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Structures quasi-périodiques pour des modèles de transport non-linéaires issus de la mécanique des fluides

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A mes parents Didier et Jocelyne
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Une page noircie d'encre se tourne, une autre encore vierge se découvre... Poulet !

RÉSUMÉ

Cette thèse s'inscrit dans le cadre de l'étude mathématique de la mécanique des fluides. Nous nous intéressons aux équations d'Euler et QGSW (*Quasi-Geostrophic Shallow-Water* en anglais) bidimensionnelles, qui prennent la forme d'équations de transport non-linéaires et non-locales. Nous étudions en particulier l'émergence de structures quasi-périodiques sous la forme de poches de tourbillon pour ces modèles.

La mécanique des fluides est une branche de la Physique dont l'objet est la description des propriétés dynamiques des fluides (généralement des liquides ou des gaz mais aussi parfois des plasmas). Rappelons que les équations d'Euler décrivent l'évolution d'un fluide homogène incompressible non-visqueux. Elles peuvent être posées en n'importe quelle dimension d'espace, mais nous nous limiterons au cas de la dimension deux qui a l'avantage de présenter une structure de transport sur la vorticité. Leur présentation sera faite en Section 1.1.1. Les équations QGSW quant à elles décrivent la circulation des océans ou de l'atmosphère sur des échelles de temps et d'espace assez larges. Elles sont obtenues à partir des équations Shallow-Water en effectuant un développement asymptotique au premier ordre par rapport au nombre de Rossby proche de l'équilibre géostrophique. Cet équilibre correspond à une compensation entre les effets de rotation et de stratification du fluide étudié. Analytiquement, les équations QGSW peuvent se voir comme une généralisation des équations d'Euler écrites en formulation vitesse-tourbillon via l'introduction d'un paramètre appelé *rayon de Rossby* relié à la fréquence de Coriolis, la constante de gravitation et la hauteur moyenne du fluide étudié. Nous renvoyons à la Section 1.1.2 pour une présentation plus détaillée de ce modèle.

Dans ce travail, nous étudions quelques propriétés dynamiques des poches de tourbillon planaires, qui sont des solutions faibles de la classe de Yudovich pour les modèles cités plus haut. Les poches décrivent l'évolution temporelle de domaines bidimensionnels bornés et l'étude de leur dynamique est réduite à celle de leur contour qui est soumis à une équation intégro-différentielle. Grâce à la structure des équations, les fonctions radiales fournissent des solutions stationnaires, en particulier les poches associées aux disques appelées *tourbillons de Rankine*. L'analyse des portraits de phase autour de ces points d'équilibre a suscité beaucoup d'intérêts. Notons qu'une activité assez riche s'est développée durant la dernière décennie autour des solutions périodiques. Dans le cas rigide, où la forme de la solution ne change pas au cours du temps (appelée V-state), de nombreuses structures dépendant de la topologie ont été mise en évidence grâce à des techniques de bifurcation. Ces solutions implicites effectuent une rotation uniforme autour

de leur centre de masse à vitesse angulaire constante. Par contre, dans le cas non-rigide, peu de résultats sont connus à ce jour. A la suite de ces travaux, une question naturelle s'est alors imposée :

Peut-on trouver des solutions quasi-périodiques (plus générales que périodiques) proches de certains de ces points d'équilibre ?

Cette thèse a pour vocation à apporter une réponse positive à cette question. Les techniques employées sont empruntées aux théories de KAM (nomée d'après ses fondateurs Kolmogorov, Arnold et Moser) et de Nash-Moser dans l'esprit des récents travaux de Massimiliano Berti et de ses collaborateurs. Rappelons que la théorie de KAM originelle décrit (sous de bonne conditions de régularité et de non-dégénérescence) la persistance de tores invariants supportant des trajectoires quasi-périodiques pour de petites perturbations de systèmes hamiltoniens intégrables en dimension finie. Cette théorie fut développée dans les années 50-60 et a commencé à être étendue aux EDP hamiltoniennes et/ou réversibles, i.e. en dimension infinie, à partir des années 80-90 avec notamment les travaux de Kuksin, Wayne, Pöschel et Bourgain. Une présentation plus exhaustive de l'utilisation des techniques KAM en EDP est faite à la Section 1.3. Le schéma de Nash-Moser, quant à lui, est un processus itératif généralisant la méthode de Newton au moyen d'opérateurs de régularisation (typiquement des projections sur un nombre fini de modes de Fourier) afin de trouver certains zéros d'une fonctionnelle. Son utilisation permet d'effectuer un théorème des fonctions implicites "à la main" dans le cas d'existence d'un inverse approché à droite satisfaisant de bonnes estimées douces avec perte fixe de régularité. Tout comme la méthode classique de Newton, l'avantage majeur de ce procédé, introduit par John Nash dans les années 50, réside dans son caractère quadratique, ce qui implique une vitesse de convergence exponentielle.

Les modèles qui nous intéressent ici et en particulier leurs formulations au niveau des poches de tourbillon sont des EDP hamiltoniennes. De plus, elles peuvent être décrites comme des perturbations quasi-linéaires de leurs linéarisations aux tourbillons de Rankine qui, elles, forment des systèmes intégrables. Nous sommes donc précisément dans le cadre adapté à l'utilisation des techniques KAM. Nous arrivons à générer des poches de tourbillon quasi-périodiques en jouant avec un paramètre qui apparaît soit naturellement dans l'équation soit géométriquement dû à des propriétés de non-invariance par changement d'échelle. Pour de bonnes valeurs de ce paramètre choisies dans un ensemble de type Cantor nous arrivons à montrer l'existence de telles structures. Voici à présent un plan succint de la thèse.

- ➤ La première partie de la thèse (Part I) est consacrée à l'étude de l'existence de poches de tourbillon quasi-périodiques proche du disque unité pour les équations QGSW. Ces structures apparaissant naturellement au niveau linéaire persistent au niveau non-linéaire modulo un choix du rayon de Rossby parmi un ensemble possible de mesure presque pleine. Il est à noté que ce choix revient à sélectionner des équations pour lesquelles on est capable de construire des solutions, mais en aucun cas les équations sont fixées à l'avance. Ce travail a été effectué avec mon directeur Taoufik Hmidi.

- La seconde partie (Part II) est dédiée à l'obtention de poches quasi-périodiques proches des tourbillons de Rankine pour les équations d'Euler posées dans le disque unité. Cette fois, c'est un paramètre géométrique, le rayon des tourbillons de Rankine, qui permet de générer les solutions quasi-périodiques. L'apparition de ce paramètre est reliée à la non-invariance du problème par dilatation. En effet, dans le plan entier, cette approche n'est pas possible dû notamment à des résonances triviales entre les fréquences à l'équilibre. L'ensemble des rayons admissibles est de type Cantor et de mesure presque pleine. L'analyse est plus simple dans ce cadre car les effets du bord se font au travers de termes réguliers. Ce travail a été fait en collaboration avec Zineb Hassainia.

Techniquement, les difficultés rencontrées dans les preuves de ces deux résultats peuvent être classifiées en trois composantes.

- La première est de nature spectrale. Comme mentionné plus haut, chaque équation est quasi-linéaire et sa linéarisation est à coefficients variables. Au cours du schéma de Nash-Moser, il nous faut construire un inverse approché à droite du linéarisé ce qui se fait en conjuguant celui-ci à un opérateur à coefficients constants en choisissant les paramètres parmi des ensembles de Cantor liés au spectre de l'opérateur. Cette procédure est assez coûteuse et est basée sur les techniques KAM. En particulier, pour les équations QGSW, le spectre est relié à des fonctions de Bessel modifiées et l'on doit faire appel à des propriétés fines de ces dernières, reliées à leurs asymptotiques, leurs représentations intégrales etc...
- La seconde est de nature fonctionnelle. En vue de l'application du schéma de Nash-Moser, nous devons montrer des estimées douces et des propriétés de symétrie pour l'inverse approché ce qui nous oblige à être attentif aux lois de produit et composition en lien avec les fonctions et opérateurs utilisés lors de la réduction du linéarisé à coefficients constants. Pour faire converger le schéma de réduction, il nous faut utiliser à certains endroits une topologie particulière sur les opérateurs Toeplitz en temps, plus forte que la topologie standard sur les opérateurs. Enfin, l'analyse est basée sur l'étude d'opérateurs à noyaux qui sont assez singuliers et requièrent donc une attention particulière. Le travail direct avec la structure du noyau et non du symbole de l'opérateur (techniques de calcul pseudo-différentiel) est en contraste avec les travaux précédents dans l'étude de l'émergence de solutions quasi-périodiques en EDP. Il est important de remarquer que, pour les équations qui nous intéressent, les singularités du noyau apparaissent comme des convolutions. C'est un point clé qui, grâce à des changements de variables, permet d'estimer le noyau. Les parties

non-singulières étant régularisantes à tous ordres, leurs estimations sont relativement simples.

- La dernière difficulté est plutôt relative à la théorie des nombres. En effet, l'application des techniques KAM implique la résolution d'équations dites *homologiques* qui nécessitent des conditions de non-résonance en lien avec l'approximation diophantienne. Afin d'assurer ces conditions, il nous faut sélectionner des paramètres admissibles en exploitant une rigidité des fréquences à l'équilibre qui se manifeste par la non-dégénérescence et la transversalité.

➤ La troisième partie de la thèse (Part III) est consacrée à l'étude de l'existence de V-states doublement-connexes analytiques pour les équations QGSW. Il s'agit de solutions dont le domaine possède un trou et qui sont en rotation uniforme. Ces poches, obtenues par des techniques de bifurcation, satisfont des conditions de hautes symétries (non explicites) et les branches de bifurcations associées émergent de l'anneau pour des vitesses angulaires bien spécifiques reliées aux fonctions de Bessel modifiées. Le point délicat de l'analyse est en lien avec des propriétés fines sur ces fonctions spéciales. Ce résultat est dans la lignée de ceux obtenus dans la dernière décennie concernant les poches périodiques.

ABSTRACT

This thesis takes place in the mathematical study of fluid mechanics. We are interested in bidimensional Euler and quasi-geostrophic shallow-water (QGSW) equations, which take the form of nonlinear and nonlocal transport-type equations. We study in particular the emergence of quasi-periodic vortex patch structures for these models.

Recall that Euler equations describe the evolution of an inviscid homogeneous and incompressible fluid. They can be set in any space dimension but we shall restrict our discussion to the dimension two since in this case the vorticity solves a transport equation. Their presentation is done in Section 1.1.1. As to the QGSW equations, they describe the circulation of the ocean and the atmosphere at large time and space scales. They are obtained from shallow-water equations by making some asymptotic expansions with respect to the Rossby number close to the quasi-geostrophic balance. This equilibrium corresponds to a balance between rotation and stratification effects. Analytically, these equations can be seen as a generalization of Euler equations written in velocity-vorticity formulation through the introduction of a parameter called *Rossby radius.* We refer to Section 1.1.2 for a detailed presentation of this model.

In this work, we study some dynamical properties of planar vortex patches, which are weak solutions in the Yudovich class for the above mentioned models. They describe the evolution of bidimensional bounded domains and the study of their dynamics is reduced to the one of their boundary which is subject to an integro-differential equation. Thanks to the structure of the equations, radial profiles provide stationary solutions, in particular vortex patches associated with the discs called *Rankine vortices.* The analysis of the phase portraits close to these equilibrium points has aroused great interest. Notice that, during the last decade, a quite rich activity has been developed around periodic solutions. In the rigid motion case, where the solution keeps the same shape (and is called *V-state*), several structures depending on the topology were found by using bifurcation theory. These implicit solutions perform a uniform rotation around their center of mass with constant angular velocity. However, very few results are known in the non-rigid case. After these works, a natural question appeared :

Can we find quasi-periodic solutions (more general than periodic)
close to some of these equilibrium points ?

This thesis answers positively to this question. The techniques involved are borrowed from KAM and Nash-Moser theories in the spirit of the recents works of Massimiliano Berti and his collaborators. Recall that the original KAM theory describes (in suitable regularity) the persistence of invariant tori supporting quasi-periodic motions for small perturbations of integrable Hamiltonian systems in finite dimension. This theory was developed in the 50-60s and started to be extended to Hamiltonian PDE, i.e. in infinite dimension, in the 80-90s. The Nash-Moser scheme is an iterative procedure generalizing the Newton's method through the use of regularizing operators. It allows to perform an implicit function theorem in case of existence of an approximate right inverse satisfying nice tame estimates with fixed loss of regularity.

The models of interest here and in particular their formulations at the level of vortex patches are Hamiltonian PDE. In addition, they can be seen as quasilinear perturbations of their linearizations at the Rankine vortices which one are integrable. Then, we are exactly in a well-adapted situation for applying

KAM techniques. We can generate quasi-periodic vortex patch solutions by playing with a parameter appearing either naturally in the equations or geometrically due to non-invariance scaling properties. For suitable selected values of this parameter among a Cantor-type set, we can generate these solutions. We shall now present a short plan of the memoir.

- ➤ The first part of the thesis (Part I) is devoted to proving the existence of time quasi-periodic vortex patches close to the unit disc for QGSW equations. These structures appearing naturally at the linear level persist at the nonlinear one modulo the choice of the Rossby radius among a massive Cantor-like set. This work has been done together with my PhD advisor Taoufik Hmidi.
- ➤ The second part (Part II) of the thesis is devoted to proving the existence of quasi-periodic in time vortex patches close to the Rankine vortices for Euler equations set in the unit disc. Here this is a geometrical parameter, the radius of the Rankine vortices, which allows to generate quasi-periodic solutions. The apparition of this parameter is related to the non-invariance by radial dilation of the problem. Indeed, in the whole plane, this approach fails in particular due to trivial resonances between the equilibrium frequencies. The set of admissible parameters is of Cantor-type with almost full Lebesgue measure. The analysis is simpler in this case since the boundary effects make appear smooth terms. This work has been done in collaboration with Zineb Hassainia.

From a technical point of view, the difficulties encountered in the proofs of the previous two results can be classified into three components.

- The first one is of spectral nature. As already mentioned, each equation is quasilinear and its linearization has variable coefficients. Along the Nash-Moser scheme, we need to construct an approximate right inverse for the linearized operator which is done by conjugating it to a constant coefficients operator provided the choice of parameters among Cantor sets related to the spectrum of the operator. This procedure is expensive and based on KAM reductions.
- The second is linked to functional analysis. In view of the Nash-Moser iteration, we need to show tame estimates and symmetry properties for the approximate inverse which forces us to pay attention to products and composition laws related to functions and operators used during the reduction of the linearized operator to constant coefficients. In order to make the reduction scheme convergent, we have to deal with a special Toeplitz in time topology for operators which is stronger than the classical one. Finally, the analysis is based on the study of integral operators whose kernels are quite singular and require particular attention.
- The last difficulty is related to number theory. Indeed, the implementation of KAM techniques implies to solve some equations called *homological* which require non-resonance conditions linked to Diophantine approximation. In order to ensure these conditions, we must select admissible parameters by exploiting the rigidity of the equilibrium frequencies through the non-degeneracy and the transversality.

- ➤ The third part (Part III) deals with the existence of analytic doubly-connected V-states for QGSW equations. These patches, obtained by bifurcation techniques, have high symmetries and the associated branches of bifurcation emerge from the annulus for very specific angular velocities related to modified Bessel functions. The delicate point in the analysis is linked to rafined properties of these special functions. This result follows the previous works in the field obtained during the past decade.

CONTENTS

Introduction

> Le progrès n'est que l'accomplissement
> des utopies.
>
> Oscar Wilde

1 Historical and mathematical contexts

Fluid mechanics is a branch of Physics studying the dynamics of fluids, namely, liquids, gases or even plasmas. The literature in the subject is huge so we may restrict our discussion to the case of an inviscid, homogeneous and incompressible fluid for which the Euler equations [61] and their generalizations are well-adapted to describe the dynamics. We refer to Section 1.1 for a mathematical presentation of the fluid models of interest in this work. Such models (or at least the Eulerian one) have been widely studied numerically, experimentally and analytically, so we may focus on some aspects which fit with our purpose.

This thesis is devoted to the study of the emergence of ordered structures for some fluid models. More precisely, we are interested in the vortex patch dynamics. Their study goes back to the works of Helmoltz [92, 91], Kirchhoff [114] and Kelvin [111]. Helmoltz introduced in [91] the notion of vorticity for Euler equations, which is in the bidimensional case a scalar function quantifying the local rotation of the mesoscopic particles. He proved that the vorticity is a solution to an active scalar equation driven by the solenoidal velocity field of the fluid. Therefore, the initial profile is transported by the flow associated to the velocity. In particular, if the initial condition is given by the characteristic function of a bounded domain, then, at later time, the solution will keep the same structure and the resulting solution is called *vortex patch*. These structures can be seen as a simple modelization of hurricanes and in this context, the nature provides some interesting examples (see Figure 1-(b) and 2). We refer to Section 1.2 for a detailed mathematical presentation of vortex patches.

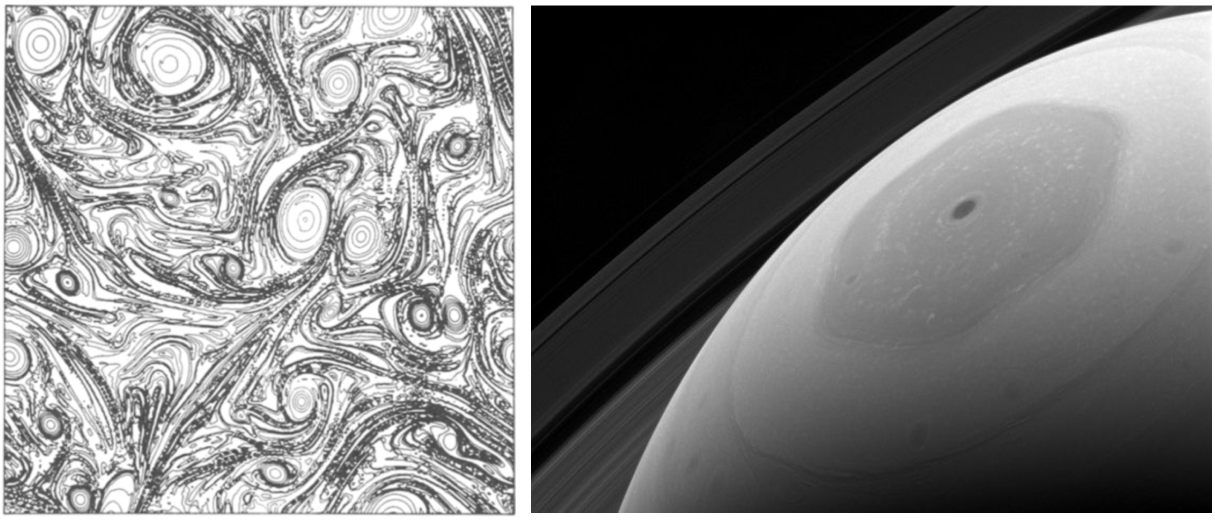
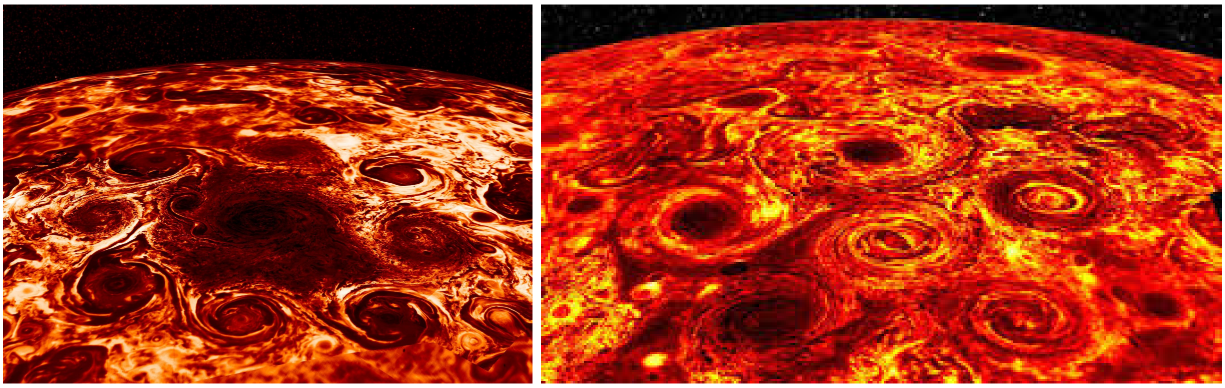

(a) (b)

Figure 1: (a) Ordered fluid structures. (b) Hexagon vortex at the north pole of Saturn (Cassini spacecraft 2017).

When renormalized by its area, then taking the diameter of the vortex patch going to zero, we find a point vortex. The point vortex system is the equivalent of the N-body problem in fluid mechanics. It was first introduced by Helmoltz in [92]. Later on, Kirchhoff [114] proved the Hamiltonian structure of this system. Then, Poincaré [129] and Gröbli [82] studied the 3-point vortex configuration and showed that it



(a) (b)

Figure 2: Pictures in false colors of the North (a) and South (b) poles of Jupiter where polygonal vortex sturctures are rotating (Juno spacecraft 2017).

is integrable. The inverse problem of desingularizing a point vortex configuration to get a vortex patch motion has been recently studied [70, 71, 90, 98] and numerical simulations like in [4] on point vortices are helpful in this task.

We shall look for the emergence of quasi-periodic solutions in the patch form for different models. Quasi-periodic functions are a generalization of periodic ones and are natural structures appearing in Hamiltonian systems. Their study goes back to the works of Kolmogorov [115], Arnold [5] and Moser [126] who proved the persistence of invariant tori supporting quasi-periodic motion for perturbations of integrable Hamiltonian systems in finite dimension. KAM theory was extended and refined for several Hamiltonian PDE with small divisors problems. For instance, it has been implemented for the 1-d semilinear wave and Schrödinger equations in several papers [39, 47, 49, 119, 131, 134, 147]. Many results were also obtained for semilinear perturbations of PDE [20, 19, 40, 60, 69, 109, 117, 118, 123]. However the case of quasi-linear or fully nonlinear perturbations were explored in [10, 8, 9, 21, 32, 67]. Many interesting results have also been obtained in the past few years on the periodic and quasi-periodic settings for the water-waves equations as in [2, 7, 29, 28, 33, 108, 128]. For Euler equations, only few results are known [11, 51]. We refer the reader to Section 1.3 for more details about KAM theory and its applications to PDE.

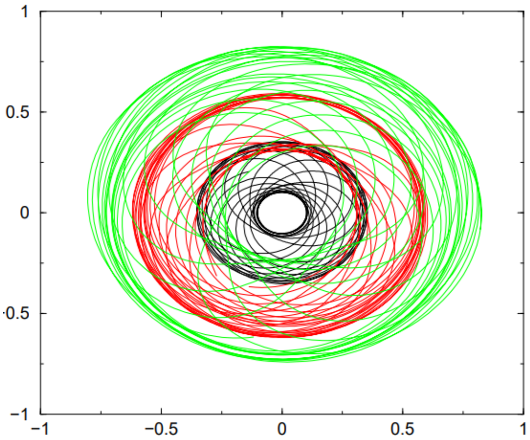


Figure 3: Numerical simulation of quasi-periodic point vortex motion [37].

1.1 Nonlinear and nonlocal transport-type fluid models

In this section, we present the various partial differential equations of interest in this PhD. These are bidimensional nonlinear fluid models which can be written as an active scalar equation, namely

$$\partial_t \boldsymbol{\omega} + \mathbf{v}\cdot\nabla\boldsymbol{\omega} = 0, \qquad (t,x)\in\mathbb{R}_+\times D$$

for a certain scalar unknown function $\boldsymbol{\omega}=\boldsymbol{\omega}(t,x)$ driven by a time dependent solenoidal vector field $\mathbf{v}=\mathbf{v}(t,x)$ related to $\boldsymbol{\omega}$ through singular integrals. The equation is set in a planar domain D taken either as the full plane or the unit disc in our discussions

$$D=\mathbb{R}^2 \qquad \text{or} \qquad D=\mathbb{D}\triangleq\Big\{(x_1,x_2)\in\mathbb{R}^2 \quad \text{s.t.} \quad x_1^2+x_2^2\leqslant 1\Big\}.$$

The divergence-free condition allows to introduce a velocity potential $\boldsymbol{\Psi}=\boldsymbol{\Psi}(t,x)$ such that

$$\mathbf{v}=\nabla^{\perp}\boldsymbol{\Psi}, \qquad \nabla^{\perp}\triangleq\begin{pmatrix}-\partial_2\\ \partial_1\end{pmatrix}.$$

In each considered model, the stream function $\boldsymbol{\Psi}$ is given by an integral in the form

$$\boldsymbol{\Psi}(t,x)=\int_D \mathbf{G}(x,y)\boldsymbol{\omega}(t,y)\,dA(y),$$

with a Green function $\mathbf{G}$ satisfying the following symmetry properties

$$\forall\theta\in\mathbb{R}, \quad \forall(x,y)\in D^2, \quad \mathbf{G}\big(e^{\mathrm{i}\theta}x,e^{\mathrm{i}\theta}y\big)=\mathbf{G}(x,y)=\mathbf{G}(y,x). \tag{1.1}$$

Here and in the sequel, we use the notation dA for the planar Lebesgue measure. Notice that along the document, we shall identify $\mathbb{C}$ with $\mathbb{R}^2$. In particular, the Euclidean structure of $\mathbb{R}^2$ is seen in the complex sense through the usual inner product defined for all $z_1=a_1+\mathrm{i}\,b_1\in\mathbb{C}$ and $z_2=a_2+\mathrm{i}\,b_2\in\mathbb{C}$ by

$$z_1\cdot z_2\triangleq\langle z_1,z_2\rangle_{\mathbb{R}^2}=\mathrm{Re}\,(z_1\overline{z_2})=a_1a_2+b_1b_2. \tag{1.2}$$

Several examples of such nonlinear transport-type equations are known in fluid mechanics :

Equations	Domain D	Unknown $\boldsymbol{\omega}$	Potential $\mathbf{G}(x,y)$
Euler	$\mathbb{R}^2$	vorticity $\boldsymbol{\omega}$	$\frac{1}{2\pi}\log(\lvert x-y\rvert)$
Euler	$\mathbb{D}$	vorticity $\boldsymbol{\omega}$	$\frac{1}{2\pi}\log\left(\left\lvert\frac{x-y}{1-x\overline{y}}\right\rvert\right)$
$(QGSW)_\lambda$ with $\lambda\in\mathbb{R}_+^*$	$\mathbb{R}^2$	potential vorticity $\boldsymbol{q}$	$-\frac{1}{2\pi}K_0\left(\lambda\lvert x-y\rvert\right)$
$(SQG)_\alpha$ with $\alpha\in(0,1)$	$\mathbb{R}^2$	temperature $\boldsymbol{\theta}$	$\frac{\Gamma\left(\frac{\alpha}{2}\right)}{\pi 2^{2-\alpha}\Gamma\left(\frac{2-\alpha}{2}\right)}\frac{1}{\lvert x-y\rvert^\alpha}$

Remark 1.1. *Observe at this stage that according to the symmetry properties in* (1.1)*, every radial initial profile generates a trivial stationary solution. In this memoir, we shall look for quasi-periodic structures living close to the stationary solutions given by discs and for periodic structures living close to the annuli.*

1.1.1 Euler equations

We first introduce the 2D-Euler equations which is the master model in fluid dynamics describing the evolution of an inviscid homogeneous incompressible fluid in a domain D. It was first introduced by Euler in [61] and writes in the following way

$$(E)\begin{cases} \partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} + \nabla p = 0, \quad \text{in } \mathbb{R}_+ \times D \\ \nabla\cdot\mathbf{v} = 0 \\ \mathbf{v}(0,\cdot) = \mathbf{v}_0. \end{cases} \tag{1.3}$$

The quantity $\mathbf{v} = \mathbf{v}(t,x)$ denotes the velocity field of the fluid. It is obtained as an average of the velocities of particles contained in a mesoscopic volume of fluid at time t and centered around a position x. The scalar function $p = p(t,x)$ denotes the pressure of the fluid at time t and position x. More generally, the gradient term can represent the sum of all conservative forces acting on the fluid. Notice that the incompressibility condition is encoded in the divergence-free condition for the vector field $\mathbf{v}$ (second equation in (1.3)). In the case $D = \mathbb{D}$, the system (1.3) is supplemented with the non penetration condition $\mathbf{v}\cdot\nu = 0$ where ν is the outward normal vector to the boundary $\partial\mathbb{D}$.

The global well-posedness theory for these equations set in the full plane goes back to the work of Wolibner [148] for smooth initial data. Later on, for classical solutions in Sobolev spaces $H^s(\mathbb{R}^2)$ ($s > 2$) the local well-posedness was proved by Kato and Ponce in [110]. Then, the question of the global existence of these solutions was solved in [15]. Notice that the global well-posedness in Hölder spaces was also obtained in [45]. The case of supercritical Besov spaces was studied in [44] and the critical cases were proved in [143] and [95]. As regards the situation in a bounded domain, the first result was given by Ebin and Marsden [58] in Sobolev and Hölder spaces, see also [57]. Their proof was very technical and based on Riemannian geometry on infinite dimensional manifolds. A simpler proof was proposed in [16].

We shall now present the *velocity-vorticity formulation* of Euler equations. In the planar case, this new system is equivalent to the Euler system (1.3) under nice decay property at infinity. We introduce the vorticity

$$\boldsymbol{\omega} \triangleq \nabla^\perp\cdot\mathbf{v} = \partial_1\mathbf{v}_2 - \partial_2\mathbf{v}_1.$$

This quantity measures the rotation effects inside the fluid. Applying the operator $\nabla^\perp\cdot$ to the first equation in (1.3), we get the following active scalar equation

$$\partial_t\boldsymbol{\omega} + \mathbf{v}\cdot\nabla\boldsymbol{\omega} = 0.$$

In the planar case, the stream function is solution of the Laplace problem

$$\Delta\boldsymbol{\Psi} = \boldsymbol{\omega}$$

and then is given by

$$\forall x \in \mathbb{R}^2, \quad \boldsymbol{\Psi}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\big(|x-y|\big)\boldsymbol{\omega}(t,y)\,dA(y).$$

In the case of the unit disc $\mathbb{D}$, the stream function $\boldsymbol{\Psi}$ solves the following Dirichlet problem

$$\begin{cases} \Delta\boldsymbol{\Psi} = \boldsymbol{\omega} \\ \boldsymbol{\Psi}|_{\partial\mathbb{D}} = 0. \end{cases}$$

Thus, by using the Green function of the unit disc, we get the expression

$$\forall x \in \mathbb{D}, \quad \boldsymbol{\Psi}(t,x) = \frac{1}{4\pi}\int_{\mathbb{D}} \log\left(\left|\frac{x-y}{1-x\overline{y}}\right|^2\right)\boldsymbol{\omega}(t,y)\,dA(y). \tag{1.4}$$

Then, the new formulation of Euler equations is given by

$$\begin{cases} \partial_t \boldsymbol{\omega} + \mathbf{v} \cdot \nabla \boldsymbol{\omega} = 0, \quad \text{in } \mathbb{R}_+ \times D \\ \mathbf{v} = \nabla^\perp \boldsymbol{\Psi} \\ \boldsymbol{\omega}(0, \cdot) = \boldsymbol{\omega}_0. \end{cases} \tag{1.5}$$

The study of weak solutions for the system (1.5) will be discussed in Section 1.2.

1.1.2 Quasi-geostrophic shallow-water equations

The quasi-geostrophic shallow-water equations $(\mathrm{QGSW})_\lambda$ is considered as one of the most common asymptotic models used to describe the large scale motion of the atmospheric and oceanic circulation and can be derived asymptotically from the rotating shallow-water equations when Rossby and Froude numbers are small enough, for more details we refer to [56, 142] and the references therein. We also refer to the formal derivation below. This model is planar and the evolution of the potential vorticity $\boldsymbol{q}$ takes the form of a nonlinear and nonlocal transport equation,

$$(\mathrm{QGSW})_\lambda \begin{cases} \partial_t \boldsymbol{q} + \mathbf{v} \cdot \nabla \boldsymbol{q} = 0, \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ \mathbf{v} = \nabla^\perp (\Delta - \lambda^2)^{-1} \boldsymbol{q}, \\ \boldsymbol{q}(0, \cdot) = \boldsymbol{q}_0. \end{cases} \tag{1.6}$$

Here $\mathbf{v}$ denotes the velocity field which is solenoidal and $\boldsymbol{q}$ is a scalar function. Physically, the parameter λ is defined by

$$\lambda \triangleq \frac{\omega_c}{\sqrt{gH}},$$

where g is the gravity constant, H is the mean active layer depth and ω_c is the Coriolis frequency, assumed to be constant. In the literature, the number $\frac{1}{\lambda}$ is called the *Rossby deformation length* or *Rossby radius* and measures the length scale at which the rotation effects are balanced by the stratification. Notice that small values of λ corresponds to a free surface which is nearly rigid and when $\lambda = 0$ we get Euler equations written in the formulation velocity-vorticity. The velocity field $\mathbf{v}$ writes $\mathbf{v} = \nabla^\perp \boldsymbol{\Psi}$ where $\boldsymbol{\Psi}$ is the stream function governed by the Helmholtz equation,

$$(\Delta - \lambda^2) \boldsymbol{\Psi}(t, \cdot) = \boldsymbol{q}(t, \cdot).$$

To invert this operator we shall make appeal to the Green function T_λ solution of the equation

$$(-\Delta + \lambda^2) T_\lambda = \delta_0 \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Using the Fourier transform yields

$$\forall \xi \in \mathbb{R}^2, \quad \widehat{T_\lambda}(\xi) = \frac{1}{|\xi|^2 + \lambda^2}.$$

Thus by Fourier inversion theorem and using a scaling argument, we find

$$T_\lambda(z) = T_1(\lambda z) \quad \text{with} \quad T_1(z) \triangleq \frac{1}{4\pi^2} \int_{\mathbb{R}^2} \frac{e^{iz\cdot\xi}}{1 + |\xi|^2} d\xi.$$

Applying a polar change of variables gives

$$T_1(z) = \frac{1}{4\pi^2} \int_0^\infty \frac{r}{1 + r^2} \int_0^{2\pi} \cos(|z| r \cos(\theta)) d\theta dr.$$

Simple arguments based on the symmetry of trigonometric functions allow to get the identity

$$\int_0^{2\pi}\cos(|z|r\cos(\theta))d\theta = 2\int_0^{\pi}\cos(|z|r\sin(\theta))d\theta.$$

Consequently, we get in view of (C.1)

$$T_1(z) = \frac{1}{2\pi}\int_0^{\infty}\frac{rJ_0(|z|r)}{1+r^2}dr,$$

where J_n denotes the Bessel function. Applying (C.8) with $\nu=\mu=0$, $a=1$ and $b=|z|$, we finally deduce the representation

$$T_1(z) = \frac{1}{2\pi}K_0(|z|).$$

Therefore, one obtains the following expression for the stream function

$$\boldsymbol{\Psi}(t,z) = -\frac{1}{2\pi}\int_{\mathbb{R}^2}K_0(\lambda|z-\xi|)\boldsymbol{q}(t,\xi)\,dA(\xi), \tag{1.7}$$

where K_0 is the zero-th order modified Bessel function of second kind which expresses as

$$K_0(z) = \log(z)F(z) + G(z), \qquad F, G \text{ analytic functions.} \tag{1.8}$$

A more precise description of the kernel K_0 can be found in (C.7). We point out that K_0 behaves like a logarithm at zero which explains the link with Euler equations whenever the parameter λ tends to 0. We mention that the well-posedness theory of classical solutions for $(QGSW)_\lambda$ equations is not properly written in the literature, but due to the similarity with Euler equations, one can easily prove it for instance in supercritical or even critical Besov regularity.

Formal derivation :

We may follow the calculation developed in [142] with the stronger assumption that the Coriolis frequency is constant. Let us consider a fluid with constant density and such that the height variation scale is small compared to the depth of the fluid. This is typically the case of the ocean and the atmosphere. Then, we can assume the hydrostatic approximation namely the gravitational force and the pressure terms compensate each other. We assume the rotation frequency of the planet to be constant equal to ω_c. Finally, we assume that the bottom of the fluid is flat and at the origin. The velocity field can be written in this context in the following way

$$\mathbf{U}(x,y,z) = \mathbf{u}(x,y) + w(x,y,z)\overrightarrow{k}, \qquad \mathbf{u}(x,y) = u(x,y)\overrightarrow{i} + v(x,y)\overrightarrow{j}.$$

We also denote $h(x,y)$ the thickness of the fluid at point (x,y) and H the average height. Newton's general law allows us to write

$$\partial_t\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \omega_c\overrightarrow{k}\wedge\mathbf{u} + g\nabla h = 0, \qquad \nabla = \nabla_{x,y}. \tag{1.9}$$

The incompressibility of the fluid provides

$$\nabla_{x,y,z}\cdot\mathbf{U} = 0, \quad \text{i.e.} \quad \partial_z w = -\nabla\cdot\mathbf{u}.$$

Integrating the last equation with respect to z yields

$$w(h) = w(0) - h\nabla\cdot\mathbf{u} = -h\nabla\cdot\mathbf{u}.$$

But at the surface, the vertical velocity corresponds to the material derivative of the position of the particle, namely

$$w(h) = \partial_t h + \mathbf{u} \cdot \nabla h.$$

Therefore, combining the foregoing formulae, we obtain the new formulation of the mass conservation

$$\partial_t h + \mathbf{u} \cdot \nabla h + h \nabla \cdot \mathbf{u} = 0. \tag{1.10}$$

The system (1.9)-(1.10) is called *rotating shallow-water* model and we shall derive from this the model of interest by using some asymptotics in a small parameter called Rossby number. For that purpose, we introduce characteristic length L and velocity U assumed to be horizontally isotropic.

$$(x, y) = O(L) \quad \text{and} \quad (u, v) = O(U).$$

Now we define the *Rossby number* R_0 and the *Rossby radius* L_d by

$$R_0 \triangleq \frac{U}{\omega_c L} \qquad \text{and} \qquad L_d \triangleq \frac{\sqrt{gH}}{\omega_c}.$$

The Rossby number is a ratio between advection and rotation term in the equation (1.9). The number L_d, also called *Rossby deformation length* is a length scale measuring the balance between rotation and stratification in (1.9).

	R_0	L_d
Ocean	0.01	25-100km
Atmosphere	0.1	1000-1500km

Then consider the adimensionalized variables

$$\big(\widehat{x}, \widehat{y}, \widehat{z}, \widehat{h}, \widehat{\mathbf{u}}, \widehat{\omega_c}\big) = \Big(\frac{x}{L}, \frac{y}{L}, \frac{z}{L}, \frac{h}{L}, \frac{\mathbf{u}}{U}, 1\Big).$$

Now we assume the quasi-geostrophic hypothesis, namely R_0 is small and the height variations are small. We write that the height h is a perturbation of its mean value H

$$h(x, y) = H + \Delta h(x, y)$$

with the following scale

$$\frac{\Delta h}{H} \sim R_0 \left(\frac{L}{L_d}\right)^2 = O(R_0).$$

Therefore, we have

$$h = H\left(1 + \frac{\Delta h}{H}\right) = H\left(1 + R_0 \frac{L^2}{L_d^2} \widehat{h}\right).$$

We assume that the advection term dominates and that the time scale can be choosen as

$$T = \frac{L}{U}.$$

Adimensionalizing the equations (1.9) and (1.10) leads to

$$R_0 \Big[\partial_{\widehat{t}} \widehat{\mathbf{u}} + \widehat{\mathbf{u}} \cdot \nabla \widehat{\mathbf{u}}\Big] + \overrightarrow{k} \wedge \widehat{\mathbf{u}} = -\nabla \widehat{h} \tag{1.11}$$

and

$$R_0 \left(\frac{L}{L_d}\right)^2 \Big[\partial_{\widehat{t}} \widehat{h} + \widehat{\mathbf{u}} \cdot \nabla \widehat{h}\Big] + \Big[1 + R_0 \left(\frac{L}{L_d}\right)^2 \widehat{h}\Big] \nabla \cdot \widehat{\mathbf{u}} = 0. \tag{1.12}$$

We expend our quantities into power series with respect to the small parameter R_0

$$\begin{cases} \widehat{h} = \widehat{h}_0 + R_0\,\widehat{h}_1 + R_0^2\,\widehat{h}_2 + ... \\ \widehat{u} = \widehat{u}_0 + R_0\,\widehat{u}_1 + R_0^2\,\widehat{u}_2 + ... \\ \widehat{v} = \widehat{v}_0 + R_0\,\widehat{v}_1 + R_0^2\,\widehat{v}_2 + ... \end{cases}$$

Taking the zero-th order terms in R_0 in (1.11), we obtain

$$\big(\widehat{u}_0, \widehat{v}_0\big) = \Big(-\partial_{\widehat{y}}\widehat{h}_0, \partial_{\widehat{x}}\widehat{h}_0\Big). \tag{1.13}$$

Denoting $\widehat{\mathbf{u}}_0 = \big(\widehat{u}_0, \widehat{v}_0\big)$, then we deduce from (1.13) the following mass conservation equation

$$\nabla \cdot \widehat{\mathbf{u}}_0 = 0.$$

At the next order, we get

$$\left(\frac{L}{L_d}\right)^2 \partial_{\widehat{t}}\widehat{h}_0 + \left(\frac{L}{L_d}\right)^2 \widehat{\mathbf{u}}_0 \cdot \nabla\widehat{h}_0 + \nabla \cdot \widehat{\mathbf{u}}_1 = 0. \tag{1.14}$$

Notice that this equation is not closed since it makes appear $\widehat{\mathbf{u}}_1$. Hence, we go to the next order in the momentum equation

$$\partial_{\widehat{t}}\widehat{\mathbf{u}}_0 + (\widehat{\mathbf{u}}_0 \cdot \nabla)\widehat{\mathbf{u}}_0 + \overrightarrow{k} \wedge \widehat{\mathbf{u}}_1 = -\nabla\widehat{h}_1.$$

To get rid of the gradient term in $\widehat{h}_1$, we introduce $\widehat{\xi}_0 \triangleq \nabla^{\perp} \cdot \widehat{\mathbf{u}}_0$, and apply the operator $\nabla^{\perp}\cdot$ to the previous equation leading to

$$\partial_{\widehat{t}}\widehat{\xi}_0 + \widehat{\mathbf{u}}_0 \cdot \nabla\widehat{\xi}_0 = -\nabla \cdot \widehat{\mathbf{u}}_1. \tag{1.15}$$

Inserting (1.14) into (1.15), we infer

$$\partial_{\widehat{t}}\widehat{\xi}_0 + \widehat{\mathbf{u}}_0 \cdot \nabla\widehat{\xi}_0 = \left(\frac{L}{L_d}\right)^2 \partial_{\widehat{t}}\widehat{h}_0 + \left(\frac{L}{L_d}\right)^2 \widehat{\mathbf{u}}_0 \cdot \nabla\widehat{h}_0.$$

Denoting $\widehat{\mathbf{\Psi}} \triangleq \widehat{h}_0$, we have by virtue of (1.13)

$$\big(\widehat{u}_0, \widehat{v}_0\big) = \Big(-\partial_{\widehat{y}}\widehat{\mathbf{\Psi}}, \partial_{\widehat{x}}\widehat{\mathbf{\Psi}}\Big), \qquad \widehat{\xi}_0 = \Delta\widehat{\mathbf{\Psi}}$$

and the previous equation becomes

$$\partial_{\widehat{t}}\left(\Delta\widehat{\mathbf{\Psi}} - \left(\frac{L}{L_d}\right)^2 \widehat{\mathbf{\Psi}}\right) + \widehat{\mathbf{u}}_0 \cdot \nabla \left(\Delta\widehat{\mathbf{\Psi}} - \left(\frac{L}{L_d}\right)^2 \widehat{\mathbf{\Psi}}\right) = 0.$$

Coming back to the dimensionalized quantities, denoting $\mathbf{v} \triangleq \mathbf{u}_0 = (u_0, v_0)$, $\lambda \triangleq \frac{1}{L_d}$ and $\boldsymbol{q} \triangleq (\Delta - \lambda^2)\mathbf{\Psi}$

$$\mathbf{v} = \nabla^{\perp}\mathbf{\Psi}, \qquad \partial_t\boldsymbol{q} + \mathbf{v} \cdot \nabla\boldsymbol{q} = 0.$$

1.2 Vortex patches : general facts and periodic rigid motion

This section provides an introduction to the vortex patch theory and more precisely to the study of uniformly rotating solutions. During the past decade, this theory has been well-developed for Euler and SQG equations using bifurcation theory. Only few theoretical results were obtained in [54] for QGSW equations. For our purpose, we may briefly deal with SQG equations in the discussion and rather focus on the other two models.

For (1.5), the global existence and uniqueness for weak solutions bounded and integrable follows from Yudovich's theory [149] for Euler equations in the plane or in the unit disc. One can adapt the theory to the case of QGSW equations due to the logarithmic behaviour of the kernel K_0 at the origin. In particular, if the initial datum is a vortex patch, that is, the characteristic function of a bounded planar domain D_0, then the solution keeps a patch form $\mathbf{1}_{D_t}$ for any time $t > 0$, where D_t is the transported domain D_0 by the flow map associated to the velocity field $\mathbf{v}$, namely

$$D_t \triangleq \mathbf{\Phi}_t(D_0), \qquad \mathbf{\Phi}_t(x) \triangleq x + \int_0^t \mathbf{v}(s, \mathbf{\Phi}_s(x))ds.$$

The boundary motion in the smooth case reduces to tackle the evolution of a curve in the complex plane surrounding a constant area domain and subject to the deformation induced by its own effect. Local/global in time persistence of the boundary regularity is a relevant subject in fluid dynamics and has attracted a lot of attention during the past decades, not only for Euler equations but also for similar active scalar equations such as generalized surface quasi-geostrophic equations or the aggregation equation. The persistence of the regularity of the boundary was proved in [35, 46, 45] for the full plane Eulerian case and in [53] for the case of the unit disc. Let us now briefly see how to write down the contour dynamics equations, more details can be found in [99, 100]. Given a smooth parametrization $z(t, \cdot) : \mathbb{T} \to \partial D_t$ of the boundary of the patch, then as particles located at the boundary move with the boundary then we get the evolution equation

$$\Big[\partial_t z(t,\theta) - \mathbf{v}(t, z(t,\theta))\Big] \cdot \mathbf{n}(t, z(t,\theta)) = 0, \tag{1.16}$$

where $\mathbf{n}(t, z(t,\theta))$ is the outward normal vector to the boundary ∂D_t of D_t at the point $z(t,\theta)$. This equation reflects the fact that the particle velocity and the boundary velocity admit the same normal components which is a classical fact for free boundary problems. As we shall see later along the document, the equation (1.16) is the starting point for our discussions. More precisely, we may start with the complex formulation of (1.16). Since one has, up to a real constant of renormalization, $\mathbf{n}(t, z(t,\theta)) = -\mathrm{i}\partial_\theta z(t,\theta)$, then we find the complex form of the contour dynamics motion,

$$\mathrm{Im}\left(\Big[\partial_t z(t,\theta) - \mathbf{v}(t, z(t,\theta))\Big]\overline{\partial_\theta z(t,\theta)}\right) = 0. \tag{1.17}$$

Notice that due to the symmetry property (1.1) of the Green kernel, any radial profile generates a stationary solution. It is a classical fact to look for periodic or quasi-periodic solutions close to these equilibrium state solutions for Hamiltonian systems like (1.5)-(1.6). Looking for particular solutions where the domain moves without any shape deformation is a traditional subject in fluid dynamics and important developments have been performed a long time ago. In the literature, these structures appear under different names: relative equilibria, V-states, long-lived structures, vortex crystals, etc... A particular class of periodic solutions is given by the rigid body rotating vortex patches around the origin described by

$$\boldsymbol{\omega}(t, \cdot) = \mathbf{1}_{D_t} \quad \text{with} \quad D_t = e^{\mathrm{i}t\Omega} D_0, \tag{1.18}$$

where Ω is a time independent angular velocity. These solutions are periodic in time with period $\frac{2\pi}{\Omega}$ or equivalently with frequency Ω. Such solutions are called V-states according to the terminology introduced by Deem and Zabusky in [52] where they numerically obtained the first examples of $\mathbf{m}$-fold uniformly rotating vortex patches solutions to Euler equations for small values of $\mathbf{m}$. Recall that a domain D_0 is called $\mathbf{m}$-fold if it is invariant under the action of the diedral group $D_{\mathbf{m}}$, with the convention that $\mathbf{1}$-fold (resp. $\mathbf{2}$-fold) means to admit one (resp. two) axis of symmetry. Consequently, a $\mathbf{m}$-fold uniformly rotating vortex patch is a solution in the form (1.18) with $\mathbf{m}$-fold inital domain D_0.

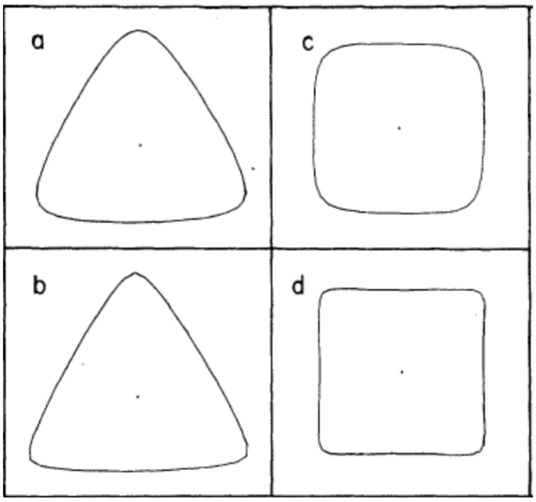


Figure 4: Numerical simulations of V-states by Deem and Zabusky in [52].

The first explicit example, discovered by Kirchhoff in [114], is the ellipse which rotates about its center of mass with the constant angular velocity

$$\Omega = \frac{ab}{(a+b)^2},$$

where a and b are the semi-axes of the ellipse. Further families of implicit solutions with higher symmetries were established by Burbea in [42] using local bifurcation tools and complex analysis. More precisely, he proved the existence of branches of $\mathbf{m}$-fold rotating solutions bifurcating from the discs at angular velocities

$$\mathbf{\Omega_m} \triangleq \frac{\mathbf{m}-1}{2\mathbf{m}}, \quad \mathbf{m} \geqslant 1. \tag{1.19}$$

Notice that the mode $\mathbf{m}=1$ corresponds to a translation of the trivial solution and the second branch, emerging at $\mathbf{\Omega_2}=\frac{1}{4}$, describes the Kirchhoff ellipses. Moreover, all the bifurcation angular velocities $\mathbf{\Omega_m}$ are in the range $(0,\frac{1}{2})$. Outside this interval, the only uniformly rotating solutions are the radial ones, as proved in the series of papers [68, 78, 93]. The boundary regularity was first discussed in [43, 88, 99] and the global bifurcation diagram was studied in [88].

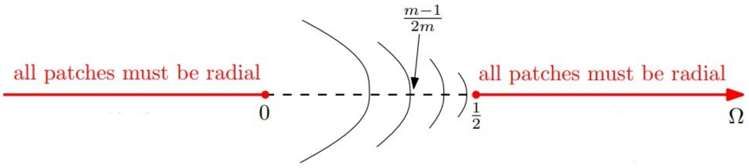


Figure 5: Local bifurcation diagram of uniformly rotating vortex patch solutions for Euler equation [78].

Note also that countable branches of rotating patches bifurcating from the ellipses at implicit angular velocities were found in [96, 97], however, the shapes have in fact less symmetry and being at most two-folds. The doubly-connected case has also been explored. To fix the terminology, a bounded open domain D_0 is said doubly-connected if

$$D_0 = D_1 \backslash \overline{D_2},$$

where D_1 and D_2 are two bounded open simply-connected domains with $\overline{D_2} \subset D_1$. This means that the boundary of D_0 is given by two interfaces, one of them is contained in the open region delimited by the second one. In [94, Thm. B], the authors proved for Euler equations that under the condition

$$1+b^{\mathbf{m}}-\frac{\mathbf{m}(1-b^2)}{2}<0,\quad b\in(0,1),\quad \mathbf{m}\in\mathbb{N}^*$$

one can find two branches of **m**-fold doubly-connected V-states bifurcating from the normalized annulus A_b, defined by

$$A_b\triangleq\Big\{z\in\mathbb{C}\quad\text{s.t.}\quad b<|z|<1\Big\}\tag{1.20}$$

at the following angular velocities

$$\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(b)\triangleq\frac{1-b^2}{4}\pm\frac{1}{2\mathbf{m}}\sqrt{\left(\frac{\mathbf{m}(1-b^2)}{2}-1\right)^2-b^{2\mathbf{m}}}.\tag{1.21}$$

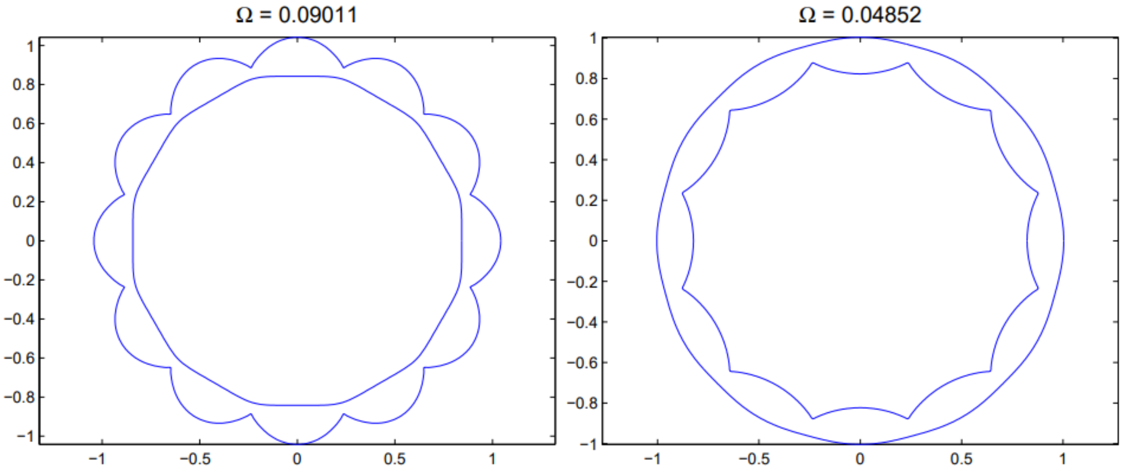


Figure 6: Exemples of 12-fold doubly-connected V-states for Euler equations [94].

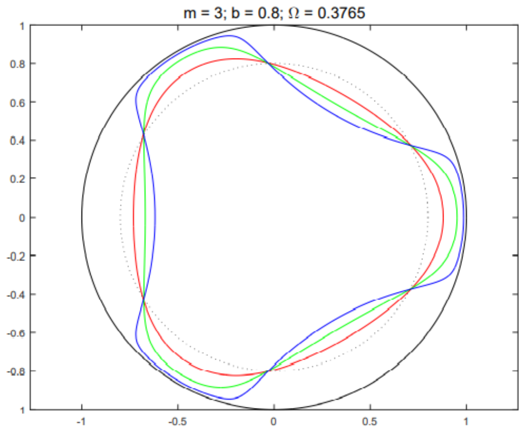


Figure 7: 3-fold V-states for Euler equations in the unit disc [86].

Let us now turn to the case where Euler equations (1.5) are set in the unit disc. The theory of weak solutions and vortex patches is still valid in this context and the persistence of the boundary regularity of vortex patches remains true, as proved in [53]. The existence of V-states close to the discs $b\mathbb{D}$ ($b\in(0,1)$), also called Rankine vortices, were obtained in [86]. These curves of solutions have **m**-fold symmetry, perform a uniform rotation and emerge at the angular velocities

$$\boldsymbol{\Omega}_{\mathbf{m}}(b) \triangleq \frac{\mathbf{m}-1+b^{2\mathbf{m}}}{2\mathbf{m}}, \quad \mathbf{m} \geqslant 1. \tag{1.22}$$

It is of paramount importance to highlight different boundary effects observable at this periodic level. First, Burbea's frequencies (1.19) are shifted to the right, implying in particular that the **1**-fold patches, which are not centered at the origin, are no longer associated to the trivial solution. Second, the numerical observations in [86] show that the bifurcation curves have oscillations, see Figure 8.

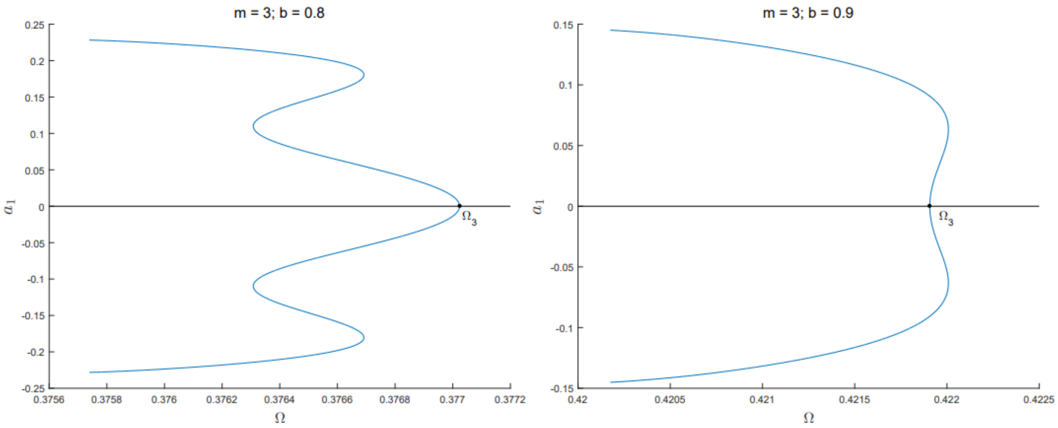


Figure 8: Oscillations in the bifucation curves for Euler equations in the unit disc [86].

In the same paper, the authors also studied the bifurcation from the annulus

$$A_{b_1,b_2} \triangleq \Big\{z \in \mathbb{C} \quad \text{s.t.} \quad b_2 < |z| < b_1\Big\}, \qquad 0 < b_2 < b_1 < 1,$$

which occurs for the following angular velocities ($b = \frac{b_2}{b_1}$)

$$\boldsymbol{\Omega}^{\pm}_{\mathbf{m}}(b_1,b_2) \triangleq \frac{1-b^2}{4} + \frac{b_1^{2\mathbf{m}}-b_2^{2\mathbf{m}}}{4\mathbf{m}} \pm \sqrt{\left[\frac{1-b^2}{2} - \frac{2-b_1^{2\mathbf{m}}-b_2^{2\mathbf{m}}}{2\mathbf{m}}\right]^2 - b^{2\mathbf{m}}\left(\frac{1-b_1^{2\mathbf{m}}}{\mathbf{m}}\right)^2}$$

provided that the following condition is fullfilled

$$\mathbf{m} > \frac{2+2b^{\mathbf{m}}-(b_1^{\mathbf{m}}+b_2^{\mathbf{m}})^2}{1-b^2}.$$

Concerning $(\mathrm{QGSW})_\lambda$ there are few results dealing with relative equilibria. Interesting numerical simulations showing the complexity and the richness of the bifurcation diagram with respect to the parameter λ was studied in [54, 55]. In [54], using bifurcation tools the authors proved analoguous results to those of Burbea. They show in particular the existence of branches of **m**-fold symmetric V-states ($\mathbf{m} \geq 2$) bifurcating from the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ with the angular velocity

$$\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) \triangleq I_1(\lambda)K_1(\lambda) - I_{\mathbf{m}}(\lambda)K_{\mathbf{m}}(\lambda), \tag{1.23}$$

where I_m and K_m are the modified Bessel functions of first and second kind. For more details about these functions, we refer to the Appendix C. Notice that in the same paper the authors explored the two-fold branch when λ is small and proved first that it is located close to the ellipse branch of Euler equations and second it is not connected (see Figure 9) and from numerical simulations they put in evidence the fragmentation of this branch in multiple connected pieces. The second bifurcation from this branch was also analyzed leading to similar results as for Euler equations.

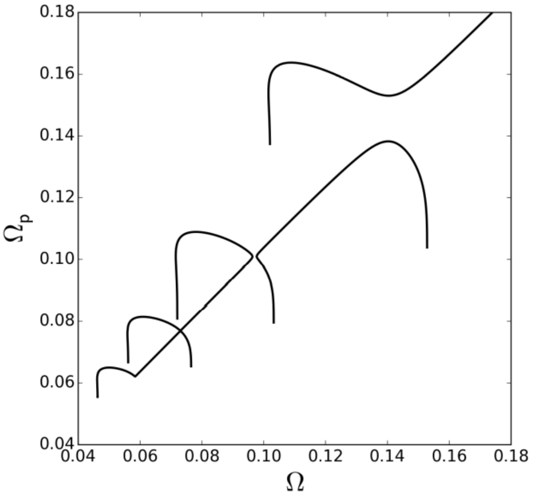


Figure 9: Disconnected bifurcation diagram for 2-fold bifurcation curves from Kirchhoff branch for QGSW equations [54].

We point out that numerical simulations showed the behaviour of the end of branches of bifurcation for the different models. The corresponding patches are called *limiting V-states* and seem to present singularities in their boundary. Nevertheless, no theorical result is known so far. We also mention that more investigations on the V-states have been implemented during the past decade by several authors in different settings like for the SQG equations [43, 77, 83, 85, 137] or for the multipole case [70, 71, 84, 90, 98].

1.3 Quasi-periodic solutions for Hamiltonian systems : KAM theory

In this section, we present the basis of KAM and Nash-Moser theories. We also discuss some recent results about the application of these theories to PDE. The methods developed in this section will be the one used all along the document in the proofs of our results. First, we give the definition of quasi-periodic functions which is the notion of interest in this study. A function $f:\mathbb{R}\to\mathbb{C}$ is said to be *quasi-periodic* if there exists a continuous function $F:\mathbb{T}^d\to\mathbb{C}$ such that

$$\forall\, t\in\mathbb{R},\quad f(t)=F(\omega t)$$

for some frequency vector $\omega\in\mathbb{R}^d$ ($d\in\mathbb{N}^*$) which is *non-resonant*, that is

$$\forall\, l\in\mathbb{Z}^d\setminus\{0\},\quad \omega\cdot l\neq 0. \tag{1.24}$$

Here and in the sequel, we denote $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$. In the case $d=1$, we recover from this definition periodic functions with frequency $\omega\in\mathbb{R}^*$. The variable living in the d-dimensional torus $\mathbb{T}^d$ will be generically denoted φ in the remainder of the document. The archetype of quasi-periodic function is given by

$$f(t)=\sum_{j=1}^{d}\mathtt{a}_j e^{\mathrm{i}\omega_j t},\qquad \mathtt{a}_j\in\mathbb{C},\qquad \omega=(\omega_1,\ldots,\omega_d)\in\mathbb{R}^d\text{ non-resonant.}$$

From a dynamical point of view, the trajectory is densely contained in a d-dimensional torus, see Figure 10. We say that this torus is *invariant* and supports a quasi-periodic motion with frequency vector ω.

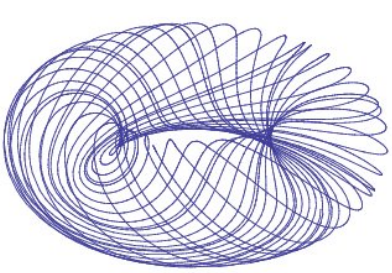


Figure 10: Quasi-periodc trajectory in an invariant torus [116].

Consider a finite dimensional Hamiltonian system with $2d$ variables associated to a Hamiltonian H

$$\left\{\begin{array}{l}\dot{p}=-\nabla_q H(p,q)\\ \dot{q}=\nabla_p H(p,q)\end{array}\right.\quad,\quad\text{i.e.}\quad\begin{pmatrix}\dot{p}\\ \dot{q}\end{pmatrix}=J\nabla_{p,q}H(p,q),\quad J\triangleq\begin{pmatrix}0 & -I_d\\ I_d & 0\end{pmatrix}\tag{1.25}$$

This is a particular class of ODE appearing naturally in Physics whenever the energy of a system is conserved and in this case, the Hamiltonian H is related to this energy. Indeed, in presence of conservative forces, the Newton's law can be written in such a form. Lagrangian/Hamiltonian formalism is an important and elegant mathematical aspect of Physics allowing to simplify the computations. The cost is a mathematical abstraction into the world of symplectic geometry. Here we discuss the finite dimensional Hamiltonian systems but the same formalism can be transposed into infinite dimension through PDE. The latter is the context for the next sections. The Hamiltonian H in (1.25) is said to be *Liouville-integrable* if there exist $(F_j)_{1\leqslant j\leqslant d}\in\mathcal{C}^\infty(\mathbb{R}^{2d},\mathbb{R})^d$ such that

- $\forall j\in[\![1,d]\!],\{F_j,H\}=0$ (i.e. the F_j are intragrals of the motion).
- $\forall(j,k)\in[\![1,d]\!]^2,\{F_j,F_k\}=0$ (i.e. the F_j are in involution).
- $(\nabla_{p,q}F_j)_{1\leqslant j\leqslant N}$ is a free family.

Notice that $\{\cdot,\cdot\}$ is the Poisson structure induced by the Hamiltonian system (1.25) and defined by

$$\{F,G\}\triangleq\sum_{i=1}^{d}\partial_{q_i}F\partial_{p_i}G-\partial_{p_i}F\partial_{q_i}G.$$

Then, the Arnold-Liouville Theorem asserts that if we assume that H is Liouville-integrable and that there exists $c\in\mathbb{R}^d$ such that the set

$$M_c\triangleq\Big\{(p,q)\in\mathbb{R}^{2d}\quad\text{s.t.}\quad\forall j\in[\![1,d]\!],\,F_j(p,q)=c_j\Big\}$$

is connected and compact. Then, there exist a neighbourhood U of M_c, a neighbourhood D of 0 in $\mathbb{R}^d$ and a symplectic (that is which preserves Hamiltonian structures) change of variables

$$\begin{array}{rccc}A: & D\times\mathbb{T}^d & \rightarrow & U\\ & (I,\vartheta) & \mapsto & (p,q)\end{array}$$

such that $H\circ A(I,\vartheta)=h(I)$ is a function of I only. Therefore, the equations become

$$\begin{cases}\dot{I}=0\\ \dot{\vartheta}=\nabla_I h(I)\triangleq\omega(I).\end{cases}$$

Thus, we can integrate the system and obtain

$$\begin{cases}I(t)=I(0)\triangleq I_0\\ \vartheta(t)=\omega(I_0)t+\vartheta(0).\end{cases}$$

Consequently, the motion is confined in the torus $\{I_0\}\times\mathbb{T}^d$ and given by the linear flow ϑ. Therefore, the nature of the motion depends on the arithmetic properties of $\omega(I)$. In particular, if $\omega(I)$ is non-resonnant, then the phase space is foliated by Lagrangian invariant tori carrying a quasi-periodic dynamics with frequency vector $\omega(I)$. Recall that a torus is said to be *Lagrangian* if the restriction of the symplectic form $dI\wedge d\vartheta$ to its tangent space vanishes and the dimension of the torus is maximal (equal to d) for this property. Notice that the variables (I,ϑ) are called *action-angle variables* and such denomination can be justified by the archetype of Hamiltonian satisfying the Arnold-Liouville conditions, namely the harmonic oscillator on $\mathbb{R}^2$

$$H(p,q)=\frac{1}{2}\big(p^2+q^2\big)$$

with

$$F_1=H,\qquad (p,q)=A(I,\vartheta)=\big(\sqrt{2I}\cos(\vartheta),\sqrt{2I}\sin(\vartheta)\big),\qquad H\circ A(I,\vartheta)=I.\tag{1.26}$$

Indeed, one can easily check that the application A is symplectic since

$$dp\wedge dq=dI\wedge d\vartheta.$$

The study of quasi-periodic solutions to perturbations of integrable Hamiltonian systems goes back to the pioneering works of Kolmogorov [115], Arnold [5] and Moser [126] where they proved, in finite dimension and under suitable non degeneracy and smoothness conditions, the persistence of invariant tori for small perturbations of integrable Hamiltonian systems. In the action-angle variables (I,ϑ), such perturbation can be written as

$$H(I,\vartheta)=h(I)+\varepsilon P(I,\vartheta),\qquad \varepsilon\ll 1.\tag{1.27}$$

The various techniques and ideas used to study such kind of problems are now gathered under the name of KAM theory, in honor of Kolmogorov, Arnold and Moser. Kolmogorov's Theorem states as follows.

Kolmogorov's Theorem :
Consider an Hamiltonian H in the form (1.27) being real-analytic on the closure of a domain $D\times\mathbb{T}^d$. Assume that for some $I^*\in D$, the following conditions hold.

1. The frequency vector $\omega(I^*)\triangleq\nabla h(I^*)$ is Diophantine, namely $\omega(I^*)\in\mathtt{DC}(\gamma,\tau)$ where for given $\gamma\in(0,1)$ and $\tau>d-1$, the Diophantine set $\mathtt{DC}(\gamma,\tau)$ is given by

$$\mathtt{DC}(\gamma,\tau)\triangleq\bigcap_{l\in\mathbb{Z}^d\setminus\{0\}}\Big\{\omega\in\mathbb{R}^d\quad\text{s.t.}\quad|\omega\cdot l|>\tfrac{\gamma}{\langle l\rangle^{\tau}}\Big\},\qquad\langle l\rangle\triangleq\max(1,|l|).\tag{1.28}$$

2. Non-degeneracy/twist condition

$$\det\left(\frac{\partial^2 h}{\partial I_i\partial I_j}(I^*)\right)_{1\leqslant i,j\leqslant d}\neq 0.$$

Then for ε small enough, the torus $\{I^*\} \times \mathbb{T}^d$ persits for the pertubed Hamiltonian system associated with H, being just slightly deformed, as a Lagrangian invariant torus carrying a quasi-periodic motion with the same frequency vector $\omega(I^*)$.

For a complete and pedagogical proof of Kolmogorov's result, we refer the reader to [132, 146]. It is based on a Newton method, where at each step, we shall remove some terms from the perturbation P which imply at the end the preservation of one invariant torus.

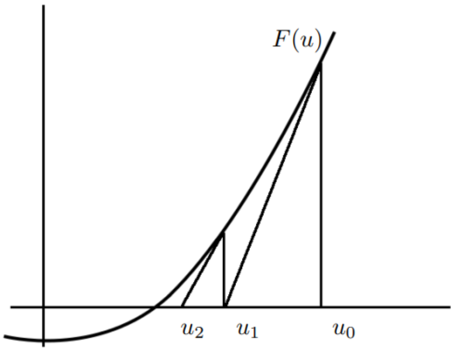


Figure 11: Three steps of the classical Newton scheme $u_{n+1} = u_n - \big(DF(u_n)\big)^{-1} F(u_n)$ [67].

We shall now discuss the hypothesis and the main points in the proof of Kolmogorov's Theorem. First, at each step of the Newton method, we compose the previous Hamiltonian by a well-chosen symplectic change of variables in order to improve the structure of the Hamiltonian system in terms of normal form plus a perturbation. This transformation is chosen as the solution of a functional equation called the *homological equation* and we shall explain the typical difficulty appearing at this level by discussing the fundamental theorem of calculus in the quasi-periodic setting, namely the inversion of the operator $\omega \cdot \partial_\varphi$. Given $g : \mathbb{T}^d \to \mathbb{R}$ with zero average, we look for a function f solution of the equation

$$\omega \cdot \partial_\varphi f = g. \tag{1.29}$$

To solve (1.29), we expand into Fourier series which yields

$$f(\varphi) = \sum_{l \in \mathbb{Z}^d \setminus \{0\}} \frac{g_l}{\omega \cdot l} e^{\mathrm{i} l \cdot \varphi}. \tag{1.30}$$

For a long time people like Poincaré thought that it was not possible to make such series convergent due to the possible smallness of the denominator. The key idea of Kolmogorov was to introduce the Diophantine conditions (1.28) for ω to control the small divisors problem and get only an algebraic loss of regularity. Such kind of non-resonance conditions are called *zero-th order Melnikov conditions.* We mention that when looking for lower dimensional invariant tori (i.e. non-Lagrangian), one should need to deal with other type of non-resonant condition called *first or second order Melnikov conditions.* We refer the reader to [125, 133]. Notice that for the Hamiltonian PDE as of interest is this document, we are led to consider this type of non-resonant conditions, see Propositions 7.5 and 13.4. Taking $\omega \in \mathtt{DC}(\gamma, \tau)$, then we can estimate f, given by (1.30), in Sobolev norm and we obtain the following estimate with loss of regularity τ

$$\|f\|_s \lesssim \gamma^{-1} \|g\|_{s+\tau}.$$

In the analytic setting, treated by Kolmogorov and Arnorld, the fixed loss of regularity τ corresponds to a shrinkness of the domain of analyticity and the quadratic convergence of the Newton method is sufficient to overcome the small divisors problem so that the final domain of analyticity is non-empty. Nevertheless, in the finitely many differentiable case, this is an important issue and the method may fail. Moser overcame this problem using what is now called "Nash-Moser procedure". The twist condition ensures that the application $I \mapsto \omega(I)$ is a local diffeomorphism which allows to consider the frequencies as independant parameters and to follow the shift of frequencies along the scheme in terms of tori. This condition, which is not satisfied by the harmonic oscillator for instance, is quite strong and actually it is now known since the works of Rüssmann [140], see also Sevryuk [141], that the minimal assumption is that the frequency map $I \mapsto \omega(I)$ is not contained in an hyperplane of $\mathbb{R}^d$. We call this fact the *non-degeneracy property.* Notice that such properties are the one used later in the proofs of the main results of this thesis, see Lemmata 5.4 and 11.4. In particular they are used to prove the so-called *Rüssmann transversality conditions* Lemmata 5.5 and 11.5, useful for measuring the Cantor sets of admissible parameters generating quasi-periodic solutions similarly to [12] and based on the application of [139, Thm. 17.1].

We shall now present the main steps concerning a Nash-Moser procedure, an iterative scheme used in this document for the construction of quasi-periodic vortex patches. The Nash-Moser iteration is a modification of the standard Newton scheme making appeal to regularizing operators $(\Pi_N)_N$ in order to solve an equation $F(U) = 0$ in a Banach scale allowing some fixed loss of derivatives at each step. This strategy was first introduced by Nash in [127] to prove the isometric embedding theorem. There exist many versions of the Nash-Moser scheme and we refer the interested reader to [3, 18, 103, 150, 151]. For our later purpose, we may present one of them adapted to the Sobolev Banach scale $(H^s)_{s\geqslant 0}$ and taken from the book of Massimiliano Berti [18, Thm. 3.6]. We consider a differentiable function F satisfying the following tame estimates : there exists $\alpha \geqslant 0$ such that for any $s \geqslant 0$, for any $(u, u', h) \in H^{s+\alpha}$,

$$\begin{cases} \|F(u)\|_s \lesssim 1 + \|u\|_{s+\alpha} \\ \|DF(u)[h]\|_s \lesssim \|h\|_{s+\alpha} \\ \|F(u) - F(u') - DF(u)[u' - u]\|_s \lesssim \|u - u'\|_{s+\alpha}^2 \end{cases}$$

Assume that there exists a right inverse T of the linearized operator with fixed loss $\tau \geqslant 0$ of derivatives, namely

$$DF(u) \circ \mathrm{T}(u)[h] = h, \qquad \|\mathrm{T}(u)[h]\|_s \lesssim \|h\|_{s+\tau}.$$

We shall also assume that there exists a Sobolev index $s_0 > \alpha + \tau$ such that

$$\|F(0)\|_{s_0+\tau} \ll 1.$$

Then introducing the family of projectors $(\Pi_N)_N$ and $\left(\Pi_N^\perp\right)_N$ defined by

$$\Pi_N\left(\sum_{j\in\mathbb{Z}} f_j e^{\mathrm{i}j\theta}\right) = \sum_{\substack{j\in\mathbb{Z}\\ |j|\leqslant N}} f_j e^{\mathrm{i}j\theta}, \qquad \Pi_N^\perp = \mathrm{Id} - \Pi_N$$

and satisfying

$$\forall t \geqslant 0, \qquad \begin{cases} \|\Pi_N u\|_{s+t} \leqslant N^t \|u\|_s \\ \|\Pi_N^\perp u\|_s \leqslant N^{-t} \|u\|_{s+t} \end{cases},$$

we can make the following scheme convergent

$$u_0 = 0, \qquad u_{n+1} = u_n - \Pi_{N_n}\mathrm{T}(u_n)F(u_n), \qquad N_n = N_0^{\left(\frac{3}{2}\right)^n}, \qquad N_0 \gg 1.$$

Indeed, one has the induction inequality for some a priori free large parameter β.

$$\|u_{n+1}-u_n\|_{s_0}\lesssim N_n^{\alpha+\tau-\frac{2}{3}\beta}\|\mathrm{T}(u_{n-1})F(u_{n-1})\|_{s_0+\beta}+N_n^{\alpha+\tau}\|u_n-u_{n-1}\|_{s_0}^2.$$

Observe that the previous inequality makes appear a fast decaying term linear in a high regularity norm ($s_0+\beta$) and a quadratic term in low regularity norm (s_0). Then, one can prove by induction that for suitable selected values of β and for some well-chosen parameter $\nu>0$, we have

$$\|\mathrm{T}(u_n)F(u_n)\|_{s_0+\beta}\leqslant N_n^{\nu}\quad\text{and}\quad\|u_{n+1}-u_n\|_{s_0}\leqslant N_n^{-\nu}.$$

The second estimate allows to make the scheme convergent and one obtains the convergence $F(u_n)\to 0$ for $n\to\infty$ due to the relation

$$\|F(u_n)\|_{s_0-\alpha}\lesssim N_n^{-\frac{2}{3}\beta}\|\mathrm{T}(u_{n-1})F(u_{n-1})\|_{s_0+\beta}+\|u_n-u_{n-1}\|_{s_0}^2.$$

The reader is refered to [18, Thm. 3.6] for the missing details hidden here to avoid too much technicality.

Later on, in the 80-90's, started the investigation for quasi-periodic solutions to PDE viewed as lower dimensional invariant tori for infinite dimensional Hamiltonian systems. Such equations can generally be written as

$$\partial_t u=J\nabla H(u),\tag{1.31}$$

where H is a functional over an infinite dimensional Hilbert function space $\mathtt{H}$ and J is an antisymmetric non-degenerate operator. Normal forms KAM methods and Nash-Moser implicit function iterative schemes were explored and developed in partial differential equations by several authors leading to important contributions and opening new perspectives. The complexity of the problem depends on the space dimension and on the structure of the equations (semilinear, quasilinear, fully nonlinear, asymptotic of the linear frequencies). The first use of KAM methods for PDE was proposed by Kuksin [119] in 1987 and Wayne [147] in 1990 regarding parameter dependant (through a potential) one dimensional NLS and NLW with Dirichlet boundary conditions. The corresponding proofs are based on KAM reducibilty techniques whose main feature is the following. Consider a quasi-periodically time dependant linear system

$$\partial_t h+A(\omega t)h=0,\tag{1.32}$$

with $h\in\mathtt{H}$. In practice (1.32) corresponds to the linearization of (1.31), up to a restriction to a closed subspace of the phase space $\mathtt{H}$, at a quasi-periodic solution with frequency $\omega\in\mathbb{R}^d$, namely $A(\omega t)=-J(D^2H)[u(\omega t),\cdot]$. One wants to find a bounded invertible linear transformation $\Phi(\varphi):\mathtt{H}\to\mathtt{H}$, depending smoothly on $\varphi\in\mathbb{T}^d$, such that under the change of unknown

$$h=\Phi(\omega t)[v],$$

the equation (1.32) reduces to

$$\partial_t v+Bv=0,\tag{1.33}$$

with $B=\mathrm{diag}_j(b_j)$ a diagonal time independant operator. Therefore, denoting $(v_j)_j$ the decomposition of v in a Hilbert basis of eigenvectors of B, the equation (1.33) decouples into

$$\partial_t v_j+b_jv_j=0,\qquad\text{i.e.}\qquad v_j(t)=e^{-b_jt}v_j(0).$$

Then, if $b_j=\mathrm{i}\mu_j\in\mathrm{i}\mathbb{R}$, one recovers the linear stability in the sense of dynamical systems related to the Floquet exponents theory. In addition, the knowledge of the asymptotic expansion of the μ_j allows to control the small divisors problems that can appear. A particular case (of interest in the sequel) of application of KAM reducibilty techniques is when

$$A = D + R, \qquad D = \mathrm{diag}_j(\mathrm{i}d_j),$$

with $(\mathrm{i}d_j)_j$ simple eigenvalues and R a perturbation. In this case, the choice of Φ is done in such a way to reduce quadratically the size of the remainder up to an additional diagonal contribution. At this stage the second order Melnikov conditions naturally appear. These are Diophantine constraints in the form

$$\forall (l, j, j') \in \mathbb{Z}^d \times \mathbb{Z}^2, \quad \left|\omega \cdot l + d_j - d_{j'}\right| > \frac{\gamma}{\langle l \rangle^\tau},$$

with γ and τ as in the statement of Kolmogorov's Theorem. Iterating the procedure allows to diagonalize the oparator A. For more details, we refer to [24] and to Section 7.3.2. The works of Kuksin and Wayne were later extended to parameter independant situations by Kuksin-Pöschel [120] for the NLS and Pöschel [134] for the nonlinear Klein-Gordon equation. Then for one dimensional equations with periodic boundary conditions the first result is due to Craig-Wayne [49] for time periodic solutions for the NLW. They introduced a completely different approach with respect to Kuksin and Wayne, with a Lyapunov-Schmidt decomposition and a multiscale approach to invert the linearized operator with tame estimates for the inverse. Then, Bourgain extended this work to the search of quasi-periodic solutions to the one dimensional semilinear NLS and NLW in [39]. The KAM reducibility approach was extended later by Chierchia-You [47] for 1D semilinear wave equations with periodic boundary conditions. The first KAM reducibility result for NLS with $x \in \mathbb{T}$ has been proved by Eliasson-Kuksin in [60]. Then, the Nash-Moser scheme was used to find periodic solutions for completely resonant nonlinear 1D wave equations with Dirichlet boundary conditions, both with analytic and differentiable nonlinearities, see [22, 23]. We also refer to [74].

