscr{U}_m^2 = \big[\partial_\theta, \Sigma_m\big]\Big(P_{N_m}^{\perp}\mathscr{R}_m + \mathscr{R}_m\Psi_m - \Psi_m\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big).$$

It is easy to check that for any Toeplitz in time operator $T(\lambda,\omega)$, we have

$$\big[\partial_\theta, T(\lambda,\omega)\big]\mathbf{e}_{l_0,j_0} = \mathrm{i}\sum_{(l,j)\in\mathbb{Z}^{d+1}} (j-j_0) T_{j_0}^{j}(\lambda,\omega,l-l_0)\mathbf{e}_{l,j},$$

which implies using the norm definition

$$\big\|[\partial_\theta, T]\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant \|T\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}}. \tag{6.332}$$

Since $\Phi_m^{-1} = \mathrm{Id} + \Sigma_m$, then applying Lemma 4.3, we obtain successively for $S \geqslant \overline{s} \geqslant s \geqslant s_0$

$$\begin{aligned} \|\mathscr{U}_m^1\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant C\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big[\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big(1 + \|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big) + \|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big] \\ &\quad + C\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big[\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big(1 + \|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big) + \|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \\ &\quad + \|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big] + \|P_{N_m}^{\perp}\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \end{aligned} \tag{6.333}$$

and

$$\begin{aligned} \|\mathscr{U}_m^2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big(\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big(1 + \|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big)\Big) \\ &\quad + \|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big(1 + \|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\Big). \end{aligned} \tag{6.334}$$

By using (6.332), (6.267) and Lemma 4.3, we obtain

$$\begin{aligned} \|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Psi_m\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}} \\ &\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2 q+\tau_2+1}^{\gamma,\mathcal{O}} \\ &\leqslant CN_m^{\tau_2 q+\tau_2+1}\delta_m(s). \end{aligned}$$

Coming back to (6.287), we obtain

$$\begin{aligned} \|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Sigma_m\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}} \\ &\leqslant CN_m^{\tau_2 q+\tau_2+1}\delta_m(s). \end{aligned}$$

Then inserting the preceding estimates and (6.299) into (6.333) we deduce that

$$\forall\, S \geqslant \overline{s} \geqslant s \geqslant s_0, \quad \widehat{\delta}_{m+1}(s) \leqslant N_m^{s-\overline{s}}\,\widehat{\delta}_m(\overline{s}) + CN_m^{\tau_2 q+\tau_2+1}\widehat{\delta}_m(s)\widehat{\delta}_m(s_0). \tag{6.335}$$

In particular, for $s = s_0$ we get by the induction assumption (6.330),

$$\begin{aligned}\widehat{\delta}_{m+1}(s_0) &\leqslant N_m^{s_0-s_h}\widehat{\delta}_m(s_h) + CN_m^{\tau_2 q+\tau_2+1}\left(\widehat{\delta}_m(s_0)\right)^2\\ &\leqslant \left(2-\tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h)N_m^{s_0-s_h} + CN_0^{2\mu_2}N_m^{\tau_2 q+\tau_2+1-2\mu_2}\left(\widehat{\delta}_0(s_h)\right)^2\\ &\leqslant \widehat{\delta}_0(s_h)\left(2N_m^{s_0-s_h} + CN_0^{2\mu_2}N_m^{\tau_2 q+\tau_2+1-2\mu_2}\widehat{\delta}_0(s_h)\right).\end{aligned}$$

If we fix s_2 and μ_2 such that

$$N_m^{s_0-s_h} \leqslant \tfrac{1}{4}N_0^{\mu_2}N_{m+1}^{-\mu_2} \quad \text{and} \quad CN_0^{2\mu_2}N_m^{\tau_2 q+\tau_2+1-2\mu_2}\widehat{\delta}_0(s_h) \leqslant \tfrac{1}{2}N_0^{\mu_2}N_{m+1}^{-\mu_2}, \tag{6.336}$$

then we find

$$\widehat{\delta}_{m+1}(s_0) \leqslant \widehat{\delta}_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.$$

Notice that (6.244) implies in particular

$$s_h \geqslant \tfrac{3}{2}\mu_2 + s_0 + 1 \quad \text{and} \quad \mu_2 \geqslant 2(\tau_2 q + \tau_2 + 1) + 1.$$

Hence, using (5.82), we see that the assumptions of (6.336) hold true provided that

$$4N_0^{-\mu_2} \leqslant 1 \qquad \text{and} \qquad 2C\widehat{\delta}_0(s_h) \leqslant N_0^{-\mu_2}.$$

Remark that these conditions are satisfied thanks to (6.290), (6.331) and (6.245). Now, we turn to the proof of the second estimate in (6.330). By (6.335) and (6.330)

$$\begin{aligned}\widehat{\delta}_{m+1}(s_h) &\leqslant \widehat{\delta}_m(s_h) + CN_m^{\tau_2 q+\tau_2+1}\widehat{\delta}_m(s_h)\widehat{\delta}_m(s_0)\\ &\leqslant \left(2-\tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h)\left(1 + CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2+1-\mu_2}\widehat{\delta}_0(s_h)\right).\end{aligned}$$

Taking the parameters s_2 and μ_2 such that

$$\left(2-\tfrac{1}{m+1}\right)\left(1 + CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2+1-\mu_2}\widehat{\delta}_0(s_h)\right) \leqslant 2 - \tfrac{1}{m+2}, \tag{6.337}$$

then we obtain

$$\widehat{\delta}_{m+1}(s_h) \leqslant \left(2-\tfrac{1}{m+2}\right)\widehat{\delta}_0(s_h),$$

which achieves the induction argument in (6.330). Now observe that (6.337) is quite similar to (6.291) using in particular $\mu_2 \geqslant 2(\tau_2 q + \tau_2) + 1$ and one may proceed following the same lines. Next let us see how to get the estimate (6.248). Recall that

$$r_j^\infty = \sum_{m=0}^{\infty} r_j^m \qquad \text{with} \qquad r_j^m = \big\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Then it is clear that

$$\big\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})} = \tfrac{\mathrm{i}}{j}\big\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{0,j}, \partial_\theta \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Therefore integration by parts leads to

$$\big\langle P_{N_m}\mathscr{R}_m \mathbf{e}_{0,j}, \partial_\theta\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})} = -\big\langle P_{N_m}\partial_\theta\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Using a duality argument $H^{s_0} - H^{-s_0}$ combined with Lemma 4.3 and (6.330), we obtain

$$\begin{aligned}\|\langle P_{N_m}\partial_\theta\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\,\widehat{\delta}_m(s_0)\\ &\leqslant C\gamma\,\widehat{\delta}_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}.\end{aligned}$$

Putting together the preceding estimates with (6.331) and Lemma A.1 yields

$$\begin{aligned}\|r_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\lesssim \gamma\,|j|^{-1}\widehat{\delta}_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\ &\lesssim |j|^{-1}\varepsilon\gamma^{-1}.\end{aligned}$$

This achieves the proof of (6.248).

(ii) We shall now work with fixed values (minimal) of μ_2 and s_h denoted respectively by μ_c and s_c, namely

$$\mu_c := \overline{\mu_2} + 2\tau_2 q + 2\tau_2 \quad \text{and} \quad s_c := \frac{3}{2}\mu_c + \overline{s}_l + 1 = \overline{s}_h + 4\tau_2 q + 4\tau_2. \tag{6.338}$$

From (6.277) and (6.286), we can write

$$\mathscr{R}_{m+1} = (\mathrm{Id} + \Sigma_m)\,U_m,$$

where

$$U_m := P_{N_m}^{\perp}\mathscr{R}_m + \mathscr{R}_m\Psi_m - \Psi_m\lfloor P_{N_m}\mathscr{R}_m\rfloor. \tag{6.339}$$

After straightforward computations, we get

$$\begin{aligned}\Delta_{12}U_m =& P_{N_m}^{\perp}\Delta_{12}\mathscr{R}_m + (\Delta_{12}\mathscr{R}_m)(\Psi_m)_{r_1} + (\mathscr{R}_m)_{r_2}(\Delta_{12}\Psi_m)\\ &- (\Delta_{12}\Psi_m)\lfloor P_{N_m}(\mathscr{R}_m)_{r_1}\rfloor - (\Psi_m)_{r_2}\lfloor P_{N_m}\Delta_{12}\mathscr{R}_m\rfloor\end{aligned} \tag{6.340}$$

and

$$\Delta_{12}\mathscr{R}_{m+1} = \Delta_{12}U_m + (\Delta_{12}\Sigma_m)(U_m)_{r_1} + (\Sigma_m)_{r_2}\Delta_{12}U_m. \tag{6.341}$$

We have used the notation $(f)_r = f(r)$. Elementary manipulations based on (6.286) give

$$\Delta_{12}\Sigma_m = \Delta_{12}\Phi_m^{-1} = -(\Phi_m^{-1})_{r_2}(\Delta_{12}\Psi_m)(\Phi_m^{-1})_{r_1}.$$

The law products of Lemma 4.3 together with (6.303) and (6.245) imply

$$\forall\, s\in[s_0,s_c],\quad \|\Delta_{12}\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \|\Delta_{12}\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}. \tag{6.342}$$

Using once again the law products of Lemma 4.3, (6.342) and (6.341) we obtain

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant& \|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|(\Delta_{12}\Psi_m)\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\end{aligned} \tag{6.343}$$

and

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant& \|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + \|(\Delta_{12}\Psi_m)\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ \|(\Delta_{12}\Psi_m)\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.344}$$

For the estimate $(U_m)_{r_1}$ (to alleviate the notation we shall remove in this part remove the subscript r_1) described by (6.339) we use the law products leading to

$$\|U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant \|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + \|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \tag{6.345}$$

and

$$\begin{aligned}\|U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant& \|\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + \|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ \|\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.346}$$

By (6.285),(6.252) and (6.299) together with (6.345) we infer

$$\|U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}. \tag{6.347}$$

Putting together the first estimate of (6.301), (6.285) and (6.252) we deduce that

$$\begin{aligned}\max_{j=1,2}\|(\Psi_m)_{r_j}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} &\lesssim N_m^{\tau_2 q+\tau_2}\delta_m(s_c)\\ &\lesssim \varepsilon\gamma^{-2}N_m^{\tau_2 q+\tau_2}.\end{aligned} \tag{6.348}$$

Hence we get in view of (6.346), (6.285) and (6.245)

$$\|U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1}. \tag{6.349}$$

Plugging (6.347) and (6.349) into (6.344) implies

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant{}& \|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.350}$$

Applying (6.287) and (6.252) gives

$$\|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c} \quad \text{and} \quad \|(\Sigma_m)_{r_2}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-2}N_m^{\tau_2 q+\tau_2}. \tag{6.351}$$

Inserting (6.351) into (6.350) allows to get

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant{}& \big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-2}N_m^{\tau_2 q+\tau_2}\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.352}$$

In a similar way, by combining (6.347), (6.351) with (6.343) we find

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant{}& \big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.353}$$

From (6.340) and the law products of Lemma 4.3 we obtain $\forall\, s\in[s_0,s_c]$,

$$\begin{aligned}\|\Delta_{12}U_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant{}& N_m^{s-s_c}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\max_{j=1,2}\|(\Psi_m)_{r_j}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|(\Psi_m)_{r_j}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + C\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\max_{j=1,2}\|(\mathscr{R}_m)_{r_j}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|(\mathscr{R}_m)_{r_j}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combining the foregoing estimate with (6.348), (6.285) and (6.299) yields

$$\begin{aligned}\|\Delta_{12}U_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant{}& N_m^{s_0-s_c}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\end{aligned} \tag{6.354}$$

and

$$\begin{aligned}\|\Delta_{12}U_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant{}& \big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + CN_m^{\tau_2 q+\tau_2}\varepsilon\gamma^{-2}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the preceding estimate with (6.352), (6.353), (6.244) and (6.245) we deduce that

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant{}& \Big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}+C\varepsilon\gamma^{-2}N_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ CN_m^{\tau_2 q+\tau_2}\varepsilon\gamma^{-2}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} + C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ &+ C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.355}$$

In a similar way, by making appeal to (6.353), (6.354) and (6.245) we find

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant N_m^{s_0-s_c}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\\&\quad+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.356}$$

We shall now estimate $\Delta_{12}\Psi_m$. Remark that

$$\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\leqslant q}}\gamma^{\alpha}\sup_{(\lambda,\omega)\in\mathcal{O}}\left(\sum_{\substack{(l,k)\in\mathbb{Z}^{d+1}\\|l|,|k|\leqslant N_m}}\langle l,k\rangle^{2(s-|\alpha|)}\sup_{j\in\mathbb{Z}}\left|\partial_{\lambda,\omega}^{\alpha}\Delta_{12}(\Psi_m)_{j+k}^{j}(\lambda,\omega,l)\right|^2\right)^{\frac{1}{2}}.$$

By virtue of (6.282), we get

$$(\Psi_m)_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}-(\varrho_m)_{j_0}^{j}(\lambda,\omega,l)r_{j_0,m}^{j}(\lambda,\omega,l) & \text{if } (l,j)\neq(0,j_0)\\ 0 & \text{if } (l,j)=(0,j_0),\end{cases}$$

where

$$(\varrho_m)_{j_0}^{j}(\lambda,\omega,l):=\frac{\chi\big((\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega))(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}\big)}{\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)}.$$

Recall from (6.258), that $\{\mathrm{i}r_{j_0,m}^{j}(\lambda,\omega,l)\}$ are the Fourier coefficients of $P_{N_m}\mathscr{R}_m$, that is

$$\mathrm{i}\,r_{j_0,m}^{j}(\lambda,\omega,l)=\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j_0},\mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})}.\tag{6.357}$$

We can write for non-zero coefficients

$$\begin{aligned}\Delta_{12}(\Psi_m)_{j+k}^{j}(\lambda,\omega,l)&=\Delta_{12}(\varrho_m)_{j+k}^{j}(\lambda,\omega,l)\big(r_{j+k,m}^{j}\big)_{r_1}(\lambda,\omega,l)\\&\quad+\big((\varrho_m)_{j+k}^{j}\big)_{r_2}(\lambda,\omega,l)\Delta_{12}r_{j+k,m}^{j}(\lambda,\omega,l).\end{aligned}$$

Hence, using Lemma 4.1-(iv)

$$\begin{aligned}\forall q'\in[\![0,q]\!],\quad \|\Delta_{12}(\Psi_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}(\varrho_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\\&\quad+\max_{i\in\{1,2\}}\|\big((\varrho_m)_{j+k}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\|\Delta_{12}r_{j+k,m}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.358}$$

From (6.357), we deduce

$$\mathrm{i}\Delta_{12}r_{j_0,m}^{j}(\lambda,\omega,l)=\langle P_{N_m}\Delta_{12}\mathscr{R}_m\mathbf{e}_{0,j_0},\mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})}.$$

One can write

$$(\varrho_m)_{j_0}^{j}(\lambda,\omega,l)=b_{l,j,j_0,m}\widehat{\chi}\Big(b_{l,j,j_0,m}B_{l,j,j_0,m}(\lambda,\omega)\Big),$$

with

$$b_{l,j,j_0,m}:=(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2},\quad B_{l,j,j_0,m}(\lambda,\omega):=\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega),\quad \widehat{\chi}(x)=\frac{\chi(x)}{x}.$$

Notice that from (6.297), one obtains

$$\forall q'\in[\![0,q]\!],\quad\|B_{l,j,j_0,m}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\langle l,j-j_0\rangle.\tag{6.359}$$

In a similar way to (6.266), one gets from (6.297)

$$\forall q'\in[\![0,q]\!],\quad\|(\varrho_m)_{j_0}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\lesssim\gamma^{-(q'+1)}\langle l,j-j_0\rangle^{\tau_2q'+\tau_2+q'}.\tag{6.360}$$

Using Taylor formula in a similar way to (6.109), we find (to simplify the notation we remove the dependence in (λ,ω))

$$\Delta_{12}(\varrho_m)_{j_0}^{j}(l)=b_{l,j,j_0,m}^2(\Delta_{12}B_{l,j,j_0,m})\int_0^1\widehat{\chi}'\Big(b_{l,j,j_0,m}\Big[(1-\tau)(B_{l,j,0,m})_{r_1}+\tau(B_{l,j,0,m})_{r_2}\Big]\Big)d\tau.$$

We shall estimate $\Delta_{12}B_{l,j,j_0,m}$. For that purpose, we use (6.279) to write

$$\mu_j^m=\mu_j^0+\sum_{n=0}^{m-1}\langle P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}.$$

We recall from Proposition 6.4 that

$$\mu_j^0(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0),\quad r^1(\lambda,\omega,i_0)=c_{i_0}(\lambda,\omega)-V_0(\lambda).$$

Therefore

$$\Delta_{12}\mu_j^m=\Delta_{12}\mu_j^0+\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}$$

and

$$\begin{aligned}\Delta_{12}B_{l,j,j_0,m}&=\Delta_{12}\big(\mu_j^m-\mu_{j_0}^m\big)\\&=(j-j_0)\Delta_{12}c_i+\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\\&\quad-\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j_0},\mathbf{e}_{0,j_0}\rangle_{L^2(\mathbb{T}^{d+1})}.\end{aligned}$$

Hence, using (6.30), one gets

$$\forall q'\in[\![0,q]\!],\quad\|\Delta_{12}B_{l,j,j_0,m}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\varepsilon|j-j_0|\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}+\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\text{O-d},q',s_0}^{\gamma,\mathcal{O}}.\tag{6.361}$$

Then, one obtains from Lemma 4.1-(vi), (6.359) and (6.361)

$$\begin{aligned}\forall q'\in[\![0,q]\!],\quad\|\Delta_{12}(\varrho_m)_{j_0}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\gamma^{-2-q'}\langle l,j-j_0\rangle^{\tau_2q'+2\tau_2+q'+1}\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-2-q'}\langle l,j-j_0\rangle^{\tau_2q'+2\tau_2+q'}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\text{O-d},q',s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.362}$$

Gathering (6.358), (6.360) and (6.362) gives for all $q'\in[\![0,q]\!]$,

$$\begin{aligned}\|\Delta_{12}(\Psi_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\gamma^{-2-q'}\langle l,k\rangle^{\tau_2q'+2\tau_2+q'+1}\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-2-q'}\langle l,k\rangle^{\tau_2q'+2\tau_2+q'}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\text{O-d},q',s_0}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-1-q'}\langle l,k\rangle^{\tau_2q'+\tau_2+q'}\|\Delta_{12}r_{j+k,m}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}.\end{aligned}$$

We deduce that for all $s\in[s_0,S]$,

$$\begin{aligned}\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\gamma^{-2-q}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+2\tau_2+1}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-2-q}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+2\tau_2}^{\gamma,\mathcal{O}}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-1-q}\|P_{N_m}\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}.\end{aligned}\tag{6.363}$$

We set

$$\overline{\delta}_m(s)=\gamma^{-1}\|\Delta_{12}\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\quad\text{and}\quad\varkappa_m(s):=\sum_{n=0}^{m-1}\overline{\delta}_n(s).\tag{6.364}$$

Then, using (6.363), (6.284) and (6.3), we get

$$\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1-q}N_m^{\tau_2}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\delta_m(\overline{s}_l) + \gamma^{-q}N_m^{\tau_2}\delta_m(\overline{s}_l)\varkappa_m(s_0) + \gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0) \tag{6.365}$$

and

$$\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1-q}N_m^{\tau_2 q+2\tau_2+1}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\delta_m(s_c) + \gamma^{-q}N_m^{\tau_2 q+2\tau_2}\delta_m(s_c)\varkappa_m(s_0) + \gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_c). \tag{6.366}$$

According to (6.331), one has

$$\delta_m(\overline{s}_l) \lesssim \varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c} \quad \text{and} \quad \sup_{m\in\mathbb{N}}\delta_m(s_c) \lesssim \varepsilon\gamma^{-2} \lesssim 1. \tag{6.367}$$

Putting together (6.367) and (6.365) and using (6.245) yields

$$\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_m^{\tau_2-\mu_c}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} + N_m^{\tau_2-\mu_c}\varkappa_m(s_0) + \gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0). \tag{6.368}$$

In a similar way, on gets by (6.367), (6.366) and (6.245)

$$\|\Delta_{12}\Psi_m\|_{\text{O-d},q,s_c}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_m^{\tau_2 q+2\tau_2+1}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} + N_m^{\tau_2 q+2\tau_2}\varkappa_m(s_0) + \gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_c). \tag{6.369}$$

Plugging (6.368) into (6.356) yields by virtue of (6.285) and (6.245)

$$\overline{\delta}_{m+1}(s_0) \leqslant N_m^{s_0-s_c}\overline{\delta}_m(s_c) + CN_m^{\tau_2 q+\tau_2-\mu_c}\overline{\delta}_m(s_0) + CN_m^{\tau_2-2\mu_c}\varkappa_m(s_0) + C\varepsilon\gamma^{-1}N_m^{\tau_2-2\mu_c}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}. \tag{6.370}$$

Therefore, inserting (6.368), (6.369) into (6.355) and using (6.245) implies

$$\overline{\delta}_{m+1}(s_c) \leqslant \Big(1 + CN_m^{\tau_2 q+\tau_2-\mu_c} + CN_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\overline{\delta}_m(s_c) + C\varepsilon\gamma^{-2-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0) + CN_m^{\tau_2 q+\tau_2-\mu_c}\varkappa_m(s_0) + C\varepsilon\gamma^{-1}N_m^{\tau_2 q+2\tau_2+1-\mu_c}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}. \tag{6.371}$$

Next, we intend to prove by induction in $m\in\mathbb{N}$ that

$$\forall\, k\leqslant m, \quad \overline{\delta}_k(s_0) \leqslant N_0^{\mu_c}N_k^{-\mu_c}\nu(s_c) \quad \text{and} \quad \overline{\delta}_k(s_c) \leqslant \Big(2-\tfrac{1}{k+1}\Big)\nu(s_c), \tag{6.372}$$

with

$$\nu(s) := \overline{\delta}_0(s) + \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.$$

The estimate (6.372) is obvious for $m=0$ by Sobolev embeddings. Now let us assume that the preceding property holds true at the order m and let us check it at the order $m+1$. Thus by applying (6.364) and Lemma A.1, we get

$$\sup_{m\in\mathbb{N}}\varkappa_m(s_0) \leqslant C\nu(s_c).$$

Putting together this estimate with the induction assumption, (6.370), (6.371),(6.338) and (6.245) yields

$$\overline{\delta}_{m+1}(s_0) \leqslant \big(2N_m^{s_0-s_c} + CN_0^{\mu_c}N_m^{\tau_2 q+\tau_2-2\mu_c}\big)\,\nu(s_c)$$

and

$$\overline{\delta}_{m+1}(s_c) \leqslant \Big(1 + CN_m^{\tau_2 q+\tau_2-\mu_c} + CN_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\Big(2-\frac{1}{m+1}\Big)\nu(s_c) + C\left(N_m^{\tau_2 q+\tau_2-\mu_c} + N_m^{\tau_2 q+2\tau_2+1-\mu_c}\right)\nu(s_c).$$

Since (6.338) implies in particular

$$\mu_c \geqslant 2\tau_2 q + 2\tau_2 + 1 \quad \text{and} \quad s_c \geqslant \tfrac{3}{2}\mu_c + s_0 + \tau_2 q + \tau_2 + 1,$$

then proceeding similarly to the proof of (6.285), we conclude that

$$\overline{\delta}_{m+1}(s_0) \leqslant N_0^{\mu_c} N_{m+1}^{-\mu_c} \nu(s_c) \quad \text{and} \quad \overline{\delta}_{m+1}(s_c) \leqslant \left(2 - \tfrac{1}{m+2}\right) \nu(s_c),$$

which achieves the induction. The next target is to estimate $\Delta_{12} r_j^{\infty}$. Then similarly to (6.295) we obtain through a duality argument, Lemma 4.3, (6.372) and Lemma A.1

$$\begin{aligned}
\|\Delta_{12} r_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty} \left\| \langle P_{N_m} \Delta_{12} \mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j} \rangle_{L^2(\mathbb{T}^{d+1})} \right\|_q^{\gamma,\mathcal{O}} \\
&\lesssim \sum_{m=0}^{\infty} \|\Delta_{12} \mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \\
&\lesssim \gamma \nu(s_c) \sum_{m=0}^{\infty} N_0^{\mu_c} N_m^{-\mu_c} \\
&\leqslant C\gamma\, \nu(s_c).
\end{aligned}$$

From the particular value of $\mathtt{p}$ in (6.177), we infer

$$s_c = \overline{s}_h + 4\tau_2 q + 4\tau_2 = \overline{s}_h + \mathtt{p}. \tag{6.373}$$

Then, applying (6.182) we obtain

$$\begin{aligned}
\|\Delta_{12} r_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\, \nu(\overline{s}_h + 4\tau_2 q + 4\tau_2) \\
&\leqslant C\varepsilon\gamma^{-1} \|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.
\end{aligned}$$

Finally, combining the previous estimate with (6.322) and (6.179) we deduce

$$\begin{aligned}
\forall j \in \mathbb{S}_0^c, \quad \|\Delta_{12} \mu_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12} \mu_j^0\|_q^{\gamma,\mathcal{O}} + \|\Delta_{12} r_j^{\infty}\|_q^{\gamma,\mathcal{O}} \\
&\lesssim \varepsilon\gamma^{-1} |j| \|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.
\end{aligned}$$

The achieves the proof of Proposition 6.5. □

6.4 Approximate inverse in the normal directions

In this section we plan to construct an approximate right inverse in the normal directions for the linearized operator $\widehat{\mathcal{L}}_\omega$ defined in (5.87) when the parameters are restricted in a Cantor like set. Our main result is the following.

Proposition 6.6. *Let $(\gamma, q, d, \tau_1, s_0, \mu_2, s_h, S)$ satisfying (4.1), (4.2) and (6.244). There exists $\sigma := \sigma(\tau_1, \tau_2, q, d) \geqslant \sigma_4$ such that if*

$$\varepsilon\gamma^{-2-q} N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.374}$$

then the following assertions hold true.

(i) Consider the operator $\mathscr{L}_\infty$ defined in Proposition 6.5, then there exists a family of linear operators $(\mathtt{T}_n)_{n\in\mathbb{N}}$ defined in $\mathcal{O}$ satisfying the estimate

$$\forall s \in [s_0, S], \quad \sup_{n\in\mathbb{N}} \|\mathtt{T}_n \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} \|\rho\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}$$

and such that for any $n \in \mathbb{N}$, in the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c \\ |l|\leqslant N_n}} \left\{ (\lambda,\omega) \in \mathcal{O} \quad \text{s.t.} \quad \left|\omega\cdot l + \mu_j^{\infty}(\lambda,\omega,i_0)\right| > \tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}} \right\},$$

we have
$$\mathscr{L}_\infty \mathtt{T}_n = \mathrm{Id} + \mathtt{E}^3_n,$$
with
$$\forall s_0 \leqslant s \leqslant \overline{s} \leqslant S, \quad \|\mathtt{E}^3_n \rho\|^{\gamma,\mathcal{O}}_{q,s} \lesssim N_n^{s-\overline{s}} \gamma^{-1} \|\rho\|^{\gamma,\mathcal{O}}_{q,\overline{s}+1+\tau_1 q+\tau_1}.$$

(ii) *There exists a family of linear operators* $\big(\mathtt{T}_{\omega,n}\big)_{n\in\mathbb{N}}$ *satisfying*
$$\forall\, s \in [s_0, S], \quad \sup_{n\in\mathbb{N}} \|\mathtt{T}_{\omega,n}\rho\|^{\gamma,\mathcal{O}}_{q,s} \lesssim \gamma^{-1}\Big(\|\rho\|^{\gamma,\mathcal{O}}_{q,s+\sigma} + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}\Big) \tag{6.375}$$
and such that in the Cantor set
$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0) := \mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0) \cap \mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0) \cap \Lambda^{\gamma,\tau_1}_{\infty,n}(i_0),$$
we have
$$\widehat{\mathcal{L}}_\omega \mathtt{T}_{\omega,n} = \mathrm{Id} + \mathtt{E}_n,$$
where $\mathtt{E}_n$ *satisfies the following estimate*
$$\begin{aligned}\forall\, s \in [s_0, S], \quad \|\mathtt{E}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0} \lesssim{}& N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|^{\gamma,\mathcal{O}}_{q,s+\sigma} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}\Big)\\ &+ \varepsilon\gamma^{-3} N_0^{\mu_2} N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}.\end{aligned}$$
Recall that $\widehat{\mathcal{L}}_\omega$, $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)$ *and* $\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)$ *are given in Propositions* 6.1, 6.2 *and* 6.5, *respectively.*

(iii) *In the Cantor set* $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$, *we have the following splitting*
$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n \quad \text{with} \quad \widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n} = \mathrm{Id} \quad \text{and} \quad \widehat{\mathtt{R}}_n = \mathtt{E}_n \widehat{\mathtt{L}}_{\omega,n},$$
where the operators $\widehat{\mathtt{L}}_{\omega,n}$ *and* $\widehat{\mathtt{R}}_n$ *are defined in* $\mathcal{O}$ *and satisfy the following estimates*
$$\begin{aligned}\forall s \in [s_0, S], \quad & \sup_{n\in\mathbb{N}} \|\widehat{\mathtt{L}}_{\omega,n}\rho\|^{\gamma,\mathcal{O}}_{q,s} \lesssim \|\rho\|^{\gamma,\mathcal{O}}_{q,s+1} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1},\\ \forall s \in [s_0, S], \quad & \|\widehat{\mathtt{R}}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0} \lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|^{\gamma,\mathcal{O}}_{q,s+\sigma} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}\Big)\\ &\qquad + \varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}.\end{aligned}$$

Proof. **(i)** From Proposition 6.5 we recall that
$$\mathscr{L}_\infty = \omega\cdot\partial_\varphi \Pi^\perp_{\mathbb{S}_0} + \mathscr{D}_\infty.$$
Then we may split this operator as follows, using the projectors defined in (4.4)
$$\begin{aligned}\mathscr{L}_\infty &= \Pi_{N_n}\omega\cdot\partial_\varphi\Pi_{N_n}\Pi^\perp_{\mathbb{S}_0} + \mathscr{D}_\infty - \Pi^\perp_{N_n}\omega\cdot\partial_\varphi\Pi^\perp_{N_n}\Pi^\perp_{\mathbb{S}_0}\\ &:= \mathtt{L}_n - \mathtt{R}_n,\end{aligned}\tag{6.376}$$
with $\mathtt{R}_n := \Pi^\perp_{N_n}\omega\cdot\partial_\varphi\Pi^\perp_{N_n}\Pi^\perp_{\mathbb{S}_0}$. From this definition and the structure of $\mathscr{D}_\infty$ in Proposition 6.5 we deduce that
$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c, \quad \mathbf{e}_{-l,-j}\mathtt{L}_n\mathbf{e}_{l,j} = \begin{cases}\mathrm{i}\big(\omega\cdot l + \mu_j^\infty\big) & \text{if } |l|\leqslant N_n\\ \mathrm{i}\,\mu_j^\infty & \text{if } |l| > N_n.\end{cases}$$
Define the diagonal operator $\mathtt{T}_n$ by
$$\begin{aligned}\mathtt{T}_n\rho(\lambda,\omega,\varphi,\theta) :={}& -\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}} \tfrac{\chi\left((\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0))\gamma^{-1}\langle l\rangle^{\tau_1}\right)}{\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)}\rho_{l,j}(\lambda,\omega)\,e^{\mathrm{i}(l\cdot\varphi+j\theta)}\\ &-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|> N_n}} \tfrac{\rho_{l,j}(\lambda,\omega)}{\mu_j^\infty(\lambda,\omega,i_0)}\,e^{\mathrm{i}(l\cdot\varphi+j\theta)},\end{aligned}$$

where χ is the cut-off function defined in (5.81) and $\big(\rho_{l,j}(\lambda,\omega)\big)_{l,j}$ are the Fourier coefficients of ρ. We recall from Proposition 6.5 that

$$\mu_j^\infty(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega)+r_j^\infty(\lambda,\omega)\quad\text{ with }\quad r^1(\lambda,\omega)=c_{i_0}(\lambda,\omega)-V_0(\lambda),$$

with the estimates

$$\forall j\in\mathbb{S}_0^c,\quad \|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim|j|,$$

where we use in part the estimate (6.274), (6.248) and (6.179). According to Lemma 3.3-(iii), (6.274), (6.248) and the smallness condition (6.245) we infer

$$|j|\lesssim\|\mu_j^\infty\|_0^{\gamma,\mathcal{O}}\leqslant\|\mu_j^\infty\|_q^{\gamma,\mathcal{O}},$$

Implementing the same arguments as for (6.267) one gets

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}. \tag{6.377}$$

Moreover, by construction

$$\mathtt{L}_n\mathtt{T}_n=\mathrm{Id}\quad\text{in }\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) \tag{6.378}$$

since $\chi(\cdot)=1$ in this set. It follows from (6.376) that

$$\begin{aligned}\forall\,(\lambda,\omega)\in\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\quad \mathscr{L}_\infty\mathtt{T}_n&=\mathrm{Id}-\mathtt{R}_n\mathtt{T}_n\\&:=\mathrm{Id}+\mathtt{E}_n^3.\end{aligned} \tag{6.379}$$

Notice that by Lemma 4.1-(ii),

$$\forall\,s_0\leqslant s\leqslant\overline{s},\quad \|\mathtt{R}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\|\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}.$$

Combining this estimate with (6.377) yields

$$\begin{aligned}\forall\,s_0\leqslant s\leqslant\overline{s},\quad \|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim N_n^{s-\overline{s}}\|\mathtt{T}_n\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}\\&\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.380}$$

(ii) Let us define

$$\mathtt{T}_{\omega,n}:=\mathscr{B}_\perp\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}, \tag{6.381}$$

where the operators $\mathscr{B}_\perp$ and Φ_∞ are defined in Propositions 6.4 and 6.5 respectively. Notice that $\mathtt{T}_{\omega,n}$ is defined in the whole range of parameters $\mathcal{O}$. Since the condition (6.374) is satisfied, then, both Propositions 6.2 and 6.5 apply and from (6.159) we obtain

$$\forall s\in[s_0,S],\quad \|\mathtt{T}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

By using (6.246) and (6.374), one gets

$$\forall s\in[s_0,S],\quad \|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Thus the point (i) of the current proposition implies

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

Applying (6.246) and (6.159) with (6.374) yields

$$\begin{aligned}\forall s\in[s_0,S],\quad \|\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the preceding three estimates gives (6.375). Now combining Propositions 6.4 and 6.5, we find that in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)$ the following decomposition holds

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty&=\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\\&=\mathscr{L}_\infty+\mathtt{E}_n^2+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty.\end{aligned}$$

It follows that in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathscr{O}^{\gamma,\tau_2}_{\infty,n}(i_0)\cap\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0)$ one has by virtue of the identity (6.379)

$$\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n,$$

which gives, using (6.381), the following identity in $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\begin{aligned}\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n}&=\mathrm{Id}+\mathscr{B}_\perp\Phi_\infty\big(\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\big)\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\\&:=\mathrm{Id}+\mathscr{B}_\perp\Phi_\infty\mathtt{E}_n^4\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\\&:=\mathrm{Id}+\mathtt{E}_n.\end{aligned}\tag{6.382}$$

The estimate of the first term of $\mathtt{E}_n^4$ is given in (6.380). For the second term of $\mathtt{E}_n^4$ we use (6.249) and (6.377) leading to

$$\begin{aligned}\|\mathtt{E}_n^2\mathtt{T}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0}&\lesssim\varepsilon\gamma^{-2}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\mathtt{T}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\\&\lesssim\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1+\tau_1q+\tau_1}.\end{aligned}\tag{6.383}$$