The case of higher dimension was first studied by Bourgain in [41] looking for time quasi-periodic solutions for the NLS on $\mathbb{T}^2$, followed by the results on the NLW on $\mathbb{T}^d$, for time periodic [38] and quasi-periodic solutions [40]. His solutions were analytic. The existence of quasi-periodic solutions for bounded perturbations (cubic or convolution-type) of the multidimensional NLS has been studied in the series of papers [60, 72, 135]. We also mention the result of Grébert-Kappeler [79], were they obtained the existence of quasi-periodic solutions for Hamiltonian perturbations of the defocusing NLS. The extension of the regularization method to finite Sobolev regularity in higher dimensions was considered by Berti and Bolle for quasi-periodic solutions on $\mathbb{T}^d$ of the NLW [26] and of the NLS [25] with an external potential. We also mention [34, 27], where they provided an abstract Nash-Moser theorem for the NLS and the NLW on compact Lie groups. More recently, Berti and Bolle obtained in [24] the existence of small amplitude time quasi-periodic solutions for the NLW with multiplicative potential and cubic nonlinearity for the NLW in any dimensional torus. KAM results have been proved for parameter dependent beam equations by Geng-You [73], Procesi [136], and, more recently, in Eliasson-Grébert-Kuksin [59] for multidimensional beam equations. We also mention the KAM result by Grébert-Thomann [81] for smoothing nonlinear perturbations of the 1d harmonic oscillator and Grébert-Paturel [80] in higher space dimension.

KAM theory was also developed for equations involving unbounded nonlinearities. In this case, the symplectic transformation at each step of the KAM iteration may lose space derivatives which destroys the convergence of the scheme. The first results in this direction for semilinear PDE with unbounded perturbations were obtained by Kuksin [117] and Kappeler-Pöschel [109] for Hamiltonian, analytic perturbations of the KdV equation on the torus. The case of quasilinear or even fully nonlinear is much harder and the pioneering works in this situation were presented by Plotnikov-Toland [128] and

Iooss-Plotnikov-Toland [108], for the existence of 2D periodic standing waves with finite and infinite depth, respectively. We also mention that in the same decade, Baldi [6] studied a forced quasilinear Kirchhoff equation on a bounded domain in $\mathbb{R}^d$ with Dirichlet boundary condition and on the d-dimensional periodic domain. The nonlinearity is there space-independent and he found time periodic solutions in this context. The first existence results for time quasi-periodic solutions for quasilinear and fully nonlinear PDE are due to Baldi-Berti-Montalto for some quasilinear and fully nonlinear perturbations of the forced Airy equation [10], of the autonomous KdV [8] and of the autonomous modified KdV [9]. These results are obtained with a Nash-Moser iteration as stated in [21], where the analysis of the linearized operator is inspired by the descent regularization procedure introduced by Plotnikov-Toland [128] via pseudo-differential calculus combined with the KAM reducibility scheme. The Nash-Moser approach was applied also by Feola-Procesi [67], who considered a class of fully nonlinear forced and reversible Schrödinger equations on the torus and proved existence and stability of quasi-periodic solutions, see also [62] for the quasi-linear Hamiltonian case and [66] for the fully nonlinear autonomous case. We refer also to the work of Giuliani [76] for quasilinear perturbations of generalized KdV equations, the result by Feola-Giuliani-Procesi [65] for Hamiltonian perturbations of the Degasperis-Procesi equation and the recent works of Berti-Kappeler-Montalto [31, 32], who provided the existence of finite dimensional invariant tori of any size for perturbations of the defocusing NLS and of KdV, respectively.

Several results have been obtained concerning the water-waves equations both for the standing and traveling waves. Let us first deal with the periodic standing waves. The case of 2D gravity in finite depth was treated by Plotnikov-Toland [128]. Then, the infinite depth situation was covered in 2D by Iooss-Plotnikov-Toland [108], see also [105, 106]. Later, the 2D gravity-capillary water-waves in infinite depth was done by Alazard-Baldi [2]. The quasi-periodic standing waves have been obtained first by Berti-Montalto [33] where the authors constructed these solutions in the gravity-capillary case for most values of the surface tension. In this case, the linear eigenvalues grow like $|j|^{\frac{3}{2}}$. Then, the more difficult pure gravity case in finite depth, where the equilibrium frequencies admit the asymptotic $|j|^{\frac{1}{2}}$ and vary exponentially with the depth parameter, has been treated in [7] with quasi-periodic solutions constructed for most values of the depth. Notice that in this case one may impose Diophantine conditions that lose also space derivatives. Let us now turn to the traveling periodic solutions. First, Levi-Civita [122] proved the 2D gravity case. Then, the 2D/3D gravity-capillary situation was studied by Craig-Nicholls [48]. And the 3D with pure gravity was developed by Iooss-Plotnikov [107, 104]. The first results about traveling quasi-periodic water-waves were very recently exposed in [29] for the gravity-capillary in finite depth with constant vorticity, [28] for the pure gravity in finite depth with constant vorticity and [63] for the pure gravity in infinite depth.

Concerning Euler equations, the first result for quasi-periodic solutions was obtained by Crouseilles-Faou [51] in the two dimensional torus and was not invloving small divisors difficulties. More recently, a quasi-periodic forcing term was used in [11] to generate quasi-periodic solutions for the 3D case.

Next we shall give more details on the general scheme performed to construct quasi-periodic solutions that was developed by Berti and Bolle in [21]. This approach is robust and flexible and will be adapted to our framework with the suitable modifications. The first step is to write in a standard way the equations using the action-angle variables for the tangential part. When we linearize the nonlinear functional around a state near the equilibrium we end with an operator with variable coefficients that we should invert approximately up to small errors provided the external parameters belong to a suitable Cantor set defined through various Diophantine conditions. To do that we first look for an approximate inverse using an isotropic torus built around the initial one. It has the advantage to transform the linearized operator via symplectic change of coordinates into a triangular system up to errors that vanish when tested against an invariant torus. Notice that the outcome is that the Hamiltonian has a good normal form structure such that we can almost decouple the dynamics in the phase space in tangential and normal modes. On the tangential part the system can be solved in a triangular way provided we can invert the linearized operator on the normal part up to a small coupling error term. This is more or less a finite dimensional KAM theory appearing here. Then, the analysis reduces to invert the linearized operator on the normal part which is a small perturbation of a diagonal infinite dimensional matrix. This is done by conjugating the linearized operator to a diagonal one with constant coefficients. This step is long and technical and most of the non-resonance conditions in the Cantor set arise during this process. This allows the construction of an approximate inverse for the linearized operator with adequate tame estimates required along Nash-Moser scheme. Finally, we point out that the use of suitable isotropic tori is a commodity but it is not essential to get the triangular structure up to small errors, we will come back on this remark later on.

2 Main contributions of the thesis

The purpose of this thesis is to gather the previous two theories looking for the emergence of quasi-periodic structures in the patch form for Euler and quasi-geostrophic shallow-water equations. We shall now present the main theorems proved during the PhD and briefly discuss the key steps of their proofs. More detailed proof structures will be given in the introduction of the corresponding parts. We mention that in parallel to this thesis, similar results have been obtained for SQG equations [87] and for Euler equations close to Kirchhoff ellipses [30].

2.1 Time quasi-periodic vortex patches for QGSW equations

In this section, we present the first contribution of this PhD concerning the existence of quasi-periodic in time solutions in the patch form close to the unit disc for the quasi-geostrophic shallow-water equations (1.6) with parameter λ. The result can be found in [101] and we refer to Theorem 3.1 for a precise statement. Fixing two real numbers λ_0 and λ_1 such that $0<\lambda_0<\lambda_1$, the parameter λ lies in the interval (λ_0,λ_1). Nevertheless, at the end it will belong to a Cantor set for which invariant torus can be constructed. Using the following polar parametrization of the boundary ∂D_t

$$z(t,\theta)\triangleq R(t,\theta)e^{\mathrm{i}(\theta-\Omega t)}\qquad\text{with}\qquad R(t,\theta)\triangleq\big(1+2r(t,\theta)\big)^{\frac{1}{2}},\tag{2.1}$$

the vortex patch equation (1.17) becomes an Hamiltonian equation satisfied by the radial deformation r. Notice that the parametrization is well-defined at least for short time when the initial patch is close to the equilibrium state given by the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ where $\mathbb{D}$ is the unit disc of $\mathbb{R}^2$. The particular choice for the radius is inspired from (1.26) to get a Hamiltonian equation for r. We emphasize that the parameter Ω is introduced to get rid of the degeneracy of the first eigenvalue associated with the linearized operator at the equilibrium state, see (2.6). The radial deformation r is subject to a nonlinear and nonlocal transport-type Hamiltonian equation in the form

$$\partial_t r=\partial_\theta\nabla H(r),\tag{2.2}$$

where the Hamiltonian H is related to the kinetic energy and the angular momentum which are prime integrals of the system. The Hamiltonian system (2.2) is reversible, namely, if $(t,\theta)\mapsto r(t,\theta)$ is a solution, then so is $(t,\theta)\mapsto r(-t,-\theta)$. The purpose is to find a reversible quasi-periodic solution of (2.2), that is to find a frequency vector $\omega\in\mathbb{R}^d$, such that the equation (2.2) admits solutions in the form $r(t,\theta)=\widehat{r}(\omega t,\theta)$ with $\widehat{r}$ being a smooth $(2\pi)^{d+1}-$periodic even function. Then $\widehat{r}$ satisfies the equation (2.2) replacing ∂_t by $\omega\cdot\partial_\varphi$, therefore, we should use the same notation for r and $\widehat{r}$. To explore quasi-periodic solutions we should first check their existence at the linear level. The linearized operator around a given small state r

is given by

$$\mathcal{L}_r = \omega\cdot\partial_\varphi + \partial_\theta\left[V_r \cdot - \mathbf{L}_r\right], \tag{2.3}$$

where V_r is a scalar function depending on λ defined in (3.6) and $\mathbf{L}_r$ is a nonlocal operator depending on λ given in (3.7). At the equilibrium state $r\equiv 0$, we find that the linearized operator is a Fourier multiplier,

$$\mathcal{L}_0 = \omega\cdot\partial_\varphi + V_0(\lambda)\partial_\theta - \partial_\theta\mathcal{K}_\lambda *_\theta \cdot, \qquad V_0(\lambda)\triangleq\Omega + I_1(\lambda)K_1(\lambda), \qquad \mathcal{K}_\lambda(\theta)\triangleq K_0\left(2\lambda\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right). \tag{2.4}$$

We refer to Appendix C for the definition of the modified Bessel functions I_1, K_1 and K_0. The equation $\mathcal{L}_0\rho=0$ is integrable and its reversible quasi-periodic solutions take the form

$$\rho(t,\theta)=\sum_{j\in\mathbb{S}}\rho_j\cos\big(j\theta-\Omega_j(\lambda)t\big), \qquad \rho_j\in\mathbb{R}, \qquad \mathbb{S}\subset\mathbb{N}^*, \qquad |\mathbb{S}|=d\in\mathbb{N}^*, \tag{2.5}$$

with frequency vector

$$\omega_{\mathrm{Eq}}(\lambda)\triangleq\big(\Omega_j(\lambda)\big)_{j\in\mathbb{S}}, \qquad \Omega_j(\lambda)\triangleq j\Big(\Omega+I_1(\lambda)K_1(\lambda)-I_j(\lambda)K_j(\lambda)\Big) \tag{2.6}$$

provided that the vector $\omega_{\mathrm{Eq}}(\lambda)$ satisfies the non-resonant condition (1.24). Observe that $\Omega_0\equiv 0$ and therefore may create trivial resonances. This can be fixed by working in a phase space with zero space average which is possible due to the structure of (3.3). Similarly, notice that for $\Omega=0$, the frequency Ω_1 vanishes. This is the reason for the introduction of Ω which is taken to be strictly positive to remedy to this defect and avoid resonances.

Observe that small divisors already appear at this level and the non-resonant condition is obtained by selecting the parameter λ such that the vector $\omega_{\mathrm{Eq}}(\lambda)$ belong to a Diophantine set in the form (1.28). Actually, this property holds true for almost all the values of λ. Our main result concerns the persistence of quasi-periodic solutions for the nonlinear model (2.2) when the perturbation is small enough and the parameter λ is subject to be in a massive Cantor set. We state here a simplified version of the result and refer the reader to Theorem 3.1 for a precise statement.

Theorem 2.1. *Given $\lambda_1>\lambda_0>0$ and ε small enough, there exists a Cantor-like set $\mathcal{C}_\infty$ with almost full Lebesgue measure in (λ_0,λ_1), such that any parameter $\lambda\in\mathcal{C}_\infty$ generates a quasi-periodic vortex patch for $(QGSW)_\lambda$ equations in the form*

$$\boldsymbol{q}(t,\cdot)=\mathbf{1}_{D_t}, \qquad D_t=\Big\{\ell e^{\mathrm{i}(\theta-\Omega t)}, \quad \theta\in[0,2\pi], \quad 0\leqslant\ell\leqslant R(t,\theta)\Big\}, \qquad R(t,\theta)=\sqrt{1+2r\big(\omega(\lambda,\varepsilon)t,\theta\big)},$$

where $r:\mathbb{T}^{d+1}\to\mathbb{R}$ is a perturbation of the equilibrium quasi-periodic solutions (2.5) *with ε-amplitudes and associated frenquency vector $\omega(\lambda,\varepsilon)$ which is an ε-perturbation of the equilibrium frequency vector $\omega_{\mathrm{Eq}}(\lambda)$ defined in* (2.6)*.*

The proof of Theorem 2.1 (or more precisely Theorem 3.1) is the content of Part I, but let us make some comments about the main steps of the proof and the novelties in there.

► Step 1. Reformulation with embedded tori and restriction to the normal directions : After a rescaling $r\mapsto\varepsilon r$, we remark that the equation (2.2) can be seen as a perturbation of the integrable system given by the linear dynamics at the equilibrium state. This allows us to hope using ideas from KAM theory as presented in Section 1.3. More precisely, we may use and adapt the method developed in [21], slightly modified in [87], and successfully implemented for instance in [7, 33]. The dynamics is decoupled into tangential and normal parts. On the tangential modes, we introduce action-angles variables

(I, ϑ), seen as symplectic polar variables for the Fourier coefficients, allowing to reformulate the problem in terms of embedded tori i. More precisely, we shall look for the zeros of a certain functional, namely to solve an equation in the form

$$\mathcal{F}(i, \alpha, \mu, \varepsilon) = 0, \qquad \mu \triangleq (\lambda, \omega). \tag{2.7}$$

It turns out that it is more convenient to introduce one degree of freedom through a parameter α which provides at the end of the scheme a solution for the original problem when it is fixed to $-\omega_{\mathrm{Eq}}(\lambda)$. At this stage one cannot apply the classical implicit function theorem because of resonances preventing the invertibility of the linearized operator at the equilibrium state. The restriction of the parameter λ to a suitable Cantor-like set related to some Diophantine conditions on the linear frequency $\omega_{\mathrm{Eq}}(\lambda)$ allows in particular to control the small divisors problem as explained before and therefore avoid the resonances. This provides an inverse at the equilibrium state but with algebraic loss of regularity. Unfortunately, this is not sufficient to apply Nash-Moser scheme requiring the construction of a right inverse with tame estimates in a small neighborhood of the equilibrium. Indeed, the linearized operator is no longer with constant coefficients as for the integrable case and its main part is not a Fourier multiplier. At this level we are dealing with a quasilinear problem where the perturbation is unbounded. Given any small reversible embedded torus i_0 and any $\alpha_0 \in \mathbb{R}^d$, we shall construct an approximate right inverse for the linear operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$. For that purpose, we conjugate the linearized operator $d_{i,\alpha}\mathcal{F}(i_0, \alpha_0)$ via a suitable linear diffeomorphism of the toroidal phase space associated to the action-angle-normal formulation. We obtain a triangular system in the action-angle-normal variables up to error terms. To solve the triangular system, we only have to invert the linearized operator in the normal directions, which is denoted by $\widehat{\mathcal{L}}_\omega$. Notice that the approach used here is slightly different from [21] where they linearized around an isotropic torus close enough to the original one and then use a symplectic change of coordinates leading to a triangular system up to small errors, essentially of "type Z" (that is vanishing at an exact solution) or highly decaying in frequency, that can be incorporated in Nash-Moser scheme. Here, and similarly to [87], we can bypass the use of isotropic torus by a slight modification of Berti-Bolle approach. Actually, according to Proposition 6.1, we can conjugate the linearized operator with the transformation described by (6.58) computed at the torus i_0 and get a triangular system with small errors mainly of "type Z". The computations are performed in a straightforward way using in a crucial way the Hamiltonian structure of the original system. The main advantage that simplifies some arguments is to require the invertibility for the linearized operator only at the torus itself and not necessary at a closer isotropic one. By this way, we can avoid the accumulation of different errors induced by the isotropic torus that one encounters for example in the estimates of the approximate inverse or in the multiple Cantor sets generated along the different reduction steps where the coefficients should be computed at the isotropic torus.

► **Step 2. Approximate inverse in the normal directions :**
Therefore, the main issue consists in the construction of an approximate inverse of the linearized operator in the normal direction. Notice that the latter expresses as

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^{\perp} \left(\mathcal{L}_{\varepsilon r} - \varepsilon \partial_\theta \mathcal{R}\right) \Pi_{\mathbb{S}_0}^{\perp},$$

where $\Pi_{\mathbb{S}_0}^{\perp}$ denotes the projector in the normal directions, $\mathcal{L}_{\varepsilon r}$ is defined through (3.5) and $\varepsilon \partial_\theta \mathcal{R}$ is a perturbation of finite rank encoding a coupling between tangential and normal dynamics. Notice that even for ε small, this operator is with non-constant coefficients since the perturbation affects the main part of the operator in a similar way to water waves [7, 33] or generalized SQG equation [87]. Nevertheless, it has constant coefficients at the equilibrium state $\varepsilon = 0$. Therefore, to invert it, the idea is to conjugate it to a constant coefficient by suitable bounded operators close to the identity. The reduction is done in decreasing positive orders in the spirit of [7, 33, 128].

First, we reduce the transport part and look at the effects on the lower orders and as regards the localization in the normal modes. This provides a diagonal operator plus a small remainder term of order 0. Then, we also reduce the remainder term. Let us begin with general remarks about these reduction procedures. Each reduction is based on KAM techniques at the level of functions or operators and makes appear small divisors problems through a countable family of non-resonant conditions. Notice that in our work, for each reduction, the final Cantor set is built on the final state, which is slightly different from [7, 33] where they perform the reduction procedure simultaneously with the Nash-Moser scheme. Also remark that one needs to consider time truncation in the Cantor set implying the addition of error terms that can be later included in the Nash-Moser procedure so that the latter runs. We mention that the KAM reductions allow to diagonalize the operator when the parameters belong to a Cantor-like set, even though all the involved transformations and operators can be extended in the whole set of parameters using standard cut-off functions for the Fourier coefficients. This extension with adequate estimates is needed later during the implementation of the Nash-Moser scheme. This is not the only way to produce suitable extensions and one expects Whitney extension to be also well adapted as in [7, 33]. In our case, we opt for the first procedure which can be easily set up and manipulated using classical functional tools. The last general comment is related to a technical point in KAM reduction. Contrary to the preceding papers such as [7, 33], we do not need to use pseudo-differential operators techniques. In fact, they can be avoided since all the involved operators can be described through their kernels and therefore instead of splitting the symbols we simply expand the kernels which sounds to be more appropriate in our context. Now let us make some specific comments on the different reductions. The reduction of the transport part basically follows the ideas in [8, 28, 29, 32, 64] conjugating the linearized operator $\mathcal{L}_{\varepsilon r}$ to an operator with constant coefficients transport part, through a suitable quasi-periodic symplectic change of coordinates $\mathscr{B}$ in the form

$$\mathscr{B}\rho(\mu,\varphi,\theta)=\Big(1+\partial_\theta\beta(\mu,\varphi,\theta)\Big)\rho\big(\mu,\varphi,\theta+\beta(\mu,\varphi,\theta)\big).$$

The symplectic form has the advantage to avoid the apparition of zero order terms at the end of the reduction. Notice that the action of this conjugation on the nonlocal term is tricky due to the structure (1.8)-(C.7) of K_0. This makes appear a singular kernel which can be carefully treated in the analysis. The projection in the normal directions is done by the operator

$$\mathscr{B}_\perp\triangleq\Pi_{\mathbb{S}_0}^{\perp}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp},$$

for which we obtained a nice duality representation linked to $\mathscr{B}$ useful for doing estimates. A similar relation is obtained for the inverse transformation. Then we obtain

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp}+\mathscr{D}_0+\mathscr{R}_0+\text{fast decaying error terms},$$

where $\mathscr{D}_0$ is a diagonal operator whose spectrum $(\mathrm{i}\mu_j^0)_j$ satisfies

$$\mu_j^0(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0),\qquad r^1=O(\varepsilon)$$

and the remainder term $\mathscr{R}_0$ is a reversible Toeplitz in time integral operator enjoying nice smallness property. The Toeplitz structure and the revesibility are crucial properties that need to be checked along the KAM reduction of the remainder term. The first one allows to work with a nice operator topology well-adapted to the convergence of the scheme. The second is required for the construction of a reversible approximate right inverse in order to construct, later in the Nash-Moser scheme, a reversible invariant tori. Also this provides pure imaginary spectrum diagonal operators ensuring the linear stability. This reduction provides a completely diagonalized operator with spectrum $(\mathrm{i}\mu_j^\infty)_j$,

$$\mu_j^\infty(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0)+r_j^\infty(\lambda,\omega,i_0),\qquad \sup_j jr_j^\infty=O(\varepsilon)$$

and easily invertible up to new restrictions of the parameters. We mention the Lipschitz dependence of these eigenvalues with respsect to the torus. Such property is required in studying the stability of Cantor sets in the Nash-Moser scheme and allows to construct a final massive Cantor set.

▶ **Step 3. Construction of a non-trivial solution :**

Now we can implement a Nash-Moser procedure to find non-trivial zeros for the nonlinear function $\mathcal{F}$ for small ε in the spirit of the papers [7, 33]. We can build by induction a sequence of approximate solutions U_n

$$U_{n+1}\triangleq U_n-\Pi_{N_n}\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n),\qquad U_n\triangleq(i_n,\alpha_n).$$

with T_n an approximate right inverse of $d_{i,\alpha}\mathcal{F}(U_n)$ obtained at step n using the above mentioned construction (**Steps 1** and **2**). At each step of the scheme, we needed to construct classical extensions to the whole set of parameters $\mathcal{O}$. This is different from the papers [7, 33] where they considered Whitney extensions. Actually, we get a precise statement in Proposition 8.1 allowing to deduce that the sequence $(U_n)_n$ converges in a strong topology towards a profile $U_\infty=\big(i_\infty,\alpha_\infty\big)$ solution of (2.7) whenever the parameters (λ,ω) are selected among a Cantor-like set $\mathcal{G}_\infty$ which is constructed as the intersection of all the Cantor sets appearing in the scheme to invert at each step the linearized operator. To find a solution to the original problem we construct a frequency curve $\lambda\mapsto\omega(\lambda,\varepsilon)$ implicitly defined by solving the equation

$$\alpha_\infty(\lambda,\omega(\lambda,\varepsilon))=-\omega_{\mathrm{Eq}}(\lambda).$$

Hence, we obtain the desired result for any value of λ in the rigidified Cantor set

$$\mathcal{C}_\infty^\varepsilon\triangleq\Big\{\lambda\in(\lambda_0,\lambda_1)\quad\text{s.t.}\quad(\lambda,\omega(\lambda,\varepsilon))\in\mathcal{G}_\infty\Big\}.$$

Then, it remains to check that this set is non-trivial. The proof of this fact is inspired from [10] with adaptations to the structure of our Cantor set, namely contructed on the final states of each KAM reduction procedure. We can prove

$$|\mathcal{C}_\infty^\varepsilon|\geqslant(\lambda_1-\lambda_0)-C\varepsilon^\delta,$$

with small δ connected to the geometry of the Cantor set and the non degeneracy of the equilibrium spectrum. There are two main ingredients to get this result. The first one is the stability of the intermediate Cantor sets following from the fast convergence of Nash-Moser scheme. However the second one is the transversality property stated in Lemma 8.3 used in the spirit of [12] and [139]. This property will be first established for the linear frequencies in Proposition 5.5, using the analyticity of the eigenvalues and their asymptotic behavior. Then the extension of the transversality assumption to the perturbed frequencies is done using perturbative arguments together with the asymptotic description of the approximate eigenvalues detailed in (8.67), (8.82) and (8.81). We emphasize that the transversality is strongly related to the non-degeneracy of the eigenvalues in the sense of the Definition 5.1 . For instance, we show that the curve $\lambda\in[\lambda_0,\lambda_1]\mapsto\big(\Omega_{j_1}(\lambda),...,\Omega_{j_d}(\lambda)\big)$ is not contained in any vectorial hyperplane. This is proved in Lemma 5.4 and follows from the asymptotic of the eigenvalues for large values of λ according to the law (C.15) combined with the invertibility of Vandermonde matrices.

2.2 Boundary effects on the emergence of quasi-periodic solutions to Euler equations

We shall now present the second result obtained during the thesis and presented in [89]. This work is based on the remark that the Euler equations set in the unit disc $\mathbb{D}$ are not invariant under dilation. Therefore, we can introduce a geometric parameter $b \in (0,1)$ such that the Rankine vortices $\mathbf{1}_{b\mathbb{D}}$ are not equivalent and provide a family of stationary solutions. Hence, as in the previous mentioned result, we can expect to play with this parameter to generate quasi-periodic solutions. For that purpose, we consider a polar parametrization of a patch boundary close to the stationary solution $\mathbf{1}_{b\mathbb{D}}$, namely

$$z(t,\theta) \triangleq R(b,t,\theta)e^{\mathrm{i}\theta}, \qquad R(b,t,\theta) \triangleq \sqrt{b^2 + 2r(t,\theta)}.$$

The parameter b is assumed to live in an interval (b_0, b_1), where

$$0 < b_0 < b_1 < 1.$$

However, as in the previous result, at the end this parameter will belong to a Cantor set for which invariant torus can be constructed. We emphasize that this ansatz no longer depends on Ω as in (2.1). Indeed, in this context the first frequency is non-degenerate according to (1.22). The radial deformation r solves a nonlinear and nonlocal transport Hamiltonian PDE taking the form

$$\partial_t r = \tfrac{1}{2}\partial_\theta \nabla E(r),$$

where E is the kinetic energy related to the stream function given by (1.4). The linearized operator close to the Rankine patch $\mathbf{1}_{b\mathbb{D}}$, i.e. at a small state r writes

$$\mathcal{L}_r = \partial_t + \partial_\theta\Big(V_r \cdot + \mathbf{L}_r - \mathbf{S}_r\Big),$$

where V_r is a scalar function depending on b, $\mathbf{L}_r$ is a nonlocal operator with logarithmic singular kernel reflecting the planar Euler action and $\mathbf{S}_r$ is a smoothing nonlocal operator. We refer to (9.3), (9.5) and (9.6) for their definitions. The boundary effects of $\mathbb{D}$ are observed through a quasilinear smoothing action in the transport part and through the smoothing operator $\mathbf{S}_r$. At the equilibrium state $r = 0$, the linearized operator is a Fourier multiplier given by

$$\mathcal{L}_0 = \partial_t + \tfrac{1}{2}\partial_\theta + \partial_\theta \mathcal{K}_{1,b} * \cdot - \partial_\theta \mathcal{K}_{2,b} * \cdot,$$

where

$$\mathcal{K}_{1,b}(\theta) \triangleq \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right) \quad \text{and} \quad \mathcal{K}_{2,b}(\theta) \triangleq \log\left(|1 - b^2 e^{\mathrm{i}\theta}|\right).$$

Now, almost every $b \in (b_0, b_1)$ generates reversible quasi-periodic solutions to $\mathcal{L}_0 \rho = 0$ in the form

$$\rho(t,\theta) = \sum_{j\in\mathbb{S}} \rho_j \cos(j\theta - \Omega_j(b)t), \qquad \rho_j \in \mathbb{R}, \qquad \mathbb{S} \subset \mathbb{N}^*, \qquad |\mathbb{S}| = d \in \mathbb{N}^*, \tag{2.8}$$

with frequency vector

$$\omega_{\mathrm{Eq}}(b) \triangleq \big(\Omega_j(b)\big)_{j\in\mathbb{S}}, \qquad \Omega_j(b) \triangleq \tfrac{1}{2}\big(j - 1 + b^{2j}\big). \tag{2.9}$$

The measure of the Cantor set in b generating these solutions is estimated using Rüssmann Lemma 5.6 requiring a lower bound on the maximal derivative of a given function up to order q_0. It is a remarkable fact that here the value of q_0 is explicit, namely $q_0 = 2\max(\mathbb{S}) + 2$ which is due to the polynomial structure of the $\Omega_j(b)$. The aim of the following result is to state that these structures persist at the nonlinear level. We mention that this is a simplified version and the interested reader may be refered to Theorem 9.1 for a complete statement.

Theorem 2.2. *Given* $0 < b_0 < b_1 < 1$ *and* ε *small enough, there exists a Cantor-like set* $\mathcal{C}_\infty$ *with almost full Lebesgue measure in* (b_0, b_1)*, such that any parameter* $b \in \mathcal{C}_\infty$ *generates a quasi-periodic vortex patch, solution of* (1.5) *set in the unit disc, in the form*

$$\boldsymbol{\omega}(t,\cdot) = \mathbf{1}_{D_t}, \qquad D_t = \Big\{\ell e^{\mathrm{i}\theta}, \quad \theta \in [0,2\pi], \quad 0 \leqslant \ell \leqslant R(b,t,\theta)\Big\}, \qquad R(b,t,\theta) = \sqrt{b^2 + 2r\big(\omega(b,\varepsilon)t, \theta\big)},$$

where $r : \mathbb{T}^{d+1} \to \mathbb{R}$ *is a perturbation of the equilibrium quasi-periodic solutions* (2.8) *with* ε*-amplitudes and associated frenquency vector* $\omega(b,\varepsilon)$ *which is an* ε*-perturbation of the equilibrium frequency vector* $\omega_{\mathrm{Eq}}(b)$ *defined in* (2.9)*.*

The proof of Theorem 2.2 (or more precisely Theorem 9.1) is the content of Part II and is similar to the previous one. Indeed, we reformulate the problem in terms of embedded tori, looking for the zeros of a certain nonlinear functional. We obtain a non-trivial solution by applying a Nash-Moser scheme, where at each step, we construct an approximate right inverse with nice tame estimates for the linearized operator. This inverse is obtained from the application of the Berti-Bolle theory reducing the problem to the search of an approximate right inverse for the normal projection of the linearized operator. This latter is obtained by using KAM reducibility techniques which imply restrictions of parameters (b,ω) to a Cantor-like set. The iterative implicit function procedure generates a solution provided that we ensure all the required restrictions of parameters along the scheme. Then we rigidify the frequency vector ω in terms of b and we estimate, through the perturbed Rüssmann conditions, the measure of the final Cantor set proving that it has almost full Lebesgue measure. The main difference with the previous result is the smoothing effects of the boundary observable on the linearized operator. In particular, in this case, the remainder term obtained after the reduction of the transport part and the projection in the normal modes is directly regularizing at every order, which simplifies the analytical study.

2.3 Doubly-connected V-states for QGSW equations

The main purpose of this section is to present the last result obtained in this thesis concerning the emergence of time periodic solutions in the patch form close to the annulus of radii 1 and b for the system $(\mathrm{QGSW})_\lambda$ with fixed $\lambda > 0$ and $b \in (0,1)$. A simplified version of the result can be written as follows, we refer the reader to Theorem 15.1 for a precise statement.

Theorem 2.3. *For fixed* $\lambda \in (0,\infty)$ *and* $b \in (0,1)$*, there exist non-trivial analytic doubly-connected V-states close to the annulus* A_b *defined in* (1.20) *for* $(QGSW)_\lambda$ *equations with* $\mathbf{m}$*-fold symmetry for any* $\mathbf{m}$ *larger than a threshold depending on* λ *and* b*.*

Notice that the proof of Theorem 2.3 (or more precisely Theorem 15.1) is the content of Part III. The related paper is [138]. More precisely, these solutions are implicitly obtained as branches of bifurcation emerging from the annulus A_b as in (1.20) for specific angular velocities related to modified Bessel functions. The proof is based on local bifurcation theory and more precisely on Crandall-Rabinowitz's Theorem B.1 in the spirit of the previous works mentioned in Section 1.2. Indeed, by using conformal mappings, we can reformulate the contour dynamics equation (1.17) in the context of uniformly rotating solutions as the zeros of a suitable nonlinear and nonlocal functional. Remarking that the annulus A_b is a solution for any angular velocity, we obtain a line of trivial solutions. Therefore, we may expect to apply bifurcation theory to find other curves of solutions provided that the linearized operator is a Fredholm operator with zero index and one dimensional kernel supplemented with a transversality assumption. Actually, the linearized operator at the equilibrium state is a Fourier multiplier and its kernel is one dimensional for explicit angular velocities $\Omega_n^{\pm}(\lambda, b)$ related to Bessel functions of imaginary argument. This last property is based on the asymptotic monotonicity for large modes n of the sequence $\big(\Omega_n^{\pm}(\lambda, b)\big)_n$ which is obtained from the asymptotic properties of the involved special functions. The bifurcation is proved in the regularity $C^{1+\alpha}$ with $\alpha \in (0,1)$ and using a regularity argument for elliptic equations, we find the analyticity of the boundaries.

PART I

Time quasi-periodic vortex patches for QGSW equations

> La façon dont on trouve n'est pas celle dont on prouve.
>
> Albert Einstein

This part is devoted to the proof of Theorem 2.1. We also refer to Theorem 3.1 below for a more precise statement. This result is the subject of the following preprint [101] which is submitted to the journal Mémoires de la Société Mathématique de France and entitled "Time quasi-periodic vortex patches for quasi-geostrophic shallow-water equations".

Abstract

We shall implement KAM theory in order to construct a large class of time quasi-periodic solutions for an active scalar model arising in fluid dynamics. More precisely, the construction of invariant tori is performed for quasi-geostrophic shallow-water equations when the *Rossby deformation length* belongs to a massive Cantor set. As a consequence, we construct pulsating vortex patches whose boundary is localized in a thin annulus for any time.

3 Introduction

We shall present here the main result obtained during the thesis and related to the existence of quasi-periodic vortex patches close to the unit disc for QGSW equations. We shall also present the main steps of its proof with more details than in Section 2.1. The contour dynamics equation stated in (1.16) can be written in a more tractable way using polar coordinates. This is meaningful at least for short time when the initial patch is close to the equilibrium state given by the Rankine vortex $\mathbf{1}_{\mathbb{D}}$ where $\mathbb{D}$ is the unit disc of $\mathbb{R}^2$. Thus the boundary ∂D_t will be parametrized as follows

$$z(t,\theta)\triangleq R(t,\theta)e^{i(\theta-\Omega t)}\qquad\text{with}\qquad R(t,\theta)\triangleq\big(1+2r(t,\theta)\big)^{\frac{1}{2}}.\tag{3.1}$$

We shall prove in Section 4.1 that the function r satisfies a nonlinear and non-local transport equation taking the form

$$\partial_t r+\Omega\partial_\theta r+F_\lambda[r]=0,\tag{3.2}$$

with

$$F_\lambda[r](t,\theta)\triangleq\int_{\mathbb{T}}K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big)d\eta$$

and

$$A_r(t,\theta,\eta)\triangleq\left|R(t,\theta)e^{i\theta}-R(t,\eta)e^{i\eta}\right|.$$

The function K_0 is a Bessel function of imaginary parts and it is defined in Appendix C. We point out that the introduction of the parameter Ω seems at this level artificial but it will be used later to fix the degeneracy of the first eigenvalue associated with the linearized operator at the equilibrium state. As we shall see in Proposition 4.1, the equation (3.2) has an Hamiltonian structure

$$\partial_t r=\partial_\theta\nabla H(r),\tag{3.3}$$

where the Hamiltonian H is related to the kinetic energy and the angular momentum which are prime integrals of the system. In the quasi-periodic setting, we should find a frequency vector $\omega\in\mathbb{R}^d$, such that the equation (3.2) admits solutions in the form $r(t,\theta)=\widehat{r}(\omega t,\theta)$ with $\widehat{r}$ being a smooth $(2\pi)^{d+1}-$periodic function. Then $\widehat{r}$ satisfies (to alleviate the notation we keep the notation r for $\widehat{r}$)

$$\omega\cdot\partial_\varphi r+\Omega\partial_\theta r+F_\lambda[r]=0. \tag{3.4}$$

To explore quasi-periodic solutions we should first check their existence at the linear level. Then according to Lemma 5.1 the linearized operator to (3.4) around a given small state r is given by the linear Hamiltonian equation,

$$\mathcal{L}_r\rho=0 \quad \text{with} \quad \mathcal{L}_r=\partial_t+\partial_\theta\left[V_r\cdot-\mathbf{L}_r\right], \tag{3.5}$$

where V_r is a scalar function defined by

$$V_r(\lambda,t,\theta)\triangleq\Omega+\frac{1}{R(t,\theta)}\int_{\mathbb{T}}K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial_\eta\left(R(t,\eta)\sin(\eta-\theta)\right)d\eta \tag{3.6}$$

and $\mathbf{L}_r$ is a non-local operator in the form

$$\mathbf{L}_r(\rho)(\lambda,t,\theta)\triangleq\int_{\mathbb{T}}K_0\left(\lambda A_r(t,\theta,\eta)\right)\rho(t,\eta)d\eta. \tag{3.7}$$

At the equilibrium state $r\equiv0$, we find that the linearized operator is a Fourier multiplier, see Lemma 5.2,

$$\mathcal{L}_0\rho=\partial_t\rho+V_0(\lambda)\partial_\theta\rho-\partial_\theta\mathcal{K}_\lambda*\rho, \tag{3.8}$$

where $*$ denotes the convolution product in the variable θ and

$$V_0(\lambda)\triangleq\Omega+I_1(\lambda)K_1(\lambda) \quad \text{and} \quad \mathcal{K}_\lambda(\theta)\triangleq K_0\left(2\lambda\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right).$$

Expanding into Fourier series

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}}\rho_j(t)e^{\mathrm{i}j\theta},$$

yields

$$\rho\in\ker(\mathcal{L}_0) \quad \Longleftrightarrow \quad \rho(t,\theta)=\sum_{j\in\mathbb{Z}}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(\lambda)t)}, \tag{3.9}$$

where the eigenvalues Ω_j are defined by

$$\Omega_j(\lambda)\triangleq j\Big(\Omega+I_1(\lambda)K_1(\lambda)-I_j(\lambda)K_j(\lambda)\Big) \tag{3.10}$$

and the Bessel functions of imaginary argument I_n and K_n are given by (C.2). It is worthy to point out that the frequency associated to the mode $j=0$ is vanishing and therefore it creates trivial resonance. This can be fixed by imposing a zero space average which can be maintained at the nonlinear level by virtue of the structure of (3.3). Hence we shall work with the phase space of real functions enjoying this property, namely,

$$L_0^2\triangleq L_0^2(\mathbb{T},\mathbb{R})=\Big\{r=\sum_{j\in\mathbb{Z}^*}r_je_j \quad \text{s.t.} \quad r_{-j}=\overline{r_j} \quad \text{and} \quad \sum_{j\in\mathbb{Z}^*}|r_j|^2<\infty\Big\}.$$

Another similar comment concerns the mode $j=1$ which vanishes for any λ when $\Omega=0$. This is why we have introduced Ω which should be strictly positive to remedy to this defect and avoid any resonance at higher frequencies. The reversibility of the system (3.3) can be also exploited to find the requested parity property of the solutions. Actually, we can check that if $(t,\theta)\mapsto r(t,\theta)$ is a solution then $(t,\theta)\mapsto r(-t,-\theta)$ is a solution too. Then the solutions to the linear problem with this symmetry are in the form

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j\cos\big(j\theta-\Omega_j(\lambda)t\big). \tag{3.11}$$

Now, in order to generate quasi-periodic solutions to the linear problem it suffices to excite a finite number of frequencies from the linear spectrum. We shall then consider the following frequency vector.

$$\omega_{\mathrm{Eq}}(\lambda)\triangleq\big(\Omega_j(\lambda)\big)_{j\in\mathbb{S}}\quad\text{with}\quad\mathbb{S}\triangleq\{j_1,\ldots,j_d\}\subset\mathbb{N}^*.$$

Notice that the vector $\omega_{\mathrm{Eq}}(\lambda)$ gives periodic solutions provided that it satisfies the non-resonant condition (1.24). This property holds true for almost all the values of λ as it is proved in Proposition 5.1. Observe that this latter is based on the equilibrium Rüssmann conditions proved in Lemma 5.5 and which make appear an index of regularity q_0 with respect to the parameter λ. Our main result concerns the persistence of quasi-periodic solutions for the nonlinear model (3.3) when the perturbation is small enough and the parameter λ is subject to be in a massive Cantor set.

Theorem 3.1. *Let $\lambda_1>\lambda_0>0$, $d\in\mathbb{N}^*$ and $\mathbb{S}\subset\mathbb{N}^*$ with $|\mathbb{S}|=d$. There exist $\varepsilon_0\in(0,1)$ small enough with the following properties : For every amplitudes $\mathtt{a}=(\mathtt{a}_j)_{j\in\mathbb{S}}\in(\mathbb{R}_+^*)^d$ satisfying*

$$|\mathtt{a}|\leqslant\varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty\subset(\lambda_0,\lambda_1)$ with asymptotically full Lebesgue measure as $\mathtt{a}\to0$, i.e.

$$\lim_{\mathtt{a}\to0}|\mathcal{C}_\infty|=\lambda_1-\lambda_0,$$

such that for any $\lambda\in\mathcal{C}_\infty$, the equation (3.3) admits a time quasi-periodic solution with diophantine frequency vector $\omega_{\mathrm{pe}}(\lambda,\mathtt{a})\triangleq(\omega_j(\lambda,\mathtt{a}))_{j\in\mathbb{S}}\in\mathbb{R}^d$ and taking the form

$$r(t,\theta)=\sum_{j\in\mathbb{S}}\mathtt{a}_j\cos\big(j\theta+\omega_j(\lambda,\mathtt{a})t\big)+\mathtt{p}(\omega_{\mathrm{pe}}t,\theta),$$

with

$$\omega_{\mathrm{pe}}(\lambda,\mathtt{a})\underset{\mathtt{a}\to0}{\longrightarrow}\big(-\Omega_j(\lambda)\big)_{j\in\mathbb{S}},$$

where $\Omega_j(\lambda)$ are the equilibrium frequencies defined in (3.10) and the perturbation $\mathtt{p}:\mathbb{T}^{d+1}\to\mathbb{R}$ is an even function satisfying for some large index of regularity $s=s(d,q_0)$

$$\|\mathtt{p}\|_{H^s(\mathbb{T}^{d+1},\mathbb{R})}\underset{\mathtt{a}\to0}{=}o(|\mathtt{a}|).$$

We shall now outline the main steps of the proof which will be developed following standard scheme as in the preceding works [8, 7, 21, 33] with different variations connected to the models structure. We mainly use techniques from KAM theory combined with Nash Moser scheme. This will be implemented along several steps which are detailed below.

► **Step 1.** *Action-angle reformulation.* We first notice that the equation (3.3) can be seen as a perturbation of the integrable system given by the linear dynamics at the equilibrium state. Indeed, by combining (3.8), (3.10) and (3.3) we may write

$$\partial_t r=\partial_\theta\mathrm{L}(\lambda)(r)+X_P(r),$$

where $\mathrm{L}(\lambda)$ and the perturbed Hamiltonian vector field X_P are defined by

$$\mathrm{L}(\lambda)(r) \triangleq -\big(\Omega + (I_1K_1)(\lambda)\big)r + \mathcal{K}_\lambda * r \quad \text{and} \quad X_P(r) \triangleq I_1(\lambda)K_1(\lambda)\partial_\theta r - \partial_\theta \mathcal{K}_\lambda * r - F_\lambda[r].$$

Since we are looking for small solutions then we find it convenient to rescale the solution $r \rightsquigarrow \varepsilon r$ with ε a small positive number and consequently the new unknown still denoted by r satisfies

$$\partial_t r = \partial_\theta \mathrm{L}(\lambda)(r) + \varepsilon X_{P_\varepsilon}(r),$$

where X_{P_ε} is the Hamiltonian vector field defined by $X_{P_\varepsilon}(r) \triangleq \varepsilon^{-2}X_P(\varepsilon r)$. Then finding quasi-periodic solutions with frequency $\omega \in \mathbb{R}^d$ amounts to solve the equation

$$\omega \cdot \partial_\varphi r = \partial_\theta \mathrm{L}(\lambda)(r) + \varepsilon X_{P_\varepsilon}(r).$$

Here we still use the same notation r for the new profile which depends in the variables $(\varphi, \theta) \in \mathbb{T}^{d+1}$. The next step consists in splitting the phase space L^2_0 into an orthogonal sum of tangential and normal subspaces as follows

$$L^2_0 = L_{\overline{\mathbb{S}}} \overset{\perp}{\oplus} L^2_\perp,$$

where $L_{\overline{\mathbb{S}}}$ is the finite dimensional subspace of real functions generated by $\{e^{\mathrm{i}j\theta}, j \in \overline{\mathbb{S}}\}$ with $\overline{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S})$. For more details on this description we refer to Section 6.1. To track the dynamics it seems to be more suitable to use the action-angle variables (I, ϑ) seen as symplectic polar variables for the Fourier coefficients of the tangential part in $L_{\overline{\mathbb{S}}}$. This leads to reformulate the problem in terms of the embedded torus,

$$\begin{array}{rrcl} i: & \mathbb{T}^d & \to & \mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp \\ & \varphi & \mapsto & (\vartheta(\varphi), I(\varphi), z(\varphi)), \end{array}$$

with

$$r(\varphi,\theta) = \underbrace{v(\vartheta(\varphi), I(\varphi))(\theta)}_{\in L_{\overline{\mathbb{S}}}} + \underbrace{z(\varphi,\theta)}_{\in L^2_\perp} \triangleq A(i(\varphi))(\theta)$$

and

$$v(\vartheta, I) \triangleq \sum_{j \in \overline{\mathbb{S}}} \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j} e_j, \qquad e_j(\theta) \triangleq e^{\mathrm{i}j\theta}\,.$$

Notice that the action and angle variables should satisfy the symmetry properties

$$\forall j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{T}.$$

Therefore we reduce the problem in the new variables to construct invariant tori with non-resonant frequency vector ω to the system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_\varepsilon}(i(\varphi)), \tag{3.12}$$

where X_{H_ε} is the Hamiltonian vector field associated to the Hamiltonian H_ε given by

$$H_\varepsilon \triangleq -\omega_{\mathrm{Eq}}(\lambda) \cdot I + \tfrac{1}{2}\langle \mathrm{L}(\lambda)z, z\rangle_{L^2(\mathbb{T})} + \varepsilon \mathcal{P}_\varepsilon,$$

with $\mathcal{P}_\varepsilon$ defined by $\mathcal{P}_\varepsilon \triangleq P_\varepsilon \circ A$. A useful trick used in [21, 125] consists in solving first the relaxed problem

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_\varepsilon^\alpha}(i(\varphi)),$$

where the vector field $X_{H_\varepsilon^\alpha}$ is associated to the modified Hamiltonian H_ε^α given by

$$H_\varepsilon^\alpha \triangleq \alpha\cdot I+\tfrac{1}{2}\langle \mathrm{L}(\lambda)z,z\rangle_{L^2(\mathbb{T})}+\varepsilon\mathcal{P}_\varepsilon.$$

The advantage of this procedure is to get one degree of freedom with the vector $\alpha\in\mathbb{R}^d$ that will be used to ensure some compatibility assumptions during the construction of an approximate inverse of the linearized operator. At the end of Nash-Moser scheme we shall adjust implicitly the frequency ω so that α coincides with the equilibrium frequency $-\omega_{\mathrm{Eq}}(\lambda)$, which enables to finally get solutions to the original Hamiltonian equation. The relaxed problem can be written in the following form

$$\mathcal{F}(i,\alpha,\lambda,\omega,\varepsilon)=0,$$

with

$$\begin{aligned}\mathcal{F}(i,\alpha,\lambda,\omega,\varepsilon)&\triangleq \omega\cdot\partial_\varphi i(\varphi)-X_{H_\varepsilon^\alpha}(i(\varphi))\\&=\begin{pmatrix}\omega\cdot\partial_\varphi\vartheta(\varphi)-\alpha-\varepsilon\partial_I\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi I(\varphi)+\varepsilon\partial_\vartheta\mathcal{P}_\varepsilon(i(\varphi))\\ \omega\cdot\partial_\varphi z(\varphi)-\partial_\theta\left(\mathrm{L}(\lambda)z(\varphi)+\varepsilon\nabla_z\mathcal{P}_\varepsilon(i(\varphi))\right)\end{pmatrix}.\end{aligned}\tag{3.13}$$

We point out that the linear torus corresponding to the linear solution

$$r(\varphi,\theta)=\sum_{j\in\overline{\mathbb{S}}}\mathtt{a}_j\,e^{\mathrm{i}\varphi_j}e^{\mathrm{i}j\theta}$$

is given in the new coordinates system by $i_{\mathrm{flat}}(\varphi)=(\varphi,0,0)$ and it is obvious that

$$\mathcal{F}\big(i_{\mathrm{flat}},-\omega_{\mathrm{Eq}}(\lambda),\lambda,-\omega_{\mathrm{Eq}}(\lambda),0\big)=0.$$

We emphasize that at this stage the classical implicit function theorem does not work because the linearized operator at the equilibrium state is not invertible due to resonances. One can avoid resonances by restricting the parameter λ to a suitable Cantor set according to some Diophantine conditions on the linear frequency $\omega_{\mathrm{Eq}}(\lambda)$ allowing in particular to control the small divisors problem. By this way we get an inverse at the equilibrium state but with algebraic loss of regularity. Unfortunately, this is not enough to apply Nash-Moser scheme which requires to construct a right inverse with tame estimates in a small neighborhood of the equilibrium and this is the challenging deal in this topic. Indeed, the linearized operator is no longer with constant coefficients as for the integrable case and its main part is not a Fourier multiplier. At this level we are dealing with a quasilinear problem where the perturbation is unbounded.
▶ **Step 2.** *Approximate inverse.* Let $\alpha_0\in\mathbb{R}^d$ (actually α_0 is a function of the parameters ω and λ) and consider an embedded torus $i_0=(\vartheta_0,I_0,z_0)$ near the flat one with the reversible structure,

$$\vartheta_0(-\varphi)=-\vartheta_0(\varphi),\quad I_0(-\varphi)=I_0(\varphi)\quad\text{and}\quad z_0(-\varphi,-\theta)=z_0(\varphi,\theta).$$

To deal with the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$, which exhibits complicated structure, and see whether we can construct an approximate inverse we should fix two important issues. One is related to the coupling structure in the new coordinates system and the second is that the linearized operator is with variable coefficients. For this aim we shall follow the approach conceived by Berti and Bolle in [21] with making suitable modifications. This approach consists in linearizing around an isotropic torus close enough to the original one and then use a symplectic change of coordinates leading to a triangular system up to small errors, essentially of "type Z" or highly decaying in frequency, that can be incorporated in Nash-Moser scheme. Therefore to invert this triangular system it suffices to get an approximate right inverse for the linearized operator in the normal direction, denoted in what follows by $\widehat{\mathcal{L}}_\omega$. We notice that in Section 6.3, and similarly to [87], we can bypass the use of isotropic torus by a slight modification of Berti-Bolle approach. Actually, according to Proposition 6.1, we can conjugate the linearized operator with the transformation described by (6.58) computed at the torus i_0 and get a triangular system with small errors mainly of "type Z". The computations are performed in a straightforward way using in a crucial way the Hamiltonian structure of the original system. The main advantage that simplifies some arguments is to require the invertibility for the linearized operator only at the torus itself and not necessary at a closer isotropic one. By this way, we can avoid the accumulation of different errors induced by the isotropic torus that one encounters for example in the estimates of the approximate inverse or in the multiple Cantor sets generated along the different reduction steps where the coefficients should be computed at the isotropic torus. The final outcome of this first step is to reduce the invertibility to finding an approximate inverse of $\widehat{\mathcal{L}}_\omega$ which takes, according to Proposition 7.1, the form

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^{\perp}\left(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\right)\Pi_{\mathbb{S}_0}^{\perp} \quad \text{with} \quad \mathcal{L}_{\varepsilon r} = \omega\cdot\partial_\varphi + \partial_\theta\left[V_{\varepsilon r}\cdot - \mathbf{L}_{\varepsilon r}\right],$$

where $\varepsilon\partial_\theta\mathcal{R}$ is a perturbation of finite rank, the function $V_{\varepsilon r}$ and the nonlocal operator $\mathbf{L}_{\varepsilon r}$ are defined in (3.6) and (3.7), respectively. At the equilibrium state (corresponding to $\varepsilon = 0$) $\widehat{\mathcal{L}}_\omega$ is diagonal and we shall see that the set of parameters (λ, ω) leading to the existence of a right inverse is almost full. Now remark that even for ε small, the perturbation affects the main part of the operator in a similar way to water waves [7, 33] or generalized SQG equation [87] and then we should construct the suitable change of coordinates in order to reduce the positive part of the operator to a diagonal operator. Later we should implement KAM scheme to diagonalize the zero-order part. This will be done in three steps.

ⓐ **Reduction of the transport part.** This procedure will be discussed in Proposition 7.2 and Proposition 7.3. We basically use KAM techniques as in [29, 64] in order to conjugate the operator $\mathcal{L}_{\varepsilon r}$, through a suitable quasi-periodic symplectic change of coordinates $\mathscr{B}$, to a transport operator with constant coefficients. Indeed, we may construct an invertible transformation

$$\mathscr{B}\rho(\varphi,\theta) = \big(1 + \partial_\theta\beta(\varphi,\theta)\big)\rho\big(\varphi, \theta + \beta(\varphi,\theta)\big)$$

and a constant $c_{i_0}(\lambda,\omega)$ such that for any given number $n \in \mathbb{N}$, if the parameter (λ, ω) belongs to the truncated set defined through the first order Melnikov condition

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \triangleq \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \Big\{(\lambda,\omega)\in\mathcal{O} \quad \text{s.t.} \quad |\omega\cdot l + jc_{i_0}(\lambda,\omega)| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},$$

then we have

$$\mathfrak{L}_{\varepsilon r} \triangleq \mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}(\lambda,\omega)\partial_\theta - \partial_\theta\mathcal{K}_\lambda * \cdot + \partial_\theta\mathfrak{R}_{\varepsilon r} + \mathtt{E}_n^0, \tag{3.14}$$

with $N_n = N_0^{(\frac{3}{2})^n}$, $N_0 \geqslant 2$, $\upsilon \in (0,1)$, $\mathcal{O} = (\lambda_0,\lambda_1)\times\mathscr{U}$, $0 < \lambda_0 < \lambda_1$ and $\mathscr{U} \triangleq B(0,R_0)$ being an open ball of $\mathbb{R}^d$ containing the curve of the linear vector frequency $\lambda \in (\lambda_0,\lambda_1) \mapsto -\omega_{\mathrm{Eq}}(\lambda)$. The operator $\mathfrak{R}_{\varepsilon r}$ is a self-adjoint Toeplitz integral operator satisfying the estimates

$$\forall s\in[s_0,S], \qquad \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\right),$$

where the off-diagonal norm $\|\cdot\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}$ is defined in (A.23) and the loss of regularity σ is connected to τ_1 and d but it is independent of the index regularity s. Concerning the operator $\mathtt{E}_n^0$, we can show that it is a small fast decaying remainder with the following estimate for low regularity

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\, N_0^{\mu_2} N_n^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}, \tag{3.15}$$

where the weighted norms $\|\cdot\|_{q,s_0}^{\gamma,\mathcal{O}}$ are defined in (A.6). For the number μ_2, it is connected to the regularity of the torus i_0 and can be taken large enough allowing to identify the contributions of $\mathtt{E}_n^0$ as small errors in the construction of the approximate inverse. The next step will be discussed in Proposition 7.4 where we explore the effect of the transport reduction on the original operator $\widehat{\mathcal{L}}_\omega$ which is localized to the normal direction. We prove that with the localized transformation $\mathscr{B}_\perp$ defined by

$$\mathscr{B}_\perp \triangleq \Pi_{\mathbb{S}_0}^\perp \mathscr{B}\Pi_{\mathbb{S}_0}^\perp,$$

one obtains in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$,

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp = \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_0 + \mathscr{R}_0 + \mathtt{E}_n^1, \tag{3.16}$$

where $\mathscr{D}_0$ is a diagonal operator whose spectrum $\{\mathrm{i}\mu_j^0, j\in\mathbb{S}_0^c\}$ satisfies

$$\mu_j^0(\lambda,\omega,i_0) \triangleq \Omega_j(\lambda) + jr^1(\lambda,\omega,i_0) \quad \text{ with } \quad \|r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon$$

and $\mathscr{R}_0$ is a remainder term taking the form of an integral operator with Toeplitz and reversibility structures with the estimates the asymptotic

$$\forall s\in[s_0,S], \quad \max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\right),$$

We remark that the operator $\mathtt{E}_n^1$ satisfies similar estimates as for $\mathtt{E}_n^0$ seen before in (3.15). Finally, we want to emphasize that the derivation of the asymptotic structure of the operator $\mathfrak{L}_{\varepsilon r}$ seen before in (3.14) requires some refined analysis. The delicate point concerns the expansion of the operator $\mathbf{L}_r$ defined in (3.7) and for this part we use the kernel structure detailed in (C.7)

$$K_0(z) = -\log\left(\tfrac{z}{2}\right)I_0(z) + \sum_{m=0}^{\infty}\tfrac{\psi(m+1)}{(m!)^2}\left(\tfrac{z}{2}\right)^{2m}.$$

with I_0 being analytic. This is different from the cases discussed before as for the water waves in [7, 33] where the kernel is given by that of Euler equations (corresponding to $\lambda=0$), that is, $K(z)=-\log\left(\frac{z}{2}\right)$. In this latter case the deformed kernel enjoys the structure

$$-\log\left(2A_r(t,\theta,\eta)\right) = -\log\left|\sin\left(\tfrac{\theta-\eta}{2}\right)\right| + \text{smooth nonhomogeneous kernel.}$$

This means that the associated operator is given by a diagonal operator of order -1 up to a smoothing non diagonal pseudo-differential operator in $OPS^{-\infty}$. In our context, this decomposition fails for $\lambda>0$ and we get a similar one but with less smoothing operator. Actually we obtain from (6.39) the splitting

$$K_0\left(2\lambda A_r(t,\theta,\eta)\right) = K_0\left(2\lambda\sin\left(\tfrac{\theta-\eta}{2}\right)\right) + \mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta) + \mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta), \tag{3.17}$$

where the kernels $\mathscr{K}_{r,1}^1$ and $\mathscr{K}_{r,1}^2$ are smooth whereas $\mathscr{K}$ is slightly singular taking the form

$$\mathscr{K}(\theta) \triangleq \sin^2\left(\tfrac{\theta}{2}\right)\log\left(\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right).$$

Therefore, the change of variable $\eta\mapsto\eta+\theta$ kills the dependence in theta in the singular part, which

allows to estimate the corresponding integral operator in H^s.
ⓑ **KAM reduction of the remainder term.** This is the main target of Section 7.3.2 and the result is stated in Proposition 7.5. The goal is to conjugate the remainder $\mathscr{R}_0$ of (3.16) and transform it into a diagonal operator. This will be developed in a standard way by constructing successive transformations through the KAM reduction allowing to replace at each step the old remainder by a new one which is much smaller provided that we make the suitable parameters extraction. This scheme can be achieved unless we solve the associated *homological equation.* To avoid resonances we should at each step make an extraction from the parameters set through the second order Melnikov conditions and the final outcome is as follows,

$$\mathscr{L}_\infty \triangleq \Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty = \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{D}_\infty,$$

where $\mathscr{D}_\infty = \big(\mathrm{i}\mu_j^\infty(\lambda,\omega,i_0)\big)_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}$ is a diagonal operator with pure imaginary spectrum and Φ_∞ is a reversible invertible operator. This reduction is possible when the parameters (λ,ω) belong to the following Cantor-like set,

$$\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0) \triangleq \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n}} \Big\{(\lambda,\omega)\in\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\quad\text{s.t.}\quad \big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)-\mu_{j_0}^\infty(\lambda,\omega,i_0)\big| > \tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}.$$

The eigenvalues admit the following asymptotic

$$\mu_j^\infty(\lambda,\omega,i_0) \triangleq \Omega_j(\lambda) + jr^1(\lambda,\omega,i_0) + r_j^\infty(\lambda,\omega,i_0),$$

where r^1 and r_j^∞ are real small coefficients with Lipschitz dependence with respect to the torus. Indeed, we have

$$\begin{aligned}\|r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}|j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-1},\\ \|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}} + \sup_{j\in\mathbb{S}_0^c}\|\Delta_{12}r_j^\infty\|_q^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma}^{\gamma,\mathcal{O}},\end{aligned}$$

for some index regularity $\overline{s}_h+\sigma$ and $\Delta_{12}r^1 = r^1(\lambda,\omega,i_1) - r^1(\lambda,\omega,i_2)$.
ⓒ **Construction of the approximate inverse.** The next step is to invert approximately the operator $\widehat{\mathcal{L}}_\omega$ detailed in Proposition 7.6. First we establish an approximate inverse for $\mathscr{L}_\infty$, on the Cantor set

$$\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0) \triangleq \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}} \Big\{(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad |\omega\cdot l+\mu_j^\infty(\lambda,\omega)| > \tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

Then, introducing the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0) \triangleq \mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)\cap\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0),$$

we are able to construct an approximate inverse of $\widehat{\mathcal{L}}_\omega$ in the following sense,

$$\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n} = \mathrm{Id} + \mathtt{E}_n \quad \text{in } \mathcal{G}_n,$$

where $\mathtt{E}_n$ is a fast frequency decaying operator as in (3.15) and $\mathtt{T}_{\omega,n}$ satisfies tame estimates uniformly in n. Therefore coming back to Section 6.3, more precisely to Theorem 6.1, this enables to construct an approximate right inverse T_0 for the full differential $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ enjoying suitable tame estimates.
In what follows we want to make some comments. The first one concerns the Lipschitz dependence of the eigenvalues with respsect to the torus. This is required in studying the stability of Cantor sets in

Nash-Moser scheme and allows to construct a final massive Cantor set. As for the second one, it concerns the KAM reduction which allows to diagonalize the operator when the parameters belong to a Cantor set like, even though all the involved transformations and operators can be extended in the whole set of parameters using standard cut-off functions for the Fourier coefficients. This extension with adequate estimates will be needed later during the implementation of Nash-Moser scheme. This is not the only way to produce suitable extensions and one expects Whitney extension to be also well adapted as in [7, 33]. In our case we privilege the first procedure which can be easily set up and manipulated using classical functional tools. The last comment is related to a technical point in KAM reduction, Contrary to the preceding papers such as [7, 33], we do not need to use pseudo-differential operators techniques in the description of the aforementioned asymptotic structures of $\mathfrak{L}_{\varepsilon r}$ and $\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp$. In fact, they can be avoided since all the involved operators can be described through their kernels and therefore instead of splitting the symbols we simply expand the kernels as in (3.17) which sounds to be more appropriate in our context.

► **Step 3.** *Nash-Moser scheme.* This is the main purpose of Section 8.1 where we construct zeros for the nonlinear function $\mathcal{F}$ defined in (3.13) for small ε following Nash-Moser scheme in the spirit of the papers [7, 33]. Let us quickly sketch this scheme. We build by induction a sequence of approximate solutions U_n

$$U_{n+1} \triangleq U_n + H_{n+1} \quad \text{with} \quad H_{n+1} \triangleq -\Pi_{N_n}\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n).$$

with T_n an approximate inverse of $d_{i,\alpha}\mathcal{F}(U_n)$ constructed in **Step 2**. Thus using Taylor Formula we may write

$$\mathcal{F}(U_{n+1}) = \Pi_{N_n}^{\perp}\mathcal{F}(U_n) - \Pi_{N_n}\big(L_n\mathrm{T}_n - \mathrm{Id}\big)\Pi_{N_n}\mathcal{F}(U_n) + \big(L_n\Pi_n^{\perp} - \Pi_{N_n}^{\perp}L_n\big)\mathrm{T}_n\Pi_{N_n}\mathcal{F}(U_n) + Q_n,$$

where Q_n is a quadratic functional. Consider the Cantor set

$$\mathcal{A}_n^{\gamma} \triangleq \bigcap_{k=0}^{n-1}\mathcal{G}_k(\gamma_{k+1},\tau_1,\tau_2,i_k),$$

with $\gamma_n \triangleq \gamma(1+2^{-n})$, then we show by induction that

$$\|U_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}}, \quad \|U_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_{n-1}^{\mu}, \quad \|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{\gamma}} \lesssim \varepsilon N_{n-1}^{-a_1} \tag{3.18}$$

for a suitable choice of the parameters $a_1, b_1, \overline{a}, \mu, \overline{\sigma} > 0$ and O_n^{γ} is an open enlargement of $\mathcal{A}_n^{\gamma}$ needed to construct classical extensions to the whole set of parameters $\mathcal{O}$. Actually, we get a precise statement in Proposition 8.1 allowing to deduce that the sequence $(U_n)_n$ converges in a strong topology towards a sufficient smooth profile $(\lambda,\omega)\in\mathcal{O}\mapsto U_\infty(\lambda,\omega) = \big(i_\infty(\lambda,\omega),\alpha_\infty(\lambda,\omega),(\lambda,\omega)\big)$ with

$$\forall(\lambda,\omega)\in\mathcal{G}_\infty^{\gamma}, \quad \mathcal{F}(U_\infty(\lambda,\omega)) = 0, \quad \mathcal{G}_\infty^{\gamma} \triangleq \bigcap_{n\in\mathbb{N}}\mathcal{A}_n^{\gamma}.$$

Moreover, we get in view of Corollary 8.1 a smooth function $\lambda\in(\lambda_0,\lambda_1)\mapsto(\lambda,\omega(\lambda,\varepsilon))$ with

$$\omega(\lambda,\varepsilon) = -\omega_{\mathrm{Eq}}(\lambda) + \overline{r}_\varepsilon(\lambda), \quad \|\overline{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \tag{3.19}$$

and

$$\forall\lambda\in\mathcal{C}_\infty^{\varepsilon}, \quad \mathcal{F}\big(U_\infty(\lambda,\omega(\lambda,\varepsilon))\big) = 0 \quad \text{with} \quad \alpha_\infty\big(\lambda,\omega(\lambda,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(\lambda),$$

where the Cantor set $\mathcal{C}_\infty^\varepsilon$ is defined by

$$\mathcal{C}_\infty^\varepsilon \triangleq \Big\{\lambda \in (\lambda_0, \lambda_1) \quad \text{s.t.} \quad \big(\lambda, \omega(\lambda,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma\Big\}. \tag{3.20}$$

This gives solutions to the original equation (3.12) provided that λ belongs to the final Cantor set $\mathcal{C}_\infty^\varepsilon$ and the last point to deal with aims to measure this set.

► **Step 4.** *Measure estimates.* The measure of the final Cantor set $\mathcal{C}_\infty^\varepsilon$ will be explored in Section 8.2. We show in Proposition 8.2 that by fixing $\gamma = \varepsilon^a$ for some small a we get

$$|\mathcal{C}_\infty^\varepsilon| \geqslant (\lambda_1 - \lambda_0) - C\varepsilon^\delta,$$

with small δ connected to the geometry of the Cantor set and the non degeneracy of the equilibrium spectrum. There are two main ingredients to get this result. The first one is the stability of the intermediate Cantor sets $(\mathcal{A}_n^\gamma)_n$ following from the fast convergence of Nash-Moser scheme. However the second one is the transversality property stated in Lemma 8.3 used in the spirit of [12] and [139]. This property will be first established for the linear frequencies in Proposition 5.5, using the analyticity of the eigenvalues and their asymptotic behavior. Then the extension of the transversality assumption to the perturbed frequencies is done using perturbative arguments together with the asymptotic description of the approximate eigenvalues detailed in (8.67), (8.82) and (8.81). We emphasize that the transversality is strongly related to the non-degeneracy of the eigenvalues in the sense of the Definition 5.1 . For instance, we show that the curve $\lambda \in [\lambda_0, \lambda_1] \mapsto \big(\Omega_{j_1}(\lambda), ..., \Omega_{j_d}(\lambda)\big)$ is not contained in any vectorial plane, that is, if there exists a constant vector $c = (c_1, .., c_d)$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad \sum_{j=1}^{d} c_k \Omega_{j_k}(\lambda) = 0,$$

then $c = 0$. This is proved in Lemma 5.4 and follows from the asymptotic of the eigenvalues for large values of λ according to the law (C.15) combined with the invertibility of Vandermonde matrices.

4 Hamiltonian formulation of the patch motion

In this section we shall set up the contour dynamics equation governing the patch motion. A particular attention will be focused on the vortex patch equation in the polar coordinates system. We shall see that the Hamiltonian structure still survives at the level of the patch dynamics, which is the starting point towards the construction of quasi-periodic solutions.

4.1 Contour dynamics equation in polar coordinates

The Rankine vortex $\mathbf{1}_{\mathbb{D}}$ (actually any radial function) is a stationary solution to $(\text{QGSW})_\lambda$. To look for ordered structure like periodic or quasi-periodic vortex patches $t \mapsto \mathbf{1}_{D_t}$ around this equilibrium state, we find it convenient to consider a polar parametrization of the boundary

$$z(t,\theta) \triangleq \Big(1 + 2r(t,\theta)\Big)^{\frac{1}{2}} e^{\mathrm{i}\theta}. \tag{4.1}$$

Here r is the radial deformation of the patch which is small, namely $|r(t,\theta)| \ll 1$. Taking $r = 0$ gives a parametrization of the unit circle $\mathbb{T}$. We shall introduce the new symplectic unknown

$$R(t,\theta) \triangleq \Big(1 + 2r(t,\theta)\Big)^{\frac{1}{2}}. \tag{4.2}$$

which will be useful to write down the equations into the Hamiltonian form. In what follows we want to explicit the contour dynamics equation with the polar coordinates. The starting point is the complex formulation of the vortex patches equation (1.17), which we recall here

$$\operatorname{Im}\Big(\Big[\partial_t z(t,\theta) - \mathbf{v}(t,z(t,\theta))\Big]\overline{\partial_\theta z(t,\theta)}\Big) = 0.$$

In order to transform it into a nonlinear PDE, we need to recover the velocity field $\mathbf{v}$ from the patch parametrization. To do so, recall that $\mathbf{v} = \nabla^\perp \mathbf{\Psi}$ with $\mathbf{\Psi}$ given by (1.7) To get an explicit form of the velocity in terms of the patch boundary we shall use the complex version of Stokes theorem

$$2\mathrm{i}\int_D \partial_{\overline{\xi}} f(\xi,\overline{\xi})dA(\xi) = \int_{\partial D} f(\xi,\overline{\xi})d\xi. \tag{4.3}$$

In view of $\mathbf{v}(t,z) = 2\mathrm{i}\partial_{\overline{z}}\mathbf{\Psi}(t,z)$, one deduces that

$$\mathbf{v}(t,z) = \frac{1}{2\pi}\int_{\partial D_t} K_0(\lambda|z-\xi|)d\xi. \tag{4.4}$$

Notice that to rigorously apply Stokes theorem one may use a regularization procedure. This is purely technical and we refer the reader to the proof of Proposition 4.1 for more details. Next we intend to write down the boundary motion in terms of the contour dynamics. First, from the polar parametrization, it is easy to check from (4.1) that

$$\operatorname{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big) = -\partial_t r(t,\theta).$$

On the other hand, using (4.4) and (C.3), we infer

$$\operatorname{Im}\Big(\mathbf{v}(t,z(t,\theta))\overline{\partial_\theta z(t,\theta)}\Big) = \int_{\mathbb{T}} K_0\left(\lambda|z(t,\theta) - z(t,\eta)|\right)\operatorname{Im}\Big(\partial_\eta z(t,\eta)\overline{\partial_\theta z(t,\theta)}\Big)d\eta.$$

Next we observe that,

$$\begin{aligned}\operatorname{Im}\Big(\partial_\eta z(t,\eta)\overline{\partial_\theta z(t,\theta)}\Big) &= \partial^2_{\theta\eta}\operatorname{Im}\Big(z(t,\eta)\overline{z(t,\theta)}\Big)\\ &= \partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big).\end{aligned}$$

Thus, by setting

$$A_r(t,\theta,\eta) \triangleq |z(t,\theta) - z(t,\eta)| = |R(t,\theta)e^{\mathrm{i}\theta} - R(t,\eta)e^{\mathrm{i}\eta}| \tag{4.5}$$

and

$$F_\lambda[r](t,\theta) \triangleq \int_{\mathbb{T}} K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial^2_{\theta\eta}\Big(R(t,\eta)R(t,\theta)\sin(\eta-\theta)\Big)d\eta, \tag{4.6}$$

we get the vortex patch equation in the polar coordinates

$$\partial_t r(t,\theta) + F_\lambda[r](t,\theta) = 0. \tag{4.7}$$

Now, we fix a parameter Ω that will be used later to get rid of trivial resonances, and we shall look for solutions in the form

$$r(t,\theta) = \tilde{r}(t,\theta+\Omega t). \tag{4.8}$$

Then elementary change of variables applied with (4.6) show that

$$F_\lambda[\tilde{r}](t,\theta+\Omega t) = F_\lambda[r](t,\theta)\,. \tag{4.9}$$

Thus, the equation (4.7) becomes (to alleviate the notation we simply use r instead of $\tilde{r}$)

$$\partial_t r(t,\theta) + \Omega\partial_\theta r(t,\theta) + F_\lambda[r](t,\theta) = 0, \tag{4.10}$$

which is a nonlinear and nonlocal transport PDE. To fix the terminology, we mean by a time quasi-periodic solution of (4.10), a solution in the form

$$r(t,\theta) = \widehat{r}(\omega t,\theta),$$

where $\widehat{r}:\ (\varphi,\theta)\in\mathbb{T}^{d+1}\mapsto\ \widehat{r}(\varphi,\theta)\in\mathbb{R}$, $\omega\in\mathbb{R}^d$ and $d\in\mathbb{N}^*$. Hence in this setting, the equation (4.10) becomes

$$\omega\cdot\partial_\varphi\widehat{r}(\varphi,\theta) + \Omega\partial_\theta\widehat{r}(\varphi,\theta) + F_\lambda[\widehat{r}](\varphi,\theta) = 0.$$

In the sequel, we shall alleviate the notation and denote $\widehat{r}$ simply by r and the foregoing equation writes

$$\forall(\varphi,\theta)\in\mathbb{T}^{d+1},\quad \omega\cdot\partial_\varphi r(\varphi,\theta) + \Omega\partial_\theta r(\varphi,\theta) + F_\lambda[r](\varphi,\theta) = 0. \tag{4.11}$$

4.2 Hamiltonian structure

We now move to a new consideration related to the analysis of the Hamiltonian structure behind the transport equation (4.10). This structure sounds to be essential if one wants to explore quasi-periodic solutions near Rankine vortices. Notice that it is a classical fact that incompressible active scalar equations such as 2D Euler equations are Hamiltonian and as we shall see in this section, we can find a suitable interpretation of this property at the level of the contour dynamics equations which is a stronger reformulation.

4.2.1 Hamiltonian reformulation

We consider the kinetic energy and the angular impulse associated to the patch $\omega(t) = \mathbf{1}_{D_t}$ and defined by

$$E(t) \triangleq -\frac{1}{2\pi}\int_{D_t}\boldsymbol{\Psi}(t,z)dA(z) \quad \text{and} \quad J(t) \triangleq \frac{1}{2\pi}\int_{D_t}|z|^2 dA(z), \tag{4.12}$$

where the stream function $\boldsymbol{\Psi}$ is defined according to (1.7). Notice that the sign convention ensures the kinetic energy to be positive. The following result dealing with the time conservation of the preceding quantities is classical and can be proved in a similar way to Euler equations.

Lemma 4.1. *The kinetic energy E and the angular impulse J are conserved during the motion,*

$$\frac{dE(t)}{dt} = 0 = \frac{dJ(t)}{dt}\cdot$$

In what follows we shall state the main result of this section on the Hamiltonian structure governing the equation (4.10).