For the estimate of $\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n$, we combine (6.246), (6.180), (6.377) and (6.374) to get

$$\begin{aligned}\|\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0}&\lesssim\|\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\\&\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\Phi_\infty\mathtt{T}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}\\&\lesssim\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2+\tau_1q+\tau_1}.\end{aligned}\tag{6.384}$$

Putting together (6.380) and (6.383) and (6.384) we find

$$\|\mathtt{E}_n^4\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim N_n^{s_0-s}\gamma^{-1}\|\rho\|^{\gamma,\mathcal{O}}_{s+2+\tau_1q+\tau_1}+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2+\tau_1q+\tau_1}.\tag{6.385}$$

Set $\Psi=\mathscr{B}_\perp\Phi_\infty$ then from (6.246), (6.159) and (6.374) we deduce that

$$\forall s\in[s_0,S],\qquad\|\Psi^{\pm1}\rho\|^{\gamma,\mathcal{O}}_{q,s}\lesssim\|\rho\|^{\gamma,\mathcal{O}}_{q,s}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0}.\tag{6.386}$$

Straightforward computations based on (6.385), (6.386) and (6.374) yields

$$\begin{aligned}\|\Psi\mathtt{E}_n^4\Psi^{-1}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}&\lesssim\|\mathtt{E}_n^4\Psi^{-1}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\\&\lesssim N_n^{s_0-s}\gamma^{-1}\|\Psi^{-1}\rho\|^{\gamma,\mathcal{O}}_{s+2+\tau_1q+\tau_1}+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\Psi^{-1}\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2+\tau_1q+\tau_1}\\&\lesssim N_n^{s_0-s}\gamma^{-1}\big(\|\rho\|^{\gamma,\mathcal{O}}_{q,s+2+\tau_1q+\tau_1}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\big)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2+\tau_1q+\tau_1}.\end{aligned}$$

Consequently, taking σ large enough, we get

$$\|\mathtt{E}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim N_n^{s_0-s}\gamma^{-1}\big(\|\rho\|^{\gamma,\mathcal{O}}_{q,s+\sigma}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}\big)+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+\sigma}.$$

(iii) According to (6.382), one can write in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathcal{L}}_\omega=\mathtt{T}_{\omega,n}^{-1}+\mathtt{E}_n\mathtt{T}_{\omega,n}^{-1}.\tag{6.387}$$

Gathering (6.381) and (6.378), one obtains in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathtt{L}}_{\omega,n}:=\mathtt{T}_{\omega,n}^{-1}=\mathscr{B}_\perp\Phi_\infty\mathtt{L}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}=\Psi\mathtt{L}_n\Psi^{-1}.$$

Hence, (6.387) can be rewritten

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n \quad \text{with} \quad \widehat{\mathtt{R}}_n := \mathtt{E}_n \widehat{\mathtt{L}}_{\omega,n}. \tag{6.388}$$

Putting together (6.376), (6.386) and (6.374), we obtain

$$\begin{aligned}\forall s \in [s_0, S], \quad \|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}} &= \|\Psi \mathtt{L}_n \Psi^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \\ &\lesssim \|\mathtt{L}_n \Psi^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathtt{L}_n\Psi^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \\ &\lesssim \|\Psi^{-1}\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\Psi^{-1}\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}} \\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \end{aligned} \tag{6.389}$$

Hence combining this estimate with (6.387) yields

$$\begin{aligned}\forall s \in [s_0, S], \quad \|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big) \\ &\quad + \varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

This achieves the proof of the third point and the proof of the proposition is now complete. □

7 Proof of the main result

This section is devoted to the proof of Theorem 1.1. For this aim we intend to implement Nash-Moser scheme in order to construct zeros for the nonlinear functional $\mathcal{F}\big(i, \alpha, \lambda, \omega, \varepsilon\big)$ defined in (5.23). We shall be able to capture the solutions when the parameters (λ, ω) belong to a suitable final Cantor set $\mathcal{G}_\infty^\gamma$ obtained as the intersection of all the Cantor sets required during the steps of the scheme to invert the linearized operator. More precisely, we get a relatively smooth function $(\lambda, \omega) \in \mathcal{O} \mapsto U_\infty(\lambda, \omega) = \big(i_\infty(\lambda, \omega), \alpha_\infty(\lambda, \omega)\big)$ such that

$$\forall (\lambda, \omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}\big(U_\infty(\lambda, \omega), \lambda, \omega, \varepsilon\big) = 0.$$

To generate solutions to the initial Hamiltonian equation (5.4) we should adjust the parameters so that $\alpha_\infty(\lambda, \omega) = -\omega_{\mathrm{Eq}}(\lambda)$, where ω_{Eq} corresponds to the equilibrium frequency vector defined in (3.32). As a consequence, nontrivial solutions are constructed when the scalar parameter λ is selected in the final Cantor set

$$\mathcal{C}_\infty^\varepsilon = \Big\{\lambda \in (\lambda_0, \lambda_1) \quad \text{s.t.} \quad (\lambda, \omega(\lambda, \varepsilon)) \in \mathcal{G}_\infty^\gamma \quad \text{with} \quad \alpha_\infty(\lambda, \omega(\lambda, \varepsilon)) = -\omega_{\mathrm{Eq}}(\lambda)\Big\}.$$

The measure of this set will be discussed in Section 7.2.

7.1 Nash-Moser scheme

In this section we implement the Nash-Moser scheme, which is a modified Newton method implemented with a suitable Banach scales and through a frequency cut-off. Basically, it consists in a recursive construction of approximate solutions to the equation $\mathcal{F}\big(i, \alpha, \lambda, \omega, \varepsilon\big) = 0$ where the functional $\mathcal{F}$ is defined in (5.23). At each step of this scheme, we need to construct an approximate inverse of the linearized operator at a state near the equilibrium by applying the reduction procedure developed in Section 6. This enables to get the result of Theorem 5.1 with the suitable tame estimates associated to the final loss of regularity $\overline{\sigma}$ that could be arranged to be large enough. We point out that $\overline{\sigma}$ depends only on the shape of the Cantor set through the parameters τ_1, τ_2, d and on the non degeneracy of the equilibrium frequency through $q = 1 + q_0$, where q_0 be defined in Lemma 3.5. However, $\overline{\sigma}$ is independent of the regularity of the solutions that we want to construct. Now, we shall fix the following parameters needed to implement Nash-Moser scheme and related to the geometry of the

Cantor sets encoded in τ_1, τ_2, d fixed by (4.2) and to the parameter $q = q_0 + 1$,

$$\left\{\begin{array}{rcl} \overline{a} & = & \tau_2 + 2 \\ \mu_1 & = & 3q(\tau_2+2) + 6\overline{\sigma} + 6 \\ a_1 & = & 6q(\tau_2+2) + 12\overline{\sigma} + 15 \\ a_2 & = & 3q(\tau_2+2) + 6\overline{\sigma} + 9 \\ \mu_2 & = & 2q(\tau_2+2) + 5\overline{\sigma} + 7 \\ s_h & = & s_0 + 4q(\tau_2+2) + 9\overline{\sigma} + 11 \\ b_1 & = & 2s_h - s_0. \end{array}\right. \tag{7.1}$$

The numbers a_1 and a_2 will describe the rate of convergence for the regularity s_0 and $s_0 + \overline{\sigma}$, respectively. They appear in the statements $(\mathcal{P}1)_n$ and $(\mathcal{P}2)_n$ in the Proposition 7.1. The parameter μ_1 controls the norm inflation at the high regularity index b_1 and appears in $(\mathcal{P}3)_n$. As to the parameter $\overline{a}$, it is linked to the thickness of a suitable enlargement of the intermediate Cantor sets, needed to construct classical extensions of our approximate solutions. Finally, the numbers μ_2 and s_h corresponds to those already encountered before in the reduction of the linearized operator and are now fixed to their minimal required values. In particular, we recall that μ_2 corresponds to the rate of convergence of the error terms emerging in the almost reducibility of the linearized operator, for instance we refer to Theorem 5.1. We should emphasize that, by taking $\overline{\sigma}$ large enough, the choice for μ_2 and s_h done in (7.1) enables to cover all the required assumptions in (6.24) and (6.244). Another assumption that we need to fix is related to γ, N_0 and ε

$$0 < a < \tfrac{1}{\mu_2+q+2}, \quad \gamma := \varepsilon^a, \quad N_0 := \gamma^{-1}. \tag{7.2}$$

This constraint is required for the measuring the final Cantor set and to check that it is massive, for more details we refer to Proposition 7.2.
We shall start with defining the finite dimensional subspaces where the approximate solutions are expected to live with controlled estimates. Consider the space,

$$E_n := \Big\{\mathfrak{I} = (\Theta, I, z) \quad \text{s.t.} \quad \Theta = \Pi_n\Theta, \quad I = \Pi_n I \quad \text{and} \quad z = \Pi_n z\Big\},$$

where Π_n is the projector defined by

$$f(\varphi,\theta) = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}} f_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\theta)} \quad \Rightarrow \quad \Pi_n f(\varphi,\theta) = \sum_{\langle l,j\rangle \leqslant N_n} f_{l,j} e^{\mathrm{i}(l\cdot\varphi + j\theta)},$$

where the sequence (N_n) is defined in (5.82). We observe that the same definition applies without ambiguity when the functions depend only on φ such as the action and the angles unknowns. The main result of this section is to prove the following induction statement.

Proposition 7.1 (Nash-Moser)**.** *Let $(\tau_1, \tau_2, q, d, s_0)$ satisfy* (4.1) *and* (4.2)*. Consider the parameters fixed by* (7.1) *and* (7.2)*. There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ we get for all $n \in \mathbb{N}$ the following properties,*

$(\mathcal{P}1)_n$ *There exists a q-times differentiable function*

$$\begin{array}{rrcl} W_n : & \mathcal{O} & \to & E_{n-1} \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ & (\lambda,\omega) & \mapsto & \big(\mathfrak{I}_n, \alpha_n - \omega, 0\big) \end{array}$$

satisfying

$$W_0 = 0 \quad \textit{and} \quad \textit{for} \quad n \geqslant 1, \quad \|W_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}.$$

By setting

$$U_0 = \Big((\varphi,0,0), \omega, (\lambda,\omega)\Big) \quad \text{and} \quad \textit{for} \quad n \in \mathbb{N}^*, \, U_n = U_0 + W_n \quad \text{and} \quad H_n = U_n - U_{n-1},$$

then

$$\forall s \in [s_0, S], \, \|H_1\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} \quad \textit{and} \quad \forall\, 2 \leqslant k \leqslant n, \, \|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_{k-1}^{-a_2}.$$

$(\mathcal{P}2)_n$ *Define*

$$i_n=(\varphi,0,0)+\mathfrak{I}_n,\quad \gamma_n=\gamma(1+2^{-n})\in[\gamma,2\gamma].$$

The embedded torus i_n *statisfies the reversibility condition*

$$\mathfrak{S}i_n(\varphi)=i_n(-\varphi),$$

where the involution $\mathfrak{S}$ *is defined in* (5.14)*. Introduce*

$$\mathcal{A}_0^\gamma=\mathcal{O}\quad \textit{and}\quad \mathcal{A}_{n+1}^\gamma=\mathcal{A}_n^\gamma\cap\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n),$$

where $\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$ *is described in Proposition* 6.6 *and consider the open sets*

$$\forall \mathtt{r}>0,\quad \mathrm{O}_n^{\mathtt{r}}:=\Big\{(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad \mathtt{dist}\big((\lambda,\omega),\mathcal{A}_n^\gamma\big)<\mathtt{r}N_n^{-\overline{a}}\Big\},$$

where $\mathtt{dist}(x,A)=\inf\limits_{y\in A}\|x-y\|$*. Then we have the following estimate*

$$\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^\gamma}\leqslant C_*\varepsilon N_{n-1}^{-a_1}.$$

$(\mathcal{P}3)_n$ $\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}.$

Remark 7.1. *Let* O *be an open subset of* $\mathcal{O}$*. Since* $\forall n\in\mathbb{N},\gamma_n\in[\gamma,2\gamma]$*, then the norms* $\|\cdot\|_{q,s}^{\gamma,\mathrm{O}}$ *and* $\|\cdot\|_{q,s}^{\gamma_n,\mathrm{O}}$*. are equivalent uniformly in* n*.*

Proof. • **Initialization :** By construction, $U_0=\Big((\varphi,0,0),\omega,(\lambda,\omega)\Big)$ and the flat torus $i_{\text{flat}}(\varphi)=(\varphi,0,0)$ satisfies obviously the reversibility condition. By (5.23), we have

$$\mathcal{F}(U_0)=\varepsilon\begin{pmatrix}-\partial_I\mathcal{P}_\varepsilon((\varphi,0,0))\\ \partial_\vartheta\mathcal{P}_\varepsilon((\varphi,0,0))\\ -\partial_\theta\nabla_z\mathcal{P}_\varepsilon((\varphi,0,0))\end{pmatrix}.$$

Using Lemma 5.2, we get

$$\forall s\geqslant 0,\quad \|\mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon,\tag{7.3}$$

up to take C_* large enough. The properties $(\mathcal{P}1)_0$, $(\mathcal{P}2)_0$ and $(\mathcal{P}3)_0$ then follow immediately since $N_{-1}=1$ and $\mathrm{O}_0^\gamma=\mathcal{O}$ and by setting $W_0=0$.

• **Induction step :** Given $n\in\mathbb{N}$, assume that $(\mathcal{P}1)_k$, $(\mathcal{P}2)_k$ and $(\mathcal{P}3)_k$ are true for all $k\in[\![0,n]\!]$ and let us check them at the next order $n+1$. Introduce the linearized operator of $\mathcal{F}$ at the state (i_n,α_n)

$$L_n:=L_n(\lambda,\omega):=d_{i,\alpha}\mathcal{F}\big(i_n(\lambda,\omega),\alpha_n(\lambda,\omega),(\lambda,\omega)\big).$$

In order to construct the next approximation U_{n+1}, we need an approximate right inverse for L_n. Its construction was performed along the preceding sections and we refer to Theorem 5.1 for a precise statement. To apply this result and get some bounds on U_{n+1} we need to establish first some intermediate results connected to the smallness condition and to some Cantor set inclusions.

► **Smallness/boundedness properties.** First of all, remark that the parameters conditions (6.3) are automatically satisfied by (7.1). Then, provided that the smallness assumption (6.374) is satisfied, Proposition 6.6 applies. It remains to check that (6.374) is satisfied. According to the first condition in (7.2) and choosing ε small enough, we can ensure

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}=\varepsilon^{1-a(\mu_2+q+2)}\leqslant\varepsilon_0\tag{7.4}$$

for some a priori fixed $\varepsilon_0>0$. Therefore the first assumption in (6.374) holds. We now turn to the second assumption. Since from (7.1) $b_1=2s_h-s_0$, then by interpolation inequality in Lemma 4.1, we have

$$\|H_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\lesssim\left(\|H_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)^{\frac12}\left(\|H_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)^{\frac12}.\tag{7.5}$$

Besides, by using $(\mathcal{P}1)_n$, we find

$$\forall s\in[s_0,S],\quad \|H_1\|_{q,s}^{\gamma,\mathcal{O}}\leqslant \tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}\quad\text{and}\quad \|H_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}. \tag{7.6}$$

Now $(\mathcal{P}3)_n$ and $(\mathcal{P}3)_{n-1}$ imply

$$\begin{aligned}\|H_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}&=\|U_n-U_{n-1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&=\|W_n-W_{n-1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&\leqslant\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}+\|W_{n-1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&\leqslant 2C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}.\end{aligned}$$

Putting together the foregoing estimates into (7.5) gives for $n\geqslant 2$,

$$\|H_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant CC_*\varepsilon\gamma^{-1}N_{n-1}^{\frac12(\mu_1-a_2)} \tag{7.7}$$

and for $n=1$,

$$\|H_1\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant \tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{7.8}$$

Now from (7.1) we infer

$$a_2\geqslant\mu_1+2. \tag{7.9}$$

Thus, by (7.2) and Lemma A.1, we get for small ε

$$\begin{aligned}\|W_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}&\leqslant\|H_1\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}+\sum_{k=2}^n\|H_k\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&\leqslant\tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}+CC_*\varepsilon\gamma^{-1}\sum_{k=0}^n N_k^{-1}\\&\leqslant\tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}+CN_0^{-1}C_*\varepsilon\gamma^{-1}\\&\leqslant C_*\varepsilon^{1-a(1+q\overline{a})}.\end{aligned}$$

One can check from (7.1) and (7.2) that

$$a\leqslant\tfrac{1}{2(1+q\overline{a})} \tag{7.10}$$

and therefore, by choosing ε small enough and since $\overline{\sigma}\geqslant\sigma$, we get

$$\begin{aligned}\|\mathfrak{I}_n\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}}\leqslant\|W_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}&\leqslant C_*\varepsilon^{\frac12}\\&\leqslant 1.\end{aligned}$$

As we have already mentioned, the parameter $\overline{\sigma}$ is the final loss of regularity constructed in Theorem 5.1 and depending only on the parameters τ_1,τ_2,q and d but it is independent of the state and the regularity. Hence it can be selected large enough such that $s_0+\overline{\sigma}\geqslant\overline{s}_h+\sigma_4$ where $\overline{s}_h$ and σ_4 are respectively defined in (6.24) and Proposition 6.5. Then using (7.6) and Sobolev embeddings, we obtain

$$\forall n\geqslant 2,\quad \|H_n\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}. \tag{7.11}$$

► **Set inclusions.** From the previous point, Propositions 6.2, 6.5 and 6.6 apply and allow us to perform the reduction of the linearized operator in the normal directions at the current step. Therefore, the sets $\mathcal{A}_k^\gamma$ for all $k\leqslant n+1$ are well-defined. We shall now prove the following inclusions needed later to establish suitable estimates for the extensions.

$$\mathcal{A}_{n+1}^\gamma\subset\mathrm{O}_{n+1}^{2\gamma}\subset\left(\mathcal{A}_{n+1}^{\frac{\gamma}{2}}\cap\mathrm{O}_n^\gamma\right). \tag{7.12}$$