Proposition 4.1. *The equation* (4.10) *is Hamiltonian and takes the form*

$$\partial_t r = \partial_\theta\nabla H(r), \tag{4.13}$$

where ∇ is the $L^2_\theta(\mathbb{T})$-gradient with respect to the $L^2_\theta(\mathbb{T})$-normalized scalar product defined by

$$\big\langle \rho_1,\rho_2\big\rangle_{L^2(\mathbb{T})}\triangleq\int_{\mathbb{T}}\rho_1(\theta)\rho_2(\theta)d\theta$$

and the hamiltonian H is defined by

$$H(r)\triangleq\tfrac12\Big(E(r)-\Omega J(r)\Big).$$

In particular, we get the conservation of the average, that is,

$$\frac{d}{dt}\int_{\mathbb{T}}r(t,\theta)d\theta=0. \tag{4.14}$$

Proof. ▶ Using Stokes formula (4.3), we may write

$$J(r)(t)=\tfrac{1}{8\mathrm{i}\pi}\int_{\partial D_t}|\xi|^2\overline{\xi}d\xi.$$

Then from the parametrization detailed in (4.1) one gets easily

$$\begin{aligned}J(r)(t)&=\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}|z(t,\theta)|^2\overline{z(t,\theta)}\partial_\theta z(t,\theta)d\theta\\&=\tfrac{1}{16\mathrm{i}}\int_{\mathbb{T}}\partial_\theta\left(R^4(t,\theta)\right)d\theta+\tfrac14\int_{\mathbb{T}}R^4(t,\theta)d\theta\\&=\tfrac14\int_{\mathbb{T}}R^4(t,\theta)d\theta.\end{aligned}$$

Consequently,

$$J(r)(t)=\tfrac14\int_{\mathbb{T}}\left(1+2r(t,\theta)\right)^2d\theta. \tag{4.15}$$

Differentiating in r one gets for $\rho\in L^2(\mathbb{T})$

$$\langle\nabla J(r),\rho\rangle_{L^2(\mathbb{T})}(t)=\int_{\mathbb{T}}(1+2r(t,\theta))\rho(\theta)d\theta,\qquad\text{i.e.}\qquad\nabla J(r)=1+2r.$$

It follows that

$$\tfrac12\Omega\partial_\theta\nabla J(r)=\Omega\partial_\theta r. \tag{4.16}$$

▶ Next, we shall compute the Gâteaux derivative of E in a given direction $\rho\in L^2(\mathbb{T})$. The first step is to express the energy

$$E(t)=-\frac{1}{2\pi}\int_{D_t}\boldsymbol{\Psi}(t,z)dA(z)$$

in terms of the boundary parametrization of ∂D_t, which shall be done by using Stokes theorem (4.3). Recall from (1.7) that the potential velocity expresses as follows

$$\boldsymbol{\Psi}(t,z)=-\tfrac{1}{2\pi}\int_{D_t}K_0(\lambda|\xi-z|)dA(\xi).$$

In order to apply Stokes theorem we shall a priori formally compute an anti-derivative of $K_0(\lambda|\xi-z|)$ with respect to $\overline{\xi}$. We shall search it in the form

$$(\overline{\xi}-\overline{z})f(\lambda|\xi-z|).$$

Then we should get

$$K_0(\lambda|\xi-z|)=\partial_{\overline{\xi}}\left((\overline{\xi}-\overline{z})f(\lambda|\xi-z|)\right)=f(\lambda|\xi-z|)+\tfrac{\lambda|\xi-z|}{2}f'(\lambda|\xi-z|).$$

Hence f is a solution on $\mathbb{R}_+^*$ of the ordinary differential equation

$$\tfrac{1}{2}xf'(x)+f(x)=K_0(x),\qquad\text{i.e.}\qquad (x^2f(x))'=2xK_0(x).\tag{4.17}$$

Using (C.5), we obtain

$$f(x)=-\tfrac{2xK_1(x)+C}{x^2},$$

where C is a constant to be fixed so that the integral converges. Using (C.6), one has on the real line

$$K_1(x)\underset{x\to0}{=}\tfrac{1}{x}+\tfrac{x}{2}\log\left(\tfrac{x}{2}\right)+o\left(x\log\left(\tfrac{x}{2}\right)\right),$$

so that

$$xK_1(x)\underset{x\to0}{=}1+\tfrac{x^2}{2}\log\left(\tfrac{x}{2}\right)+o\left(x^2\log\left(\tfrac{x}{2}\right)\right).$$

Making the choice $C=-2$ we get

$$f(x)=-\tfrac{2(xK_1(x)-1)}{x^2},\tag{4.18}$$

which behaves like a logarithm near 0 and thus it is integrable. But notice that Stokes theorem requires some smoothness on the integrated function to be applied. Consequently, we shall rather consider for $\epsilon>0$ the smooth quantity

$$F_\epsilon(\xi,z)\triangleq\partial_{\overline{\xi}}\left((\overline{\xi}-\overline{z})f\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right)\right).$$

Then applying Stokes theorem (4.3) yields

$$2\mathrm{i}\int_{D_t}F_\epsilon(\xi,z)dA(\xi)=\int_{\partial D_t}(\overline{\xi}-\overline{z})f\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right)d\xi.$$

According to the structure of f described above, a simple application of dominated convergence theorem gives

$$\lim_{\epsilon\to0}\int_{\partial D_t}(\overline{\xi}-\overline{z})f\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right)d\xi=\int_{\partial D_t}(\overline{\xi}-\overline{z})f\left(\lambda|\xi-z|\right)d\xi.$$

Now observe that by virtue of (4.17), we can write

$$F_\epsilon(\xi,z)=K_0\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right)-\frac{\lambda\epsilon^2}{2\sqrt{|\xi-z|^2+\epsilon^2}}f'\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right).$$

Using the fact that $f'(x)$ is equivalent to $\frac{1}{x}$ at 0, we obtain by dominated convergence theorem

$$\lim_{\epsilon\to0}\int_{D_t}\frac{\epsilon^2}{\sqrt{|\xi-z|^2+\epsilon^2}}f'\left(\lambda\sqrt{|\xi-z|^2+\epsilon^2}\right)dA(\xi)=0.$$

Thus, still by dominated convergence theorem, we get

$$\lim_{\epsilon\to0}\int_{D_t}F_\epsilon(\xi,z)dA(\xi)=\int_{D_t}K_0(\lambda|\xi-z|)dA(\xi).$$

Gathering the foregoing computations implies

$$\mathbf{\Psi}(t,z)=\tfrac{1}{4\mathrm{i}\pi}\int_{\partial D_t}(\overline{\xi}-\overline{z})f(\lambda|\xi-z|)d\xi.$$

Therefore using the parametrization (4.1) we find

$$\mathbf{\Psi}(t,z(t,\theta))=\tfrac{1}{\mathrm{i}\lambda^2}\int_{\mathbb{T}}\frac{(\overline{z}(t,\eta)-\overline{z}(t,\theta))\left[\lambda|z(t,\theta)-z(t,\eta)|K_1\left(\lambda|z(t,\eta)-z(t,\theta)|\right)-1\right]}{|z(t,\eta)-z(t,\theta)|^2}\partial_\eta z(t,\eta)d\eta.$$

Making appeal to f and removing the time dependence, we get

$$\mathbf{\Psi}(z(\theta))=\tfrac{\mathrm{i}}{2}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)d\eta. \tag{4.19}$$

At this stage we need to look for an anti-derivative with respect to $\overline{z}$ of $\frac{-1}{2}(\overline{\xi}-\overline{z})f(\lambda|\xi-z|)$ in the form

$$(\overline{\xi}-\overline{z})^2 g(\lambda|\xi-z|).$$

Therefore we deduce the constraint

$$\tfrac{-1}{2}(\overline{\xi}-\overline{z})f(\lambda|\xi-z|)=\partial_{\overline{z}}\left((\overline{\xi}-\overline{z})^2g(\lambda|\xi-z|)\right)=-(\overline{\xi}-\overline{z})\Big(2g(\lambda|\xi-z|)+\tfrac{\lambda|\xi-z|}{2}g'(\lambda|\xi-z|)\Big).$$

Hence, g should be a solution on $\mathbb{R}_+^*$ of the ordinary differential equation

$$\tfrac{x}{2}g'(x)+2g(x)=\tfrac{f(x)}{2},\qquad \text{i.e.}\qquad (x^4g(x))'=x^3f(x)=2x-2x^2K_1(x). \tag{4.20}$$

Using once again (C.5) yields

$$g(x)=\tfrac{x^2+2x^2K_2(x)+C}{x^4},$$

where C is again a constant used to cancel the violent singularity. From (C.6) and (C.2), one obtains the asymptotic

$$K_2(x)\underset{x\to0}{=}\tfrac{2}{x^2}-\tfrac{1}{2}+O\left(x^2\log(x)\right).$$

Thus

$$x^2K_2(x)\underset{x\to0}{=}2-\tfrac{x^2}{2}+O(x^4\log(x)).$$

Then by choosing $C=-4$ we deduce that the function below

$$g(x)=\tfrac{x^2+2x^2K_2(x)-4}{x^4}$$

is integrable. Hence, applying once again Stokes theorem (4.3) together with a regularization procedure as above, we infer

$$\begin{aligned}E(r)(t)&=\tfrac{1}{4\pi^2\lambda^4}\int_{\partial D_t}\int_{\partial D_t}\frac{(\overline{\xi}-\overline{z})^2\left[\lambda^2|\xi-z|^2\left(1+2K_2(\lambda|\xi-z|)\right)-4\right]}{|\xi-z|^4}dzd\xi\\&=\tfrac{1}{\lambda^4}\int_{\mathbb{T}}\int_{\mathbb{T}}\tfrac{(\overline{z}(t,\eta)-\overline{z}(t,\theta))^2[\lambda|z(t,\eta)-z(t,\theta)|(1+2K_2(\lambda|z(t,\eta)-z(t,\theta)|))-4]}{|z(t,\eta)-z(t,\theta)|^2}\partial_\theta z(t,\theta)\partial_\eta z(t,\eta)d\eta d\theta.\end{aligned}$$

Hence using g and removing the dependence in time, we find

$$E(r)=\tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))^2g(\lambda|z(\theta)-z(\eta)|)\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta. \tag{4.21}$$

The next goal is to compute the derivative of E with respect to r in the direction ρ, which is straightforward

$$
\begin{aligned}
\langle \nabla E(r),\rho\rangle_{L^2(\mathbb{T})} = &\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right) g\left(\lambda|z(\theta)-z(\eta)|\right) \left(\tfrac{\rho(\theta)e^{-\mathrm{i}\theta}}{R(\theta)}-\tfrac{\rho(\eta)e^{-\mathrm{i}\eta}}{R(\eta)}\right)\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&+\tfrac{\lambda}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta)-\overline{z}(\eta))^2}{|z(\theta)-z(\eta)|} g'\left(\lambda|z(\theta)-z(\eta)|\right)\tfrac{\rho(\theta)}{R(\theta)}\mathrm{Re}\left((z(\theta)-z(\eta))e^{-\mathrm{i}\theta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&+\tfrac{\lambda}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta)-\overline{z}(\eta))^2}{|z(\theta)-z(\eta)|} g'\left(\lambda|z(\theta)-z(\eta)|\right)\tfrac{\rho(\eta)}{R(\eta)}\mathrm{Re}\left((z(\eta)-z(\theta))e^{-\mathrm{i}\eta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&+\tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right)^2 g\left(\lambda|z(\theta)-z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta)d\theta d\eta\\
&+\tfrac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right)^2 g\left(\lambda|z(\theta)-z(\eta)|\right)\partial_\theta z(\theta)\partial_\eta\left(\tfrac{\rho(\eta)e^{\mathrm{i}\eta}}{R(\eta)}\right)d\theta d\eta.
\end{aligned}
$$

By exchanging in the double integral θ and η, we deduce

$$
\begin{aligned}
\langle \nabla E(r),\rho\rangle_{L^2(\mathbb{T})} = 2&\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right) g\left(\lambda|z(\theta)-z(\eta)|\right)\tfrac{\rho(\theta)e^{-\mathrm{i}\theta}}{R(\theta)}\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&+\lambda\int_{\mathbb{T}}\int_{\mathbb{T}} \tfrac{(\overline{z}(\theta)-\overline{z}(\eta))^2}{|z(\theta)-z(\eta)|} g'\left(\lambda|z(\theta)-z(\eta)|\right)\tfrac{\rho(\theta)}{R(\theta)}\mathrm{Re}\left((z(\theta)-z(\eta))e^{-\mathrm{i}\theta}\right)\partial_\theta z(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&+\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right)^2 g\left(\lambda|z(\theta)-z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta)d\theta d\eta.
\end{aligned}
$$

An integration by parts in the last integral leads to

$$
\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathbb{T}} \left(\overline{z}(\theta)-\overline{z}(\eta)\right)^2 g\left(\lambda|z(\theta)-z(\eta)|\right)\partial_\theta\left(\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\right)\partial_\eta z(\eta)d\theta d\eta\\
&=-2\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))g(\lambda|z(\theta)-z(\eta)|)\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\partial_\theta\overline{z}(\theta)\partial_\eta z(\eta)d\theta d\eta\\
&\quad-\lambda\int_{\mathbb{T}}\int_{\mathbb{T}}\tfrac{(\overline{z}(\theta)-\overline{z}(\eta))^2}{|z(\theta)-z(\eta)|}g'(\lambda|z(\theta)-z(\eta)|)\tfrac{\rho(\theta)e^{\mathrm{i}\theta}}{R(\theta)}\mathrm{Re}\left[(z(\theta)-z(\eta))\partial_\theta\overline{z}(\theta)\right]\partial_\eta z(\eta)d\theta d\eta.
\end{aligned}
$$

Using the identities

$$e^{\mathrm{i}\theta}\partial_\theta\overline{z}(\theta)-e^{-\mathrm{i}\theta}\partial_\theta z(\theta)=-2\mathrm{i}R(\theta)$$

and

$$\mathrm{Re}\left[(z(\theta)-z(\eta))\partial_\theta\overline{z}(\theta)\right]e^{\mathrm{i}\theta}-\partial_\theta z(\theta)\mathrm{Re}\left[(z(\theta)-z(\eta))e^{-\mathrm{i}\theta}\right]=-\mathrm{i}R(\theta)(z(\theta)-z(\eta)),$$

we infer

$$
\begin{aligned}
\langle \nabla E(r),\rho\rangle_{L^2(\mathbb{T})} \quad = \quad &\tfrac{4}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))g(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta\\
&+\tfrac{\lambda}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))|z(\theta)-z(\eta)|g'(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta.
\end{aligned}
$$

Applying (4.20), we find

$$\langle \nabla E(r),\rho\rangle_{L^2(\mathbb{T})}=\tfrac{1}{\mathrm{i}}\int_{\mathbb{T}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)\rho(\theta)d\theta d\eta,$$

which implies by virtue of (4.19)

$$\nabla E(r)=\tfrac{1}{\mathrm{i}}\int_{\mathbb{T}}(\overline{z}(\theta)-\overline{z}(\eta))f(\lambda|z(\theta)-z(\eta)|)\partial_\eta z(\eta)d\eta=-2\boldsymbol{\Psi}(z(\theta)).$$

Now, using the complex notation we deduce that

$$\begin{aligned}\partial_\theta\boldsymbol{\Psi}(z(\theta))&=\nabla\boldsymbol{\Psi}(z(\theta))\cdot\partial_\theta z(\theta)\\&=\mathrm{Im}\Big(\mathbf{v}(z(\theta))\overline{\partial_\theta z(\theta)}\Big)\\&=F_\lambda[r](\theta),\end{aligned}$$

where we used (1.2) and the facts that $\nabla^\perp\boldsymbol{\Psi}=\mathbf{v}$ and $\boldsymbol{\Psi}$ is real-valued. Recall that the functional $F_\lambda[r]$ was introduced in (4.6). Hence

$$\partial_\theta\nabla E(r)=-2\partial_\theta\boldsymbol{\Psi}(z(\theta))=-2F_\lambda[r](\theta).$$

Finally we get

$$\tfrac{1}{2}\partial_\theta\nabla E(r)=-F_\lambda[r](\theta).\tag{4.22}$$

The conservation of the average is easy to check from the Hamiltonian equation. Therefore the proof of Proposition 4.1 is achieved. □

4.2.2 Symplectic structure and reversibility

The main concern is to investigate the symplectic structure together with the reversibility property associated to the Hamiltonian equation (4.13). These properties will be used in a crucial way to fix the symmetry in the function spaces and by this way remove from the phase space the trivial resonances. For more details we refer to Section A.1 and Section 6.

According to Proposition 4.1, it seems more convenient to work with the phase space

$$L^2_0(\mathbb{T})\triangleq\left\{r=\sum_{j\in\mathbb{Z}^*}r_je_j\quad\text{s.t.}\quad r_{-j}=\overline{r_j}\quad\text{and}\quad\sum_{j\in\mathbb{Z}^*}|r_j|^2<\infty\right\},\qquad e_j(\theta)\triangleq e^{\mathrm{i}j\theta}.\tag{4.23}$$

The symplectic structure on $L^2_0(\mathbb{T})$ induced by (4.13) is given by the symplectic 2-form

$$\mathcal{W}(r,h)\triangleq\int_{\mathbb{T}}\partial_\theta^{-1}r(\theta)h(\theta)d\theta\quad\text{with}\quad\partial_\theta^{-1}r(\theta)=\sum_{j\in\mathbb{Z}^*}\frac{r_j}{\mathrm{i}j}e^{\mathrm{i}j\theta}.\tag{4.24}$$

Then for a given function H, its symplectic gradient X_H is defined through the identity

$$dH(r)[\cdot]=\mathcal{W}(X_H(r),\cdot).\tag{4.25}$$

Using the Fourier expansion

$$r(\theta)=\sum_{j\in\mathbb{Z}^*}r_je^{\mathrm{i}j\theta}\quad\text{with}\quad r_{-j}=\overline{r_j},$$

we easily find that the symplectic form $\mathcal{W}$ writes

$$\mathcal{W}(r,h)=\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}r_jh_{-j}=\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}r_j\overline{h_j},$$

that is

$$\mathcal{W}=\frac{1}{2}\sum_{j\in\mathbb{Z}^*}\frac{1}{\mathrm{i}j}dr_j\wedge dr_{-j}=\sum_{j\in\mathbb{N}^*}\frac{1}{\mathrm{i}j}dr_j\wedge dr_{-j},\tag{4.26}$$

where for all $j\in\mathbb{Z}^*$, the exterior product $dr_j\wedge dr_{-j}$ is defined by

$$dr_j\wedge dr_{-j}(r,h)=r_jh_{-j}-r_{-j}h_j.$$

To define the reversibility, we shall introduce the involution $\mathscr{S}$

$$(\mathscr{S}r)(\theta)\triangleq r(-\theta),\tag{4.27}$$

which satisfies the obvious properties

$$\mathscr{S}^2=\mathrm{Id}\qquad\text{and}\qquad\partial_\theta\circ\mathscr{S}=-\mathscr{S}\circ\partial_\theta.\tag{4.28}$$

The following elementary result is useful and can be easily checked from the structure of the Hamiltonian. Actually, it suffices to make changes of variables.

Lemma 4.2. *The Hamiltonian H and its associated vector field $X_H=\partial_\theta\nabla H$ satisfy*

$$H\circ\mathscr{S}=H\quad\text{and}\quad X_H\circ\mathscr{S}=-\mathscr{S}\circ X_H.$$

5 Linearization and frequencies structure

This section is devoted to some aspects of the linearized operator associated to the evolution equation (4.10) or its Hamiltonian version (4.13). We shall in particular compute it at any state close to the equilibrium and reveal some of its main general feature. As we shall see, the radial shape is very special and gives rise to a Fourier multiplier and thus the spectral properties follow immediately. This latter case serves as a toy model to check the emergence of quasi-periodic solutions at the linear level provided that the Rossby radius λ belongs to a Cantor set, see Proposition 5.1 . However, around this ideal state the situation is roughly uncontrolled and the operator is no longer diagonal and its spectral study is extremely delicate due to resonances that prevent to diagonalize the operator. To deal with this problem we will implement some important tools borrowed from KAM theory as we shall see in Section 7.

5.1 Linearized operator

The main goal of this section is to compute the differential of the nonlinear operator in (4.10) for any small state r. The computations will be conducted at a formal level by simply computing Gateaux derivatives which are related to Frechet derivatives. This formal part can be justified rigorously in a classical way for the suitable functional setting fixed in Section A.1.

5.1.1 The general form

In what follows we shall derive a formula for the linearized operator associated to the equation (4.13). We shall see that it can be split into a transport part with variable coefficients and a nonlocal operator of order zero. More precisiely, we shall establish the following lemma.

Lemma 5.1. *The linearized equation of* (4.13) *at a given small state r is given by the time-dependent linear Hamiltonian equation,*

$$\partial_t\rho(t,\theta)=\partial_\theta\Big(-V_r(\lambda,t,\theta)\rho(t,\theta)+\mathbf{L}_r\rho(\lambda,t,\theta)\Big),$$

where V_r is a scalar function defined by

$$V_r(\lambda,t,\theta)\triangleq\Omega+\tfrac{1}{R(t,\theta)}\int_{\mathbb{T}}K_0\left(\lambda A_r(t,\theta,\eta)\right)\partial_\eta\left(R(t,\eta)\sin(\eta-\theta)\right)d\eta \tag{5.1}$$

and $\mathbf{L}_r$ is a nonlocal operator given by

$$\mathbf{L}_r(\rho)(\lambda,t,\theta)\triangleq\int_{\mathbb{T}}K_0\big(\lambda A_r(t,\theta,\eta)\big)\rho(t,\eta)d\eta. \tag{5.2}$$

We recall that K_0, A_r and R are defined by (C.7), (4.5) *and* (4.2)*, respectively.*
Moreover, if $r(-t,-\theta)=r(t,\theta)$, then

$$V_r(\lambda,-t,-\theta)=V_r(\lambda,t,\theta). \tag{5.3}$$

Proof. Throughout the proof, we shall remove the time dependency of the involved quantities except when it is relevant to keep it. The computations of the Gâteaux derivative of F_λ defined by (4.6) at a point r in the direction ρ are straightforward and standard and we shall only sketch the main lines. Notice that the functional F_λ is smooth in a suitable functional setting and therefore its differential should be recovered from its Gâteaux derivative. First, we observe that the function A_r defined in (4.5) can be written in the form

$$\begin{aligned}A_r(\theta,\eta)&=\big(R^2(\theta)+R^2(\eta)-2R(\theta)R(\eta)\cos(\eta-\theta)\big)^{\frac12}\\&=\Big((R(\theta)-R(\eta))^2+4R(\theta)R(\eta)\sin^2\Big(\tfrac{\eta-\theta}{2}\Big)\Big)^{\frac12}.\end{aligned} \tag{5.4}$$

This identity (5.4) will be of constant use in the sequel. Second, after straightforward computations, we obtain from (4.6),

$$\begin{aligned}d_rF_\lambda[r](\rho)&=\partial_\tau F_\lambda[r+\tau\rho]_{|\tau=0}\\&=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3+\mathcal{I}_4,\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_1&\triangleq\lambda\rho(\theta)\int_{\mathbb{T}}B_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)d\eta,\\\mathcal{I}_2&\triangleq\lambda\int_{\mathbb{T}}\rho(\eta)B_r(\eta,\theta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)d\eta,\\\mathcal{I}_3&\triangleq\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\Big(\rho(\theta)\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\Big)d\eta,\\\mathcal{I}_4&\triangleq\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\Big(\rho(\eta)\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\Big)d\eta,\end{aligned}$$

with

$$B_r(\theta,\eta)\triangleq\frac{R(\theta)-R(\eta)\cos(\eta-\theta)}{R(\theta)A_r(\theta,\eta)}. \tag{5.5}$$

Next, we shall compute $\mathcal{I}_1+\mathcal{I}_3$. To do that, we split $\mathcal{I}_3$ into two terms as follows,

$$\begin{aligned}\mathcal{I}_3&=\partial_\theta\rho(\theta)\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta\\&\quad+\rho(\theta)\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta\\&\triangleq\partial_\theta\rho(\theta)\overline{V}_r(\lambda,\theta)+\rho(\theta)\overline{\mathcal{I}}_3.\end{aligned}$$

An integration by parts in $\overline{\mathcal{I}}_3$ allows to get,

$$\overline{\mathcal{I}}_3=-\lambda\int_{\mathbb{T}}\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)R(\eta)\partial_\theta\left(\tfrac{\sin(\eta-\theta)}{R(\theta)}\right)d\eta.$$

Putting together the preceding identities yields

$$\mathcal{I}_1+\mathcal{I}_3=\partial_\theta\rho(\theta)\overline{V_r}(\lambda,\theta)+\rho(\theta)V_1(\lambda,\theta)\tag{5.6}$$

with

$$\begin{aligned}V_1(\lambda,\theta)&\triangleq\lambda\int_{\mathbb{T}}B_r(\theta,\eta)\partial^2_{\theta\eta}\big(R(\theta)R(\eta)\sin(\eta-\theta)\big)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta\\&\quad-\lambda\int_{\mathbb{T}}\partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta.\end{aligned}\tag{5.7}$$

Differentiating term by term $\overline{V_r}$ with respect to θ gives

$$\begin{aligned}\partial_\theta\overline{V_r}(\lambda,\theta)&=\lambda\int_{\mathbb{T}}\partial_\theta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta\\&\quad+\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial^2_{\theta\eta}\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)d\eta\\&\triangleq\mathcal{J}_1+\mathcal{J}_2.\end{aligned}$$

Integrating by parts in $\mathcal{J}_2$ yields

$$\mathcal{J}_2=-\lambda\int_{\mathbb{T}}R(\eta)\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\theta\left(\tfrac{\sin(\eta-\theta)}{R(\theta)}\right)d\eta.$$

Combining the preceding identities allows to deduce that

$$\partial_\theta\overline{V_r}(\lambda,\theta)=\lambda\int_{\mathbb{T}}K_0'\left(\lambda A_r(\theta,\eta)\right)\Big[\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)-\partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)\Big]d\eta.$$

Next we shall check the following identity

$$\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)=B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)-\partial_\eta A_r(\theta,\eta).\tag{5.8}$$

Indeed, by (5.4) and (5.5), one finds

$$\partial_\theta A_r(\theta,\eta)\partial_\eta\left(\tfrac{R(\eta)\sin(\eta-\theta)}{R(\theta)}\right)=\partial_\theta R(\theta)B_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta))-\tfrac{R(\eta)\sin(\eta-\theta)\partial_\eta(R(\eta)\sin(\eta-\theta))}{A_r(\theta,\eta)}$$

and

$$\begin{aligned}B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right) &= \partial_\theta R(\theta)B_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta))\\ &\quad -\tfrac{(R(\theta)-R(\eta)\cos(\eta-\theta))\partial_\eta(R(\eta)\cos(\eta-\theta))}{A_r(\theta,\eta)}.\end{aligned}$$

Putting together the foregoing identities leads to

$$\frac{\partial_\theta A_r(\theta,\eta)\partial_\eta(R(\eta)\sin(\eta-\theta))}{R(\theta)} = B_r(\theta,\eta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)+g(\theta,\eta),$$

where

$$\begin{aligned}g(\theta,\eta) &\triangleq \tfrac{1}{A_r(\theta,\eta)}\left[(R(\theta)-R(\eta)\cos(\eta-\theta))\partial_\eta(R(\eta)\cos(\eta-\theta))-R(\eta)\sin(\eta-\theta)\partial_\eta(R(\eta)\sin(\eta-\theta))\right]\\ &= \tfrac{R(\theta)\partial_\eta(R(\eta)\cos(\eta-\theta))-R(\eta)\partial_\eta R(\eta)}{A_r(\theta,\eta)}\\ &= -\partial_\eta A_r(\theta,\eta).\end{aligned}$$

This achieves the proof of (5.8). From the periodicity we get

$$\int_{\mathbb{T}}\lambda\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)d\eta = \int_{\mathbb{T}}\partial_\eta\left[K_0\left(\lambda A_r(\theta,\eta)\right)\right]d\eta = 0$$

and thus we get the following important identity

$$\partial_\theta\overline{V_r}(\lambda,\theta) = V_1(\lambda,\theta).$$

Plugging this into (5.6) allows to get

$$\mathcal{I}_1+\mathcal{I}_3 = \partial_\theta\left(\overline{V_r}(\lambda,\theta)\rho(\theta)\right).$$

Notice that it is easy to check that if $r(-t,-\theta)=r(t,\theta)$, then

$$\overline{V_r}(\lambda,-t,-\theta) = \overline{V_r}(\lambda,t,\theta). \tag{5.9}$$

The next task is to compute $\mathcal{I}_2+\mathcal{I}_4$. Using integration by parts in $\mathcal{I}_4$ gives,

$$\mathcal{I}_4 = -\lambda\int_{\mathbb{T}}\rho(\eta)\partial_\eta A_r(\theta,\eta)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\theta\left(\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\right)d\eta.$$

From the symmetry property $A_r(\theta,\eta)=A_r(\eta,\theta)$ and by exchanging the roles of θ and η in (5.8), one deduces

$$B_r(\eta,\theta)\partial^2_{\theta\eta}\left(R(\theta)R(\eta)\sin(\eta-\theta)\right)-\partial_\eta A_r(\theta,\eta)\partial_\theta\left(\tfrac{R(\theta)\sin(\eta-\theta)}{R(\eta)}\right) = -\partial_\theta A_r(\theta,\eta).$$

Therefore we obtain

$$\mathcal{I}_2+\mathcal{I}_4 = -\partial_\theta\left(\int_{\mathbb{T}}\rho(\eta)K_0\left(\lambda A_r(\theta,\eta)\right)d\eta\right) \triangleq -\partial_\theta\mathbf{L}_r(\rho)(\lambda,\theta).$$

Finally, by setting

$$V_r(\lambda,t,\theta)\triangleq\Omega+\overline{V_r}(\lambda,t,\theta)$$

and combining the preceding identities, we end the proof of Lemma 5.1. □

5.1.2 The integrable case

The main purpose here is to explore the structure of the linearized operator at the equilibrium state. We shall see that the radial shape is reflected on the structure the linearized operator which is a Fourier multiplier (of a convolution type). More precisely, we have the following result.

Lemma 5.2. *The following properties hold true.*

1. *The linearized equation of* (4.13) *at the equilibrium state* ($r=0$) *writes,*

$$\partial_t\rho=\partial_\theta\mathrm{L}(\lambda)\rho=\partial_\theta\nabla H_{\mathrm{L}}(\rho),\tag{5.10}$$

where $\mathrm{L}(\lambda)$ *is the self-adjoint operator defined by* $\mathrm{L}(\lambda)\triangleq-V_0(\lambda)+\mathcal{K}_\lambda*_\theta$ *with*

$$V_0(\lambda)\triangleq\Omega+I_1(\lambda)K_1(\lambda)\tag{5.11}$$

and

$$\mathcal{K}_\lambda(\theta)\triangleq K_0\left(2\lambda\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right).\tag{5.12}$$

We refer to the Appendix A for the definitions of the modified Bessel functions I_1, K_1 *and* K_0. *Moreover, the Hamiltonian* H_{L} *is quadratic and takes the form*

$$H_{\mathrm{L}}(\rho)\triangleq\tfrac{1}{2}\langle\mathrm{L}(\lambda)\rho,\rho\rangle_{L^2(\mathbb{T})}.$$

2. *The solutions to* (5.10) *with zero space average are given by*

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(\lambda)t)},\tag{5.13}$$

with

$$\Omega_j(\lambda)\triangleq j\big[\Omega+(I_1K_1)(\lambda)-(I_jK_j)(\lambda)\big].\tag{5.14}$$

and for $\rho(\theta)=\displaystyle\sum_{j\in\mathbb{Z}^*}\rho_je^{\mathrm{i}j\theta}$ *we have*

$$\mathrm{L}(\lambda)\rho(\theta)=-\sum_{j\in\mathbb{Z}^*}\tfrac{\Omega_j(\lambda)}{j}\rho_je^{\mathrm{i}j\theta}\quad\textit{and}\quad H_{\mathrm{L}}\rho=-\sum_{j\in\mathbb{Z}^*}\tfrac{\Omega_j(\lambda)}{2j}|\rho_j|^2,\tag{5.15}$$

Before proceeding with the proof we want to give some remarks.

Remark 5.1.
- *When* $\Omega=0$ *the eigenvalue* $\Omega_1(\lambda)$ *vanishes for any* λ *due to the rotation invariance of the equation and the use of the free parameter* Ω *is to avoid this degeneracy. However the trivial resonance* $\Omega_0(\lambda)=0$ *can be removed by imposing the zero space average which is preserved by the nonlinear dynamics from the Hamiltonian structure as we have seen before in* (4.14).
- *The solutions to the linear equation at the equilibrium are aperiodic and if we excite only a finite number of frequencies with non-resonances assumption we get quasi-periodic solutions. We will make a precise comment later on Proposition* 5.1.

Proof. **1**. First observe that from (5.4), one deduces for $r=0$ that $A_0(\theta,\eta)=2\left|\sin\left(\frac{\eta-\theta}{2}\right)\right|$. Then we obtain from (5.1) and (5.2),

$$\begin{aligned}\mathbf{L}_0\rho(\lambda,\theta) &= \int_{\mathbb{T}} \rho(\eta) K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right) d\eta\\ &= \mathcal{K}_\lambda * \rho(\theta),\end{aligned}$$

with $\mathcal{K}_\lambda$ defined in (5.12) and using the change of variables $\eta \mapsto \eta+\theta$ we obtain

$$\begin{aligned}V_0(\lambda,\theta) &= \Omega + \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\cos(\eta-\theta) d\eta\\ &= \Omega + \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)\cos(\eta) d\eta\\ &\triangleq V_0(\lambda).\end{aligned}$$

We remark that if we write $e_j(\theta) = e^{\mathrm{i}j\theta}$, then direct computations yield

$$\begin{aligned}(\mathcal{K}_\lambda * e_j)(\theta) &= \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right) e^{\mathrm{i}j(\theta-\eta)} d\eta\\ &= e_j(\theta)\int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right) e^{-\mathrm{i}j\eta} d\eta.\end{aligned}$$

Since the function $\eta \mapsto K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)$ is even, we deduce using the change of variables $\eta = 2\tau+\pi$ and the formula (C.9) that

$$\begin{aligned}\int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right) e^{-\mathrm{i}j\eta} d\eta &= \int_{\mathbb{T}} K_0\left(2\lambda\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)\cos(j\eta) d\eta\\ &= \tfrac{(-1)^j}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} K_0(2\lambda\cos(\tau))\cos(2j\tau)d\tau\\ &= (I_jK_j)(\lambda).\end{aligned}$$

Hence, the Fourier coefficients of $\mathcal{K}_\lambda$ are

$$(\mathcal{K}_\lambda)_j = (I_jK_j)(\lambda). \tag{5.16}$$

Similar arguments as before with $j=1$ allow to get

$$V_0(\lambda) = \Omega + (I_1K_1)(\lambda).$$

Recall that $\mathcal{K}_\lambda$ is even and then we find that $\mathrm{L}(\lambda)$ is self-adjoint in $L^2(\mathbb{T})$.

2. Starting from the Fourier expansion $\rho(t,\theta) = \sum\limits_{j\in\mathbb{Z}^*} \rho_j(t)e^{\mathrm{i}j\theta}$, then we can easily ensure from direct computations using the previous results, that ρ solves the equation (5.10) if and only if

$$\dot{\rho_j} = -\mathrm{i}\Omega_j(\lambda)\rho_j \quad \text{with} \quad \Omega_j(\lambda) = j\big[\Omega + (I_1K_1)(\lambda) - (I_jK_j)(\lambda)\big],$$

and therefore

$$\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*} \rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(\lambda)t)}.$$

Concerning the identities (5.15) they can be obtained from straightforward computations. This ends the proof of Lemma 5.2. ☐

5.2 Structure of the linear frequencies

The main target in this section is to explore some interesting structures of the equilibrium frequencies. We shall in particular focus on their monotonicity and detail some asymptotic behavior for large modes. Another important discussion will be devoted to the non-degeneracy of these frequencies through the so-called Rüssmann conditions. This is the cornerstone step in measuring the final Cantor set giving rise to quasi-periodic solutions for the linear/nonlinear problems. Actually, in the nonlinear case the final Cantor appears as a perturbation of the Cantor set constructed from the equilibrium eigenvalues and therefore perturbative arguments based on their non-degeneracy are very useful and will be performed in Section 8.2.

5.2.1 Monotonicity and asymptotic behaviour

Our purpose is to establish some useful properties related to the monotonicity and the asymptotic behavior for large modes of the eigenvalues of the linearized operator at the equilibrium state. Notice that their explicit values are detailed in (5.14). Our result reads as follows.

Lemma 5.3. *Let $\Omega > 0$ and $\lambda \in \mathbb{R}$, then the frequencies $(\Omega_j(\lambda))_{j\in\mathbb{Z}^*}$ satisfy the following properties.*

(i) For any $j \in \mathbb{Z}^*$, $\lambda > 0$ we have $\Omega_{-j}(\lambda) = -\Omega_j(\lambda)$.

(ii) For any $\lambda > 0$, the sequences $(\Omega_j(\lambda))_{j\in\mathbb{N}^*}$ and $\left(\frac{\Omega_j(\lambda)}{j}\right)_{j\in\mathbb{N}^*}$ are strictly increasing.

(iii) For any $\lambda > 0$, the following expansion holds

$$\Omega_j(\lambda) \underset{j\to\infty}{=} V_0(\lambda)j - \tfrac{1}{2} + \tfrac{\lambda^2}{4j^2} + O_\lambda\left(\tfrac{1}{j^4}\right), \tag{5.17}$$

where $V_0(\lambda)$ is defined in (5.11).

(iv) For any $j \in \mathbb{Z}^*$, $\lambda > 0$ we have

$$|\Omega_j(\lambda)| \geqslant \Omega|j|.$$

(v) Given $0 < \lambda_0 < \lambda_1$, there exists $C_0 > 0$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall j, j_0 \in \mathbb{Z}, \quad |\Omega_j(\lambda) \pm \Omega_{j_0}(\lambda)| \geqslant C_0|j \pm j_0|.$$

(vi) Given $0 < \lambda_0 < \lambda_1$ and $q_0 \in \mathbb{N}$, there exists $C_0 > 0$ such that

$$\forall\, j, j_0 \in \mathbb{Z}^*, \quad \max_{q\in[\![0,q_0]\!]} \sup_{\lambda\in[\lambda_0,\lambda_1]} |\partial_\lambda^q\left(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\right)| \leqslant C_0|j - j_0|.$$

Proof. **(i)** It is an immediate consequence of (5.14) and (C.3).
(ii) The monotonicity of the sequence $\left(\frac{\Omega_j(\lambda)}{j}\right)_{j\in\mathbb{N}^*}$ is proved in [54, Prop. 5.9. (1)], see also the Appendix C. It follows that the sequence $(\Omega_j(\lambda))_{j\in\mathbb{N}^*}$ is strictly increasing as the product of two strictly increasing positive sequences.
(iii) It is an immediate consequence of (5.14) and the asymptotic expansion (C.14)
(iv) Recall that $j \mapsto \Omega_j(\lambda)$ is odd and vanishes at $j = 0$. Then it suffices to check the result for $j \in \mathbb{N}^*$. According to the Appendix C, the sequence $j \mapsto (I_jK_j)(\lambda)$ is decreasing and therefore

$$\forall \lambda > 0, \quad (I_1K_1)(\lambda) - (I_jK_j)(\lambda) \geqslant 0. \tag{5.18}$$

It follows that

$$\forall\,\lambda>0,\quad |\Omega_j(\lambda)|\geqslant \Omega j.$$

(v) By the oddness of $j\mapsto\Omega_j(\lambda)$ it is enough to establish the estimate for $j,j_0\in\mathbb{N}$. We shall first focus on the estimate of the difference $\Omega_j(\lambda)-\Omega_{j_0}(\lambda)$. Without loss of generality we can assume that $j>j_0\geqslant 1$, (The case $j=j_0$ is obvious and the case $j_0=0$ brings us back to the previous point). One may write by (5.14) that for $\lambda>0$,

$$\begin{aligned}\Omega_j(\lambda)-\Omega_{j_0}(\lambda)&=(j-j_0)\Big(\Omega+I_1(\lambda)K_1(\lambda)-I_j(\lambda)K_j(\lambda)\Big)\\&\quad+j_0\Big(I_{j_0}(\lambda)K_{j_0}(\lambda)-I_j(\lambda)K_j(\lambda)\Big).\end{aligned}\tag{5.19}$$

Combining this identity with the estimate (5.18) yields

$$\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\geqslant(j-j_0)\Omega+j_0\Big(I_{j_0}(\lambda)K_{j_0}(\lambda)-I_j(\lambda)K_j(\lambda)\Big).\tag{5.20}$$

We need to get refined estimate for the last term of the right hand side. For this goal we use the formulae (C.10) to write

$$(I_nK_n)(\lambda)=\tfrac12\int_0^\infty J_0\big(2\lambda\sinh(\tfrac t2)\big)e^{-nt}dt.\tag{5.21}$$

This allows to construct for a fixed λ a smooth extension $n\in(0,\infty)\mapsto(I_nK_n)(\lambda)$. Thus differentiating term by term using change of variable we get for any $m\in\mathbb{N}$

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}|\partial_n^m(I_nK_n)(\lambda)|&\leqslant\tfrac12\int_0^\infty t^me^{-nt}dt\\&\leqslant\tfrac{m!}{2n^{m+1}},\end{aligned}\tag{5.22}$$

where we have used the classical estimates for Bessel functions (applied with $n=q=0$)

$$\sup_{\substack{n,q\in\mathbb{N}\\x\in\mathbb{R}}}|J_n^{(q)}(x)|\leqslant 1,\tag{5.23}$$

which follows easily from the integral representation (C.1). In particular, for $m=1$ we find that for for any $n\geqslant1$

$$\sup_{\lambda\in\mathbb{R}}\Big|\partial_n(I_nK_n)(\lambda)\Big|\leqslant\tfrac{1}{2n^2}.$$

Therefore applying Taylor Formula we infer for $j>j_0\geqslant1$

$$\begin{aligned}\sup_{\lambda\in\mathbb{R}}\Big|(I_jK_j)(\lambda)-(I_{j_0}K_{j_0})(\lambda)\Big|&\leqslant\tfrac12\int_{j_0}^j\frac{dn}{n^2}\\&\leqslant\tfrac{|j-j_0|}{2\,j\,j_0}.\end{aligned}\tag{5.24}$$

Inserting this estimate into (5.20) gives

$$\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\geqslant(j-j_0)\Big(\Omega-\tfrac{1}{2j}\Big).$$

Therefore for $j>N=\left[\Omega^{-1}\right]$ and $j>j_0\geqslant1$ we get

$$\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\geqslant\tfrac12\Omega\,(j-j_0).\tag{5.25}$$

Now for $j \neq j_0 \in [\![1, N]\!]$ we get from the point (ii) that the map $\lambda \in [\lambda_0, \lambda_1] \mapsto \Omega_j(\lambda) - \Omega_{j_0}(\lambda)$ does not vanish and therefore we can find by a compactness argument a constant $C > 0$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C|j - j_0|.$$

Taking $C_0 = \min(C, \frac{1}{2}\Omega)$ and combining the preceding inequality with (5.25) we obtain

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall j \geqslant j_0 \geqslant 1, \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C_0|j - j_0|.$$

Finally we get

$$\forall \lambda \in [\lambda_0, \lambda_1], \forall j, j_0 \in \mathbb{N}, \quad |\Omega_j(\lambda) - \Omega_{j_0}(\lambda)| \geqslant C_0|j - j_0|.$$

Let us now move to the estimate $\Omega_j(\lambda) + \Omega_{j_0}(\lambda)$ for $j, j_0 \in \mathbb{N}$. Since both quantities are positive then using the point (iv) yields

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad |\Omega_j(\lambda) + \Omega_{j_0}(\lambda)| = \Omega_j(\lambda) + \Omega_{j_0}(\lambda) \geqslant \Omega(j + j_0) \geqslant C_0(j + j_0).$$

This completes the proof of the desired estimate.
(vi) Let $q_0 \in \mathbb{N}^*$. let $q \in [\![0, q_0]\!]$. Differentiating q times (5.19) in λ, one obtains

$$\begin{aligned}\partial_\lambda^q\big(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\big) = (j - j_0)\Big(\partial_\lambda^q \Omega + \partial_\lambda^q\big(I_1(\lambda)K_1(\lambda)\big) - \partial_\lambda^\alpha\big(I_j(\lambda)K_j(\lambda)\big)\Big)\\ + j_0 \partial_\lambda^q\Big(I_{j_0}(\lambda)K_{j_0}(\lambda) - I_j(\lambda)K_j(\lambda)\Big).\end{aligned} \tag{5.26}$$

Similarly, we get by differentiating q times in λ the identity (5.21)

$$\partial_\lambda^q(I_nK_n)(\lambda) = 2^{q-1}\lambda^q \int_0^\infty J_0^{(q)}\big(2\lambda \sinh(\tfrac{t}{2})\big) \sinh^q(\tfrac{t}{2}) e^{-nt} dt. \tag{5.27}$$

From (5.23) we deduce for any $\lambda \in [\lambda_0, \lambda_1]$,

$$|\partial_\lambda^q(I_nK_n)(\lambda)| \leqslant 2^{q-1}\lambda_1^q \int_0^\infty \sinh^q(\tfrac{t}{2}) e^{-nt} dt.$$

Then using the inequality $\sinh x \leqslant \frac{e^x}{2}$ for $x \geqslant 0$ we get for $n > \frac{q}{2}$

$$\begin{aligned}|\partial_\lambda^q(I_nK_n)(\lambda)| &\leqslant \tfrac{\lambda_1^q}{2} \int_0^\infty e^{\left(\frac{q}{2} - n\right)t} dt\\ &\leqslant \tfrac{\lambda_1^q}{2n - q}.\end{aligned} \tag{5.28}$$

By compactness argument, we deduce that

$$\sup_{j \in \mathbb{N}} \max_{q \in [\![0, q_0]\!]} \sup_{\lambda \in [\lambda_0, \lambda_1]} \left|\partial_\lambda^q\big(I_j(\lambda)K_j(\lambda)\big)\right| \leqslant C. \tag{5.29}$$

Differentiating in n (5.27) yields

$$\partial_\lambda^q \partial_n(I_nK_n)(\lambda) = -2^{q-1}\lambda^q \int_0^\infty J_0^{(q)}\big(2\lambda \sinh(t/2)\big) \sinh^q(t/2) t e^{-nt} dt.$$

Therefore applying similar arguments used to show (5.28) gives for $2n > q$

$$\begin{aligned}|\partial_\lambda^q\partial_n(I_nK_n)(\lambda)| &\leqslant \tfrac{\lambda_1^q}{2}\int_0^\infty te^{-\left(n-\frac{q}{2}\right)t}dt\\ &\leqslant \frac{2\lambda_1^q}{(2n-q)^2}\cdot\end{aligned}\tag{5.30}$$

Then Taylor Formula allows to get for $j, j_0 > \frac{q}{2}$

$$\sup_{\lambda\in[\lambda_0,\lambda_1]}|\partial_\lambda^q(I_jK_j - I_{j_0}K_{j_0})(\lambda)| \leqslant C\tfrac{|j-j_0|}{jj_0}\cdot\tag{5.31}$$

Setting $N = \lfloor\frac{q_0}{2}\rfloor + 1$, one obtains for any $j, j_0 \geqslant N$

$$\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|j_0\partial_\lambda^q(I_jK_j - I_{j_0}K_{j_0})(\lambda)| \leqslant C|j-j_0|\cdot$$

By compactness argument, one obtains for any $j, j_0 \in [\![1, N]\!]$

$$\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|j_0\partial_\lambda^q(I_jK_j - I_{j_0}K_{j_0})(\lambda)| \leqslant C|j-j_0|\cdot$$

Now for the remaining case $j_0 \in [\![1, N]\!]$ and $j \geqslant N$ one has gathering the previous two estimates

$$\begin{aligned}\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|j_0\partial_\lambda^q(I_jK_j - I_{j_0}K_{j_0})(\lambda)| &\leqslant N\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|\partial_\lambda^q(I_jK_j - I_NK_N)(\lambda)|\\ &\quad + N\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|\partial_\lambda^q(I_NK_N - I_{j_0}K_{j_0})(\lambda)|\\ &\leqslant C|j-N| + C|N-j_0| \leqslant C|j-j_0|\cdot\end{aligned}$$

Thus we can find $C > 0$ such that for any $j, j_0 \in \mathbb{N}^*$

$$\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|j_0\partial_\lambda^q(I_jK_j - I_{j_0}K_{j_0})(\lambda)| \leqslant C|j-j_0|\cdot$$

Putting together (5.26), (5.29) and (5.14) yields

$$\max_{q\in[\![0,q_0]\!]}\sup_{\lambda\in[\lambda_0,\lambda_1]}|\partial_\lambda^q\left(\Omega_j(\lambda) - \Omega_{j_0}(\lambda)\right)| \leqslant C|j-j_0|.$$

This ends the proof of Lemma 5.3. □

5.2.2 Non-degeneracy and transversality

Fix finitely many tangential sites

$$\mathbb{S} \triangleq \{j_1, \dots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad d \geqslant 1 \quad \text{and} \quad 1 \leqslant j_1 < \cdots < j_d.$$

We consider the linear vector frequency at the equilibrium state

$$\omega_{\mathrm{Eq}}(\lambda) \triangleq (\Omega_j(\lambda))_{j\in\mathbb{S}},\tag{5.32}$$

where $\Omega_j(\lambda)$ is defined by (5.14). The main purpose is to study some Diophantine structure of the curve $\lambda \in (\lambda_0, \lambda_1) \mapsto \omega_{\mathrm{Eq}}(\lambda)$ for fixed $0 < \lambda_0 < \lambda_1$. In particular, we shall focus on the non-degeneracy and the transversality conditions of these eigenvalues which are essential in getting non trivial Cantor set from which quasi-periodic solutions emerge at the linear and nonlinear levels. Notice that the approach that we shall implement here has been developed before in several papers such as [7, 12, 139]. Before exploring these properties we need to fix some definitions.

Definition 5.1. *Given two numbers $\lambda_0 < \lambda_1$ and $d \in \mathbb{N}^*$, a vector-valued function $f = (f_1, ..., f_d) : [\lambda_0, \lambda_1] \to \mathbb{R}^d$ is called non-degenerate if, for any vector $c = (c_1, ..., c_d) \in \mathbb{R}^d \setminus \{0\}$, the function $f \cdot c = f_1c_1 + ... + f_dc_d$ is not identically zero on the whole interval $[\lambda_0, \lambda_1]$. This means that the curve of f is not contained in an hyperplane.*

Now we shall prove the following result on the non-degeneracy of the linear frequencies which is related to the asymptotic behavior of Bessel functions $(I_jK_j)(\lambda)$ for large values of λ. This property will be crucial to check a suitable transversality assumption.

Lemma 5.4. *Let $\Omega \in \mathbb{R}^*$ and $0 < \lambda_0 < \lambda_1$, then the frequency curve ω_{Eq} defined by* (5.32) *and the vector-valued function $\lambda \mapsto (\Omega + I_1K_1, \omega_{\mathrm{Eq}}) \in \mathbb{R}^{d+1}$ are non degenerate on $[\lambda_0, \lambda_1]$ in the sense of the Definition* 5.1.

Proof. ▶ Let us start with checking the non-degeneracy of ω_{Eq}. For this aim, we shall argue by contradiction and assume the existence of a fixed vector $c = (c_k)_{0 \leqslant k \leqslant d} \in \mathbb{R}^d$ such that

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad \sum_{k=1}^{d} c_k \Omega_{j_k}(\lambda) = 0. \tag{5.33}$$

Since for all $j \in \mathbb{N}^*$, the application $\lambda \mapsto (I_jK_j)(\lambda)$ admits a holomorphic extension in the open connected set $\big\{\lambda \in \mathbb{C}, \mathrm{Re}(\lambda) > 0\big\}$ (see Appendix C) then by the continuation principle we obtain

$$\forall \lambda > 0, \quad \sum_{k=1}^{d} c_k j_k (I_{j_k}K_{j_k})(\lambda) = \left(\sum_{k=1}^{d} c_k j_k\right) \big((I_1K_1)(\lambda) + \Omega\big). \tag{5.34}$$

Using the asymptotic expansion (C.15) obtained for I_jK_j with large λ, we first get

$$\forall j \in \mathbb{N}^*, \quad \lim_{\lambda \to \infty} (I_jK_j)(\lambda) = 0.$$

Then taking the limit in (5.34) as $\lambda \to \infty$ implies

$$\Omega \sum_{k=1}^{d} c_k j_k = 0.$$

Since we assumed that $\Omega \neq 0$, then necessary we find that $\displaystyle\sum_{k=1}^{d} c_k j_k = 0$ which implies in turn according to (5.34)

$$\forall \lambda > 0, \quad \sum_{k=1}^{d} c_k j_k (I_{j_k}K_{j_k})(\lambda) = 0.$$

Applying once again the expansion (C.15) yields

$$\forall m \in [\![1, d]\!], \quad \sum_{k=1}^{d} c_k j_k \alpha_{j_k,m} = 0. \tag{5.35}$$

We consider the matrix $A_d = (A_{m,k})_{1 \leqslant m,k \leqslant d} \in M_d(\mathbb{R})$ defined by

$$\forall (m,k) \in [\![1, d]\!]^2, \quad A_{m,k} \triangleq j_k \alpha_{j_k,m}.$$

Then the system (5.35) is equivalent to $A_d c = 0$ with $c = \begin{pmatrix} c_1 \\ \vdots \\ c_d \end{pmatrix}$. To get the desired result, $c = 0$, it suffices to check that $\det\big(A_d\big) \neq 0$. Using the expression of the coefficients $\alpha_{j_k,m}$ in (C.16) one deduces that

$$\alpha_{j_k,m} = a_m(\mu_{j_k} - 1)Q_m\big(\mu_{j_k}\big), \quad a_m = (-1)^m \frac{(2m)!}{4^m\big(m!\big)^2}, \quad \mu_j = 4j^2, \tag{5.36}$$

with $Q_1(X) = 1$ and for $m \geq 2$

$$Q_m(X) = \prod_{\ell=2}^{m} \big(X - (2\ell - 1)^2\big).$$

Remark that Q_m is a unitary polynomial of degree $m - 1$. Using the homogeneity of the determinant with respect to each column and row we find

$$\det\big(A_d\big) = \prod_{m,k=1}^{d} a_m(\mu_{j_k} - 1)\,\det\big(B_d\big),$$

with B_d the matrix given by

$$B_d \triangleq \begin{pmatrix} Q_1(\mu_{j_1}) & \cdots & Q_1(\mu_{j_d}) \\ \vdots & & \vdots \\ Q_d(\mu_{j_1}) & \cdots & Q_d(\mu_{j_d}) \end{pmatrix}.$$

Therefore we infer that A_d is nonsingular if $\det\big(B_d\big) \neq 0$. On the other hand, the computation of $\det\big(B_d\big)$ can be done in a similar way to Vandermonde determinant. Indeed, define the polynomial given by the determinant

$$P(X) \triangleq \begin{vmatrix} Q_1(\mu_{j_1}) & \cdots & Q_1(\mu_{j_{d-1}}) & Q_1(X) \\ \vdots & & \vdots & \vdots \\ Q_d(\mu_{j_1}) & \cdots & Q_d(\mu_{j_{d-1}}) & Q_d(X) \end{vmatrix}.$$

Then P is a polynomial of degree $d - 1$ and vanishes at all the points $X = \mu_{j_k}$ for $k \in [\![1, d-1]\!]$. Consequently, we get

$$\det\big(B_d\big) = P(\mu_{j_d}) = \det\big(B_{d-1}\big) \prod_{k=1}^{d-1} \big(\mu_{j_d} - \mu_{j_k}\big).$$

Therefore, iterating this identity yields

$$\det\big(B_d\big) = \prod_{1 \leqslant k < \ell \leqslant d-1} \big(\mu_{j_\ell} - \mu_{j_k}\big).$$

Since $\mu_{j_\ell} \neq \mu_{j_k}$ for $\ell \neq k$ we get $\det\big(B_d\big) \neq 0$ which achieves the proof of the first point.

► Next we move to the second point of the lemma and show that if

$$\forall \lambda \in [\lambda_0, \lambda_1], \quad c_0\Big(\Omega + (I_1K_1)(\lambda)\Big) + \sum_{k=1}^{d} c_k j_k\Big(\Omega + (I_1K_1)(\lambda) - (I_{j_k}K_{j_k})(\lambda)\Big) = 0,$$

then necessary $c_0 = ... = c_d = 0$. As before we can extend by analyticity the preceding identity to $(0, \infty)$ By checking the terms in $\frac{1}{\lambda}$ in the preceding identity using (C.15) we find immediately that $c_0 = 0$. Therefore the system reduces to (5.33) and then we may apply the result of the first point in order to get

$c_1 = ... = c_d = 0$. This completes the proof of Lemma 5.4. □

The next goal is to check that Rüssemann transversality conditions are satisfied for the linear frequencies of the equilibrium state. Namely, we shall prove the following result in the spirit of the papers [7, 12, 139].

Lemma 5.5. [Transversality] *Given $0 < \lambda_0 < \lambda_1$, there exist $q_0 \in \mathbb{N}$ and $\rho_0 > 0$ such that the following results hold true. Recall that ω_{Eq} and Ω_j are defined in* (5.32) *and* (5.14) *respectively.*

(i) For any $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{\lambda\in[\lambda_0,\lambda_1]} \max_{q\in[\![0,q_0]\!]} |\partial_\lambda^q \omega_{\mathrm{Eq}}(\lambda)\cdot l| \geqslant \rho_0\langle l\rangle.$$

(ii) For any $(l,j) \in (\mathbb{Z}^d\times\mathbb{N})\setminus\{(0,0)\}$

$$\inf_{\lambda\in[\lambda_0,\lambda_1]} \max_{q\in[\![0,q_0]\!]} \Big|\partial_\lambda^q\Big(\omega_{\mathrm{Eq}}(\lambda)\cdot l \pm j\big(\Omega + (I_1K_1)(\lambda)\big)\Big)\Big| \geqslant \rho_0\langle l\rangle.$$

(iii) For any $(l,j)\in\mathbb{Z}^d\times(\mathbb{N}^*\setminus\mathbb{S})$

$$\inf_{\lambda\in[\lambda_0,\lambda_1]} \max_{q\in[\![0,q_0]\!]} \big|\partial_\lambda^q\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l \pm \Omega_j(\lambda)\right)\big| \geqslant \rho_0\langle l\rangle.$$

(iv) For any $l\in\mathbb{Z}^d, j,j'\in\mathbb{N}^*\setminus\mathbb{S}$ with $(l,j)\neq(0,j')$, we have

$$\inf_{\lambda\in[\lambda_0,\lambda_1]} \max_{q\in[\![0,q_0]\!]} \big|\partial_\lambda^q\big(\omega_{\mathrm{Eq}}(\lambda)\cdot l + \Omega_j(\lambda)\pm\Omega_{j'}(\lambda)\big)\big| \geqslant \rho_0\langle l\rangle.$$

Proof. **(i)** We argue by contradiction by assuming that for any $q_0\in\mathbb{N}$ and $\rho_0>0$, there exist $l\in\mathbb{Z}^d\setminus\{0\}$ and $\lambda\in[\lambda_0,\lambda_1]$ such that

$$\max_{q\in[\![0,q_0]\!]} |\partial_\lambda^q(\omega_{\mathrm{Eq}}(\lambda)\cdot l)| < \rho_0\langle l\rangle.$$

It follows that for any $m\in\mathbb{N}$, and by taking $q_0 = m$ and $\rho_0 = \frac{1}{m+1}$, there exist $l_m\in\mathbb{Z}^d\setminus\{0\}$ and $\lambda_m\in[\lambda_0,\lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]} |\partial_\lambda^q\omega_{\mathrm{Eq}}(\lambda_m)\cdot l_m| < \tfrac{\langle l_m\rangle}{m+1}$$

and therefore

$$\forall q\in\mathbb{N},\quad \forall m\geqslant q,\qquad \Big|\partial_\lambda^q\omega_{\mathrm{Eq}}(\lambda_m)\cdot\tfrac{l_m}{\langle l_m\rangle}\Big| < \tfrac{1}{m+1}\cdot \tag{5.37}$$

Since the sequences $\left(\frac{l_m}{\langle l_m\rangle}\right)_m$ and $(\lambda_m)_m$ are bounded, then by compactness and up to an extraction we can assume that

$$\lim_{m\to\infty}\tfrac{l_m}{\langle l_m\rangle} = \bar{c}\neq 0 \quad\text{and}\quad \lim_{m\to\infty}\lambda_m = \bar{\lambda}.$$

Hence, passing to the limit in (5.37) as $m\to\infty$ leads to

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\omega_{\mathrm{Eq}}(\bar{\lambda})\cdot\bar{c} = 0.$$

Thus, we conclude that the real analytic function $\lambda\mapsto\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}$ is identically zero which contradicts the non-degeneracy condition stated in Lemma 5.4.

(ii) We shall first check the result for the case $l=0$ and $j\in\mathbb{N}^*$. Obviously, one has from the monotonicity of $\lambda \mapsto I_1(\lambda)K_1(\lambda)$ stated in Appendix C,

$$\begin{aligned}\inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{q\in[\![0,q_0]\!]}\Big|\partial_\lambda^q\Big(j\big(\Omega+(I_1K_1)(\lambda)\big)\Big)\Big| &\geqslant \Omega+(I_1K_1)(\lambda_1)\\ &\geqslant \rho_0\langle l\rangle,\end{aligned}$$

for some $\rho_0>0$. Now let us consider $l\in\mathbb{Z}^d\backslash\{0\}$ and $j\in\mathbb{N}$. Then we may write according to the triangle and Cauchy-Schwarz inequalities combined with the boundedness of $\omega_{\textrm{Eq}}$ and the monotonicity of $\lambda\mapsto I_1(\lambda)K_1(\lambda)$ stated in Appendix C,

$$\Big|\omega_{\textrm{Eq}}(\lambda)\cdot l\pm j\big(\Omega+I_1(\lambda)K_1(\lambda)\big)\Big|\geqslant j\big(\Omega+I_1(\lambda_1)K_1(\lambda_1)\big)-|\omega_{\textrm{Eq}}(\lambda)\cdot l|\geqslant c_0 j-C\langle l\rangle\geqslant\langle l\rangle$$

provided that $j\geqslant C_0\langle l\rangle$ for some $C_0>0$. Therefore we reduce the proof to indices j and l with

$$0\leqslant j<C_0\langle l\rangle,\quad j\in\mathbb{N}\quad\text{and}\quad l\in\mathbb{Z}^d\backslash\{0\}. \tag{5.38}$$

Arguing by contradiction as in the previous case, we may assume the existence of sequences $l_m\in\mathbb{Z}^d\setminus\{0\}$, $j_m\in\mathbb{N}$ satisfying (5.38) and $\lambda_m\in[\lambda_0,\lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]}\left|\partial_\lambda^q\left(\omega_{\textrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|}\pm j_m\tfrac{\Omega+(I_1K_1)(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}$$

and therefore

$$\forall q\in\mathbb{N},\quad\forall m\geqslant q,\quad\left|\partial_\lambda^q\left(\omega_{\textrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|}\pm j_m\tfrac{\Omega+(I_1K_1)(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}\cdot \tag{5.39}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$, $\left(\frac{j_m}{|l_m|}\right)_m$ and $(\lambda_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty}\tfrac{l_m}{|l_m|}=\bar{c}\neq0,\quad\lim_{m\to\infty}\tfrac{j_m}{|l_m|}=\bar{d}\quad\text{and}\quad\lim_{m\to\infty}\lambda_m=\bar{\lambda}.$$

Hence, by letting $m\to\infty$ in (5.39), using that $\lambda\mapsto(I_1K_1)(\lambda)$ is smooth, we find

$$\forall q\in\mathbb{N},\quad\partial_\lambda^q\Big(\omega_{\textrm{Eq}}(\lambda)\cdot\bar{c}\pm\bar{d}\left(\Omega+(I_1K_1)(\lambda)\right)\Big)_{|\lambda=\bar{\lambda}}=0.$$

Thus, the real analytic function $\lambda\mapsto\omega_{\textrm{Eq}}(\lambda)\cdot\bar{c}\pm\bar{d}\left(\Omega+I_1(\lambda)K_1(\lambda)\right)$ with $(\bar{c},\bar{d})\neq(0,0)$ is identically zero and this contradicts Lemma 5.4.

(iii) Consider $(l,j)\in\mathbb{Z}^d\times(\mathbb{N}^*\setminus\mathbb{S})$. Then applying the triangle inequality and Lemma 5.3-(iv), yields

$$\begin{aligned}|\omega_{\textrm{Eq}}(\lambda)\cdot l\pm\Omega_j(\lambda)|&\geqslant|\Omega_j(\lambda)|-|\omega_{\textrm{Eq}}(\lambda)\cdot l|\\&\geqslant\Omega j-C|l|\geqslant\langle l\rangle\end{aligned}$$

provided $j\geqslant C_0\langle l\rangle$ for some $C_0>0$. Thus as before we shall restrict the proof to indices j and l with

$$0\leqslant j<C_0\langle l\rangle,\quad j\in\mathbb{N}^*\setminus\mathbb{S}\quad\text{and}\quad l\in\mathbb{Z}^d\backslash\{0\}. \tag{5.40}$$

Proceeding by contradiction as in the previous case, we may assume the existence of sequences $l_m\in\mathbb{Z}^d\backslash\{0\}$, $j_m\in\mathbb{N}\setminus\mathbb{S}$ satisfying (5.40) and $\lambda_m\in[\lambda_0,\lambda_1]$ such that

$$\max_{q\in[\![0,m]\!]}\left|\partial_\lambda^q\left(\omega_{\textrm{Eq}}(\lambda)\cdot\tfrac{l_m}{|l_m|}\pm\tfrac{\Omega_{j_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}$$

and therefore

$$\forall q\in\mathbb{N},\quad \forall m\geqslant q,\qquad \left|\partial_\lambda^q\left(\omega_{\rm Eq}(\lambda)\cdot\tfrac{l_m}{|l_m|}\pm\tfrac{\Omega_{j_m}(\lambda)}{|l_m|}\right)_{|\lambda=\lambda_m}\right|<\tfrac{1}{m+1}. \tag{5.41}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$ and $(\lambda_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty}\tfrac{l_m}{|l_m|}=\bar{c}\neq 0\quad\text{and}\quad \lim_{m\to\infty}\lambda_m=\bar{\lambda}.$$

Now we shall distinguish two cases.

► Case ❶ : $(l_m)_m$ is bounded. In this case, by (5.40) we find that $(j_m)_m$ is bounded too and thus up to an extraction we may assume $\lim\limits_{m\to\infty}l_m=\bar{l}$ and $\lim\limits_{m\to\infty}j_m=\bar{j}$. Since $(j_m)_m$ and $(|l_m|)_m$ are sequences of integers, then they are necessary stationary. In particular, the condition (5.40) implies $\bar{l}\neq 0$. Hence, taking the limit $n\to\infty$ in (5.41), yields

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\left(\omega_{\rm Eq}(\lambda)\cdot\bar{l}\pm\Omega_{\bar{j}}(\lambda)\right)_{|\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda\mapsto\omega_{\rm Eq}(\lambda)\cdot\bar{l}\pm\Omega_{\bar{j}}(\lambda)$ with $(\bar{l},1)\neq(0,0)$ is identically zero which contradicts Lemma 5.4.

► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty}|l_m|=\infty$. We have two sub-cases.

• Sub-case ① : $(j_m)_m$ is bounded. In this case and up to an extraction we can assume that it converges. Then, taking the limit $m\to\infty$ in (5.41), we find

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\omega_{\rm Eq}(\bar{\lambda})\cdot\bar{c}=0.$$

As before we conclude that function $\lambda\mapsto\omega_{\rm Eq}(\lambda)\cdot\bar{c}$ with $\bar{c}\neq 0$ is identically zero which contradicts Lemma 5.4.

• Sub-case ② : $(j_m)_m$ is unbounded. Then up to an extraction we can assume that $\lim\limits_{m\to\infty}j_m=\infty$. We write according to (5.14)

$$\tfrac{\Omega_{j_m}(\lambda)}{|l_m|}=\tfrac{j_m}{|l_m|}\Big(\Omega+(I_1K_1)(\lambda)-(I_{j_m}K_{j_m})(\lambda)\Big). \tag{5.42}$$

By (5.40), the sequence $\left(\frac{j_m}{|l_m|}\right)_n$ is bounded, thus up to an extraction we can assume that it converges to $\bar{d}$. Using the first inequality of (5.22) we deduce that

$$\forall\, m\in\mathbb{N},\quad \sup_{\lambda\in\mathbb{R}}\big|(I_{j_m}K_{j_m})(\lambda)\big|\leq\tfrac{1}{2j_m},$$

which implies that

$$\lim_{m\to\infty}\sup_{\lambda\in\mathbb{R}}(I_{j_m}K_{j_m})(\lambda)=0.$$

Moreover by (5.28), we have

$$\lim_{m\to\infty}\sup_{\lambda\in[\lambda_0,\lambda_1]}|\partial_\lambda^q(I_{j_m}K_{j_m})(\lambda)|=0. \tag{5.43}$$

Taking the limit in (5.42) and using (5.43) yields

$$\lim_{m\to\infty}\tfrac{\partial_\lambda^q\Omega_{j_m}(\lambda_m)}{|l_m|}=\partial_\lambda^q\Big(\bar{d}\big(\Omega+\big(I_1K_1\big)(\lambda)\big)\Big)_{|\lambda=\bar{\lambda}}.$$

Consequently, taking the limit $m \to \infty$ in (5.41), we have

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \Big(\omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c} \pm \bar{d}\big(\Omega + \big(I_1 K_1\big)(\lambda)\big)\Big)_{|\lambda=\overline{\lambda}} = 0.$$

By continuation the analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{c} \pm \bar{d}\big(\Omega + I_1(\lambda)K_1(\lambda)\big)$ with $(\bar{c}, \bar{d}) \neq 0$ is identically zero which contradicts Lemma 5.4.

(iv) Consider $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l,j) \neq (0,j')$. Then applying the triangle inequality combined with Lemma 5.3-(v), we infer

$$|\omega_{\mathrm{Eq}}(\lambda) \cdot l + \Omega_j(\lambda) \pm \Omega_{j'}(\lambda)| \geqslant |\Omega_j(\lambda) \pm \Omega_{j'}(\lambda)| - |\omega_{\mathrm{Eq}}(\lambda) \cdot l| \geqslant C_0 |j \pm j'| - C|l| \geqslant \langle l \rangle$$

provided that $|j \pm j'| \geqslant c_0 \langle l \rangle$ for some $c_0 > 0$. Then it remains to check the proof for indices satisfying

$$|j \pm j'| < c_0 \langle l \rangle, \quad l \in \mathbb{Z}^d \backslash \{0\} \quad \text{and} \quad j, j' \in \mathbb{N}^* \setminus \mathbb{S}. \tag{5.44}$$

Reasoning by contradiction as in the previous cases, we get for all $m \in \mathbb{N}$, real numbers $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m, j'_m \in \mathbb{N}^* \setminus \mathbb{S}$ satisfying (5.44) and $\lambda_m \in [\lambda_0, \lambda_1]$ such that

$$\max_{q \in [\![0,m]\!]} \left| \partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot \tfrac{l_m}{|l_m|} + \tfrac{\Omega_{j_m}(\lambda) \pm \Omega_{j'_m}(\lambda)}{|l_m|} \right)_{|\lambda=\lambda_m} \right| < \tfrac{1}{m+1}$$

implying in turn that

$$\forall q \in \mathbb{N}, \quad \forall m \geqslant q, \quad \left| \partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot \tfrac{l_m}{|l_m|} + \tfrac{\Omega_{j_m}(\lambda) \pm \Omega_{j'_m}(\lambda)}{|l_m|} \right)_{|\lambda=\lambda_m} \right| < \tfrac{1}{m+1} \cdot \tag{5.45}$$

Up to an extraction we can assume that $\lim\limits_{m \to \infty} \frac{l_m}{|l_m|} = \bar{c} \neq 0$ and $\lim\limits_{m \to \infty} \lambda_m = \bar{\lambda}$.
As before we shall distinguish two cases.

► Case ❶ : $(l_m)_m$ is bounded. We shall only focus on the most delicate case associated to the difference $\Omega_{j_m} - \Omega_{j'_m}$. Up to an extraction we may assume that $\lim\limits_{m \to \infty} l_m = \bar{l} \neq 0$. Now according to (5.44) we have two sub-cases to discuss depending whether the sequences $(j_m)_m$ and $(j'_m)_m$ are simultaneously bounded or unbounded.

• Sub-case ① : $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case, up to an extraction we may assume that these sequences are stationary $j_m = \bar{j}$ and $j'_m = \bar{j}'$ with $\bar{j}, \bar{j}' \in \mathbb{N}^* \setminus \mathbb{S}$. Hence taking the limit as $m \to \infty$ in (5.45), we infer

$$\forall q \in \mathbb{N}, \quad \partial_\lambda^q \left(\omega_{\mathrm{Eq}}(\lambda) \cdot \bar{l} + \Omega_{\bar{j}}(\lambda) - \Omega_{\bar{j}'}(\lambda) \right)_{|\lambda=\overline{\lambda}} = 0.$$

Thus, the analytic function $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda) \cdot \bar{l} + \Omega_{\bar{j}}(\lambda) - \Omega_{\bar{j}'}(\lambda)$ is identically zero. If $\bar{j} = \bar{j}'$ then this contradicts Lemma 5.4 since $\bar{l} \neq 0$. However in the case $\bar{j} \neq \bar{j}' \in \mathbb{N}^* \setminus \mathbb{S}$ this still contradicts this lemma applied with the vector frequency $(\omega_{\mathrm{Eq}}, \Omega_{\bar{j}}, \Omega_{\bar{j}'})$ instead of ω_{Eq}.

• Sub-case ② : $(j_m)_m$ and $(j'_m)_m$ are both unbounded and without loss of generality we can assume that $\lim\limits_{m \to \infty} j_m = \lim\limits_{m \to \infty} j'_m = \infty$. From (5.31) combined with (5.44) and the boundedness of $(l_m)_m$ we deduce that

$$\left| \partial_\lambda^q (I_{j_m} K_{j_m} - I_{j'_m} K_{j'_m})(\lambda_m) \right| \leqslant \tfrac{C}{j_m j'_m},$$

which implies in turn

$$\lim_{m \to \infty} j'_m \, \partial_\lambda^q (I_{j_m} K_{j_m} - I_{j'_m} K_{j'_m})(\lambda_m) = 0. \tag{5.46}$$

Coming back to (5.14) we get the splitting

$$\begin{aligned}\Omega_{j_m}(\lambda)-\Omega_{j_m'}(\lambda)=&(j_m-j_m')\big(\Omega+(I_1K_1)(\lambda)\big)-(j_m-j_m')(I_{j_m}K_{j_m})(\lambda)\\&+j_m'\Big((I_{j_m'}K_{j_m'})(\lambda)-(I_{j_m}K_{j_m})(\lambda)\Big).\end{aligned}\tag{5.47}$$

Therefore by applying (5.43) and (5.46) we get for any $q\in\mathbb{N}$,

$$\lim_{m\to\infty}\partial_\lambda^q\Big(\Omega_{j_m}(\lambda)-\Omega_{j_m'}(\lambda)-(j_m-j_m')\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{|\lambda=\lambda_m}=0.$$

Using once again (5.44) and up to an extraction we have $\lim\limits_{m\to\infty}\frac{j_m-j_m'}{|l_m|}=\bar{d}$. Thus

$$\lim_{m\to\infty}|l_m|^{-1}\partial_\lambda^q\left(\Omega_{j_m}(\lambda)-\Omega_{j_m'}(\lambda)\right)_{|\lambda=\lambda_m}=\bar{d}\,\partial_\lambda^q\left(\Omega+(I_1K_1)(\lambda)\right)_{|\lambda=\bar{\lambda}}.$$

By taking the limit as $m\to\infty$ in (5.45), we find

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\Big(\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}+\bar{d}\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{|\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda\mapsto\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}+\bar{d}(\Omega+I_1(\lambda)K_1(\lambda))$ with $(\bar{c},\bar{d})\neq0$ is vanishing which contradicts Lemma 5.4. Now we shall move to the second case.

▶ Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty}|l_m|=\infty$.
We shall distinguish three sub-cases.

• Sub-case ①. The sequences $(j_m)_m$ and $(j_m')_m$ are bounded. In this case and up to an extraction they will converge and then taking the limit in (5.45) yields,

$$\forall q\in\mathbb{N},\quad \partial_\lambda^q\omega_{\mathrm{Eq}}(\bar{\lambda})\cdot\bar{c}=0.$$

which leads to a contradiction as before.

• Sub-case ②. The sequences $(j_m)_m$ and $(j_m')_m$ are both unbounded. This is similar to the sub-case ② of the case ❶.

• Sub-case ③. The sequence $(j_m)_m$ is unbounded and $(j_m')_m$ is bounded (the symmetric case is similar). Without loss of generality we can assume that $\lim\limits_{m\to\infty}j_m=\infty$ and $j_m'=\bar{j}$. By (5.44) and up to an extraction one gets $\lim\limits_{m\to\infty}\frac{j_m\pm j_m'}{|l_m|}=\bar{d}$. One may use (5.14) combined with (5.43) and (5.46) in order to get for any $q\in\mathbb{N}$,

$$\begin{aligned}\lim_{m\to\infty}|l_m|^{-1}\partial_\lambda^q\Big(\Omega_{j_m}(\lambda)\pm\Omega_{j_m'}(\lambda)-(j_m\pm j_m')\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{|\lambda=\lambda_m}&=\\ \lim_{m\to\infty}\partial_\lambda^q\left(\tfrac{(j_m\pm j_m')}{|l_m|}(I_{j_m}K_{j_m})(\lambda)\pm\tfrac{j_m'}{|l_m|}\Big((I_{j_m}K_{j_m})(\lambda)-(I_{j_m'}K_{j_m'})(\lambda)\Big)\right)_{|\lambda=\lambda_m}&=0.\end{aligned}$$

Hence, taking the limit in (5.45) implies

$$\forall q\in\mathbb{N},\partial_\lambda^q\Big(\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}+\bar{d}\big(\Omega+(I_1K_1)(\lambda)\big)\Big)_{\lambda=\bar{\lambda}}=0.$$

Thus, the analytic function $\lambda\mapsto\omega_{\mathrm{Eq}}(\lambda)\cdot\bar{c}+\bar{d}\big(\Omega+I_1(\lambda)K_1(\lambda)\big)$ is identically zero with $(\bar{c},\bar{d})\neq0$ which contradicts Lemma 5.4.This completes the proof of Lemma 5.5. □

5.2.3 Linear quasi-periodic solutions

Notice that all the solutions of (5.10) taking the form (5.13) are either periodic, quasi-periodic or almost periodic in time, with linear frequencies of oscillations $\Omega_j(\lambda)$ defined by (5.14) These different notions depend on the irrationality properties of the frequencies $\{\Omega_j(\lambda)\}$ and on the cardinality of the Fourier-space support (finite for quasi-periodic functions and possibly infinite for almost periodic ones). Remark that we have the implications

$$\text{Periodic} \quad \Rightarrow \quad \text{Quasi-periodic} \quad \Rightarrow \quad \text{Almost periodic.}$$

We shall prove here the existence of quasi-periodic solutions for the linear equation (5.10) when λ belongs to a massive Cantor set.

Proposition 5.1. *Let $\lambda_1 > \lambda_0 > 0$, $d \in \mathbb{N}^*$ and $\mathbb{S} \subset \mathbb{N}^*$ with $|\mathbb{S}| = d$. Then, there exists a Cantor-like set $\mathcal{C} \subset [\lambda_0, \lambda_1]$ satisfying $|\mathcal{C}| = \lambda_1 - \lambda_0$ and such that for all $\lambda \in \mathcal{C}$, every function in the form*

$$\rho(t,\theta) = \sum_{j\in\mathbb{S}} \rho_j \cos(j\theta - \Omega_j(\lambda)t), \quad \rho_j \in \mathbb{R}^* \tag{5.48}$$

is a time quasi-periodic reversible solution to the equation (5.10) *with the vector frequency*

$$\omega_{\mathrm{Eq}}(\lambda) \triangleq \big(\Omega_j(\lambda)\big)_{j\in\mathbb{S}}.$$

Proof. It is easy to check that any function in the form (5.48) is a reversible solution to (5.10), that is a solution satisfying the property

$$r(-t,-\theta) = r(t,\theta).$$

Then, it remains to check the non-resonance condition (1.24) for the frequency vector ω_{Eq} for almost every $\lambda \in [\lambda_0, \lambda_1]$. For that purpose, we consider $\tau_1 > 0, \gamma \in (0,1)$ and define the set $\mathcal{C}_\gamma$ by

$$\mathcal{C}_\gamma \triangleq \bigcap_{l\in\mathbb{Z}^d\setminus\{0\}} \Big\{\lambda \in [\lambda_0, \lambda_1] \quad \text{s.t.} \quad |\omega_{\mathrm{Eq}}(\lambda)\cdot l| > \tfrac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}.$$

Therefore its complement set takes the form

$$[\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma = \bigcup_{l\in\mathbb{Z}^d\setminus\{0\}} \mathcal{R}_l \quad \text{where} \quad \mathcal{R}_l \triangleq \Big\{\lambda \in [\lambda_0, \lambda_1] \quad \text{s.t.} \quad |\omega_{\mathrm{Eq}}(\lambda)\cdot l| \leqslant \tfrac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}.$$

It follows that

$$\Big|[\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma\Big| \leqslant \sum_{l\in\mathbb{Z}^d\setminus\{0\}} |\mathcal{R}_l|.$$

Now applying Lemma 5.6 together with Lemma 5.5-(i), one obtains

$$|\mathcal{R}_l| \lesssim \gamma^{\frac{1}{q_0}} \langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}.$$

Then by imposing

$$\tau_1 > (d-1)q_0 - 1,$$

one gets a convergent series with

$$\big|[\lambda_0, \lambda_1] \setminus \mathcal{C}_\gamma\big| \leqslant C\gamma^{\frac{1}{q_0}}.$$

Now, we define the Cantor set

$$\mathcal{C} \triangleq \bigcup_{\gamma>0} \mathcal{C}_\gamma.$$

Then one gets easily for any $\gamma > 0$

$$\lambda_1 - \lambda_0 - C\gamma^{\frac{1}{q_0}} \leqslant |\mathcal{C}_\gamma| \leqslant |\mathcal{C}| \leqslant \lambda_1 - \lambda_0.$$

Passing to the limit as $\gamma \to 0$ yields

$$|\mathcal{C}| = \lambda_1 - \lambda_0,$$

which achieves the proof of Proposition 5.1. □

In the previous proof, we used the following Lemma whose proof can be found in [139, Thm. 17.1]. Notice that in all the document, we use the notation $|A|$ as the Lebesgue measure of a given measurable set A.

Lemma 5.6. *Let* $q_0 \in \mathbb{N}^*$, $a, b \in \mathbb{R}$ *with* $a < b$ *and* $\mathtt{m}, \mathtt{M} \in (0, \infty)$. *Let* $f \in C^{q_0}([a, b], \mathbb{R})$ *such that*

$$\inf_{x\in[a,b]} \max_{q\in[\![0,q_0]\!]} |f^{(q)}(x)| \geqslant \mathtt{m}.$$

Then, there exists $C = C(a, b, q_0, \|f\|_{C^{q_0}([a,b],\mathbb{R})}) > 0$ *such that*

$$\Big|\{x \in [a, b] \quad \text{s.t.} \quad |f(x)| \leqslant \mathtt{M}\}\Big| \leqslant C\frac{\mathtt{M}^{\frac{1}{q_0}}}{\mathtt{m}^{1+\frac{1}{q_0}}}.$$

6 Hamiltonian toolkit and approximate inverse

In this section, we shall reformulate the problem into the form of searching for zeros of a functional $\mathcal{F}$. We first rescale the equation by introducing a small parameter ε. This allows us to see the Hamiltonian equation (4.13) as a perturbation of the equilibirum one (5.10). The latter being integrable and admitting quasi-periodic solutions in view of Lemma 5.2-2 and Lemma 5.1, we can hope using KAM techniques to find quasi-periodic solutions to the first one. This approach has been intensively used before in [7, 8, 29, 28, 33]. We select finitely-many tangential sites $\mathbb{S}$ and decompose the phase space into tangential and normal subspaces described by the selection of Fourier modes belonging to $\mathbb{S}$ or not. On the tangential part, containing the main part of the quasi-periodic solutions, we introduce action-angle variables allowing to reformulate the problem in terms of embedded invariant tori. We shall also be concerned with some regularity aspects for the perturbed Hamiltonian vector field appearing in $\mathcal{F}$ and needed during the Nash-Moser scheme. Finally, we construct an approximate right inverse for the linearized operator associated to $\mathcal{F}$.

Next, with the result of Lemma 5.2 we can easily check that the equation (4.13) can be written in the form

$$\partial_t r = \partial_\theta \mathrm{L}(\lambda)(r) + X_P(r),$$

where X_P is the Hamiltonian vector field defined by

$$X_P(r) \triangleq I_1(\lambda)K_1(\lambda)\partial_\theta r - \partial_\theta \mathcal{K}_\lambda * r - F_\lambda[r]. \tag{6.1}$$

Remind that $F_\lambda[r]$ is introduced in (4.6) and the convolution kernel is stated in (5.12). To measure the smallness condition it seems to be more convenient to introduce a small parameter ε and rescale the Hamiltonian as done for instance in the papers [7, 33]. To do that we rescale the solution as follows

$r \mapsto \varepsilon r$ with r bounded. Therefore the Hamiltonian equation takes the form

$$\partial_t r = \partial_\theta \mathrm{L}(\lambda)(r) + \varepsilon X_{P_\varepsilon}(r), \tag{6.2}$$

where X_{P_ε} is the rescaled Hamiltonian vector field defined by $X_{P_\varepsilon}(r) \triangleq \varepsilon^{-2} X_P(\varepsilon r)$. Notice that (6.2) can be recast in the Hamiltonian form

$$\partial_t r = \partial_\theta \nabla \mathcal{H}_\varepsilon(r), \tag{6.3}$$

where the rescaled Hamiltonian $\mathcal{H}_\varepsilon(r)$ is given by

$$\begin{aligned}\mathcal{H}_\varepsilon(r) &\triangleq \varepsilon^{-2} H(\varepsilon r)\\ &\triangleq H_{\mathrm{L}}(r) + \varepsilon P_\varepsilon(r),\end{aligned} \tag{6.4}$$

with H_{L} being the quadratic Hamiltonian defined in Lemma 5.2 and $\varepsilon P_\varepsilon(r)$ is composed with terms of higher order more than cubic.

6.1 Action-angle reformulation

Let us consider finitely many Fourier-frenquencies, called tangential sites, gathered in the tangantial set $\mathbb{S}$ defined by

$$\mathbb{S} \triangleq \{j_1, \ldots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < j_2 < \ldots < j_d.$$

We now define the symmetrized tangential sets $\overline{\mathbb{S}}$ and $\mathbb{S}_0$ by

$$\overline{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S}) = \{\pm j,\ j \in \mathbb{S}\} \quad \text{and} \quad \mathbb{S}_0 \triangleq \overline{\mathbb{S}} \cup \{0\}. \tag{6.5}$$

Recall from (5.32) that we denote the unperturbed tangential frequency vector by

$$\omega_{\mathrm{Eq}}(\lambda) = \big(\Omega_j(\lambda)\big)_{j \in \mathbb{S}}, \tag{6.6}$$

where $\Omega_j(\lambda)$ are given by (5.14). Since the application $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda)$ is continuous then $\omega_{\mathrm{Eq}}\big([\lambda_0, \lambda_1]\big)$ is a compact subset of $\mathbb{R}^d$. In particular, there exists $R_0 > 0$ such that

$$\omega_{\mathrm{Eq}}\big([\lambda_0, \lambda_1]\big) \subset \mathscr{U} \triangleq B(0, R_0).$$

Therefore, the parameters set $\mathcal{O}$ is defined as

$$\mathcal{O} \triangleq (\lambda_0, \lambda_1) \times \mathscr{U}. \tag{6.7}$$

For $s \in \mathbb{R}$, we decompose the phase space of $L^2_0(\mathbb{T})$ as the direct sum

$$\begin{aligned} L^2_0(\mathbb{T}) =& L_{\overline{\mathbb{S}}} \overset{\perp}{\oplus} L^2_\perp, \\ L_{\overline{\mathbb{S}}} \triangleq \Big\{ v = \sum_{j \in \overline{\mathbb{S}}} r_j e_j,\ \overline{r_j} = r_{-j} \Big\}, &\quad L^2_\perp \triangleq \Big\{ z = \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} z_j e_j \in L^2,\ \overline{z_j} = z_{-j} \Big\}, \end{aligned} \tag{6.8}$$

where $e_j(\theta) = e^{\mathrm{i} j \theta}$. We denote by $\Pi_{\overline{\mathbb{S}}}, \Pi^\perp_{\mathbb{S}_0}$ the corresponding orthogonal projectors defined by

$$r = v + z, \quad v \triangleq \Pi_{\overline{\mathbb{S}}} r \triangleq \sum_{j \in \overline{\mathbb{S}}} r_j e_j, \quad z \triangleq \Pi^\perp_{\mathbb{S}_0} r \triangleq \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} r_j e_j\,, \tag{6.9}$$

where v and z are called the tangential and normal variables, respectively. Fix some small amplitudes $(\mathtt{a}_j)_{j\in\mathbb{S}} \in (\mathbb{R}_+^*)^d$ and set $\mathtt{a}_{-j} = \mathtt{a}_j$. We shall now introduce the action-angle variables on the tangential set $H_{\overline{\mathbb{S}}}$ by making the following symplectic polar change of coordinates

$$\forall\, j \in \overline{\mathbb{S}}, \quad r_j = \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j}, \tag{6.10}$$

where

$$\forall\, j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{T}. \tag{6.11}$$

Thus, any function of the phase space L_0^2 decomposes as

$$r = A(\vartheta, I, z) \triangleq v(\vartheta, I) + z \quad \text{where} \quad v(\vartheta, I) \triangleq \sum_{j\in\overline{\mathbb{S}}} \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j} e_j\,. \tag{6.12}$$

In these coordinates the solutions (5.48) of the linear system (5.10) simply read as $v(-\omega_{\mathrm{Eq}}(\lambda)t, I)$ where ω_{Eq} is defined in (6.6) and $I \in \mathbb{R}^d$ such that the quantity under the square root is positive. The involution $\mathscr{S}$ defined in (4.27) now reads in the new variables

$$\mathfrak{S} : (\vartheta, I, z) \mapsto (-\vartheta, I, \mathscr{S} z) \tag{6.13}$$

and the symplectic 2-form in (4.26) becomes after straightforward computations using (6.10) and (6.11)

$$\mathcal{W} = \sum_{j\in\mathbb{S}} d\vartheta_j \wedge dI_j + \frac{1}{2\mathrm{i}} \sum_{j\in\mathbb{Z}\setminus\mathbb{S}_0} \frac{1}{j} dr_j \wedge dr_{-j} = \Big(\sum_{j\in\mathbb{S}} d\vartheta_j \wedge dI_j\Big) \oplus \mathcal{W}_{|L^2_\perp}, \tag{6.14}$$

where $\mathcal{W}_{|L^2_\perp}$ denotes the restriction of $\mathcal{W}$ to $L^2_\perp$. This proves that the transformation A is symplectic. The next goal is to study the Hamiltonian system generated by the Hamiltonian $\mathcal{H}_\varepsilon$ in (6.4), in the action-angle and normal coordinates $(\vartheta, I, z) \in \mathbb{T}^\nu \times \mathbb{R}^\nu \times L^2_\perp$. We consider the Hamiltonian H_ε defined by

$$H_\varepsilon \triangleq \mathcal{H}_\varepsilon \circ A, \tag{6.15}$$

where A is the map described before in (6.12). Since $\mathrm{L}(\lambda)$ in (5.15) is a Fourier multiplier keeping invariant the subspaces $L_{\overline{\mathbb{S}}}$ and $L^2_\perp$, then the quadratic Hamiltonian H_{L} in (5.15) in the variables (ϑ, I, z) reads, up to an additive constant which can be removed since it does not change the dynamics in view of (4.13),

$$H_{\mathrm{L}} \circ A = -\sum_{j\in\mathbb{S}} \Omega_j(\lambda) I_j + \tfrac{1}{2} \langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})} = -\omega_{\mathrm{Eq}}(\lambda) \cdot I + \tfrac{1}{2} \langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})}, \tag{6.16}$$

where $\omega_{\mathrm{Eq}} \in \mathbb{R}^d$ is the unperturbed tangential frequency vector defined by (5.14). According to (6.4) and (6.16), one deduces that the Hamiltonian H_ε in (6.15) has the form

$$H_\varepsilon = \mathcal{N} + \varepsilon \mathcal{P}_\varepsilon \quad \text{with} \quad \mathcal{N} \triangleq -\omega_{\mathrm{Eq}}(\lambda) \cdot I + \tfrac{1}{2} \langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})} \quad \text{and} \quad \mathcal{P}_\varepsilon \triangleq P_\varepsilon \circ A. \tag{6.17}$$

We look for an embedded invariant torus

$$\begin{aligned} i :\ \mathbb{T}^d &\ \rightarrow\ \mathbb{R}^d \times \mathbb{R}^d \times L^2_\perp \\ \varphi &\ \mapsto\ i(\varphi) \triangleq (\vartheta(\varphi), I(\varphi), z(\varphi)) \end{aligned} \tag{6.18}$$

of the Hamiltonian vector field

$$X_{H_\varepsilon} \triangleq (\partial_I H_\varepsilon, -\partial_\vartheta H_\varepsilon, \Pi_{\mathbb{S}_0}^\perp \partial_\theta \nabla_z H_\varepsilon) \tag{6.19}$$

filled by quasi-periodic solutions with Diophantine frequency vector ω. Remark that for the value $\varepsilon = 0$, the Hamiltonian system reduces to the linear equation

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_0}(i(\varphi))$$

which admits the trivial solution given by the flat torus $i_{\text{flat}}(\varphi) = (\varphi, 0, 0)$ provided that $\omega = -\omega_{\text{Eq}}(\lambda)$. In what follows we shall consider the modified Hamiltonian equation indexed with a parameter $\alpha \in \mathbb{R}^d$,

$$H_\varepsilon^\alpha \triangleq \mathcal{N}_\alpha + \varepsilon \mathcal{P}_\varepsilon \quad \text{where} \quad \mathcal{N}_\alpha \triangleq \alpha \cdot I + \frac{1}{2}\langle \mathrm{L}(\lambda)\, z, z\rangle_{L^2(\mathbb{T})}. \tag{6.20}$$

For the value $\alpha = -\omega_{\text{Eq}}(\lambda)$ we have $H_\varepsilon^\alpha = H_\varepsilon$. The parameter α will play the role of a Lagrangian multiplier in order to satisfy a compatibility condition during the approximate inverse process. Notice that the initial problem is reduced to finding the zeros of the nonlinear operator

$$\mathcal{F}(i, \alpha, \mu, \varepsilon) \triangleq \omega \cdot \partial_\varphi i(\varphi) - X_{H_\varepsilon^\alpha}(i(\varphi)) = \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \alpha - \varepsilon \partial_I \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \partial_\theta \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \partial_\theta \big[\mathrm{L}(\lambda) z(\varphi) + \varepsilon \nabla_z \mathcal{P}_\varepsilon\big(i(\varphi)\big)\big] \end{pmatrix}, \tag{6.21}$$

where $\mu \triangleq (\lambda, \omega)$ and where $\mathcal{P}_\varepsilon$ is defined in (6.4). We point out that we can easily check that the Hamiltonian H_ε^α is reversible in the sense of the Definition A.2, that is,

$$H_\varepsilon^\alpha \circ \mathfrak{S} = H_\varepsilon^\alpha, \tag{6.22}$$

where the involution $\mathfrak{S}$ is defined in (6.13). Thus, we shall look for reversible solutions of

$$\mathcal{F}(i, \alpha, \mu, \varepsilon) = 0,$$

that is, solutions satisfying

$$\mathfrak{S} i(\varphi) = i(-\varphi),$$

or equivalently,

$$\vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathscr{S} z)(\varphi). \tag{6.23}$$

We define the periodic component $\mathfrak{I}$ of the torus i by

$$\mathfrak{I}(\varphi) \triangleq i(\varphi) - (\varphi, 0, 0) = \big(\Theta(\varphi), I(\varphi), z(\varphi)\big) \quad \text{with} \quad \Theta(\varphi) \triangleq \vartheta(\varphi) - \varphi.$$

We define the weighted Sobolev norm of $\mathfrak{I}$ as

$$\|\mathfrak{I}\|_{q,s}^{\gamma,\mathcal{O}} \triangleq \|\Theta\|_{q,s}^{\gamma,\mathcal{O}} + \|I\|_{q,s}^{\gamma,\mathcal{O}} + \|z\|_{q,s}^{\gamma,\mathcal{O}}.$$

6.2 Hamiltonian regularity

This section is devoted to some regularity aspects of the Hamiltonian vector field introduced in (6.1), together with the rescaled one associated to the Hamiltonian described in (6.17). First, we shall give a useful decomposition of the nonlocal operator appearing in Lemma 5.1.

Lemma 6.1. *We have the following decomposition of the operator $\mathbf{L}_r$ defined in* (5.2).

$$\mathbf{L}_r = \mathcal{K}_\lambda * \cdot + \mathbf{L}_{r,1}, \tag{6.24}$$

where $\mathcal{K}_\lambda$ *is introduced in* (5.12) *and* $\mathbf{L}_{r,1}$ *is an integral operator with kernel* $\mathbb{K}_{r,1}$ *taking the following form*

$$\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\eta)\triangleq\mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^{1}(\lambda,\varphi,\theta,\eta)+\mathscr{K}_{r,1}^{2}(\lambda,\varphi,\theta,\eta),\tag{6.25}$$

with $\mathscr{K}$ *defined by*

$$\mathscr{K}(\theta)\triangleq\sin^2\left(\tfrac{\theta}{2}\right)\log\left(\left|\sin\left(\tfrac{\theta}{2}\right)\right|\right)\tag{6.26}$$

and $\mathscr{K}_{r,1}^{1}$, $\mathscr{K}_{r,1}^{2}$ *smooth kernels. Moreover, the kernel* $\mathbb{K}_{r,1}$ *satisfies the following symmetry property*

$$r(-\varphi,-\theta)=r(\varphi,\theta)\quad\Rightarrow\quad\mathbb{K}_{r,1}(\lambda,-\varphi,-\theta,-\eta)=\mathbb{K}_{r,1}(\lambda,\varphi,\theta,\eta).\tag{6.27}$$

In addition,

$$\|\partial_\theta\mathcal{K}_\lambda*r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s}^{\gamma,\mathcal{O}}.\tag{6.28}$$

Proof. We start by proving (6.28). According to (5.16), the Fourier coefficients of $\partial_\theta\mathcal{K}_\lambda$ are $\big(\mathrm{i}jI_j(\lambda)K_j(\lambda)\big)_{j\in\mathbb{Z}}$. Hence

$$\|\partial_\theta\mathcal{K}_\lambda*r\|_{H^s}^2=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}\langle l,j\rangle^{2s}j^2I_{|j|}^2(\lambda)K_{|j|}^2(\lambda)|r_{l,j}|^2\leqslant\tfrac14\|r\|_{H^s}^2.$$

Notice that the last inequality is obtained by the decay property of the product I_jK_j on $\mathbb{R}_+^*$, (C.3) and (C.13). Thus we deduce that

$$\|\partial_\theta\mathcal{K}_\lambda*r\|_{H^s}\leqslant\tfrac12\|r\|_{H^s}\leqslant\|r\|_{H^s}.$$

Now, from (5.16), we infer that

$$\partial_\theta\mathcal{K}_\lambda*r=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}\mathrm{i}jI_j(\lambda)K_j(\lambda)r_{l,j}(\lambda,\omega)\mathbf{e}_{l,j}.\tag{6.29}$$

At this stage we need to explore the regularity of the multiplier with respect to λ. By using (C.9), we write

$$I_j(\lambda)K_j(\lambda)=\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}}K_0\left(2\lambda\cos(\tau)\right)\cos(2j\tau)d\tau.$$

From (C.7), we have the decomposition

$$K_0(z)=-\log(z/2)I_0(z)+f(z),\tag{6.30}$$

with I_0 being the modified Bessel function of the first kind and f an analytic function. By the morphism property of the logarithm, we get

$$\begin{aligned}I_j(\lambda)K_j(\lambda)&=-\log(\lambda)\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}}I_0(2\lambda\cos(\tau))\cos(2j\tau)d\tau\\&\quad-\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}}\log(\cos(\tau))\cos(2j\tau)d\tau\\&\quad-\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}}\log(\cos(\tau))\left(I_0(2\lambda\cos(\tau))-1\right)\cos(2j\tau)d\tau\\&\quad+\tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}}f(2\lambda\cos(\tau))\cos(2j\tau)d\tau\\&\triangleq\mathcal{I}_{1,j}(\lambda)+\mathcal{I}_{2,j}+\mathcal{I}_{3,j}(\lambda)+\mathcal{I}_{4,j}(\lambda).\end{aligned}$$

Since I_0 and f are analytic, then the above expressions are smooth with respect to the parameter

$\lambda \in (\lambda_0, \lambda_1) \subset \mathbb{R}_+^*$. An integration by parts in $\mathcal{I}_{1,j}(\lambda)$ and $\mathcal{I}_{4,j}(\lambda)$ yields

$$\forall\, i \in \{1,4\}, \quad \sup_{j\in\mathbb{Z}} \left(|j| \max_{n\in[\![0,q]\!]} \|\partial_\lambda^{(n)} \mathcal{I}_{i,j}\|_{L^\infty([\lambda_0,\lambda_1])} \right) \lesssim 1.$$

Looking at the definition of I_0 in (C.2), we see that we have uniformly in $\lambda \in [\lambda_0, \lambda_1]$,

$$\forall\, n \in [\![0,q]\!], \quad \partial_\lambda^{(n)} \left(I_0\left(2\lambda\cos(\tau)\right) - 1 \right) = O\left(\cos(\tau)\right).$$

Hence, an integration by parts in $\mathcal{I}_{3,j}(\lambda)$ yields

$$\sup_{j\in\mathbb{Z}} \left(|j| \max_{n\in[\![0,q]\!]} \|\partial_\lambda^{(n)} \mathcal{I}_{3,j}\|_{L^\infty([\lambda_0,\lambda_1])} \right) \lesssim 1.$$

It remains to study the integral $\mathcal{I}_{2,j}$. One can easily check from the above decomposition that

$$\mathcal{I}_{2,j} = \lim_{\lambda\to 0^+} I_j(\lambda) K_j(\lambda).$$

Using (C.13), we then find

$$\mathcal{I}_{2,j} = \frac{1}{2j}\cdot$$

Putting together the preceding estimates, we obtain

$$\sup_{j\in\mathbb{Z}} \left(|j| \max_{n\in[\![0,q]\!]} \|\partial_\lambda^{(n)} \left(I_j K_j\right) \|_{L^\infty([\lambda_0,\lambda_1])} \right) \lesssim 1.$$

Then coming back to (6.29) and using Leibniz formula, we obtain (6.28). Now we turn to the proof of (6.24). According to (5.4) we may write

$$\begin{aligned} A_r(\varphi,\theta,\eta) &= 2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right| \left(\left(\frac{R(\varphi,\eta)-R(\varphi,\theta)}{2\sin\left(\frac{\eta-\theta}{2}\right)} \right)^2 + R(\varphi,\eta)R(\varphi,\theta) \right)^{\frac{1}{2}} \\ &\triangleq 2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right| v_{r,1}(\varphi,\theta,\eta). \end{aligned} \tag{6.31}$$

Notice that $v_{r,1}$ is smooth when r is smooth and small enough, and $v_{0,1} = 1$. More precisely, Lemma A.1-(v) combined with Lemma A.2 allow to get

$$\begin{aligned} &\sup_{\eta\in\mathbb{T}} \|v_{r,1}(\ast,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare}) - 1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}, \\ \forall k\in\mathbb{N}^*, &\sup_{\eta\in\mathbb{T}} \|(\partial_\theta^k v_{r,1})(\ast,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \end{aligned} \tag{6.32}$$

Here and in the sequel, the symbols $\ast, \cdot, {\scriptstyle\blacksquare}$ denote the variables $\mu = (\lambda,\omega), \varphi, \theta$, respectively. Then from the identity (6.30) we infer

$$\begin{aligned} K_0(\lambda A_r(\varphi,\theta,\eta)) = K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right) + \log\left(\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\left[I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right) - I_0\big(\lambda A_r(\varphi,\theta,\eta)\big)\right] \\ - \log\big(v_{r,1}(\varphi,\theta,\eta)\big) I_0\big(\lambda A_r(\varphi,\theta,\eta)\big) + f\big(\lambda A_r(\varphi,\theta,\eta)\big) - f\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right). \end{aligned} \tag{6.33}$$

By virtue of the expansion (C.2), we can write

$$I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right) - I_0\big(\lambda A_r(\varphi,\theta,\eta)\big) = \sin^2\left(\tfrac{\eta-\theta}{2}\right) \mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta),$$

with $\mathscr{K}_{r,1}^1$ being smooth and vanishing at $r=0$. More precisely, we have the expansion

$$\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta)=\sum_{m=1}^{\infty}\tfrac{(2\lambda)^{2m}}{(m!)^2}\sin^{2m-2}\left(\tfrac{\eta-\theta}{2}\right)\left(1-v_{r,1}^{2m}(\varphi,\theta,\eta)\right). \tag{6.34}$$

Now our aim is to establish the following estimate.

$$\forall\, k\in\mathbb{N},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathscr{K}_{r,1}^1)(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{6.35}$$

For this goal we apply Taylor Formula at the order 2,

$$\begin{aligned}I_0\big(\lambda A_r(\varphi,\theta,\eta)\big)-I_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)&=2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)I_0'\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\\&+4\lambda^2\sin^2\left(\tfrac{\eta-\theta}{2}\right)\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)^2\int_0^1(1-t)I_0''\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\big(1-t+tv_{r,1}(\varphi,\theta,\eta)\big)\right)dt.\end{aligned}$$

Consequently, the kernel $\mathscr{K}_{r,1}^1$ can be rewritten into the form

$$\begin{aligned}\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta)&=2\lambda\Big(1-v_{r,1}(\varphi,\theta,\eta)\Big)\frac{I_0'\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)}{\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|}\\&-4\lambda^2\Big(v_{r,1}(\varphi,\theta,\eta)-1\Big)^2\int_0^1(1-t)I_0''\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\big(1-t+tv_{r,1}(\varphi,\theta,\eta)\big)\right)dt.\end{aligned} \tag{6.36}$$

Using the structure (C.2) and Lemma A.1-(iv)-(v) combined with (6.32) we deduce the estimate (6.35). Coming back to (6.33) and set

$$\begin{aligned}\mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta)&=\log(\lambda)\sin^2\left(\tfrac{\eta-\theta}{2}\right)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta)-\log(v_{r,1}(\varphi,\theta,\eta))I_0(\lambda A_r(\varphi,\theta,\eta))\\&+f(\lambda A_r(\varphi,\theta,\eta))-f\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right).\end{aligned} \tag{6.37}$$

Then, by virtue of the product laws and the composition laws of Lemma A.1 combined with (6.32), (6.35) and the fact that f is analytic and even, we get

$$\forall k\in\mathbb{N},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathscr{K}_{r,1}^2)(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{6.38}$$

Consequently we obtain the decomposition

$$K_0(\lambda A_r(\varphi,\theta,\eta))=K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta)+\mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta), \tag{6.39}$$

where $\mathscr{K}$ is defined by (6.26) and the functions $\mathscr{K}_{r,1}^1$ and $\mathscr{K}_{r,1}^2$ satisfy the estimates (6.35) and (6.38). We can obviously check that $\mathscr{K}$ is an even function satisfying

$$\mathscr{K},\,\partial_\theta\mathscr{K}\in L^\infty(\mathbb{T},\mathbb{R})\subset L^1(\mathbb{T},\mathbb{R})\quad\text{ and }\quad\partial_\theta^2\mathscr{K}\in L^1(\mathbb{T},\mathbb{R})\setminus L^\infty(\mathbb{T},\mathbb{R}). \tag{6.40}$$

Introduce the kernel $\mathbb{K}_{r,1}$ as in (6.25). Hence, putting together (6.35), (6.38) and (6.40), we obtain

$$\forall k\in\{0,1\},\quad \sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\mathbb{K}_{r,1})(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+1+k}^{\gamma,\mathcal{O}}. \tag{6.41}$$

The symmetry property (6.27) is obtained by straightforward computations. From (5.2), (5.12), (6.39) and (6.25) we deduce the decomposition (6.24). ☐

The main result of this section reads as follows.

Lemma 6.2. *Let* (γ, q, s_0, s) *satisfying* (A.2). *There exists* $\varepsilon_0 \in (0,1)$ *such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0,$$

then the vector field X_P *defined in* (6.1) *satisfies the following estimates*

(i) $\|X_P(r)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(ii) $\|d_r X_P(r)[\rho]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

Proof. Remarking that $X_P(0)=0$ and $d_r X_P(0)=0$, it suffices to prove the point (iii) and we immediately obtain (i)-(ii) by applying Taylor formula. From Lemma 5.1 and its proof we find

$$d_r F_\lambda[r]\rho = \partial_\theta\left((V_r-\Omega)\rho\right) - \partial_\theta \mathcal{K}_\lambda * \rho - \partial_\theta \mathbf{L}_{r,1}\rho.$$

Thus, we get according to the definition (6.1)

$$d_r X_P(r)\rho = \partial_\theta \mathbf{L}_{r,1}\rho - \partial_\theta\left((V_r - V_0)\rho\right). \tag{6.42}$$

Coming back to (5.1) and using the kernel decomposition (6.39) together with the product laws, the composition laws in Lemma A.1 and the smallness condition, we deduce for any $s \geqslant s_0$,

$$\|V_r - V_0\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.43}$$

Therefore, we obtain from the product laws, (6.43) and the smallness property on r,

$$\begin{aligned}\|\partial_\theta\left((V_r-V_0))\rho\right)\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|V_r-V_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|V_r-V_0\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Differentiating in r the identity (6.42) yields,

$$d_r^2 X_P(r)[\rho_1,\rho_2] = \partial_\theta\left(d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\right) - \partial_\theta\left(\big(d_r V_r(r)[\rho_2]\big)\rho_1\right). \tag{6.44}$$

For the first member of the right-hand side we first recall from (6.25) that

$$\mathbf{L}_{r,1}\rho(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\big[\mathscr{K}(\eta-\theta)\mathscr{K}_{r,1}^1(\lambda,\varphi,\theta,\eta) + \mathscr{K}_{r,1}^2(\lambda,\varphi,\theta,\eta)\big]d\eta.$$

Hence, by differentiation and change of variables, we obtain

$$\begin{aligned}d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1(\varphi,\theta) &= \int_{\mathbb{T}} \rho_1(\varphi,\eta)\big[\mathscr{K}(\eta-\theta)\big(d_r\mathscr{K}_{r,1}^1\big)[\rho_2](\varphi,\theta,\eta) + d_r\mathscr{K}_{r,1}^2\big)[\rho_2](\varphi,\theta,\eta)\big]d\eta \\ &= \int_{\mathbb{T}} \rho_1(\varphi,\theta+\eta)\big[\mathscr{K}(\eta)\big(d_r\mathscr{K}_{r,1}^1\big)[\rho_2](\varphi,\theta,\theta+\eta) + d_r\mathscr{K}_{r,1}^2\big)[\rho_2](\varphi,\theta,\theta+\eta)\big]d\eta.\end{aligned} \tag{6.45}$$

Coming back to (6.36), we emphasize that the dependence in r of the functional $\mathscr{K}_{r,1}^1$ is smooth since the function $v_{r,1}$, introduced in (6.31), depends smoothly in r. In addition $d_r\mathscr{K}_{r,1}^1$ can be easily related to $d_r v_{r,1}$. From straightforward calculations we see that, for the sake of simple notation we remove the dependence in the parameters and φ,

$$d_r v_{r,1}(r)[\rho](\theta,\eta)=\frac{1}{v_{r,1}(\theta,\eta)}\left(\frac{R(\theta)-R(\eta)}{\sin^2\left(\frac{\eta-\theta}{2}\right)}\left(\frac{\rho(\theta)}{R(\theta)}-\frac{\rho(\eta)}{R(\eta)}\right)+\frac{\rho(\theta)R^2(\eta)+\rho(\eta)R^2(\theta)}{2R(\theta)R(\eta)}\right). \tag{6.46}$$

Therefore using (6.32) combined with the product laws stated in Lemma A.1, Lemma A.2 and the smallness condition of Lemma 6.2, we find that

$$\sup_{\eta\in\mathbb{T}}\|d_r v_{r,1}(r)[\rho](*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.47}$$

Similarly to (6.47), one gets from (6.36) and (6.37),

$$\forall\, i\in\{1,2\},\quad\sup_{\eta\in\mathbb{T}}\|d_r\mathscr{K}_{r,1}^{i}[\rho](*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.48}$$

Inserting (6.48) into (6.45) and using once again the product laws and the smallness condition we obtain,

$$\begin{aligned}\|\partial_\theta d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim&\|d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ \lesssim&\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.49}$$

Next we shall move to the estimate of the last member of (6.44). Differentiating the definition of V_r in the proof of Lemma 5.1, we infer

$$\begin{aligned}d_rV_r(r)[\rho_2](\theta)=&\int_{\mathbb{T}}K_0\left(\lambda A_r(\theta,\eta)\right)\partial_\eta\left(\tfrac{\rho_2(\eta)R^2(\theta)-\rho_2(\theta)R^2(\eta)}{R^3(\theta)R(\eta)}\sin(\eta-\theta)\right)d\eta\\ &+\tfrac{\lambda}{R(\theta)}\int_{\mathbb{T}}\tfrac{(R(\theta)-R(\eta))\left(\frac{\rho_2(\theta)}{R(\theta)}-\frac{\rho_2(\eta)}{R(\eta)}\right)}{A_r(\theta,\eta)}K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta\\ &+2\lambda\int_{\mathbb{T}}\tfrac{\rho_2(\theta)R^2(\eta)+\rho_2(\eta)R^2(\theta)}{R^2(\theta)R(\eta)A_r(\theta,\eta)}\sin^2\left(\tfrac{\eta-\theta}{2}\right)K_0'\left(\lambda A_r(\theta,\eta)\right)\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta\\ \triangleq&\,\mathcal{I}_1(\theta)+\mathcal{I}_2(\theta)+\mathcal{I}_3(\theta).\end{aligned}$$

The estimate of $\mathcal{I}_1$ is obtained using the decomposition (6.39) together with the estimates (6.35), (6.38) and the Lemma A.1-(iv)-(v). We get

$$\|\mathcal{I}_1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.50}$$

For the terms $\mathcal{I}_2$ and $\mathcal{I}_3$ the computations are straightforward and we shall only extract their main parts and give the suitable estimates. For this goal we differentiate (C.7), leading to

$$K_0'(z)=\frac{-1}{z}+\log(z)F(z)+G(z),$$

with F and G being entire functions. Hence, applying (6.31), we deduce that $\mathcal{I}_2$ takes the form

$$\mathcal{I}_2(\theta)=-\frac{1}{4}\int_{\mathbb{T}}\frac{(R(\theta)-R(\eta))\left(\frac{\rho_2(\theta)}{R(\theta)}-\frac{\rho_2(\eta)}{R(\eta)}\right)}{R^2(\theta)v_{r,1}^2(\theta,\eta)\sin^2\left(\frac{\eta-\theta}{2}\right)}\partial_\eta(R(\eta)\sin(\eta-\theta))d\eta+\text{l.o.t.}$$

Hence we proceed as for (6.48) and one finds

$$\|\mathcal{I}_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.51}$$

As for the last term $\mathcal{I}_3$, we write

$$\mathcal{I}_3(\theta)=-\frac{1}{2}\int_{\mathbb{T}}\tfrac{\rho_2(\theta)R^2(\eta)+\rho_2(\eta)R^2(\theta)}{R^2(\theta)R(\eta)v_{r,1}^2(\theta,\eta)}\partial_\eta\big(R(\eta)\sin(\eta-\theta)\big)d\eta+\text{l.o.t.}$$

Then, we get

$$\|\mathcal{I}_3\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.52}$$

Putting together (6.50), (6.51) and (6.52) yields

$$\|d_rV_r(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{6.53}$$

Therefore we obtain according to the product laws in Lemma A.1, (6.53) and the smallness condition,

$$\begin{aligned}\big\|\partial_\theta\big(d_rV_r(r)[\rho_2]\rho_1\big)\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\big\|d_rV_r(r)[\rho_2]\big\|_{q,s+1}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s_0}^{\gamma,\mathcal{O}}+\big\|d_rV_r(r)[\rho_2]\big\|_{q,s_0}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combining the latter estimate with (6.44) and (6.49) allows to get

$$\|d_r^2X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$$

Using Sobolev embeddings we get the desired result. This achieves the proof of Lemma 6.2. ☐

As an application of Lemma 6.2, we shall establish tame estimates for the Hamiltonian vector field

$$X_{\mathcal{P}_\varepsilon}=(\partial_I\mathcal{P}_\varepsilon,-\partial_\vartheta\mathcal{P}_\varepsilon,\Pi_{\mathbb{S}}^\perp\partial_\theta\nabla_z\mathcal{P}_\varepsilon)$$

defined through (6.17) and (6.19).

Lemma 6.3. *Let* (γ,q,s_0,s) *satisfy* (A.2). *There exists* $\varepsilon_0\in(0,1)$ *such that if*

$$\varepsilon\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant1,$$

then the perturbed Hamiltonian vector field $X_{\mathcal{P}_\varepsilon}$ *satisfies the following estimates,*

(i) $\|X_{\mathcal{P}_\varepsilon}(i)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim1+\|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}$.

(ii) $\big\|d_iX_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath}]\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{\imath}\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\widehat{\imath}\|_{q,s_0+2}^{\gamma,\mathcal{O}}$.

(iii) $\big\|d_i^2X_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath},\widehat{\imath}]\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{\imath}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\widehat{\imath}\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\Big(\|\widehat{\imath}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\Big)^2$.

Proof. These estimates can be recovered from Lemma 6.2 combined with the following estimate on the action-angle change of variables introduced in (6.12)

$$\forall\alpha,\beta\in\mathbb{N}^d,\quad\|\partial_\vartheta^\alpha\partial_I^\beta v(\vartheta,I)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim1+\|\mathfrak{I}\|_{q,s}^{\gamma,\mathcal{O}}. \tag{6.54}$$

This estimate follows from Lemma A.1-(iv)-(v) provided that $\|\vartheta\|_{q,s_0}^{\gamma,\mathcal{O}},\|I\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant1$. This latter condition is satisfied due to the smallness condition in the Lemma. For more details, we refer to [33, Lem. 5.1]. ☐

6.3 Berti-Bolle approach for the approximate inverse

In this section, we shall follow the remarkable procedure developed by Berti and Bolle in [21] to construct an approximate right inverse for the linearized operator

$$d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)[\widehat{\imath},\widehat{\alpha}]=\omega\cdot\partial_\varphi\widehat{\imath}-d_iX_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))[\widehat{\imath}]-(\widehat{\alpha},0,0),\tag{6.55}$$

where $\mathcal{F}$ is the nonlinear functional defined in (6.21). This construction is crucial for the Nash-Moser scheme that we shall perform later in Section 8. From (6.18), we denote by i_0 an embedded torus with

$$i_0(\varphi)=(\vartheta_0(\varphi),I_0(\varphi),z_0(\varphi))\quad\text{and}\quad\mathfrak{I}_0(\varphi)=i_0(\varphi)-(\varphi,0,0).$$

Throughout this section, we shall assume the following smallness condition : the application $(\lambda,\omega)\mapsto\mathfrak{I}_0(\lambda,\omega)$ is q-times differentiable on $\mathcal{O}$ and there exists $\varepsilon_0\in(0,1)$ (small enough) such that

$$\|\mathfrak{I}_0\|_{q,s_0+2}^{\gamma,\mathcal{O}}+\|\alpha_0-\omega\|_q^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.\tag{6.56}$$

We mainly follow the same approach as in [21] which reduces the search of an approximate right inverse of (6.55) to the search of an approximate right inverse in the normal directions. The main difference with [21] is to be able to bypass the use of the isotropic torus in a similar way to the recent paper [87].

6.3.1 Triangularization up to error terms

Given a linear operator $A\in\mathcal{L}(\mathbb{R}^d,L^2_\perp)$, we define the transposed operator $A^\top:L^2_\perp\to\mathbb{R}^d$ by the duality relation

$$\forall(u,v)\in L^2_\perp\times\mathbb{R}^d,\quad\langle A^\top u,v\rangle_{\mathbb{R}^d}=\langle u,Av\rangle_{L^2(\mathbb{T})}.\tag{6.57}$$

We introduce the following change of coordinates $G_0:(\phi,y,w)\to(\vartheta,I,z)$ of the phase space $\mathbb{T}^d\times\mathbb{R}^d\times L^2_\perp$ defined by

$$\begin{pmatrix}\vartheta\\ I\\ z\end{pmatrix}\triangleq G_0\begin{pmatrix}\phi\\ y\\ w\end{pmatrix}\triangleq\begin{pmatrix}\vartheta_0(\phi)\\ I_0(\phi)+L_1(\phi)y+L_2(\phi)w\\ z_0(\phi)+w\end{pmatrix},\tag{6.58}$$

where

$$L_1(\phi)\triangleq[\partial_\phi\vartheta_0(\phi)]^{-\top},\tag{6.59}$$

$$L_2(\phi)\triangleq[(\partial_\vartheta\widetilde{z}_0)(\vartheta_0(\phi))]^\top\partial_\theta^{-1},\tag{6.60}$$

$$\widetilde{z}_0(\vartheta)\triangleq z_0(\vartheta_0^{-1}(\vartheta)),\tag{6.61}$$

provided that $\vartheta_0:\mathbb{R}^d\to\mathbb{R}^d$ is a diffeomorphism. Notice that one recovers the torus i_0 by taking in the new coordinates, the flat torus $i_{\text{flat}}(\varphi)=(\varphi,0,0)$ namely

$$G_0(i_{\text{flat}}(\varphi))=i_0(\varphi).$$

Next, we shall adopt the notation $\mathtt{u}=(\phi,y,w)$ to denote the new coordinates induced by G_0 in (6.58) and we simply set $\mathtt{u}_0(\varphi)=i_{\text{flat}}(\varphi)$. Now, to measure to which extent an embedded torus $i_0(\mathbb{T})$ is close to be invariant for the Hamiltonian vector field $X_{H_\varepsilon^{\alpha_0}}$, we shall make appeal to the error function

$$Z(\varphi)\triangleq(Z_1,Z_2,Z_3)(\varphi)\triangleq\mathcal{F}(i_0,\alpha_0)(\varphi)=\omega\cdot\partial_\varphi i_0(\varphi)-X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))\,.\tag{6.62}$$

We say that a quantity is of "type Z" is it is $O(Z)$, and particular it is vanishing at an exact solution. In the next Proposition, we study the conjugation of the linear operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ by the linear change of variables induced by G_0 defined in (6.58),

$$DG_0(\varphi,0,0)\begin{pmatrix}\widehat{\phi}\\ \widehat{y}\\ \widehat{w}\end{pmatrix}\triangleq\begin{pmatrix}\partial_\varphi\vartheta_0(\varphi) & 0 & 0\\ \partial_\varphi I_0(\varphi) & L_1(\varphi) & L_2(\varphi)\\ \partial_\varphi z_0(\varphi) & 0 & I\end{pmatrix}\begin{pmatrix}\widehat{\phi}\\ \widehat{y}\\ \widehat{w}\end{pmatrix}. \tag{6.63}$$

Proposition 6.1. *The conjugation of the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ by the linear change of variables $DG_0(\mathtt{u}_0)$ writes as follows*

$$[DG_0(\mathtt{u}_0)]^{-1}d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)D\widetilde{G}_0(\mathtt{u}_0)[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]=\mathbb{D}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]+\mathbb{E}[\widehat{\phi},\widehat{y},\widehat{w}], \tag{6.64}$$

where $\widetilde{G}_0$ is defined by

$$\widetilde{G}_0(\mathtt{u},\alpha)\triangleq(G_0(\mathtt{u}),\alpha)$$

and where

(i) the operator $\mathbb{D}$ admits a triangular structure in the variables $(\widehat{\phi},\widehat{y},\widehat{w})$ in the form

$$\mathbb{D}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]\triangleq\begin{pmatrix}\omega\cdot\partial_\varphi\widehat{\phi}-\big[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}+L_1^\top(\varphi)\widehat{\alpha}\big]\\ \omega\cdot\partial_\varphi\widehat{y}+\mathtt{B}(\varphi)\widehat{\alpha}\\ \omega\cdot\partial_\varphi\widehat{w}-\partial_\theta\big[\mathsf{K}_{11}(\varphi)\widehat{y}+\mathsf{K}_{02}(\varphi)\widehat{w}+L_2^\top(\varphi)\widehat{\alpha}\big]\end{pmatrix},$$

$\mathtt{B}(\varphi)$ and $\mathsf{K}_{20}(\varphi)$ are $d\times d$ real matrices given by

$$\mathtt{B}(\varphi)\triangleq[\partial_\varphi\vartheta_0(\varphi)]^\top\partial_\varphi I_0(\varphi)L_1^\top(\varphi)+[\partial_\varphi z_0(\varphi)]^\top L_2^\top(\varphi), \tag{6.65}$$

$$\mathsf{K}_{20}(\varphi)\triangleq\varepsilon L_1^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi), \tag{6.66}$$

$\mathsf{K}_{02}(\varphi)$ is a linear self-adjoint operator of $L^2_\perp$ in the form

$$\begin{aligned}\mathsf{K}_{02}(\varphi)\triangleq{}&(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))+\varepsilon L_2^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_2(\varphi)\\ &+\varepsilon L_2^\top(\varphi)(\partial_{zI}\mathcal{P}_\varepsilon)(i_0(\varphi))+\varepsilon(\partial_I\nabla_z\mathcal{P}_\varepsilon)(i_0(\varphi))L_2(\varphi)\end{aligned} \tag{6.67}$$

and $\mathsf{K}_{11}(\varphi)\in\mathcal{L}(\mathbb{R}^d,L^2_\perp)$ is given by

$$\mathsf{K}_{11}(\varphi)\triangleq\varepsilon L_2^\top(\varphi)(\partial_{II}\mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi)+\varepsilon(\partial_I\nabla_z\mathcal{P}_\varepsilon)(i_0(\varphi))L_1(\varphi). \tag{6.68}$$

(ii) the operator $\mathbb{E}$ is an error term in the form

$$\begin{aligned}\mathbb{E}[\widehat{\phi},\widehat{y},\widehat{w}]\triangleq{}&[DG_0(\mathtt{u}_0)]^{-1}\partial_\varphi Z(\varphi)\widehat{\phi}\\ &+\begin{pmatrix}0\\ \mathtt{A}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}\big]-R_{10}(\varphi)\widehat{y}-R_{01}(\varphi)\widehat{w}\\ 0\end{pmatrix},\end{aligned}$$

where $\mathtt{A}(\varphi)$ and $R_{10}(\varphi)$ are $d\times d$ matrices defined by

$$\mathtt{A}(\varphi)\triangleq[\partial_\varphi\vartheta_0(\varphi)]^\top\partial_\varphi I_0(\varphi)-[\partial_\varphi I_0(\varphi)]^\top\partial_\varphi\vartheta_0(\varphi)+[\partial_\varphi z_0(\varphi)]^\top\partial_\theta^{-1}\partial_\varphi z_0(\varphi), \tag{6.69}$$

$$R_{10}(\varphi)\triangleq[\partial_\varphi Z_1(\varphi)]^\top L_1(\varphi) \tag{6.70}$$

and $R_{01}(\varphi) \in \mathcal{L}(L^2_\perp, \mathbb{R}^d)$ *with*

$$R_{01}(\varphi) \triangleq [\partial_\varphi Z_1(\varphi)]^\top L_2(\varphi) - [\partial_\varphi Z_3(\varphi)]^\top \partial_\theta^{-1}. \tag{6.71}$$

Proof. Under the map G_0, the nonlinear functional $\mathcal{F}$ in (6.21) is transformed into

$$\mathcal{F}(G_0(\mathtt{u}(\varphi)), \alpha) = \omega \cdot \partial_\varphi\big(G_0(\mathtt{u}(\varphi))\big) - X_{H_\varepsilon^\alpha}\big(G_0(\mathtt{u}(\varphi))\big). \tag{6.72}$$

Differentiating (6.72) at $(\mathtt{u}_0, \alpha_0)$ in the direction $(\widehat{\mathtt{u}}, \widehat{\alpha})$ gives

$$\begin{aligned} d_{(\mathtt{u},\alpha)}(\mathcal{F} \circ G_0)(\mathtt{u}_0, \alpha_0)[(\widehat{\mathtt{u}}, \widehat{\alpha})](\varphi) &= \omega \cdot \partial_\varphi\big(DG_0(\mathtt{u}_0)\widehat{\mathtt{u}}\big) - \partial_\phi\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{\phi} \\ &\quad - \partial_y\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{y} - \partial_w\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{w} - \begin{pmatrix} \widehat{\alpha} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{6.73}$$

From the expression of $DG_0(\mathtt{u}_0)$ in (6.63), we obtain

$$\begin{aligned} \omega \cdot \partial_\varphi\big(DG_0(\mathtt{u}_0)[\widehat{\mathtt{u}}](\varphi)\big) &= DG_0(\mathtt{u}_0)\, \omega \cdot \partial_\varphi \widehat{\mathtt{u}} + \partial_\varphi\big(\omega \cdot \partial_\varphi i_0\big)\widehat{\phi} \\ &\quad + \begin{pmatrix} 0 \\ (\omega \cdot \partial_\varphi L_1(\varphi))\widehat{y} + \big(\omega \cdot \partial_\varphi L_2(\varphi)\big)\widehat{w} \\ 0 \end{pmatrix}. \end{aligned} \tag{6.74}$$

In view of (6.59) and (6.62) we have

$$\begin{aligned} \omega \cdot \partial_\varphi L_1(\varphi) &= -[\partial_\varphi \vartheta_0(\varphi)]^{-\top}(\omega \cdot \partial_\varphi[\partial_\varphi \vartheta_0(\varphi)]^\top)[\partial_\varphi \vartheta_0(\varphi)]^{-\top} \\ &= -[\partial_\varphi \vartheta_0(\varphi)]^{-\top}\Big(\big[\partial_\varphi Z_1(\varphi)\big]^\top + \big[\partial_\varphi\big((\partial_I H_\varepsilon^\alpha)(i_0(\varphi))\big)\big]^\top\Big)[\partial_\varphi \vartheta_0(\varphi)]^{-\top}. \end{aligned} \tag{6.75}$$

By (6.60) we can easily check that

$$\partial_\varphi z_0(\varphi) = (\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\varphi))\partial_\phi \vartheta_0(\varphi) \tag{6.76}$$

and thus, we may write the operator $L_2(\varphi)$ in term of the matrix $L_1(\varphi)$,

$$L_2(\varphi) = [\partial_\varphi \vartheta_0(\varphi)]^{-\top}[\partial_\varphi z_0(\varphi)]^\top \partial_\theta^{-1} = L_1(\varphi)[\partial_\varphi z_0(\varphi)]^\top \partial_\theta^{-1}. \tag{6.77}$$

Then, by (6.75), (6.77) we have

$$\begin{aligned} \omega \cdot \partial_\varphi L_2(\varphi) &= -[\partial_\varphi \vartheta_0(\varphi)]^{-\top}(\omega \cdot \partial_\varphi[\partial_\varphi \vartheta_0(\varphi)]^\top)[\partial_\varphi \vartheta_0(\varphi)]^{-\top}[\partial_\varphi z_0(\varphi)]^\top \partial_\theta^{-1} \\ &\quad + [\partial_\varphi \vartheta_0(\varphi)]^{-\top}[\partial_\varphi(\omega \cdot \partial_\varphi z_0)(\varphi)]^\top \partial_\theta^{-1} \end{aligned}$$

and from (6.62) we get

$$\begin{aligned} \omega \cdot \partial_\varphi L_2(\varphi) &= -[\partial_\varphi \vartheta_0(\varphi)]^{-\top}\Big(\big[\partial_\varphi Z_1(\varphi)\big]^\top + \big[\partial_\varphi\big((\partial_I H_\varepsilon^\alpha)(i_0(\varphi))\big)\big]^\top\Big)L_2(\varphi) \\ &\quad + [\partial_\varphi \vartheta_0(\varphi)]^{-\top}\Big(\big[\partial_\varphi Z_3(\varphi)\big]^\top - \big[\partial_\varphi\big((\nabla_z H_\varepsilon^\alpha)(i_0(\varphi))\big)\big]^\top \partial_\theta\Big)\partial_\theta^{-1}. \end{aligned} \tag{6.78}$$

Putting together (6.74), (6.75) and (6.78) we conclude that

$$\begin{aligned}\omega\cdot\partial_\varphi\big(DG_0(\mathtt{u}_0)[\widehat{\mathtt{u}}](\varphi)\big) &= DG_0(\mathtt{u}_0)\,\omega\cdot\partial_\varphi\widehat{\mathtt{u}}+\partial_\varphi\big(\omega\cdot\partial_\varphi i_0\big)\widehat{\phi}\\ &\quad-\begin{pmatrix}0\\ [\partial_\varphi\vartheta_0(\varphi)]^{-\top}\big[\mathcal{C}_I(\varphi)L_1(\varphi)+R_{10}(\varphi)\big]\widehat{y}\\ 0\end{pmatrix}\\ &\quad-\begin{pmatrix}0\\ [\partial_\varphi\vartheta_0(\varphi)]^{-\top}\big[\mathcal{C}_I(\varphi)L_2(\varphi)+\mathcal{C}_z(\varphi)+R_{01}(\varphi)\big]\widehat{w}\\ 0\end{pmatrix},\end{aligned}\tag{6.79}$$

where $R_{10}(\varphi)$ and $R_{01}(\varphi)$ are given by (6.70)-(6.71) and

$$\begin{aligned}\mathcal{C}_I(\varphi)&\triangleq\big[\partial_\varphi\big((\partial_I H_\varepsilon^{\alpha_0})(i_0(\varphi))\big)\big]^\top\\ &=[\partial_\varphi I_0(\varphi)]^\top(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))+[\partial_\varphi\vartheta_0(\varphi)]^\top(\partial_{\vartheta I}H_\varepsilon^{\alpha})(i_0(\varphi))\\ &\quad+[\partial_\varphi z_0(\varphi)]^\top(\partial_I\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)),\end{aligned}\tag{6.80}$$

$$\begin{aligned}\mathcal{C}_z(\varphi)&\triangleq\big[\partial_\varphi\big((\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))\big)\big]^\top\\ &=[\partial_\varphi I_0(\varphi)]^\top(\partial_{zI}H_\varepsilon^{\alpha_0})(i_0(\varphi))+[\partial_\varphi\vartheta_0(\varphi)]^\top(\partial_{z\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))\\ &\quad+[\partial_\varphi z_0(\varphi)]^\top(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)).\end{aligned}\tag{6.81}$$

On the other hand, in view of (6.21) and (6.58) we obtain

$$\partial_\phi\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{\phi}=\partial_\varphi\big[X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))\big]\widehat{\phi},\tag{6.82}$$

$$\partial_y\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{y}=\begin{pmatrix}(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}\\ -(\partial_{I\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}\\ \partial_\theta\big[(\partial_I\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}\big]\end{pmatrix},\tag{6.83}$$

$$\partial_w\big[X_{H_\varepsilon^{\alpha_0}}\big(G_0(\mathtt{u}(\varphi))\big)\big]_{\mathtt{u}=\mathtt{u}_0}\widehat{w}=\begin{pmatrix}(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)\widehat{w}+(\partial_{zI}H_\varepsilon^{\alpha_0})(i_0(\varphi))\widehat{w}\\ -(\partial_{I\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)\widehat{w}-(\partial_{z\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))\widehat{w}\\ \partial_\theta\big[(\partial_I\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)\widehat{w}+(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))\widehat{w}\big]\end{pmatrix}.\tag{6.84}$$

Plugging (6.79), (6.82), (6.83) and (6.84) into (6.73) we find

$$\begin{aligned}d_{(\mathtt{u},\alpha)}(\mathcal{F}\circ G_0)(\mathtt{u}_0,\alpha_0)[(\widehat{\mathtt{u}},\widehat{\alpha})]&=DG_0(\mathtt{u}_0)\,\omega\cdot\partial_\varphi\widehat{\mathtt{u}}+\partial_\varphi\big[\mathcal{F}(i_0(\varphi))\big]\widehat{\phi}\\ +&\begin{pmatrix}-(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}\\ (\partial_{I\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}-[\partial_\varphi\vartheta_0(\varphi)]^{-\top}[\mathcal{C}_I(\varphi)L_1(\varphi)+R_{10}(\varphi)]\widehat{y}\\ -\partial_\theta(\partial_I\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\widehat{y}\end{pmatrix}\\ +&\begin{pmatrix}-(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)\widehat{w}-(\partial_{zI}H_\varepsilon^{\alpha_0})(i_0(\varphi))\widehat{w}\\ \big[(\partial_{I\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)+(\partial_{z\vartheta}H_\varepsilon^{\alpha_0})(i_0(\varphi))\big]\widehat{w}\\ -\partial_\theta\big[(\partial_I\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)\widehat{w}+\partial_\theta(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi))\widehat{w}\big]\end{pmatrix}\\ -&\begin{pmatrix}0\\ [\partial_\varphi\vartheta_0(\varphi)]^{-\top}\big[\mathcal{C}_I(\varphi)L_2(\varphi)\widehat{w}+\mathcal{C}_z(\varphi)\widehat{w}+R_{01}(\varphi)\widehat{w}\big]\\ 0\end{pmatrix}-\begin{pmatrix}\widehat{\alpha}\\ 0\\ 0\end{pmatrix}.\end{aligned}\tag{6.85}$$

According to (6.63) and using the identities (6.76) and (6.77), the inverse of the linear operator $DG_0(\mathtt{u}_0)$

is given by

$$[DG_0(\mathtt{u}_0)]^{-1}=\begin{pmatrix}[\partial_\varphi\vartheta_0(\varphi)]^{-1} & 0 & 0\\ -\mathtt{B}(\varphi) & [\partial_\varphi\vartheta_0(\varphi)]^\top & -[\partial_\varphi z_0(\varphi)]^\top\partial_\theta^{-1}\\ -(\partial_\vartheta\widetilde{z}_0)(\vartheta_0(\varphi)) & 0 & I\end{pmatrix}\tag{6.86}$$

where $\mathtt{B}(\varphi)$ is given by (6.65). Applying $[DG_0(\mathtt{u}_0)]^{-1}$ to (6.85) and using the identities (6.80), (6.81) and the fact that

$$\mathtt{B}(\varphi)=\mathtt{A}(\varphi)[\partial_\varphi\vartheta_0(\varphi)]^{-1}+[\partial_\varphi I_0(\varphi)]^\top,\tag{6.87}$$

where $\mathtt{A}(\varphi)$ is defined in (6.69), we obtain

$$\begin{aligned}&[DG_0(\mathtt{u}_0)]^{-1}d_{(\mathtt{u},\alpha)}(\mathcal{F}\circ G_0)(\mathtt{u}_0,\alpha_0)[\widehat{\mathtt{u}},\widehat{\alpha}]=\omega\cdot\partial_\varphi\widehat{\mathtt{u}}+[DG_0(\mathtt{u}_0)]^{-1}\partial_\varphi\big[\mathcal{F}(i_0(\varphi))\big]\widehat{\phi}\\ &+\begin{pmatrix}-K_{20}(\varphi)\widehat{y}\\ \mathtt{A}(\varphi)K_{20}(\varphi)\widehat{y}-R_{10}(\varphi)\widehat{y}\\ -\partial_\theta K_{11}(\varphi)\widehat{y}\end{pmatrix}+\begin{pmatrix}-K_{11}^\top(\varphi)\widehat{w}\\ \mathtt{A}(\varphi)K_{11}^\top(\varphi)\widehat{w}-R_{01}(\varphi)\widehat{w}\\ -\partial_\theta K_{02}(\varphi)\widehat{w}\end{pmatrix}+\begin{pmatrix}-L_1^\top(\phi)\widehat{\alpha}\\ \mathtt{B}(\varphi)\widehat{\alpha}\\ -\partial_\theta L_2^\top(\varphi)\widehat{\alpha}\end{pmatrix},\end{aligned}$$

with

$$\begin{aligned}K_{20}(\varphi)&\triangleq L_1^\top(\varphi)(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\,,\\ K_{11}(\varphi)&\triangleq L_2^\top(\varphi)(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)+(\partial_I\nabla_zH_\varepsilon^{\alpha_0})(i_0(\varphi))L_1(\varphi)\,,\\ K_{02}(\varphi)&\triangleq(\partial_z\nabla_zH_\varepsilon^{\alpha_0})(i_0(\varphi))+L_2^\top(\varphi)(\partial_{II}H_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi)+L_2^\top(\varphi)(\partial_{zI}H_\varepsilon^{\alpha_0})(i_0(\varphi))\\ &\quad+(\partial_I\nabla_zH_\varepsilon^{\alpha_0})(i_0(\varphi))L_2(\varphi).\end{aligned}$$

Finally, by (6.20) we conclude the desired identity and this ends the proof of Proposition 6.1. ☐

Now we recall the following result, for the proof we refer to [33, Lem. 5.6 and 5.7].

Lemma 6.4. *The following assertions hold true.*

(i) The operator $DG_0(\mathtt{u}_0)$ *and* $[DG_0(\mathtt{u}_0)]^{-1}$ *satisfy for all* $\widehat{\mathtt{u}}=(\widehat{\phi},\widehat{y},\widehat{w})$*,*

$$\forall s\in[s_0,S],\quad\|[DG_0(\mathtt{u}_0)]^{\pm1}[\widehat{\mathtt{u}}]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{\mathtt{u}}\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{\mathtt{u}}\|_{q,s_0}^{\gamma,\mathcal{O}}\,.$$

(ii) The operators R_{10} *and* R_{01}*, defined in* (6.70) *and* (6.71)*, satisfy the estimates*

$$\begin{aligned}\forall s\in[s_0,S],\quad\|R_{10}\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|Z\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|Z\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+1}^{\gamma,\mathcal{O}},\\ \forall s\in[s_0,S],\quad\|R_{01}\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|Z\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|Z\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\,.\end{aligned}$$

(iii) The operators K_{20} *and* K_{11}*, defined in* (6.66) *and* (6.68)*, satisfy the estimates*

$$\begin{aligned}\forall s\in[s_0,S],\quad&\|\mathsf{K}_{20}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\big(1+\|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\big)\,,\\ \forall s\in[s_0,S],\quad&\|\mathsf{K}_{11}\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\big(\|\widehat{y}\|_{q,s+3}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\|\widehat{y}\|_{q,s_0+3}^{\gamma,\mathcal{O}}\big)\,,\\ \forall s\in[s_0,S],\quad&\|\mathsf{K}_{11}^\top\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\big(\|\widehat{w}\|_{q,s+3}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+3}^{\gamma,\mathcal{O}}\big)\,.\end{aligned}$$

(iv) The matrices $\mathtt{A}$ *and* $\mathtt{B}$ *defined in* (6.69) *and* (6.65) *satisfy*

$$\forall s\in[s_0,S],\quad\|\mathtt{A}\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathtt{B}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}.$$

Notice that the matrix $\mathtt{A}(\varphi)$ measures the defect of the symplectic structure. In the following, we shall see that it is of order $O(Z)$. Notice that according to (6.69) and [21, Lem. 5], the coefficients $\mathtt{A}_{jk}$ of the matrix $\mathtt{A}$ can be written

$$\mathtt{A}_{jk}(\varphi)=\partial_{\varphi_k}I_0(\varphi)\cdot\partial_{\varphi_j}\vartheta_0(\varphi)-\partial_{\varphi_k}\vartheta_0(\varphi)\cdot\partial_{\varphi_j}I_0(\varphi)+\langle\partial_\theta^{-1}\partial_{\varphi_k}z_0(\varphi),\partial_{\varphi_j}z_0(\varphi)\rangle_{L^2(\mathbb{T})},\tag{6.88}$$

and satisfy

$$\omega\cdot\partial_\varphi\mathtt{A}_{jk}(\varphi)=\mathcal{W}\big(\partial_\varphi Z(\varphi)\underline{e}_k,\partial_\varphi i_0(\varphi)\underline{e}_j\big)+\mathcal{W}\big(\partial_\varphi i_0(\varphi)\underline{e}_k,\partial_\varphi Z(\varphi)\underline{e}_j\big),\tag{6.89}$$

where $\mathcal{W}$ is the symplectic form defined in (4.24) and $(\underline{e}_1,\ldots,\underline{e}_d)$ denotes the canonical basis of $\mathbb{R}^d$. In order to estimate $\mathtt{A}_{jk}(\varphi)$, we shall discuss the invertibility of the operator $\omega\cdot\partial_\varphi$. This task was accomplished in several papers [7, 21, 33, 87] and we shall outline here the main lines.
Let $\gamma\in(0,1]$ and $\tau_1>0$ be defined as in (A.2). We introduce the Diophantine Cantor set

$$\mathtt{DC}(\gamma,\tau_1)\triangleq\bigcap_{l\in\mathbb{Z}^d\setminus\{0\}}\Big\{\omega\in\mathbb{R}^d\quad\text{s.t.}\quad|\omega\cdot l|>\frac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}$$

and for $N\in\mathbb{N}^*$ we define the truncated Diophantine Cantor set

$$\mathtt{DC}_N(\gamma,\tau_1)\triangleq\bigcap_{\substack{l\in\mathbb{Z}^d\setminus\{0\}\\|l|\leqslant N}}\Big\{\omega\in\mathbb{R}^d\quad\text{s.t.}\quad|\omega\cdot l|>\frac{\gamma}{\langle l\rangle^{\tau_1}}\Big\}.\tag{6.90}$$

Given $f:\mathcal{O}\times\mathbb{T}^{d+1}\to\mathbb{R}$ a smooth function with zero φ-average, that can be expanded in Fourier series as follows

$$f=\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\l\neq0}}f_{l,j}(\lambda,\omega)\mathbf{e}_{l,j},\qquad\mathbf{e}_{l,j}(\varphi,\theta)\triangleq e^{\mathrm{i}(l\cdot\varphi+j\theta)}.$$

If $\omega\in\mathtt{DC}(\gamma,\tau_1)$, then the equation $\omega\cdot\partial_\varphi u=f$ has a periodic solution $u:\mathbb{T}^{d+1}\to\mathbb{R}$ given by

$$u(\lambda,\varphi,\theta)=-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\l\neq0}}\frac{f_{l,j}(\lambda)}{\omega\cdot l}\mathbf{e}_{l,j}(\varphi,\theta).$$

For all $\omega\in\mathcal{O}$, we define the smooth extension of u by

$$(\omega\cdot\partial_\varphi)_{\text{ext}}^{-1}f\triangleq-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\l\neq0}}\frac{\chi\big(\gamma^{-1}\langle l\rangle^{\tau_1}\,\omega\cdot l\big)f_{l,j}(\lambda)}{\omega\cdot l}\mathbf{e}_{l,j}.\tag{6.91}$$

where $\chi\in\mathscr{C}^\infty(\mathbb{R},[0,1])$ is an even positive cut-off function such that

$$\chi(\xi)=\begin{cases}0 & \text{if}\quad|\xi|\leqslant\frac13\\ 1 & \text{if}\quad|\xi|\geqslant\frac12.\end{cases}\tag{6.92}$$

Notice that this operator is well-defned in the whole set of parameters $\mathcal{O}$ and coincides with the formal inverse of $(\omega\cdot\partial_\varphi)^{-1}$ when the frequency ω belongs to $\mathtt{DC}(\gamma,\tau_1)$. The next result is the fundamental theorem of calculus in the quasi-periodic setting.

Lemma 6.5. *Let $\gamma\in(0,1],q\in\mathbb{N}^*$. Then for any $s\geqslant q$ we have*

$$\big\|(\omega\cdot\partial_\varphi)_{\text{ext}}^{-1}f\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|f\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

In addition, for any $N \in \mathbb{N}^*$ *and for any* $\omega \in \mathtt{DC}_N(\gamma,\tau_1)$ *we have*

$$(\omega\cdot\partial_\varphi)(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}\Pi_N = \Pi_N,$$

where Π_N *is the orthogonal projection defined by*

$$\Pi_N \sum_{(l,j)\in\mathbb{Z}^{d+1}} f_{l,j}\mathbf{e}_{l,j} = \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ |l|\leqslant N}} f_{l,j}\mathbf{e}_{l,j}.$$

Proof. The proof of the first point can be done using Faá di Bruno's formula in a similar way to [7, Lem. 2.5]. We also refer to the proof of Proposition 7.2.
By construction, one has for $\omega \in \mathtt{DC}_N(\gamma,\tau_1)$ and $|l| \leqslant N$,

$$\chi\big((\omega\cdot l)\gamma^{-1}\langle l\rangle^{\tau_1}\big) = 1,$$

Thus, according to the explicit extension (6.91),

$$\begin{aligned}(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}\Pi_N h &= -\mathrm{i}\sum_{\substack{l\in\mathbb{Z}^d\setminus\{0\}\\ |l|\leqslant N}} \frac{\chi\big((\omega\cdot l)\gamma^{-1}\langle l\rangle^{\tau_1}\big)h_l(\lambda)}{\omega\cdot l}\mathbf{e}_{l,0}\\ &= -\mathrm{i}\sum_{\substack{l\in\mathbb{Z}^d\setminus\{0\}\\ |l|\leqslant N}} \frac{h_l(\lambda)}{\omega\cdot l}\mathbf{e}_{l,0}.\end{aligned} \tag{6.93}$$

Therefore, we obtain

$$\begin{aligned}(\omega\cdot\partial_\varphi)(\omega\cdot\partial_\varphi)^{-1}_{\text{ext}}\Pi_N h &= \sum_{\substack{l\in\mathbb{Z}^d\setminus\{0\}\\ |l|\leqslant N}} h_l(\lambda)\mathbf{e}_{l,0}\\ &= \Pi_N h.\end{aligned}$$

This concludes the proof of the lemma. □

For later purposes we need to fix some notation that will be adopted in the sequel. Take $N_0 \geqslant 2$ and define the sequence

$$N_{-1} = 1, \quad \forall n\in\mathbb{N}, \quad N_n = N_0^{\left(\frac{3}{2}\right)^n}. \tag{6.94}$$

Next, we shall split the coefficients of the matrix $\mathtt{A} = \mathtt{A}(\varphi)$ defined in (6.69) as

$$\mathtt{A}_{kj} = \mathtt{A}_{kj}^{(n)} + \mathtt{A}_{kj}^{(n),\perp}, \qquad \mathtt{A}_{kj}^{(n)} \triangleq \Pi_{N_n}\mathtt{A}_{kj}, \qquad \mathtt{A}_{kj}^{(n),\perp} \triangleq \Pi_{N_n}^{\perp}\mathtt{A}_{kj}. \tag{6.95}$$

The proof of the following lemma is quite similar to Lemma 5.3. in [11] with the a minor difference in the weighted norms.

Lemma 6.6. *Let* $n \in \mathbb{N}$, *then the following results hold true.*

(i) The function $\mathtt{A}_{kj}^{(n),\perp}$ *satisfies*

$$\forall b\geqslant 0, \quad \forall s\in[s_0,S], \quad \|\mathtt{A}_{kj}^{(n),\perp}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\|\mathfrak{I}_0\|_{q,s+2+b}^{\gamma,\mathcal{O}}.$$

(ii) There exist functions $\mathtt{A}_{kj}^{(n),\text{ext}}$ *defined for any* $(\lambda,\omega)\in\mathcal{O}$, *q-times differentiable with respect to* λ *and*

satisfying the estimate

$$\forall s\in[s_0,S],\quad \|\mathtt{A}_{kj}^{(n),\mathrm{ext}}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\big(\|Z\|^{\gamma,\mathcal{O}}_{s+\tau_1q+\tau_1+1}+\|Z\|^{\gamma,\mathcal{O}}_{q,s_0+1}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\tau_1q+\tau_1+1}\big)\,.$$

Moreover, $\mathtt{A}_{kj}^{(n),\mathrm{ext}}$ *coincides with* $\mathtt{A}_{kj}^{(n)}$ *in the Cantor set* $\mathtt{DC}_{N_n}(\gamma,\tau_1)$.

Proof. **(i)** Follows immediately from (6.88), (6.95) and Lemma A.1-(ii).
(ii) Applying the projector to the identity (6.89) we obtain

$$\omega\cdot\partial_\varphi\mathtt{A}_{jk}^{(n)}(\varphi)=\Pi_{N_n}\big[\mathcal{W}\big(\partial_\varphi Z(\varphi)\underline{e}_k,\partial_\varphi i_0(\varphi)\underline{e}_j\big)+\mathcal{W}\big(\partial_\varphi i_0(\varphi)\underline{e}_k,\partial_\varphi Z(\varphi)\underline{e}_j\big)\big].$$

Then, by (6.56) and Lemma A.1-(ii)-(iv), we get

$$\big\|\Pi_{N_n}\big[\mathcal{W}\big(\partial_\varphi Z(\varphi)\underline{e}_k,\partial_\varphi i_0(\varphi)\underline{e}_j\big)+\mathcal{W}\big(\partial_\varphi i_0(\varphi)\underline{e}_k,\partial_\varphi Z(\varphi)\underline{e}_j\big)\big]\big\|_s^{q,\kappa}\lesssim_s\big(\|Z\|^{q,\kappa}_{s+1}+\|Z\|^{q,\kappa}_{s_0+1}\|\mathfrak{I}_0\|^{q,\kappa}_{s+1}\big)\,.$$

We define the the function $\mathtt{A}_{kj}^{(n),\mathrm{ext}}$ as

$$\mathtt{A}_{kj}^{(n),\mathrm{ext}}(\varphi)\triangleq(\omega\cdot\partial_\varphi)^{-1}_{\mathrm{ext}}\Pi_{N_n}\big[\mathcal{W}\big(\partial_\varphi Z(\varphi)\underline{e}_k,\partial_\varphi i_0(\varphi)\underline{e}_j\big)+\mathcal{W}\big(\partial_\varphi i_0(\varphi)\underline{e}_k,\partial_\varphi Z(\varphi)\underline{e}_j\big)\big].$$

Applying Lemma 6.5 concludes the proof of the Lemma.

□

6.3.2 Construction of the approximate inverse

This section is devoted to the construction of an approximate right inverse of the operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ that will be discussed in Theorem 6.1. One first may observe according to Proposition 6.1-(ii) and Lemmata 6.4 and 6.6, that the operator $\mathbb{E}$ vanishes at an exact solution up to fast decaying remainder terms. As a consequence, getting an approximate inverse for the full operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ amounts simply to invert the operator $\mathbb{D}$ up to small errors of type "Z" mixed with fast frequency decaying error. Let us consider the triangular system given by

$$\mathbb{D}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]=\begin{pmatrix}g_1\\ g_2\\ g_3\end{pmatrix},\tag{6.96}$$

where $\mathbb{D}$ is defined in Proposition 6.1-(i). The system (6.96) writes more explicitly in the following way

$$\begin{cases}\omega\cdot\partial_\varphi\widehat{\phi}=g_1+[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}+L_1^\top(\varphi)\widehat{\alpha}]\\ \omega\cdot\partial_\varphi\widehat{y}=g_2-\mathtt{B}(\varphi)\widehat{\alpha}\\ \big(\omega\cdot\partial_\varphi-\partial_\theta\mathsf{K}_{02}(\varphi)\big)\widehat{w}=g_3+\partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y}+L_2^\top(\varphi)\widehat{\alpha}].\end{cases}\tag{6.97}$$

The strategy to solve the above system in the variables $(\widehat{\phi},\widehat{y},\widehat{w})$ is first to solve the second action-component equation, then to solve the third normal-component equation and finally to solve the first angle-component equation. Due to the fact that the Cantor set should be truncated then we need to solve approximately the system (6.97) and for this aim we need the following statement.

Lemma 6.7. *The following results hold true.*

(i) There exists a function $\mathsf{g}:\mathbb{Z}^d\setminus\{0\}\to\{-1,1\}$ *such that*

$$\forall l\in\mathbb{Z}^d\setminus\{0\},\quad \mathsf{g}(-l)=-\mathsf{g}(l).$$

(ii) For all $(\lambda,\omega)\in\mathcal{O}$ *the operator* $\omega\cdot\partial_\varphi$ *can be split as follows*

$$\omega\cdot\partial_\varphi=\mathcal{D}_{(n)}+\mathcal{D}_{(n)}^{\perp},$$

with

$$\mathcal{D}_{(n)}\triangleq\omega\cdot\partial_\varphi\,\Pi_{N_n}+\Pi_{N_n,\mathtt{g}}^{\perp}\tag{6.98}$$

$$\mathcal{D}_{(n)}^{\perp}\triangleq\omega\cdot\partial_\varphi\,\Pi_{N_n}^{\perp}-\Pi_{N_n,\mathtt{g}}^{\perp}\,,\tag{6.99}$$

where

$$\Pi_{N_n,\mathtt{g}}^{\perp}\sum_{(l,j)\in\mathbb{Z}^{d+1}}f_{l,j}\mathbf{e}_{l,j}\triangleq\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\|l|>N_n}}\mathtt{g}(l)f_{l,j}\mathbf{e}_{l,j}.$$

(iii) The operator $\mathcal{D}_{(n)}^{\perp}$ *satisfies*

$$\forall b\geqslant0,\quad\forall s\in[s_0,S],\quad\|\mathcal{D}_{(n)}^{\perp}h\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N_n^{-b}\|h\|_{q,s+b+1}^{\gamma,\mathcal{O}}.$$

(iv) There exists a family of linear operators $\big([\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}\big)_n$ *satisfying, for any* $h\in W^{q,\infty,\gamma}(\mathcal{O},H_0^s(\mathbb{T}^{d+1}))$,

$$\forall s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}h\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|h\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

Moreover, for all $\omega\in\mathtt{DC}_{N_n}(\gamma,\tau_1)$ *one has the identity*

$$\mathcal{D}_{(n)}[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}=\textnormal{Id}.\tag{6.100}$$

Proof. **(i)** The function $\mathtt{g}:\mathbb{Z}^d\setminus\{0\}\to\{-1,1\}$ is defined, for all $l=(l_1,\cdots,l_d)\in\mathbb{Z}^d\setminus\{0\}$, as the sign of the first non-zero component in the vector l. Thus, it satisfies

$$\forall l\in\mathbb{Z}^d\setminus\{0\},\quad\mathtt{g}(-l)=-\mathtt{g}(l).$$

(ii) Immediate.
(iii) Follows immediately from Lemma A.1-(ii).
(iv) We define the operator $[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}$ as

$$[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}\triangleq(\omega\cdot\partial_\varphi)_{\textnormal{ext}}^{-1}\Pi_{N_n}+\Pi_{N_n,\frac{1}{\mathtt{g}}}^{\perp}.\tag{6.101}$$

From (6.93), (6.98), (6.99) and (6.101) we get, for all $\omega\in\mathtt{DC}_{N_n}(\gamma,\tau_1)$,

$$\begin{aligned}\mathcal{D}_{(n)}[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}&=\omega\cdot\partial_\varphi\,\Pi_{N_n}\big[(\omega\cdot\partial_\varphi)_{\textnormal{ext}}^{-1}\Pi_{N_n}+\Pi_{N_n,\frac{1}{\mathtt{g}}}^{\perp}\big]+\Pi_{N_n,\mathtt{g}}^{\perp}\big[(\omega\cdot\partial_\varphi)_{\textnormal{ext}}^{-1}\Pi_{N_n}+\Pi_{N_n,\frac{1}{\mathtt{g}}}^{\perp}\big]\\&=\omega\cdot\partial_\varphi\,(\omega\cdot\partial_\varphi)_{\textnormal{ext}}^{-1}\Pi_{N_n}+\Pi_{N_n}^{\perp}.\end{aligned}$$

Applying Lemma 6.5-(ii) we conclude that

$$\mathcal{D}_{(n)}[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}=\Pi_{N_n}+\Pi_{N_n}^{\perp}=\textnormal{Id}.$$

The estimate on $[\mathcal{D}_{(n)}]_{\textnormal{ext}}^{-1}$ follows from (6.101), Lemma A.1-(ii) and Lemma 6.5. □

Consider the linearized operator restricted to the normal directions $\widehat{\mathcal{L}}_\omega$ and defined by

$$\widehat{\mathcal{L}}_\omega \triangleq \Pi_{\mathbb{S}_0}^\perp\big(\omega\cdot\partial_\varphi - \partial_\theta \mathsf{K}_{02}(\varphi)\big)\Pi_{\mathbb{S}_0}^\perp, \tag{6.102}$$

which appears in the last equation of (6.97). The construction of an approximate right inverse of this operator is the heart part of this work and will be discussed in Proposition 7.6. Here we give only a partial statement.

Proposition 6.2. *Let* $(\gamma, q, d, \tau_1, \tau_2, s_0, s_h, \mu_2, S)$ *satisfy* (A.2) (A.1) *and* (7.235). *There exist* $\varepsilon_0 > 0$ *and* $\sigma = \sigma(\tau_1, \tau_2, q, d) > 0$ *such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}} \leqslant 1, \tag{6.103}$$

then there exist a family of linear operator $(\mathtt{T}_{\omega,n})_n$ *satisfying*

$$\forall\, s\in\,[s_0,S],\quad \sup_{n\in\mathbb{N}}\|\mathtt{T}_{\omega,n}h\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|h\|_{q,s+\sigma}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|h\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right) \tag{6.104}$$

and a family of Cantor sets $\{\mathcal{G}_n = \mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)\}_n$, *satisfying the inclusion*

$$\mathcal{G}_n \subset (\lambda_0,\lambda_1)\times \mathtt{DC}_{N_n}(\gamma,\tau_1)$$

such that in each set $\mathcal{G}_n$ *we have the splitting*

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n,$$

with

$$\widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n} = \mathrm{Id}, \tag{6.105}$$

where the operators $\widehat{\mathtt{L}}_{\omega,n}$ *and* $\widehat{\mathtt{R}}_n$ *are defined in the whole set* $\mathcal{O}$ *with the estimates*

$$\begin{aligned}
\forall\, s\in[s_0,S],\quad & \|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}},\\
\forall\, b\in[0,S],\quad & \|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b}\gamma^{-1}\Big(\|\rho\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\
&\qquad + \varepsilon\overline{\gamma}^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.
\end{aligned}$$

For the splitting below which follows from the foregoing results we refer to (6.45) in [87]. Consider the linear operator $\mathbb{L}_{\mathrm{ext}}$ defined by

$$\mathbb{L}_{\mathrm{ext}} = \mathbb{D}_n + \mathbb{E}_n^{\mathrm{ext}} + \mathscr{P}_n + \mathscr{Q}_n, \tag{6.106}$$

where, for any $(\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha})\in\mathbb{T}^d\times\mathbb{R}^d\times L^2_\perp\times\mathbb{R}^d$

$$\mathbb{D}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]\triangleq\begin{pmatrix}\mathcal{D}_{(n)}\widehat{\phi}-\mathsf{K}_{20}(\varphi)\widehat{y}-\mathsf{K}_{11}^\top(\varphi)\widehat{w}-L_1^\top(\varphi)\widehat{\alpha}\\ \mathcal{D}_{(n)}\widehat{y}+\mathsf{B}(\varphi)\widehat{\alpha}\\ \widehat{\mathtt{L}}_{\omega,n}\widehat{w}-\partial_\theta\big[\mathsf{K}_{11}(\varphi)\widehat{y}+L_2^\top(\varphi)\widehat{\alpha}\big]\end{pmatrix},\tag{6.107}$$

$$\begin{aligned}\mathbb{E}_n^{\mathrm{ext}}[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]&\triangleq[DG_0(\mathtt{u}_0(\varphi))]^{-1}\partial_\varphi Z(\varphi)\widehat{\phi}-\begin{pmatrix}0\\ R_{10}(\varphi)\widehat{y}+R_{01}(\varphi)\widehat{w}\\ 0\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ \mathtt{A}^{(n),\mathrm{ext}}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}\big]\\ 0\end{pmatrix},\end{aligned}\tag{6.108}$$

$$\mathscr{P}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]\triangleq\begin{pmatrix}\mathcal{D}_{(n)}^\perp\widehat{\phi}\\ \mathcal{D}_{(n)}^\perp\widehat{y}+\mathtt{A}^{(n),\perp}(\varphi)\big[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^\top(\varphi)\widehat{w}\big]\\ 0\end{pmatrix},\tag{6.109}$$

$$\mathscr{Q}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]\triangleq\begin{pmatrix}0\\ 0\\ \widehat{\mathtt{R}}_n[\widehat{w}]\end{pmatrix}.\tag{6.110}$$

Then, the operator $\mathbb{L}_{\mathrm{ext}}$ is defined on the whole set $\mathcal{O}$ and when it is restricted to the Cantor set $\mathcal{G}_n$ it coincides with the conjugated linearized operator obtained in (6.64), that is,

$$\mathbb{L}_{\mathrm{ext}}=[DG_0(\mathtt{u}_0)]^{-1}d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)D\widetilde{G}_0(\mathtt{u}_0)\quad\text{in }\mathcal{G}_n.\tag{6.111}$$

In the next result, we give some useful estimates for the different terms appearing in $\mathbb{L}_{\mathrm{ext}}$ needed to obtain good tame estimates for the approximate inverse.