Notice that the first inclusion is obvious by construction since $\mathrm{O}_{n+1}^{2\gamma}$ is an enlargement of $\mathcal{A}_{n+1}^\gamma$. It remains to prove the last inclusion. We have the inclusion

$$\forall k\in[\![0,n]\!],\quad \mathrm{O}_{k+1}^{2\gamma}\subset\mathrm{O}_k^\gamma. \tag{7.13}$$

Indeed, since by construction $\mathcal{A}_{k+1}^{\gamma}\subset\mathcal{A}_{k}^{\gamma}$ then taking $(\lambda,\omega)\in \mathrm{O}_{k+1}^{2\gamma}$ we have the following estimates

$$\begin{aligned}\mathtt{dist}\big((\lambda,\omega),\mathcal{A}_{k}^{\gamma}\big)&\leqslant \mathtt{dist}\big((\lambda,\omega),\mathcal{A}_{k+1}^{\gamma}\big)\\&<2\gamma N_{k+1}^{-\overline{a}}=2\gamma N_{k}^{-\overline{a}}N_{0}^{-\frac{1}{2}\overline{a}}\\&<\gamma N_{k}^{-\overline{a}},\end{aligned}$$

provided that $2N_{0}^{-\frac{1}{2}\overline{a}}<1$, which is true up to take N_0 large enough, that is in view of (7.2) for ε small enough. We shall now prove by induction in k that

$$\forall k\in[\![0,n+1]\!],\quad \mathrm{O}_{k}^{2\gamma}\subset\mathcal{A}_{k}^{\frac{\gamma}{2}}.\tag{7.14}$$

The case $k=0$ is trivial since $\mathrm{O}_{0}^{2\gamma}=\mathcal{O}=\mathcal{A}_{0}^{\frac{\gamma}{2}}$. Let us now assume that (7.14) is true for the index $k\in[\![0,n]\!]$ and let us check it at the next order. From (7.13) and (7.14), we obtain

$$\mathrm{O}_{k+1}^{2\gamma}\subset\mathrm{O}_{k}^{\gamma}\subset\mathrm{O}_{k}^{2\gamma}\subset\mathcal{A}_{k}^{\frac{\gamma}{2}}.$$

Therefore, we are left to check that

$$\mathrm{O}_{k+1}^{2\gamma}\subset\mathcal{G}_{k}\Big(\frac{\gamma_{k+1}}{2},\tau_1,\tau_2,i_k\Big).$$

Let $(\lambda,\omega)\in\mathcal{O}_{k+1}^{2\gamma}$, then by construction, there exists $(\lambda',\omega')\in\mathcal{A}_{k+1}^{\gamma}$ such that

$$\mathtt{dist}\,\big((\lambda,\omega),(\lambda',\omega')\big)<2\gamma N_{k+1}^{-\overline{a}}.$$

Hence, for all $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$ with $|l|\leqslant N_k$, we have by left triangle and Cauchy-Schwarz inequalities together with $(\lambda',\omega')\in\Lambda_{\infty,k}^{\gamma_{k+1},\tau_1}(i_k)$

$$\begin{aligned}\big|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_k)\big|&\geqslant\big|\omega'\cdot l+\mu_j^{\infty}(\lambda',\omega',i_k)\big|-|\omega-\omega'||l|-\big|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\big|\\&>\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}-2\gamma N_kN_{k+1}^{-\overline{a}}-\big|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\big|\\&>\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}-2\gamma N_{k+1}^{1-\overline{a}}-\big|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\big|.\end{aligned}$$

Using the Mean Value Theorem and the definition of $\mathrm{O}_{k+1}^{2\gamma}$ yields

$$\begin{aligned}\big|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\big|&\leqslant|(\lambda,\omega)-(\lambda',\omega')|\gamma^{-1}\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\\&\leqslant 2N_{k+1}^{-\overline{a}}\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

On the other hand,

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\leqslant\|\mu_j^{\infty}(i_k)-\Omega_j\|_q^{\gamma,\mathcal{O}}+\|\Omega_j\|_q^{\gamma,\mathcal{O}}.$$

Using the asymptotic (3.17) and the smoothness of $\lambda\mapsto I_j(\lambda)K_j(\lambda)$ for all $j\in\mathbb{N}^*$, one has

$$\|\Omega_j\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

Since (6.245) is satisfied by the previous point, we can apply (6.247) and obtain

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)-\Omega_j\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

Hence

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

It follows that

$$\big|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\big|\leqslant C\langle j\rangle N_{k+1}^{-\overline{a}}\leqslant C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}}.$$

Since $|l| \leqslant N_k \leqslant N_{k+1}$ and $\gamma_{k+1} \geqslant \gamma$, we obtain

$$\begin{aligned}\left|\omega \cdot l + \mu_j^\infty(\lambda, \omega, i_k)\right| &\geqslant \tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}} \\ &\geqslant \tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\left(1 - CN_{k+1}^{\tau_1+1-\overline{a}}\right).\end{aligned}$$

From (7.1) and (4.2) we infer

$$\overline{a} \geqslant \tau_2 + 2 \geqslant \tau_1 + 2 \tag{7.15}$$

and we can take N_0 sufficiently large to ensure

$$CN_{k+1}^{\tau_1+1-\overline{a}} \leqslant CN_0^{-1} < \tfrac{1}{2},$$

allowing to finally get

$$\left|\omega \cdot l + \mu_j^\infty(\lambda, \omega, i_k)\right| > \tfrac{\gamma_{k+1}\langle j\rangle}{2\langle l\rangle^{\tau_1}}.$$

This shows that, $(\lambda, \omega) \in \Lambda_{\infty,k}^{\frac{\gamma_{k+1}}{2},\tau_1}(i_k)$. Let us now check that $(\lambda, \omega) \in \mathcal{O}_{\infty,k}^{\frac{\gamma_{k+1}}{2},\tau_1}(i_k)$. For all $(l, j) \in \mathbb{Z}^d \times \mathbb{S}_0^c$ with $|l| \leqslant N_k$, we have by Cauchy-Schwarz inequality together with $(\lambda', \omega') \in \mathcal{O}_{\infty,k}^{\gamma_{k+1},\tau_1}(i_k)$

$$\begin{aligned}|\omega \cdot l + jc_{i_k}(\lambda, \omega)| &\geqslant \left|\omega' \cdot l + jc_{i_k}(\lambda', \omega')\right| - |\omega - \omega'||l| - |j|\left|c_{i_k}(\lambda, \omega) - c_{i_k}(\lambda', \omega')\right| \\ &> \tfrac{4\gamma_{k+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} - 2\gamma N_{k+1}^{1-\overline{a}} - \langle j\rangle\left|c_{i_k}(\lambda, \omega) - c_{i_k}(\lambda', \omega')\right|.\end{aligned}$$

Using the Mean Value Theorem and the definition of $\mathrm{O}_{k+1}^{2\gamma}$ yields

$$\left|c_{i_k}(\lambda, \omega) - c_{i_k}(\lambda', \omega')\right| \leqslant CN_{k+1}^{-\overline{a}}\|c_{i_k}\|_q^{\gamma,\mathcal{O}}.$$

Since (6.25) is satisfied by the previous point, we can apply (6.26) leading to

$$\begin{aligned}\|c_{i_k}\|_q^{\gamma,\mathcal{O}} &\leqslant \|c_{i_k} - V_0\|_q^{\gamma,\mathcal{O}} + \|V_0\|_q^{\gamma,\mathcal{O}} \\ &\leqslant C.\end{aligned}$$

Thus

$$\left|c_{i_k}(\lambda, \omega) - c_{i_k}(\lambda', \omega')\right| \leqslant C\gamma\gamma^{-1}N_{k+1}^{-\overline{a}} \leqslant C\gamma N_{k+1}^{1-\overline{a}}.$$

Therefore, we obtain from the definition of γ_k and $\upsilon \in (0, 1)$

$$\begin{aligned}|\omega \cdot l + jc_{i_k}(\lambda, \omega)| &> \tfrac{4\gamma_{k+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}} \\ &\geqslant \tfrac{4\gamma_{k+1}^\upsilon\langle j\rangle}{2^\upsilon\langle l\rangle^{\tau_1}}\left(2^\upsilon - C2^\upsilon N_{k+1}^{\tau_1+1-\overline{a}}\right).\end{aligned}$$

By the choice of $\overline{a}$ made in (7.15), we can ensure, up to take N_0 sufficiently large,

$$CN_{k+1}^{\tau_1+1-\overline{a}} \leqslant CN_0^{-1} < 1 - 2^{-\upsilon},$$

so that

$$|\omega \cdot l + jc_{i_k}(\lambda, \omega)| > \tfrac{4\gamma_{k+1}^\upsilon\langle j\rangle}{2^\upsilon\langle l\rangle^{\tau_1}}.$$

As a consequence, $(\lambda, \omega) \in \mathcal{O}_{\infty,k}^{\frac{\gamma_{k+1}}{2},\tau_1}(i_k)$. Let us now check that $(\lambda, \omega) \in \mathscr{O}_{\infty,k}^{\frac{\gamma_{k+1}}{2},\tau_1,\tau_2}(i_k)$. For all $(l, j, j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2$ with $|l| \leqslant N_k$, we have by the triangle and Cauchy-Schwarz inequalities together with $(\lambda', \omega') \in \mathscr{O}_{\infty,k}^{\gamma_{k+1},\tau_1,\tau_2}(i_k)$

$$\begin{aligned}\left|\omega \cdot l + \mu_j^\infty(\lambda, \omega, i_k) - \mu_{j_0}^\infty(\lambda, \omega, i_k)\right| &\geqslant \left|\omega' \cdot l + \mu_j^\infty(\lambda', \omega', i_k) - \mu_{j_0}^\infty(\lambda', \omega', i_k)\right| - |\omega - \omega'||l| \\ &\quad - \left|\mu_j^\infty(\lambda, \omega, i_k) - \mu_{j_0}^\infty(\lambda, \omega, i_k) + \mu_{j_0}^\infty(\lambda', \omega', i_k) - \mu_j^\infty(\lambda', \omega', i_k)\right| \\ &> \tfrac{2\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} - 2\gamma N_{k+1}^{1-\overline{a}} \\ &\quad - \left|\mu_j^\infty(\lambda, \omega, i_k) - \mu_{j_0}^\infty(\lambda, \omega, i_k) + \mu_{j_0}^\infty(\lambda', \omega', i_k) - \mu_j^\infty(\lambda', \omega', i_k)\right|.\end{aligned}$$

We recall by virtue of Proposition 6.5 that

$$\mu_j^\infty(\lambda,\omega,i_k)=\mu_j^0(\lambda,\omega,i_k)+r_j^\infty(\lambda,\omega,i_k).$$

Thus

$$\begin{aligned}&\big|\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)+\mu_{j_0}^\infty(\lambda',\omega',i_k)-\mu_j^\infty(\lambda',\omega',i_k)\big|\\&\leqslant\big|\mu_j^0(\lambda,\omega,i_k)-\mu_{j_0}^0(\lambda,\omega,i_k)+\mu_{j_0}^0(\lambda',\omega',i_k)-\mu_j^0(\lambda',\omega',i_k)\big|\\&\quad+\big|r_j^\infty(\lambda,\omega,i_k)-r_j^\infty(\lambda',\omega',i_k)\big|+\big|r_{j_0}^\infty(\lambda,\omega,i_k)-r_{j_0}^\infty(\lambda',\omega',i_k)\big|.\end{aligned}$$

According to the Mean Value Theorem, (6.264) and the definition of $\mathrm{O}_{k+1}^{2\gamma}$ we find

$$\big|\mu_j^0(\lambda,\omega,i_k)-\mu_{j_0}^0(\lambda,\omega,i_k)+\mu_{j_0}^0(\lambda',\omega',i_k)-\mu_j^0(\lambda',\omega',i_k)\big|\leqslant\gamma CN_{k+1}^{1-\overline{a}}\langle j-j_0\rangle.$$

Applying once again the Mean Value Theorem, (6.248), (7.4) and the definition of $\mathrm{O}_{n+1}^{2\gamma}$ yields

$$\big|r_j^\infty(\lambda,\omega,i_k)-r_j^\infty(\lambda',\omega',i_k)\big|\leqslant C\gamma N_{k+1}^{-\overline{a}}\varepsilon\gamma^{-2}\leqslant\gamma CN_{k+1}^{1-\overline{a}}\langle j-j_0\rangle.$$

Putting together the foregoing estimates and the facts that $|l|\leqslant N_k$ and $\gamma_{k+1}\geqslant\gamma$ we infer

$$\big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)\big|\geqslant\tfrac{\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big(2-CN_{k+1}^{\tau_2+1-\overline{a}}\Big).$$

By virtue of (7.15) and taking N_0 sufficiently large we get

$$CN_n^{\tau_2+1-\overline{a}}\leqslant CN_0^{-1}<1.$$

This implies

$$\big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)\big|>\tfrac{2\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

As a consequence, we deduce that $(\lambda,\omega)\in\mathscr{O}_{\infty,k}^{\frac{\gamma_{k+1}}{2},\tau_1,\tau_2}(i_n)$. Finally, $(\lambda,\omega)\in\mathcal{G}_k\big(\frac{\gamma_{k+1}}{2},\tau_1,\tau_2,i_k\big)$ and therefore $(\lambda,\omega)\in\mathcal{A}_{k+1}^{\frac{\gamma}{2}}$. This achieves the induction proof of (7.14).

► **Construction of the next approximation.** We are now going to construct the next approximation U_{n+1} by using a modified Nash-Moser scheme. The assumption (6.374) being satisfied, we can apply Theorem 5.1 with L_n and obtain the existence of an operator $\mathrm{T}_n:=\mathrm{T}_n(\lambda,\omega)$ well-defined in the whole set of parameters $\mathcal{O}$ and satisfying the following estimates

$$\forall s\in[s_0,S],\quad\|\mathrm{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\Big(\|\rho\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\tag{7.16}$$

and

$$\|\mathrm{T}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.\tag{7.17}$$

Moreover, when it is restricted to the Cantor set $\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$, T_n is an approximate right inverse of L_n with suitable tame estimates needed later, see Theorem 5.1. Next we define,

$$\widetilde{U}_{n+1}:=U_n+\widetilde{H}_{n+1}\quad\text{with}\quad\widetilde{H}_{n+1}:=(\widehat{\mathfrak{I}}_{n+1},\widehat{\alpha}_{n+1},0):=-\boldsymbol{\Pi}_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\in E_n\times\mathbb{R}^d\times\mathbb{R}^{d+1},$$

where $\boldsymbol{\Pi}_n$ is defined by

$$\boldsymbol{\Pi}_n(\mathfrak{I},\alpha,0)=(\Pi_n\mathfrak{I},\alpha,0)\quad\text{and}\quad\boldsymbol{\Pi}_n^\perp(\mathfrak{I},\alpha,0)=(\Pi_n\mathfrak{I},0,0).\tag{7.18}$$

Notice that the projectors Π_n are reversibility preserving due to the symmetry with respect to the Fourier modes. Then, using the reversibility of T_n together with (5.24) and Lemma 2.2, one deduces from $\mathfrak{S}i_n(\varphi)=i_n(-\varphi)$ that

$$\mathfrak{S}\widehat{\mathfrak{I}}_{n+1}(\varphi)=\widehat{\mathfrak{I}}_{n+1}(-\varphi).\tag{7.19}$$

Note that U_n is defined in the full set $\mathcal{O}$ and so does $\widetilde{U}_{n+1}$. Nevertheless, we will not be working with this natural extension but rather with a suitable localized version of it around the Cantor set $\mathcal{A}_{n+1}^{\gamma}$. Doing so, we shall get a nice decay property allowing the scheme to converge. Now, introduce the quadratic function

$$Q_n = \mathcal{F}(U_n + \widetilde{H}_{n+1}) - \mathcal{F}(U_n) - L_n \widetilde{H}_{n+1}, \tag{7.20}$$

then simple transformations give

$$\begin{aligned}
\mathcal{F}(\widetilde{U}_{n+1}) &= \mathcal{F}(U_n) - L_n \mathbf{\Pi}_n \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + Q_n \\
&= \mathcal{F}(U_n) - L_n \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + L_n \mathbf{\Pi}_n^{\perp} \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + Q_n \\
&= \mathcal{F}(U_n) - \Pi_n L_n \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + (L_n \mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp} L_n) \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + Q_n \\
&= \Pi_n^{\perp} \mathcal{F}(U_n) - \Pi_n (L_n \mathrm{T}_n - \mathrm{Id}) \Pi_n \mathcal{F}(U_n) + (L_n \mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp} L_n) \mathrm{T}_n \Pi_n \mathcal{F}(U_n) + Q_n.
\end{aligned} \tag{7.21}$$

In the sequel we shall prove

$$\|\mathcal{F}(U_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{\gamma}} \leqslant C_* \varepsilon N_n^{-a_1},$$

with U_{n+1} a suitable extension of $\widetilde{U}_{n+1}|_{\mathrm{O}_{n+1}^{\gamma}}$.

► **Estimates of $\mathcal{F}(\widetilde{U}_{n+1})$.** We shall now estimate $\mathcal{F}(\widetilde{U}_{n+1})$ with the norm $\|\cdot\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}$ by using (7.21). The localization in $\mathrm{O}_{n+1}^{2\gamma}$ is required for the classical extension in the next point, see (7.48).
➢ *Estimate of* $\Pi_n^{\perp}\mathcal{F}(U_n)$. We apply Taylor formula combined with (5.23) and Lemma 5.2 together with (7.3) and $(\mathcal{P}1)_n$. Therefore, we obtain

$$\begin{aligned}
\forall s \geqslant s_0, \quad \|\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_n^{\gamma}} &\leqslant \|\mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{F}(U_n) - \mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathrm{O}_n^{\gamma}} \\
&\lesssim \varepsilon + \|W_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}.
\end{aligned} \tag{7.22}$$

As a consequence, (7.2) and $(\mathcal{P}1)_n$ imply

$$\gamma^{-1}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} \leqslant 1. \tag{7.23}$$

From Lemma 4.1-(ii) and (7.22), we get

$$\begin{aligned}
\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} &\leqslant N_n^{s_0-b_1}\|\mathcal{F}(U_n)\|_{q,b_1}^{\gamma,\mathrm{O}_n^{\gamma}} \\
&\lesssim N_n^{\overline{\sigma}-b_1}\Big(\varepsilon + \|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big).
\end{aligned} \tag{7.24}$$

Now, $(\mathcal{P}3)_n$ together (5.82) and (7.2) yield

$$\begin{aligned}
\varepsilon + \|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \varepsilon\left(1 + C_*\gamma^{-1}N_{n-1}^{\mu_1}\right) \\
&\leqslant 2C_*\varepsilon N_n^{\frac{2}{3}\mu_1+1}.
\end{aligned} \tag{7.25}$$

By putting together (7.25) and (7.24) and by making appeal to (7.13), we infer for any $n \in \mathbb{N}$,

$$\begin{aligned}
\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} &\leqslant \|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} \\
&\lesssim C_*\varepsilon N_n^{s_0+\frac{2}{3}\mu_1+1-b_1}.
\end{aligned} \tag{7.26}$$

Remark that one also obtains, combining (7.22) and (7.25),

$$\|\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}} \leqslant C_*\varepsilon N_n^{\overline{\sigma}+\frac{2}{3}\mu_1+1}. \tag{7.27}$$

➢ *Estimate of* $\Pi_n(L_n\mathrm{T}_n - \mathrm{Id})\Pi_n\mathcal{F}(U_n)$. In view of (7.14), one has

$$\mathrm{O}_{n+1}^{2\gamma} \subset \mathcal{A}_{n+1}^{\frac{\gamma}{2}} \subset \mathcal{G}_n\Big(\frac{\gamma_{n+1}}{2}, \tau_1, \tau_2, i_n\Big).$$

Then, applying Theorem 5.1, we can write

$$\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)=\mathscr{E}_{1,n}+\mathscr{E}_{2,n}+\mathscr{E}_{3,n},$$

with

$$\begin{aligned}\mathscr{E}_{1,n}&:=\Pi_n\mathcal{E}_1^{(n)}\Pi_n\mathcal{F}(U_n),\\ \mathscr{E}_{2,n}&:=\Pi_n\mathcal{E}_2^{(n)}\Pi_n\mathcal{F}(U_n),\\ \mathscr{E}_{3,n}&:=\Pi_n\mathcal{E}_3^{(n)}\Pi_n\mathcal{F}(U_n)\end{aligned}$$

where $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ satisfy the estimates (5.116), (5.117) and (5.118) respectively. By (7.13), we get

$$\|\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant\|\mathscr{E}_{1,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+\|\mathscr{E}_{2,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+\|\mathscr{E}_{3,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}.\tag{7.28}$$

We shall first focus on $\mathscr{E}_{1,n}$. We need the following interpolation-type inequality

$$\begin{aligned}\|\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}&\leqslant\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}+\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\\ &\leqslant N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+N_n^{s_0-b_1}\|\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}.\end{aligned}\tag{7.29}$$

Combining (5.116), (7.29), $(\mathcal{P}_1)_n$, (7.4) and (7.27), we obtain

$$\begin{aligned}\|\mathscr{E}_{1,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}&\lesssim\gamma^{-1}\|\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\left(1+\|\mathfrak{I}_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)\\ &\lesssim\gamma^{-1}N_n^{\overline{\sigma}}\left(N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+N_n^{s_0-b_1}\|\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\right)\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}\left(1+\|W_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)\\ &\lesssim C_*\varepsilon\left(N_n^{2\overline{\sigma}-\frac{4}{3}a_1}+N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-\frac{2}{3}a_1-b_1}\right).\end{aligned}\tag{7.30}$$

We now turn to $\mathscr{E}_{2,n}$ and $\mathscr{E}_{3,n}$. Applying (5.117) with $b=b_1-s_0$ and using (7.4), $(\mathcal{P}_2)_n$ and $(\mathcal{P}_3)_n$, we get

$$\begin{aligned}\|\mathscr{E}_{2,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}&\lesssim\gamma^{-1}N_n^{s_0-b_1}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}+\varepsilon\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\right)\\ &\lesssim\gamma^{-1}N_n^{s_0-b_1}\left(\|\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}+\varepsilon N_n^{\overline{\sigma}}\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}\right)\\ &\lesssim C_*\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-b_1}+C_*\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-\frac{2}{3}a_1-b_1}\\ &\lesssim C_*\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-b_1}.\end{aligned}\tag{7.31}$$

Using the same techniques together with (5.118), (5.82), (7.2) and (7.4), we infer

$$\begin{aligned}\|\mathscr{E}_{3,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}&\lesssim N_n^{s_0-b_1}\gamma^{-2}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\right)\\ &\quad+\varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{\gamma}}\\ &\lesssim C_*\varepsilon\left(N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-b_1}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}\right).\end{aligned}\tag{7.32}$$

Putting together (7.28), (7.30), (7.31) and (7.31), we obtain

$$\|\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant CC_*\varepsilon\left(N_n^{2\overline{\sigma}-\frac{4}{3}a_1}+N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-b_1}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}\right).\tag{7.33}$$

For $n=0$, we deduce from (7.3),(7.4) and by slight modifications of the preceding computations

$$\begin{aligned}\|\Pi_0(L_0\mathrm{T}_0-\mathrm{Id})\Pi_0\mathcal{F}(U_0)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}}&\leqslant\|\mathscr{E}_{1,0}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+\|\mathscr{E}_{2,0}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}+\|\mathscr{E}_{3,0}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}}\\ &\lesssim\varepsilon^2\gamma^{-1}+\varepsilon\gamma^{-1}+\left(\varepsilon\gamma^{-2}N_0^{s_0-b_1}+\varepsilon^2\gamma^{-4}\right)\\ &\lesssim\varepsilon\gamma^{-2}.\end{aligned}\tag{7.34}$$

➢ *Estimate of* $\big(L_n\mathbf{\Pi}_n^\perp-\Pi_n^\perp L_n\big)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)$. Combining (5.60) and (5.23), we get for $H=(\widehat{\mathfrak{I}},\widehat{\alpha})$ with $\widehat{\mathfrak{I}}=(\widehat{\Theta},\widehat{I},\widehat{z})$,

$$L_nH=\omega\cdot\partial_\varphi\widehat{\mathfrak{I}}-(0,0,\partial_\theta\mathrm{L}(\lambda)\widehat{z})-\varepsilon d_iX_{\mathcal{P}_\varepsilon}(i_n)\widehat{\mathfrak{I}}-(\widehat{\alpha},0,0). \tag{7.35}$$

Using (7.18) and the fact that $\omega\cdot\partial_\varphi$ and $\partial_\theta\mathrm{L}(\lambda)$ are diagonal leading to $[\Pi_n^\perp,\omega\cdot\partial_\varphi]=[\Pi_n^\perp,\partial_\theta\mathrm{L}(\lambda)]=0$, one has for $H=(\widehat{\mathfrak{I}},\widehat{\alpha})$,

$$\Big(L_n\mathbf{\Pi}_n^\perp-\Pi_n^\perp L_n\Big)H=-\varepsilon[d_iX_{\mathcal{P}_\varepsilon}(i_n),\Pi_n^\perp]\widehat{\mathfrak{I}}.$$

In view of Lemma 5.2-(ii), Lemma 4.3, (7.13) and $(\mathcal{P}1)_n$ we get

$$\left\|\Big(L_n\mathbf{\Pi}_n^\perp-\Pi_n^\perp L_n\Big)H\right\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\lesssim\varepsilon N_n^{s_0-b_1}\left(\|\widehat{\mathfrak{I}}\|_{q,b_1+1}^{\gamma,\mathrm{O}_n^\gamma}+\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\widehat{\mathfrak{I}}\|_{q,s_0+1}^{\gamma,\mathrm{O}_n^\gamma}\right).$$

Consequently,

$$\begin{aligned}\mathtt{N}_{\mathrm{com}}(s_0):=\left\|\Big(L_n\mathbf{\Pi}_n^\perp-\Pi_n^\perp L_n\Big)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\right\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}&\lesssim\varepsilon N_n^{s_0-b_1}\|\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,b_1+1}^{\gamma,\mathrm{O}_n^\gamma}\\&\quad+\varepsilon N_n^{s_0-b_1}\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s_0+1}^{\gamma,\mathrm{O}_n^\gamma}.\end{aligned}$$

Hence, gathering (7.16), Lemma 4.1, Sobolev embeddings, (7.4), (7.2) and $(\mathcal{P}1)_n$ yields

$$\begin{aligned}\mathtt{N}_{\mathrm{com}}(s_0)&\lesssim\varepsilon\gamma^{-1}N_n^{s_0-b_1}\|\Pi_n\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}+1}^{\gamma,\mathrm{O}_n^\gamma}+\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}+1}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}\\&\quad+\varepsilon\gamma^{-1}N_n^{s_0-b_1}\|\mathfrak{I}_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}+1}^{\gamma,\mathrm{O}_n^\gamma}+\|\mathfrak{I}_n\|_{q,s_0+\overline{\sigma}+1}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}\right)\\&\lesssim\varepsilon N_n^{s_0+2-b_1}\left(\|\mathcal{F}(U_n)\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}+\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}\right).\end{aligned}$$

Applying Lemma 4.1-(ii), $(\mathcal{P}2)_n$ and (5.82), we infer

$$\begin{aligned}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}&\leqslant N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^\gamma}\\&\leqslant C_\ast\varepsilon N_n^{\overline{\sigma}}N_{n-1}^{-a_1}\\&\leqslant C_\ast\varepsilon N_n^{\overline{\sigma}-\frac{2}{3}a_1}.\end{aligned}$$

Added to (7.1), (7.27) and $(\mathcal{P}3)_n$, we obtain for $n\in\mathbb{N}$,

$$\|(L_n\mathbf{\Pi}_n^\perp-\Pi_n^\perp L_n)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant CC_\ast\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+3-b_1}. \tag{7.36}$$

➢ *Estimate of* Q_n. We apply Taylor formula together with (7.20) leading to

$$Q_n=\int_0^1(1-t)d_{i,\alpha}^2\mathcal{F}(U_n+t\widetilde{H}_{n+1})[\widetilde{H}_{n+1},\widetilde{H}_{n+1}]dt.$$

Thus, (7.35) and Lemma 5.2-(iii) allow to get

$$\|Q_n\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\lesssim\varepsilon\left(1+\|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}}+\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\right)\left(\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\right)^2. \tag{7.37}$$

Combining (7.14), (7.16), (7.22) and (7.23), we find for all $s\in[s_0,S]$

$$\begin{aligned}\|\widetilde{H}_{n+1}\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}&=\|\mathbf{\Pi}_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\\&\lesssim\gamma^{-1}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,s+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}+\|\mathfrak{I}_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^\gamma}\right)\\&\lesssim\gamma^{-1}\left(N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_n^\gamma}+N_n^{2\overline{\sigma}}\|\mathfrak{I}_n\|_{q,s}^{\gamma,\mathcal{O}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^\gamma}\right)\\&\lesssim\gamma^{-1}N_n^{2\overline{\sigma}}\left(\varepsilon+\|W_n\|_{q,s}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{7.38}$$

In the same way, according to (7.17), $(\mathcal{P}1)_n$ and $(\mathcal{P}2)_n$, we infer

$$\begin{aligned}\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} &\lesssim \gamma^{-1}N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\\ &\lesssim C_*\varepsilon\gamma^{-1}N_n^{\overline{\sigma}}N_{n-1}^{-a_1}.\end{aligned} \tag{7.39}$$

Choosing ε small enough and using $(\mathcal{P}1)_n$ and (7.39), we find

$$\begin{aligned}\|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}}+\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} &\leqslant C_*\varepsilon\gamma^{-1}+N_n^2\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\\ &\leqslant 1+C\varepsilon\gamma^{-1}N_n^{\overline{\sigma}+2}N_{n-1}^{-a_1}\\ &\leqslant 1+C\varepsilon\gamma^{-1}N_{n-1}^{3+\frac{3}{2}\overline{\sigma}-a_1}.\end{aligned}$$

Now notice that (7.1) implies

$$a_1\geqslant 3+\tfrac{3}{2}\overline{\sigma}. \tag{7.40}$$

Therefore, we obtain

$$\|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}}+\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant 2.$$

Hence, plugging this estimate and (7.39) into (7.37) and using (7.2) and (7.4), we find

$$\begin{aligned}\|Q_n\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} &\lesssim \varepsilon\left(\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\right)^2\\ &\leqslant \varepsilon N_n^4\left(\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\right)^2\\ &\lesssim \varepsilon C_*N_n^{2\overline{\sigma}+4}N_{n-1}^{-2a_1}.\end{aligned}$$

By using (5.82), we deduce when $n\geqslant 1$,

$$\|Q_n\|_{q,s_0}^{\gamma,\mathcal{O}_{n+1}^{2\gamma}}\leqslant CC_*\varepsilon N_n^{2\overline{\sigma}+4-\frac{4}{3}a_1}. \tag{7.41}$$

For $n=0$, we come back to (7.38) and (7.3) to obtain for all $s\in[s_0,S]$

$$\begin{aligned}\|\widetilde{H}_1\|_{q,s}^{\gamma,\mathrm{O}_1^{2\gamma}} &\lesssim \gamma^{-1}\|\Pi_0\mathcal{F}(U_0)\|_{q,s+\overline{\sigma}}^{\gamma,\mathrm{O}_0^{2\gamma}}\\ &\lesssim C_*\varepsilon\gamma^{-1}.\end{aligned} \tag{7.42}$$

Finally, the inequality (7.41) becomes for $n=0$,

$$\|Q_0\|_{q,s_0}^{\gamma,\mathrm{O}_0^{2\gamma}}\lesssim C_*\varepsilon^3\gamma^{-2}. \tag{7.43}$$

➢ *Conclusion.* Inserting (7.26), (7.33), (7.36) and (7.41), into (7.21) implies for $n\in\mathbb{N}^*$,

$$\|\mathcal{F}(\widetilde{U}_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant CC_*\varepsilon\left(N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-b_1}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}+N_n^{2\overline{\sigma}+4-\frac{4}{3}a_1}\right).$$

The parameters conditions stated in (7.1) give

$$\begin{cases} s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+2+a_1 &\leqslant& b_1\\ \overline{\sigma}+\frac{1}{3}a_1+2 &\leqslant& \mu_2\\ 2\overline{\sigma}+5 &\leqslant& \frac{1}{3}a_1.\end{cases} \tag{7.44}$$

Thus, by taking N_0 large enough, that is ε small enough, we obtain for $n\in\mathbb{N}$,

$$\begin{cases} CN_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-b_1} &\leqslant& \frac{1}{3}N_n^{-a_1}\\ CN_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1} &\leqslant& \frac{1}{3}N_n^{-a_1}\\ CN_n^{2\overline{\sigma}+4-\frac{4}{3}a_1} &\leqslant& \frac{1}{3}N_n^{-a_1},\end{cases} \tag{7.45}$$

which implies in turn that when $n \in \mathbb{N}^*$,

$$\|\mathcal{F}(\widetilde{U}_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} \leqslant C_* \varepsilon N_n^{-a_1}. \tag{7.46}$$

However, when $n = 0$, we plug (7.26), (7.34), (7.36) and (7.43) into (7.21) in order to get

$$\|\mathcal{F}(\widetilde{U}_1)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}} \leqslant CC_* \varepsilon \left(N_0^{s_0+2\overline{\sigma}+\frac{3}{2}\mu_1+1-b_1} + \varepsilon\gamma^{-2} + \varepsilon^2\gamma^{-2} \right).$$

From (7.45), one already has

$$CN_0^{s_0+2\overline{\sigma}+\frac{3}{2}\mu_1+1-b_1} \leqslant \tfrac{1}{3} N_0^{-a_1}.$$

Therefore, we need at this level to take ε small enough to ensure

$$C\left(\varepsilon\gamma^{-2} + \varepsilon^2\gamma^{-2}\right) \leqslant \tfrac{2}{3} N_0^{-a_1}.$$

This occurs since (7.2) and (7.1) imply

$$0 < a < \tfrac{1}{2+a_1}.$$

Hence

$$\|\mathcal{F}(\widetilde{U}_1)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}} \leqslant C_* \varepsilon N_0^{-a_1}.$$

This completes the proof of the estimates in $(\mathcal{P}2)_{n+1}$.