Proposition 6.3. *Let $(\gamma,q,d,\tau_1,s_0,\mu_2)$ satisfy* (A.2) *and* (7.235) *and assume the conditions* (6.56) *and* (6.103)*. Then, denoting $\widehat{\mathtt{v}}=(\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha})$, the following assertions hold true.*

(i) The operator $\mathbb{E}_n^{\mathrm{ext}}$ satisfies the estimate

$$\|\mathbb{E}_n^{\mathrm{ext}}[\widehat{\mathtt{v}}]\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\|Z\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\|\widehat{\mathtt{v}}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.$$

(ii) The operator $\mathscr{P}^{(n)}$ satisfies the estimate

$$\forall b\geqslant 0,\quad\|\mathscr{P}_n[\widehat{\mathtt{v}}]\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim N_n^{-b}\big(\|\widehat{\mathtt{v}}\|_{q,s_0+\sigma+b}^{\gamma,\mathcal{O}}+\varepsilon\|\mathfrak{I}_0\|_{q,s_0+\sigma+b}^{\gamma,\mathcal{O}}\|\widehat{\mathtt{v}}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\big).$$

(iii) The operator $\mathscr{Q}_n$ satisfies the estimate

$$\begin{aligned}\forall\,b\in[0,S],\quad\|\mathscr{Q}_n\widehat{\mathtt{v}}\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{-b}\gamma^{-1}\Big(\|\widehat{w}\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\sigma}^{\gamma,\mathcal{O}}\|\widehat{w}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\ &\quad+\varepsilon\overline{\gamma}^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\widehat{w}\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

(iv) There exists a family of operators $\big([\mathbb{D}_n]_{\mathrm{ext}}^{-1}\big)_n$ such that for all $g\triangleq(g_1,g_2,g_3)$ satisfying the reversibility property

$$g_1(\varphi)=g_1(-\varphi)\,,\quad g_2(\varphi)=-g_2(-\varphi)\,,\quad g_3(\varphi)=-(\mathscr{S}g_3)(-\varphi)\,,$$

the function $[\mathbb{D}_n]_{\mathrm{ext}}^{-1}g$ satisfies the estimate

$$\forall\,s\in[s_0,S],\quad\|[\mathbb{D}_n]_{\mathrm{ext}}^{-1}g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\big(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\big)$$

and for all $(\lambda,\omega)\in\mathcal{G}_n$ *one has*

$$\mathbb{D}_n\,[\mathbb{D}_n]_{\rm ext}^{-1}=\mathrm{Id}.$$

Proof. **(i)** The estimate of $\mathbb{E}_n^{\rm ext}$ is obtained from (6.108), Lemma 6.4, Lemma A.1-(iv) and Lemma 6.6-(ii).
(ii) From (6.109), Lemma 6.7-(iii), Lemma A.1-(iv), Lemma 6.6-(i), Lemma 6.4-(ii) we obtain the estimate on $\mathscr{P}_n$.
(iii) It is a consequence of (6.110) and Proposition 6.2.
(iv) The proof can be found in [87, Prop. 6.3] and for the sake of completeness we shall sketch the main ideas. We intend to look for an exact inverse of $\mathbb{D}_n$ by solving the system

$$\mathbb{D}_n[\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha}]=\begin{pmatrix}g_1\\ g_2\\ g_3\end{pmatrix},\tag{6.112}$$

where (g_1,g_2,g_3) satisfy the reversibility property

$$g_1(\varphi)=g_1(-\varphi)\,,\quad g_2(\varphi)=-g_2(-\varphi)\,,\quad g_3(\varphi)=-(\mathscr{S}g_3)(-\varphi),\tag{6.113}$$

with $\mathscr{S}$ being the involution defined in (4.27). Note that in view of (6.107), the system (6.112) writes

$$\begin{cases}\mathcal{D}_{(n)}\widehat{\phi}=g_1+[\mathsf{K}_{20}(\varphi)\widehat{y}+\mathsf{K}_{11}^{\top}(\varphi)\widehat{w}+L_1^{\top}(\varphi)\widehat{\alpha}]\\ \mathcal{D}_{(n)}\widehat{y}=g_2-\mathtt{B}(\varphi)\widehat{\alpha}\\ \widehat{\mathtt{L}}_{\omega,n}\widehat{w}=g_3+\partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y}+L_2^{\top}(\varphi)\widehat{\alpha}].\end{cases}\tag{6.114}$$

We first consider the second action-component equation in (6.114), namely

$$\mathcal{D}_{(n)}\widehat{y}=g_2-\mathtt{B}(\varphi)\widehat{\alpha}.$$

In view of (6.113), (6.65) and (6.88), g_2 and $\mathtt{B}$ are odd in the variable φ. Thus, the φ-average of the right hand side of this equation is zero. Then, by Lemma 6.7-(iv) its solution in the Cantor set $\mathtt{DC}_{N_n}(\gamma,\tau_1)$ is given by

$$\widehat{y}\triangleq[\mathcal{D}_{(n)}]_{\rm ext}^{-1}\big(g_2-\mathtt{B}(\varphi)\widehat{\alpha}\big)\,.\tag{6.115}$$

Then we turn to the third normal-component equation in (6.114), namely

$$\widehat{\mathtt{L}}_{\omega,n}\widehat{w}=g_3+\partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y}+L_2^{\top}(\varphi)\widehat{\alpha}].$$

By Proposition 6.2, this equation admits as a solution

$$\widehat{w}\triangleq\mathtt{T}_{\omega,n}\big(g_3+\partial_\theta[\mathsf{K}_{11}(\varphi)\widehat{y}+L_2^{\top}(\varphi)\widehat{\alpha}]\big)\,.\tag{6.116}$$

Finally, we solve the first angle-equation in (6.114), which, substituting (6.115), (6.116), becomes

$$\mathcal{D}_{(n)}\widehat{\phi}=g_1+M_1(\varphi)\widehat{\alpha}+M_2(\varphi)g_2+M_3(\varphi)g_3\,,\tag{6.117}$$

where

$$M_1(\varphi)\triangleq L_1^{\top}(\varphi)-M_2(\varphi)\mathtt{B}(\varphi)+M_3(\varphi)\partial_\theta L_2^{\top}(\varphi)\,,\tag{6.118}$$

$$M_2(\varphi)\triangleq\mathsf{K}_{20}(\varphi)[\mathcal{D}_{(n)}]_{\rm ext}^{-1}+\mathsf{K}_{11}^{\top}(\varphi)\mathtt{T}_{\omega,n}\partial_\vartheta\mathsf{K}_{11}(\varphi)[\mathcal{D}_{(n)}]_{\rm ext}^{-1}\,,\tag{6.119}$$

$$M_3(\varphi)\triangleq\mathsf{K}_{11}^{\top}(\varphi)\mathtt{T}_{\omega,n}\,.\tag{6.120}$$

To solve the equation (6.117) we choose $\widehat{\alpha}$ such that the right hand side has zero φ-average. Notice that Lemma 6.4, (6.56), (6.104) and Lemma 6.7-(ii) imply

$$\forall s\in[s_0,S],\quad \|M_2g_2\|_{q,s}^{\gamma,\mathcal{O}}+\|M_3g_3\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).\tag{6.121}$$

By Lemma 6.4-(iii), (6.56), the ϕ-averaged matrix is $\langle M_1\rangle=\mathrm{Id}+O(\varepsilon\gamma^{-1})$. Therefore, for $\varepsilon\gamma^{-1}$ small enough, $\langle M_1\rangle$ is invertible and $\langle M_1\rangle^{-1}=\mathrm{Id}+O(\varepsilon\gamma^{-1})$. We thus define

$$\widehat{\alpha}\triangleq-\langle M_1\rangle^{-1}(\langle g_1\rangle+\langle M_2g_2\rangle+\langle M_3g_3\rangle).\tag{6.122}$$

Remark that $\widehat{\alpha}$ satisfies

$$\|\widehat{\alpha}\|_{q}^{\gamma,\mathcal{O}}\lesssim\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\tag{6.123}$$

Coming back to (6.115) and using (6.123), (6.56) together with Lemma 6.7-(iv) and Lemma 6.4-(iv), we obtain

$$\forall s\in[s_0,S],\quad \|\widehat{y}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).\tag{6.124}$$

Putting together (6.116), (6.104), Lemma 6.4-(iii), (6.123), (6.124) and (6.56), one should get, up to redefining the value of σ,

$$\forall s\in[s_0,S],\quad \|\widehat{w}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).\tag{6.125}$$

With the choice (6.122) of $\widehat{\alpha}$, the equation (6.117) admits as a solution

$$\widehat{\phi}\triangleq[\mathcal{D}_{(n)}]_{\mathrm{ext}}^{-1}\big(g_1+M_1(\varphi)\widehat{\alpha}+M_2(\varphi)g_2+M_3(\varphi)g_3\big).\tag{6.126}$$

Putting together (6.126), Lemma 6.7-(ii), (6.123) and (6.121), one obtains

$$\forall s\in[s_0,S],\quad \|\widehat{\phi}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).\tag{6.127}$$

In conclusion, we have obtained a solution $(\widehat{\phi},\widehat{y},\widehat{w},\widehat{\alpha})\triangleq[\mathbb{D}_n]_{\mathrm{ext}}^{-1}g$ of the linear system (6.112) satisfying in virtue of (6.123), (6.127), (6.125) and (6.124),

$$\forall s\in[s_0,S],\quad \|[\mathbb{D}_n]_{\mathrm{ext}}^{-1}g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|g\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right).$$

Notice that the relation

$$\mathbb{D}_n[\mathbb{D}_n]_{\mathrm{ext}}^{-1}=\mathrm{Id}\quad\text{in }\mathcal{G}_n$$

is a direct consequence of (6.100) and (6.105). □

The last point is to prove that the operator

$$\mathrm{T}_0\triangleq\mathrm{T}_0(i_0)\triangleq(D\widetilde{G}_0)(\mathrm{u}_0)\circ[\mathbb{D}_n]_{\mathrm{ext}}^{-1}\circ(DG_0)(\mathrm{u}_0)^{-1}\tag{6.128}$$

is an approximate right inverse for $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$.

Theorem 6.1 (Approximate inverse). *Let $(\gamma,q,d,\tau_1,\tau_2,s_0,s_h,\mu_2,S)$ satisfy* (A.2), (A.1), (7.235) *and* (7.3). *Then there exists $\overline{\sigma}=\overline{\sigma}(\tau_1,\tau_2,d,q)>0$ such that if the smallness conditions* (6.56) *and* (6.103) *hold, then the operator T_0 defined in* (6.128) *is reversible and satisfies for all $g=(g_1,g_2,g_3)$, with* (6.113),

$$\forall s\in[s_0,S],\quad \|\mathrm{T}_0g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|g\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right).\tag{6.129}$$

Moreover T_0 *is an almost-approximate right inverse of* $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ *in the Cantor set* $\mathcal{G}_n$*. More precisely, for all* $(\lambda,\omega)\in\mathcal{G}_n$ *one has*

$$d_{i,\alpha}\mathcal{F}(i_0)\circ\mathrm{T}_0-\mathrm{Id}=\mathcal{E}_1^{(n)}+\mathcal{E}_2^{(n)}+\mathcal{E}_3^{(n)}, \tag{6.130}$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ *and* $\mathcal{E}_3^{(n)}$ *are defined in the set* $\mathcal{O}$ *with the estimates*

$$\|\mathcal{E}_1^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\mathcal{F}(i_0,\alpha_0)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}, \tag{6.131}$$

$$\forall b\geqslant 0,\quad \|\mathcal{E}_2^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}N_n^{-b}\big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}+\varepsilon\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\big)\,, \tag{6.132}$$

$$\begin{aligned}\forall b\in[0,S],\quad \|\mathcal{E}_3^{(n)}g\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{-b}\gamma^{-2}\Big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\\&\quad+\varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.\end{aligned} \tag{6.133}$$

Proof. The estimate (6.129) is a consequence of (6.128), Proposition 6.3-(iv) and Lemma 6.4-(i). Then, according to (6.106) and (6.111), in the Cantor set $\mathcal{G}_n$ we have the decomposition

$$\begin{aligned}d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)&=DG_0(\mathrm{u}_0)\circ\mathbb{L}_{\mathrm{ext}}\circ D[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\\&=DG_0(\mathrm{u}_0)\circ\mathbb{D}_n\circ D[\widetilde{G}_0(\mathrm{u}_0)]^{-1}+DG_0(\mathrm{u}_0)\circ\mathbb{E}_n^{\mathrm{ext}}\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\\&\quad+DG_0(\mathrm{u}_0)\circ\mathscr{P}_n\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}+DG_0(\mathrm{u}_0)\circ\mathscr{Q}_n\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}.\end{aligned}$$

By applying T_0, defined in (6.128), to the last identity we get for all $(\lambda,\omega)\in\mathcal{G}_n$

$$d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)\circ\mathrm{T}_0-\mathrm{Id}=\mathcal{E}_1^{(n)}+\mathcal{E}_2^{(n)}+\mathcal{E}_3^{(n)},$$

with

$$\begin{aligned}\mathcal{E}_1^{(n)}&\triangleq DG_0(\mathrm{u}_0)\circ\mathbb{E}_n^{\mathrm{ext}}\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\circ\mathrm{T}_0,\\\mathcal{E}_2^{(n)}&\triangleq DG_0(\mathrm{u}_0)\circ\mathscr{P}_n\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\circ\mathrm{T}_0,\\\mathcal{E}_3^{(n)}&\triangleq DG_0(\mathrm{u}_0)\circ\mathscr{Q}_n\circ[\widetilde{G}_0(\mathrm{u}_0)]^{-1}\circ\mathrm{T}_0.\end{aligned}$$

The estimates on $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ come from (6.129), Proposition 6.3 and Lemma 6.4-(i).

□

7 Reduction of the linearized operator in the normal directions

In this section, we fix a torus $i_0=(\vartheta_0,I_0,z_0)$ close to the flat one and satisfying the reversibility condition (6.23), that is

$$\vartheta_0(-\varphi)=-\vartheta_0(\varphi),\quad I_0(-\varphi)=I_0(\varphi),\quad z_0(-\varphi)=(\mathscr{S}z_0)(\varphi). \tag{7.1}$$

As in the previous section, we denote $\mathfrak{I}_0(\varphi)=i_0(\varphi)-(\varphi,0,0)$. Our main goal here is to explore the invertibility of the operator

$$\widehat{\mathcal{L}}_\omega=\widehat{\mathcal{L}}_\omega(i_0)=\Pi_{\mathbb{S}_0}^{\perp}\left(\omega\cdot\partial_\varphi-\partial_\theta\mathsf{K}_{02}(\varphi)\right)\Pi_{\mathbb{S}_0}^{\perp} \tag{7.2}$$

defined through (6.102) and (6.67) with the suitable tame estimates for the inverse. For a precise statement we refer to Proposition 7.6. Notice that this operator will be described as a quasilinear perturbation of the diagonal operator stated in Lemma 5.1 and we expect that suitable standard reductions can be performed to conjugate it to a diagonal one provided that the exterior parameters are subject to live in a Cantor set allowing to prevent resonances. For this aim, we shall implement with suitable adaptions the strategy developed in the works [7, 33]. We distinguish two long reduction steps. First, we perform a quasi-periodic

change of variables such that in the new coordinates system the transport part is straightened to a constant coefficient operator. The construction of this transformation is based on a KAM reducibility procedure as in [64]. The outcome of this first step is a new operator whose positive part is diagonal with a small nonlocal perturbation of order -1. Then the second step consists in applying KAM scheme in order to reduce the remainder and conjugate the resulting operator from step 1 into a diagonal one up to small errors. The proof follows basically a common procedure that can be found for instance in [17]. We point out that our results differ slightly from the preceding ones in [7, 33], especially at the level of Cantor sets which are constructed over the final targets.
We shall use throughout the proofs some frequency cut-offs with respect to the sequence defined in (6.94), with N_0 a constant needed to be large enough. In the current section, the numbers $N_0 \geqslant 2$ and $\gamma \in (0,1)$ are a priori free parameters, but during the Nash-Moser scheme, see Proposition 8.1, they will be adjusted with respect to ε according to the relations

$$N_0 = \gamma^{-1} \quad \text{and} \quad \gamma = \varepsilon^a \quad \text{for some } a > 0.$$

We shall set the following parameters required along the different reductions that we intend to perform,

$$\begin{aligned} s_l &\triangleq s_0 + \tau_1 q + \tau_1 + 2, & \overline{\mu}_2 &\triangleq 4\tau_1 q + 6\tau_1 + 3, \\ \overline{s}_l &\triangleq s_l + \tau_2 q + \tau_2, & \overline{s}_h &\triangleq \tfrac{3}{2}\overline{\mu}_2 + s_l + 1, \end{aligned} \tag{7.3}$$

supplemented with the assumptions (A.2) and (A.1).

7.1 Localization on the normal directions

According to Theorem 6.1, the construction of an approximate inverse for $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ is based on Proposition 6.2 dealing with finding an approximate right inverse for the operator $\widehat{\mathcal{L}}_\omega$. This program will be achieved along several steps and in the first one we shall describe its asymptotic structure around the linearized operator at the equilibrium state described in Lemma 5.1. More precisely, we shall prove the following result.

Proposition 7.1. *Let (γ, q, d, s_0) satisfy* (A.2)*. Then the operator $\widehat{\mathcal{L}}_\omega$ defined in* (7.2) *takes the form*

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^{\perp}\left(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\right)\Pi_{\mathbb{S}_0}^{\perp}, \quad \mathcal{L}_{\varepsilon r} = \omega\cdot\partial_\varphi + \partial_\theta\left(V_{\varepsilon r}\cdot\right) - \partial_\theta\mathbf{L}_{\varepsilon r},$$

where $V_{\varepsilon r}$ and $\mathbf{L}_{\varepsilon r}$ are defined in Lemma 5.1*, and from* (6.12) *we have*

$$\begin{aligned} r(\varphi) &= A\big(\vartheta_0(\varphi),\, I_0(\varphi), z_0(\varphi)\big) \\ &= v\big(\vartheta_0(\varphi), I_0(\varphi)\big) + z_0(\varphi), \end{aligned}$$

supplemented with the reversibility assumption

$$r(\lambda,\omega,-\varphi,-\theta) = r(\lambda,\omega,\varphi,\theta). \tag{7.4}$$

*Moreover, $\mathcal{R}$ is an integral operator in the sense of the Definition A.*3*, whose kernel J satisfies the symmetry property*

$$J(\lambda,\omega,-\varphi,-\theta,-\eta) = J(\lambda,\omega,\varphi,\theta,\eta). \tag{7.5}$$

and under the assumption

$$\|\mathfrak{I}_0\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant 1, \tag{7.6}$$

we have for all $s \geqslant s_0$,

(i) The function r satisfies the estimates,

$$\|r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s}^{\gamma,\mathcal{O}} \tag{7.7}$$

and

$$\|\Delta_{12}r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}i\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}}. \tag{7.8}$$

(ii) The kernel J satisfies the following estimates for all $\ell\in\mathbb{N}$,

$$\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^\ell J)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s+3+\ell}^{\gamma,\mathcal{O}} \tag{7.9}$$

and

$$\sup_{\eta\in\mathbb{T}}\|\Delta_{12}(\partial_\theta^\ell J)(*,\cdot,\centerdot,\eta+\centerdot)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}i\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0+3}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}. \tag{7.10}$$

Here $*,\cdot,\centerdot$ stand for $(\lambda,\omega),\varphi,\theta$, respectively and $\mathfrak{I}_\ell(\varphi)=i_\ell(\varphi)-(\varphi,0,0)$. In addition, for any function f, $\Delta_{12}f\triangleq f(i_1)-f(i_2)$ refers for the difference of f taken at two different states i_1 and i_2 satisfying (7.6).

Proof. To alleviate the notation we shall at several stages of the proof remove the dependence of the involved functions/operators with respect to (λ,ω) and keep it when we deem it relevant. Recall that the operator $\widehat{\mathcal{L}}_\omega$ is defined in (7.2). To describe $\mathsf{K}_{02}(\varphi)$ we follow [7, 33]. First, we observe from (6.67) and (6.20) that

$$\mathsf{K}_{02}(\varphi)=\mathrm{L}(\lambda)+\varepsilon\partial_w\nabla_w(\mathcal{P}_\varepsilon(i_0(\varphi))+\varepsilon\mathcal{R}(\varphi),$$

with

$$\mathcal{R}(\varphi)=\mathcal{R}_1(\varphi)+\mathcal{R}_2(\varphi)+\mathcal{R}_3(\varphi),$$

where

$$\begin{aligned}\mathcal{R}_1(\varphi)&\triangleq L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi),\\ \mathcal{R}_2(\varphi)&\triangleq L_2^\top(\varphi)\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)&\triangleq\partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi).\end{aligned}$$

As we shall see, all the operators $\mathcal{R}_1(\varphi)$, $\mathcal{R}_2(\varphi)$ and $\mathcal{R}_3(\varphi)$ have a finite-dimensional rank. This property is obvious for the operator $L_2(\varphi)$ defined in (6.60), which sends in view of (6.57) the space $L^2_\perp$ to $\mathbb{R}^d$ and therefore for any $\rho\in L^2_\perp$ we write

$$L_2(\varphi)[\rho]=\sum_{k=1}^d\big\langle L_2(\varphi)[\rho],\underline{e}_k\big\rangle_{\mathbb{R}^d}\underline{e}_k=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}\underline{e}_k,$$

with $(\underline{e}_k)_{k=1}^d$ being the canonical basis of $\mathbb{R}^d$. Hence

$$\begin{aligned}\mathcal{R}_1(\varphi)[\rho]&=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_1(\varphi)[\underline{e}_k]\qquad\text{with}\qquad A_1(\varphi)=L_2^\top(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)[\rho]&=\sum_{k=1}^d\big\langle\rho,L_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_3(\varphi)[\underline{e}_k]\qquad\text{with}\qquad A_3(\varphi)=\partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi)).\end{aligned}$$

In a similar way, by setting $A_2(\varphi) \triangleq \partial_z \nabla_I \mathcal{P}_\varepsilon(i_0(\varphi)) : L^2_\perp \to \mathbb{R}^d$, then we may write

$$\mathcal{R}_2(\varphi)[\rho] = \sum_{k=1}^{d} \big\langle \rho, A_2^\top(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})} L_2^\top(\varphi)[\underline{e}_k].$$

Define

$$g_{k,1}(\varphi,\theta) = g_{k,3}(\varphi,\theta) = \chi_{k,2}(\varphi,\theta) \triangleq L_2^\top(\varphi)[\underline{e}_k](\theta), \quad g_{k,2}(\varphi,\theta) \triangleq A_2^\top(\varphi)[\underline{e}_k](\theta)$$

and

$$\chi_{k,1}(\varphi,\theta) \triangleq A_1(\varphi)[\underline{e}_k](\theta), \quad \chi_{k,3}(\varphi,\theta) \triangleq A_3(\varphi)[\underline{e}_k](\theta),$$

then we can see that the operator $\mathcal{R}$ takes the integral form

$$\begin{aligned}\mathcal{R}\rho(\varphi,\theta) &= \sum_{k'=1}^{3}\sum_{k=1}^{d} \langle \rho(\varphi,\cdot), g_{k,k'}(\varphi,\cdot)\rangle_{L^2(\mathbb{T})} \chi_{k,k'}(\varphi,\theta)\\ &= \int_{\mathbb{T}} \rho(\varphi,\eta) J(\varphi,\theta,\eta)\, d\eta,\end{aligned}$$

with

$$J(\varphi,\theta,\eta) \triangleq \sum_{k'=1}^{3}\sum_{k=1}^{d} g_{k,k'}(\varphi,\eta)\chi_{k,k'}(\varphi,\theta).$$

Now we remark that by construction $g_{k,k'}, \chi_{k,k'} \in L^2_\perp$ with

$$\|g_{k,k'}\|_{q,s}^{\gamma,\mathcal{O}} + \|\chi_{k,k'}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}_0\|_{q,s+3}^{\gamma,\mathcal{O}} \tag{7.11}$$

and straightforward computations yield

$$\|d_i g_{k,k'}[\widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}} + \|d_i \chi_{k,k'}[\widehat{i}]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\widehat{i}\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+4}^{\gamma,\mathcal{O}} \|\widehat{i}\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{7.12}$$

On the other hand, one has from direct computations that

$$\forall \ell \in \mathbb{N}, \quad (\partial_\theta^\ell J)(\varphi,\theta,\eta+\theta) = \sum_{k'=1}^{3}\sum_{k=1}^{d} g_{k,k'}(\varphi,\eta+\theta)(\partial_\theta^\ell \chi_{k,k'})(\varphi,\theta).$$

Hence, we may combine (7.11) with Lemma A.1-(iv) and (7.6) allowing to get

$$\begin{aligned}\sup_{\eta\in\mathbb{T}} \|(\partial_\theta^\ell J)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \sum_{k'=1}^{3}\sum_{k=1}^{d} \|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}} \|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\ &\quad + \sum_{k'=1}^{3}\sum_{k=1}^{d} \|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}} \|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\ &\lesssim 1 + \|\mathfrak{I}_0\|_{q,s+3+\ell}^{\gamma,\mathcal{O}},\end{aligned}$$

where we have used the interpolation inequality: for $s \geqslant s_0$

$$\begin{aligned}\|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}} \|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}} &\lesssim \|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}} \|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + \|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}} \|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}.\end{aligned}$$

In addition, to estimate the difference we simply write

$$\begin{aligned}\forall\ell\in\mathbb{N},\quad \Delta_{12}(\partial_\theta^\ell J)(\varphi,\theta,\eta+\theta)&=\sum_{k'=1}^{3}\sum_{k=1}^{d}\Delta_{12}g_{k,k'}(\varphi,\eta+\theta)(\partial_\theta^\ell(\chi_{k,k'})_{r_1})(\varphi,\theta)\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}(g_{k,k'})_{r_2}(\varphi,\eta+\theta)(\Delta_{12}\partial_\theta^\ell\chi_{k,k'})(\varphi,\theta).\end{aligned}$$

By applying the mean value theorem combined with (7.12), (7.6) and interpolation inequalities

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|\Delta_{12}(\partial_\theta^\ell J)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\sum_{k'=1}^{3}\sum_{k=1}^{d}\|\Delta_{12}g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|\Delta_{12}g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s+\ell}^{\gamma,\mathcal{O}}\\&\quad+\sum_{k'=1}^{3}\sum_{k=1}^{d}\|g_{k,k'}(*,\cdot,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\chi_{k,k'}(*,\cdot,\bullet)\|_{q,s_0+\ell}^{\gamma,\mathcal{O}}\\&\lesssim\|\Delta_{12}i\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0+3}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s+3+\ell}^{\gamma,\mathcal{O}}.\end{aligned}$$

The symmetry property detailed in (7.5) is a consequence of the definition of r and the reversibility condition (7.1) imposed on the torus i_0. Consequently, putting together (6.4) and (6.12) gives

$$\begin{aligned}\mathsf{K}_{02}(\varphi)&=\mathrm{L}(\lambda)\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\partial_z\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))+\varepsilon\mathcal{R}(\varphi)\\&=\mathrm{L}(\lambda)\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r P_\varepsilon(A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi)\\&=\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r\mathcal{H}_\varepsilon(A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi)\\&=\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r H(\varepsilon A(i_0(\varphi)))\Pi_{\mathbb{S}_0}^{\perp}+\varepsilon\mathcal{R}(\varphi).\end{aligned}$$

Recall from (6.12) that

$$r(\varphi,\cdot)=A(i_0(\varphi)),\tag{7.13}$$

then according to the general form of the linearized operator stated in Lemma 5.1 one has

$$-\partial_\theta\partial_r\nabla_r H(\varepsilon r(\varphi,\cdot))=\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r},$$

which implies in turn

$$-\mathsf{K}_{02}(\varphi)=\Pi_{\mathbb{S}_0}^{\perp}\big(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r}-\varepsilon\mathcal{R}(\varphi)\big)\Pi_{\mathbb{S}_0}^{\perp}.$$

Plugging this identity into (7.2) gives the desired result. Next, using (7.13), (6.12) and (6.54), we obtain

$$\begin{aligned}\|r\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|v(\vartheta_0,I_0)\|_{q,s}^{\gamma,\mathcal{O}}+\|z_0\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim 1+\|\mathfrak{I}_0\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

We shall now move to the proof of the bound (7.8). First, we observe from (6.12) that

$$\|\Delta_{12}r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}v(\vartheta,I)\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}z\|_{q,s}^{\gamma,\mathcal{O}}.$$

Therefore, Taylor Formula with (6.54) and product laws allow to get

$$\|\Delta_{12}v(\vartheta,I)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}(I,\vartheta)\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}(I,\vartheta)\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}},$$

which implies that

$$\|\Delta_{12}r\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}i\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j=1,2}\|\mathfrak{I}_j\|_{q,s}^{\gamma,\mathcal{O}}.$$

This achieves the proof of Proposition 7.1. □

7.2 Reduction of order 1

In this section, we perform the reduction of the transport part of the linearized operator $\mathcal{L}_{\varepsilon r}$ described in Proposition 7.1. More precisely, we conjugate the operator $\mathcal{L}_{\varepsilon r}$ by a quasi-periodic symplectic change of variables $\mathscr{B}$ leading to a transport part with constant coefficients depending only on the torus i_0 and the parameters ε, λ and ω. To get a precise information on the remainder, which is of order -1 in θ, we need to describe the action of this conjugation on the nonlocal term using the kernel structure rather than pseudo-differential theory. The reduction to a constant coefficient operator is based on KAM scheme through the construction of successive quasi-periodic symplectic change of coordinates. This will be implemented in the spirit of [8, 64]. Here we need to extend their construction to the framework of symplectic change of coordinates with C^q regularity. We point out that similar results with slight variations have been established in [11, 28, 32].

7.2.1 Reduction of the transport part

Now we shall state the main result of this section concerning the reduction of the transport part of the linearized operator $\mathcal{L}_{\varepsilon r}$.

Proposition 7.2. *Let* $(\gamma,q,d,\tau_1,s_0,S,s_l,\overline{s}_h,\overline{\mu}_2)$ *satisfy* (A.2), (A.1) *and* (7.3). *Let* $\upsilon\in\left(0,\frac{1}{q+2}\right]$. *We set*

$$\sigma_1\triangleq s_0+\tau_1q+2\tau_1+4. \tag{7.14}$$

For any $(\mu_2,\mathtt{p},s_h)$ *satisfying*

$$\mu_2\geqslant\overline{\mu}_2\triangleq4\tau_1q+6\tau_1+3,\quad \mathtt{p}\geqslant0,\quad s_h\geqslant\max\left(\frac{3}{2}\mu_2+s_l+1,\overline{s}_h+\mathtt{p}\right), \tag{7.15}$$

there exists $\varepsilon_0>0$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_1}^{\gamma,\mathcal{O}}\leqslant1, \tag{7.16}$$

there exist

$$c_{i_0}\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{R})\quad\textit{and}\quad\beta\in\bigcap_{s\in[s_0,S]}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\rm odd})$$

such that with $\mathscr{B}$ *defined in* (A.12) *one gets the following results.*

(i) The function c_{i_0} *satisfies the following estimate,*

$$\|c_{i_0}-V_0\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon, \tag{7.17}$$

where V_0 *is defined in Lemma* 5.2.

(ii) The transformations $\mathscr{B}^{\pm 1}, \mathcal{B}^{\pm 1}, \beta$ and $\widehat{\beta}$ satisfy the following estimates for all $s \in [s_0, S]$

$$\|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \tag{7.18}$$

and

$$\|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\beta\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right). \tag{7.19}$$

(iii) Let $n \in \mathbb{N}$, then in the truncated Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) = \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\} \\ |l|\leqslant N_n}} \left\{(\lambda,\omega) \in \mathcal{O} \quad \text{s.t.} \quad \big|\omega\cdot l + jc_{i_0}(\lambda,\omega)\big| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\},$$

we have

$$\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi + \partial_\theta\big(V_{\varepsilon r}\,\cdot\,\big)\big)\mathscr{B} = \omega\cdot\partial_\varphi + c_{i_0}\partial_\theta + \mathtt{E}_n^0,$$

with $\mathtt{E}_n^0 = \mathtt{E}_n^0(\lambda,\omega,i_0)$ a linear operator satisfying

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{7.20}$$

(iv) Given two tori i_1 and i_2 both satisfying (7.16), we have

$$\|\Delta_{12}c_i\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} \tag{7.21}$$

and

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} + \|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}. \tag{7.22}$$

Before giving the proof, some remarks are in order.

Remark 7.1.

- *The final Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is constructed over the limit coefficient c_{i_0} but it is still truncated in the time frequency, that is $|l| \leqslant N_n$, leading to a residual remainder with enough decay. This induces a suitable stability property that is crucial during the Nash-Moser scheme achieved with the nonlinear functional.*
- *Notice that, since $4\gamma^{\upsilon} \geqslant \gamma$, then looking at $j = 0$ we find that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is contained in the Diophantine Cantor set $(\lambda_0,\lambda_1)\times\mathtt{DC}_{N_n}(\gamma,\tau_1)$ introduced in (6.90).*
- *The parameter υ is introduced for technical reasons appearing later in the measure estimates of the final Cantor set and it will be fixed in (8.64).*
- *The constant 4 used in the definition of the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ is useful to ensure the inclusion of this set in all the Cantor sets built in the KAM procedure (see (7.81) in the proof below) and also to establish some inclusions related to the final Cantor set (see the proof of Lemma 8.2).*
- *We emphasize here that the functions β and $\widehat{\beta}$ are odd in the sense*

  $$\beta(\lambda,\omega,-\varphi,-\theta) = -\beta(\lambda,\omega,\varphi,\theta) \quad \text{and} \quad \widehat{\beta}(\lambda,\omega,-\varphi,-\theta) = -\widehat{\beta}(\lambda,\omega,\varphi,\theta) \tag{7.23}$$

  which will be crucial later to get the Toeplitz structure of the new remainder term emerging after this reduction.

Proof. Since we are looking at a state near the disc, we can split $V_{\varepsilon r}$ defined by (5.1) according to

$$V_{\varepsilon r}(\lambda,\varphi,\theta) = V_0(\lambda) + f_0(\lambda,\varphi,\theta), \tag{7.24}$$

with f_0 being a perturbation term of small size. We refer to (7.47) for a more precise quantification of this smallness. The proof is an iteration process introducing at each step a linear quasi-periodic symplectic change of coordinates. This transformation is linked to the remainder term of the previous step. Roughly speaking, if the latter is of size ε, then we choose the change of coordinates in such a way that we extract the main diagonal part of the previous remainder and keep a new perturbation term of size ε^2. The choice of the transformation is done through the resolution of an homological equation requiring non-resonance conditions capted by a suitable selection of the parameters of the system. Thus, by iteration, we can construct a final Cantor set gathering all the parameters restrictions of all steps in which we completely reduced the transport operator into a constant coefficient one. We shall now explain a typical step of the procedure Later, we shall implement the scheme.

(i)-(ii) ▶ KAM step. Let us consider a transport operator in the form,

$$\omega\cdot\partial_\varphi+\partial_\theta\Big(V+f\Big)$$

for suitable parameters (λ,ω) that belong to a subset $\mathcal{O}_-^\gamma\subset\mathcal{O}$, where $\mathcal{O}$ is the ambient set and

$$V=V(\lambda,\omega)\quad\text{and}\quad f=f(\lambda,\omega,\varphi,\theta),$$

where f enjoys the following symmetry condition

$$f(\lambda,\omega,-\varphi,-\theta)=f(\lambda,\omega,\varphi,\theta). \tag{7.25}$$

To alleviate the notations we shall use during the proof the variable $\mu\triangleq(\lambda,\omega)$. We consider a symplectic quasi-periodic change of coordinates close to the identity taking the form

$$\begin{aligned}\mathscr{G}\rho(\mu,\varphi,\theta)&\triangleq\big(1+\partial_\theta g(\mu,\varphi,\theta)\big)\mathcal{G}\rho(\mu,\varphi,\theta)\\&\triangleq\big(1+\partial_\theta g(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+g(\mu,\varphi,\theta)\big),\end{aligned}\tag{7.26}$$

where $g:\mathcal{O}\times\mathbb{T}^{d+1}\to\mathbb{R}$ is a small function which will be later linked to f. Then, by using Lemma A.3, we can write for any $N\geqslant 2$

$$\mathscr{G}^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta\left(V+f\right)\Big)\mathscr{G}=\omega\cdot\partial_\varphi+\partial_\theta\mathcal{G}^{-1}\left(V+\omega\cdot\partial_\varphi g+V\partial_\theta g+\Pi_N f+\Pi_N^\perp f+f\partial_\theta g\right). \tag{7.27}$$

Recall that the projections Π_N are defined in (A.5). The basic idea is to obtain after this transformation a new transport operator in the form

$$\mathscr{G}^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta\left(V+f\right)\Big)\mathscr{G}=\omega\cdot\partial_\varphi+\partial_\theta\big(V_++f_+\big), \tag{7.28}$$

where

$$V_+=V_+(\mu)\quad\text{and}\quad f_+=f_+(\mu,\varphi,\theta),$$

with f_+ quadratically smaller than f. In order to get rid of the terms wich are not small of quadratic in f in the right hand-side of (7.27), we shall select g as a solution of the following *homological equation*

$$\omega\cdot\partial_\varphi g+V\partial_\theta g+\Pi_N f=\langle f\rangle_{\varphi,\theta}, \tag{7.29}$$

where

$$\langle f\rangle_{\varphi,\theta}(\mu)\triangleq\int_{\mathbb{T}^{d+1}}f(\mu,\varphi,\theta)d\varphi d\theta.$$

To find a solution to the *homological equation* (7.29), we use Fourier decomposition and look for g in the form

$$g(\mu,\varphi,\theta)\triangleq \mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N}}\tfrac{f_{l,j}(\mu)}{\omega\cdot l+jV(\mu)}e^{\mathrm{i}(l\cdot\varphi+j\theta)}.\tag{7.30}$$

The denominators appearing in the Fourier decomposition of g may be small and generate problems in the convergence of the series in (7.30) for large N. This is a well-known phenomenon in KAM theory called "small divisors problem". To overcome this difficulty, one has to avoid the resonances and, following the ideas of Kolmogorov, we introduce Diophantine conditions gathered in the following Cantor set

$$\mathcal{O}_{+}^{\gamma}\triangleq\bigcap_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N}}\Big\{\mu\in\mathcal{O}_{-}^{\gamma}\quad\text{s.t.}\quad\big|\omega\cdot l+jV(\mu)\big|>\tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.\tag{7.31}$$

Such a selection of the external parameters allows us to control the size of the denominators in (7.30). As we shall see in (7.39), the quantification of this control, linked to the parameters γ and τ_1, allows to get suitable estimates for g with some loss of regularity uniform with respect to N. Before performing this estimate, we shall first construct an extension of g to the whole set $\mathcal{O}$. In what follows, we still denote g this extension. This is done by extending the Fourier coefficients of g using the cut-off function χ defined in (6.92). More precisely, we define

$$\begin{aligned}g_{l,j}(\mu)&\triangleq\mathrm{i}\tfrac{\chi\left((\omega\cdot l+jV(\mu))(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}\right)}{\omega\cdot l+jV(\mu)}f_{l,j}(\mu)\\&\triangleq\widetilde{g_{l,j}}(\mu)f_{l,j}(\mu).\end{aligned}\tag{7.32}$$

Notice that the extension g is a solution to (7.29) only when the parameters are restricted to the Cantor set $\mathcal{O}_{+}^{\gamma}$. Then, we define

$$V_{+}\triangleq V+\langle f\rangle_{\varphi,\theta}\quad\text{and}\quad f_{+}\triangleq\mathscr{G}^{-1}\big(\Pi_N^{\perp}f+f\partial_\theta g\big),$$

so that in restriction to the Cantor set $\mathcal{O}_{+}^{\gamma}$, the identity (7.28) holds. Remark that V_{+} and f_{+} are well-defined in the whole set of parameters $\mathcal{O}$ and the function g is smooth since it is generated by a finite number of frequencies. According to (7.25), we obtain that g is odd. As a consequence,

$$g\in\bigcap_{s\geqslant 0}W^{q,\infty,\gamma}(\mathcal{O},H^{s}_{\mathrm{odd}}).\tag{7.33}$$

Our next task is to estimate the Fourier coefficients $\widetilde{g_{l,j}}$ defined by (7.32). Notice that we can write them in the following form

$$\begin{aligned}\widetilde{g_{l,j}}(\mu)&=\mathrm{i}\,a_{l,j}\widehat{\chi}(a_{l,j}A_{l,j}(\mu)),\quad\widehat{\chi}(x)\triangleq\tfrac{\chi(x)}{x}\\A_{l,j}(\mu)&\triangleq\omega\cdot l+jV(\mu),\quad a_{l,j}\triangleq(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}.\end{aligned}\tag{7.34}$$

Since $\widehat{\chi}$ is C^∞ with bounded derivatives and $\widehat{\chi}(0)=0$, then applying Lemma A.1-(vi), we obtain

$$\forall q'\in[\![0,q]\!],\quad\|\widetilde{g_{l,j}}\|_{q'}^{\gamma,\mathcal{O}}\lesssim a_{l,j}^{2}\|A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\Big(1+a_{l,j}^{q'-1}\|A_{l,j}\|_{L^\infty(\mathcal{O})}^{q'-1}\Big).$$

Direct computations lead to

$$\begin{aligned}\forall(l,j)\in\mathbb{Z}^{d+1},\,\forall\alpha\in\mathbb{N}^{d+1},\quad|\alpha|\leqslant q,\quad\sup_{\mu\in\mathcal{O}}\big|\partial_\mu^{\alpha}A_{l,j}(\mu)\big|&\lesssim\langle l,j\rangle\max\Big(1,\sup_{\mu\in\mathcal{O}}\big|\partial_\mu^{\alpha}V(\mu)\big|\Big)\\&\lesssim\gamma^{-|\alpha|}\langle l,j\rangle\max\big(1,\|V\|_{q}^{\gamma,\mathcal{O}}\big).\end{aligned}$$

Assuming

$$\|V\|_{q}^{\gamma,\mathcal{O}}\leqslant C,\tag{7.35}$$

we then obtain

$$\forall q'\in[\![0,q]\!],\quad\forall(l,j)\in\mathbb{Z}^{d+1},\quad\|A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\langle l,j\rangle.\tag{7.36}$$

Added to the fact that $0\leqslant a_{l,j}\leqslant\gamma^{-\upsilon}\langle l\rangle^{\tau_1}$, we then find that

$$\forall q'\in[\![0,q]\!],\quad\|\widetilde{g_{l,j}}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\gamma^{-\upsilon(q'+1)}\langle l,j\rangle^{\tau_1q'+\tau_1+q'}.\tag{7.37}$$

Our choice of υ in Proposition 7.2 implies in particular that

$$\upsilon\leqslant\tfrac{1}{q+1}.\tag{7.38}$$

Therefore, we deduce from (7.32) and Leibniz rule that for all $\alpha\in\mathbb{N}^{d+1}$ with $|\alpha|\leqslant q$

$$\begin{aligned}\gamma^{2|\alpha|}\|\partial_\mu^\alpha g(\mu,\cdot,\bullet)\|_{H^{s-|\alpha|}}^2&\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\\langle l,j\rangle\leqslant N}}\sum_{\substack{\beta\in\mathbb{N}^{d+1}\\\beta\leqslant\alpha}}\gamma^{2|\alpha|-2|\beta|}\big|\partial_\mu^{\alpha-\beta}\widetilde{g_{l,j}}(\mu)\big|^2\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2s-2|\alpha|}\\&\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\\langle l,j\rangle\leqslant N}}\sum_{\substack{\beta\in\mathbb{N}^{d+1}\\\beta\leqslant\alpha}}\Big(\|\widetilde{g_{l,j}}\|_{|\alpha|-|\beta|}^{\gamma,\mathcal{O}}\Big)^2\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2s-2|\alpha|}\\&\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}\\\langle l,j\rangle\leqslant N}}\sum_{\substack{\beta\in\mathbb{N}^{d+1}\\\beta\leqslant\alpha}}\gamma^{-2}\gamma^{2|\beta|}\big|\partial_\mu^\beta f_{l,j}(\mu)\big|^2\langle l,j\rangle^{2(s+\tau_1q+\tau_1-|\beta|)}.\end{aligned}$$

As a consequence, by interverting the summation symbols, we find

$$\|g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\Pi_Nf\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\tag{7.39}$$

Assume now that

$$\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.\tag{7.40}$$

Then added to (7.39) and Lemma A.1-(ii), we get

$$\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-1}N^{\tau_1q+\tau_1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant C\varepsilon_0.$$

On the other hand if we assume

$$\|f\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right),$$

then (7.39) gives

$$\begin{aligned}\|g\|_{q,2s_0+1}^{\gamma,\mathcal{O}}&\lesssim\gamma^{-1}\|f\|_{q,2s_0+\tau_1q+\tau_1+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,2s_0+\tau_1q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_h+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Notice that to obtain the last inequality we used the fact that (7.15) and (7.14) imply

$$2s_0+\tau_1q+\tau_1+2\leqslant s_h+\sigma_1.$$

Using interpolation inequality and (7.16), one gets for some $\overline{\theta}\in(0,1)$

$$\begin{aligned}\|g\|_{q,2s_0}^{\gamma,\mathcal{O}}&\lesssim\left(\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|g\|_{q,2s_0+1}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}\\&\lesssim\varepsilon_0.\end{aligned}\tag{7.41}$$

Thus, taking ε_0 small enough, we can ensure the smallness condition in Lemma A.4 and get that the linear operator $\mathscr{G}$ is invertible. Now, we introduce

$$u\triangleq\Pi_N^{\perp}f+f\partial_\theta g.$$

By the triangle inequality, Lemma A.1-(ii) and (7.39), we obtain for all $s\in[s_0,S]$

$$\begin{aligned}\|u\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Pi_N^{\perp}f\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta g\|_{q,s}^{\gamma,\mathcal{O}}+\|f\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta g\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|\Pi_N^{\perp}f\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|g\|_{q,s+1}^{\gamma,\mathcal{O}}+\|f\|_{q,s}^{\gamma,\mathcal{O}}\|g\|_{q,s_0+1}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|\Pi_N^{\perp}f\|_{q,s}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combined with Lemma A.4, Lemma A.1-(ii) and (7.40), we get for all $s\in[s_0,S]$

$$\begin{aligned}\|f_+\|_{q,s}^{\gamma,\mathcal{O}}&=\|\mathcal{G}^{-1}(u)\|_{q,s}^{\gamma,\mathcal{O}}\\&\leqslant\|u\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|u\|_{q,s}^{\gamma,\mathcal{O}}\|\widehat{g}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\widehat{g}\|_{q,s}^{\gamma,\mathcal{O}}\|u\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|u\|_{q,s}^{\gamma,\mathcal{O}}+C\Big(\|u\|_{q,s}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}+\|g\|_{q,s}^{\gamma,\mathcal{O}}\|u\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\leqslant\|\Pi_N^{\perp}f\|_{q,s}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Using once again Lemma A.1-(ii), we find for $S\geqslant\overline{s}\geqslant s\geqslant s_0$

$$\|f_+\|_{q,s}^{\gamma,\mathcal{O}}\leqslant N^{s-\overline{s}}\|f\|_{q,\overline{s}}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_1q+\tau_1+1}\|f\|_{q,s_0}^{\gamma,\mathcal{O}}\|f\|_{q,s}^{\gamma,\mathcal{O}}.\tag{7.42}$$

► **KAM scheme.** Let us now assume that we have constructed V_m and f_m, well-defined in the whole set of parameters $\mathcal{O}$ and satisfying the assumptions (7.35) and (7.40). We shall now construct the corresponding quantity at the next order, namely V_{m+1} and f_{m+1}, still satisfying (7.35) and (7.40). For this aim, we shall implement the KAM step with (V,f,V_+,f_+,N) replaced by $(V_m,f_m,V_{m+1},f_{m+1},N_m)$. More precisely, we will shall prove by induction the existence of a sequence $\{V_m,f_m\}_{m\in\mathbb{N}}$ such that

$$\delta_m(s_l)\leqslant\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\quad\text{ and }\quad\delta_m(s_h)\leqslant\Big(2-\tfrac{1}{m+1}\Big)\,\delta_0(s_h)\tag{7.43}$$

and

$$\|V_m\|_q^{\gamma,\mathcal{O}}\leqslant C\quad\text{ and }\quad N_m^{\tau_1q+\tau_1+1}\delta_m(s_0)\leqslant\varepsilon_0,\tag{7.44}$$

with f_m satisfying the following symmetry condition

$$f_m(\mu,-\varphi,-\theta)=f_m(\mu,\varphi,\theta)\tag{7.45}$$

and where we denote

$$\delta_m(s)\triangleq\gamma^{-1}\|f_m\|_{q,s}^{\gamma,\mathcal{O}}.$$

Recall that the parameters s_l and s_h were introduced in (7.3) and (7.15).

➢ *Initialization.* We shall first check that the estimates (7.43) and (7.44) are satisfied for $m=0$. In which case the functions V_0 and f_0 are defined by (5.11) and (7.24). By (6.43) and (7.7) we infer

$$\begin{aligned}\delta_0(s)&=\gamma^{-1}\|V_{\varepsilon r}-V_0\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.46}$$

Thus, the notation (7.15) and the smallness condition (7.16) imply that

$$N_0^{\mu_2}\delta_0(s_h)\leqslant C\varepsilon_0.\tag{7.47}$$

In addition, by (7.4) and (5.3), we deduce that f_0 satisfies the following symmetry condition

$$f_0(\lambda,-\varphi,-\theta)=f_0(\lambda,\varphi,\theta).\tag{7.48}$$

We set $\mathcal{O}_0^\gamma=\mathcal{O}$ and consider $N_0\geqslant 2$. Our next task is to check that the assumptions (7.35) and (7.40) are satisfied by V_0 and f_0. First recall that V_0 is defined by

$$V_0(\lambda)=\Omega+I_1(\lambda)K_1(\lambda).$$

Using the smooth regularity of (C.10), we obtain

$$\|V_0\|_q^{\gamma,\mathcal{O}}\leqslant C.\tag{7.49}$$

Therefore, the required boundedness property (7.35) is satisfied with $V=V_0$. Now by (7.15), we have

$$\mu_2\geqslant\tau_1 q+\tau_1+2.\tag{7.50}$$

Hence, using (7.47), we obtain

$$\begin{aligned}\gamma^{-1}N_0^{\tau_1q+\tau_1+1}\|f_0\|_{q,s_0}^{\gamma,\mathcal{O}}&=N_0^{\tau_1q+\tau_1+1}\delta_0(s_0)\\&\leqslant N_0^{\tau_1q+\tau_1+1-\mu_2}N_0^{\mu_2}\delta_0(s_h)\\&\leqslant C\varepsilon_0N_0^{-1}.\end{aligned}$$

By taking N_0 large enough we get

$$CN_0^{-1}\leqslant 1,\tag{7.51}$$

so that

$$\gamma^{-1}N_0^{\tau_1q+\tau_1+1}\|f_0\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.$$

Hence, the assumption (7.40) is satisfied for $f=f_0$. This ends the initialization step.

➢ *Iteration.* let us now assume that we have constructed V_m and f_m enjoying the properties (7.43), (7.44) and (7.45). We shall see how to construct V_{m+1} and f_{m+1}. According to the KAM step, we consider a symplectic quasi-periodic change of variables $\mathscr{G}_m$ taking the form

$$\begin{aligned}\mathscr{G}_m\rho(\mu,\varphi,\theta)&\triangleq\big(1+\partial_\theta g_m(\mu,\varphi,\theta)\big)\mathcal{G}_m\rho(\mu,\varphi,\theta)\\&=\big(1+\partial_\theta g_m(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+g_m(\mu,\varphi,\theta)\big),\end{aligned}$$

with

$$g_m(\mu,\varphi,\theta) \triangleq \mathrm{i} \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} \frac{\chi\Big((\omega\cdot l+jV_m(\mu))(\gamma^{\upsilon}\langle j\rangle)^{-1}\langle l\rangle^{\tau_1}\Big)}{\omega\cdot l+jV_m(\mu)}(f_m)_{l,j}(\mu)\, e^{\mathrm{i}(l\cdot\varphi+j\theta)}, \tag{7.52}$$

where χ is the cut-off function introduced in (6.92) and N_m is defined in (6.94). As explained in the KAM step, g_m is well-defined on the whole set of parameters $\mathcal{O}$ and solves the *homological equation*

$$\omega\cdot\partial_\varphi g_m + V_m\partial_\theta g_m + \Pi_{N_m} f_m = \langle f_m\rangle_{\varphi,\theta}$$

when restricted to the Cantor set

$$\mathcal{O}_{m+1}^{\gamma} \triangleq \bigcap_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} \Big\{\mu=(\lambda,\omega)\in\mathcal{O}_m^{\gamma}\quad \text{s.t.}\quad \big|\omega\cdot l+jV_m(\mu)\big|>\tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}. \tag{7.53}$$

Hence, in the Cantor set $\mathcal{O}_{m+1}^{\gamma}$, the following reduction holds

$$\mathscr{G}_m^{-1}\Big(\omega\cdot\partial_\varphi+\partial_\theta(V_m+f_m)\Big)\mathscr{G}_m=\omega\cdot\partial_\varphi+\partial_\theta(V_{m+1}+f_{m+1}),$$

with V_{m+1} and f_{m+1} defined by

$$\begin{cases} V_{m+1}\triangleq V_m+\langle f_m\rangle_{\varphi,\theta}\\ f_{m+1}\triangleq \mathcal{G}_m^{-1}\Big(\Pi_{N_m}^{\perp}f_m+f_m\partial_\theta g_m\Big). \end{cases} \tag{7.54}$$

In view of (7.45), the function f_m is even and therefore g_m is odd. Consequently, we deduce through elementary manipulations that f_{m+1} is also even. This allows us to follow the symmetry persistence along the scheme. Besides, in a similar way to (7.33), one obtains

$$g_m\in\bigcap_{s\geqslant0}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\mathrm{odd}}). \tag{7.55}$$

Now, we set

$$\mathscr{B}_{-1}\triangleq\mathscr{G}_{-1}\triangleq\mathrm{Id}\quad\text{ and }\quad\forall m\in\mathbb{N},\ \mathscr{B}_m\triangleq\mathscr{G}_0\circ\mathscr{G}_1\circ\ldots\circ\mathscr{G}_m.$$

One easily finds that

$$\begin{aligned}\mathscr{B}_m\rho(\mu,\varphi,\theta)&=\big(1+\partial_\theta\beta_m(\mu,\varphi,\theta)\big)\mathcal{B}_m\rho(\mu,\varphi,\theta)\\ &=\big(1+\partial_\theta\beta_m(\mu,\varphi,\theta)\big)\rho\big(\mu,\varphi,\theta+\beta_m(\mu,\varphi,\theta)\big),\end{aligned}$$

where the sequence $(\beta_m)_{m\in\mathbb{N}}$ is defined by $\beta_{-1}\triangleq g_{-1}\triangleq0$ and

$$\beta_0\triangleq g_0\quad\text{ and }\quad\beta_m(\mu,\varphi,\theta)\triangleq\beta_{m-1}(\mu,\varphi,\theta)+g_m\big(\mu,\varphi,\theta+\beta_{m-1}(\mu,\varphi,\theta)\big). \tag{7.56}$$

A trivial induction based on (7.55) yields

$$\beta_m\in\bigcap_{s\geqslant0}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\mathrm{odd}}). \tag{7.57}$$

According to Sobolev embeddings, (7.54) and the induction assumption (7.43), we infer

$$\begin{aligned}\|V_m - V_{m-1}\|_q^{\gamma,\mathcal{O}} &= \|\langle f_{m-1}\rangle_{\varphi,\theta}\|_q^{\gamma,\mathcal{O}}\\ &\leqslant \|f_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &= \gamma\delta_{m-1}(s_0)\\ &\leqslant \gamma\delta_0(s_h)N_0^{\mu_2}N_{m-1}^{-\mu_2}.\end{aligned}\tag{7.58}$$

As a consequence, by using the triangle inequality, (7.47) and choosing ε_0 small enough we deduce

$$\begin{aligned}\|V_m\|_q^{\gamma,\mathcal{O}} &\leqslant \|V_{m-1}\|_q^{\gamma,\mathcal{O}} + \gamma\delta_0(s_h)N_0^{\mu_2}N_{m-1}^{-\mu_2}\\ &\leqslant \|V_0\|_q^{\gamma,\mathcal{O}} + \gamma\delta_0(s_h)N_0^{\mu_2}\left(\sum_{k=0}^{m-1}N_k^{-\mu_2}\right)\\ &\leqslant \|V_0\|_q^{\gamma,\mathcal{O}} + \sum_{k=0}^{\infty}N_k^{-\mu_2}.\end{aligned}$$

Now, remark that (7.15) implies in particular

$$\mu_2 \geqslant \tau_1 q + \tau_1 + 2.$$

Hence, by the induction hypothesis (7.43), (7.47), (7.50) and (7.51), we have

$$\begin{aligned}\delta_m(s_0)N_m^{\tau_1 q+\tau_1+1} &\leqslant \delta_0(s_h)N_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\\ &\leqslant \varepsilon_0 N_0^{-1}\\ &\leqslant \varepsilon_0.\end{aligned}\tag{7.59}$$

Using (7.49) and the previous estimate, we deduce that

$$\sup_{m\in\mathbb{N}}\|V_m\|_q^{\gamma,\mathcal{O}} \leqslant C \quad\text{and}\quad \delta_m(s_0)N_m^{\tau_1 q+\tau_1+1} \leqslant \varepsilon_0.\tag{7.60}$$

Thus, the KAM step applies and, in particular, the estimate (7.42) becomes

$$\delta_{m+1}(s) \leqslant N_m^{s-\overline{s}}\delta_m(\overline{s}) + CN_m^{\tau_1 q+\tau_1+1}\delta_m(s)\delta_m(s_0).\tag{7.61}$$

If we apply (7.61) with $s = s_l$ and $\overline{s} = s_h$, we obtain

$$\delta_{m+1}(s_l) \leqslant N_m^{s_l-s_h}\delta_m(s_h) + CN_m^{\tau_1 q+\tau_1+1}\delta_m(s_l)\delta_m(s_0).$$

Using the induction assumption (7.43) and the fact that $s_l \geqslant s_0$ yields

$$\begin{aligned}\delta_{m+1}(s_l) &\leqslant N_m^{s_l-s_h}\delta_m(s_h) + CN_m^{\tau_1 q+\tau_1+1}(\delta_m(s_l))^2\\ &\leqslant \left(2-\tfrac{1}{m+1}\right)N_m^{s_l-s_h}\delta_0(s_h) + CN_0^{2\mu_2}N_m^{\tau_1 q+\tau_1+1-2\mu_2}(\delta_0(s_h))^2\\ &\leqslant 2N_m^{s_l-s_h}\delta_0(s_h) + CN_0^{2\mu_2}N_m^{\tau_1 q+\tau_1+1-2\mu_2}(\delta_0(s_h))^2.\end{aligned}$$

The conditions (7.15) imply

$$s_h \geqslant \tfrac{3}{2}\mu_2 + s_l + 1, \quad\text{and}\quad \mu_2 \geqslant 2(\tau_1 q + \tau_1 + 1) + 1.$$

Also, using the fact that $N_0 \geqslant 2$ and choosing ε_0 small enough, we get in view of (7.47),

$$4N_0^{-\mu_2} \leqslant 1 \quad \text{and} \quad 2C\delta_0(s_h)N_0^{\mu_2} \leqslant 1.$$

As a consequence, one has

$$N_m^{s_l-s_h} \leqslant \tfrac{1}{4}N_0^{\mu_2}N_{m+1}^{-\mu_2} \quad \text{and} \quad CN_0^{2\mu_2}N_m^{\tau_1 q+\tau_1+1-2\mu_2}\delta_0(s_h) \leqslant \tfrac{1}{2}N_0^{\mu_2}N_{m+1}^{-\mu_2}, \tag{7.62}$$

which implies in turn

$$\delta_{m+1}(s_l) \leqslant \delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.$$

This proves the first statement of the induction in (7.43) and we now turn to the proof of the second statement. Applying (7.61) with $s = \overline{s} = s_h$ and using the induction (7.43), we get

$$\begin{aligned}\delta_{m+1}(s_h) &\leqslant \delta_m(s_h)\left(1 + CN_m^{\tau_1 q+\tau_1+1}\delta_m(s_0)\right)\\ &\leqslant \left(2 - \tfrac{1}{m+1}\right)\delta_0(s_h)\left(1 + CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h)\right).\end{aligned}$$

Notice that if the condition

$$\left(2 - \tfrac{1}{m+1}\right)\left(1 + CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h)\right) \leqslant 2 - \tfrac{1}{m+2} \tag{7.63}$$

holds true, then

$$\delta_{m+1}(s_h) \leqslant \left(2 - \tfrac{1}{m+2}\right)\delta_0(s_h),$$

which achieves the induction argument of (7.43). Notice that (7.63) is equivalent to

$$\left(2 - \tfrac{1}{m+1}\right)CN_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\delta_0(s_h) \leqslant \tfrac{1}{(m+1)(m+2)}.$$

Using (7.50), the preceding condition holds true if

$$CN_0^{\mu_2}N_m^{-1}\delta_0(s_h) \leqslant \tfrac{1}{(m+1)(m+2)}. \tag{7.64}$$

Since $N_0 \geqslant 2$, then in view of (6.94) there exists a small enough constant $c_0 > 0$ such that

$$\forall m \in \mathbb{N}, \quad c_0 N_m^{-1} \leqslant \tfrac{1}{(m+1)(m+2)}.$$

Consequently, (7.64) is ensured provided that

$$CN_0^{\mu_2}\delta_0(s_h) \leqslant c_0. \tag{7.65}$$

Choosing ε_0 small enough and making use of (7.47), we obtain

$$\begin{aligned}CN_0^{\mu_2}\delta_0(s_h) &\leqslant C\varepsilon_0\\ &\leqslant c_0.\end{aligned}$$

Hence, the condition (7.65) is satisfied and the proof of (7.43) is now achieved.

➢ *Persistence of the regularity.* Putting together (7.61), applied with $\overline{s} = s \in [s_0, S]$, (7.43) and (7.50), we infer

$$\begin{aligned}\delta_{m+1}(s) &\leqslant \delta_m(s)\left(1+CN_m^{\tau_1 q+\tau_1+1}\delta_m(s_0)\right)\\ &\leqslant \delta_m(s)\left(1+C\delta_0(s_h)N_0^{\mu_2}N_m^{\tau_1 q+\tau_1+1-\mu_2}\right)\\ &\leqslant \delta_m(s)\left(1+CN_m^{-1}\right).\end{aligned}$$

Gathering this estimate with (7.46), implies, up to a trivial induction,

$$\begin{aligned}\delta_m(s) &\leqslant \delta_0(s)\prod_{k=0}^{\infty}\left(1+CN_k^{-1}\right)\\ &\leqslant C\delta_0(s)\\ &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.66}$$

Then, (7.39), interpolation inequality in Lemma A.1 and (7.43) give

$$\begin{aligned}\|g_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\delta_m(s+\tau_1 q+\tau_1)\\ &\leqslant C\left(\delta_m(s_0)\right)^{\overline{\theta}(s)}\left(\delta_m(s+\tau_1 q+\tau_1+1)\right)^{1-\overline{\theta}(s)}\\ &\leqslant C\delta_0^{\overline{\theta}(s)}(s_h)\delta_0^{1-\overline{\theta}(s)}(s+\tau_1 q+\tau_1+1)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2},\end{aligned}$$

with $\overline{\theta}(s)\triangleq\frac{1}{s+\tau_1 q+\tau_1+1-s_0}$. From (7.66), (7.16) and (7.46), we deduce

$$\begin{aligned}\|g_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_h+1}^{\gamma,\mathcal{O}}\right)\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2}\\ &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)N_0^{\overline{\theta}(s)\mu_2}N_m^{-\overline{\theta}(s)\mu_2}.\end{aligned}\tag{7.67}$$

Using (7.56) and (A.16), we get for all $s\in[s_0,S]$

$$\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\left(1+\|\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\|g_m\|_{q,s}^{\gamma,\mathcal{O}}.\tag{7.68}$$

If we apply this estimate with $s=s_0$ and use Sobolev embeddings, we deduce

$$\|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\|\beta_{m-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\left(1+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

The previous two expressions make appear recurrent relation for the weighted norms of the sequence $(\beta_m)_m$. To get good estimate for β_m, we shall make use of the following result which is quite easy to prove by induction : Given three positive sequences $(a_n)_{n\in\mathbb{N}},(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ satisfying

$$\forall\, n\in\mathbb{N},\quad a_{n+1}\leqslant b_n a_n+c_n,$$

we have

$$\begin{aligned}\forall\, n\geqslant 2,\quad a_n &\leqslant a_0\prod_{i=0}^{n-1}b_i+\sum_{k=0}^{n-2}c_k\prod_{i=k+1}^{n-1}b_i+c_{n-1}\\ &\leqslant\Big(a_0+\sum_{k=0}^{n-1}c_k\Big)\prod_{i=0}^{n-1}b_i.\end{aligned}\tag{7.69}$$

In particular, if $\prod_{n=0}^{\infty} b_n$ and $\sum_{n=0}^{\infty} c_n$ converge then

$$\sup_{n\in\mathbb{N}} a_n \leqslant \Big(a_0 + \sum_{n=0}^{\infty} c_n\Big) \prod_{n=0}^{\infty} b_i. \tag{7.70}$$

Since the conditions (7.15) and (7.14) imply

$$s_0 + \tau_1 q + \tau_1 + 2 \leqslant s_h + \sigma_1 \quad \text{and} \quad \overline{\theta}(s_0)\mu_2 \geqslant 1, \tag{7.71}$$

then, from (7.67) and (7.16), we deduce

$$\begin{aligned}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant C\varepsilon\gamma^{-1} N_0^{\mu_2}\Big(1 + \|\mathfrak{I}_0\|_{q,s_0+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\Big) N_m^{-\overline{\theta}(s_0)\mu_2}\\ &\leqslant C\varepsilon_0 N_m^{-1}.\end{aligned}$$

Choosing ε_0 small enough to ensure $C\varepsilon_0 \leqslant 1$, N_0 sufficiently large to ensure $\sum_{m=0}^{\infty} N_m^{-1} < \infty$ and we can apply (7.70) together with the fact that $\beta_0 = g_0$ to obtain

$$\begin{aligned}\sup_{m\in\mathbb{N}} \|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant \left(\|\beta_0\|_{q,s_0}^{\gamma,\mathcal{O}} + C\sum_{k=0}^{\infty} \|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right) \prod_{k=0}^{\infty} \left(1 + C\|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\leqslant \left(1 + C\sum_{k=0}^{\infty} N_k^{-1}\right) \prod_{k=0}^{\infty} \left(1 + N_k^{-1}\right)\\ &\leqslant C.\end{aligned} \tag{7.72}$$

Hence the sequence $\big(\|\beta_m\|_{q,s_0}^{\gamma,\mathcal{O}}\big)_{m\in\mathbb{N}}$ is bounded and inserting this information in (7.68) gives for all $s \in [s_0, S]$

$$\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\beta_{m-1}\|_{q,s}^{\gamma,\mathcal{O}} \left(1 + C\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right) + C\|g_m\|_{q,s}^{\gamma,\mathcal{O}}.$$

Similarly to what preceeds, if we apply (7.70) and (7.67), we infer

$$\begin{aligned}\sup_{m\in\mathbb{N}} \|\beta_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \left(\|\beta_0\|_{q,s}^{\gamma,\mathcal{O}} + C\sum_{k=0}^{\infty} \|g_k\|_{q,s}^{\gamma,\mathcal{O}}\right) \prod_{k=0}^{\infty} \left(1 + C\|g_k\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\left(1 + N_0^{\overline{\theta}(s)\mu_2} \sum_{k=0}^{\infty} N_k^{-\overline{\theta}(s)\mu_2}\right).\end{aligned}$$

From Lemma A.5 we get

$$\forall s \in [s_0, S], \quad N_0^{\overline{\theta}(s)\mu_2} \sum_{k=0}^{\infty} N_k^{-\overline{\theta}(s)\mu_2} \lesssim 1$$

which implies in turn

$$\forall s \in [s_0, S], \quad \sup_{m\in\mathbb{N}} \|\beta_m\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1}\left(1 + \|\mathfrak{I}_0\|_{q,s+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}\right). \tag{7.73}$$

From the condition (7.3) we have $s_l = s_0 + \tau_1 q + \tau_1 + 2$, and consequently we deduce from (A.16), (7.72), (7.39) and (7.43),

$$\begin{aligned}\|\beta_m-\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}}&\leqslant C\|g_m\|_{q,s_0+2}^{\gamma,\mathcal{O}}\left(1+\|\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\right)\\&\leqslant C\|g_m\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant C\delta_m(s_l)\\&\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h).\end{aligned}\tag{7.74}$$

Applying once again Lemma A.5, we deduce that

$$\sum_{m=0}^{\infty}\|\beta_m-\beta_{m-1}\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant C\delta_0(s_h).$$

Hence there exists $\beta\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_0+2})$ such that

$$\beta_m\underset{m\to\infty}{\longrightarrow}\beta\quad\text{(strongly) in}\quad W^{q,\infty,\gamma}(\mathcal{O},H^{s_0+2}).$$

By (7.73) the sequence $(\beta_m)_{m\in\mathbb{N}}$ is bounded in $W^{q,\infty,\gamma}(\mathcal{O},H^s)$, then by a weak-compactness argument we find that $\beta\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$. Using (7.73), we obtain

$$\begin{aligned}\forall s\in[s_0,S],\quad\|\beta\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\liminf_{m\to\infty}\|\beta_m\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\tau_1q+\tau_1+2}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.75}$$

We then can consider the quasi-periodic symplectic change of variables $\mathscr{B}$ associated with β and defined by

$$\begin{aligned}\mathscr{B}\rho(\lambda,\omega,\varphi,\theta)&=(1+\partial_\theta\beta(\lambda,\omega,\varphi,\theta))\,\mathcal{B}\rho(\lambda,\omega,\varphi,\theta)\\&=(1+\partial_\theta\beta(\lambda,\omega,\varphi,\theta))\,\rho(\lambda,\omega,\varphi,\theta+\beta(\lambda,\omega,\varphi,\theta)).\end{aligned}$$

By (7.75), (7.71) and (7.16), we have

$$\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_0+\tau_1q+\tau_1+2}^{\gamma,\mathcal{O}}\right)\lesssim\varepsilon_0.\tag{7.76}$$

Proceeding as for (7.41), using interpolation (7.75), (7.76), (7.16) and the fact that $2s_0+\tau_1q+\tau_1+3\leqslant s_h+\sigma_1$, one obtains

$$\|\beta\|_{q,2s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon_0.$$

Therefore, choosing ε_0 small enough, we deduce in view of Lemma A.4 that $\mathscr{B}$ is an invertible operator. Moreover, by (A.17) and (7.75), we get

$$\|\mathscr{B}^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\tau_1q+\tau_1+3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{7.77}$$

In addition, by (7.57), and Sobolev embeddings (to get pointwise convergence), we find

$$\beta\in\bigcap_{s\in[s_0,S]}W^{q,\infty,\gamma}(\mathcal{O},H^s_{\mathrm{odd}}).$$

We also have an estimate of the rate of convergence for the sequence $(\beta_m)_m$ towards β,

$$\begin{aligned}\|\beta-\beta_m\|_{q,s_0+2}^{\gamma,\mathcal{O}} &\leqslant \sum_{k=m}^{\infty}\|\beta_{k+1}-\beta_k\|_{q,s_0+2}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}\sum_{k=m+1}^{\infty}N_k^{-\mu_2}.\end{aligned}\tag{7.78}$$

From Lemma A.5, one obtains

$$\sum_{k=m}^{\infty}N_k^{-\mu_2}\underset{m\to\infty}{=}O\left(N_m^{-\mu_2}\right).\tag{7.79}$$

Gathering (7.79), (7.78) and (7.46), we get

$$\begin{aligned}\|\beta-\beta_m\|_{q,s_0+2}^{\gamma,\mathcal{O}} &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{m+1}^{-\mu_2}.\end{aligned}\tag{7.80}$$

▶ KAM conclusion

By (7.58), we have

$$\begin{aligned}\sum_{m=0}^{\infty}\|V_{m+1}-V_m\|_q^{\gamma,\mathcal{O}} &\leqslant \gamma\delta_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\ &\lesssim \gamma\delta_0(s_h).\end{aligned}$$

We deduce that the sequence $(V_m)_{m\in\mathbb{N}}$ is convergent in $W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ and let us denote by c_{i_0} its limit. Moreover, we have by (7.79), (7.46) and (7.16)

$$\begin{aligned}\|c_{i_0}-V_0\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\|V_{m+1}-V_m\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)\lesssim\varepsilon\Big(1+\|\mathfrak{I}_0\|_{q,s_h+1}^{\gamma,\mathcal{O}}\Big)\\ &\lesssim\varepsilon.\end{aligned}$$

Now, we introduce the truncated Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0\}\\ |l|\leqslant N_n}}\Big\{\mu\triangleq(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+jc_{i_0}(\mu)\big|>\tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

In what follows, we shall prove that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ satisfies the inclusion

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\bigcap_{m=0}^{n+1}\mathcal{O}_m^{\gamma}=\mathcal{O}_{n+1}^{\gamma},$$

where the intermediate Cantor sets are defined in (7.53). For this aim, we shall argue by induction. We first remark that by construction $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}\triangleq\mathcal{O}_0^{\gamma}$. Now assume that $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}_m^{\gamma}$ for $m\leqslant n$ and let us check that

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\subset\mathcal{O}_{m+1}^{\gamma}.\tag{7.81}$$

Putting together (7.58) and (7.79) we infer

$$\begin{aligned}\|V_m - c_{i_0}\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{l=m}^{\infty} \|V_{l+1} - V_l\|_q^{\gamma,\mathcal{O}}\\ &\leqslant \gamma\delta_0(s_h) N_0^{\mu_2} \sum_{l=m}^{\infty} N_l^{-\mu_2}\\ &\lesssim \gamma\delta_0(s_h) N_0^{\mu_2} N_m^{-\mu_2}. \end{aligned}\tag{7.82}$$

Given $\mu \in \mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ and $(l,j) \in \mathbb{Z}^d \times \mathbb{Z} \setminus \{(0,0)\}$ such that $0 \leqslant |l| \leqslant N_m$, we have then $|l| \leqslant N_n$ and by triangle inequality,

$$\begin{aligned}|\omega \cdot l + jV_m(\mu)| &\geqslant |\omega \cdot l + jc_{i_0}(\mu)| - |j||V_m(\mu) - c_{i_0}(\mu)|\\ &\geqslant \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\langle j\rangle\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\ &\geqslant \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}} - C\langle j\rangle\gamma^{\upsilon}\varepsilon_0\langle l\rangle^{-\mu_2}.\end{aligned}$$

Since (7.15) implies $\mu_2 \geqslant \tau_1$, then taking $\varepsilon_0 \leqslant \frac{1}{C}$, we deduce from the previous estimate

$$|\omega \cdot l + jV_m(\mu)| > \tfrac{\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Consequently, $\mu \in \mathcal{O}_{m+1}^{\gamma}$ and the inclusion (7.81) holds.
(iii) We can write for all $n \in \mathbb{N}$,

$$\begin{aligned}\mathscr{B}^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B} &= \left(\mathscr{B}^{-1} - \mathscr{B}_n^{-1}\right)\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B}\\ &\quad + \mathscr{B}_n^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\left(\mathscr{B} - \mathscr{B}_n\right)\\ &\quad + \mathscr{B}_n^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B}_n.\end{aligned}$$

In view of (7.81) and the definition of $\mathscr{B}_n$, we have in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathscr{B}_n^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B}_n = \omega \cdot \partial_\varphi + \partial_\theta\left(V_{n+1} + f_{n+1}\right).$$

Therefore, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, the following decomposition holds

$$\mathscr{B}^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B} = \omega \cdot \partial_\varphi + c_{i_0}\partial_\theta + \mathtt{E}_n^0(i_0),$$

where

$$\begin{aligned}\mathtt{E}_n^0(i_0) &\triangleq \left(V_{n+1} - c_{i_0}\right)\partial_\theta + \partial_\theta\left(f_{n+1}\cdot\right) + \left(\mathscr{B}^{-1} - \mathscr{B}_n^{-1}\right)\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\mathscr{B}\\ &\quad + \mathscr{B}^{-1}\Big(\omega \cdot \partial_\varphi + \partial_\theta\left(V_0 + f_0\right)\Big)\left(\mathscr{B} - \mathscr{B}_n\right)\\ &\triangleq \mathtt{E}_{n,1}^0 + \mathtt{E}_{n,2}^0 + \mathtt{E}_{n,3}^0 + \mathtt{E}_{n,4}^0.\end{aligned}$$

By the product laws in Lemma A.1, (7.82) and (7.46) we have

$$\begin{aligned}\|\mathtt{E}_{n,1}^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \|V_{n+1} - c_{i_0}\|_q^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.83}$$

From (7.43) and since (7.3) implies in particular $s_l \geqslant s_0+1$, we obtain

$$\begin{aligned}\|\mathtt{E}^0_{n,2}\rho\|^{\gamma,\mathcal{O}}_{q,s_0} &\lesssim \gamma\delta_{n+1}(s_0+1)\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\\ &\lesssim \gamma\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}.\end{aligned} \tag{7.84}$$

We now turn to the estimate of $\mathtt{E}^0_{n,4}$. First remark that by the product laws in Lemma A.1, we have

$$\begin{aligned}\|\omega\cdot\partial_\varphi\rho+\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0} &\leqslant \|\omega\cdot\partial_\varphi\rho\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0}\\ &\lesssim \|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\left(1+\|V_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).\end{aligned}$$

But combining (7.24), (7.49), (7.66) and (7.16), we obtain

$$\begin{aligned}\|V_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{q,s_0} &\leqslant \|V_0\|^{\gamma,\mathcal{O}}_{q}+\|f_0\|^{\gamma,\mathcal{O}}_{q,s_0}\\ &\leqslant C+C\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s_0+1}\\ &\leqslant C.\end{aligned}$$

Therefore, we get

$$\|\omega\cdot\partial_\varphi\rho+\partial_\theta\left(V_{\varepsilon r}\rho\right)\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}. \tag{7.85}$$

Putting together (7.85), (7.77) and (7.16), gives

$$\|\mathtt{E}^0_{n,4}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\|(\mathscr{B}-\mathscr{B}_n)\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}. \tag{7.86}$$

Applying Taylor Formula, we may write

$$\begin{aligned}(\mathscr{B}-\mathscr{B}_n)\rho(\theta) &= (1+\partial_\theta\beta(\theta))\rho(\theta+\beta(\theta))-(1+\partial_\theta\beta_n(\theta))\rho(\theta+h_n(\theta))\\ &= (1+\partial_\theta\beta(\theta))\left[\rho(\theta+\beta(\theta))-\rho(\theta+\beta_n(\theta))\right]+\partial_\theta\left(\beta-\beta_n\right)(\theta)\rho(\theta+\beta_n(\theta))\\ &\triangleq (1+\partial_\theta\beta(\theta))(\beta-\beta_n)(\theta)\mathscr{I}_n(\theta)+\partial_\theta(\beta-\beta_n)(\theta)\mathcal{B}_n\rho(\theta),\end{aligned}$$

where

$$\mathscr{I}_n\rho(\theta)\triangleq\int_0^1\left(\partial_\theta\rho\right)\left(\theta+\beta_n(\theta)+t(\beta(\theta)-\beta_n(\theta))\right)dt.$$

Hence, we get by the product laws, A.16 and (7.80)

$$\begin{aligned}\|\partial_\theta\left(\beta-\beta_n\right)\mathcal{B}_n\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1} &\lesssim \|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+2}\|\rho\|_{q,s_0+1}\left(1+\|\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}.\end{aligned}$$

Using the product laws together with (7.80), (7.72) and (7.76) we find

$$\begin{aligned}\left\|(1+\partial_\theta\beta)\left(\beta-\beta_n\right)\mathscr{I}_n\rho\right\|^{\gamma,\mathcal{O}}_{q,s_0+1} &\lesssim \left(1+\|\beta\|^{\gamma,\mathcal{O}}_{q,s_0+2}\right)\|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}\\ &\quad\times\left(1+\|\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}+\|\beta-\beta_n\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}.\end{aligned}$$