► **Extension and verification of** $(\mathcal{P}1)_{n+1} - (\mathcal{P}3)_{n+1}$. We shall now construct an extention of $\widetilde{H}_{n+1}$ living in the whole set of parameters and enjoying suitable decay properties. This is done by using the C^∞ cut-off function $\chi_{n+1} : \mathcal{O} \to [0,1]$ defined by

$$\chi_{n+1}(\lambda,\omega) = \begin{cases} 1 & \text{in } \mathrm{O}_{n+1}^{\gamma} \\ 0 & \text{in } \mathcal{O} \setminus \mathrm{O}_{n+1}^{2\gamma} \end{cases}$$

and satisfying the additional growth conditions

$$\forall \alpha \in \mathbb{N}^d, \quad |\alpha| \in [\![0,q]\!], \quad \|\partial_{\lambda,\omega}^{\alpha} \chi_{n+1}\|_{L^\infty(\mathcal{O})} \lesssim \left(\gamma^{-1} N_n^{\overline{a}}\right)^{|\alpha|}. \tag{7.47}$$

Next, we shall deal with the extension H_{n+1} of $\widetilde{H}_{n+1}$ defined by

$$H_{n+1}(\lambda,\omega) := \begin{cases} \chi_{n+1}(\lambda,\omega)\widetilde{H}_{n+1}(\lambda,\omega) & \text{in} \quad \mathrm{O}_{n+1}^{2\gamma} \\ 0 & \text{in} \quad \mathcal{O} \setminus \mathrm{O}_{n+1}^{2\gamma} \end{cases} \tag{7.48}$$

and the extension U_{n+1} of $\widetilde{U}_{n+1}$ by

$$U_{n+1} := U_n + H_{n+1}. \tag{7.49}$$

We remark that

$$H_{n+1} = \widetilde{H}_{n+1} \quad \text{and} \quad \mathcal{F}(U_{n+1}) = \mathcal{F}(\widetilde{U}_{n+1}) \quad \text{in} \quad \mathrm{O}_{n+1}^{\gamma}.$$

Looking at the first component of (7.49), one can write with obvious notations

$$i_{n+1} = i_n + \mathfrak{I}_{n+1}.$$

By the induction assumption $(\mathcal{P}2)_n$, (7.48) and (7.19), one has

$$\mathfrak{S} i_n(\varphi) = i_n(-\varphi) \quad \text{and} \quad \mathfrak{S}\mathfrak{I}_{n+1}(\varphi) = \mathfrak{I}_{n+1}(-\varphi).$$

Thus

$$\mathfrak{S} i_{n+1}(\varphi) = i_{n+1}(-\varphi). \tag{7.50}$$

Using Lemma 4.1-(iv) together with (7.47) and the fact that $H_{n+1} = 0$ in $\mathcal{O} \setminus \mathrm{O}_{n+1}^{2\gamma}$, we obtain

$$\forall s \geqslant s_0, \quad \|H_{n+1}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim N_n^{q\overline{a}} \|\widetilde{H}_{n+1}\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}. \tag{7.51}$$

Applying (7.51) and (7.39) we deduce that for $n \in \mathbb{N}^*$,

$$\begin{aligned}\|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant CN_n^{q\overline{a}}\|\widetilde{H}_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\\ &\leqslant CN_n^{q\overline{a}+\overline{\sigma}}\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\\ &\leqslant CC_*\varepsilon\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}-\frac{2}{3}a_1}.\end{aligned}$$

From (7.1), we have

$$a_2 = \tfrac{2}{3}a_1 - q\overline{a} - 2\overline{\sigma} - 1 \geqslant 1. \tag{7.52}$$

Therefore, choosing ε small enough, we obtain

$$\begin{aligned}\|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{-a_2}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_n^{-a_2}.\end{aligned} \tag{7.53}$$

As to the case $n=0$, we combine (7.51) and (7.42) to obtain, up to take C_* large enough,

$$\|H_1\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{7.54}$$

We now set

$$W_{n+1} := W_n + H_{n+1}, \tag{7.55}$$

then by construction, we infer

$$U_{n+1} = U_0 + W_{n+1}.$$

Moreover, applying $(\mathcal{P}1)_n$, (7.54) and (7.53) and Lemma A.1, we infer

$$\begin{aligned}\|W_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \|H_1\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} + \sum_{k=2}^{n+1}\|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + C_*\varepsilon\gamma^{-1}\sum_{k=0}^{\infty}N_k^{-1}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + CN_0^{-1}C_*\varepsilon\gamma^{-1}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}.\end{aligned}$$

This completes the proof of $(\mathcal{P}1)_{n+1}$. Now gathering (7.38), (7.51) and $(\mathcal{P}3)_n$ allows to write

$$\begin{aligned}\|W_{n+1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} + CN_n^{q\overline{a}}\|H_{n+1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1} + CC_*\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}}\left(\varepsilon + \|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}_n^{\gamma}}\right)\\ &\leqslant CC_*\varepsilon\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}+1+\frac{2}{3}\mu_1}.\end{aligned}$$

From (7.1), we can ensure the condition

$$q\overline{a} + 2\overline{\sigma} + 2 = \tfrac{\mu_1}{3}, \tag{7.56}$$

in order to get

$$\begin{aligned}\|W_{n+1}\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}_{n+1}^{\gamma}} &\leqslant CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{\mu_1}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_n^{\mu_1}\end{aligned}$$

by taking ε small enough and using (7.2). This proves $(\mathcal{P}3)_{n+1}$ and the proof of Proposition 7.1 is now complete. □

Once this sequence of approximate solutions is constructed, we may obtain a non-trivial solution by passing to the limit. This is possible due the decay properties given in Proposition 7.1. Actually, we obtain the following corollary.

Corollary 7.1. *There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the following assertions hold true. We consider the Cantor set $\mathcal{G}_\infty^\gamma$, depending on ε through γ, and defined by*

$$\mathcal{G}_\infty^\gamma := \bigcap_{n\in\mathbb{N}} \mathcal{A}_n^\gamma.$$

There exists a function

$$\begin{array}{rrcl} U_\infty : & \mathcal{O} & \to & \big(\mathbb{T}^d \times \mathbb{R}^d \times H_{\mathbb{S}}^\perp\big) \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ & (\lambda,\omega) & \mapsto & \big(i_\infty(\lambda,\omega), \alpha_\infty(\lambda,\omega), (\lambda,\omega)\big) \end{array}$$

such that

$$\forall (\lambda,\omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}(U_\infty(\lambda,\omega)) = 0.$$

In addition, i_∞ is reversible and $\alpha_\infty \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{R}^d)$ with

$$\alpha_\infty(\lambda,\omega) = \omega + \mathrm{r}_\varepsilon(\lambda,\omega) \quad \text{and} \quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{7.57}$$

Moreover, there exists a q-times differentiable function $\lambda \in (\lambda_0, \lambda_1) \mapsto \omega(\lambda,\varepsilon) \in \mathbb{R}^d$ with

$$\omega(\lambda,\varepsilon) = -\omega_{\mathrm{Eq}}(\lambda) + \bar{r}_\varepsilon(\lambda), \quad \|\bar{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \tag{7.58}$$

and

$$\forall \lambda \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\Big(U_\infty\big(\lambda, \omega(\lambda,\varepsilon)\big)\Big) = 0 \quad \text{and} \quad \alpha_\infty\big(\lambda, \omega(\lambda,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(\lambda),$$

where the Cantor set $\mathcal{C}_\infty^\varepsilon$ is defined by

$$\mathcal{C}_\infty^\varepsilon = \Big\{\lambda \in (\lambda_0,\lambda_1) \quad \text{s.t.} \quad \big(\lambda, \omega(\lambda,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma\Big\}. \tag{7.59}$$

Proof. Putting together (7.55) and (7.53), we infer

$$\|W_{n+1} - W_n\|_{q,s_0}^{\gamma,\mathcal{O}} = \|H_{n+1}\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant \|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_n^{-a_2}.$$

Thus, the telescopic series associated with the sequence $(W_n)_{n\in\mathbb{N}}$ is convergent, so the sequence itself converges. We denote its limit

$$W_\infty := \lim_{n\to\infty} W_n := (\mathfrak{I}_\infty, \alpha_\infty - \omega, 0, 0)$$

and

$$U_\infty := \big(i_\infty, \alpha_\infty, (\lambda,\omega)\big) = U_0 + W_\infty.$$

Passing to the limit in (7.50), one obtains the reversibility property

$$\mathfrak{S} i_\infty(\varphi) = i_\infty(-\varphi).$$

By the point $(\mathcal{P}2)_n$ of Proposition 7.1, we have for small ε

$$\forall (\lambda,\omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}\Big(i_\infty(\lambda,\omega), \alpha_\infty(\lambda,\omega), (\lambda,\omega), \varepsilon\Big) = 0, \tag{7.60}$$

with $\mathcal{F}$ the functional defined in (5.23). We highlight that the Cantor set $\mathcal{G}_\infty^\gamma$ depends on ε through γ and (7.2). By the point $(\mathcal{P}1)_n$ of the Proposition 7.1, we have

$$\alpha_\infty(\lambda,\omega) = \omega + \mathrm{r}_\varepsilon(\lambda,\omega) \quad \text{with} \quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}.$$

We now prove the second result and check the existence of solutions to the original Hamiltonian equation. First recall that the open set $\mathcal{O}$ is defined in (4.3) by

$$\mathcal{O}=(\lambda_0,\lambda_1)\times\mathscr{U}\quad\text{with}\quad\mathscr{U}=B(0,R_0)\quad\text{for some large } R_0>0,$$

where the ball $\mathscr{U}$ is taken to contain the equilibrium frequency vector $\lambda\mapsto\omega_{\rm Eq}(\lambda)$. According to (7.57), we deduce that for any $\lambda\in(\lambda_0,\lambda_1)$, the mapping $\omega\mapsto\alpha_\infty(\lambda,\omega)$ is invertible from $\mathscr{U}$ into its image $\alpha_\infty(\lambda,\mathscr{U})$ and we have

$$\widehat{\omega}=\alpha_\infty(\lambda,\omega)=\omega+\mathrm{r}_\varepsilon(\lambda,\omega)\Leftrightarrow\omega=\alpha_\infty^{-1}(\lambda,\widehat{\omega})=\widehat{\omega}+\widehat{\mathrm{r}}_\varepsilon(\lambda,\widehat{\omega}).$$

This gives the identity

$$\widehat{\mathrm{r}}_\varepsilon(\lambda,\widehat{\omega})=-\mathrm{r}_\varepsilon(\lambda,\omega),$$

which implies in turn after using successive differentiation and (7.57) that $\widehat{\mathrm{r}}_\varepsilon$ satisfies the estimate

$$\|\widehat{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_0^{q\overline{a}}.\tag{7.61}$$

We now set

$$\omega(\lambda,\varepsilon):=\alpha_\infty^{-1}(\lambda,-\omega_{\rm Eq}(\lambda))=-\omega_{\rm Eq}(\lambda)+\overline{\mathrm{r}}_\varepsilon(\lambda)\quad\text{with}\quad\overline{\mathrm{r}}_\varepsilon(\lambda):=\widehat{\mathrm{r}}_\varepsilon\big(\lambda,-\omega_{\rm Eq}(\lambda)\big).$$

As a consequence of (7.60), if we denote

$$\mathcal{C}_\infty^\varepsilon:=\Big\{\lambda\in(\lambda_0,\lambda_1)\quad\text{s.t.}\quad\big(\lambda,\omega(\lambda,\varepsilon)\big)\in\mathcal{G}_\infty^\gamma\Big\},$$

then we have

$$\forall\lambda\in\mathcal{C}_\infty^\varepsilon,\quad\mathcal{F}\Big(U_\infty\big(\lambda,\omega(\lambda,\varepsilon)\big)\Big)=0.$$

This gives a nontrivial reversible solution for the original Hamiltonian equation provided that $\lambda\in\mathcal{C}_\infty^\varepsilon$. Since all the derivatives up to order q of $\omega_{\rm Eq}$ are uniformly bounded on $[\lambda_0,\lambda_1]$, see Lemma 3.3-(vi), then by chain rule and (7.61), we obtain

$$\|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_0^{q\overline{a}}\quad\text{and}\quad\|\omega(\cdot,\varepsilon)\|_q^{\gamma,\mathcal{O}}\lesssim1+\varepsilon\gamma^{-1}N_0^{q\overline{a}}\lesssim1.\tag{7.62}$$

This ends the proof of Corollary 7.1. □

7.2 Measure of the final Cantor set

The purpose of this final section is to give a lower bound of the Lebesgue measure of the Cantor set $\mathcal{C}_\infty^\varepsilon$ constructed in Corollary 7.1 via (7.59). We show that this set is massive and asymptotically when $\varepsilon\to0$ it tends to be of full measure in (λ_0,λ_1). Note that Corollary 7.1 allows us to write the Cantor set $\mathcal{C}_\infty^\varepsilon$ in the following form

$$\mathcal{C}_\infty^\varepsilon:=\bigcap_{n\in\mathbb{N}}\mathcal{C}_n^\varepsilon\quad\text{where}\quad\mathcal{C}_n^\varepsilon:=\Big\{\lambda\in(\lambda_0,\lambda_1)\quad\text{s.t}\quad\big(\lambda,\omega(\lambda,\varepsilon)\big)\in\mathcal{A}_n^\gamma\Big\}.\tag{7.63}$$

The sets $\mathcal{A}_n^\gamma$ and the perturbed frequency vector $\omega(\lambda,\varepsilon)$ are respectively defined in Proposition 7.1 and in (7.57). The main result of this section reads as follows.

Proposition 7.2. *Let q_0 be defined as in Lemma 3.5 and assume that (7.1) and (7.2) hold with $q=q_0+1$. Assume the additional conditions*

$$\begin{cases}\tau_1>dq_0\\ \tau_2>\tau_1+dq_0\\ \upsilon=\frac{1}{q_0+3}.\end{cases}\tag{7.64}$$

Then, there exists $C>0$ such that

$$\big|\mathcal{C}_\infty^\varepsilon\big|\geqslant(\lambda_1-\lambda_0)-C\varepsilon^{\frac{a\upsilon}{q_0}}.$$

In particular,

$$\lim_{\varepsilon\to0}\big|\mathcal{C}_\infty^\varepsilon\big|=\lambda_1-\lambda_0.$$

The remainder of this section is devoted to the proof of Proposition 7.2. We shall begin by giving the proof using some a priori results. These results will be proved later in Lemmas 7.1, 7.2 and 7.3. We first give a short insight about the strategy to prove Proposition 7.2. The idea is to measure the complementary set of $\mathcal{C}_\infty^\varepsilon$ in (λ_0, λ_1). To proceed with, we write

$$(\lambda_0, \lambda_1) \setminus \mathcal{C}_\infty^\varepsilon = \big((\lambda_0, \lambda_1) \setminus \mathcal{C}_0^\varepsilon\big) \sqcup \bigsqcup_{n=0}^{\infty} \big(\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon\big). \tag{7.65}$$

The measure of each set which appears, we estimate it by using Lemma 3.6. We shall now give the proof of Proposition 7.2.

Proof. By choosing R_0 large enough, one can ensure using (7.58) that

$$\forall \lambda \in (\lambda_0, \lambda_1), \quad \omega(\lambda, \varepsilon) \in \mathscr{U} = B(0, R_0).$$

Indeed, $\mathscr{U}$ contains by construction the curve $\lambda \in (\lambda_0, \lambda_1) \mapsto \pm\omega_{\mathrm{Eq}}(\lambda)$ and by (7.58) and (7.2), one has

$$\sup_{\lambda \in (\lambda_0, \lambda_1)} |\omega(\lambda, \varepsilon) + \omega_{\mathrm{Eq}}(\lambda)| \leqslant \|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma, \mathcal{O}} \leqslant C\varepsilon\gamma^{-1}N_0^{q\overline{a}} = C\varepsilon^{1-a(1+q\overline{a})}.$$

Now, the conditions (7.1) and (7.2) imply in particular

$$0 < a < \frac{1}{1 + q\overline{a}}\cdot$$

Hence, by taking ε small enough, we find

$$\sup_{\lambda \in (\lambda_0, \lambda_1)} |\omega(\lambda, \varepsilon) + \omega_{\mathrm{Eq}}(\lambda)| \leqslant \|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma, \mathcal{O}} \leqslant 1.$$

As a consequence,

$$\mathcal{C}_0^\varepsilon = (\lambda_0, \lambda_1).$$

By (7.65), we can write

$$\begin{aligned}\Big|(\lambda_0, \lambda_1) \setminus \mathcal{C}_\infty^\varepsilon\Big| &\leqslant \sum_{n=0}^{\infty} \Big|\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon\Big| \\ &:= \sum_{n=0}^{\infty} \mathcal{S}_n.\end{aligned} \tag{7.66}$$

According to the notation introduced in Proposition 6.5 and Proposition 6.4 one may write

$$\begin{aligned}\mu_j^{\infty,n}(\lambda, \varepsilon) &:= \mu_j^\infty\big(\lambda, \omega(\lambda, \varepsilon), i_n\big) \\ &= \Omega_j(\lambda) + jr^{1,n}(\lambda, \varepsilon) + r_j^{\infty,n}(\lambda, \varepsilon),\end{aligned} \tag{7.67}$$

with

$$\begin{aligned}r^{1,n}(\lambda, \varepsilon) &:= c_n(\lambda, \varepsilon) - \Omega - I_1(\lambda)K_1(\lambda), \\ c_n(\lambda, \varepsilon) &:= c_{i_n}(\lambda, \omega(\lambda, \varepsilon)), \\ r_j^{\infty,n}(\lambda, \varepsilon) &:= r_j^\infty\big(\lambda, \omega(\lambda, \varepsilon), i_n\big).\end{aligned}$$

Coming back to (7.63) and using the Cantor sets introduced in Proposition 6.5, Proposition 6.6 and Proposition 6.2 one obtains by construction that for any $n \in \mathbb{N}$,

$$\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\} \\ |l| \leqslant N_n}} \mathcal{R}_{l,j}^{(0)}(i_n) \bigcup_{\substack{(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2 \\ |l| \leqslant N_n}} \mathcal{R}_{l,j,j_0}(i_n) \bigcup_{\substack{(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c \\ |l| \leqslant N_n}} \mathcal{R}_{l,j}^{(1)}(i_n), \tag{7.68}$$

with

$$\begin{aligned}
\mathcal{R}_{l,j}^{(0)}(i_n) &:= \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)|\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},\\
\mathcal{R}_{l,j,j_0}(i_n) &:= \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)|\leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\},\\
\mathcal{R}_{l,j}^{(1)}(i_n) &:= \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)|\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.
\end{aligned}$$

Notice that using the inclusion

$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C})$$

and the fact that $q=q_0+1$, one gets that for all $n\in\mathbb{N}$ and $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$, the curves

$$\begin{aligned}
&\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+c_n(\lambda,\varepsilon),\\
&\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon),\\
&\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)
\end{aligned}$$

have a C^{q_0} regularity. Then, applying Lemma 3.6 combined with Lemma 7.3 gives for any $n\in\mathbb{N}$,

$$\begin{aligned}
\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big| &\lesssim \gamma^{\frac{\upsilon}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\
\Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\
\Big|\mathcal{R}_{l,j,j_0}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j-j_0\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}.
\end{aligned}\tag{7.69}$$

Let us now move to the estimate of $\mathcal{S}_0$ and $\mathcal{S}_1$ defined in (7.66) that should be treated differently from the other terms. This is related to the discussion done at the beginning of the proof of Lemma 7.1 dealing with the validity of the estimate (7.74). By using Lemma 7.2, we find for all $k\in\{0,1\}$,

$$\mathcal{S}_k\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|j|\leqslant C_0\langle l\rangle,|l|\leqslant N_k}}\Big|\mathcal{R}_{l,j}^{(0)}(i_k)\Big|+\sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|j-j_0|\leqslant C_0\langle l\rangle,|l|\leqslant N_k\\\min(|j|,|j_0|)\leqslant c_2\gamma_1^{-\upsilon}\langle l\rangle^{\tau_1}}}\Big|\mathcal{R}_{l,j,j_0}(i_k)\Big|+\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|j|\leqslant C_0\langle l\rangle,|l|\leqslant N_k}}\Big|\mathcal{R}_{l,j}^{(1)}(i_k)\Big|.\tag{7.70}$$