Gathering the foregoing estimates leads to

$$\|(\mathscr{B}-\mathscr{B}_n)\rho\|^{\gamma,\mathcal{O}}_{q,s_0+1}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{7.87}$$

Plugging (7.87) into (7.86) gives

$$\|\mathtt{E}^0_{n,4}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{7.88}$$

Proceeding in a similar way as before using in particular the identity (A.14) and (A.18) we find

$$\|\mathtt{E}^0_{n,3}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}. \tag{7.89}$$

Putting together (7.83), (7.84), (7.88), (7.89) allows to get

$$\|\mathtt{E}^0_{n}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0+2}.$$

(iv) ▶ Estimate of $\Delta_{12}\beta$. First notice that, since $\beta_{-1}=0$, then

$$\Delta_{12}\beta=\sum_{m=0}^{\infty}\Delta_{12}(\beta_m-\beta_{m-1}). \tag{7.90}$$

The triangle inequality allows us to write

$$\|\Delta_{12}\beta\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant\sum_{m=0}^{\infty}\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}. \tag{7.91}$$

According to Taylor Formula and (7.56), we infer

$$\begin{aligned}\Delta_{12}\beta_m(\theta)=\Delta_{12}\beta_{m-1}(\theta)&+(\mathscr{B}_{m-1})_{r_1}(\Delta_{12}g_m)(\theta)\\&+\Delta_{12}\beta_{m-1}(\theta)\int_0^1\big(\partial_\theta(g_m)_{r_2}\big)\big(\theta+(\beta_{m-1})_{r_2}(\theta)+t\Delta_{12}\beta_{m-1}(\theta)\big)dt.\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_{12}(\beta_m-\beta_{m-1})(\theta)&=(\mathscr{B}_{m-1})_{r_1}(\Delta_{12}g_m)(\theta)\\&\quad+\Delta_{12}\beta_{m-1}(\theta)\int_0^1\big(\partial_\theta(g_m)_{r_2}\big)\big(\theta+(\beta_{m-1})_{r_2}(\theta)+t\Delta_{12}\beta_{m-1}(\theta)\big)dt.\end{aligned}$$

Consequently, using the law product in Lemma A.1, Lemma A.4 and Sobolev embeddings we obtain

$$\begin{aligned}\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant{}&\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\left(1+C\|(\beta_{m-1})_{r_1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)+\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,s_0}\|(\beta_{m-1})_{r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\\&+C\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+1}\left(1+\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\\&+C\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\Big(\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\Big)\\&+C\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\left(1+\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\end{aligned}$$

and for all $s\in[s_0,\overline{s}_h+\mathtt{p}]$

$$\begin{aligned}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s}\leqslant{}&\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,s}\left(1+C\|(\beta_{m-1})_{r_1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)+\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,s_0}\|(\beta_{m-1})_{r_1}\|^{\gamma,\mathcal{O}}_{q,s}\\&+C\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s+1}\left(1+\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\\&+C\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\left(\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,s}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s}\right)\\&+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s}\Big(1+C\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\left(1+\|(\beta_{m-1})_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\Big).\end{aligned}$$

Notice that (7.15) implies in particular $\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+3\leqslant s_h+\sigma_1$. Therefore, using (7.67) and (7.16), we get

$$\begin{aligned}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+1}&\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+3}\right)\\&\leqslant C.\end{aligned}\tag{7.92}$$

Notice that the previous estimate is sufficient to easily get rid of most of terms in the estimates of $\Delta_{12}\beta_m$ and $\Delta_{12}(\beta_m-\beta_{m-1})$, but not enough to make the series (7.90) convergent. For this purpose, we shall refine the estimates. By (7.67), (7.15) and (7.16), we have

$$\begin{aligned}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|_{q,\overline{s}_h+\mathtt{p}+1}&\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+3}\right)N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}.\end{aligned}\tag{7.93}$$

Combining (7.73) and (7.16)

$$\begin{aligned}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(\beta_m)_{r_k}\|_{q,\overline{s}_h+\mathtt{p}}&\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+3}\right)\\&\leqslant C.\end{aligned}\tag{7.94}$$

Hence, using (7.92), (7.94) and Sobolev embeddings, the previous two estimates can be reduced to

$$\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant C\left(\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+1}\right),\tag{7.95}$$

$$\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant C\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,s_0}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\left(1+C\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right)\tag{7.96}$$

and

$$\begin{aligned}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}&\leqslant C\left(\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,s_0}\|(g_m)_{r_2}\|_{q,\overline{s}_h+\mathtt{p}+1}\right)\\&\quad+\|\Delta_{12}\beta_{m-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\left(1+C\|(g_m)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).\end{aligned}\tag{7.97}$$

From (7.96), using (7.70) and the fact that $\beta_0=g_0$, we deduce that

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant\left(\|\Delta_{12}g_0\|^{\gamma,\mathcal{O}}_{q,s_0}+C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,s_0}\right)\prod_{k=0}^{\infty}\left(1+\|(g_k)_{r_2}\|^{\gamma,\mathcal{O}}_{q,s_0+1}\right).$$

Adding (7.93), we obtain

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,s_0}.$$

Similarly, (7.97), (7.70), (7.93) and the previous estimate allow to get

$$\sup_{m\in\mathbb{N}}\|\Delta_{12}\beta_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\leqslant C\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}.$$

Putting together the previous bounds, (7.95) and (7.93) gives

$$\|\Delta_{12}(\beta_m-\beta_{m-1})\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\lesssim\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}+\varepsilon\gamma^{-1}N_0^{\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}N_m^{-\overline{\theta}(\overline{s}_h+\mathtt{p}+1)\mu_2}\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}.\tag{7.98}$$

Thus, the main delicate point is to estimate $\Delta_{12}g_m$. First remark that according to (7.34) and (7.52), we can make the splitting

$$\begin{aligned} g_m(\mu,\varphi,\theta) &= \mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} a_{l,j}\widehat{\chi}\big(a_{l,j}(A_{l,j})_{r_2}(\mu)\big)(\Delta_{12}f_m)_{l,j}(\mu)\mathbf{e}_{l,j}\\ &\quad+\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} a_{l,j}\Delta_{12}\widehat{\chi}\big(a_{l,j}A_{l,j}(\mu)\big)((f_m)_{r_1})_{l,j}(\mu)\mathbf{e}_{l,j}\\ &\triangleq \mathbf{I}_1+\mathbf{I}_2. \end{aligned}$$

Similarly to (7.39), one obtains

$$\|\mathbf{I}_1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\Pi_{N_m}\Delta_{12}f_m\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}. \tag{7.99}$$

We shall now estimate the second term. Applying Taylor Formula, we get

$$\begin{aligned} \mathbf{I}_2 &= \mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}} a_{l,j}^2(\Delta_{12}A_{l,j})\int_0^1\widehat{\chi}'\Big(a_{l,j}\Big[\tau(A_{l,j})_{r_1}(\mu)+(1-\tau)(A_{l,j})_{r_2}(\mu)\Big]\Big)d\tau((f_m)_{r_1})_{l,j}\mathbf{e}_{l,j}\\ &\triangleq\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\setminus\{0\}\\ \langle l,j\rangle\leqslant N_m}}\widetilde{h}_{l,j}(\mu)((f_m)_{r_1})_{l,j}(\mu)\mathbf{e}_{l,j}. \end{aligned}\tag{7.100}$$

Remark that direct computations yield

$$\forall q'\in[\![0,q]\!],\quad \|\Delta_{12}A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\lesssim\langle l,j\rangle\|\Delta_{12}V_m\|_{q'}^{\gamma,\mathcal{O}}. \tag{7.101}$$

Since that $\widehat{\chi}'\in C^\infty$ with $\widehat{\chi}'(0)=0$, then applying Lemma A.1-(iv)-(vi) together with (7.36) and (7.101), we get

$$\begin{aligned} \forall q'\in[\![0,q]\!],\quad \|\widetilde{h_{l,j}}\|_{q'}^{\gamma,\mathcal{O}} &\lesssim a_{l,j}^3\|\Delta_{12}A_{l,j}\|_{q'}^{\gamma,\mathcal{O}}\left(\|(A_{l,j})_{r_1}\|_{q'}^{\gamma,\mathcal{O}}+\|(A_{l,j})_{r_2}\|_{q'}^{\gamma,\mathcal{O}}\right)\\ &\quad\times\left(1+a_{l,j}^{q'-1}\left(\|(A_{l,j})_{r_1}\|_{L^\infty(\mathcal{O})}+\|(A_{l,j})_{r_2}\|_{L^\infty(\mathcal{O})}\right)^{q'-1}\right)\\ &\lesssim\gamma^{-\upsilon(q'+2)}\langle l,j\rangle^{\tau_1 q'+2\tau_1+q'+1}\|\Delta_{12}V_m\|_{q'}^{\gamma,\mathcal{O}}. \end{aligned}$$

By assumption in Proposition 7.2, we have

$$\upsilon\leqslant\tfrac{1}{q+2} \tag{7.102}$$

and using Leibniz rule, we deduce that

$$\|\mathbf{I}_2\|_q^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|\Pi_{N_m}(f_m)_{r_1}\|_{q,s+\tau_1 q+2\tau_1+1}^{\gamma,\mathcal{O}}. \tag{7.103}$$

Putting together (7.99) and (7.103), we obtain for all $s\geqslant s_0$

$$\begin{aligned} \|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim\gamma^{-1}\|\Pi_{N_m}\Delta_{12}f_m\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}+\gamma^{-1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|\Pi_{N_m}(f_m)_{r_1}\|_{q,s+\tau_1 q+2\tau_1+1}^{\gamma,\mathcal{O}}\\ &\lesssim\gamma^{-1}N_m^{\tau_1 q+\tau_1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}+\gamma^{-2}N_m^{\tau_1 q+2\tau_1+1}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\|(f_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}. \end{aligned}\tag{7.104}$$

Therefore, estimating $\Delta_{12}g_m$ can be done through the estimate of $\Delta_{12}f_m$. To do so, we shall argue by induction. For that purpose, we shall consider a parameter $\widetilde{\mathtt{p}}$ (which can depend on the parameter $\mathtt{p}$, see for instance (7.118)) satisfying the following constraint

$$\overline{s}_h+\widetilde{\mathtt{p}}+3\leqslant s_h+\sigma_1. \tag{7.105}$$

We denote

$$u_m\triangleq\Pi_{N_m}^{\perp}f_m+f_m\partial_\theta g_m.$$

Then, we can write

$$\Delta_{12}f_{m+1}=(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m+\left(\Delta_{12}\mathcal{G}_m^{-1}\right)(u_m)_{r_2},$$

with

$$\Delta_{12}u_m=\Pi_{N_m}^{\perp}\Delta_{12}f_m+\Delta_{12}f_m\partial_\theta(g_m)_{r_1}+(f_m)_{r_2}\partial_\theta\Delta_{12}g_m.$$

By the triangle inequality, we have for all $s\geqslant s_0$

$$\|\Delta_{12}f_{m+1}\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}+\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}. \tag{7.106}$$

Therefore, combining (A.16), (A.18), (7.39) and Lemma A.1-(ii), we get for all $s\geqslant s_0$

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(\widehat{g_m})_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|(\widehat{g_m})_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\|(g_m)_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+C\|(g_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+C\gamma^{-1}N_m^{\tau_1q+\tau_1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\&\quad+C\gamma^{-1}N_m^{\tau_1q+\tau_1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Using (7.66),(7.105) and (7.16), one gets

$$\begin{aligned}\gamma^{-1}\sup_{m\in\mathbb{N}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+1}^{\gamma,\mathcal{O}}&\leqslant C\varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{I}_k\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+2}^{\gamma,\mathcal{O}}\right)\\&\leqslant C.\end{aligned}\tag{7.107}$$

Therefore, from (7.43) and (7.107), we get for all $s\in[s_0,\overline{s}_h]$

$$\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\left(1+CN_0^{\overline{\mu}_2}N_m^{\tau_1q+\tau_1-\overline{\mu}_2}\right)+CN_m^{\tau_1q+\tau_1}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

At this level we need to give a suitable estimate for $\Delta_{12}u_m$. For this aim, we apply the product laws in Lemma A.1, ensuring that for all $s\geqslant s_0$

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}&+C\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta(g_m)_{r_1}\|_{q,s_0}^{\gamma,\mathcal{O}}+C\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta(g_m)_{r_1}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\|(f_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+C\|(f_m)_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence we deduce by (7.39) and Lemma A.1-(ii),

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}&+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+C\gamma^{-1}N_m^{\tau_1q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\\&+C\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+C\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Added to (7.104), we finally obtain for all $s \geqslant s_0$

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\Pi_{N_m}^{\perp}\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}} &+ C\gamma^{-1}N_m^{\tau_1 q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &+ C\gamma^{-1}N_m^{\tau_1 q+\tau_1+1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\\ &+ C\gamma^{-2}N_m^{\tau_1 q+2\tau_1+1}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

Consequently, we find from (7.43), Lemma A.1-(ii) and (7.107),

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant N_m^{s_0-\overline{s}_h-\widetilde{\mathtt{p}}}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} &+ CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &+ CN_0^{2\overline{\mu}_2}N_m^{\tau_1 q+2\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}\end{aligned}$$

and

$$\begin{aligned}\|\Delta_{12}u_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} \leqslant \|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\Big(1+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\Big) &+ CN_m^{\tau_1 q+\tau_1+1}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &+ CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+2\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}},\end{aligned}$$

where we use the notation

$$\delta_0^{1,2}(s) \triangleq \gamma^{-1}\max_{k\in\{1,2\}}\|(f_0)_{r_k}\|_{q,s}^{\gamma,\mathcal{O}}.$$

It follows from the preceding estimates that,

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant CN_m^{\tau_1 q+\tau_1}\|\Delta_{12}u_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant CN_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} + CN_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.108}$$

In a similar way, direct computations yield

$$\begin{aligned}\|(\mathcal{G}_m^{-1})_{r_1}\Delta_{12}u_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} \leqslant \|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}&\Big(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\Big)\\ &+ C\left(N_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}+N_m^{\tau_1 q+\tau_1+1}\right)\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &+ CN_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_q^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.109}$$

By a new use of Taylor Formula, we can write

$$(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}(\theta) = \Delta_{12}\widehat{g_m}(\theta)\int_0^1 \partial_\theta(u_m)_{r_2}\Big(\theta+(\widehat{g_m})_{r_2}(\theta)+t\Delta_{12}\widehat{g_m}(\theta)\Big)dt.$$

Applying Lemma A.1 and (7.39), we deduce for all $s \geqslant s_0$

$$\begin{aligned}\|u_m\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Pi_{N_m}^{\perp}f_m\|_{q,s}^{\gamma,\mathcal{O}} + C\|f_m\|_{q,s}^{\gamma,\mathcal{O}}\|\partial_\theta g_m\|_{q,s_0}^{\gamma,\mathcal{O}} + C\|f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta g_m\|_{q,s}^{\gamma,\mathcal{O}}\\ &\leqslant \|f_m\|_{q,s}\left(1+CN_m^{\tau_1 q+\tau_1+1}\|f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\|f_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.110}$$

Using once again the product laws in Lemma A.1 combined with (A.16) yield for all $s \geqslant s_0$

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s_0+1}\big(1+\|(\widehat{g_m})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\\ &\quad + C\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s+1}^{\gamma,\mathcal{O}}\big(1+\|(\widehat{g_m})_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\\ &\quad + C\|\Delta_{12}\widehat{g_m}\|_{q,s_0}\|(u_m)_{r_2}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big(\|(\widehat{g_m})_{r_2}\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}}\big)\,.\end{aligned}$$

In view of (A.19), (7.93) and Sobolev embeddings, one gets for all $s \in [s_0, \overline{s}_h + \mathtt{p}]$

$$\begin{aligned}\|\Delta_{12}\widehat{g_m}\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\left(\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(g_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the previous estimates, (7.92) and (A.18) gives for all $s \in [s_0, \overline{s}_h]$

$$\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\|\Delta_{12}g_m\|_{q,s}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s_0+1}+C\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|(u_m)_{r_2}\|_{q,s+1}^{\gamma,\mathcal{O}}.$$

Thus, by virtue of (7.104), (7.110), we get for all $s \in [s_0, \overline{s}_h]$

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\gamma^{-1}N_m^{\tau_1 q+\tau_1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\quad + C\gamma^{-1}N_m^{\tau_1 q+\tau_1}\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\quad + C\gamma^{-2}N_m^{\tau_1 q+2\tau_1+1}\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s_0+1}^{\gamma,\mathcal{O}}\max_{k\in\{1,2\}}\|(f_m)_{r_k}\|_{q,s+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence, (7.43), (7.66) and (7.107) allow to get (since $s_l \geqslant s_0+1$)

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{2\overline{\mu}_2}N_m^{\tau_1 q+2\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}\end{aligned}\tag{7.111}$$

and

$$\begin{aligned}\|(\Delta_{12}\mathcal{G}_m^{-1})(u_m)_{r_2}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} &\leqslant CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\\ &\quad + CN_m^{\tau_1 q+\tau_1}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+2\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.112}$$

Gathering (7.106), (7.108) and (7.111) implies (since $N_m^{-\widetilde{\mathtt{p}}} \leqslant 1$)

$$\begin{aligned}\|\Delta_{12}f_{m+1}\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}\|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.113}$$

In a similar war, we get in view of (7.106), (7.109) and (7.112)

$$\begin{aligned}\|\Delta_{12}f_{m+1}\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}} &\leqslant \|\Delta_{12}f_m\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}}^{\gamma,\mathcal{O}}\Big(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h-\widetilde{\mathtt{p}}}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\Big)\\ &\quad + C\left(N_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}+N_m^{\tau_1 q+\tau_1+1}\right)\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}f_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad + CN_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.114}$$

In the sequel, we shall use the following notations

$$\overline{\delta}_m(s)=\gamma^{-1}\|\Delta_{12}f_m\|_{q,s}^{\gamma,\mathcal{O}}\quad\text{and}\quad\varkappa_m=\gamma^{-1}\|\Delta_{12}V_m\|_{q}^{\gamma,\mathcal{O}}.$$

Notice that

$$\Delta_{12}V_{m+1}=\Delta_{12}V_m+\langle\Delta_{12}f_m\rangle_{\varphi,\theta}\quad\text{and}\quad\Delta_{12}V_0=0.$$

Then, by using Sobolev embeddings, we obtain

$$\varkappa_m\leqslant\sum_{k=0}^{m-1}\overline{\delta}_k(s_0).\tag{7.115}$$

We shall now prove by induction that, for all $\widetilde{\mathtt{p}}$ satisfying the condition (7.105), we have

$$\forall k\leqslant m,\quad\overline{\delta}_k(s_0)\leqslant N_0^{\overline{\mu}_2}N_k^{-\overline{\mu}_2}\nu(\overline{s}_h+\widetilde{\mathtt{p}})\quad\text{and}\quad\overline{\delta}_k(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\left(2-\tfrac{1}{k+1}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}}),\tag{7.116}$$

with

$$\nu(s)\triangleq\overline{\delta}_0(s)+\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{s_0+2}.$$

First remark that the property (7.116) is trivially satisfied for $m=0$ according to Sobolev embeddings. We now assume that (7.116) is true at the order m and let us check it at the next order. By the induction assumption (7.116) and (7.115), one obtains the following estimate

$$\sup_{m\in\mathbb{N}}\varkappa_m\leqslant C\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\tag{7.117}$$

Using (7.113), (7.117) and hypothesis of induction (7.116), we find

$$\begin{aligned}\overline{\delta}_{m+1}(s_0)&\leqslant N_m^{s_0+\tau_1q+\tau_1-\overline{s}_h}\overline{\delta}_m(\overline{s}_h+\widetilde{\mathtt{p}})+CN_0^{\overline{\mu}_2}N_m^{2(\tau_1q+\tau_1)+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\overline{\delta}_m(s_0)\\&\quad+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\varkappa_m\\&\leqslant\left[2N_m^{s_0+\tau_1q+\tau_1-\overline{s}_h}+CN_0^{2\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right]\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\end{aligned}$$

Then, in view of (7.15), we infer

$$\begin{aligned}2N_m^{s_0+\tau_1q+\tau_1-\overline{s}_h}&=2N_m^{-\frac{3}{2}\overline{\mu}_2-3}=2N_m^{-3}N_{m+1}^{-\overline{\mu}_2}\\&\leqslant 2N_0^{-3}N_{m+1}^{-\overline{\mu}_2}\\&\leqslant\frac{1}{2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.\end{aligned}$$

To prove the last inequality, we remark that since $N_0\geqslant 2$ and $\overline{\mu}_2\geqslant 0$ (according to (7.3)), then

$$4\leqslant N_0^{\overline{\mu}_2+3}.$$

Similarly, from the expression of $\overline{\mu}_2$ in (7.15) and using (6.94) one obtains

$$\begin{aligned}CN_0^{2\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)&\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\mu}_2}N_m^{2\tau_1q+3\tau_1+1-\frac{1}{2}\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{2\tau_1q+3\tau_1+1+\frac{1}{2}\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\overline{\mu}_2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.\end{aligned}$$

Hence, choosing ε_0 small enough and using (7.16) we deduce that

$$CN_0^{2\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-2\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\leqslant \frac{1}{2}N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}.$$

Gathering the preceding estimates gives

$$\overline{\delta}_{m+1}(s_0)\leqslant N_0^{\overline{\mu}_2}N_{m+1}^{-\overline{\mu}_2}\nu(\overline{s}_h+\widetilde{\mathtt{p}}).$$

This ends the proof of the first statement in (7.116). As for the second one, we shall first write in view of (7.114),

$$\begin{aligned}\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant \overline{\delta}_m(\overline{s}_h+\widetilde{\mathtt{p}})&\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h)\right)\\ &+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2(\tau_1 q+\tau_1)+1-\overline{\mu}_2}\right)\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\overline{\delta}_m(s_0)\\ &+CN_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)\varkappa_m.\end{aligned}$$

Notice that since $\overline{s}_h+\widetilde{\mathtt{p}}+2\leqslant s_h+\sigma_1$, then by (7.46) and (7.16), one has

$$\begin{aligned}\delta_0^{1,2}(\overline{s}_h+\widetilde{\mathtt{p}}+1)&\lesssim \varepsilon\gamma^{-1}\left(1+\max_{k\in\{1,2\}}\|\mathfrak{J}_k\|_{q,\overline{s}_h+\widetilde{\mathtt{p}}+2}^{\gamma,\mathcal{O}}\right)\\ &\lesssim \varepsilon\gamma^{-1}.\end{aligned}$$

It follows from (7.116) and (7.117),

$$\begin{aligned}\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant &\left(2-\tfrac{1}{m+1}\right)\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}})\\ &+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\right)N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\varepsilon\gamma^{-1}\nu(\overline{s}_h+\widetilde{\mathtt{p}}).\end{aligned}$$

Proceeding as for (7.63), taking ε_0 small enough and thanks to (7.15), we obtain

$$\begin{aligned}&\left(2-\tfrac{1}{m+1}\right)\left(1+N_m^{s_0+\tau_1 q+\tau_1-\overline{s}_h}+CN_0^{\overline{\mu}_2}N_m^{\tau_1 q+\tau_1+1-\overline{\mu}_2}\right)\\ &+C\left(N_m^{\tau_1 q+\tau_1+1}+N_0^{\overline{\mu}_2}N_m^{2\tau_1 q+3\tau_1+1-\overline{\mu}_2}\right)N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\varepsilon\gamma^{-1}\\ &\leqslant 2-\tfrac{1}{m+2},\end{aligned}$$

so that

$$\overline{\delta}_{m+1}(\overline{s}_h+\widetilde{\mathtt{p}})\leqslant\left(2-\tfrac{1}{m+2}\right)\nu(\overline{s}_h+\widetilde{\mathtt{p}}).$$

This completes the proof of the second statement in (7.116).
➢ *Conclusion.* From (7.104), we get for $s=s_0$.

$$\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim \overline{\delta}_m(s_0+\tau_1 q+\tau_1)+\varkappa_m\delta_m(s_0+\tau_1 q+2\tau_1+1).$$

By interpolation inequality in Lemma A.1, (7.43) applied with $\mu_2=\overline{\mu}_2$, (7.116) applied with $\widetilde{\mathtt{p}}=0$ and Sobolev embeddings, we have for some $\overline{\theta}\in(0,1)$

$$\begin{aligned}\overline{\delta}_m(s_0+\tau_1 q+\tau_1)&\leqslant \overline{\delta}_m(s_0+\tau_1 q+2\tau_1+1)\\ &\lesssim \overline{\delta}_m(s_0)^{\overline{\theta}}\overline{\delta}_m(\overline{s}_h)^{1-\overline{\theta}}\\ &\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h)\end{aligned}$$

and

$$\begin{aligned}\delta_m(s_0+\tau_1 q+2\tau_1+1) &\lesssim \delta_m(s_0)^{\overline{\theta}}\delta_m(\overline{s}_h)^{1-\overline{\theta}}\\ &\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\delta_0(\overline{s}_h)\\ &\lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}.\end{aligned}$$

Therefore

$$\|\Delta_{12}g_m\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h).$$

Now from (7.104), we have

$$\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}+1}^{\gamma,\mathcal{O}} \lesssim \overline{\delta}_m(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)+\varkappa_m\delta_m(\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+2).$$

Applying (7.116) with

$$\widetilde{\mathtt{p}}=\mathtt{p}+\tau_1 q+\tau_1+1, \tag{7.118}$$

which is possible since from (7.3), (7.15) and (7.14), one has $\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+4\leqslant s_h+\sigma_1$, we find

$$\begin{aligned}\overline{\delta}_m(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1) &\leqslant 2\nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)\\ &\leqslant 2\overline{\delta}_0(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)+2\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\end{aligned}$$

Implementing a similar proof to (6.43) based on the kernel decomposition (6.39), the composition laws and (7.8), we find

$$\begin{aligned}\forall s\geqslant s_0,\quad \overline{\delta}_0(s) &= \gamma^{-1}\|\Delta_{12}V_{\varepsilon r}\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0+1}^{\gamma,\mathcal{O}}\max_{\ell=1,2}\|r_\ell\|_{q,s+1}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

On the other hand, since

$$\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+3\leqslant s_h+\sigma_1,$$

one may obtain through combining (7.66) and (7.16)

$$\begin{aligned}\delta_m(\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+2) &\leqslant C\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+2\tau_1+3}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\varepsilon\gamma^{-1}.\end{aligned}$$

Thus, by interpolation inequality in Lemma A.1, we finally obtain for some $\overline{\theta}\in(0,1)$

$$\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim N_0^{\overline{\theta}\overline{\mu}_2}N_m^{-\overline{\theta}\overline{\mu}_2}\nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1). \tag{7.119}$$

Choosing N_0 sufficiently large, then the composition law in Lemma A.1 allows to get

$$\begin{aligned}\sum_{k=0}^{\infty}\|\Delta_{12}g_k\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \nu(\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+1)N_0^{\overline{\theta}\overline{\mu}_2}\sum_{k=0}^{\infty}N_m^{-\overline{\theta}\overline{\mu}_2}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.120}$$

Finally, gathering (7.91), (7.98), (7.119) and (7.120), we get

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}.$$

Putting together this estimate, (A.19) and (7.94) yields

$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}} &\lesssim \|\Delta_{12}\beta\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\tau_1 q+\tau_1+2}.\end{aligned}$$

▶ **Estimate on $\Delta_{12}c_i$.** Since $V_0=\Omega+I_1K_1$ is independent of r, then

$$\Delta_{12}c_i=\sum_{m=0}^{\infty}\Delta_{12}(V_{m+1}-V_m).$$

Therefore we obtain in view of (7.54), Sobolev embeddings and (7.116) applied with $\widetilde{\mathtt{p}}=0$,

$$\begin{aligned}\|\Delta_{12}(V_{m+1}-V_m)\|^{\gamma,\mathcal{O}}_{q} &= \|\langle\Delta_{12}f_m\rangle_{\varphi,\theta}\|^{\gamma,\mathcal{O}}_{q}\\ &\leqslant C\gamma\overline{\delta}_m(s_0)\\ &\leqslant C\gamma N_0^{\overline{\mu}_2}N_m^{-\overline{\mu}_2}\nu(\overline{s}_h).\end{aligned}$$

Hence by the composition law in Lemma A.1, Lemma A.5 and (7.8) one may find

$$\begin{aligned}\|\Delta_{12}c_i\|^{\gamma,\mathcal{O}}_{q} &\leqslant \sum_{m=0}^{\infty}\|\Delta_{12}(V_{m+1}-V_m)\|^{\gamma,\mathcal{O}}_{q}\\ &\leqslant C\gamma\nu(\overline{s}_h)N_0^{\overline{\mu}_2}\sum_{m=0}^{\infty}N_m^{-\overline{\mu}_2}\\ &\leqslant C\varepsilon\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+2}.\end{aligned}$$

This achieves the proof of Proposition 7.2. □

7.2.2 Action on the nonlocal term

In this section, we shall analyze the conjugation action by $\mathscr{B}$ on the nonlocal term appearing in the linearized operator $\mathcal{L}_{\varepsilon r}$ described in Proposition 7.1. The main result reads as follows.

Proposition 7.3. *Let $(\gamma,q,d,\tau_1,s_0,\overline{s}_h,\sigma_1,S)$ satisfy* (A.2), (A.1), (7.3) *and* (7.14). *We set*

$$\sigma_2\triangleq s_0+\sigma_1+3. \tag{7.121}$$

For any $(\mu_2,\mathtt{p},s_h)$ satisfying the condition (7.15)*, there exists $\varepsilon_0>0$ such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0 \quad\text{and}\quad \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s_h+\sigma_2}\leqslant 1, \tag{7.122}$$

then in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)$, we have

$$\mathfrak{L}_{\varepsilon r}\triangleq\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}^0_n,$$

where $\mathcal{K}_\lambda$ is defined in (5.12)*, $\mathtt{E}^0_n$ is introduced in Proposition 7.2 and $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving self-adjoint integral operator satisfying*

$$\forall s\in[s_0,S],\quad \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\mathtt{O\text{-}d},q,s}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+\sigma_2}\right). \tag{7.123}$$

In addition, if i_1 *and* i_2 *are two tori satisfying the smallness property* (7.122)*, then*

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_2}^{\gamma,\mathcal{O}}. \tag{7.124}$$

Proof. We recall from Proposition 7.1 and Lemma 5.1. that

$$\mathcal{L}_{\varepsilon r}=\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r},$$

where $\mathbf{L}_{\varepsilon r}$ is a nonlocal operator defined by

$$\mathbf{L}_{\varepsilon r}(\rho)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)K_0(\lambda A_{\varepsilon r}(\varphi,\theta,\eta))d\eta,$$

with

$$A_{\varepsilon r}(\varphi,\theta,\eta)=\Big(\big(R(\varphi,\eta)-R(\varphi,\theta)\big)^2+4R(\varphi,\eta)R(\varphi,\theta)\sin^2\left(\tfrac{\eta-\theta}{2}\right)\Big)^{\frac{1}{2}}$$

and

$$R(\varphi,\theta)=\big(1+2\varepsilon r(\varphi,\theta)\big)^{\frac{1}{2}}.$$

Notice that we have removed the dependance in (λ,ω) from the functions in order to alleviate the notation. Hence by Proposition 7.2, Lemma A.3-(i) and (6.24), we have in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\begin{aligned}\mathfrak{L}_{\varepsilon r}&\triangleq\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}\\&=\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)\big)\mathscr{B}-\mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r}\mathscr{B}\\&=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r}\mathscr{B}+\mathtt{E}_n^0\\&=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\Big(\mathcal{B}^{-1}\left(\mathcal{K}_\lambda*\cdot\right)\mathscr{B}+\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\Big)+\mathtt{E}_n^0.\end{aligned} \tag{7.125}$$

From a direct computation using (5.12) combined with (A.14) and (A.12), we find

$$\mathcal{B}^{-1}\big(\mathcal{K}_\lambda*\mathscr{B}\rho\big)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)K_0\big(\lambda\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta)\big)d\eta,$$

where

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta)\triangleq2\left|\sin\left(\tfrac{\eta-\theta}{2}+\widehat{h}(\varphi,\theta,\eta)\right)\right|,$$

with

$$\widehat{h}(\varphi,\theta,\eta)\triangleq\tfrac{\widehat{\beta}(\varphi,\eta)-\widehat{\beta}(\varphi,\theta)}{2}.$$

Using elementary trigonometric identities, we can write

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta)=2\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|v_{\widehat{\beta},2}(\varphi,\theta,\eta), \tag{7.126}$$

with

$$v_{\widehat{\beta},2}(\varphi,\theta,\eta)\triangleq\cos\left(\widehat{h}(\varphi,\theta,\eta)\right)+\tfrac{\sin\left(\widehat{h}(\varphi,\theta,\eta)\right)}{\tan\left(\frac{\eta-\theta}{2}\right)}.$$

Notice that $v_{0,2}=1$ and one may write

$$v_{\widehat{\beta},2}(\theta,\eta)=1+\big(\cos\big(\widehat{h}(\theta,\eta)\big)-1\big)+\tfrac{\widehat{h}(\theta,\eta)}{\tan\left(\frac{\eta-\theta}{2}\right)}+\left(\tfrac{\sin(\widehat{h}(\theta,\eta))}{\widehat{h}(\theta,\eta)}-1\right)\tfrac{\widehat{h}(\theta,\eta)}{\tan\left(\frac{\eta-\theta}{2}\right)}$$

and then using Lemma A.1-(iv)-(v), Lemma A.2 and (7.19), we obtain

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\big\|v_{\widehat{\beta},2}(*,\cdot,\bullet,\eta+\bullet)-1\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+1}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right),\\ \forall k\in\mathbb{N}^*,\ \sup_{\eta\in\mathbb{T}}\big\|(\partial_\theta^k v_{\widehat{\beta},2})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+k+1}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1+k}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.127}$$

Proceeding as for (6.39), one obtains the decomposition

$$K_0(\lambda\mathscr{A}_{\widehat{\beta}}(\lambda,\omega,\varphi,\theta,\eta))=K_0\left(2\lambda\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)+\mathscr{K}(\eta-\theta)\mathscr{K}^1_{\widehat{\beta},2}(\varphi,\theta,\eta)+\mathscr{K}^2_{\widehat{\beta},2}(\varphi,\theta,\eta)$$

with similar estimates to (6.35) and (6.38), that is, for all $k\in\mathbb{N}$,

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\left(\big\|(\partial_\theta^k\mathscr{K}^1_{\widehat{\beta},2})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}+\big\|(\partial_\theta^k\mathscr{K}^2_{\widehat{\beta},2})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}\right)&\lesssim\|\widehat{\beta}\|_{q,s+1+k}^{\gamma,\mathcal{O}}\\ &\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1+k}^{\gamma,\mathcal{O}}\right),\end{aligned}\tag{7.128}$$

where the symbols $*,\cdot,\bullet$ stand for $(\lambda,\omega),\varphi,\theta$, respectively. Now we shall denote by $\mathbf{L}_{\varepsilon r,2}$ the integral operator with the kernel $\mathbb{K}_{\varepsilon r,2}$ defined by

$$\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)\triangleq\mathscr{K}(\eta-\theta)\mathscr{K}^1_{\widehat{\beta},2}(\varphi,\theta,\eta)+\mathscr{K}^2_{\widehat{\beta},2}(\varphi,\theta,\eta).\tag{7.129}$$

Then we find the decomposition

$$\mathcal{B}^{-1}\left(\mathcal{K}_\lambda*\cdot\right)\mathscr{B}=\mathcal{K}_\lambda*\cdot+\mathbf{L}_{\varepsilon r,2}.$$

Inserting this identity into (7.125) allows to get

$$\mathfrak{L}_{\varepsilon r}=\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}^0_n,$$

with

$$\mathfrak{R}_{\varepsilon r}\triangleq-\mathbf{L}_{\varepsilon r,2}-\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}.\tag{7.130}$$

Observe that by (7.4) and (7.23) we can easily check that the kernel $\mathbb{K}_{\varepsilon r,2}$ satisfies the following symmetry property

$$\mathbb{K}_{\varepsilon r,2}(-\varphi,-\theta,-\eta)=\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)\in\mathbb{R},\tag{7.131}$$

which implies in turn, according to Lemma A.7, that $\mathbf{L}_{\varepsilon r,2}$ is a real and reversibility preserving operator. Moreover, one obtains from (7.128)

$$\max_{k\in\{0,1,2\}}\big\|(\partial_\theta^k\mathbb{K}_{\varepsilon r,2})(*,\cdot,\bullet,\eta+\bullet)\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right).\tag{7.132}$$

Our next purpose is to highlight some properties of the operator $\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}$ which takes the integral form

$$\big(\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\big)\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)d\eta,\tag{7.133}$$

where the kernel $\widehat{\mathbb{K}}_{\varepsilon r,1}$ is related to the kernel $\mathbb{K}_{\varepsilon r,1}$ defined in (6.25) through the formula,

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)\triangleq\mathbb{K}_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\tag{7.134}$$

It is quite easy to check from (6.27) and (7.23), that

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(-\varphi,-\theta,-\eta)=\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)\in\mathbb{R}. \tag{7.135}$$

According to (6.25), one gets the decomposition

$$\widehat{\mathbb{K}}_{\varepsilon r,1}(\varphi,\theta,\eta)=\widehat{\mathscr{K}}(\varphi,\theta,\eta)\widehat{\mathscr{K}}_{\varepsilon r,1}^{1}(\varphi,\theta,\eta)+\widehat{\mathscr{K}}_{\varepsilon r,1}^{2}(\varphi,\theta,\eta), \tag{7.136}$$

with

$$\begin{aligned}\widehat{\mathscr{K}}(\varphi,\theta,\eta)&\triangleq\mathscr{K}\big(\eta-\theta+\widehat{\beta}(\varphi,\eta)-\widehat{\beta}(\varphi,\theta)\big),\\ \widehat{\mathscr{K}}_{\varepsilon r,1}^{1}(\varphi,\theta,\eta)&\triangleq\mathscr{K}_{\varepsilon r,1}^{1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big),\\ \widehat{\mathscr{K}}_{\varepsilon r,1}^{2}(\varphi,\theta,\eta)&\triangleq\mathscr{K}_{\varepsilon r,1}^{2}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\end{aligned}$$

Coming back to (6.26) and using the morphism property of the logarithm, combined with (7.126) we deduce that

$$\begin{aligned}\widehat{\mathscr{K}}(\varphi,\theta,\eta)&=\sin^{2}\left(\tfrac{\eta-\theta}{2}\right)v_{\widehat{\beta},2}^{2}(\varphi,\theta,\eta)\left(\log\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|+\log\left|v_{\widehat{\beta},2}(\varphi,\theta,\eta)\right|\right)\\ &=\mathscr{K}\left(\eta-\theta\right)+\mathscr{K}\left(\eta-\theta\right)\left(v_{\widehat{\beta},2}^{2}(\varphi,\theta,\eta)-1\right)\\ &\quad+\sin^{2}\left(\tfrac{\eta-\theta}{2}\right)v_{\widehat{\beta},2}^{2}(\varphi,\theta,\eta)\log\left|v_{\widehat{\beta},2}(\varphi,\theta,\eta)\right|.\end{aligned}$$

Combining Lemma A.1-(iv)-(v), (7.127), (7.19) gives for any $\eta\in\mathbb{T}$

$$\begin{aligned}\max_{k\in\{0,1,2\}}\left\|(\partial_{\theta}^{k}\widehat{\mathscr{K}})(*,\cdot,\centerdot,\eta+\centerdot)\right\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\widehat{\beta}\|_{q,s+3}^{\gamma,\mathcal{O}}\left(1-\log\left|\sin\left(\frac{\eta}{2}\right)\right|\right)-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|+1\\ &\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+\sigma_{2}}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\frac{\eta}{2}\right)\right|\right)-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|+1.\end{aligned} \tag{7.137}$$

The next goal is to prove that

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\left\|(\partial_{\theta}^{k}\widehat{\mathscr{K}}_{\varepsilon r,1}^{1})(*,\cdot,\centerdot,\eta+\centerdot)\right\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_{0}\|_{q,s+\sigma_{2}}^{\gamma,\mathcal{O}}\right). \tag{7.138}$$

For this aim we first write from (6.36) and (C.2)

$$\begin{aligned}\mathscr{K}_{\varepsilon r,1}^{1}(\varphi,\theta,\eta)&=4\lambda^{2}\big(1-v_{\varepsilon r,1}(\varphi,\theta,\eta)\big)\tilde{I}_{\lambda}(\eta-\theta)\\ &\quad-4\lambda^{2}(v_{\varepsilon r,1}(\varphi,\theta)-1)^{2}\int_{0}^{1}(1-t)I_{0}''\left(2\lambda\sin\left(\tfrac{\eta-\theta}{2}\right)\left(1-t+tv_{\varepsilon r,1}(\varphi,\theta,\eta)\right)\right)dt\\ &\triangleq4\lambda^{2}\big(1-v_{\varepsilon r,1}(\varphi,\theta,\eta)\big)\tilde{I}_{\lambda}(\eta-\theta)+G(\varphi,\theta,\eta),\end{aligned} \tag{7.139}$$

with

$$\begin{aligned}\tilde{I}_{\lambda}(\eta)&\triangleq\frac{I_{0}'\left(2\lambda\left|\sin\left(\frac{\eta}{2}\right)\right|\right)}{2\lambda\left|\sin\left(\frac{\eta}{2}\right)\right|}\\ &=\frac{1}{2}\sum_{m=0}^{\infty}\frac{\lambda^{2m}\sin^{2m}\left(\frac{\eta}{2}\right)}{m!(m+1)!}.\end{aligned}$$

Then we get the decomposition

$$\widehat{\mathscr{K}}_{\varepsilon r,1}^{1}(\varphi,\theta,\eta)=4\lambda^{2}\big[1-\widehat{v}_{\varepsilon r,1}(\varphi,\theta,\eta)\big]\widehat{I}_{\lambda}(\varphi,\theta,\eta)+\widehat{G}(\varphi,\theta,\eta),$$

with

$$\begin{aligned}\widehat{v}_{\varepsilon r,1}(\varphi,\theta,\eta)&\triangleq v_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big),\\ \widehat{I}_{\lambda}(\varphi,\theta,\eta)&\triangleq \tilde{I}_{\lambda}\big(\eta+\widehat{\beta}(\varphi,\eta)-\theta-\widehat{\beta}(\varphi,\theta)\big),\\ \widehat{G}(\varphi,\theta,\eta)&\triangleq G\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\end{aligned}$$

It follows that

$$\begin{aligned}\widehat{\mathscr{K}^1_{\varepsilon r,1}}(\varphi,\theta,\theta+\eta)=&4\lambda^2\big[1-v_{\varepsilon r,1}\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\theta+\eta+\widehat{\beta}(\varphi,\theta+\eta)\big)\big]\tilde{I}_{\lambda}\big(\eta+\widehat{\beta}(\varphi,\theta+\eta)-\widehat{\beta}(\varphi,\theta)\big)\\&+G\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\theta+\widehat{\beta}(\varphi,\eta+\theta)\big).\end{aligned}\tag{7.140}$$

Notice that $(\lambda,\eta)\mapsto\tilde{I}_{\lambda}(\eta)$ is $\mathcal{C}^{\infty}$, then using Lemma A.1-(v) and (7.19) yields for any $k\in\mathbb{N}$

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|(\partial_{\theta}^{k}\widehat{I}_{\lambda})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim 1+\|\widehat{\beta}\|_{q,s+k}^{\gamma,\mathcal{O}}\\&\lesssim 1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+k}^{\gamma,\mathcal{O}}.\end{aligned}$$

Now using (6.31), Lemma A.1-(v), (7.18), (7.7), (7.127) and proceeding as in (6.32) we obtain

$$\begin{aligned}\sup_{\eta\in\mathbb{T}}\|\widehat{v}_{\varepsilon r,1}\big(*,\cdot,\bullet,\eta+\bullet\big)-1\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\|r\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon^2\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)\end{aligned}$$

and

$$\begin{aligned}\max_{k\in\{1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_{\theta}^{k}\widehat{v}_{\varepsilon r,1})\big(*,\cdot,\bullet,\eta+\bullet\big)\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\|r\|_{q,s+3}^{\gamma,\mathcal{O}}+\varepsilon^2\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+3}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Arguing as above using the structure of G detailed in (7.139) allows to get

$$\sup_{\eta\in\mathbb{T}}\|\widehat{G}\big(*,\cdot,\bullet,\eta+\bullet\big)-1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)$$

and

$$\max_{k\in\{1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_{\theta}^{k}\widehat{G}\big(*,\cdot,\bullet,\eta+\bullet\big)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).$$

Thus applying the product laws in Lemma A.1 and using the preceding estimates combined with (7.140) imply

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|\big(\partial_{\theta}^{k}\widehat{\mathscr{K}^1_{\varepsilon r,1}}\big)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right),\tag{7.141}$$

which gives in particular (7.138). The estimate of the last term $\mathscr{K}^2_{\varepsilon r,1}$ in (7.136), which is connected to (6.37), can be treated in a similar way to the estimate (7.141) and one finds

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\left\|(\partial_{\theta}^{k}\widehat{\mathscr{K}^2_{\varepsilon r,1}})\big(*,\cdot,\bullet,\eta+\bullet\big)\right\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\ \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right).\tag{7.142}$$

Consequently, putting together (7.136),(7.137), (7.138) and (7.142) yields

$$\max_{k\in\{0,1,2\}}\big\|(\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1})(*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right). \tag{7.143}$$

By (7.130) we infer that $\mathfrak{R}_{\varepsilon r}$ is an integral operator of kernel $\mathbb{K}_{\varepsilon r}$ given by

$$\mathbb{K}_{\varepsilon r}\triangleq-\widehat{\mathbb{K}}_{\varepsilon r,1}-\mathbb{K}_{\varepsilon r,2}.$$

Therefore, by virtue of Lemma A.7 combined with (7.132) and (7.143) we find, taking $\sigma_2=s_0+\sigma_1+3$,

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\max_{k\in\{0,1,2\}}\int_{\mathbb{T}}\left(\|(\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1})(*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\|_{q,s+s_0}^{\gamma,\mathcal{O}}+\|(\partial_\theta^k\mathbb{K}_{\varepsilon r,2})(*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\|_{q,s+s_0}^{\gamma,\mathcal{O}}\right)d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+\sigma_1+3}^{\gamma,\mathcal{O}}\right)\int_{\mathbb{T}}\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right)d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Notice that by (7.135), (7.131), the kernel $\mathbb{K}_{\varepsilon r}$ satisfies the following symmetry property

$$\mathbb{K}_{\varepsilon r}(-\varphi,-\theta,-\eta)=\mathbb{K}_{\varepsilon r}(\varphi,\theta,\eta)\in\mathbb{R}, \tag{7.144}$$

which implies in view of Lemma A.7 that $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving Toeplitz in time integral operator. It remains to estimate the quantity $\max\limits_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}$. This is will be implemented as before and we shall here sketch the main ideas. First we observe that for $k\in\{0,1\}$ the kernel of $\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}$ is given by

$$\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}=-\Delta_{12}\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1}-\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}.$$

To estimate $\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}$ we shall use (7.129) leading to

$$\Delta_{12}\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)=\mathscr{K}(\theta-\eta)\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)+\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^2(\varphi,\theta,\eta) \tag{7.145}$$

and

$$\begin{aligned}\Delta_{12}\partial_\theta\mathbb{K}_{\varepsilon r,2}(\varphi,\theta,\eta)=\mathscr{K}(\theta-\eta)\Delta_{12}\partial_\theta\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)+\mathscr{K}'(\theta-\eta)\Delta_{12}\mathscr{K}_{\widehat{\beta},2}^1(\varphi,\theta,\eta)\\+\Delta_{12}\partial_\theta\mathscr{K}_{\widehat{\beta},2}^2(\varphi,\theta,\eta).\end{aligned} \tag{7.146}$$

Observe from (7.126) that the preceding kernels can be expressed with respect to $\widehat{\beta}$. Then proceeding in a similar way to (6.48) we obtain

$$\forall\, i\in\{1,2\},\qquad\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\|d_{\widehat{\beta}}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i[\rho](*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\widehat{\beta}\|_{q,s+2}^{\gamma,\mathcal{O}}. \tag{7.147}$$

Applying Taylor Formula yields for all $i\in\{1,2\}$ and for all $k\in\{0,1\}$,

$$\Delta_{12}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i(\varphi,\theta,\theta+\eta)=\int_0^1 d_{\widehat{\beta}}\partial_\theta^k\mathscr{K}_{(1-\tau)\widehat{\beta}_2+\tau\widehat{\beta}_1,2}^i[\widehat{\beta}_1-\widehat{\beta}_2](\varphi,\theta,\theta+\eta)\,d\tau.$$

It follows from (7.147) that for all $i\in\{1,2\}$ and for all $k\in\{0,1\}$

$$\big\|\Delta_{12}\partial_\theta^k\mathscr{K}_{\widehat{\beta},2}^i(*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\big\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\widehat{\beta}_2-\widehat{\beta}_1\|_{q,s+2}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2-\widehat{\beta}_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\int_0^1\|(1-\tau)\widehat{\beta}_2+\tau\widehat{\beta}_1\|_{q,s+2}^{\gamma,\mathcal{O}}\,d\tau.$$

Therefore, by our previous choice of σ_2, we obtain in view of (7.19), (7.22) (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0$) and the smallness condition (7.122),

$$\begin{aligned}\forall i\in\{1,2\},\quad \max_{k\in\{0,1\}}\big\|\Delta_{12}\partial_\theta^k\mathscr{K}^i_{\widehat{\beta},2}(*,\cdot,\blacksquare,\eta+\blacksquare)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}\Big(1+\varepsilon\gamma^{-1}\big(1+\|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}\big)\Big)\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\end{aligned}$$

Inserting this estimate into (7.145) and (7.146) yields

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r,2}(*,\cdot,\blacksquare,\eta+\blacksquare)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\tag{7.148}$$

Using similar techniques based on Taylor Formula, one can estimate $\Delta_{12}\partial_\theta\widehat{\mathbb{K}}_{\varepsilon r,1}$. We use in particular the identity (7.136) combined with (6.48), (7.19), (7.22) and the smallness condition (7.122) allowing to get

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\widehat{\mathbb{K}}_{\varepsilon r,1}(*,\cdot,\blacksquare,\eta+\blacksquare)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\tag{7.149}$$

Putting together (7.148) and (7.149) gives

$$\max_{k\in\{0,1\}}\sup_{\eta\in\mathbb{T}}\big\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}(*,\cdot,\blacksquare,\eta+\blacksquare)\big\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.$$

Comibining this estimate with Lemma A.7 yields

$$\begin{aligned}\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|^{\gamma,\mathcal{O}}_{\mathrm{O\text{-}d},q,\overline{s}_h+\mathtt{p}}&\lesssim\max_{k\in\{0,1\}}\int_{\mathbb{T}}\|\Delta_{12}\partial_\theta^k\mathbb{K}_{\varepsilon r}(*,\cdot,\blacksquare,\eta+\blacksquare)\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}d\eta\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_2}.\end{aligned}$$

This completes the proof of the Proposition 7.3. $\square$

7.3 Diagonalization up to small errors

The main goal of this section is to diagonalize, up to small errors, the operator $\widehat{\mathcal{L}}_\omega$ discussed in Proposition 7.1 and given by

$$\widehat{\mathcal{L}}_\omega=\Pi^\perp_{\mathbb{S}_0}(\mathcal{L}_{\varepsilon r}-\varepsilon\partial_\theta\mathcal{R})\Pi^\perp_{\mathbb{S}_0}.$$

This will be performed in two main steps. First, we shall explore the effect of the frequency localization in the normal direction on the transport reduction discussed in Section 7.2. We essentially get the same structure up to a small perturbation of finite-dimensional rank. Then, in the second step we shall implement a KAM reducibility scheme in order to reduce the remainder to a diagonal one modulo small fast decaying operators. This will be performed through the use of a suitable strong topology on continuous operators given by (A.23). With this topology one has tame estimates and the Toeplitz structure of the remainder is very important in this part. The reduction will be conducted by assuming non resonance conditions stemming from the second order Melnikov conditions needed in the resolution of adequate homological equations during the scheme.

7.3.1 Projection in the normal directions

In this section, we study the effects of the reduction of the transport part when the linearized operator is localized in the normal directions. Notice that the change of coordinates does not stabilize the normal subspace and as we shall see the defect of the commutation can be modeled by projectors of finite ranks.

Let us define

$$\mathscr{B}_\perp \triangleq \Pi_{\mathbb{S}_0}^\perp \mathscr{B} \Pi_{\mathbb{S}_0}^\perp,$$

where the transformation $\mathscr{B}$ is introduced in (A.12) and constructed in Proposition 7.2. Recall that the projection $\Pi_{\mathbb{S}_0}^\perp$ and the space $L^2_\perp$ were respectively defined in (6.9) and (6.8). We also recall the following notations

$$\mathbf{e}_{l,j}(\varphi,\theta) = e^{\mathrm{i}(l\cdot\varphi + j\theta)} \quad \text{and} \quad e_m(\theta) = e^{\mathrm{i}m\theta}.$$

In the sequel, we may use the following notation

$$H^s_\perp \triangleq H^s \cap L^2_\perp.$$

The first main result of this section reads as follows.

Lemma 7.1. *Let $\mathscr{B}$ the transformation constructed in Proposition* 7.2, *then under the condition* (7.122) *and* (7.15), *the following assertions hold.*

(i) For all $s \in [s_0, S]$, the operator $\mathscr{B}_\perp : W^{q,\infty,\gamma}\big(\mathcal{O}, H^s_\perp\big) \to W^{q,\infty,\gamma}\big(\mathcal{O}, H^s_\perp\big)$ is continuous and invertible, with

$$\|\mathscr{B}_\perp^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{7.150}$$

In addition, we have the representations

$$\mathscr{B}_\perp\rho = \mathscr{B}\rho - \sum_{m\in\mathbb{S}_0}\big\langle \rho, \big(\mathcal{B}^{-1} - \mathrm{Id}\big)e_m\big\rangle_{L^2_\theta(\mathbb{T})} e_m$$

and

$$\mathscr{B}_\perp^{-1}\rho = \mathscr{B}^{-1}\rho - \sum_{m\in\mathbb{S}_0}\big\langle \rho, \big(\mathcal{B} - \mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})} \mathscr{B}^{-1}e_m,$$

where

$$\mathbf{A}(\varphi) \triangleq \Big(\big\langle e_m, \mathcal{B}e_k\big\rangle_{L^2_\theta(\mathbb{T})}\Big)_{\substack{m\in\mathbb{S}_0\\ k\in\mathbb{S}_0}}, \quad \mathbf{A}^{-1}(\varphi) \triangleq \Big(\alpha_{k,m}\Big)_{\substack{m\in\mathbb{S}_0\\ k\in\mathbb{S}_0}}, \quad g_m(\varphi,\theta) \triangleq \sum_{k\in\mathbb{S}_0}\alpha_{k,m}(\varphi)e_k(\theta),$$

with the estimate

$$\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m} - \delta_{km}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\Big(1 + \|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\Big).$$

(ii) Given two tori i_1 and i_2 satisfying the smallness condition (7.122), *one has*

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}. \tag{7.151}$$

Proof. **(i)** The first estimate concerning $\mathscr{B}_\perp$ follows easily from the continuity of the orthogonal projector $\Pi_{\mathbb{S}_0}^\perp$ on the space $L^2_\perp$, combined with (7.18). For the representation of $\mathscr{B}_\perp$, take $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s_\perp)$ and set

$$\mathscr{B}_\perp\rho = \Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\rho = \Pi_{\mathbb{S}_0}^\perp\mathscr{B}\rho \triangleq g.$$

Next, we write the following splitting

$$\mathscr{B}\rho = g + h \quad \text{with} \quad \Pi_{\mathbb{S}_0}h = h. \tag{7.152}$$

Notice that the projector $\Pi_{\mathbb{S}_0}$ is defined by

$$\Pi_{\mathbb{S}_0}\rho=\sum_{j\in\mathbb{S}_0}\rho_j e_j=\Pi_{\overline{\mathbb{S}}}\rho+\langle\rho\rangle_\theta,$$

where $\Pi_{\overline{\mathbb{S}}}$ is defined in (6.9) and $\langle\cdot\rangle_\theta$ denotes the average in the variable θ. Therefore

$$h(\varphi,\theta)=\sum_{m\in\mathbb{S}_0}h_m(\varphi)e_m(\theta),$$

supplemented with the orthogonal conditions

$$\forall\, k\in\mathbb{S}_0,\quad \langle\mathscr{B}\rho-h,e_k\rangle_{L^2_\theta(\mathbb{T})}=0.$$

This implies

$$h(\varphi,\theta)=\sum_{m\in\mathbb{S}_0}\langle\mathscr{B}\rho,e_m\rangle_{L^2_\theta(\mathbb{T})}e_m(\theta).$$

Using Lemma A.3-(iii) leads to

$$h(\varphi,\theta)=\sum_{m\in\mathbb{S}_0}\langle\rho,\mathcal{B}^{-1}e_m\rangle_{L^2_\theta(\mathbb{T})}e_m(\theta).$$

Inserting this identity into (7.152) yields

$$\mathscr{B}_\perp\rho=g=\mathscr{B}\rho-\sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{B}^{-1}e_m\big\rangle_{L^2_\theta(\mathbb{T})}e_m.$$

Since $\forall m\in\mathbb{S}_0$, $\big\langle\rho,e_m\big\rangle_{L^2_\theta(\mathbb{T})}=0$, then

$$\mathscr{B}_\perp\rho=g=\mathscr{B}\rho-\sum_{m\in\mathbb{S}_0}\big\langle\rho,\big(\mathcal{B}^{-1}-\mathrm{Id}\big)e_m\big\rangle_{L^2_\theta(\mathbb{T})}e_m.$$

This ensures the desired representation of $\mathscr{B}_\perp$.

Next, we intend to establish similar representation for $\mathscr{B}_\perp^{-1}$. Let $g\in W^{q,\infty,\gamma}(\mathcal{O},H^s_\perp)$ and we need to solve the equation

$$f\in W^{q,\infty,\gamma}(\mathcal{O},H^s_\perp),\quad \mathscr{B}_\perp f=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}f=g.$$

This is equivalent to

$$\mathscr{B}f=g+h,\quad\text{with}\quad \Pi_{\mathbb{S}_0}h=h\quad\text{and}\quad \Pi_{\mathbb{S}_0}f=0.$$

Then we get

$$f=\mathscr{B}^{-1}\big(g+h\big),\quad\text{with}\quad \Pi_{\mathbb{S}_0}h=h\quad\text{and}\quad \Pi_{\mathbb{S}_0}f=0. \tag{7.153}$$

The condition $\Pi_{\mathbb{S}_0}f=0$ is equivalent to,

$$\forall k\in\mathbb{S}_0,\quad \big\langle\mathscr{B}^{-1}\big(g+h\big),e_k\big\rangle_{L^2_\theta(\mathbb{T})}=0.$$

Therefore using Lemma A.3-(iii) the latter equation reads

$$\forall k\in\mathbb{S}_0,\quad \big\langle g+h,\widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})}=0\quad\text{with}\quad \widehat{e}_k(\varphi,\theta)\triangleq\mathcal{B}e_k(\varphi,\theta)=e^{\mathrm{i}k(\theta+\beta(\varphi,\theta))},$$

which will fix h. Indeed, by expanding $h(\varphi,\theta)=\sum_{m\in\mathbb{S}_0}a_m(\varphi)e_m(\theta)$, we can transform the preceding system into

$$\forall k\in\mathbb{S}_0,\quad \sum_{m\in\mathbb{S}_0}a_m(\varphi)\big\langle e_m,\widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})}=-\big\langle g,\widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})}. \tag{7.154}$$

Define the matrix

$$\mathbf{A}(\varphi)\triangleq\big(c_{m,k}(\varphi)\big)_{(m,k)\in\mathbb{S}_0^2},\quad c_{m,k}(\varphi)\triangleq\langle e_m,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}=\int_{\mathbb{T}}e^{\mathrm{i}((m-k)\theta-k\beta(\varphi,\theta))}d\theta. \tag{7.155}$$

Notice that according to (7.23) and the change of variables $\theta\mapsto-\theta$, one obtains

$$\forall(m,k)\in\mathbb{S}_0^2,\quad\forall\varphi\in\mathbb{T}^d,\quad c_{m,k}(-\varphi)=c_{-m,-k}(\varphi)=\overline{c_{m,k}(\varphi,\theta)}. \tag{7.156}$$

One can check by slight adaptation of the composition law in Lemma A.1 and using the smallness condition (7.122) and (7.19)

$$\begin{aligned}\|c_{m,m}-1\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\int_{\mathbb{T}}\|e^{-\mathrm{i}m\beta(\cdot,\theta))}-1\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}d\theta\\&\lesssim\|\beta\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{7.157}$$

For $k\neq m\in\mathbb{S}_0$ we use integration by parts,

$$c_{m,k}(\varphi)=\tfrac{k}{\mathrm{i}(m-k)}\int_{\mathbb{T}}e^{\mathrm{i}((m-k)\theta-k\beta(\varphi,\theta))}\partial_\theta\beta(\varphi,\theta)d\theta.$$

Then using product laws and composition laws in Lemma A.1 combined with (7.19) yield

$$\begin{aligned}\sup_{\substack{(m,k)\in\mathbb{S}_0^2\\m\neq k}}\|c_{m,k}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Finally, we get that

$$\mathbf{A}(\varphi)=\mathrm{Id}+\mathbf{R}(\varphi)\quad\text{with}\quad\|\mathbf{R}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{7.158}$$

Hence under the smallness condition $\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\ll1$ following from (7.122), combined with the product laws in Lemma A.1 we get that $\mathbf{A}$ is invertible with

$$\begin{aligned}\|\mathbf{A}^{-1}-\mathrm{Id}\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{7.159}$$

Therefore the system (7.154) is invertible and one gets a unique solution given by

$$a_m(\varphi)=-\sum_{k\in\mathbb{S}_0}\alpha_{m,k}(\varphi)\big\langle g,\widehat{e}_k\big\rangle_{L^2_\theta(\mathbb{T})}\quad\text{with}\quad\mathbf{A}^{-1}(\varphi)\triangleq\Big(\alpha_{m,k}(\varphi)\Big)_{(m,k)\in\mathbb{S}_0^2}. \tag{7.160}$$

We claim that the coefficients of $\mathbf{A}^{-1}$ admit the same symmetry conditions as (7.156), that is

$$\forall(m,k)\in\mathbb{S}_0^2,\quad\forall\varphi\in\mathbb{T}^d,\quad\alpha_{m,k}(-\varphi)=\alpha_{-m,-k}(\varphi)=\overline{\alpha_{m,k}(\varphi)}. \tag{7.161}$$

This can be done through the series expansion $A^{-1}=\displaystyle\sum_{n\in\mathbb{N}}(-1)^n(A-\mathrm{Id})^n$ together with the fact that the entries of the monomials $(A-\mathrm{Id})^n$ satisfy in turn (7.156). Next, using the product laws yields

$$\sup_{m\in\mathbb{S}_0}\|a_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\sup_{k\in\mathbb{S}_0}\Big(\|\mathbf{A}^{-1}\|_{q,s}^{\gamma,\mathcal{O}}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|\mathbf{A}^{-1}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^{s}_\varphi}^{\gamma,\mathcal{O}}\Big).\tag{7.162}$$

Notice that one gets from (7.159)

$$\sup_{(m,k)\in\mathbb{S}_0^2}\|\alpha_{k,m}-\delta_{km}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right),$$

where δ_{km} denotes the Kronecker symbol. Let us now move to the estimate of the partial scalar product containing g in (7.162). Using the product laws in Lemma A.1 with Cauchy-Schwarz inequality gives

$$\begin{aligned}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\Big(\|g(\cdot,\theta)\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta(\cdot,\theta)}\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|g(\cdot,\theta)\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta(\cdot,\theta)}\|_{q,H^{s}_\varphi}^{\gamma,\mathcal{O}}\Big)d\theta\\&\lesssim\|g\|_{q,L^2_\theta H^s_\varphi}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta}\|_{q,L^2_\theta H^{s_0}_\varphi}^{\gamma,\mathcal{O}}+\|g\|_{q,L^2_\theta H^{s_0}_\varphi}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta}\|_{q,L^2_\theta H^{s}_\varphi}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}+\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\|e^{\mathrm{i}\beta}\|_{q,s}^{\gamma,\mathcal{O}}.\end{aligned}$$

Then applying the composition law as in (7.157) combined with with (7.19) and the smallness condition (7.122) gives

$$\begin{aligned}\|\langle g,\widehat{e}_k\rangle_{L^2_\theta(\mathbb{T})}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\|\beta\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right)\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Plugging this estimate into (7.162) and using (7.19), (7.159) combined with the smallness condition (7.122) and Sobolev embeddings implies

$$\begin{aligned}\sup_{m\in\mathbb{S}_0}\|a_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\right)\|g\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\|h\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\sum_{m\in\mathbb{S}_0}\|a_m\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Coming back to (7.153) and using (7.18), we get

$$\begin{aligned}\|f\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g+h\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g+h\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

It follows that

$$\|\mathscr{B}_\perp^{-1}g\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|g\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1+1}^{\gamma,\mathcal{O}}\|g\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In addition, from (7.160) and (7.153) we deduce the formula

$$\begin{aligned}\mathscr{B}_{\perp}^{-1}g(\varphi,\theta)&=\mathscr{B}^{-1}g(\varphi,\theta)-\sum_{\substack{m\in\mathbb{S}_0\\k\in\mathbb{S}_0}}\alpha_{m,k}(\varphi)\big\langle g,\mathcal{B}e_k\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m(\theta)\\&=\mathscr{B}^{-1}g(\varphi,\theta)-\sum_{m\in\mathbb{S}_0}\big\langle g,\mathcal{B}g_m\big\rangle_{L^2_\theta(\mathbb{T})}\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta),\end{aligned}\tag{7.163}$$

with

$$g_m(\varphi,\theta)\triangleq\sum_{k\in\mathbb{S}_0}\alpha_{m,k}(\varphi)e_k(\theta).\tag{7.164}$$

From (7.161) and the symmetry of $\mathbb{S}_0$, we infer

$$\forall m\in\mathbb{S}_0,\quad\forall(\varphi,\theta)\in\mathbb{T}^{d+1},\quad g_m(-\varphi,-\theta)=g_{-m}(\varphi,\theta)=\overline{g_m(\varphi,\theta)}.\tag{7.165}$$

Since $\Pi_{\mathbb{S}_0}^{\perp}g=g$ and $\Pi_{\mathbb{S}_0}^{\perp}g_m=0$ then $\big\langle g,g_m\big\rangle_{L^2_\theta(\mathbb{T})}=0$ and therefore

$$\big\langle g,\mathcal{B}g_m\big\rangle_{L^2_\theta(\mathbb{T})}=\big\langle g,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}.$$

Plugging this identity into (7.163) yields

$$\mathscr{B}_{\perp}^{-1}g=\mathscr{B}^{-1}g-\sum_{m\in\mathbb{S}_0}\big\langle g,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m.$$

(ii) Coming back to the definition of $c_{m,k}$ in (7.155), one can write

$$\forall(m,k)\in\mathbb{S}_0^2,\quad\Delta_{12}c_{m,k}=\langle e_m,(\Delta_{12}\mathcal{B})e_k\rangle_{L^2_\theta(\mathbb{T})}.$$

Hence, using Taylor Formula and (7.22), we have

$$\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}c_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}.$$

From (7.164), one has

$$\Delta_{12}g_m=\sum_{k\in\mathbb{S}_0}\Delta_{12}\alpha_{m,k}\,e_k.$$

Thus

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}\alpha_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}.\tag{7.166}$$

Using Neumann series, we can write

$$\mathbf{A}^{-1}(\varphi)=\mathrm{Id}+\sum_{n=1}^{\infty}(-1)^n\mathbf{R}^n(\varphi).$$

Therefore, the product laws in Lemma A.1 combined with (7.158) and the smallness condition (7.122) lead to

$$\begin{aligned}\|\Delta_{12}\mathbf{A}^{-1}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\sum_{n=1}^{\infty}\|\Delta_{12}\mathbf{R}^n\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\lesssim\|\Delta_{12}\mathbf{R}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

As a consequence,

$$\max_{(m,k)\in\mathbb{S}_0^2}\|\Delta_{12}\alpha_{m,k}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}.\tag{7.167}$$

Gathering (7.167) and (7.166) finally gives

$$\max_{m\in\mathbb{S}_0}\|\Delta_{12}g_m\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1+1}^{\gamma,\mathcal{O}}.$$

This achieves the proof of Lemma 7.1. □

In Lemma 7.1, the parameter $\mathtt{p}$ is subject to the constraint (7.15) and from now on, we shall fix it to the value

$$\mathtt{p}\triangleq 4\tau_2 q+4\tau_2.\tag{7.168}$$

This particular choice is determined through some constraints in the proof of the remainder reduction. More precisely, it appears in (7.364). Next we shall establish the second main result of this section.

Proposition 7.4. *Let* $(\gamma,q,d,\tau_1,s_0,s_h,\overline{s}_h,\mu_2,\mathtt{p},\sigma_2,S)$ *satisfy the assumptions* (A.2)*,* (A.1)*,* (7.3)*,* (7.15)*,* (7.121) *and* (7.168)*. Consider the operator* $\widehat{\mathcal{L}}_\omega$ *defined in Proposition* 7.1*.*
There exist $\varepsilon_0>0$ *and* $\sigma_3=\sigma_3(\tau_1,q,d,s_0)\geqslant\sigma_2$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_3}^{\gamma,\mathcal{O}}\leqslant 1,\tag{7.169}$$

then the following assertions hold true.

(i) For any $n\in\mathbb{N}^*$*, in the Cantor set* $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *introduced in Proposition* 7.2*, we have*

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp&=\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{R}_0+\mathtt{E}_n^1\\&\triangleq\big(\omega\cdot\partial_\varphi+\mathscr{D}_0\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{R}_0+\mathtt{E}_n^1\\&\triangleq\mathscr{L}_0+\mathtt{E}_n^1,\end{aligned}$$

where $\mathscr{D}_0$ *is a reversible Fourier multiplier given by*

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad\mathscr{D}_0\mathbf{e}_{l,j}=\mathrm{i}\mu_j^0\,\mathbf{e}_{l,j},$$

with

$$\mu_j^0(\lambda,\omega,i_0)\triangleq\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0)\qquad\textit{and}\qquad r^1(\lambda,\omega,i_0)\triangleq c_{i_0}(\lambda,\omega)-V_0(\lambda)$$

and such that

$$\|r^1\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\quad\text{and}\quad\|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.\tag{7.170}$$

(ii) The operator $\mathtt{E}_n^1$ *satisfies the following estimate*

$$\|\mathtt{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\tag{7.171}$$

(iii) $\mathscr{R}_0$ *is a real and reversible Toeplitz in time operator satisfying* $\mathscr{R}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ *with*

$$\forall s\in[s_0,S],\quad\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\tag{7.172}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{7.173}$$

(iv) The operator $\mathscr{L}_0$ satisfies

$$\forall s\in[s_0,S],\quad \|\mathscr{L}_0\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{7.174}$$

Proof. **(i)** We shall first start with finding a suitable expansion for $\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp$. Using the expression of $\widehat{\mathcal{L}}_\omega$ given in Proposition 7.1 and the decomposition $\mathrm{Id}=\Pi_{\mathbb{S}_0}+\Pi_{\mathbb{S}_0}^\perp$ we write

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp&=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp(\mathcal{L}_{\varepsilon r}-\varepsilon\partial_\theta\mathcal{R})\mathscr{B}_\perp\\&=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathcal{R}\mathscr{B}_\perp.\end{aligned}$$

According to the definitions of $\mathfrak{L}_{\varepsilon r}$ and $\mathcal{L}_{\varepsilon r}$ seen in Proposition 7.3 and in Lemma 5.1 and using (6.24), one has in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathcal{L}_{\varepsilon r}\mathscr{B}=\mathscr{B}\mathfrak{L}_{\varepsilon r}\quad\text{and}\quad\mathcal{L}_{\varepsilon r}=\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}-\partial_\theta\mathcal{K}_\lambda*\cdot$$

and therefore

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp,$$

where we have used the identities

$$\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp=\mathscr{B}_\perp^{-1}\quad\text{and}\quad[\Pi_{\mathbb{S}_0}^\perp,T]=0=[\Pi_{\mathbb{S}_0},T],$$

for any Fourier multiplier T. The structure of $\mathfrak{L}_{\varepsilon r}$ is detailed in Proposition 7.3, and from this we deduce that

$$\begin{aligned}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0\big)\Pi_{\mathbb{S}_0}^\perp\\&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\mathscr{B}_\perp\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

It follows that

$$\begin{aligned}\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&\quad+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

Consequently, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, one has the following reduction

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=&\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\\triangleq&\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta-\partial_\theta\mathcal{K}_\lambda*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{R}_0+\mathtt{E}_n^1,\end{aligned}\tag{7.175}$$

where we set

$$\mathtt{E}_n^1\triangleq\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.$$

Notice that the estimates (7.170) are simple reformulations of (7.17) and (7.21) since $\Delta_{12}r^1=\Delta_{12}c_i$.

(ii) By using (7.150), (7.18), the continuity of the projectors, (7.20) and (7.169), one obtains

$$\begin{aligned}\|\mathrm{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \|\mathscr{B}\mathrm{E}_n^0\Pi_{\mathbb{S}_0}^{\perp}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\mathrm{E}_n^0\Pi_{\mathbb{S}_0}^{\perp}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\end{aligned}$$

(iii) Now, we shall prove the following estimates,

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\tag{7.176}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\tag{7.177}$$

To do that, we shall study separately the different terms appearing in (7.175) in the definition of $\mathscr{R}_0$. Notice that in the various estimates below, we use the notation σ_3 to denote some loss of regularity. This index depends only on τ_1,q,d,s_0 and may change increasingly from one line to another and it is always taken greater than the σ_2 introduced in Proposition 7.3.