Plugging (7.69) into (7.70) yields for all $k\in\{0,1\}$,

$$\begin{aligned}
\mathcal{S}_k\lesssim\gamma^{\frac{1}{q_0}}&\Bigg(\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}+\sum_{\substack{|j-j_0|\leqslant C_0\langle l\rangle\\\min(|j|,|j_0|)\leqslant c_2\gamma^{-\upsilon}\langle l\rangle^{\tau_1}}}|j-j_0|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}\Bigg)\\
&+\gamma^{\frac{\upsilon}{q_0}}\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
\max_{k\in\{0,1\}}\mathcal{S}_k&\lesssim\gamma^{\frac{1}{q_0}}\Bigg(\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg)+\gamma^{\frac{\upsilon}{q_0}}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}\\
&\lesssim\gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.
\end{aligned}\tag{7.71}$$

Notice that the last estimate is obtained provided that we choose the parameters τ_1 and τ_2 in the following way in order to make the series convergent

$$\tau_1>d\,q_0\quad\text{and}\quad\tau_2>\tau_1+d\,q_0.\tag{7.72}$$

This condition is exactly what we required in (7.64). Concerning the estimate of $\mathcal{S}_n$ for $n \geqslant 2$ in (7.66) we may use Lemma 7.1 and Lemma 7.2, in order to get

$$\mathcal{S}_n \leqslant \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |j|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n}} \left|\mathcal{R}_{l,j}^{(0)}(i_n)\right| + \sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |j-j_0|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n\\ \min(|j|,|j_0|)\leqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}}} \left|\mathcal{R}_{l,j,j_0}(i_n)\right| + \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |j|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n}} \left|\mathcal{R}_{l,j}^{(1)}(i_n)\right|.$$

Remark that if $|j-j_0| \leqslant C_0\langle l\rangle$ and $\min(|j|,|j_0|) \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\max(|j|,|j_0|) = \min(|j|,|j_0|) + |j-j_0| \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1} + C_0\langle l\rangle \lesssim \gamma^{-\upsilon}\langle l\rangle^{\tau_1}.$$

Therefore, (7.69) implies

$$\mathcal{S}_n \lesssim \gamma^{\frac{1}{q_0}}\left(\sum_{|l|>N_{n-1}} \langle l\rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-\upsilon}\sum_{|l|>N_{n-1}} \langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\right) + \gamma^{\frac{\upsilon}{q_0}}\sum_{|l|>N_{n-1}} \langle l\rangle^{-\frac{\tau_1}{q_0}}.$$

Under the assumption, we obtain (7.72)

$$\sum_{n=2}^{\infty} \mathcal{S}_n \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}. \tag{7.73}$$

Plugging (7.73) and (7.71) into (7.66) gives

$$\left|(\lambda_0,\lambda_1)\setminus\mathcal{C}_\infty^\varepsilon\right| \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}$$

provided that the condition (7.72) is satisfied. The condition (7.64) implies that

$$\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right) = \frac{\upsilon}{q_0}.$$

We then find, since $\gamma = \varepsilon^a$ according to (7.2),

$$\left|(\lambda_0,\lambda_1)\setminus\mathcal{C}_\infty^\varepsilon\right| \lesssim \varepsilon^{\frac{a\upsilon}{q_0}}.$$

This completes the proof of Proposition 7.2. □

Now we are left to prove Lemma 7.1 and Lemma 7.2 used in the proof of Proposition 7.2.

Lemma 7.1. *Let $n \in \mathbb{N}\setminus\{0,1\}$ and $l \in \mathbb{Z}^d$ such that $|l| \leqslant N_{n-1}$. Then the following assertions hold true.*

(i) For $j \in \mathbb{Z}$ with $(l,j) \neq (0,0)$, we get $\mathcal{R}_{l,j}^{(0)}(i_n) = \varnothing$.

(ii) For $(j,j_0) \in (\mathbb{S}_0^c)^2$ with $(l,j) \neq (0,j_0)$, we get $\mathcal{R}_{l,j,j_0}(i_n) = \varnothing$.

(iii) For $j \in \mathbb{S}_0^c$, we get $\mathcal{R}_{l,j}^{(1)}(i_n) = \varnothing$.

(iv) For any $n \in \mathbb{N}\setminus\{0,1\}$,

$$\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon = \bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ N_{n-1}<|l|\leqslant N_n}} \mathcal{R}_{l,j}^{(0)}(i_n) \cup \bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ N_{n-1}<|l|\leqslant N_n}} \mathcal{R}_{l,j,j_0}(i_n) \cup \bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ N_{n-1}<|l|\leqslant N_n}} \mathcal{R}_{l,j}^{(1)}(i_n).$$

Proof. In all the proof, we shall use the following estimate coming from (7.11), namely, for all $n \geqslant 2$,

$$\begin{aligned}\|i_n - i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} &\leqslant \|U_n - U_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant \|H_n\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}.\end{aligned} \tag{7.74}$$

The fact that the previous estimate is valid only for $n \geqslant 2$ is the reason why we had to treat the cases of $\mathcal{S}_0$ and $\mathcal{S}_1$ sparately in the proof of Proposition 7.2.

(i) We begin by proving that if $|l| \leqslant N_{n-1}$ and $(l,j) \neq (0,0)$, then $\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{R}_{l,j}^{(0)}(i_{n-1})$. Assume for a while this inclusion and let us check how this implies that $\mathcal{R}_{l,j}^{(0)}(i_n) = \varnothing$. In view of (7.68) one obtains

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{R}_{l,j}^{(0)}(i_{n-1}) \subset \mathcal{C}_{n-1}^{\varepsilon} \setminus \mathcal{C}_n^{\varepsilon}.$$

Now (7.68) implies in particular $\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{C}_n^{\varepsilon} \setminus \mathcal{C}_{n+1}^{\varepsilon}$ and thus we conclude

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \big(\mathcal{C}_n^{\varepsilon} \setminus \mathcal{C}_{n+1}^{\varepsilon}\big) \cap \big(\mathcal{C}_{n-1}^{\varepsilon} \setminus \mathcal{C}_n^{\varepsilon}\big) = \varnothing.$$

We now turn to the proof of the inclusion. Let us consider $\lambda \in \mathcal{R}_{l,j}^{(0)}(i_n)$. By construction, we get in particular that $\lambda \in \mathcal{C}_n^{\varepsilon} \subset \mathcal{C}_{n-1}^{\varepsilon}$. Moreover, by the triangle inequality, we obtain

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l + jc_{n-1}(\lambda,\varepsilon)\big| &\leqslant \big|\omega(\lambda,\varepsilon)\cdot l + jc_n(\lambda,\varepsilon)\big| + |j|\big|c_n(\lambda,\varepsilon) - c_{n-1}(\lambda,\varepsilon)\big| \\ &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C|j|\|c_{i_n} - c_{i_{n-1}}\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore, combining (6.30), (7.74), (7.2) and the fact tht $\sigma_4 \geqslant 2$, we infer

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l + jc_{n-1}(\lambda,\varepsilon)\big| &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon\langle j\rangle\|i_n - i_{n-1}\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} \\ &\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon^{2-a}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

In view of the definition of γ_n in Proposition 7.1-$(\mathcal{P}2)_n$ one gets

$$\exists c_0 > 0, \quad \forall n \in \mathbb{N}, \quad \gamma_{n+1}^{\upsilon} - \gamma_n^{\upsilon} \leqslant -c_0\,\gamma^{\upsilon}2^{-n}.$$

Now remark that (7.64), (7.1) and (7.2) imply

$$2 - a - a\upsilon > 1 \quad \text{and} \quad a_2 > \tau_1, \tag{7.75}$$

and therefore one gets $\sup_{n\in\mathbb{N}} 2^n N_{n-1}^{-a_2+\tau_1} < \infty$. It follows that, for ε small enough and $|l| \leqslant N_{n-1}$,

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l + jc_{n-1}(\lambda,\varepsilon)\big| &\leqslant \tfrac{4\gamma_n^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\tfrac{\langle j\rangle\gamma^{\upsilon}}{2^n\langle l\rangle^{\tau_1}}\Big(-4c_0 + C\varepsilon 2^n N_{n-1}^{-a_2+\tau_1}\Big) \\ &\leqslant \tfrac{4\gamma_n^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.\end{aligned}$$

Consequently $\lambda \in \mathcal{R}_{l,j}^{(0)}(i_{n-1})$ and this achieves the proof.

(ii) Let $(j,j_0) \in (\mathbb{S}_0^c)^2$ and $(l,j) \neq (0,j_0)$. If $j = j_0$ then by construction $\mathcal{R}_{l,j_0,j_0}(i_n) = \mathcal{R}_{l,0}^{(0)}(i_n)$ and then the result follows from the point (i). Now let us discuss the case when $j \neq j_0$. Similarly to the point (i), in order to get the result it is enough to check that $\mathcal{R}_{l,j,j_0}(i_n) \subset \mathcal{R}_{l,j,j_0}(i_{n-1})$. Let $\lambda \in \mathcal{R}_{l,j,j_0}(i_n)$ then from the definition of this set introduced in (7.68) we deduce that $\lambda \in \mathcal{C}_n^{\varepsilon} \subset \mathcal{C}_{n-1}^{\varepsilon}$ and

$$\big|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(\lambda,\varepsilon) - \mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} + \varrho_{j,j_0}^n(\lambda,\varepsilon), \tag{7.76}$$

where we set

$$\varrho_{j,j_0}^n(\lambda,\varepsilon) := \big|\mu_j^{\infty,n}(\lambda,\varepsilon) - \mu_{j_0}^{\infty,n}(\lambda,\varepsilon) - \mu_j^{\infty,n-1}(\lambda,\varepsilon) + \mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|.$$

Then coming back to (7.67), one gets

$$\begin{aligned}\varrho_{j,j_0}^n(\lambda,\varepsilon) \leqslant |j-j_0|\big|r^{1,n}(\lambda,\varepsilon) - r^{1,n-1}(\lambda,\varepsilon)\big| &+ \big|r_j^{\infty,n}(\lambda,\varepsilon) - r_j^{\infty,n-1}(\lambda,\varepsilon)\big| \\ &+ \big|r_{j_0}^{\infty,n}(\lambda,\varepsilon) - r_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|.\end{aligned} \tag{7.77}$$

In view of (6.179), (7.74), (7.2) and the fact that $\sigma_4 \geqslant \sigma_3$, one obtains

$$\begin{aligned}\left|r^{1,n}(\lambda,\varepsilon) - r^{1,n-1}(\lambda,\varepsilon)\right| &\lesssim \varepsilon\|i_n - i_{n-1}\|_{q,\overline{s}_h+\sigma_3}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon^2\gamma^{-1}N_{n-1}^{-a_2}\\ &\lesssim \varepsilon^{2-a}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

In a similar line, using (6.250), (7.74) and (7.2) yields

$$\begin{aligned}\left|r_j^{\infty,n}(\lambda,\varepsilon) - r_j^{\infty,n-1}(\lambda,\varepsilon)\right| &\lesssim \varepsilon\gamma^{-1}\|i_n - i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon^2\gamma^{-2}N_{n-1}^{-a_2}\\ &\lesssim \varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Inserting the preceding two estimates into (7.77) gives

$$\varrho_{j,j_0}^n(\lambda,\varepsilon) \lesssim \varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}. \tag{7.78}$$

Putting together (7.78) and (7.76) and using $\gamma_{n+1} = \gamma_n - \varepsilon^a 2^{-n-1}$, we deduce

$$\begin{aligned}\left|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(\lambda,\varepsilon) - \mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\right| \leqslant{}& \tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} - \varepsilon^a\langle j-j_0\rangle 2^{-n}\langle l\rangle^{-\tau_2}\\ &+ C\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Since $|l| \leqslant N_{n-1}$,we can write

$$-\varepsilon^a 2^{-n}\langle l\rangle^{-\tau_2} + C\varepsilon^{2(1-a)}N_{n-1}^{-a_2} \leqslant \varepsilon^a 2^{-n}\langle l\rangle^{-\tau_2}\Big(-1 + C\varepsilon^{2-3a}2^n N_{n-1}^{-a_2+\tau_2}\Big).$$

Now remark that (7.1) and (7.2) yield in particular

$$a_2 > \tau_2 \quad \text{and} \quad a < \tfrac{2}{3}. \tag{7.79}$$

Hence, we find for ε small enough

$$\forall\, n \in \mathbb{N}, \quad -1 + C\varepsilon^{2-3a}2^n N_{n-1}^{-a_2+\tau_2} \leqslant 0$$

and therefore

$$\left|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(\lambda,\varepsilon) - \mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\right| \leqslant \tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Consequently, $\lambda \in \mathcal{R}_{l,j,j_0}(i_{n-1})$ and the proof of the second point is now achieved.
(iii) Let $j \in \mathbb{S}_0^c$. In particular, one has $(l,j) \neq (0,0)$. We shall first prove that if $|l| \leqslant N_{n-1}$ and then $\mathcal{R}_{l,j}^{(1)}(i_n) \subset \mathcal{R}_{l,j}^{(1)}(i_{n-1})$. As in the point (i) this implies that $\mathcal{R}_{l,j}^{(1)}(i_n) = \varnothing$. Remind that the set $\mathcal{R}_{l,j}^{(1)}(i_n)$ is defined below (7.68). Consider $\lambda \in \mathcal{R}_{l,j}^{(1)}(i_n)$ then by construction $\lambda \in \mathcal{C}_n^\varepsilon \subset \mathcal{C}_{n-1}^\varepsilon$. Now by the triangle inequality we may write in view of (6.251) and (7.74) and the choice $\gamma = \varepsilon^a$

$$\begin{aligned}\left|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(\lambda,\varepsilon)\right| &\leqslant \left|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n}(\lambda,\varepsilon)\right| + |\mu_j^{\infty,n}(\lambda,\varepsilon) - \mu_j^{\infty,n-1}(\lambda,\varepsilon)|\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon\gamma^{-1}|j|\|i_n - i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon^{2(1-a)}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Since $\gamma_{n+1} = \gamma_n - \varepsilon^a 2^{-n-1}$ and $|l| \leqslant N_{n-1}$, then

$$\left|\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(\lambda,\varepsilon)\right| \leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}} + \tfrac{\langle j\rangle\varepsilon^a}{2^{n+1}\langle l\rangle^{\tau_1}}\Big(-1 + \varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\Big).$$

Notice that (7.79) implies in particular

$$a_2 > \tau_1 \quad \text{and} \quad a < \tfrac{2}{3} \tag{7.80}$$

and taking ε small enough we find that

$$\forall\, n\in\mathbb{N},\quad -1+\varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\leqslant 0,$$

which implies in turn that

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)\big|\leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Consequently, $\lambda\in\mathcal{R}_{l,j}^{(1)}(i_{n-1})$ and this ends the proof of the third point.
(iv) It is an immediate consequence of (7.68) and the points (i)-(ii) and (iii) of Lemma 7.1. □

The next result deals with necessary conditions such that the sets in (7.68) are nonempty.

Lemma 7.2. *There exists ε_0 such that for any $\varepsilon\in[0,\varepsilon_0]$ and $n\in\mathbb{N}$ the following assertions hold true.*

(i) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}$. If $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(ii) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. If $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$, then $|j-j_0|\leqslant C_0\langle l\rangle$.

(iii) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$. If $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(iv) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. There exists $c_2>0$ such that if $\min(|j|,|j_0|)\geqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\mathcal{R}_{l,j,j_0}(i_n)\subset\mathcal{R}_{l,j-j_0}^{(0)}(i_n).$$

Proof. **(i)** Assume $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$, then we can find $\lambda\in(\lambda_0,\lambda_1)$ such that, using triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned}|c_n(\lambda,\varepsilon)||j|&\leqslant 4|j|\gamma_{n+1}^{\upsilon}\langle l\rangle^{-\tau_1}+|\omega(\lambda,\varepsilon)\cdot l|\\&\leqslant 4|j|\gamma_{n+1}^{\upsilon}+C\langle l\rangle\\&\leqslant 8\varepsilon^{a\upsilon}|j|+C\langle l\rangle,\end{aligned}$$

where we have used $\gamma=\varepsilon^a$ and the fact that $(\lambda,\varepsilon)\mapsto\omega(\lambda,\varepsilon)$ is bounded. Notice that

$$c_n(\lambda,\varepsilon)=I_1(\lambda)K_1(\lambda)+r^{1,n}(\lambda,\varepsilon)\quad\text{and}\quad\inf_{\lambda\in(\lambda_0,\lambda_1)}I_1(\lambda)K_1(\lambda):=c_1>0.$$

Then, from (6.26), (6.248) and Proposition 7.1 $(\mathcal{P}1)_n$, we obtain

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad\sup_{n\in\mathbb{N}}\sup_{\lambda\in(\lambda_0,\lambda_1)}|\partial_\lambda^k r^{1,n}(\lambda,\varepsilon)|&\leqslant\gamma^{-k}\sup_{n\in\mathbb{N}}\|r^{1,n}\|_q^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-k}\\&\lesssim\varepsilon^{1-ak}.\end{aligned}\tag{7.81}$$

Thus, by choosing ε small enough, we can ensure by (7.81)

$$\inf_{n\in\mathbb{N}}\inf_{\lambda\in(\lambda_0,\lambda_1)}|c_n(\lambda,\varepsilon)|\geqslant\tfrac{c_1}{2}.$$

Hence, by taking ε small enough we find that $|j|\leqslant C_0\langle l\rangle$ for some $C_0>0$.
(ii) In the case $j=j_0$ we get by definition $\mathcal{R}_{l,j_0,j_0}(i_n)=\mathcal{R}_{l,0}^{(0)}(i_n)$, and then we use the point (i). In what follows we take $j\neq j_0$ and we assume that $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$ then there exists $\lambda\in(\lambda_0,\lambda_1)$ such that

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)|&\leqslant 2\gamma_{n+1}|j-j_0|\langle l\rangle^{-\tau_2}+|\omega(\lambda,\varepsilon)\cdot l|\\&\leqslant 2\gamma_{n+1}|j-j_0|+C\langle l\rangle\\&\leqslant 4\varepsilon^a|j-j_0|+C\langle l\rangle.\end{aligned}$$

Similarly to (7.81), we can prove

$$\forall k\in[\![0,q]\!],\quad \sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}\sup_{\lambda\in(\lambda_0,\lambda_1)}|j||\partial_\lambda^k r_j^{\infty,n}(\lambda,\varepsilon)|\leqslant\gamma^{-k}\sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^{\infty,n}\|_q^{\gamma,\mathcal{O}}$$
$$\lesssim\varepsilon\gamma^{-1-k}$$
$$\lesssim\varepsilon^{1-a(1+k)}. \tag{7.82}$$

By using the triangle inequality, Lemma 3.3-(v), (7.81) and (7.82) we get for $j\neq j_0$,

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)|&\geqslant|\Omega_j(\lambda)-\Omega_{j_0}(\lambda)|-|r^{1,n}(\lambda,\varepsilon)||j-j_0|-|r_j^{\infty,n}(\lambda,\varepsilon)|-|r_{j_0}^{\infty,n}(\lambda,\varepsilon)|\\&\geqslant\big(C_0-C\varepsilon^{1-a}\big)|j-j_0|\\&\geqslant\tfrac{C_0}{2}|j-j_0|\end{aligned}$$

provided that ε is small enough. Putting together the previous inequalities yields for ε small enough $|j-j_0|\leqslant C_0\langle l\rangle$, for some $C_0>0$.