► *Study of the term* $\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}$. One gets easily according to (7.123) and (7.124)

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^k\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\end{aligned}\tag{7.178}$$

and

$$\begin{aligned}\|\Delta_{12}\big(\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.179}$$

► *Study of the term* $\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}$. Using the first point of Proposition 7.4 yields

$$\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}=\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}-\mathcal{T}_0\mathcal{S}_1,\tag{7.180}$$

where

$$\mathcal{T}_0\rho=\sum_{m\in\mathbb{S}_0}\big\langle\rho,\big(\mathcal{B}-\textnormal{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\quad\textnormal{and}\quad\mathcal{S}_1\triangleq\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}.\tag{7.181}$$

To estimate the first term, we use Proposition 7.3

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^k\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.182}$$

As for the second term, we write

$$\begin{aligned}\mathcal{T}_0\mathcal{S}_1\rho &= \sum_{m\in\mathbb{S}_0}\big\langle\mathcal{S}_1\rho,\big(\mathcal{B}-\textnormal{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\ &= \sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{S}_1^\star\big(\mathcal{B}-\textnormal{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

where $\mathcal{S}_1^\star$ is the $L^2_\theta(\mathbb{T})$-adjoint of $\mathcal{S}_1$. This is an integral operator taking the form

$$\begin{aligned}\big(\mathcal{T}_0\mathcal{S}_1\rho\big)(\varphi,\theta)&=\int_{\mathbb{T}}\mathcal{K}_1(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\ \mathcal{K}_1(\varphi,\theta,\eta)&\triangleq\sum_{m\in\mathbb{S}_0}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

Recall from Proposition 7.3 that $\mathfrak{R}_{\varepsilon r}$ is self-adjoint and using Lemma A.3 we have the identities $\mathscr{B}^\star=\mathcal{B}^{-1}$ and $\mathcal{B}^\star=\mathscr{B}^{-1}$, then

$$\mathcal{S}_1^\star=-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\mathcal{B}^{-1}.\tag{7.183}$$

Therefore, combining (7.165), (7.23) and (7.144) imply

$$\mathcal{K}_1(-\varphi,-\theta,-\eta)=-\mathcal{K}_1(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.184}$$

Applying Lemma A.7 combined with the product laws yields for any $k\in\mathbb{N}$

$$\begin{aligned}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_1\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_1)(*,\cdot,{\scriptscriptstyle\blacksquare},\eta+{\scriptscriptstyle\blacksquare})\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\ &\lesssim\sum_{m\in\mathbb{S}_0}\Big(\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s_0+k}^{\gamma,\mathcal{O}}+\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big).\end{aligned}\tag{7.185}$$

Remark that (7.183) implies

$$\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m=-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m.$$

Hence according to Lemma A.6 combined with Proposition 7.3 we find

$$\begin{aligned}\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\partial_\theta\Pi_{\mathbb{S}_0}\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\quad+\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s_0+\sigma_3}^{\gamma,\mathcal{O}}\right)\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s+1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.186}$$

Using (7.18) together with Lemma 7.1 and the smallness condition (7.169) leads to

$$\begin{aligned}\|\big(\mathrm{Id}-\mathcal{B}^{-1}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g_m\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m}\|_{q,H^s_\varphi}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\sup_{k,m\in\mathbb{S}_0}\|\alpha_{k,m}\|_{q,H^{s_0}_\varphi}^{\gamma,\mathcal{O}}\\ &\lesssim1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.187}$$

Therefore, inserting this estimate into (7.186) and using (7.169) allow to get

$$\|\mathcal{S}_1^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).$$

Plugging this estimate into (7.185) and using (7.18) ensure

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_1\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\tag{7.188}$$

Consequently, by combining (7.180), (7.182) and (7.188), we find

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\tag{7.189}$$

We now turn to the difference estimate. From (7.180), it is obvious that

$$\Delta_{12}\big(\mathscr{B}_{\perp}^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\big)=\Pi_{\mathbb{S}_0}\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}-\Delta_{12}(\mathcal{T}_0\mathcal{S}_1). \tag{7.190}$$

To estimate the first term, we use (7.124)

$$\begin{aligned}\|\Pi_{\mathbb{S}_0}\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}\partial_\theta\mathfrak{R}_{\varepsilon r}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.191}$$

As for the second term, we notice that $\Delta_{12}\big(\mathcal{T}_0\mathcal{S}_1\big)$ is an integral operator whose kernel $\Delta_{12}\mathcal{K}_1$ is

$$\begin{aligned}\Delta_{12}\mathcal{K}_1(\varphi,\theta,\eta)=\sum_{m\in\mathbb{S}_0}&\Delta_{12}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\textrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}_{r_1}e_m\big)(\varphi,\theta)\\&+\big(\mathcal{S}_1^\star\big(\mathcal{B}-\textrm{Id}\big)g_m\big)_{r_2}(\varphi,\eta)\big(\Delta_{12}\mathscr{B}e_m\big)(\varphi,\theta).\end{aligned}$$

Hence, using Lemma A.7-(ii) together with the product laws we deduce that

$$\begin{aligned}\|\Delta_{12}\mathcal{T}_0\mathcal{S}_1\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}\mathcal{K}_1(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\sum_{m\in\mathbb{S}_0}\|\Delta_{12}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\textrm{Id}\big)g_m\big)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}_{r_1}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\\&\quad+\|\big(\mathcal{S}_1^\star\big(\mathcal{B}-\textrm{Id}\big)g_m\big)_{r_2}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}\mathscr{B}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Notice that by Taylor Formula and (7.22) (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0$), one has

$$\sup_{m\in\mathbb{S}_0}\|\Delta_{12}\mathscr{B}^{-1}e_m\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}. \tag{7.192}$$

On the other hand, we have

$$\Delta_{12}\mathcal{S}_1^\star=-\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\mathcal{B}_{r_1}^{-1}-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1},$$

leading to

$$\begin{aligned}\Delta_{12}\big(\mathcal{S}_1^\star\big(\mathcal{B}-\textrm{Id}\big)g_m\big)=&-\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\textrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}-\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1}\big(\mathcal{B}_{r_1}-\textrm{Id}\big)g_{m,r_1}\\&+\mathcal{S}_{1,r_2}^\star\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}+\mathcal{S}_{1,r_2}^\star\big(\mathcal{B}_{r_2}-\textrm{Id}\big)\Delta_{12}g_m.\end{aligned}$$

According to Lemma A.6, we obtain

$$\|\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\textrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}\mathfrak{R}_{\varepsilon r}\|_{\textrm{O-d},q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\|\big(\textrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}.$$

From (7.187), one has

$$\|\big(\textrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim1+\varepsilon\gamma^{-1}\|\mathfrak{I}_1\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Thus, from (7.124) and (7.169), we infer

$$\|\Pi_{\mathbb{S}_0}^{\perp}\Delta_{12}\mathfrak{R}_{\varepsilon r}\partial_\theta\Pi_{\mathbb{S}_0}\big(\textrm{Id}-\mathcal{B}_{r_1}^{-1}\big)g_{m,r_1}\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{7.193}$$

Applying Lemma A.6, (7.123) and (7.169) we deduce that

$$\begin{aligned}\|\mathcal{S}^{\star}_{1,r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}&\lesssim\|\mathfrak{R}_{\varepsilon r_2}\|_{\textnormal{O-d},q,\overline{s}_h+s_0}\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}\\&\lesssim\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}.\end{aligned}$$

To estimate the right hand side member, it suffices to use (7.18) and (7.169), leading to

$$\|\mathcal{B}_{r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}\lesssim\|\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}.$$

By Taylor Formula, we may write

$$\Delta_{12}\mathcal{B}\rho(\theta)=\Delta_{12}\beta(\theta)\int_0^1\partial_\theta\rho\big(\theta+\beta_2(\theta)+t\Delta_{12}\beta(\theta)\big)dt.$$

It follows from the product laws in Lemma A.1, (7.22) and (7.169) that

$$\begin{aligned}\|\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+1}&\lesssim\|\Delta_{12}\beta\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}\|g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+2}\\&\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\end{aligned}$$

Thus

$$\|\mathcal{S}^{\star}_{1,r_2}\big(\Delta_{12}\mathcal{B}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.194}$$

In the same way, using Taylor Formula together with (7.22), we get

$$\|\Pi_{\mathbb{S}_0}^{\perp}\mathfrak{R}_{\varepsilon r_2}\partial_\theta\Pi_{\mathbb{S}_0}\Delta_{12}\mathcal{B}^{-1}\big(\mathcal{B}_{r_1}-\textnormal{Id}\big)g_{m,r_1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.195}$$

By Lemma A.6, (7.123) and (7.169), one finds

$$\begin{aligned}\|\mathcal{S}^{\star}_{1,r_2}\big(\mathcal{B}_{r_2}-\textnormal{Id}\big)\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}&\lesssim\|\mathfrak{R}_{\varepsilon r_2}\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}+s_0}\|\big(\mathcal{B}_{r_2}-\textnormal{Id}\big)\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}\\&\lesssim\|\big(\mathcal{B}_{r_2}-\textnormal{Id}\big)\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}.\end{aligned}$$

Applying (7.18) and (7.169), we obtain

$$\|\big(\mathcal{B}_{r_2}-\textnormal{Id}\big)\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}\lesssim\|\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0+1}.$$

Using (7.151) (applied with $\mathtt{p}=s_0+1$), we finally get

$$\|\mathcal{S}^{\star}_{1,r_2}\big(\mathcal{B}_{r_2}-\textnormal{Id}\big)\Delta_{12}g_m\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+s_0}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.196}$$

Gathering (7.192), (7.193), (7.194), (7.195) and (7.196) implies

$$\|\Delta_{12}\big(\mathcal{T}_0\mathcal{S}_1\big)\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.197}$$

Putting together (7.190), (7.191) and (7.197), one obtains

$$\|\Delta_{12}\big(\mathscr{B}_{\perp}^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^{\perp}\big)\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.198}$$

► *Study of the term* $\mathscr{B}_{\perp}^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}$. We first write,

$$\mathscr{B}_{\perp}^{-1}\Pi_{\mathbb{S}_0}^{\perp}\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)-\partial_\theta\mathbf{L}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\triangleq\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp},$$

with

$$\mathcal{S}_2 = \big(\big(V_{\varepsilon r} - c_{i_0}\big)\cdot - \mathbf{L}_{\varepsilon r,1}\big)\,\Pi_{\mathbb{S}_0}.$$

Notice that to get the above identity we have used the identity

$$\Pi_{\mathbb{S}_0}^{\perp}\partial_\theta(c_{i_0}\cdot)\Pi_{\mathbb{S}_0} = 0.$$

Recall from (6.24) and (6.25) that

$$\mathbf{L}_{\varepsilon r,1}\rho(\varphi,\theta) = \int_{\mathbb{T}} \mathbb{K}_{\varepsilon r,1}(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta.$$

Then from elementary computations we find

$$\mathcal{S}_2\rho(\varphi,\theta) = \int_{\mathbb{T}} \mathcal{K}_2(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,$$

with

$$\begin{aligned}\mathcal{K}_2(\varphi,\theta,\eta) &\triangleq \big(V_{\varepsilon r}(\varphi,\theta) - c_{i_0}\big)D_{\mathbb{S}_0}(\theta-\eta) - \int_{\mathbb{T}} \mathbb{K}_{\varepsilon r,1}(\varphi,\theta,\eta')D_{\mathbb{S}_0}(\eta'-\eta)d\eta',\\ D_{\mathbb{S}_0}(\theta) &\triangleq \sum_{n\in\mathbb{S}_0} e^{in\theta}.\end{aligned}$$

Combining (6.27), (7.4), (5.3) and the change of variables $\eta' \mapsto \eta'$, one gets

$$\mathcal{K}_2(-\varphi,-\theta,-\eta) = \mathcal{K}_2(\varphi,\theta,\eta) \in \mathbb{R}. \tag{7.199}$$

Proceeding as in (7.180) we obtain

$$\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} = \mathscr{B}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp} - \mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}. \tag{7.200}$$

It follows that

$$\big\|\partial_\theta^k\mathscr{B}_{\perp}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \big\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} + \big\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}. \tag{7.201}$$

The expression of the first term is similar to that of (7.133), namely, one has

$$\big(\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\rho(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\widehat{\mathcal{K}}_2(\varphi,\theta,\eta)d\eta,$$

with

$$\widehat{\mathcal{K}}_2(\varphi,\theta,\eta) \triangleq \mathcal{K}_2\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).$$

Combining (7.199) and (7.23), one gets

$$\widehat{\mathcal{K}}_2(-\varphi,-\theta,-\eta) = \widehat{\mathcal{K}}_2(\varphi,\theta,\eta) \in \mathbb{R}. \tag{7.202}$$

Then coming back to (7.134) and arguing as for (7.143), we find

$$\sup_{k\in\{0,1,2\}} \big\|(\partial_\theta^k\widehat{\mathcal{K}}_2)\big(*,\cdot,\blacksquare,\eta+\blacksquare\big)\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right)\left(1-\log\big|\sin\big(\eta/2\big)\big|\right). \tag{7.203}$$

By virtue of Lemma A.7 and (7.203) we obtain

$$\begin{aligned}\sup_{k\in\{0,1\}}\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\sup_{k\in\{0,1\}}\int_{\mathbb{T}}\|(\partial_\theta^{k+1}\widehat{\mathcal{K}}_2)(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.204}$$

Notice that from (7.181), we can write

$$\begin{aligned}\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\rho&=\sum_{m\in\mathbb{S}_0}\big\langle\partial_\theta\mathcal{S}_2\mathscr{B}\rho,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\&=-\sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

where $\mathcal{S}_2^\star$ is the adjoint of $\mathcal{S}_2$ and is given by

$$\mathcal{S}_2^\star=\Pi_{\mathbb{S}_0}\big(\big(V_{\varepsilon r}-c_{i_0}\big)\cdot-\mathbf{L}_{\varepsilon r,1}\big).\tag{7.205}$$

This is an integral operator taking the form

$$\begin{aligned}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\rho\big)(\varphi,\theta)&=\int_{\mathbb{T}}\mathcal{K}_3(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\\mathcal{K}_3(\varphi,\theta,\eta)&\triangleq\sum_{m\in\mathbb{S}_0}\big(\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

According to (7.165), (7.23), (7.205), (7.4), (5.3) and (6.27), one gets

$$\mathcal{K}_3(-\varphi,-\theta,-\eta)=-\mathcal{K}_3(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.206}$$

On the other hand, applying Lemma A.7 combined with the product laws yield for any $k\in\mathbb{N}$

$$\begin{aligned}&\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_3)(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\sum_{m\in\mathbb{S}_0}\Big(\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s_0+k}^{\gamma,\mathcal{O}}+\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Applying (7.18) we find

$$\|\mathcal{B}^{-1}\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathcal{S}_2^\star\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Now, from (7.205), the product laws and Lemma A.6, we find

$$\begin{aligned}\|\mathcal{S}_2^\star\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim&\|V_{\varepsilon r}-c_{i_0}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|V_{\varepsilon r}-c_{i_0}\|_{q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&+\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

From the composition law and (7.17), one has

$$\begin{aligned}\|V_{\varepsilon r}-c_{i_0}\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|V_{\varepsilon r}-V_0\|_{q,s}^{\gamma,\mathcal{O}}+\|V_0-c_{i_0}\|_{q}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

According to Lemma A.7 and (6.41), we deduce that

$$\begin{aligned}\|\mathbf{L}_{\varepsilon r,1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}} \|\mathbb{K}_{\varepsilon r,1}(*,\cdot,\bullet,\eta+\bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned}$$

Using (7.169), one gets

$$\|\mathcal{S}_2^{\star}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Combining this with (7.18) allows to get

$$\begin{aligned}\|\mathcal{S}_2^{\star}\partial_\theta\big(\mathcal{B}-\mathrm{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|g_m\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim 1+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore,

$$\max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right). \tag{7.207}$$

Plugging the estimates (7.204) and (7.207) into (7.201) we find

$$\begin{aligned}\max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \max_{k\in\{0,1\}}\left\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} + \max_{k\in\{0,1\}}\left\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\right\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{7.208}$$

We now turn to the estimate of the difference. Coming back to (7.200), one can write

$$\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big) = \Delta_{12}\big(\mathscr{B}^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big) - \Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big).$$

It follows that

$$\left\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \left\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} + \left\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}. \tag{7.209}$$

Arguing as for (7.149), one obtains

$$\|\Delta_{12}(\partial_\theta\widehat{\mathcal{K}}_2)(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}\left(1-\log\left|\sin\left(\tfrac{\eta}{2}\right)\right|\right).$$

Then, using Lemma A.7 implies

$$\begin{aligned}\left\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widehat{\mathcal{K}}_2)(*,\cdot,\bullet,\eta+\bullet)\|_{q,\overline{s}_h+\mathtt{p}+s_0}d\eta\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.210}$$

On the other hand, proceeding as for (7.197), and using in particular (7.21),

$$\left\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{S}_2\mathscr{B}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{7.211}$$

Putting together (7.210), (7.211) and (7.209), ensures that

$$\left\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{S}_2\mathscr{B}\Pi_{\mathbb{S}_0}^{\perp}\big)\right\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{7.212}$$

► *Study of the term* $\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp$. Using the relation $\mathrm{Id}=\Pi_{\mathbb{S}_0}+\Pi_{\mathbb{S}_0}^\perp$, we can write

$$\begin{aligned}\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp&=\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}_\perp-\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}_\perp\\&=\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\\&=\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp+\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.\end{aligned}\tag{7.213}$$

Hence

$$\begin{aligned}\|\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant&\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}+\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\\&+\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}+\|\partial_\theta^k\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.214}$$

Recall that from Proposition 7.1 that $\mathcal{R}$ is an integral operator of kernel J and therefore direct computations give

$$\big(\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\rho\big)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\widehat{J}(\varphi,\theta,\eta)d\eta,\tag{7.215}$$

with

$$\widehat{J}(\varphi,\theta,\eta)\triangleq J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta+\widehat{\beta}(\varphi,\eta)\big).\tag{7.216}$$

Combining (7.23) and (7.5), one gets

$$\widehat{J}(-\varphi,-\theta,-\eta)=\widehat{J}(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.217}$$

Using the composition law and (7.9), we obtain

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\widehat{J})\big(*,\cdot,\blacksquare,\eta+\blacksquare\big)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Thus, applying Lemma A.7-(ii) implies

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^{k+1}\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\max_{k\in\{0,1,2\}}\int_{\mathbb{T}}\|(\partial_\theta^k\widehat{J})\big(*,\cdot,\blacksquare,\eta+\blacksquare\big)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.218}$$

On the other hand we notice from (7.215) that we get the structure

$$\big(\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\rho\big)(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\widetilde{J}(\varphi,\theta,\eta)d\eta,\tag{7.219}$$

with

$$\widetilde{J}(\varphi,\theta,\eta)\triangleq\int_{\mathbb{T}}J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta'\big)D_{\mathbb{S}_0}(\eta'-\eta)d\eta'.\tag{7.220}$$

Combining (7.23), (7.5) and the change of variables $\eta'\mapsto-\eta'$, one finds

$$\widetilde{J}(-\varphi,-\theta,-\eta)=\widetilde{J}(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.221}$$

Using the change of variables $\eta'\mapsto\eta'+\theta$ yields

$$\widetilde{J}(\varphi,\theta,\eta+\theta)\triangleq\int_{\mathbb{T}}J\big(\varphi,\theta+\widehat{\beta}(\varphi,\theta),\eta'+\theta\big)D_{\mathbb{S}_0}(\eta'-\eta)d\eta'.$$

Then by the composition law, we infer

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k\widetilde{J})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.$$

Consequently, we find in view of Lemma A.7

$$\begin{aligned}\max_{k\in\{0,1\}}\|\partial_\theta^k\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\max_{k\in\{0,1,2\}}\int_{\mathbb{T}}\|(\partial_\theta^k\widetilde{J})(*,\cdot,\bullet,\eta+\bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.222}$$

If wet set

$$\mathcal{S}_3=\partial_\theta\mathcal{R}\mathscr{B}\quad\text{or}\quad\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B},$$

then using (7.181), we deduce that

$$\begin{aligned}\mathcal{T}_0\mathcal{S}_3\rho&=\sum_{m\in\mathbb{S}_0}\big\langle\mathcal{S}_3\rho,\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m\\&=\sum_{m\in\mathbb{S}_0}\big\langle\rho,\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big\rangle_{L^2_\theta(\mathbb{T})}\mathscr{B}^{-1}e_m,\end{aligned}$$

with $\mathcal{S}_3^\star$ is the adjoint of $\mathcal{S}_3$ given by

$$\mathcal{S}_3^\star=-\mathcal{B}^{-1}\mathcal{R}^\star\partial_\theta\quad\text{or}\quad-\mathcal{B}^{-1}\Pi_{\mathbb{S}_0}\mathcal{R}^\star\partial_\theta,\tag{7.223}$$

and $\mathcal{R}^\star$ the adjoint of $\mathcal{R}$ which is an integral operator with kernel

$$J^\star(\varphi,\theta,\eta)\triangleq\sum_{k'=1}^{3}\sum_{k=1}^{d}g_{k,k'}(\varphi,\theta)\chi_{k,k'}(\varphi,\eta),\tag{7.224}$$

where we use the notations of the proof of Proposition 7.1. Notice that similarly to (7.9) and (7.5), the kernel $J^\star$ satisfies

$$\max_{k\in\{0,1,2\}}\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^kJ^\star)(*,\cdot,\bullet,\eta+\bullet)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim 1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\tag{7.225}$$

and

$$J^\star(-\varphi,-\theta,-\eta)=J^\star(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.226}$$

Now, we have the integral representation

$$\begin{aligned}\big(\mathcal{T}_0\mathcal{S}_3\rho\big)(\varphi,\theta)&=\int_{\mathbb{T}}\mathcal{K}_4(\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta,\\\mathcal{K}_4(\varphi,\theta,\eta)&\triangleq\sum_{m\in\mathbb{S}_0}\big(\mathcal{S}_3^\star\big(\mathcal{B}-\mathrm{Id}\big)g_m\big)(\varphi,\eta)\big(\mathscr{B}^{-1}e_m\big)(\varphi,\theta).\end{aligned}$$

Then by virtue of (7.165), (7.23), (7.205) and (7.226) we obtain

$$\mathcal{K}_4(-\varphi,-\theta,-\eta)=-\mathcal{K}_4(\varphi,\theta,\eta)\in\mathbb{R}.\tag{7.227}$$

Applying Lemma A.7 combined with the product laws, we get for all $k\in\{0,1\}$

$$\begin{aligned}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_3\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|(\partial_\theta^k\mathcal{K}_4)(*,\cdot,\scriptstyle\blacksquare\displaystyle,\eta+\scriptstyle\blacksquare\displaystyle)\|_{q,s+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\sum_{m\in\mathbb{S}_0}\Big(\|\mathcal{S}_3^\star\big(\mathcal{B}-\textnormal{Id}\big)g_m\|_{q,s+s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s_0+k}^{\gamma,\mathcal{O}}+\|\mathcal{S}_3^\star\big(\mathcal{B}-\textnormal{Id}\big)g_m\|_{q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{B}^{-1}e_m\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Consequently, using Lemma A.7 and (7.225), we get

$$\begin{aligned}\|\mathcal{R}^\star\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|J^\star(*,\cdot,\scriptstyle\blacksquare\displaystyle,\eta+\scriptstyle\blacksquare\displaystyle)\|_{q,s+s_0}^{\gamma,\mathcal{O}}\\&\lesssim1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Applying (7.18), Lemma A.6 and the previous estimate implies

$$\begin{aligned}\|\mathcal{S}_3^\star\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathcal{R}^\star\partial_\theta\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\mathcal{R}^\star\partial_\theta\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\Big(\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big)+\|\mathcal{R}^\star\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\Big)\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\mathcal{R}^\star\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Thus

$$\begin{aligned}\|\mathcal{S}_3^\star\big(\mathcal{B}-\textnormal{Id}\big)g_m\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim\|g_m\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|g_m\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence

$$\max_{k\in\{0,1\}}\|\partial_\theta^k\mathcal{T}_0\mathcal{S}_3\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}.\tag{7.228}$$

Putting together (7.214), (7.218), (7.222) and (7.228) allows to get

$$\max_{k\in\{0,1\}}\varepsilon\big\|\partial_\theta^k\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big).\tag{7.229}$$

We now move to the estimate of the difference. From (7.213), one has

$$\begin{aligned}\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\leqslant\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\\&\quad+\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{R}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\big(\mathcal{T}_0\partial_\theta\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.230}$$

Combining Lemma A.7 with Taylor Formula, (7.215), (7.216), (7.10) and (7.22) one obtains

$$\begin{aligned}\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widehat{J})(*,\cdot,\scriptstyle\blacksquare\displaystyle,\eta+\scriptstyle\blacksquare\displaystyle)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.231}$$

In the same spirit, (7.219) and (7.220) give

$$\begin{aligned}\|\Delta_{12}\big(\partial_\theta\mathcal{B}^{-1}\mathcal{R}\Pi_{\mathbb{S}_0}\mathscr{B}\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}&\lesssim\int_{\mathbb{T}}\|\Delta_{12}(\partial_\theta\widetilde{J})(*,\cdot,\scriptstyle\blacksquare\displaystyle,\eta+\scriptstyle\blacksquare\displaystyle)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\&\lesssim\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.232}$$

According to the structure of $J^\star$ detailed in (7.224) one can check that $J^\star$ satisfies similar estimates as

(7.10). Then using (7.22), one finds in a similar way to (7.197),

$$\begin{aligned}\|\Delta_{12}(\mathcal{T}_0\mathcal{S}_3)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} &\lesssim \int_{\mathbb{T}}\|\Delta_{12}\mathcal{K}_4(*,\cdot,\scriptstyle\blacksquare\displaystyle,\eta+\scriptstyle\blacksquare\displaystyle)\|_{q,\overline{s}_h+\mathtt{p}+s_0}^{\gamma,\mathcal{O}}d\eta\\ &\lesssim \|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.233}$$

Hence, putting together (7.231), (7.232), (7.233) and (7.230) gives

$$\varepsilon\|\Delta_{12}\big(\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp\big)\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}.\tag{7.234}$$

On the other hand, gathering (7.175), (7.144), (7.184), (7.202), (7.206), (7.227), (7.217) and (7.221) together with Lemma A.7, we find that $\mathscr{R}_0$ is a real and reversible Toeplitz in time integral operator.
In addition, (7.175), (7.178), (7.189), (7.208) and (7.229) give (7.176).
Furthermore, (7.175), (7.179), (7.198), (7.212) and (7.234) imply (7.177).
(iv) Using Lemma A.6 together with (7.172), (6.28), (7.17) and (7.169), one obtains for all $s\in[s_0,S]$

$$\begin{aligned}\|\mathscr{L}_0\rho\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant \|\big(\omega\cdot\partial_\varphi+c_{i_0}\partial_\theta+\partial_\theta\mathcal{K}_\lambda*\cdot)\big)\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\rho\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\mathscr{R}_0\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

This ends the proof of Proposition 7.4. ☐

7.3.2 KAM reduction of the remainder term

The goal of this section is to conjugate $\mathscr{L}_0$ defined in Proposition 7.4 to a diagonal operator, up to a fast decaying small remainder. This will be achieved through standard KAM reducibility techniques in the spirit of Proposition 7.2 but well-adapted to the operators setting. This will be implemented by taking advantage of the exterior parameters which are restricted to a suitable Cantor set that prevents the resonances in the second Melnikov assumption. Notice that one gets from this study some estimates on the distribution of the eigenvalues and their stability with respect to the torus parametrization. This is considered as the key step not only to get an approximate inverse but also to achieve Nash-Moser scheme with a final massive Cantor set. The main result of this section reads as follows.

Proposition 7.5. *Let* $(\gamma,q,d,\tau_1,\tau_2,s_0,\overline{s}_l,\overline{\mu}_2,S)$ *satisfy* (A.2), (A.1) *and* (7.3)*. For any* (μ_2,s_h) *satisfying*

$$\mu_2\geqslant\overline{\mu}_2+2\tau_2q+2\tau_2\quad\text{and}\quad s_h\geqslant\frac{3}{2}\mu_2+\overline{s}_l+1,\tag{7.235}$$

there exist $\varepsilon_0\in(0,1)$ *and* $\sigma_4=\sigma_4(\tau_1,\tau_2,q,d)\geqslant\sigma_3$*, with* σ_3 *defined in Proposition* 7.4*, such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\leqslant 1,\tag{7.236}$$

then the following assertions hold true.

(i) There exists a family of invertible linear operator $\Phi_\infty:\mathcal{O}\to\mathcal{L}\big(H_\perp^s\big)$ *satisfying the estimates*

$$\forall s\in[s_0,S],\qquad\|\Phi_\infty^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{7.237}$$

There exists a diagonal operator $\mathscr{L}_\infty=\mathscr{L}_\infty(\lambda,\omega,i_0)$ *taking the form*

$$\mathscr{L}_\infty\triangleq\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty$$

where $\mathscr{D}_\infty=\mathscr{D}_\infty(\lambda,\omega,i_0)$ *is a reversible Fourier multiplier operator given by,*

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad \mathscr{D}_\infty\mathbf{e}_{l,j}=\mathrm{i}\mu_j^\infty\,\mathbf{e}_{l,j},$$

with

$$\forall j\in\mathbb{S}_0^c,\quad \mu_j^\infty(\lambda,\omega,i_0)\triangleq\mu_j^0(\lambda,\omega,i_0)+r_j^\infty(\lambda,\omega,i_0),\quad \|r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\tag{7.238}$$

and

$$\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1},\tag{7.239}$$

such that in the Cantor set

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\triangleq\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n\\ (l,j)\neq(0,j_0)}}\left\{(\lambda,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0),\,\big|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)-\mu_{j_0}^\infty(\lambda,\omega,i_0)\big|>\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\right\}$$

we have

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty=\mathscr{L}_\infty+\mathtt{E}_n^2,$$

and the linear operator $\mathtt{E}_n^2$ *satisfies the estimate*

$$\|\mathtt{E}_n^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-2}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\tag{7.240}$$

Notice that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *was introduced in Proposition* 7.2, *the operator* $\mathscr{L}_0$ *and the frequencies* $\big(\mu_j^0(\lambda,\omega,i_0)\big)_{j\in\mathbb{S}_0^c}$ *were stated in Proposition* 7.4.

(ii) Given two tori i_1 *and* i_2 *both satisfying* (7.236), *then*

$$\forall j\in\mathbb{S}_0^c,\quad \|\Delta_{12}r_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\tag{7.241}$$

and

$$\forall j\in\mathbb{S}_0^c,\quad \|\Delta_{12}\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}|j|\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.\tag{7.242}$$

Proof. **(i)** We shall introduce the quantity

$$\delta_0(s)\triangleq\gamma^{-1}\|\mathscr{R}_0\|_{\textrm{O-d},q,s}^{\gamma,\mathcal{O}},$$

where $\mathscr{R}_0$ is the remainder seen in Proposition 7.4. By applying (7.172), we deduce that

$$\delta_0(s)\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\right).\tag{7.243}$$

Therefore with the notation of (7.235), (7.236) and the fact that $\sigma_4\geqslant\sigma_3$ we obtain

$$\begin{aligned}N_0^{\mu_2}\delta_0(s_h)&\leqslant CN_0^{\mu_2}\varepsilon\gamma^{-2}\\&\leqslant C\varepsilon_0.\end{aligned}\tag{7.244}$$

▶ **KAM step.** Recall from Proposition 7.4 that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ one has

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\mathscr{L}_0+\mathtt{E}_n^1,$$

where the operator $\mathscr{L}_0$ has the following structure

$$\mathscr{L}_0 = \big(\omega\cdot\partial_\varphi + \mathscr{D}_0\big)\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{R}_0, \tag{7.245}$$

with $\mathscr{D}_0$ a diagonal operator of pure imaginary spectrum and $\mathscr{R}_0$ a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}_0\Pi_{\mathbb{S}_0}^{\perp} = \mathscr{R}_0$. Similarly to the reduction of the transport part, we shall first expose a typical step of the iteration process of the KAM scheme whose goal is to reduce to a diagonal part $\mathscr{R}_0$. Notice that the scheme is flexible and has been used in the literature to deal with various equations. Assume that we have a linear operator $\mathscr{L}$ taking the following form in restriction to some Cantor set $\mathscr{O}$ one has

$$\mathscr{L} = \big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{R},$$

where $\mathscr{D}$ is real and reversible diagonal Toeplitz in time operator, that is,

$$\mathscr{D}\mathbf{e}_{l,j} = \mathrm{i}\mu_j(\lambda,\omega)\,\mathbf{e}_{l,j} \quad \text{and} \quad \mu_{-j}(\lambda,\omega) = -\mu_j(\lambda,\omega). \tag{7.246}$$

The operator $\mathscr{R}$ is assumed to be a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}\Pi_{\mathbb{S}_0}^{\perp} = \mathscr{R}$. Consider a linear invertible transformation close to the identity

$$\Phi = \Pi_{\mathbb{S}_0}^{\perp} + \Psi : \mathcal{O} \to \mathcal{L}(H_{\perp}^{s}),$$

where Ψ is small and depends on $\mathscr{R}$. Then straightforward calculations show that in $\mathscr{O}$

$$\begin{aligned}\Phi^{-1}\mathscr{L}\Phi &= \Phi^{-1}\Big(\Phi\,(\omega\cdot\partial_\varphi + \mathscr{D})\,\Pi_{\mathbb{S}_0}^{\perp} + \big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp} + \mathscr{D}, \Psi\big] + \mathscr{R} + \mathscr{R}\Psi\Big)\\ &= \big(\omega\cdot\partial_\varphi + \mathscr{D}\big)\Pi_{\mathbb{S}_0}^{\perp} + \Phi^{-1}\Big(\big[(\omega\cdot\partial_\varphi + \mathscr{D})\Pi_{\mathbb{S}_0}^{\perp}, \Psi\big] + P_N\mathscr{R} + P_N^{\perp}\mathscr{R} + \mathscr{R}\Psi\Big),\end{aligned}$$

where the projector P_N was defined in (A.25). The main idea consists in replacing the remainder $\mathscr{R}$ with another quadratic one up to a diagonal part and provided that the parameters (λ,ω) belongs to a Cantor set connected to non-resonance conditions associated to the *homological equation*. Iterating this scheme will generate new remainders which become smaller and smaller up to new contributions on the diagonal part and with more extraction on the parameters. Then by passing to the limit we expect to diagonalize completely the operators provided that the parameters belong to a limit Cantor set. Notice that the Cantor set should be truncated in the time mode in order to get a stability form required later in Nash-Moser scheme and during the measure of the final Cantor set. This will induce a diagonalization up to small fast decaying remainders modeled by the operators $\mathtt{E}_n^2$ in Proposition 7.5. Now the first step is to impose the following homological equation,

$$\big[(\omega\cdot\partial_\varphi + \mathscr{D})\Pi_{\mathbb{S}_0}^{\perp}, \Psi\big] + P_N\mathscr{R} = \lfloor P_N\mathscr{R}\rfloor, \tag{7.247}$$

where $\lfloor P_N\mathscr{R}\rfloor$ is the diagonal part of the operator $P_N\mathscr{R}$. We emphasize that the notation $\lfloor\mathcal{R}\rfloor$ with a general operator $\mathcal{R}$ is defined as follows, for all $(l_0,j_0)\in\mathbb{Z}^d\times\mathbb{S}_0^c$,

$$\mathcal{R}\mathbf{e}_{l_0,j_0} = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\mathcal{R}_{l_0,j_0}^{l,j}\mathbf{e}_{l,j} \Longrightarrow \lfloor\mathcal{R}\rfloor\mathbf{e}_{l_0,j_0} = \mathcal{R}_{l_0,j_0}^{l_0,j_0}\mathbf{e}_{l_0,j_0} = \big\langle\mathcal{R}\mathbf{e}_{l_0,j_0},\mathbf{e}_{l_0,j_0}\big\rangle_{L^2(\mathbb{T}^{d+1})}\mathbf{e}_{l_0,j_0}. \tag{7.248}$$

Remind the notation $\mathbf{e}_{l_0,j_0}(\varphi,\theta) = e^{\mathrm{i}(l_0\cdot\varphi + j_0\theta)}$. The Fourier coefficients of Ψ are defined through

$$\Psi\mathbf{e}_{l_0,j_0} = \sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\mathbf{e}_{l,j}, \quad \Psi_{l_0,j_0}^{l,j}\in\mathbb{C}.$$

From direct computations based on the above Fourier decomposition, we infer

$$\big[\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^{\perp},\Psi\big]\mathbf{e}_{l_0,j_0}=\mathrm{i}\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\;\omega\cdot(l-l_0)\;\mathbf{e}_{l,j}$$

and using the diagonal structure of $\mathscr{D}$,

$$[\mathscr{D}_0,\Psi]\mathbf{e}_{l_0,j_0}=\mathrm{i}\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c}\Psi_{l_0,j_0}^{l,j}\left(\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\right)\mathbf{e}_{l,j}.$$

By hypothesis, $\mathscr{R}$ is a real and reversible Toeplitz in time operator. Hence its Fourier coefficients write in view of Proposition A.1,

$$\mathscr{R}_{l_0,j_0}^{l,j}\triangleq\mathrm{i}\,r_{j_0}^{j}(\lambda,\omega,l_0-l)\in\mathrm{i}\,\mathbb{R}\quad\text{ and }\quad\mathscr{R}_{-l_0,-j_0}^{-l,-j}=-\mathscr{R}_{l_0,j_0}^{l,j}.\tag{7.249}$$

Consequently Ψ is a solution of (7.247) if and only if

$$\Psi\mathbf{e}_{l_0,j_0}=\sum_{\substack{|l-l_0|\leqslant N\\|j-j_0|\leqslant N}}\Psi_{l_0,j_0}^{l,j}\mathbf{e}_{l,j}$$

and

$$\Psi_{l_0,j_0}^{l,j}\Big(\omega\cdot(l-l_0)+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\Big)=\begin{cases}-r_{j_0}^{j}(\lambda,\omega,l_0-l) & \text{if } (l,j)\neq(l_0,j_0)\\ 0 & \text{if } (l,j)=(l_0,j_0).\end{cases}$$

In particular, we get that Ψ is a Toeplitz in time operator with $\Psi_{j_0}^{j}(l_0-l)\triangleq\Psi_{l_0,j_0}^{l,j}$. Moreover, for $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ with $|l|,|j-j_0|\leqslant N$, one obtains

$$\Psi_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}\frac{-r_{j_0}^{j}(\lambda,\omega,l)}{\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)} & \text{if } (l,j)\neq(0,j_0)\\ 0 & \text{if } (l,j)=(0,j_0),\end{cases}\tag{7.250}$$

provided that the denominator is non zero. In addition, from $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}\Pi_{\mathbb{S}_0}^{\perp}=\mathscr{R}$, one easily gets

$$\forall\, l\in\mathbb{Z}^d,\ \forall\, j\text{ or }\ j_0\in\mathbb{S}_0,\quad r_{j_0}^{j}(\lambda,\omega,l)=0.$$

Therfore, we should impose the compatibility condition

$$\forall\, l\in\mathbb{Z}^d,\ \forall\, j\text{ or }\ j_0\in\mathbb{S}_0,\quad \Psi_{j_0}^{j}(\lambda,\omega,l)=0.$$

This implies that $\Pi_{\mathbb{S}_0}^{\perp}\Psi\Pi_{\mathbb{S}_0}^{\perp}=\Psi$. To justify the formula given by (7.250) we need to avoid resonances and restrict the parameters to the following open set according to the so-called second Melnikov condition,

$$\mathscr{O}_{+}^{\gamma}=\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N\\(l,j)\neq(0,j_0)}}\Big\{(\lambda,\omega)\in\mathscr{O}\quad\text{s.t.}\quad|\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}.$$

In view of this restriction, the identity (7.250) is well defined and to extend Ψ to the whole set $\mathcal{O}$ we shall use the cut-off function χ of (6.92). We set

$$\Psi_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}-\varrho_{j_0}^{j}(\lambda,\omega,l)\;r_{j_0}^{j}(\lambda,\omega,l), & \text{if}\quad (l,j)\neq(0,j_0)\\ 0, & \text{if}\quad (l,j)=(0,j_0),\end{cases}\tag{7.251}$$

with

$$\varrho_{j_0}^{j}(\lambda,\omega,l)\triangleq\frac{\chi\big((\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega))(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}\big)}{\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)}\cdot \tag{7.252}$$

To simplify the notation, in the sequel we shall still write Ψ to denote this extension. Note that the extension (7.251) is smooth and its restriction to the Cantor set $\mathscr{O}_+^{\gamma}$ coincides with Ψ. On the other hand, (7.249) and (7.251) imply that $\Psi_{j_0}^{j}(l)\in\mathbb{R}$. In addition, (7.252) combined with (7.246) give

$$\Psi_{-j_0}^{-j}(-l)=\Psi_{j_0}^{j}(l).$$

Consequently, in view of Proposition A.1, we deduce that Ψ is a real and reversibility preserving operator. Now consider,

$$\mathscr{D}_+=\mathscr{D}+\lfloor P_N\mathscr{R}\rfloor,\qquad \mathscr{R}_+=\Phi^{-1}\big(-\Psi\lfloor P_N\mathscr{R}\rfloor+P_N^{\perp}\mathscr{R}+\mathscr{R}\Psi\big) \tag{7.253}$$

and

$$\mathscr{L}_+\triangleq\big(\omega\cdot\partial_\varphi+\mathscr{D}_++\mathscr{R}_+\big)\Pi_{\mathbb{S}_0}^{\perp}.$$

Therefore, in restriction to the Cantor set $\mathscr{O}_+^{\gamma}$, we can write

$$\mathscr{L}_+=\Phi^{-1}\mathscr{L}\Phi.$$

Our next task is to estimate $\varrho_{j_0}^{j}$ defined by (7.252). Notice that this quantity can be written in the following form

$$\begin{aligned}&\varrho_{j_0}^{j}(\lambda,\omega,l)=a_{l,j,j_0}\widehat{\chi}\big(a_{l,j,j_0}A_{l,j,j_0}(\lambda,\omega)\big),\quad \widehat{\chi}(x)=\frac{\chi(x)}{x},\\ &A_{l,j,j_0}(\lambda,\omega)=\omega\cdot l+\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega),\quad a_{l,j,j_0}=(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2},\end{aligned} \tag{7.254}$$

where $\widehat{\chi}(x)\triangleq\frac{\chi(x)}{x}$ is $\mathcal{C}^{\infty}$ with bounded derivatives. Assume now the following estimate

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\big|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j(\lambda,\omega)-\mu_{j_0}(\lambda,\omega)\right)\big|\leqslant C\,|j-j_0|. \tag{7.255}$$

Then, we find

$$\forall\,(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2,\qquad \max_{\substack{\alpha\in\mathbb{N}^{d+1}\\|\alpha|\in[\![0,q]\!]}}\sup_{(\lambda,\omega)\in\mathcal{O}}\big|\partial_{\lambda,\omega}^{\alpha}A_{l,j,j_0}(\lambda,\omega)\big|\leqslant C\,\langle l,j-j_0\rangle. \tag{7.256}$$

In a similar way to (7.37), using Lemma A.1-(vi) and (7.256), we obtain

$$\forall\alpha\in\mathbb{N}^{d+1},\ |\alpha|\in[\![0,q]\!],\qquad \sup_{(\lambda,\omega)\in\mathcal{O}}\big|\partial_{\lambda,\omega}^{\alpha}\varrho_{j_0}^{j}(\lambda,\omega,l)\big|\leqslant C\gamma^{-(|\alpha|+1)}\langle l,j-j_0\rangle^{\tau_2|\alpha|+\tau_2+|\alpha|}. \tag{7.257}$$

Similarly to (7.39), using Leibniz rule, we get

$$\|\Psi\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-1}\|P_N\mathscr{R}\|_{\text{O-d},q,s+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}. \tag{7.258}$$

We also assume that the following smallness condition holds

$$\gamma^{-1}\|\mathscr{R}\|_{\text{O-d},q,s_0+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}\leqslant C\varepsilon_0. \tag{7.259}$$

Hence, by virtue of (7.258), we get

$$\begin{aligned}\|\Psi\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} &\leqslant C\gamma^{-1}\|\mathscr{R}\|_{\text{O-d},q,s_0+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}}\\ &\leqslant C\varepsilon_0. \end{aligned}\tag{7.260}$$

As a consequence, up to taking ε_0 small enough, the operator Φ is invertible and

$$\Phi^{-1}=\sum_{n=0}^{\infty}(-1)^n\Psi^n\triangleq \mathrm{Id}+\Sigma.$$

According to the product laws in Lemma A.1, Lemma A.6, (7.258) and (7.260) one gets

$$\begin{aligned}\|\Sigma\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|\Psi\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+\sum_{n=1}^{\infty}\left(C\|\Psi\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)^n\right)\\ &\leqslant C\,\gamma^{-1}N^{\tau_2 q+\tau_2}\|\mathscr{R}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}. \end{aligned}\tag{7.261}$$

Therefore, we conclude with the assumption (7.259) that Φ^{-1} satisfies the following estimate

$$\|\Phi^{-1}-\mathrm{Id}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\gamma^{-1}N^{\tau_2 q+\tau_2}\|\mathscr{R}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.\tag{7.262}$$

From (7.253), we can write

$$\mathscr{R}_+=P_N^{\perp}\mathscr{R}+\Phi^{-1}\mathscr{R}\Psi-\Psi\left\lfloor P_N\mathscr{R}\right\rfloor+\Sigma\big(P_N^{\perp}\mathscr{R}-\Psi\left\lfloor P_N\mathscr{R}\right\rfloor\big).$$

Thus, by virtue of Lemma A.6 and (7.262), we infer

$$\begin{aligned}\|\mathscr{R}_+\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \|P_N^{\perp}\mathscr{R}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+C\|\Sigma\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(\|P_N^{\perp}\mathscr{R}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\Psi\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)\\ &\quad+C\left(1+\|\Sigma\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)\left(\|\Psi\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\Psi\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\right). \end{aligned}\tag{7.263}$$

By Lemma A.6, (7.258), (7.260) and (7.262), we get for all $S\geqslant\overline{s}\geqslant s\geqslant s_0$,

$$\|\mathscr{R}_+\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant N^{s-\overline{s}}\|\mathscr{R}\|_{\text{O-d},q,\overline{s}}^{\gamma,\mathcal{O}}+C\gamma^{-1}N^{\tau_2 q+\tau_2}\|\mathscr{R}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.\tag{7.264}$$

▶ **Initialization** We shall verify that the assumptions (7.255) and (7.259) required along the KAM step to get the final form (7.264) are satisfied for $\mathscr{L}=\mathscr{L}_0$ in (7.245). Indeed, (7.255) is an immediate consequence of Lemma 5.3-(vi), that is

$$\exists C>0,\ \forall (j,j_0)\in\mathbb{Z}^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\ \sup_{\lambda\in(\lambda_0,\lambda_1)}|\partial_\lambda^\alpha\left(\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\right)|\leqslant C\,|j-j_0|.\tag{7.265}$$

Thus, applying (7.170) we obtain

$$\exists C>0,\ \forall (j,j_0)\in\mathbb{Z}^2,\qquad \max_{|\alpha|\in[\![0,q]\!]}\ \sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^\alpha\left(\mu_j^0(\lambda,\omega)-\mu_{j_0}^0(\lambda,\omega)\right)\right|\leqslant C\,|j-j_0|.$$

Concerning the second assumption (7.259), we may combine (7.172) and (7.236) to find

$$\begin{aligned}\gamma^{-1}\|\mathscr{R}_0\|_{\text{O-d},q,s_0+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}} &\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\right)\\ &\leqslant C\varepsilon_0. \end{aligned}$$

▶ **KAM iteration**. Let $m \in \mathbb{N}$ and consider a linear operator

$$\mathscr{L}_m \triangleq \big(\omega \cdot \partial_\varphi + \mathscr{D}_m + \mathscr{R}_m\big)\Pi_{\mathbb{S}_0}^{\perp} \tag{7.266}$$

with $\mathscr{D}_m$ a diagonal real reversible operator and $\mathscr{R}_m$ a real and reversible Toeplitz in time operator of zero order satisfying $\Pi_{\mathbb{S}_0}^{\perp}\mathscr{R}_m\Pi_{\mathbb{S}_0}^{\perp} = \mathscr{R}_m$. We assume that both assumptions (7.255) and (7.259) are satisfied for $\mathscr{D}_m$ and $\mathscr{R}_m$. Remark that for $m = 0$ we take the operator $\mathscr{L}_0$ defined in (7.245). Let $\Phi_m = \mathrm{Id} + \Psi_m$ be a linear invertible operator such that

$$\Phi_m^{-1}\mathscr{L}_m\Phi_m \triangleq \big(\omega \cdot \partial_\varphi + \mathscr{D}_{m+1} + \mathscr{R}_{m+1}\big)\Pi_{\mathbb{S}_0}^{\perp}, \tag{7.267}$$

with Ψ_m satisfying the homological equation

$$\big[\big(\omega \cdot \partial_\varphi + \mathscr{D}_m\big)\Pi_{\mathbb{S}_0}^{\perp}, \Psi_m\big] + P_{N_m}\mathscr{R}_m = \lfloor P_{N_m}\mathscr{R}_m \rfloor.$$

Recall that N_m was defined in (6.94). The diagonal parts $(\mathscr{D}_m)_{m\in\mathbb{N}}$ and the remainders $(\mathscr{R}_m)_{m\in\mathbb{N}}$ are defined similarly to (7.253) by the recursive formulas,

$$\mathscr{D}_{m+1} = \mathscr{D}_m + \lfloor P_{N_m}\mathscr{R}_m \rfloor \quad \text{and} \quad \mathscr{R}_{m+1} = \Phi_m^{-1}\left(-\Psi_m \lfloor P_{N_m}\mathscr{R}_m \rfloor + P_{N_m}^{\perp}\mathscr{R}_m + \mathscr{R}_m\Psi_m\right). \tag{7.268}$$

Remark that $\mathscr{D}_m$ and $\lfloor P_{N_m}\mathscr{R}_m \rfloor$ are Fourier multiplier Toeplitz operators that can be identified to their spectra $(\mathrm{i}\mu_j^m)_{j\in\mathbb{S}_0^c}$ and $(\mathrm{i}r_j^m)_{j\in\mathbb{S}_0^c}$, namely

$$\forall (l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c, \quad \mathscr{D}_m \mathbf{e}_{l,j} = \mathrm{i}\mu_j^m\, \mathbf{e}_{l,j} \quad \text{and} \quad \lfloor P_{N_m}\mathscr{R}_m \rfloor \mathbf{e}_{l,j} = \mathrm{i}r_j^m\, \mathbf{e}_{l,j}. \tag{7.269}$$

By construction, we find

$$\mu_j^{m+1} = \mu_j^m + r_j^m. \tag{7.270}$$

In a similar way to (7.250) we obtain

$$(\Psi_m)_{j_0}^j(\lambda,\omega,l) = \begin{cases} \frac{-r_{j_0,m}^j(\lambda,\omega,l)}{\omega\cdot l + \mu_j^m(\lambda,\omega) - \mu_{j_0}^m(\lambda,\omega)} & \text{if } (l,j) \neq (0,j_0) \\ 0 & \text{if } (l,j) = (0,j_0), \end{cases} \tag{7.271}$$

where the collection $\{r_{j_0,m}^j(\lambda,\omega,l)\}$ describes the Fourier coefficients of $\mathscr{R}_m$, that is,

$$\mathscr{R}_m \mathbf{e}_{l_0,j_0} = \mathrm{i} \sum_{(l,j)\in\mathbb{Z}^{d+1}} r_{j_0,m}^j(\lambda,\omega,l_0 - l)\mathbf{e}_{l,j}.$$

Now we shall define the open Cantor set where the preceding formula is meaningful,

$$\mathscr{O}_{m+1}^{\gamma} = \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2 \\ |l|\leqslant N_m \\ (l,j)\neq(0,j_0)}} \Big\{(\lambda,\omega) \in \mathscr{O}_m^{\gamma} \quad \text{s.t.} \quad |\omega\cdot l + \mu_j^m(\lambda,\omega) - \mu_{j_0}^m(\lambda,\omega)| > \tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}. \tag{7.272}$$

Similarly to (7.251) and (7.252) we can extend (7.271) as follows

$$(\Psi_m)_{j_0}^j(\lambda,\omega,l) = \begin{cases} -\frac{\chi\left((\omega\cdot l + \mu_j^m(\lambda,\omega) - \mu_{j_0}^m(\lambda,\omega))(\gamma|j-j_0|)^{-1}\langle l\rangle^{\tau_2}\right) r_{j_0,m}^j(\lambda,\omega,l)}{\omega\cdot l + \mu_j^m(\lambda,\omega) - \mu_{j_0}^m(\lambda,\omega)} & \text{if } (l,j) \neq (0,j_0) \\ 0 & \text{if } (l,j) = (0,j_0). \end{cases} \tag{7.273}$$

We point out that working with this extension for Ψ_m allows to extend both $\mathscr{D}_{m+1}$ and the remainder $\mathscr{R}_{m+1}$ provided that the operators $\mathscr{D}_m$ and $\mathscr{R}_m$ are defined in the whole range of parameters. Thus the operator defined by the right-hand side in (7.267) can be extended to the whole set $\mathcal{O}$ and we denote this extension by $\mathscr{L}_{m+1}$, that is,

$$\big(\omega\cdot\partial_\varphi+\mathscr{D}_{m+1}+\mathscr{R}_{m+1}\big)\Pi_{\mathbb{S}_0}^{\perp}\triangleq\mathscr{L}_{m+1}. \tag{7.274}$$

This enables to construct by induction the sequence of operators $(\mathscr{L}_{m+1})$ in the full set $\mathcal{O}$. Similarly the operator $\Phi_m^{-1}\mathscr{L}_m\Phi_m$ admits an extension in $\mathcal{O}$ induced by the extension of $\Phi_m^{\pm1}$. However, by construction the identity $\mathscr{L}_{m+1}=\Phi_m^{-1}\mathscr{L}_m\Phi_m$ in (7.267) occurs in the Cantor set $\mathscr{O}_{m+1}^{\gamma}$ and may fail outside this set. We define

$$\delta_m(s)\triangleq\gamma^{-1}\|\mathscr{R}_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \tag{7.275}$$

and we want to prove by induction in $m\in\mathbb{N}$ that

$$\forall\, m\in\mathbb{N},\quad \forall s\in[s_0,\overline{s}_l],\ \delta_m(s)\leqslant\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\quad\text{ and }\quad\delta_m(s_h)\leqslant\left(2-\tfrac{1}{m+1}\right)\delta_0(s_h), \tag{7.276}$$

with $\overline{s}_l$ and s_h fixed by (7.3) and (7.235). Moreover, we should check the validity of the assumptions (7.255) and (7.259) for $\mathscr{D}_{m+1}$ and $\mathscr{R}_{m+1}$. Notice that by Sobolev embeddings, it is sufficient to prove the first inequality with $s=\overline{s}_l$. The property is obvious for $m=0$. Now, assume that the property (7.276) is true for $m\in\mathbb{N}$ and let us check it at the next order. We write

$$\Phi_m^{-1}=\mathrm{Id}+\Sigma_m\quad\text{with}\quad\Sigma_m=\sum_{n=1}^{\infty}(-1)^n\Psi_m^n. \tag{7.277}$$

Thus similarly to (7.261), using in particular (7.258) and (7.260) we deduce successively

$$\begin{aligned}\|\Sigma_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant\|\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\left(1+\sum_{n=0}^{\infty}\big(C\|\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\big)^n\right)\\&\leqslant\delta_m(s_0+\tau_2q+\tau_2)\left(1+\sum_{n=0}^{\infty}\big(C\delta_m(s_0+\tau_2q+\tau_2)\big)^n\right)\end{aligned}$$

and for any $s\in[s_0,S]$,

$$\begin{aligned}\|\Sigma_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Psi_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+\sum_{n=0}^{\infty}\big(C\|\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\big)^n\right)\\&\leqslant N_m^{\tau_2q+\tau_2}\delta_m(s)\left(1+\sum_{n=0}^{\infty}\big(C\delta_m(s_0+\tau_2q+\tau_2)\big)^n\right).\end{aligned}$$

Hence, from the induction assumption, the fact that $N_m\geqslant N_0$ and since (7.3) implies in particular $s_0+\tau_2q+\tau_2\leqslant\overline{s}_l$, we obtain

$$\begin{aligned}\|\Sigma_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}&\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\left(1+\sum_{n=0}^{\infty}\big(CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\big)^n\right)\\&\leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\left(1+\sum_{n=0}^{\infty}\big(C\delta_0(s_h)\big)^n\right)\end{aligned}$$

and for any $s \in [s_0, S]$,

$$\begin{aligned}\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant N_m^{\tau_2 q+\tau_2}\delta_m(s)\left(1+\sum_{n=0}^{\infty}\left(CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h)\right)^n\right)\\ &\leqslant N_m^{\tau_2 q+\tau_2}\delta_m(s)\left(1+\sum_{n=0}^{\infty}\left(C\delta_0(s_h)\right)^n\right).\end{aligned}$$

It follows from the condition (7.244) that

$$\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant CN_0^{\mu_2}N_m^{-\mu_2}\delta_0(s_h) \quad \text{and} \quad \|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant CN_m^{\tau_2 q+\tau_2}\delta_m(s). \tag{7.278}$$

One also gets

$$\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant C\delta_m(s+\tau_2 q+\tau_2). \tag{7.279}$$

From KAM step (7.264) and Sobolev embeddings, we infer

$$\delta_{m+1}(\overline{s}_l) \leqslant N_m^{\overline{s}_l-s_h}\delta_m(s_h)+CN_m^{\tau_2 q+\tau_2}\left(\delta_m(\overline{s}_l)\right)^2.$$

Using the induction assumption (7.276) yields

$$\begin{aligned}\delta_{m+1}(\overline{s}_l) &\leqslant N_m^{\overline{s}_l-s_h}\left(2-\tfrac{1}{m+1}\right)\delta_0(s_h)+CN_m^{\tau_2 q+\tau_2}\delta_0^2(s_h)N_0^{2\mu_2}N_m^{-2\mu_2}\\ &\leqslant 2N_m^{\overline{s}_l-s_h}\delta_0(s_h)+CN_m^{\tau_2 q+\tau_2}\delta_0^2(s_h)N_0^{2\mu_2}N_m^{-2\mu_2}.\end{aligned}$$

At this level we need to select the parameters $\overline{s}_l, s_h$ and μ_2 in such a way

$$N_m^{\overline{s}_l-s_h} \leqslant \frac{1}{4}N_0^{\mu_2}N_{m+1}^{-\mu_2} \qquad \text{and} \qquad CN_m^{\tau_2 q+\tau_2}\delta_0(s_h)N_0^{2\mu_2}N_m^{-2\mu_2} \leqslant \frac{1}{2}N_0^{\mu_2}N_{m+1}^{-\mu_2} \tag{7.280}$$

leading to

$$\delta_{m+1}(\overline{s}_l) \leqslant \delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.$$

The conditions (7.235) imply in particular

$$s_h \geqslant \frac{3}{2}\mu_2+\overline{s}_l+1 \quad \text{and} \quad \mu_2 \geqslant 2(\tau_2 q+\tau_2)+1.$$

Then, using (6.94), we conclude that the assumptions of (7.280) hold true provided that

$$4N_0^{-\mu_2} \leqslant 1 \qquad \text{and} \qquad 2CN_0^{\mu_2}\delta_0(s_h) \leqslant 1, \tag{7.281}$$

which follow from (7.244), since the first condition $4N_0^{-\mu_2} \leqslant 1$ is automatically satisfied because $N_0 \geqslant 2$ and $\mu_2 \geqslant 2$, according to (7.235). Therefore, under the assumptions (7.235) we get the first statement of the induction in (7.276). The next goal is to establish the second estimate in (7.276). By KAM step (7.264) combined with the induction assumptions (7.276) we deduce that

$$\begin{aligned}\delta_{m+1}(s_h) &\leqslant \delta_m(s_h)+CN_m^{\tau_2 q+\tau_2}\delta_m(s_0)\delta_m(s_h)\\ &\leqslant \left(2-\tfrac{1}{m+1}\right)\delta_0(s_h)\left(1+CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2-\mu_2}\delta_0(s_h)\right).\end{aligned}$$

Thus if one has

$$\left(2-\tfrac{1}{m+1}\right)\left(1+CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2-\mu_2}\delta_0(s_h)\right) \leqslant 2-\tfrac{1}{m+2}, \tag{7.282}$$

then we get

$$\delta_{m+1}(s_h)\leqslant\left(2-\tfrac{1}{m+2}\right)\delta_0(s_h),$$

which ends the induction argument of (7.276). Remark that with the choice $\mu_2\geqslant 2(\tau_2 q+\tau_2)$ fixed in (7.235), the condition (7.282) is staisfied if

$$CN_0^{\mu_2}N_m^{-\tau_2 q-\tau_2}\delta_0(s_h)\leqslant\tfrac{1}{(2m+1)(m+2)}.\tag{7.283}$$

Since $N_0\geqslant 2$ we may find a constant $c_0>0$ small enough such that

$$\forall\, m\in\mathbb{N},\quad c_0N_m^{-1}\leq\tfrac{1}{(2m+1)(m+2)}.$$

Consequently, (7.283) is satisfied provided that

$$CN_0^{\mu_2}N_m^{-\tau_2 q-\tau_2+1}\delta_0(s_h)\leqslant c_0.\tag{7.284}$$

By virtue of the assumption (A.1) we get in particular

$$\tau_2 q+\tau_2-1\geqslant 0.\tag{7.285}$$

Thus (7.284) is satisfied in view of (7.244). To conclude the induction proof of (7.276) it remains to check that the assumptions (7.255) and (7.259) are satisfied for $\mathscr{D}_{m+1}$ and $\mathscr{R}_{m+1}$. First, the assumption (7.259) is a consequence of the first inequality of (7.276) applied at the order $m+1$ with the regularity index $s=s_0+\tau_2 q+\tau_2\leqslant\overline{s}_l$ supplemented with (7.244). Concerning the validity of (7.255) for $\mathscr{D}_{m+1}$, we combine (7.269), (7.270) and (7.248), in order to find

$$\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}}=\big\|\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{l,j},\mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})}\big\|_q^{\gamma,\mathcal{O}}.$$

From the Topeplitz structure of $\mathscr{R}_m$ we may write

$$\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}}=\big\|\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\big\|_q^{\gamma,\mathcal{O}}.$$

By a duality argument combined with Lemma A.6 and (7.275) we infer

$$\begin{aligned}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}}&\lesssim\big\|\mathscr{R}_m\mathbf{e}_{0,j}\|_{q,s_0}^{\gamma,\mathcal{O}}\,\langle j\rangle^{-s_0}\\&\lesssim\big\|\mathscr{R}_m\big\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathbf{e}_{0,j}\|_{H^{s_0}}\,\langle j\rangle^{-s_0}\\&\lesssim\big\|\mathscr{R}_m\big\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}=\gamma\delta_m(s_0).\end{aligned}\tag{7.286}$$

Hence we deduce from (7.276), (7.243) and (7.236)

$$\begin{aligned}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}}&\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_m^{-\mu_2}.\end{aligned}\tag{7.287}$$

As the assumption (7.255) is satisfied with $\mathscr{D}_m$, that is,

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\quad\max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\big|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)\right)\big|\leqslant C\,|j-j_0|,\tag{7.288}$$

then we obtain by (7.287)

$$\forall\,(j,j_0)\in(\mathbb{S}_0^c)^2,\quad \max_{|\alpha|\in[\![0,q]\!]}\sup_{(\lambda,\omega)\in\mathcal{O}}\left|\partial_{\lambda,\omega}^{\alpha}\left(\mu_j^{m+1}(\lambda,\omega)-\mu_{j_0}^{m+1}(\lambda,\omega)\right)\right|\leqslant C\big(1+\varepsilon\gamma^{-1-q}N_0^{\mu_2}N_m^{-\mu_2}\big)\,|j-j_0|.$$

Consequently, the convergence of the series $\sum N_m^{-\mu_2}$ gives the required assumption with the same constant C independently of m. This completes the induction principle. In what follows, we shall provide some estimates for Ψ_m that will be used later to study the string convergence. Using (7.258) combined with Lemma A.6 and $s_0+\tau_2q+\tau_2+1\leqslant\overline{s}_l$ we find

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}&\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s_0+\tau_2q+\tau_2+1}^{\gamma,\mathcal{O}}\\&\leqslant C\,\delta_m(\overline{s}_l).\end{aligned}\tag{7.289}$$

Thus (7.276) and (7.244) yield

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}&\leqslant C\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\&\leqslant C\,\varepsilon\gamma^{-2}N_0^{\mu_2}\,N_m^{-\mu_2}.\end{aligned}\tag{7.290}$$

Next, we discuss the persistence of higher regularity. Let $s\in[s_0,S]$, then from (7.264), (7.276) and (7.244) and (7.285)

$$\begin{aligned}\delta_{m+1}(s)&\leqslant\delta_m(s)\Big(1+CN_m^{\tau_2q+\tau_2}\delta_m(s_0)\Big)\\&\leqslant\delta_m(s)\Big(1+CN_0^{\mu_2}N_m^{\tau_2q+\tau_2-\mu_2}\delta_0(s_h)\Big)\\&\leqslant\delta_m(s)\big(1+CN_m^{-1}\big).\end{aligned}$$

Combining this estimate with (6.94) and (7.243) yields

$$\begin{aligned}\forall\,s\geqslant s_0,\,\forall\,m\in\mathbb{N},\quad\delta_m(s)&\leqslant\delta_0(s)\prod_{n=0}^{\infty}\big(1+CN_n^{-1}\big)\\&\leqslant C\delta_0(s)\\&\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.291}$$

Using (7.258) combined with Lemma A.6, applying in particular interpolation inequalities, leads to

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+\tau_2}^{\gamma,\mathcal{O}}\\&\leqslant C\,\delta_m(s+\tau_2q+\tau_2)\\&\leqslant C\,\delta_m^{\overline{\theta}}(s_0)\delta_m^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1),\end{aligned}\tag{7.292}$$

with $\overline{\theta}=\frac{1}{s-s_0+\tau_2q+\tau_2+1}$. Inserting (7.276) and (7.291) into (7.292) and using (7.244) give

$$\begin{aligned}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant C\,\delta_0^{\overline{\theta}}(s_2)\delta_0^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1)N_0^{\mu_2\overline{\theta}}N_m^{-\mu_2\overline{\theta}}\\&\leqslant C\,\varepsilon_0{}^{\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1)N_m^{-\mu_2\overline{\theta}}.\end{aligned}\tag{7.293}$$

We point out that one also finds from (7.279), the second inequality of (7.292) and (7.291) that

$$\forall\,s\in[s_0,S],\quad\sup_{m\in\mathbb{N}}\left(\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\right)\leqslant C\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right).\tag{7.294}$$

▶ **KAM conclusion**. Let us examine the sequence of operators $\left(\widehat{\Phi}_m\right)_{m\in\mathbb{N}}$ defined by

$$\widehat{\Phi}_0 \triangleq \Phi_0 \quad \text{and} \qquad \forall m \geqslant 1,\ \widehat{\Phi}_m \triangleq \Phi_0 \circ \Phi_1 \circ ... \circ \Phi_m. \tag{7.295}$$

It is obvious from the identity $\Phi_m = \mathrm{Id} + \Psi_m$ that $\widehat{\Phi}_{m+1} = \widehat{\Phi}_m + \widehat{\Phi}_m \Psi_{m+1}$. Applying the product laws yields

$$\|\widehat{\Phi}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \leqslant \|\widehat{\Phi}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\Big(1 + C\|\Psi_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\Big).$$

By iterating this inequality and using (7.290) we infer

$$\begin{aligned}\|\widehat{\Phi}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} &\leqslant \|\Phi_0\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \prod_{n=1}^{m+1}\Big(1 + C\|\Psi_n\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\Big)\\ &\leqslant \prod_{n=0}^{\infty}\Big(1 + C\,\varepsilon_0 N_n^{-\mu_2}\Big).\end{aligned}$$

Using the first condition of (7.244) and (6.94) imply

$$\|\widehat{\Phi}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \leqslant \prod_{n=0}^{\infty}\Big(1 + C\,\varepsilon_0 4^{-(\frac{3}{2})^n}\Big)$$

and since the infinite product converges, we obtain for ε_0 small enough

$$\sup_{m\in\mathbb{N}} \|\widehat{\Phi}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \leqslant 2. \tag{7.296}$$

Now we shall estimate the difference $\widehat{\Phi}_{m+1} - \widehat{\Phi}_m$ and for this aim we use the product laws combined with (7.290) and (7.296)

$$\begin{aligned}\|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} &\leqslant C\|\widehat{\Phi}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\|\Psi_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\\ &\leqslant C\,\delta_0(s_h) N_0^{\mu_2}\, N_{m+1}^{-\mu_2}.\end{aligned} \tag{7.297}$$

Applying Lemma A.5 gives

$$\sum_{m=0}^{\infty} \|\widehat{\Phi}_{m+1} - \widehat{\Phi}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \leqslant C\,\delta_0(s_h). \tag{7.298}$$

Therefore, by a completeness argument we deduce that the series $\sum_{m\in\mathbb{N}}(\widehat{\Phi}_{m+1} - \widehat{\Phi}_m)$ converges to an element Φ_∞. In addition, we get in view of (7.297) and Lemma A.5

$$\begin{aligned}\|\widehat{\Phi}_m - \Phi_\infty\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} &\leqslant \sum_{j=m}^{\infty} \|\widehat{\Phi}_{j+1} - \widehat{\Phi}_j\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1}\\ &\leqslant C\,\delta_0(s_h) N_0^{\mu_2} \sum_{j=m}^{\infty} N_{j+1}^{-\mu_2}\\ &\leqslant C\,\delta_0(s_h) N_0^{\mu_2} N_{m+1}^{-\mu_2}.\end{aligned} \tag{7.299}$$

Remark that one also finds from (7.296)

$$\|\Phi_\infty\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0+1} \leqslant 2. \tag{7.300}$$

Using (7.298) combined with (7.292) for $m=0$ and (7.235)

$$\begin{aligned}\|\Phi_\infty-\mathrm{Id}\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}&\leqslant\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}+\|\Psi_0\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\\&\leqslant C\,\delta_0(s_h).\end{aligned}\tag{7.301}$$

Let us now check the convergence with higher order norms. Take $s\in[s_0,S]$, then using the product laws, (7.290), (7.293) and (7.296) we infer

$$\begin{aligned}\|\widehat{\Phi}_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+C\|\Psi_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)+C\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\&\leqslant\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+C\,\varepsilon_0\,N_{m+1}^{-\mu_2}\right)+C\,\delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1)N_m^{-\mu_2\overline{\theta}}.\end{aligned}\tag{7.302}$$

According to the first condition of (7.244) and (6.94) one finds

$$\begin{aligned}\prod_{n=0}^{\infty}\left(1+C\,\varepsilon_0\,N_n^{-\mu_2}\right)&\leqslant\prod_{n=0}^{\infty}\Big(1+C\,\varepsilon_0 4^{-(\frac{3}{2})^n}\Big)\\&\leqslant 2,\end{aligned}$$

where the last inequality holds if ε_0 is chosen small enough. Applying (7.70) together with (7.302) and Lemma A.5 and using (7.258) yield

$$\begin{aligned}\sup_{m\in\mathbb{N}}\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant C\Big(\|\Phi_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}(s+\tau_2q+\tau_2+1)\Big)\\&\leqslant C\Big(1+\delta_0(s+\tau_2q+\tau_2)+\delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\delta_0^{1-\overline{\theta}}\big(s+\tau_2q+\tau_2+1\big)\Big).\end{aligned}$$

Interpolation inequalities and (7.244) allow to get

$$\sup_{m\in\mathbb{N}}\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\Big(1+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big).\tag{7.303}$$

The next task is to estimate the difference $\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{q,s}^{\gamma,\mathcal{O}}$. By the product laws combined with the first inequality in (7.290), (7.293), (7.296) and (7.303) we obtain

$$\begin{aligned}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant C\Big(\|\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\widehat{\Phi}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_{m+1}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\Big)\\&\leqslant C\,\delta_0(s_h)N_0^{\mu_2}\,N_{m+1}^{-\mu_2}\Big(1+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big)\\&\quad+C\,\delta_0^{\overline{\theta}}(s_h)N_0^{\mu_2\overline{\theta}}\,\delta_0^{1-\overline{\theta}}\big(s+\tau_2q+\tau_2+1\big)N_{m+1}^{-\mu_2\overline{\theta}}.\end{aligned}$$

Thus, we obtain in view of Lemma A.5

$$\begin{aligned}\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant C\,\delta_0(s_h)\Big(1+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big)\\&\quad+C\,\delta_0^{\overline{\theta}}(s_h)\delta_0^{1-\overline{\theta}}\big(s+\tau_2q+\tau_2+1\big).\end{aligned}$$

Combining the interpolation inequalities with the second condition in (7.244) gives

$$\sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant C\Big(\delta_0(s_h)+\delta_0\big(s+\tau_2q+\tau_2+1\big)\Big).\tag{7.304}$$

From this latter inequality combined with (7.244) and (7.303) we infer

$$\begin{aligned}\|\Phi_\infty\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\widehat{\Phi}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\leqslant C\Big(1+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big).\end{aligned}\tag{7.305}$$

On the other hand, using (7.304) and the second inequality in (7.292) with $m=0$, one can check that

$$\begin{aligned}\|\Phi_\infty-\text{Id}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\|\widehat{\Phi}_{m+1}-\widehat{\Phi}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\Psi_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\leqslant C\Big(\delta_0(s_h)+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big).\end{aligned}\tag{7.306}$$

Therefore, Lemma A.6 together with (7.300), (7.305) and Sobolev embeddings give

$$\begin{aligned}\|\Phi_\infty\rho\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\Phi_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\Phi_\infty\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\Big(1+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.307}$$

Applying (7.172) and (7.275) we obtain

$$\begin{aligned}\delta_0\big(s+\tau_2 q+\tau_2+1\big) &= \gamma^{-1}\|\mathscr{R}_0\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.308}$$

Plugging (7.308) into (7.307) and using (7.236) combined with Sobolev embeddings and (7.235) yield

$$\begin{aligned}\|\Phi_\infty\rho\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\right)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.309}$$

In a similar way to (7.307) we get by Lemma A.6 combined with (7.306) and (7.301)

$$\begin{aligned}\big\|\big(\Phi_\infty-\text{Id}\big)\rho\big\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\Phi_\infty-\text{Id}\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\Phi_\infty-\text{Id}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \delta_0(s_h)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\Big(\delta_0(s_h)+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\Big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \delta_0(s_h)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\delta_0\big(s+\tau_2 q+\tau_2+1\big)\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Hence we find from (7.308) and (7.236) combined with Sobolev embeddings and (7.243)

$$\begin{aligned}\big\|\big(\Phi_\infty-\text{Id}\big)\rho\big\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \big(\varepsilon\gamma^{-2}+\delta_0(s_h)\big)\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-2}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.310}$$

The estimates Φ_∞^{-1} and $\Phi_\infty^{-1}-\widehat{\Phi}_n^{-1}$ follow from the same type pf arguments.
➢ In what follows we plan to study the asymptotic of the eigenvalues. Summing up in m the estimates (7.287) and using Lemma A.5, we find

$$\begin{aligned}\sum_{m=0}^{\infty}\|\mu_j^{m+1}-\mu_j^m\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\ &\leqslant C\gamma\delta_0(s_h).\end{aligned}\tag{7.311}$$

Thus for each $j \in \mathbb{S}_0^c$ the sequence $(\mu_j^m)_{m\in\mathbb{N}}$ converges in the space $W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ to an element denoted by $\mu_j^\infty \in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$. Moreover, for any $m \in \mathbb{N}$, we find in view of (7.287)

$$\begin{aligned}\|\mu_j^\infty - \mu_j^m\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{n=m}^{\infty} \|\mu_j^{n+1} - \mu_j^n\|_q^{\gamma,\mathcal{O}}\\ &\leqslant C\gamma\,\delta_0(s_h)N_0^{\mu_2}\sum_{n=m}^{\infty} N_n^{-\mu_2}.\end{aligned}$$

Applying Lemma A.5

$$\sup_{j\in\mathbb{S}_0^c}\|\mu_j^\infty - \mu_j^m\|_q^{\gamma,\mathcal{O}} \leqslant C\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \tag{7.312}$$

Therefore, we deduce

$$\begin{aligned}\mu_j^\infty &= \mu_j^0 + \sum_{m=0}^{\infty}\big(\mu_j^{m+1} - \mu_j^m\big)\\ &\triangleq \mu_j^0 + r_j^\infty,\end{aligned} \tag{7.313}$$

where (μ_j^0) is described in Proposition 7.4 and takes the form

$$\mu_j^0(\lambda,\omega,i_0) = \Omega_j(\lambda) + j\big(c_{i_0}(\lambda,\omega) - I_1(\lambda)K_1(\lambda)\big).$$

Hence (7.311), (7.243) and (7.236) yield

$$\begin{aligned}\|r_j^\infty\|_q^{\gamma,\mathcal{O}} &\leqslant C\,\gamma\,\delta_0(s_h)\\ &\leqslant C\,\varepsilon\gamma^{-1}\end{aligned}$$

and this gives the first result in (7.239). We define the diagonal operator $\mathscr{D}_\infty$, acting on the normal modes, as follows

$$\forall (l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c, \quad \mathscr{D}_\infty \mathbf{e}_{l,j} = \mathrm{i}\mu_j^\infty \mathbf{e}_{l,j}. \tag{7.314}$$

By the norm definition we obtain

$$\|\mathscr{D}_m - \mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} = \sup_{j\in\mathbb{S}_0^c}\|\mu_j^m - \mu_j^\infty\|_q^{\gamma,\mathcal{O}},$$

which gives by virtue of (7.312)

$$\|\mathscr{D}_m - \mathscr{D}_\infty\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}. \tag{7.315}$$

➢ The next goal is to prove that the Cantor set $\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ defined in Proposition 7.5 satisfies

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) \subset \bigcap_{m=0}^{n+1}\mathscr{O}_m^\gamma = \mathscr{O}_{n+1}^\gamma,$$

where the intermediate Cantor sets are defined in (7.272). For this aim we shall proceed by finite induction on m with n fixed. First, we get by construction $\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) \subset \mathcal{O} \triangleq \mathscr{O}_0^\gamma$. Now assume that

$\mathscr{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)\subset\mathscr{O}^{\gamma}_{m}$ for $m\leqslant n$ and let us check that

$$\mathscr{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)\subset\mathscr{O}^{\gamma}_{m+1}. \tag{7.316}$$

Let $(\lambda,\omega)\in\mathscr{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)$ and $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ such that $0\leqslant|l|\leqslant N_m$ and $(l,j)\neq(0,j_0)$. Then, the triangle inequality, (7.312), (7.235) and (7.244) imply

$$\begin{aligned}|\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)|&\geqslant|\omega\cdot l+\mu_j^\infty(\lambda,\omega)-\mu_{j_0}^\infty(\lambda,\omega)|-2\sup_{j\in\mathbb{S}_0^c}\|\mu_j^m-\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\\&\geqslant\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-2\gamma\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}\\&\geqslant\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-2\gamma\varepsilon_0\langle l\rangle^{-\mu_2}\langle j-j_0\rangle.\end{aligned}$$

Thus for ε_0 small enough and by (7.235) (implying that $\mu_2\geqslant\tau_2$) we get

$$\big|\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)\big|>\tfrac{\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}$$

which shows that $(\lambda,\omega)\in\mathscr{O}^{\gamma}_{m+1}$ and therefore the inclusion (7.316) is satisfied.

➢ Next we shall discuss the convergence of the sequence $(\mathscr{L}_m)_{m\in\mathbb{N}}$ introduced in (7.266) towards the diagonal operator $\mathscr{L}_\infty\triangleq\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty$, where $\mathscr{D}_\infty$ is detailed in (7.314). Applying (7.315) and (7.276)

$$\begin{aligned}\|\mathscr{L}_m-\mathscr{L}_\infty\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,s_0}&\leqslant\|\mathscr{D}_m-\mathscr{D}_\infty\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,s_0}+\|\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,s_0}\\&\leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_m^{-\mu_2},\end{aligned}\tag{7.317}$$

which gives in particular that

$$\lim_{m\to\infty}\|\mathscr{L}_m-\mathscr{L}_\infty\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,s_0}=0. \tag{7.318}$$

By virtue of (7.295) and (7.267) one gets

$$\begin{aligned}\forall(\lambda,\omega)\in\mathscr{O}^{\gamma}_{n+1},\quad\widehat{\Phi}_n^{-1}\mathscr{L}_0\widehat{\Phi}_n&=\big(\omega\cdot\partial_\varphi+\mathscr{D}_{n+1}+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^\perp\\&=\mathscr{L}_\infty+\big(\mathscr{D}_{n+1}-\mathscr{D}_\infty+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

It follows that any $(\lambda,\omega)\in\mathscr{O}^{\gamma}_{n+1}$

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty&=\mathscr{L}_\infty+\big(\mathscr{D}_{n+1}-\mathscr{D}_\infty+\mathscr{R}_{n+1}\big)\Pi_{\mathbb{S}_0}^\perp\\&\quad+\Phi_\infty^{-1}\mathscr{L}_0\left(\Phi_\infty-\widehat{\Phi}_n\right)+\left(\Phi_\infty^{-1}-\widehat{\Phi}_n^{-1}\right)\mathscr{L}_0\widehat{\Phi}_n\\&\triangleq\mathscr{L}_\infty+\mathtt{E}^2_{n,1}+\mathtt{E}^2_{n,2}+\mathtt{E}^2_{n,3}\triangleq\mathscr{L}_\infty+\mathtt{E}^2_n.\end{aligned}$$

For the estimate $\mathtt{E}^2_{n,1}$ we use (7.315) combined with (7.275), (7.276), (7.243) and (7.236)

$$\begin{aligned}\|\mathtt{E}^2_{n,1}\|^{\gamma,\mathcal{O}}_{\textnormal{O-d},q,s_0}&\leqslant C\,\gamma\,\delta_0(s_h)N_0^{\mu_2}N_{n+1}^{-\mu_2}\\&\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}.\end{aligned}\tag{7.319}$$

According to Lemma A.25 with (7.319) we obtain

$$\|\mathtt{E}^2_{n,1}\rho\|^{\gamma,\mathcal{O}}_{q,s_0}\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|^{\gamma,\mathcal{O}}_{q,s_0}.$$

Now let us move to the estimates of $\mathtt{E}_{n,2}^2$ and $\mathtt{E}_{n,3}^2$. They can be treated in a similar way. Therefore we shall restrict the discussion to the term $\mathtt{E}_{n,2}^2$. Using (7.237) yields

$$\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \big\|\mathscr{L}_0\big(\Phi_\infty - \widehat{\Phi}_n\big)\rho\big\|_{q,s_0}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s_0+\sigma_4}^{\gamma,\mathcal{O}}\big\|\mathscr{L}_0\big(\Phi_\infty - \widehat{\Phi}_n\big)\rho\big\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{7.320}$$

Therefore we get from (7.174) combined with (7.236)

$$\begin{aligned}\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \big\|\mathscr{L}_0\big(\Phi_\infty - \widehat{\Phi}_n\big)\rho\big\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \big\|\big(\Phi_\infty - \widehat{\Phi}_n\big)\rho\big\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Applying (7.299) with Lemma A.25, (7.236) and (7.243) allow to get

$$\begin{aligned}\|\mathtt{E}_{n,2}^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim \big\|\Phi_\infty - \widehat{\Phi}_n\big\|_{\text{O-d},q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\leqslant C\,\delta_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\leqslant C\,\varepsilon\gamma^{-2}N_0^{\mu_2}N_{m+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Notice that for $\mathtt{E}_{n,3}^2$ we get the same estimate as the preceding one. Consequently, putting together the foregoing estimates yields (7.240).