(iii) First remark that the case $j=0$ is trivial. Now for $j\neq0$ we assume that $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$ then there exists $\lambda\in(\lambda_0,\lambda_1)$ such that

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)|&\leqslant\gamma_{n+1}|j|\langle l\rangle^{\tau_1}+|\omega(\lambda,\varepsilon)\cdot l|\\&\leqslant2\varepsilon^a|j|+C\langle l\rangle.\end{aligned}$$

Using the definition (7.67) combined with the triangle inequality, Lemma 3.3-(iv), (7.81) and (7.82), we get

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)|&\geqslant\Omega|j|-|j||r^{1,n}(\lambda,\varepsilon)|-|r_j^{\infty,n}(\lambda,\varepsilon)|\\&\geqslant\Omega|j|-C\varepsilon^{1-a}|j|.\end{aligned}$$

Combining the previous two inequalities and the second condition in (7.80) implies

$$\big(\Omega-C\varepsilon^{1-a}-2\varepsilon^a\big)|j|\leqslant C\langle l\rangle.$$

Thus, by taking ε small enough we obtain $|j|\leqslant C_0\langle l\rangle$, for some $C_0>0$.

(iv) First notice that the case $j=j_0$ is trivial and follows from the definition (7.68). Let $j\neq j_0$ and $\lambda\in\mathcal{R}_{l,j,j_0}(i_n)$, then by definition

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|\leqslant\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Combining (7.67) and (3.14) with the triangle inequality we infer

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\big|&\leqslant\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|\\&\quad+|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)|+\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|.\end{aligned}$$

Thus, we find

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\big|&\leqslant\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}+|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)|\\&\quad+\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|.\end{aligned}\tag{7.83}$$

Without loss of generality, we can assume that $|j_0|\geqslant|j|$ and remind that $j\neq j_0$. Then, from (3.24) and (3.22), we easily find

$$\begin{aligned}|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)|&\leqslant|j|\big|I_j(\lambda)K_j(\lambda)-I_{j_0}(\lambda)K_{j_0}(\lambda)\big|+|j-j_0|\big|I_{j_0}(\lambda)K_{j_0}(\lambda)\big|\\&\leqslant\tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.\end{aligned}$$

Applying (6.248), we find for $j \neq j_0 \in \mathbb{S}_0^c$,

$$\begin{aligned}\big|r_j^{\infty,n}(\lambda,\varepsilon) - r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big| \leqslant & C\varepsilon^{1-a}\big(|j|^{-1} + |j_0|^{-1}\big)\\ \leqslant & C\varepsilon^{1-a}\tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.\end{aligned}$$

Plugging the preceding estimates into (7.83) yields

$$\big|\omega(\lambda,\varepsilon)\cdot l + (j-j_0)c_n(\lambda,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} + C\tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.$$

Therefore, if we assume $\min(|j|,|j_0|) \geqslant \frac{1}{2}C\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$ and $\tau_2 > \tau_1$, then we deduce

$$\big|\omega(\lambda,\varepsilon)\cdot l + (j-j_0)c_n(\lambda,\varepsilon)\big| \leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j-j_0\rangle}{\langle l\rangle^{\tau_1}}.$$

This ends the proof of the lemma by taking $c_2 = \frac{C}{2}$. □

We shall now establish that the perturbed frequencies $\omega(\lambda,\varepsilon)$ satisfy the Rüssemann conditions. This is done by a perturbation argument from the equilibrium linear frequencies $\omega_{\mathrm{Eq}}(\lambda)$ for which we already know by Lemma 3.5 that they satisfy the transversality conditions.

Lemma 7.3. *Let q_0, C_0 and ρ_0 as in Lemma 3.5. There exist $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in [0,\varepsilon_0]$ the following assertions hold true.*

(i) For all $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l\right)\big| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(ii) For all $(l,j) \in \mathbb{Z}^{d+1} \setminus \{(0,0)\}$ such that $|j| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l + jc_n(\lambda,\varepsilon)\big)\big| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iii) For all $(l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c$ such that $|j| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n}(\lambda,\varepsilon)\big)\big| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iv) For all $(l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2$ such that $|j-j_0| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l + \mu_j^{\infty,n}(\lambda,\varepsilon) - \mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\big| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

Proof. **(i)** From the triangle and Cauchy-Schwarz inequalities together with (7.62), (7.2) and Lemma 3.5-(i), we deduce

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l\right)\big| \geqslant & \max_{k\in[\![0,q_0]\!]}\big|\partial_\lambda^k\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l\right)\big| - \max_{k\in[\![0,q]\!]}\big|\partial_\lambda^k\left(\overline{\mathrm{r}}_\varepsilon(\lambda)\cdot l\right)\big|\\ \geqslant & \rho_0\langle l\rangle - C\varepsilon\gamma^{-1-q}N_0^{q\overline{a}}\langle l\rangle\\ \geqslant & \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle\\ \geqslant & \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

provided that ε is small enough and

$$1 - a(1+q+q\overline{a}) > 0. \tag{7.84}$$

Notice that the condition (7.84) is automatically satisfied by (7.2) and (7.1).
(ii) As before, using the triangle and Cauchy-Schwarz inequalities combined with (7.62), (7.81), Lemma 3.5-(ii) and the fact that $|j| \leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)\right)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l+jI_1(\lambda)K_1(\lambda)\right)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\left(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+jr^{1,n}(\lambda,\varepsilon)\right)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-aq}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough and with the condition (7.84).
(iii) As before, performing the triangle and Cauchy-Schwarz inequalities combined with (7.62), (7.81), (7.82), Lemma 3.5-(iii) and the fact that $|j| \leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l+\Omega_j(\lambda)\right)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\big(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+jr^{1,n}(\lambda,\varepsilon)+r_j^{\infty,n}(\lambda,\varepsilon)\big)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough with the condition (7.84).
(iv) Arguing as in the preceding point, using (7.81), (7.82), Lemma 3.5-(iv)-(v) and the fact that $0<|j-j_0| \leqslant C_0\langle l\rangle$ (notice that the case $j=j_0$ is trivial), we have

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega_{\mathrm{Eq}}(\lambda)\cdot l+\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\big)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\big(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+(j-j_0)r^{1,n}(\lambda,\varepsilon)+r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j-j_0|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough. This ends the proof of Lemma 7.3. □

A Modified Bessel functions

In this short section we shall collect some properties about Bessel and modified Bessel functions that were used in the preceding sections. We refer to [53] for an almost exhaustive presentation of these special functions.
We define first the Bessel functions of order $\nu\in\mathbb{C}$ by

$$J_\nu(z)=\sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)},\quad |\arg(z)|<\pi.$$

Notice that when $\nu\in\mathbb{N}$ we have the following integral representation, see [47, p. 115].

$$J_\nu(x)=\frac{1}{\pi}\int_0^\pi\cos\big(x\sin\theta-\nu\theta\big)d\theta. \tag{A.1}$$

We shall also introduce the Bessel functions of imaginary argument also called modified Bessel functions of first and second kind

$$I_\nu(z)=\sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)},\quad |\arg(z)|<\pi \tag{A.2}$$

and

$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}, \quad \nu \in \mathbb{C}\setminus\mathbb{Z}, \quad |\arg(z)| < \pi.$$

For $j \in \mathbb{Z}$, we define $K_j(z) = \lim_{\nu\to j} K_\nu(z)$. We give now useful properties of modified Bessel functions.

► *Symmetry and positivity,* see [2, p. 375]

$$\forall j \in \mathbb{N}, \quad \forall \lambda \in \mathbb{R}_+^*, \quad I_{-j}(\lambda) = I_j(\lambda) \in \mathbb{R}_+^* \quad \text{and} \quad K_{-j}(\lambda) = K_j(\lambda) \in \mathbb{R}_+^*. \tag{A.3}$$

► *Anti-derivative,* see [2, p. 376]. If we set $\mathcal{Z}_\nu(z) = I_\nu(z)$ or $e^{i\nu\pi}K_\nu(z)$, then for all $\nu \in \mathbb{R}$, we have

$$\frac{d}{dz}\left(z^{\nu+1}\mathcal{Z}_{\nu+1}(z)\right) = z^{\nu+1}\mathcal{Z}_\nu(z). \tag{A.4}$$

► *Power series expansion for K_j,* see [2, p. 375].

$$\begin{aligned} K_j(z) = \tfrac{1}{2}\left(\tfrac{z}{2}\right)^{-j}\sum_{k=0}^{j-1}\tfrac{(j-k-1)!}{k!}\left(\tfrac{-z^2}{4}\right)^k + (-1)^{n+1}\log\left(\tfrac{z}{2}\right)I_j(z) \\ + \tfrac{1}{2}\left(\tfrac{-z}{2}\right)^j\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(j+k+1)\right)\tfrac{\left(\frac{z^2}{4}\right)^k}{k!(j+k)!}, \end{aligned} \tag{A.5}$$

where $\psi(1) = -\gamma$ (Euler's constant) and $\forall m \in \mathbb{N}^*$, $\psi(m+1) = \sum_{k=1}^{m}\frac{1}{k} - \gamma$.

In particular we have the expansion

$$K_0(z) = -\log\left(\frac{z}{2}\right)I_0(z) + \sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2m}}{(m!)^2}\psi(m+1). \tag{A.6}$$

► *Integral representation for K_ν,* see [47, p. 133] For all $a, b > 0$ for any $\nu, \mu \in \mathbb{C}$ satisfying $-1 < \mathrm{Re}(\nu) < 2\mathrm{Re}(\mu) + \frac{3}{2}$ one has

$$\int_0^\infty \frac{x^{\nu+1}J_\nu(bx)}{(x^2+a^2)^{\mu+1}}dx = \frac{a^{\nu-\mu}b^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(ab). \tag{A.7}$$

► *Nicholson's integral representation,* see [53, p. 441]. Let $j \in \mathbb{N}$ then

$$(I_jK_j)(z) = \tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} K_0(2z\cos(\tau))\cos(2j\tau)d\tau. \tag{A.8}$$

Another similar representation can be found in [47, p. 140]

$$(I_jK_j)(\lambda) = \frac{1}{2}\int_0^\infty J_0\big(2\lambda\sinh(t/2)\big)e^{-jt}dt. \tag{A.9}$$

► *Holomorphic property of the product I_jK_j.* Let $j \in \mathbb{N}$ then the function $z \mapsto (I_jK_j)(z)$ is holomorphic on the half plane $\mathrm{Re}(z) > 0$.

► *Monotonicity of $I_\nu K_\nu$,* see for instance [9, 22].The map $(\lambda,\nu) \in (\mathbb{R}_+^*)^2 \mapsto I_\nu(\lambda)K_\nu(\lambda)$ is strictly decreasing in each variable.

► *Asymptotic expansion of small argument,* see for instance [2, p. 375].

$$\forall j \in \mathbb{N}^*, \quad I_j(\lambda) \underset{\lambda\to 0}{\sim} \tfrac{\left(\frac{1}{2}\lambda\right)^j}{\Gamma(j+1)} \quad \text{and} \quad K_j(\lambda) \underset{\lambda\to 0}{\sim} \tfrac{\Gamma(j)}{2\left(\frac{1}{2}\lambda\right)^j}. \tag{A.10}$$

► *Asymptotic expansion of large argument for the product I_jK_j,* see for instance [2, p. 378].

$$\forall N \in \mathbb{N}^*, \quad I_j(\lambda)K_j(\lambda) \underset{\lambda\to\infty}{\sim} \frac{1}{2\lambda}\left(1 + \sum_{m=1}^{N}\tfrac{\alpha_{j,m}}{(2\lambda)^{2m}}\right), \tag{A.11}$$

with

$$\alpha_{j,m} = (-1)^m \frac{(2m)!}{4^m \big(m!\big)^2} P_m(\mu_j), \quad P_m(X) = \prod_{\ell=1}^{m} \big(X - (2\ell-1)^2\big), \quad \mu_j = 4j^2. \tag{A.12}$$

In particular,

$$I_j(\lambda)K_j(\lambda) \underset{\lambda\to\infty}{\longrightarrow} 0. \tag{A.13}$$

► *Asymptotic expansion of high order for the product I_jK_j*, for more details see [43].

$$I_j(\lambda)K_j(\lambda) \underset{j\to\infty}{\sim} \tfrac{1}{2j} \left(\sum_{m=0}^{\infty} \tfrac{b_m(\lambda)}{j^m}\right)\left(\sum_{m=0}^{\infty} (-1)^m \tfrac{b_m(\lambda)}{j^m}\right), \tag{A.14}$$

where for each $m \in \mathbb{N}$, $b_m(\lambda)$ is a polynomial of degree m in λ^2 defined by

$$b_0(\lambda) = 1 \quad \text{and} \quad \forall m \in \mathbb{N}^*,\, b_m(\lambda) = \sum_{k=1}^{m} (-1)^{m-k} \tfrac{S(m,k)}{k!} \left(\tfrac{\lambda^2}{4}\right)^k$$

and the $S(m,k)$ are Stirling numbers of second kind defined recursively by

$$\forall (m,k) \in \mathbb{N}^* \times \mathbb{N}, \quad S(m,k) = S(m-1,k-1) + kS(m-1,k),$$

with

$$S(0,0) = 1,\ \forall m \in \mathbb{N}^*, S(m,1) = 1 \text{ and } S(m,0) = 0 \text{ and if } m < k \text{ then } S(m,k) = 0.$$

We shall also prove the following result which has been frequently used before.

Lemma A.1. *Let $N_0 \geqslant 2$. Consider the sequence $(N_m)_{m\in\mathbb{N}}$ defined by* (5.82). *Then for all $\alpha > 0$, we have*

$$\sum_{k=m}^{\infty} N_k^{-\alpha} \underset{m\to\infty}{\sim} N_m^{-\alpha}.$$

Proof. We consider the positive decaying function

$$t \in \mathbb{R}_+^* \mapsto N_0^{-\alpha\left(\frac{3}{2}\right)^t} = \exp\left(-\alpha \ln(N_0) e^{t\ln\left(\frac{3}{2}\right)}\right),$$

and apply to it a series-integral comparison, namely

$$\sum_{k=m+1}^{\infty} N_k^{-\alpha} \leqslant \int_m^{\infty} \exp\left(-\alpha\ln(N_0)e^{t\ln\left(\frac{3}{2}\right)}\right) dt = \int_0^{\infty} \exp\left(-\alpha\ln(N_0)e^{u\ln\left(\frac{3}{2}\right)}e^{m\ln\left(\frac{3}{2}\right)}\right) du.$$

Now remark that

$$N_m^{\alpha} \exp\left(-\alpha\ln(N_0)e^{u\ln\left(\frac{3}{2}\right)}e^{m\ln\left(\frac{3}{2}\right)}\right) = \exp\left(\alpha\ln(N_0)\left(1 - e^{u\ln\left(\frac{3}{2}\right)}\right)e^{m\ln\left(\frac{3}{2}\right)}\right).$$

Since

$$\forall u \in \mathbb{R}_+^*, \quad 1 - e^{u\ln\left(\frac{3}{2}\right)} < 0,$$

then we deduce that

$$\forall u \in \mathbb{R}_+^*, \quad N_m^{\alpha} \exp\left(-\alpha\ln(N_0)e^{u\ln\left(\frac{3}{2}\right)}e^{m\ln\left(\frac{3}{2}\right)}\right) \underset{m\to\infty}{\longrightarrow} 0$$

and

$$\forall u \in \mathbb{R}_+^*, \forall m \in \mathbb{N}, \quad 0 \leqslant N_m^{\alpha} \exp\left(-\alpha\ln(N_0)e^{u\ln\left(\frac{3}{2}\right)}e^{m\ln\left(\frac{3}{2}\right)}\right) \leqslant N_0^{\alpha} \exp\left(-\alpha\ln(N_0)e^{u\ln\left(\frac{3}{2}\right)}\right) \in L^1(\mathbb{R}_+).$$

Applying dominated convergence theorem, we obtain

$$\sum_{k=m+1}^{\infty} N_k^{-\alpha} \underset{m\to\infty}{=} o\left(N_m^{-\alpha}\right).$$

As a consequence

$$\sum_{k=m}^{\infty} N_k^{-\alpha} = N_m^{-\alpha} + \sum_{k=m+1}^{\infty} N_k^{-\alpha} \underset{m\to\infty}{\sim} N_m^{-\alpha}.$$

□

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