➢ The goal now is to prove (7.239). We set

$$\widehat{\delta}_m(s) \triangleq \max\left(\gamma^{-1}\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}, \gamma^{-1}\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\right).$$

Then we shall prove by induction on $m \in \mathbb{N}$ that

$$\widehat{\delta}_m(s_0) \leqslant \widehat{\delta}_0(s_h)N_0^{\mu_2}N_m^{-\mu_2} \quad \text{and} \quad \widehat{\delta}_m(s_h) \leqslant \left(2 - \tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h). \tag{7.321}$$

According to Sobolev embeddings, the property is trivially satisfied for $m = 0$. Notice that from (7.172) and (7.236) one gets

$$\begin{aligned}\widehat{\delta}_0(s_h) &\lesssim \varepsilon\gamma^{-2}\left(1 + \|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\right)\\ &\lesssim \varepsilon\gamma^{-2}.\end{aligned} \tag{7.322}$$

We assume that (7.321) is satisfied at the order m and let us check it at the order $m+1$. Applying ∂_θ to the second identity in (7.268) and using (7.277) we obtain the expression

$$\begin{aligned}\partial_\theta\mathscr{R}_{m+1} &= \Phi_m^{-1}\Big(P_{N_m}^\perp\partial_\theta\mathscr{R}_m + \partial_\theta\mathscr{R}_m\Psi_m - \Psi_m\partial_\theta\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor - \big[\partial_\theta, \Psi_m\big]\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big)\\ &\quad + \big[\partial_\theta, \Sigma_m\big]\Big(P_{N_m}^\perp\mathscr{R}_m + \mathscr{R}_m\Psi_m - \Psi_m\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big).\\ &\triangleq \mathcal{U}_m^1 + \mathcal{U}_m^2\end{aligned}$$

with

$$\mathcal{U}_m^2 = \big[\partial_\theta, \Sigma_m\big]\Big(P_{N_m}^\perp\mathscr{R}_m + \mathscr{R}_m\Psi_m - \Psi_m\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor\Big).$$

It is easy to check that for any Toeplitz in time operator $T(\lambda,\omega)$, we have

$$\big[\partial_\theta, T(\lambda,\omega)\big]\mathbf{e}_{l_0,j_0} = \mathrm{i}\sum_{(l,j)\in\mathbb{Z}^{d+1}}(j - j_0)T_{j_0}^j(\lambda,\omega,l - l_0)\mathbf{e}_{l,j},$$

which implies using the norm definition

$$\big\|[\partial_\theta,T]\big\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant\|T\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}}.\tag{7.323}$$

Since $\Phi_m^{-1}=\text{Id}+\Sigma_m$, then applying Lemma A.6, we obtain successively for $S\geqslant\overline{s}\geqslant s\geqslant s_0$

$$\begin{aligned}\|\mathcal{U}_m^1\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\leqslant{}& C\|\Sigma_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left[\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\left(1+\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)+\|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right]\\&+C\|\Sigma_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\left[\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)+\|\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\right.\\&\left.+\|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\right]+\|P_{N_m}^{\perp}\partial_\theta\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\end{aligned}\tag{7.324}$$

and

$$\begin{aligned}\|\mathcal{U}_m^2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim{}&\|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\left(\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}+\|\mathscr{R}_m\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\left(1+\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right)\right)\\&+\|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\mathscr{R}_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\left(1+\|\Psi_m\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{7.325}$$

By using (7.323), (7.258) and Lemma A.6, we obtain

$$\begin{aligned}\|\left[\partial_\theta,\Psi_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Psi_m\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}}\\&\leqslant C\gamma^{-1}\|P_{N_m}\mathscr{R}_m\|_{\text{O-d},q,s+\tau_2q+\tau_2+1}^{\gamma,\mathcal{O}}\\&\leqslant CN_m^{\tau_2q+\tau_2+1}\delta_m(s).\end{aligned}$$

Coming back to (7.278), we obtain

$$\begin{aligned}\|\left[\partial_\theta,\Sigma_m\right]\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\leqslant\|\Sigma_m\|_{\text{O-d},q,s+1}^{\gamma,\mathcal{O}}\\&\leqslant CN_m^{\tau_2q+\tau_2+1}\delta_m(s).\end{aligned}$$

Then inserting the preceding estimates and (7.290) into (7.324) we deduce that

$$\forall\,S\geqslant\overline{s}\geqslant s\geqslant s_0,\quad\widehat{\delta}_{m+1}(s)\leqslant N_m^{s-\overline{s}}\,\widehat{\delta}_m(\overline{s})+CN_m^{\tau_2q+\tau_2+1}\widehat{\delta}_m(s)\widehat{\delta}_m(s_0).\tag{7.326}$$

In particular, for $s=s_0$ we get by the induction assumption (7.321),

$$\begin{aligned}\widehat{\delta}_{m+1}(s_0)&\leqslant N_m^{s_0-s_h}\widehat{\delta}_m(s_h)+CN_m^{\tau_2q+\tau_2+1}\left(\widehat{\delta}_m(s_0)\right)^2\\&\leqslant\left(2-\tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h)N_m^{s_0-s_h}+CN_0^{2\mu_2}N_m^{\tau_2q+\tau_2+1-2\mu_2}\left(\widehat{\delta}_0(s_h)\right)^2\\&\leqslant\widehat{\delta}_0(s_h)\left(2N_m^{s_0-s_h}+CN_0^{2\mu_2}N_m^{\tau_2q+\tau_2+1-2\mu_2}\widehat{\delta}_0(s_h)\right).\end{aligned}$$

If we fix s_2 and μ_2 such that

$$N_m^{s_0-s_h}\leqslant\tfrac{1}{4}N_0^{\mu_2}N_{m+1}^{-\mu_2}\quad\text{and}\quad CN_0^{2\mu_2}N_m^{\tau_2q+\tau_2+1-2\mu_2}\widehat{\delta}_0(s_h)\leqslant\tfrac{1}{2}N_0^{\mu_2}N_{m+1}^{-\mu_2},\tag{7.327}$$

then we find

$$\widehat{\delta}_{m+1}(s_0)\leqslant\widehat{\delta}_0(s_h)N_0^{\mu_2}N_{m+1}^{-\mu_2}.$$

Notice that (7.235) implies in particular

$$s_h\geqslant\tfrac{3}{2}\mu_2+s_0+1\quad\text{and}\quad\mu_2\geqslant2(\tau_2q+\tau_2+1)+1.$$

Hence, using (6.94), we see that the assumptions of (7.327) hold true provided that

$$4N_0^{-\mu_2} \leqslant 1 \qquad \text{and} \qquad 2C\widehat{\delta}_0(s_h) \leqslant N_0^{-\mu_2}.$$

Remark that these conditions are satisfied thanks to (7.281), (7.322) and (7.236). Now, we turn to the proof of the second estimate in (7.321). By (7.326) and (7.321)

$$\begin{aligned}\widehat{\delta}_{m+1}(s_h) &\leqslant \widehat{\delta}_m(s_h) + CN_m^{\tau_2 q+\tau_2+1}\widehat{\delta}_m(s_h)\widehat{\delta}_m(s_0)\\ &\leqslant \left(2-\tfrac{1}{m+1}\right)\widehat{\delta}_0(s_h)\left(1+CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2+1-\mu_2}\widehat{\delta}_0(s_h)\right).\end{aligned}$$

Taking the parameters s_2 and μ_2 such that

$$\left(2-\tfrac{1}{m+1}\right)\left(1+CN_0^{\mu_2}N_m^{\tau_2 q+\tau_2+1-\mu_2}\widehat{\delta}_0(s_h)\right) \leqslant 2-\tfrac{1}{m+2}, \tag{7.328}$$

then we obtain

$$\widehat{\delta}_{m+1}(s_h) \leqslant \left(2-\tfrac{1}{m+2}\right)\widehat{\delta}_0(s_h),$$

which achieves the induction argument in (7.321). Now observe that (7.328) is quite similar to (7.282) using in particular $\mu_2 \geqslant 2(\tau_2 q+\tau_2)+1$ and one may proceed following the same lines. Next let us see how to get the estimate (7.239). Recall that

$$r_j^\infty = \sum_{m=0}^{\infty} r_j^m \qquad \text{with} \qquad r_j^m = -\mathrm{i}\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Then it is clear that

$$\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})} = \tfrac{\mathrm{i}}{j}\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j}, \partial_\theta\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Therefore integration by parts leads to

$$\big\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j}, \partial_\theta\mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})} = -\big\langle P_{N_m}\partial_\theta\mathscr{R}_m\mathbf{e}_{0,j}, \mathbf{e}_{0,j}\big\rangle_{L^2(\mathbb{T}^{d+1})}.$$

Using a duality argument $H^{s_0}-H^{-s_0}$ combined with Lemma A.6 and (7.321), we obtain

$$\begin{aligned}\|\langle P_{N_m}\partial_\theta\mathscr{R}_m\mathbf{e}_{0,j}, \mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\,\widehat{\delta}_m(s_0)\\ &\leqslant C\gamma\,\widehat{\delta}_0(s_h)N_0^{\mu_2}N_m^{-\mu_2}.\end{aligned}$$

Putting together the preceding estimates with (7.322) and Lemma A.5 yields

$$\begin{aligned}\|r_j^\infty\|_q^{\gamma,\mathcal{O}} &\lesssim \gamma\,|j|^{-1}\widehat{\delta}_0(s_h)N_0^{\mu_2}\sum_{m=0}^{\infty}N_m^{-\mu_2}\\ &\lesssim |j|^{-1}\varepsilon\gamma^{-1}.\end{aligned}$$

This achieves the proof of (7.239).

(ii) We shall now work with fixed values (minimal) of μ_2 and s_h denoted respectively by μ_c and s_c, namely

$$\mu_c \triangleq \overline{\mu_2} + 2\tau_2 q + 2\tau_2 \quad \text{and} \quad s_c \triangleq \frac{3}{2}\mu_c + \overline{s}_l + 1 = \overline{s}_h + 4\tau_2 q + 4\tau_2. \tag{7.329}$$

From (7.268) and (7.277), we can write

$$\mathscr{R}_{m+1}=(\mathrm{Id}+\Sigma_m)\,U_m,$$

where

$$U_m\triangleq P_{N_m}^{\perp}\mathscr{R}_m+\mathscr{R}_m\Psi_m-\Psi_m\big\lfloor P_{N_m}\mathscr{R}_m\big\rfloor. \tag{7.330}$$

After straightforward computations, we get

$$\begin{aligned}\Delta_{12}U_m=&P_{N_m}^{\perp}\Delta_{12}\mathscr{R}_m+(\Delta_{12}\mathscr{R}_m)(\Psi_m)_{r_1}+(\mathscr{R}_m)_{r_2}(\Delta_{12}\Psi_m)\\&-(\Delta_{12}\Psi_m)\big\lfloor P_{N_m}(\mathscr{R}_m)_{r_1}\big\rfloor-(\Psi_m)_{r_2}\big\lfloor P_{N_m}\Delta_{12}\mathscr{R}_m\big\rfloor\end{aligned} \tag{7.331}$$

and

$$\Delta_{12}\mathscr{R}_{m+1}=\Delta_{12}U_m+(\Delta_{12}\Sigma_m)(U_m)_{r_1}+(\Sigma_m)_{r_2}\Delta_{12}U_m. \tag{7.332}$$

We have used the notation $(f)_r=f(r)$. Elementary manipulations based on (7.277) give

$$\Delta_{12}\Sigma_m=\Delta_{12}\Phi_m^{-1}=-(\Phi_m^{-1})_{r_2}(\Delta_{12}\Psi_m)(\Phi_m^{-1})_{r_1}.$$

The product laws of Lemma A.6 together with (7.294) and (7.236) imply

$$\forall\, s\in[s_0,s_c],\quad \|\Delta_{12}\Sigma_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}. \tag{7.333}$$

Using once again the product laws of Lemma A.6, (7.333) and (7.332) we obtain

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\leqslant{}&\|\Delta_{12}U_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|(\Delta_{12}\Psi_m)\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\\&+\|(\Sigma_m)_{r_2}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\end{aligned} \tag{7.334}$$

and

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\leqslant{}&\|\Delta_{12}U_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}+\|(\Delta_{12}\Psi_m)\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\\&+\|(\Delta_{12}\Psi_m)\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\|(U_m)_{r_1}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|(\Sigma_m)_{r_2}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\\&+\|(\Sigma_m)_{r_2}\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\|\Delta_{12}U_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.335}$$

For the estimate $(U_m)_{r_1}$ (to alleviate the notation we shall remove in this part remove the subscript r_1) described by (7.330) we use the product laws leading to

$$\|U_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\leqslant\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}} \tag{7.336}$$

and

$$\begin{aligned}\|U_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\leqslant{}&\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}+\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\\&+\|\mathscr{R}_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\|\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.337}$$

By (7.276), (7.243) and (7.290) together with (7.336) we infer

$$\|U_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\leqslant C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}. \tag{7.338}$$

Putting together the first estimate of (7.292), (7.276) and (7.243) we deduce that

$$\begin{aligned}\max_{j=1,2}\|(\Psi_m)_{r_j}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}&\lesssim N_m^{\tau_2q+\tau_2}\delta_m(s_c)\\&\lesssim\varepsilon\gamma^{-2}N_m^{\tau_2q+\tau_2}.\end{aligned}\tag{7.339}$$

Hence we get in view of (7.337), (7.276) and (7.236)

$$\|U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\leqslant C\varepsilon\gamma^{-1}.\tag{7.340}$$

Plugging (7.338) and (7.340) into (7.335) implies

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\leqslant{}&\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+\|(\Sigma_m)_{r_2}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\\&+\|(\Sigma_m)_{r_2}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}.\end{aligned}\tag{7.341}$$

Applying (7.278) and (7.243) gives

$$\|(\Sigma_m)_{r_2}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\leqslant C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\quad\text{and}\quad\|(\Sigma_m)_{r_2}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\leqslant C\varepsilon\gamma^{-2}N_m^{\tau_2q+\tau_2}.\tag{7.342}$$

Inserting (7.342) into (7.341) allows to get

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\leqslant{}&\big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\varepsilon\gamma^{-2}N_m^{\tau_2q+\tau_2}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}.\end{aligned}\tag{7.343}$$

In a similar way, by combining (7.338), (7.342) with (7.334) we find

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\leqslant{}&\big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}.\end{aligned}\tag{7.344}$$

From (7.331) and the product laws of Lemma A.6 we obtain $\forall\, s\in[s_0,s_c]$,

$$\begin{aligned}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}\leqslant{}&N_m^{s-s_c}\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}\max_{j=1,2}\|(\Psi_m)_{r_j}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\max_{j=1,2}\|(\Psi_m)_{r_j}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}+C\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}\max_{j=1,2}\|(\mathscr{R}_m)_{r_j}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\max_{j=1,2}\|(\mathscr{R}_m)_{r_j}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s}.\end{aligned}$$

Combining the foregoing estimate with (7.339), (7.276) and (7.290) yields

$$\begin{aligned}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\leqslant{}&N_m^{s_0-s_c}\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\end{aligned}\tag{7.345}$$

and

$$\begin{aligned}\|\Delta_{12}U_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}\leqslant{}&\big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\big)\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+CN_m^{\tau_2q+\tau_2}\varepsilon\gamma^{-2}\|\Delta_{12}\mathscr{R}_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}\\&+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_c}+C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s_0}.\end{aligned}$$

Putting together the preceding estimate with (7.343), (7.344), (7.235) and (7.236) we deduce that

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}} \leqslant & \Big(1+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}+C\varepsilon\gamma^{-2}N_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ & + CN_m^{\tau_2 q+\tau_2}\varepsilon\gamma^{-2}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}+C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}\\ & + C\varepsilon\gamma^{-1}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}. \end{aligned} \tag{7.346}$$

In a similar way, by making appeal to (7.344), (7.345) and (7.236) we find

$$\begin{aligned}\|\Delta_{12}\mathscr{R}_{m+1}\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}} \leqslant & N_m^{s_0-s_c}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}}+C\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ & + C\varepsilon\gamma^{-1}N_0^{\mu_c}N_m^{-\mu_c}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}. \end{aligned} \tag{7.347}$$

We shall now estimate $\Delta_{12}\Psi_m$. Remark that

$$\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}=\sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\ |\alpha|\leqslant q}}\gamma^{\alpha}\sup_{(\lambda,\omega)\in\mathcal{O}}\Bigg(\sum_{\substack{(l,k)\in\mathbb{Z}^{d+1}\\ |l|,|k|\leqslant N_m}}\langle l,k\rangle^{2(s-|\alpha|)}\sup_{j\in\mathbb{Z}}\Big|\partial_{\lambda,\omega}^{\alpha}\Delta_{12}(\Psi_m)_{j+k}^{j}(\lambda,\omega,l)\Big|^2\Bigg)^{\frac{1}{2}}.$$

By virtue of (7.273), we get

$$(\Psi_m)_{j_0}^{j}(\lambda,\omega,l)=\begin{cases}-(\varrho_m)_{j_0}^{j}(\lambda,\omega,l)r_{j_0,m}^{j}(\lambda,\omega,l) & \textnormal{if } (l,j)\neq(0,j_0)\\ 0 & \textnormal{if } (l,j)=(0,j_0),\end{cases}$$

where

$$(\varrho_m)_{j_0}^{j}(\lambda,\omega,l)\triangleq\frac{\chi\big((\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega))(\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}\big)}{\omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega)}.$$

Recall from (7.249), that $\{\mathrm{i}r_{j_0,m}^{j}(\lambda,\omega,l)\}$ are the Fourier coefficients of $P_{N_m}\mathscr{R}_m$, that is

$$\mathrm{i}\,r_{j_0,m}^{j}(\lambda,\omega,l)=\langle P_{N_m}\mathscr{R}_m\mathbf{e}_{0,j_0},\mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})}. \tag{7.348}$$

We can write for non-zero coefficients

$$\begin{aligned}\Delta_{12}(\Psi_m)_{j+k}^{j}(\lambda,\omega,l)= & \Delta_{12}(\varrho_m)_{j+k}^{j}(\lambda,\omega,l)\big(r_{j+k,m}^{j}\big)_{r_1}(\lambda,\omega,l)\\ & + \big((\varrho_m)_{j+k}^{j}\big)_{r_2}(\lambda,\omega,l)\Delta_{12}r_{j+k,m}^{j}(\lambda,\omega,l).\end{aligned}$$

Hence, using Lemma A.1-(iv)

$$\begin{aligned}\forall q'\in[\![0,q]\!],\quad \|\Delta_{12}(\Psi_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\lesssim & \|\Delta_{12}(\varrho_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\\ & + \max_{i\in\{1,2\}}\|\big((\varrho_m)_{j+k}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\|\Delta_{12}r_{j+k,m}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.349}$$

From (7.348), we deduce

$$\mathrm{i}\Delta_{12}r_{j_0,m}^{j}(\lambda,\omega,l)=\langle P_{N_m}\Delta_{12}\mathscr{R}_m\mathbf{e}_{0,j_0},\mathbf{e}_{l,j}\rangle_{L^2(\mathbb{T}^{d+1})}.$$

One can write

$$(\varrho_m)_{j_0}^{j}(\lambda,\omega,l)=b_{l,j,j_0,m}\widehat{\chi}\Big(b_{l,j,j_0,m}B_{l,j,j_0,m}(\lambda,\omega)\Big),$$

with

$$b_{l,j,j_0,m} \triangleq (\gamma\langle j-j_0\rangle)^{-1}\langle l\rangle^{\tau_2}, \quad B_{l,j,j_0,m}(\lambda,\omega) \triangleq \omega\cdot l+\mu_j^m(\lambda,\omega)-\mu_{j_0}^m(\lambda,\omega), \quad \widehat{\chi}(x)=\frac{\chi(x)}{x}\cdot$$

Notice that from (7.288), one obtains

$$\forall q'\in[\![0,q]\!], \quad \|B_{l,j,j_0,m}\|_{q'}^{\gamma,\mathcal{O}} \lesssim \langle l,j-j_0\rangle. \tag{7.350}$$

In a similar way to (7.257), one gets from (7.288)

$$\forall q'\in[\![0,q]\!], \quad \|(\varrho_m)_{j_0}^j(*,l)\|_{q'}^{\gamma,\mathcal{O}} \lesssim \gamma^{-(q'+1)}\langle l,j-j_0\rangle^{\tau_2 q'+\tau_2+q'}. \tag{7.351}$$

Using Taylor formula in a similar way to (7.100), we find (to simplify the notation we remove the dependence in (λ,ω))

$$\Delta_{12}(\varrho_m)_{j_0}^j(l)=b_{l,j,j_0,m}^2(\Delta_{12}B_{l,j,j_0,m})\int_0^1\widehat{\chi}'\Big(b_{l,j,j_0,m}\Big[(1-\tau)(B_{l,j,0,m})_{r_1}+\tau(B_{l,j,0,m})_{r_2}\Big]\Big)d\tau.$$

We shall estimate $\Delta_{12}B_{l,j,j_0,m}$. For that purpose, we use (7.270) to write

$$\mu_j^m=\mu_j^0+\sum_{n=0}^{m-1}\langle P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}.$$

We recall from Proposition 7.4 that

$$\mu_j^0(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0), \quad r^1(\lambda,\omega,i_0)=c_{i_0}(\lambda,\omega)-V_0(\lambda).$$

Therefore

$$\Delta_{12}\mu_j^m=\Delta_{12}\mu_j^0+\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}$$

and

$$\begin{aligned}\Delta_{12}B_{l,j,j_0,m}&=\Delta_{12}\big(\mu_j^m-\mu_{j_0}^m\big)\\&=(j-j_0)\Delta_{12}c_i+\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j},\mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\\&\quad-\sum_{n=0}^{m-1}\langle\Delta_{12}P_{N_n}\mathscr{R}_n\mathbf{e}_{0,j_0},\mathbf{e}_{0,j_0}\rangle_{L^2(\mathbb{T}^{d+1})}.\end{aligned}$$

Hence, using (7.21), one gets

$$\forall q'\in[\![0,q]\!], \quad \|\Delta_{12}B_{l,j,j_0,m}\|_{q'}^{\gamma,\mathcal{O}} \lesssim \varepsilon|j-j_0|\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}+\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\textnormal{O-d},q',s_0}^{\gamma,\mathcal{O}}. \tag{7.352}$$

Then, one obtains from Lemma A.1-(vi), (7.350) and (7.352)

$$\begin{aligned}\forall q'\in[\![0,q]\!], \quad \|\Delta_{12}(\varrho_m)_{j_0}^j(*,l)\|_{q'}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-2-q'}\langle l,j-j_0\rangle^{\tau_2 q'+2\tau_2+q'+1}\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}\\&\quad+\gamma^{-2-q'}\langle l,j-j_0\rangle^{\tau_2 q'+2\tau_2+q'}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\textnormal{O-d},q',s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.353}$$

Gathering (7.349), (7.351) and (7.353) gives for all $q' \in [\![0,q]\!]$,

$$\begin{aligned}\|\Delta_{12}(\Psi_m)_{j+k}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-2-q'}\langle l,k\rangle^{\tau_2 q'+2\tau_2+q'+1}\|\Delta_{12}i\|_{q',\overline{s}_h+2}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\\ &\quad+\gamma^{-2-q'}\langle l,k\rangle^{\tau_2 q'+2\tau_2+q'}\max_{i\in\{1,2\}}\|\big(r_{j+k,m}^{j}\big)_{r_i}(*,l)\|_{q'}^{\gamma,\mathcal{O}}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\textnormal{O-d},q',s_0}^{\gamma,\mathcal{O}}\\ &\quad+\gamma^{-1-q'}\langle l,k\rangle^{\tau_2 q'+\tau_2+q'}\|\Delta_{12}r_{j+k,m}^{j}(*,l)\|_{q'}^{\gamma,\mathcal{O}}.\end{aligned}$$

We deduce that for all $s\in[s_0,S]$,

$$\begin{aligned}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-2-q}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\|P_{N_m}\mathscr{R}_m\|_{\textnormal{O-d},q,s+\tau_2 q+2\tau_2+1}^{\gamma,\mathcal{O}}\\ &\quad+\gamma^{-2-q}\|P_{N_m}\mathscr{R}_m\|_{\textnormal{O-d},q,s+\tau_2 q+2\tau_2}^{\gamma,\mathcal{O}}\sum_{n=0}^{m-1}\|P_{N_n}\Delta_{12}\mathscr{R}_n\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &\quad+\gamma^{-1-q}\|P_{N_m}\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s+\tau_2 q+\tau_2}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.354}$$

We set

$$\overline{\delta}_m(s)=\gamma^{-1}\|\Delta_{12}\mathscr{R}_m\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\quad\text{and}\quad\varkappa_m(s)\triangleq\sum_{n=0}^{m-1}\overline{\delta}_n(s).\tag{7.355}$$

Then, using (7.354), (7.275) and (7.3), we get

$$\begin{aligned}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-1-q}N_m^{\tau_2}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\delta_m(\overline{s}_l)+\gamma^{-q}N_m^{\tau_2}\delta_m(\overline{s}_l)\varkappa_m(s_0)\\ &\quad+\gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0)\end{aligned}\tag{7.356}$$

and

$$\begin{aligned}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-1-q}N_m^{\tau_2 q+2\tau_2+1}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\delta_m(s_c)+\gamma^{-q}N_m^{\tau_2 q+2\tau_2}\delta_m(s_c)\varkappa_m(s_0)\\ &\quad+\gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_c).\end{aligned}\tag{7.357}$$

According to (7.322), one has

$$\delta_m(\overline{s}_l)\lesssim\varepsilon\gamma^{-2}N_0^{\mu_c}N_m^{-\mu_c}\quad\text{and}\quad\sup_{m\in\mathbb{N}}\delta_m(s_c)\lesssim\varepsilon\gamma^{-2}\lesssim 1.\tag{7.358}$$

Putting together (7.358) and (7.356) and using (7.236) yields

$$\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_m^{\tau_2-\mu_c}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}+N_m^{\tau_2-\mu_c}\varkappa_m(s_0)+\gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0).\tag{7.359}$$

In a similar way, on gets by (7.358), (7.357) and (7.236)

$$\begin{aligned}\|\Delta_{12}\Psi_m\|_{\textnormal{O-d},q,s_c}^{\gamma,\mathcal{O}} &\lesssim \varepsilon\gamma^{-1}N_m^{\tau_2 q+2\tau_2+1}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}+N_m^{\tau_2 q+2\tau_2}\varkappa_m(s_0)\\ &\quad+\gamma^{-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_c).\end{aligned}\tag{7.360}$$

Plugging (7.359) into (7.347) yields by virtue of (7.276) and (7.236)

$$\begin{aligned}\overline{\delta}_{m+1}(s_0) &\leqslant N_m^{s_0-s_c}\overline{\delta}_m(s_c)+CN_m^{\tau_2 q+\tau_2-\mu_c}\overline{\delta}_m(s_0)+CN_m^{\tau_2-2\mu_c}\varkappa_m(s_0)\\ &\quad+C\varepsilon\gamma^{-1}N_m^{\tau_2-2\mu_c}\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.361}$$

Therefore, inserting (7.359), (7.360) into (7.346) and using (7.236) implies

$$\begin{aligned}\overline{\delta}_{m+1}(s_c) \leqslant \Big(1 + CN_m^{\tau_2 q+\tau_2-\mu_c} + CN_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\overline{\delta}_m(s_c) + C\varepsilon\gamma^{-2-q}N_m^{\tau_2 q+\tau_2}\overline{\delta}_m(s_0)\\ + CN_m^{\tau_2 q+\tau_2-\mu_c}\varkappa_m(s_0) + C\varepsilon\gamma^{-1}N_m^{\tau_2 q+2\tau_2+1-\mu_c}\|\Delta_{12} i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.362}$$

Next, we intend to prove by induction in $m \in \mathbb{N}$ that

$$\forall\, k \leqslant m, \quad \overline{\delta}_k(s_0) \leqslant N_0^{\mu_c} N_k^{-\mu_c}\nu(s_c) \quad \text{and} \quad \overline{\delta}_k(s_c) \leqslant \Big(2 - \tfrac{1}{k+1}\Big)\,\nu(s_c), \tag{7.363}$$

with

$$\nu(s) \triangleq \overline{\delta}_0(s) + \varepsilon\gamma^{-1}\|\Delta_{12} i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}.$$

The estimate (7.363) is obvious for $m = 0$ by Sobolev embeddings. Now let us assume that the preceding property holds true at the order m and let us check it at the order $m+1$. Thus by applying (7.355) and Lemma A.5, we get

$$\sup_{m\in\mathbb{N}} \varkappa_m(s_0) \leqslant C\nu(s_c).$$

Putting together this estimate with the induction assumption, (7.361), (7.362), (7.329) and (7.236) yields

$$\overline{\delta}_{m+1}(s_0) \leqslant \big(2N_m^{s_0-s_c} + CN_0^{\mu_c}N_m^{\tau_2 q+\tau_2-2\mu_c}\big)\,\nu(s_c)$$

and

$$\begin{aligned}\overline{\delta}_{m+1}(s_c) \leqslant \Big(1 + CN_m^{\tau_2 q+\tau_2-\mu_c} + CN_m^{s_0-s_c+\tau_2 q+\tau_2}\Big)\left(2 - \frac{1}{m+1}\right)\nu(s_c)\\ + C\left(N_m^{\tau_2 q+\tau_2-\mu_c} + N_m^{\tau_2 q+2\tau_2+1-\mu_c}\right)\nu(s_c).\end{aligned}$$

Since (7.329) implies in particular

$$\mu_c \geqslant 2\tau_2 q + 2\tau_2 + 1 \quad \text{and} \quad s_c \geqslant \tfrac{3}{2}\mu_c + s_0 + \tau_2 q + \tau_2 + 1,$$

then proceeding similarly to the proof of (7.276), we conclude that

$$\overline{\delta}_{m+1}(s_0) \leqslant N_0^{\mu_c}N_{m+1}^{-\mu_c}\nu(s_c) \quad \text{and} \quad \overline{\delta}_{m+1}(s_c) \leqslant \Big(2 - \tfrac{1}{m+2}\Big)\,\nu(s_c),$$

which achieves the induction. The next target is to estimate $\Delta_{12} r_j^\infty$. Then similarly to (7.286) we obtain through a duality argument, Lemma A.6, (7.363) and Lemma A.5

$$\begin{aligned}\|\Delta_{12} r_j^\infty\|_q^{\gamma,\mathcal{O}} &\leqslant \sum_{m=0}^{\infty}\Big\|\langle P_{N_m}\Delta_{12}\mathscr{R}_m \mathbf{e}_{0,j}, \mathbf{e}_{0,j}\rangle_{L^2(\mathbb{T}^{d+1})}\Big\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \sum_{m=0}^{\infty}\big\|\Delta_{12}\mathscr{R}_m\big\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \gamma\nu(s_c)\sum_{m=0}^{\infty} N_0^{\mu_c}N_m^{-\mu_c}\\ &\leqslant C\gamma\,\nu(s_c).\end{aligned}$$

From the particular value of $\mathtt{p}$ in (7.168), we infer

$$s_c = \overline{s}_h + 4\tau_2 q + 4\tau_2 = \overline{s}_h + \mathtt{p}. \tag{7.364}$$

Then, applying (7.173) we obtain

$$\begin{aligned}\|\Delta_{12}r_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\leqslant C\gamma\,\nu(\overline{s}_h+4\tau_2 q+4\tau_2)\\ &\leqslant C\varepsilon\gamma^{-1}\,\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.\end{aligned}$$

Finally, combining the previous estimate with (7.313) and (7.170) we deduce

$$\begin{aligned}\forall j\in\mathbb{S}_0^c,\quad \|\Delta_{12}\mu_j^{\infty}\|_q^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\mu_j^{0}\|_q^{\gamma,\mathcal{O}}+\|\Delta_{12}r_j^{\infty}\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}|j|\|\Delta_{12}i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}.\end{aligned}$$

The achieves the proof of Proposition 7.5. □

7.4 Approximate inverse in the normal directions

In this section we plan to construct an approximate right inverse in the normal directions for the linearized operator $\widehat{\mathcal{L}}_{\omega}$ defined in (6.102) when the parameters are restricted in a Cantor like set. Our main result is the following.

Proposition 7.6. *Let* $(\gamma,q,d,\tau_1,s_0,\mu_2,s_h,S)$ *satisfying* (A.2), (A.1) *and* (7.235). *There exists* $\sigma\triangleq\sigma(\tau_1,\tau_2,q,d)\geqslant\sigma_4$ *such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}}\leqslant 1,\tag{7.365}$$

then the following assertions hold true.

(i) Consider the operator $\mathscr{L}_{\infty}$ *defined in Proposition* 7.5, *then there exists a family of linear operators* $\big(\mathtt{T}_n\big)_{n\in\mathbb{N}}$ *defined in* $\mathcal{O}$ *satisfying the estimate*

$$\forall s\in[s_0,S],\quad \sup_{n\in\mathbb{N}}\|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}$$

and such that for any $n\in\mathbb{N}$, *in the Cantor set*

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}}\Big\{(\lambda,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_0)\big|>\tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},$$

we have

$$\mathscr{L}_{\infty}\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3,$$

with

$$\forall s_0\leqslant s\leqslant\overline{s}\leqslant S,\quad \|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}.$$

(ii) There exists a family of linear operators $\big(\mathtt{T}_{\omega,n}\big)_{n\in\mathbb{N}}$ *satisfying*

$$\forall\, s\in\,[s_0,S],\quad \sup_{n\in\mathbb{N}}\|\mathtt{T}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right)\tag{7.366}$$

and such that in the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)\triangleq\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),$$

we have

$$\widehat{\mathcal{L}}_{\omega}\mathtt{T}_{\omega,n}=\mathrm{Id}+\mathtt{E}_n,$$

where $\mathtt{E}_n$ satisfies the following estimate

$$\begin{aligned}\forall\, s\in[s_0,S],\quad \|\mathtt{E}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}} &\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\ &\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

Recall that $\widehat{\mathcal{L}}_\omega$, $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ and $\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ are given in Propositions 7.1, 7.2 *and* 7.5, *respectively.*

(iii) In the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$, we have the following splitting

$$\widehat{\mathcal{L}}_\omega=\widehat{\mathtt{L}}_{\omega,n}+\widehat{\mathtt{R}}_n\quad\text{with}\quad\widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n}=\mathrm{Id}\quad\text{and}\quad\widehat{\mathtt{R}}_n=\mathtt{E}_n\widehat{\mathtt{L}}_{\omega,n},$$

where the operators $\widehat{\mathtt{L}}_{\omega,n}$ and $\widehat{\mathtt{R}}_n$ are defined in $\mathcal{O}$ and satisfy the following estimates

$$\begin{aligned}\forall s\in[s_0,S],\quad &\sup_{n\in\mathbb{N}}\|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}},\\ \forall s\in[s_0,S],\quad &\|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\ &\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

Proof. **(i)** From Proposition 7.5 we recall that

$$\mathscr{L}_\infty=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty.$$

Then we may split this operator as follows, using the projectors defined in (A.5)

$$\begin{aligned}\mathscr{L}_\infty&=\Pi_{N_n}\omega\cdot\partial_\varphi\Pi_{N_n}\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty-\Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp\\ &\triangleq\mathtt{L}_n-\mathtt{R}_n,\end{aligned}\tag{7.367}$$

with $\mathtt{R}_n\triangleq\Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp$. From this definition and the structure of $\mathscr{D}_\infty$ in Proposition 7.5 we deduce that

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad\mathbf{e}_{-l,-j}\mathtt{L}_n\mathbf{e}_{l,j}=\begin{cases}\mathrm{i}\big(\omega\cdot l+\mu_j^\infty\big)&\text{if }|l|\leqslant N_n\\ \mathrm{i}\,\mu_j^\infty&\text{if }|l|>N_n.\end{cases}$$

Define the diagonal operator $\mathtt{T}_n$ by

$$\begin{aligned}\mathtt{T}_n\rho(\lambda,\omega,\varphi,\theta)\triangleq&-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\tfrac{\chi\big((\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0))\gamma^{-1}\langle l\rangle^{\tau_1}\big)}{\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_0)}\rho_{l,j}(\lambda,\omega)\,e^{\mathrm{i}(l\cdot\varphi+j\theta)}\\ &-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|>N_n}}\tfrac{\rho_{l,j}(\lambda,\omega)}{\mu_j^\infty(\lambda,\omega,i_0)}\,e^{\mathrm{i}(l\cdot\varphi+j\theta)},\end{aligned}$$

where χ is the cut-off function defined in (6.92) and $\big(\rho_{l,j}(\lambda,\omega)\big)_{l,j}$ are the Fourier coefficients of ρ. We recall from Proposition 7.5 that

$$\mu_j^\infty(\lambda,\omega,i_0)=\Omega_j(\lambda)+jr^1(\lambda,\omega,i_0)+r_j^\infty(\lambda,\omega,i_0)\qquad\text{with}\qquad r^1(\lambda,\omega,i_0)=c_{i_0}(\lambda,\omega)-V_0(\lambda),$$

with the estimates

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim|j|,$$

where we use in part the estimate (7.265), (7.239) and (7.170). According to Lemma 5.3-(iii), (7.265), (7.239) and the smallness condition (7.236) we infer

$$|j|\lesssim \|\mu_j^\infty\|_0^{\gamma,\mathcal{O}}\leqslant \|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}.$$

Implementing the same arguments as for (7.258) one gets

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \gamma^{-1}\|\rho\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}. \tag{7.368}$$

Moreover, by construction

$$\mathtt{L}_n\mathtt{T}_n=\mathrm{Id}\quad\text{in }\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) \tag{7.369}$$

since $\chi(\cdot)=1$ in this set. It follows from (7.367) that

$$\begin{aligned}\forall\,(\lambda,\omega)\in\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\quad \mathscr{L}_\infty\mathtt{T}_n&=\mathrm{Id}-\mathtt{R}_n\mathtt{T}_n\\&\triangleq \mathrm{Id}+\mathtt{E}_n^3.\end{aligned} \tag{7.370}$$

Notice that by Lemma A.1-(ii),

$$\forall\, s_0\leqslant s\leqslant \overline{s},\quad \|\mathtt{R}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\|\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}.$$

Combining this estimate with (7.368) yields

$$\begin{aligned}\forall\, s_0\leqslant s\leqslant \overline{s},\quad \|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim N_n^{s-\overline{s}}\|\mathtt{T}_n\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}\\&\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.371}$$

(ii) Let us define

$$\mathtt{T}_{\omega,n}\triangleq \mathscr{B}_\perp\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}, \tag{7.372}$$

where the operators $\mathscr{B}_\perp$ and Φ_∞ are defined in Propositions 7.4 and 7.5 respectively. Notice that $\mathtt{T}_{\omega,n}$ is defined in the whole range of parameters $\mathcal{O}$. Since the condition (7.365) is satisfied, then, both Propositions 7.2 and 7.5 apply and from (7.150) we obtain

$$\forall s\in[s_0,S],\quad \|\mathtt{T}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

By using (7.237) and (7.365), one gets

$$\forall s\in[s_0,S],\quad \|\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

Thus the point (i) of the current proposition implies

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim \gamma^{-1}\|\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}.$$

Applying (7.237) and (7.150) with (7.365) yields

$$\begin{aligned}\forall s\in[s_0,S],\quad \|\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}&\lesssim \|\mathscr{B}_\perp^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathscr{B}_\perp^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the preceding three estimates gives (7.366). Now combining Propositions 7.4 and 7.5, we find that in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathcal{O}^{\gamma,\tau_1,\tau_2}_{\infty,n}(i_0)$ the following decomposition holds

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty&=\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\\&=\mathscr{L}_\infty+\mathtt{E}_n^2+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty.\end{aligned}$$

It follows that in the Cantor set $\mathcal{O}^{\gamma,\tau_1}_{\infty,n}(i_0)\cap\mathscr{O}^{\gamma,\tau_2}_{\infty,n}(i_0)\cap\Lambda^{\gamma,\tau_1}_{\infty,n}(i_0)$ one has by virtue of the identity (7.370)

$$\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n,$$

which gives, using (7.372), the following identity in $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\begin{aligned}\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n}&=\mathrm{Id}+\mathscr{B}_\perp\Phi_\infty\big(\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\big)\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\\&\triangleq\mathrm{Id}+\mathscr{B}_\perp\Phi_\infty\mathtt{E}_n^4\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\\&\triangleq\mathrm{Id}+\mathtt{E}_n.\end{aligned}\tag{7.373}$$

The estimate of the first term of $\mathtt{E}_n^4$ is given in (7.371). For the second term of $\mathtt{E}_n^4$ we use (7.240) and (7.368) leading to

$$\begin{aligned}\|\mathtt{E}_n^2\mathtt{T}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\varepsilon\gamma^{-2}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\mathtt{T}_n\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.374}$$

For the estimate of $\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n$, we combine (7.237), (7.171), (7.368) and (7.365) to get

$$\begin{aligned}\|\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\|\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\Phi_\infty\mathtt{T}_n\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\end{aligned}\tag{7.375}$$

Putting together (7.371) and (7.374) and (7.375) we find

$$\|\mathtt{E}_n^4\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim N_n^{s_0-s}\gamma^{-1}\|\rho\|_{s+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\tag{7.376}$$

Set $\Psi=\mathscr{B}_\perp\Phi_\infty$ then from (7.237), (7.150) and (7.365) we deduce that

$$\forall s\in[s_0,S],\qquad\|\Psi^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{7.377}$$

Straightforward computations based on (7.376), (7.377) and (7.365) yields

$$\begin{aligned}\|\Psi\mathtt{E}_n^4\Psi^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim\|\mathtt{E}_n^4\Psi^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim N_n^{s_0-s}\gamma^{-1}\|\Psi^{-1}\rho\|_{s+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\Psi^{-1}\rho\|_{q,s_0+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}\\&\lesssim N_n^{s_0-s}\gamma^{-1}\big(\|\rho\|_{q,s+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Consequently, taking σ large enough, we get

$$\|\mathtt{E}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim N_n^{s_0-s}\gamma^{-1}\big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\big)+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.$$

(iii) According to (7.373), one can write in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathcal{L}}_\omega=\mathtt{T}_{\omega,n}^{-1}+\mathtt{E}_n\mathtt{T}_{\omega,n}^{-1}. \tag{7.378}$$

Gathering (7.372) and (7.369), one obtains in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathtt{L}}_{\omega,n}\triangleq\mathtt{T}_{\omega,n}^{-1}=\mathscr{B}_\perp\Phi_\infty\mathtt{L}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}=\Psi\mathtt{L}_n\Psi^{-1}.$$

Hence, (7.378) can be rewritten

$$\widehat{\mathcal{L}}_\omega=\widehat{\mathtt{L}}_{\omega,n}+\widehat{\mathtt{R}}_n\quad\text{with}\quad\widehat{\mathtt{R}}_n\triangleq\mathtt{E}_n\widehat{\mathtt{L}}_{\omega,n}. \tag{7.379}$$

Putting together (7.367), (7.377) and (7.365), we obtain

$$\begin{aligned}\forall s\in[s_0,S],\quad\|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}&=\|\Psi\mathtt{L}_n\Psi^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\\&\lesssim\|\mathtt{L}_n\Psi^{-1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\mathtt{L}_n\Psi^{-1}\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\\&\lesssim\|\Psi^{-1}\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\Psi^{-1}\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{7.380}$$

Hence combining this estimate with (7.378) yields

$$\begin{aligned}\forall s\in[s_0,S],\quad\|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim N_n^{s_0-s}\gamma^{-1}\Big(&\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\\&+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}$$

This achieves the proof of the third point and the proof of the proposition is now complete. ☐

8 Proof of the main result

This section is devoted to the proof of Theorem 3.1. For this aim we intend to implement Nash-Moser scheme in order to construct zeros for the nonlinear functional $\mathcal{F}\big(i,\alpha,\lambda,\omega,\varepsilon\big)$ defined in (6.21). We shall be able to capture the solutions when the parameters (λ,ω) belong to a suitable final Cantor set $\mathcal{G}_\infty^\gamma$ obtained as the intersection of all the Cantor sets required during the steps of the scheme to invert the linearized operator. More precisely, we get a relatively smooth function $(\lambda,\omega)\in\mathcal{O}\mapsto U_\infty(\lambda,\omega)=\big(i_\infty(\lambda,\omega),\alpha_\infty(\lambda,\omega)\big)$ such that

$$\forall(\lambda,\omega)\in\mathcal{G}_\infty^\gamma,\quad\mathcal{F}\big(U_\infty(\lambda,\omega),\lambda,\omega,\varepsilon\big)=0.$$

To generate solutions to the initial Hamiltonian equation (6.2) we should adjust the parameters so that $\alpha_\infty(\lambda,\omega)=-\omega_{\mathrm{Eq}}(\lambda)$, where ω_{Eq} corresponds to the equilibrium frequency vector defined in (5.32). As a consequence, nontrivial solutions are constructed when the scalar parameter λ is selected in the final Cantor set

$$\mathcal{C}_\infty^\varepsilon=\Big\{\lambda\in(\lambda_0,\lambda_1)\quad\text{s.t.}\quad(\lambda,\omega(\lambda,\varepsilon))\in\mathcal{G}_\infty^\gamma\quad\text{with}\quad\alpha_\infty(\lambda,\omega(\lambda,\varepsilon))=-\omega_{\mathrm{Eq}}(\lambda)\Big\}.$$

The measure of this set will be discussed in Section 8.2.

8.1 Nash-Moser scheme

In this section we implement the Nash-Moser scheme, which is a modified Newton method implemented with a suitable Banach scales and through a frequency cut-off. Basically, it consists in a recursive construction of approximate solutions to the equation $\mathcal{F}\big(i,\alpha,\lambda,\omega,\varepsilon\big)=0$ where the functional $\mathcal{F}$ is defined in (6.21). At each step of this scheme, we need to construct an approximate inverse of the linearized operator at a state near the equilibrium by applying the reduction procedure developed in Section 7. This enables to get the result of Theorem 6.1 with the suitable tame estimates associated to the final loss of regularity $\overline{\sigma}$ that could be arranged to be large enough. We point out that $\overline{\sigma}$ depends only on the shape of the Cantor set through the parameters τ_1,τ_2,d and on the non degeneracy of the equilibrium frequency through $q=1+q_0$, where q_0 be defined in Lemma 5.5. However, $\overline{\sigma}$ is independent of the regularity of the solutions that we want to construct. Now, we shall fix the following parameters needed to implement Nash-Moser scheme and related to the geometry of the Cantor sets encoded in τ_1,τ_2,d fixed by (A.1) and to the parameter $q=q_0+1$,

$$\begin{cases}\overline{a} &=& \tau_2+2\\ \mu_1 &=& 3q(\tau_2+2)+6\overline{\sigma}+6\\ a_1 &=& 6q(\tau_2+2)+12\overline{\sigma}+15\\ a_2 &=& 3q(\tau_2+2)+6\overline{\sigma}+9\\ \mu_2 &=& 2q(\tau_2+2)+5\overline{\sigma}+7\\ s_h &=& s_0+4q(\tau_2+2)+9\overline{\sigma}+11\\ s_m &=& 2s_h-s_0.\end{cases}\tag{8.1}$$

The numbers a_1 and a_2 will describe the rate of convergence for the regularity s_0 and $s_0+\overline{\sigma}$, respectively. They appear in the statements $(\mathcal{P}1)_n$ and $(\mathcal{P}2)_n$ in the Proposition 8.1. The parameter μ_1 controls the norm inflation at the high regularity index s_m and appears in $(\mathcal{P}3)_n$. As for the parameter $\overline{a}$, it is linked to the thickness of a suitable enlargement of the intermediate Cantor sets, needed to construct classical extensions of our approximate solutions. Finally, the numbers μ_2 and s_h correspond to those already encountered before in the reduction of the linearized operator and are now fixed to their minimal required values. In particular, we recall that μ_2 corresponds to the rate of convergence of the error terms emerging in the almost reducibility of the linearized operator, for instance we refer to Theorem 6.1. We should emphasize that, by taking $\overline{\sigma}$ large enough, the choice for μ_2 and s_h done in (8.1) enables to cover all the required assumptions in (7.15) and (7.235). Another assumption that we need to fix is related to γ,N_0 and ε

$$0<a<\tfrac{1}{\mu_2+q+2},\quad \gamma\triangleq\varepsilon^a,\quad N_0\triangleq\gamma^{-1}.\tag{8.2}$$

This constraint is required for the measuring the final Cantor set and to check that it is massive, for more details we refer to Proposition 8.2.

We shall start with defining the finite dimensional subspaces where the approximate solutions are expected to live with controlled estimates. Consider the space,

$$E_n\triangleq\Big\{\mathfrak{I}=(\Theta,I,z)\quad\text{s.t.}\quad\Theta=\Pi_n\Theta,\quad I=\Pi_nI\quad\text{and}\quad z=\Pi_nz\Big\},$$

where Π_n is the projector defined by

$$f(\varphi,\theta)=\sum_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}f_{l,j}e^{\mathrm{i}(l\cdot\varphi+j\theta)}\quad\Rightarrow\quad\Pi_nf(\varphi,\theta)=\sum_{\langle l,j\rangle\leqslant N_n}f_{l,j}e^{\mathrm{i}(l\cdot\varphi+j\theta)},$$

where the sequence (N_n) is defined in (6.94). We observe that the same definition applies without ambiguity when the functions depend only on φ such as the action and the angles unknowns. The main result of this section is to prove the following induction statement.

Proposition 8.1 (Nash-Moser). *Let* (τ_1,τ_2,q,d,s_0) *satisfy* (A.2) *and* (A.1). *Consider the parameters fixed by* (8.1) *and* (8.2). *There exist* $C_*>0$ *and* $\varepsilon_0>0$ *such that for any* $\varepsilon\in[0,\varepsilon_0]$ *we get for all* $n\in\mathbb{N}$

the following properties,

$(\mathcal{P}1)_n$ *There exists a q-times differentiable function*

$$\begin{array}{rccl} W_n : & \mathcal{O} & \to & E_{n-1}\times\mathbb{R}^d\times\mathbb{R}^{d+1} \\ & (\lambda,\omega) & \mapsto & \big(\mathfrak{I}_n,\alpha_n-\omega,0\big) \end{array}$$

satisfying

$$W_0=0 \quad \textit{and} \quad \textit{for} \quad n\geqslant 1, \quad \|W_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}.$$

By setting

$$U_0=\Big((\varphi,0,0),\omega,(\lambda,\omega)\Big) \quad \text{and} \quad \textit{for} \quad n\in\mathbb{N}^*,\, U_n=U_0+W_n \quad \text{and} \quad H_n=U_n-U_{n-1},$$

then

$$\forall s\in[s_0,S],\ \|H_1\|_{q,s}^{\gamma,\mathcal{O}}\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} \quad \textit{and} \quad \forall\, 2\leqslant k\leqslant n,\ \|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{k-1}^{-a_2}.$$

$(\mathcal{P}2)_n$ *Define*

$$i_n=(\varphi,0,0)+\mathfrak{I}_n, \quad \gamma_n=\gamma(1+2^{-n})\in[\gamma,2\gamma].$$

The embedded torus i_n satisfies the reversibility condition

$$\mathfrak{S}i_n(\varphi)=i_n(-\varphi),$$

where the involution $\mathfrak{S}$ is defined in (6.13)*. Introduce*

$$\mathcal{A}_0^{\gamma}=\mathcal{O} \quad \textit{and} \quad \mathcal{A}_{n+1}^{\gamma}=\mathcal{A}_n^{\gamma}\cap\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n),$$

where $\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$ is described in Proposition 7.6 *and consider the open sets*

$$\forall \mathtt{r}>0, \quad \mathrm{O}_n^{\mathtt{r}}\triangleq\Big\{(\lambda,\omega)\in\mathcal{O} \quad \text{s.t.} \quad \mathtt{dist}\big((\lambda,\omega),\mathcal{A}_n^{2\gamma}\big)<\mathtt{r}N_n^{-\overline{a}}\Big\},$$

where $\mathtt{dist}(x,A)=\displaystyle\inf_{y\in A}\|x-y\|$. Then we have the following estimate

$$\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\leqslant C_*\varepsilon N_{n-1}^{-a_1}.$$

$(\mathcal{P}3)_n$ $\|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}$.

Remark 8.1. *Let O be an open subset of $\mathcal{O}$. Since $\forall n\in\mathbb{N},\gamma_n\in[\gamma,2\gamma]$, then the norms $\|\cdot\|_{q,s}^{\gamma,\mathrm{O}}$ and $\|\cdot\|_{q,s}^{\gamma_n,\mathrm{O}}$. are equivalent uniformly in n.*

Proof. • **Initialization :** By construction, $U_0=\Big((\varphi,0,0),\omega,(\lambda,\omega)\Big)$. Notice that the flat torus $i_{\text{flat}}(\varphi)=(\varphi,0,0)$ satisfies obviously the reversibility condition. By (6.21), we have

$$\mathcal{F}(U_0)=\varepsilon\begin{pmatrix} -\partial_I\mathcal{P}_{\varepsilon}((\varphi,0,0)) \\ \partial_{\vartheta}\mathcal{P}_{\varepsilon}((\varphi,0,0)) \\ -\partial_{\theta}\nabla_z\mathcal{P}_{\varepsilon}((\varphi,0,0)) \end{pmatrix}.$$

Using Lemma 6.3, we get

$$\forall s\geqslant 0, \quad \|\mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon, \tag{8.3}$$

up to taking C_* large enough. The properties $(\mathcal{P}1)_0$, $(\mathcal{P}2)_0$ and $(\mathcal{P}3)_0$ then follow immediately since $N_{-1}=1$ and $\mathrm{O}_0^{2\gamma}=\mathcal{O}$ and by setting $W_0=0$.

• **Induction step :** Given $n\in\mathbb{N}$, assume that $(\mathcal{P}1)_k$, $(\mathcal{P}2)_k$ and $(\mathcal{P}3)_k$ are true for all $k\in[\![0,n]\!]$ and let us check them at the next order $n+1$. Introduce the linearized operator of $\mathcal{F}$ at the state (i_n,α_n)

$$L_n\triangleq L_n(\lambda,\omega)\triangleq d_{i,\alpha}\mathcal{F}\big(i_n\big(\lambda,\omega),\alpha_n(\lambda,\omega),(\lambda,\omega)\big).$$

In order to construct the next approximation U_{n+1}, we need an approximate right inverse for L_n. Its construction was performed along the preceding sections and we refer to Theorem 6.1 for a precise statement. To apply this result and get some bounds on U_{n+1} we need to establish first some intermediate results connected to the smallness condition and to some Cantor set inclusions.

▶ **Smallness/boundedness properties.** First of all, remark that the parameters conditions (7.3) are automatically satisfied by (8.1). Then, provided that the smallness assumption (7.365) is satisfied, Proposition 7.6 applies. It remains to check that (7.365) is satisfied. According to the first condition in (8.2) and choosing ε small enough, we can ensure

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}=\varepsilon^{1-a(\mu_2+q+2)}\leqslant\varepsilon_0 \tag{8.4}$$

for some a priori fixed $\varepsilon_0>0$. Therefore the first assumption in (7.365) holds. We now turn to the second assumption. Since from (8.1) $s_m=2s_h-s_0$, then by interpolation inequality in Lemma A.1, we have

$$\|H_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\lesssim\left(\|H_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)^{\frac12}\left(\|H_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)^{\frac12}. \tag{8.5}$$

Besides, by using $(\mathcal{P}1)_n$, we find

$$\forall s\in[s_0,S],\quad \|H_1\|_{q,s}^{\gamma,\mathcal{O}}\leqslant\tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}\quad\text{and}\quad\|H_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}. \tag{8.6}$$

Now $(\mathcal{P}3)_n$ and $(\mathcal{P}3)_{n-1}$ imply

$$\begin{aligned}\|H_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}&=\|U_n-U_{n-1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&=\|W_n-W_{n-1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&\leqslant\|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}+\|W_{n-1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\\&\leqslant 2C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1}.\end{aligned}$$

Putting together the foregoing estimates into (8.5) gives for $n\geqslant2$,

$$\|H_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant CC_*\varepsilon\gamma^{-1}N_{n-1}^{\frac12(\mu_1-a_2)} \tag{8.7}$$

and for $n=1$,

$$\|H_1\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant\tfrac12 C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{8.8}$$

Now from (8.1) we infer

$$a_2\geqslant\mu_1+2. \tag{8.9}$$

Thus, by (8.2) and Lemma A.5, we get for small ε

$$\begin{aligned}\|W_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \|H_1\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}} + \sum_{k=2}^{n}\|H_k\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + CC_*\varepsilon\gamma^{-1}\sum_{k=0}^{n}N_k^{-1}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + CN_0^{-1}C_*\varepsilon\gamma^{-1}\\ &\leqslant C_*\varepsilon^{1-a(1+q\overline{a})}.\end{aligned}$$

One can check from (8.1) and (8.2) that

$$a \leqslant \tfrac{1}{2(1+q\overline{a})} \tag{8.10}$$

and therefore, by choosing ε small enough and since $\overline{\sigma} \geqslant \sigma$, we get

$$\begin{aligned}\|\mathfrak{I}_n\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}} \leqslant \|W_n\|_{q,s_h+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant C_*\varepsilon^{\frac{1}{2}}\\ &\leqslant 1.\end{aligned}$$

As we have already mentioned, the parameter $\overline{\sigma}$ is the final loss of regularity constructed in Theorem 6.1 and depends only on the parameters τ_1, τ_2, q and d but it is independent of the state and the regularity. Hence it can be selected large enough such that $s_0 + \overline{\sigma} \geqslant \overline{s}_h + \sigma_4$ where $\overline{s}_h$ and σ_4 are respectively defined in (7.15) and Proposition 7.5. Then using (8.6) and Sobolev embeddings, we obtain

$$\forall n \geqslant 2, \quad \|H_n\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} \leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}. \tag{8.11}$$

► **Set inclusions.** From the previous point, Propositions 7.2, 7.5 and 7.6 apply and allow us to perform the reduction of the linearized operator in the normal directions at the current step. Therefore, the sets $\mathcal{A}_k^{\gamma}$ for all $k \leqslant n+1$ are well-defined. We shall now prove the following inclusions needed later to establish suitable estimates for the extensions.

$$\mathcal{A}_{n+1}^{2\gamma} \subset \mathrm{O}_{n+1}^{4\gamma} \subset \left(\mathcal{A}_{n+1}^{\gamma} \cap \mathrm{O}_n^{2\gamma}\right). \tag{8.12}$$

Notice that the first inclusion is obvious by construction since $\mathrm{O}_{n+1}^{4\gamma}$ is an enlargement of $\mathcal{A}_{n+1}^{2\gamma}$. It remains to prove the last inclusion. We have the inclusion

$$\forall k \in [\![0,n]\!], \quad \mathrm{O}_{k+1}^{4\gamma} \subset \mathrm{O}_k^{2\gamma}. \tag{8.13}$$

Indeed, since by construction $\mathcal{A}_{k+1}^{2\gamma} \subset \mathcal{A}_k^{2\gamma}$ then taking $(\lambda,\omega) \in \mathrm{O}_{k+1}^{4\gamma}$ we have the following estimates

$$\begin{aligned}\mathtt{dist}\big((\lambda,\omega),\mathcal{A}_k^{2\gamma}\big) &\leqslant \mathtt{dist}\big((\lambda,\omega),\mathcal{A}_{k+1}^{2\gamma}\big)\\ &< 4\gamma N_{k+1}^{-\overline{a}} = 4\gamma N_k^{-\overline{a}}N_0^{-\frac{1}{2}\overline{a}}\\ &< 2\gamma N_k^{-\overline{a}},\end{aligned}$$

provided that $2N_0^{-\frac{1}{2}\overline{a}} < 1$, which is true up to taking N_0 large enough, that is in view of (8.2) for ε small enough. We shall now prove by induction in k that

$$\forall k \in [\![0,n+1]\!], \quad \mathrm{O}_k^{4\gamma} \subset \mathcal{A}_k^{\gamma}. \tag{8.14}$$

The case $k=0$ is trivial since $\mathrm{O}_0^{4\gamma}=\mathcal{O}=\mathcal{A}_0^{\gamma}$. Let us now assume that (8.14) is true for the index $k\in[\![0,n]\!]$ and let us check it at the next order. From (8.13) and (8.14), we obtain

$$\mathrm{O}_{k+1}^{4\gamma}\subset\mathrm{O}_k^{2\gamma}\subset\mathrm{O}_k^{4\gamma}\subset\mathcal{A}_k^{\gamma}.$$

Therefore, we are left to check that

$$\mathrm{O}_{k+1}^{4\gamma}\subset\mathcal{G}_k\Big(\gamma_{k+1},\tau_1,\tau_2,i_k\Big).$$

Let $(\lambda,\omega)\in\mathcal{O}_{k+1}^{4\gamma}$, then by construction, there exists $(\lambda',\omega')\in\mathcal{A}_{k+1}^{2\gamma}$ such that

$$\mathtt{dist}\left((\lambda,\omega),(\lambda',\omega')\right)<4\gamma N_{k+1}^{-\overline{a}}.$$

Hence, for all $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$ with $|l|\leqslant N_k$, we have by left triangle and Cauchy-Schwarz inequalities together with $(\lambda',\omega')\in\Lambda_{\infty,k}^{2\gamma_{k+1},\tau_1}(i_k)$

$$\begin{aligned}\left|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_k)\right|&\geqslant\left|\omega'\cdot l+\mu_j^{\infty}(\lambda',\omega',i_k)\right|-|\omega-\omega'||l|-\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|\\&>\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}-4\gamma N_kN_{k+1}^{-\overline{a}}-\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|\\&>\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}-4\gamma N_{k+1}^{1-\overline{a}}-\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|.\end{aligned}$$

Using the Mean Value Theorem and the definition of $\mathrm{O}_{k+1}^{4\gamma}$ yields

$$\begin{aligned}\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|&\leqslant|(\lambda,\omega)-(\lambda',\omega')|\gamma^{-1}\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\\&\leqslant4N_{k+1}^{-\overline{a}}\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

On the other hand,

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\leqslant\|\mu_j^{\infty}(i_k)-\Omega_j\|_q^{\gamma,\mathcal{O}}+\|\Omega_j\|_q^{\gamma,\mathcal{O}}.$$

Using the asymptotic (5.17) and the smoothness of $\lambda\mapsto I_j(\lambda)K_j(\lambda)$ for all $j\in\mathbb{N}^*$, one has

$$\|\Omega_j\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

Since (7.236) is satisfied by the previous point, we can apply (7.238) and obtain

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)-\Omega_j\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

Hence

$$\forall j\in\mathbb{S}_0^c,\quad\|\mu_j^{\infty}(i_k)\|_q^{\gamma,\mathcal{O}}\leqslant C|j|.$$

It follows that

$$\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|\leqslant C\langle j\rangle N_{k+1}^{-\overline{a}}\leqslant C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}}.$$

Since $|l|\leqslant N_k\leqslant N_{k+1}$ and $\gamma_{k+1}\geqslant\gamma$, we obtain

$$\begin{aligned}\left|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_k)\right|&\geqslant\tfrac{2\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}-C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}}\\&\geqslant\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\left(2-CN_{k+1}^{\tau_1+1-\overline{a}}\right).\end{aligned}$$

From (8.1) and (A.1) we infer

$$\overline{a}\geqslant\tau_2+2\geqslant\tau_1+2\tag{8.15}$$

and we can take N_0 sufficiently large to ensure

$$CN_{k+1}^{\tau_1+1-\overline{a}}\leqslant CN_0^{-1}<1,$$

allowing to finally get

$$\left|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_k)\right|>\tfrac{\gamma_{k+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

This shows that, $(\lambda,\omega)\in\Lambda_{\infty,k}^{\gamma_{k+1},\tau_1}(i_k)$. Let us now check that $(\lambda,\omega)\in\mathcal{O}_{\infty,k}^{\gamma_{k+1},\tau_1}(i_k)$. For all $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$ with $|l|\leqslant N_k$, we have by Cauchy-Schwarz inequality together with $(\lambda',\omega')\in\mathcal{O}_{\infty,k}^{2\gamma_{k+1},\tau_1}(i_k)$

$$\begin{aligned}|\omega\cdot l+jc_{i_k}(\lambda,\omega)|&\geqslant|\omega'\cdot l+jc_{i_k}(\lambda',\omega')|-|\omega-\omega'||l|-|j|\,|c_{i_k}(\lambda,\omega)-c_{i_k}(\lambda',\omega')|\\&>\tfrac{4\gamma_{k+1}^{\upsilon}2^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}-4\gamma N_{k+1}^{1-\overline{a}}-\langle j\rangle\,|c_{i_k}(\lambda,\omega)-c_{i_k}(\lambda',\omega')|\,.\end{aligned}$$

Using the Mean Value Theorem and the definition of $\mathrm{O}_{k+1}^{4\gamma}$ yields

$$|c_{i_k}(\lambda,\omega)-c_{i_k}(\lambda',\omega')|\leqslant CN_{k+1}^{-\overline{a}}\|c_{i_k}\|_q^{\gamma,\mathcal{O}}.$$

Since (7.16) is satisfied by the previous point, we can apply (7.17) leading to

$$\begin{aligned}\|c_{i_k}\|_q^{\gamma,\mathcal{O}}&\leqslant\|c_{i_k}-V_0\|_q^{\gamma,\mathcal{O}}+\|V_0\|_q^{\gamma,\mathcal{O}}\\&\leqslant C.\end{aligned}$$

Thus

$$|c_{i_k}(\lambda,\omega)-c_{i_k}(\lambda',\omega')|\leqslant C\gamma\gamma^{-1}N_{k+1}^{-\overline{a}}\leqslant C\gamma N_{k+1}^{1-\overline{a}}.$$

Therefore, we obtain from the definition of γ_k and $\upsilon\in(0,1)$

$$\begin{aligned}|\omega\cdot l+jc_{i_k}(\lambda,\omega)|&>\tfrac{4\gamma_{k+1}^{\upsilon}2^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}-C\gamma\langle j\rangle N_{k+1}^{1-\overline{a}}\\&\geqslant\tfrac{4\gamma_{k+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\left(2^{\upsilon}-CN_{k+1}^{\tau_1+1-\overline{a}}\right).\end{aligned}$$

By the choice of $\overline{a}$ made in (8.15), we can ensure, up to taking N_0 sufficiently large,

$$2^{\upsilon}-CN_{k+1}^{\tau_1+1-\overline{a}}\geqslant2^{\upsilon}-CN_0^{-1}>1,$$

so that

$$|\omega\cdot l+jc_{i_k}(\lambda,\omega)|>\tfrac{4\gamma_{k+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

As a consequence, $(\lambda,\omega)\in\mathcal{O}_{\infty,k}^{\gamma_{k+1},\tau_1}(i_k)$. Let us now check that $(\lambda,\omega)\in\mathscr{O}_{\infty,k}^{\gamma_{k+1},\tau_1,\tau_2}(i_k)$. For all $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ with $|l|\leqslant N_k$, we have by the triangle and Cauchy-Schwarz inequalities together with $(\lambda',\omega')\in\mathscr{O}_{\infty,k}^{2\gamma_{k+1},\tau_1,\tau_2}(i_k)$

$$\begin{aligned}\left|\omega\cdot l+\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_{j_0}^{\infty}(\lambda,\omega,i_k)\right|&\geqslant\left|\omega'\cdot l+\mu_j^{\infty}(\lambda',\omega',i_k)-\mu_{j_0}^{\infty}(\lambda',\omega',i_k)\right|-|\omega-\omega'||l|\\&\quad-\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_{j_0}^{\infty}(\lambda,\omega,i_k)+\mu_{j_0}^{\infty}(\lambda',\omega',i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|\\&>\tfrac{4\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-4\gamma N_{k+1}^{1-\overline{a}}\\&\quad-\left|\mu_j^{\infty}(\lambda,\omega,i_k)-\mu_{j_0}^{\infty}(\lambda,\omega,i_k)+\mu_{j_0}^{\infty}(\lambda',\omega',i_k)-\mu_j^{\infty}(\lambda',\omega',i_k)\right|.\end{aligned}$$

We recall by virtue of Proposition 7.5 that

$$\mu_j^{\infty}(\lambda,\omega,i_k)=\mu_j^0(\lambda,\omega,i_k)+r_j^{\infty}(\lambda,\omega,i_k).$$

Thus

$$\begin{aligned}&\left|\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)+\mu_{j_0}^\infty(\lambda',\omega',i_k)-\mu_j^\infty(\lambda',\omega',i_k)\right|\\&\leqslant\left|\mu_j^0(\lambda,\omega,i_k)-\mu_{j_0}^0(\lambda,\omega,i_k)+\mu_{j_0}^0(\lambda',\omega',i_k)-\mu_j^0(\lambda',\omega',i_k)\right|\\&\quad+\left|r_j^\infty(\lambda,\omega,i_k)-r_j^\infty(\lambda',\omega',i_k)\right|+\left|r_{j_0}^\infty(\lambda,\omega,i_k)-r_{j_0}^\infty(\lambda',\omega',i_k)\right|.\end{aligned}$$

According to the Mean Value Theorem, (7.255) and the definition of $\mathrm{O}_{k+1}^{4\gamma}$ we find

$$\left|\mu_j^0(\lambda,\omega,i_k)-\mu_{j_0}^0(\lambda,\omega,i_k)+\mu_{j_0}^0(\lambda',\omega',i_k)-\mu_j^0(\lambda',\omega',i_k)\right|\leqslant\gamma CN_{k+1}^{1-\overline{a}}\langle j-j_0\rangle.$$

Applying once again the Mean Value Theorem, (7.239), (8.4) and the definition of $\mathrm{O}_{n+1}^{4\gamma}$ yields

$$\left|r_j^\infty(\lambda,\omega,i_k)-r_j^\infty(\lambda',\omega',i_k)\right|\leqslant C\gamma N_{k+1}^{-\overline{a}}\varepsilon\gamma^{-2}\leqslant\gamma CN_{k+1}^{1-\overline{a}}\langle j-j_0\rangle.$$

Putting together the foregoing estimates and the facts that $|l|\leqslant N_k$ and $\gamma_{k+1}\geqslant\gamma$ we infer

$$\left|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)\right|\geqslant\tfrac{\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\left(4-CN_{k+1}^{\tau_2+1-\overline{a}}\right).$$

By virtue of (8.15) and taking N_0 sufficiently large we get

$$CN_n^{\tau_2+1-\overline{a}}\leqslant CN_0^{-1}<1.$$

This implies

$$\left|\omega\cdot l+\mu_j^\infty(\lambda,\omega,i_k)-\mu_{j_0}^\infty(\lambda,\omega,i_k)\right|>\tfrac{2\gamma_{k+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

As a consequence, we deduce that $(\lambda,\omega)\in\mathscr{O}_{\infty,k}^{\gamma_{k+1},\tau_1,\tau_2}(i_n)$. Finally, $(\lambda,\omega)\in\mathcal{G}_k\big(\gamma_{k+1},\tau_1,\tau_2,i_k\big)$ and therefore $(\lambda,\omega)\in\mathcal{A}_{k+1}^\gamma$. This achieves the induction proof of (8.14).

► **Construction of the next approximation.** We are now going to construct the next approximation U_{n+1} by using a modified Nash-Moser scheme. The assumption (7.365) being satisfied, we can apply Theorem 6.1 with L_n and obtain the existence of an operator $\mathrm{T}_n\triangleq\mathrm{T}_n(\lambda,\omega)$ well-defined in the whole set of parameters $\mathcal{O}$ and satisfying the following estimates

$$\forall s\in[s_0,S],\quad\|\mathrm{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\left(\|\rho\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)\tag{8.16}$$

and

$$\|\mathrm{T}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.\tag{8.17}$$

Moreover, when it is restricted to the Cantor set $\mathcal{G}_n(\gamma_{n+1},\tau_1,\tau_2,i_n)$, T_n is an approximate right inverse of L_n with suitable tame estimates needed later, see Theorem 6.1. Next we define,

$$\widetilde{U}_{n+1}\triangleq U_n+\widetilde{H}_{n+1}\quad\text{ with }\quad\widetilde{H}_{n+1}\triangleq(\widehat{\mathfrak{I}}_{n+1},\widehat{\alpha}_{n+1},0)\triangleq-\mathbf{\Pi}_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\in E_n\times\mathbb{R}^d\times\mathbb{R}^{d+1},$$

where $\mathbf{\Pi}_n$ is defined by

$$\mathbf{\Pi}_n(\mathfrak{I},\alpha,0)=(\Pi_n\mathfrak{I},\alpha,0)\quad\text{ and }\quad\mathbf{\Pi}_n^\perp(\mathfrak{I},\alpha,0)=(\Pi_n\mathfrak{I},0,0).\tag{8.18}$$

Notice that the projectors Π_n are reversibility preserving due to the symmetry with respect to the Fourier modes. Then, using the reversibility of T_n together with (6.22) and Lemma 4.2, one deduces from

$\mathfrak{S}i_n(\varphi)=i_n(-\varphi)$ that

$$\mathfrak{S}\widehat{\mathfrak{I}}_{n+1}(\varphi)=\widehat{\mathfrak{I}}_{n+1}(-\varphi). \tag{8.19}$$

Note that U_n is defined in the full set $\mathcal{O}$ and so does $\widetilde{U}_{n+1}$. Nevertheless, we will not be working with this natural extension but rather with a suitable localized version of it around the Cantor set $\mathcal{A}_{n+1}^{\gamma}$. Doing so, we shall get a nice decay property allowing the scheme to converge. Now, introduce the quadratic function

$$Q_n=\mathcal{F}(U_n+\widetilde{H}_{n+1})-\mathcal{F}(U_n)-L_n\widetilde{H}_{n+1}, \tag{8.20}$$

then simple transformations give

$$\begin{aligned}\mathcal{F}(\widetilde{U}_{n+1})&=\mathcal{F}(U_n)-L_n\mathbf{\Pi}_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+Q_n\\&=\mathcal{F}(U_n)-L_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+L_n\mathbf{\Pi}_n^{\perp}\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+Q_n\\&=\mathcal{F}(U_n)-\Pi_nL_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+Q_n\\&=\Pi_n^{\perp}\mathcal{F}(U_n)-\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)+(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)+Q_n.\end{aligned} \tag{8.21}$$

In the sequel we shall prove

$$\|\mathcal{F}(U_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}}\leqslant C_*\varepsilon N_n^{-a_1},$$

with U_{n+1} a suitable extension of $\widetilde{U}_{n+1}|_{\mathrm{O}_{n+1}^{2\gamma}}$.

▶ **Estimates of $\mathcal{F}(\widetilde{U}_{n+1})$.** We shall now estimate $\mathcal{F}(\widetilde{U}_{n+1})$ with the norm $\|\cdot\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}$ by using (8.21). The localization in $\mathrm{O}_{n+1}^{4\gamma}$ is required for the classical extension in the next point, see (8.48).

➢ *Estimate of* $\Pi_n^{\perp}\mathcal{F}(U_n)$. We apply Taylor formula combined with (6.21) and Lemma 6.3 together with (8.3) and $(\mathcal{P}1)_n$. Therefore, we obtain

$$\begin{aligned}\forall s\geqslant s_0,\quad\|\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_n^{2\gamma}}&\leqslant\|\mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathcal{F}(U_n)-\mathcal{F}(U_0)\|_{q,s}^{\gamma,\mathrm{O}_n^{2\gamma}}\\&\lesssim\varepsilon+\|W_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}.\end{aligned} \tag{8.22}$$

As a consequence, (8.2) and $(\mathcal{P}1)_n$ imply

$$\gamma^{-1}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\leqslant 1. \tag{8.23}$$

From Lemma A.1-(ii) and (8.22), we get

$$\begin{aligned}\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}&\leqslant N_n^{s_0-s_m}\|\mathcal{F}(U_n)\|_{q,s_m}^{\gamma,\mathrm{O}_n^{\gamma}}\\&\lesssim N_n^{\overline{\sigma}-s_m}\left(\varepsilon+\|W_n\|_{q,s_m+\overline{\sigma}}^{2\gamma,\mathcal{O}}\right).\end{aligned} \tag{8.24}$$

Now, $(\mathcal{P}3)_n$ together (6.94) and (8.2) yield

$$\begin{aligned}\varepsilon+\|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}&\leqslant\varepsilon\left(1+C_*\gamma^{-1}N_{n-1}^{\mu_1}\right)\\&\leqslant 2C_*\varepsilon N_n^{\frac{2}{3}\mu_1+1}.\end{aligned} \tag{8.25}$$

By putting together (8.25) and (8.24) and by making appeal to (8.13), we infer for any $n\in\mathbb{N}$,

$$\begin{aligned}\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}&\leqslant\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\\&\lesssim C_*\varepsilon N_n^{s_0+\frac{2}{3}\mu_1+1-s_m}.\end{aligned} \tag{8.26}$$

Remark that one also obtains, combining (8.22) and (8.25),

$$\|\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}} \leqslant C_\ast\varepsilon N_n^{\overline{\sigma}+\frac{2}{3}\mu_1+1}. \tag{8.27}$$

➢ *Estimate of* $\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)$. In view of (8.14), one has

$$\mathrm{O}_{n+1}^{4\gamma}\subset\mathcal{A}_{n+1}^{\gamma}\subset\mathcal{G}_n\Big(\gamma_{n+1},\tau_1,\tau_2,i_n\Big).$$

Then, applying Theorem 6.1, we can write

$$\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)=\mathscr{E}_{1,n}+\mathscr{E}_{2,n}+\mathscr{E}_{3,n},$$

with

$$\begin{aligned}
\mathscr{E}_{1,n}&\triangleq\Pi_n\mathcal{E}_1^{(n)}\Pi_n\mathcal{F}(U_n),\\
\mathscr{E}_{2,n}&\triangleq\Pi_n\mathcal{E}_2^{(n)}\Pi_n\mathcal{F}(U_n),\\
\mathscr{E}_{3,n}&\triangleq\Pi_n\mathcal{E}_3^{(n)}\Pi_n\mathcal{F}(U_n)
\end{aligned}$$

where $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ and $\mathcal{E}_3^{(n)}$ satisfy the estimates (6.131), (6.132) and (6.133) respectively. By (8.13), we get

$$\|\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\leqslant\|\mathscr{E}_{1,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\mathscr{E}_{2,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\mathscr{E}_{3,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}. \tag{8.28}$$

We shall first focus on $\mathscr{E}_{1,n}$. We need the following interpolation-type inequality

$$\begin{aligned}
\|\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}&\leqslant\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\Pi_n^{\perp}\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\\
&\leqslant N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}+N_n^{s_0-s_m}\|\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}.
\end{aligned} \tag{8.29}$$

Combining (6.131), (8.29), $(\mathcal{P}_1)_n$, (8.4) and (8.27), we obtain

$$\begin{aligned}
\|\mathscr{E}_{1,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}&\lesssim\gamma^{-1}\|\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\Big(1+\|\mathfrak{I}_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\\
&\lesssim\gamma^{-1}N_n^{\overline{\sigma}}\Big(N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}+N_n^{s_0-s_m}\|\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\Big)\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\Big(1+\|W_n\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\\
&\lesssim C_\ast\varepsilon\Big(N_n^{2\overline{\sigma}-\frac{4}{3}a_1}+N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-\frac{2}{3}a_1-s_m}\Big).
\end{aligned} \tag{8.30}$$

We now turn to $\mathscr{E}_{2,n}$ and $\mathscr{E}_{3,n}$. Applying (6.132) with $b=s_m-s_0$ and using (8.4), $(\mathcal{P}_2)_n$ and $(\mathcal{P}_3)_n$, we get

$$\begin{aligned}
\|\mathscr{E}_{2,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}&\lesssim\gamma^{-1}N_n^{s_0-s_m}\Big(\|\Pi_n\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}+\varepsilon\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\Big)\\
&\lesssim\gamma^{-1}N_n^{s_0-s_m}\Big(\|\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}+\varepsilon N_n^{\overline{\sigma}}\|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\Big)\\
&\lesssim C_\ast\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-s_m}+C_\ast\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-\frac{2}{3}a_1-s_m}\\
&\lesssim C_\ast\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-s_m}.
\end{aligned} \tag{8.31}$$

Using the same techniques together with (6.133), (6.94), (8.2) and (8.4), we infer

$$\begin{aligned}\|\mathscr{E}_{3,n}\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}} &\lesssim N_n^{s_0-s_m}\gamma^{-2}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\right)\\ &\quad+\varepsilon\gamma^{-4}N_0^{\mu_2}N_n^{-\mu_2}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\\ &\lesssim C_*\varepsilon\left(N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+2-s_m}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}\right).\end{aligned}\tag{8.32}$$

Putting together (8.28), (8.30), (8.31) and (8.31), we obtain

$$\|\Pi_n(L_n\mathrm{T}_n-\mathrm{Id})\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\leqslant CC_*\varepsilon\left(N_n^{2\overline{\sigma}-\frac{4}{3}a_1}+N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-s_m}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}\right).\tag{8.33}$$

For $n=0$, we deduce from (8.3),(8.4) and by slight modifications of the preceding computations

$$\begin{aligned}\|\Pi_0(L_0\mathrm{T}_0-\mathrm{Id})\Pi_0\mathcal{F}(U_0)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{4\gamma}} &\leqslant \|\mathscr{E}_{1,0}\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}}+\|\mathscr{E}_{2,0}\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}}+\|\mathscr{E}_{3,0}\|_{q,s_0}^{\gamma,\mathrm{O}_1^{2\gamma}}\\ &\lesssim \varepsilon^2\gamma^{-1}+\varepsilon\gamma^{-1}+\left(\varepsilon\gamma^{-2}N_0^{s_0-s_m}+\varepsilon^2\gamma^{-4}\right)\\ &\lesssim \varepsilon\gamma^{-2}.\end{aligned}\tag{8.34}$$

➢ *Estimate of* $\big(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n\big)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)$. Combining (6.55) and (6.21), we get for $H=(\widehat{\mathfrak{I}},\widehat{\alpha})$ with $\widehat{\mathfrak{I}}=(\widehat{\Theta},\widehat{I},\widehat{z})$,

$$L_nH=\omega\cdot\partial_\varphi\widehat{\mathfrak{I}}-(0,0,\partial_\theta\mathrm{L}(\lambda)\widehat{z})-\varepsilon d_iX_{\mathcal{P}_\varepsilon}(i_n)\widehat{\mathfrak{I}}-(\widehat{\alpha},0,0).\tag{8.35}$$

Using (8.18) and the fact that $\omega\cdot\partial_\varphi$ and $\partial_\theta\mathrm{L}(\lambda)$ are diagonal leading to $[\Pi_n^{\perp},\omega\cdot\partial_\varphi]=[\Pi_n^{\perp},\partial_\theta\mathrm{L}(\lambda)]=0$, one has for $H=(\widehat{\mathfrak{I}},\widehat{\alpha})$,

$$\big(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n\big)H=-\varepsilon[d_iX_{\mathcal{P}_\varepsilon}(i_n),\Pi_n^{\perp}]\widehat{\mathfrak{I}}.$$

In view of Lemma 6.3-(ii), Lemma A.6, (8.13) and $(\mathcal{P}1)_n$ we get

$$\big\|\big(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n\big)H\big\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\lesssim\varepsilon N_n^{s_0-s_m}\left(\|\widehat{\mathfrak{I}}\|_{q,s_m+2}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\widehat{\mathfrak{I}}\|_{q,s_0+1}^{\gamma,\mathrm{O}_n^{2\gamma}}\right).$$

Consequently,

$$\begin{aligned}\mathtt{N}_{\mathrm{com}}(s_0)\triangleq\big\|\big(L_n\mathbf{\Pi}_n^{\perp}-\Pi_n^{\perp}L_n\big)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\big\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{2\gamma}} &\lesssim\varepsilon N_n^{s_0-s_m}\|\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s_m+2}^{\gamma,\mathrm{O}_n^{\gamma}}\\ &\quad+\varepsilon N_n^{s_0-s_m}\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s_0+1}^{\gamma,\mathrm{O}_n^{\gamma}}.\end{aligned}$$

Hence, gathering (8.16), Lemma A.1, Sobolev embeddings, (8.4), (8.2) and $(\mathcal{P}1)_n$ yields

$$\begin{aligned}\mathtt{N}_{\mathrm{com}}(s_0) &\lesssim\varepsilon\gamma^{-1}N_n^{s_0-s_m}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}+2}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}+1}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\\ &\quad+\varepsilon\gamma^{-1}N_n^{s_0-s_m}\|\mathfrak{I}_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}+1}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|\mathfrak{I}_n\|_{q,s_0+\overline{\sigma}+1}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\right)\\ &\lesssim\varepsilon N_n^{s_0+2-s_m}\left(\|\mathcal{F}(U_n)\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}+\|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\right).\end{aligned}$$

Applying Lemma A.1-(ii), $(\mathcal{P}2)_n$ and (6.94), we infer

$$\begin{aligned}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}} &\leqslant N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\\ &\leqslant C_*\varepsilon N_n^{\overline{\sigma}}N_{n-1}^{-a_1}\\ &\leqslant C_*\varepsilon N_n^{\overline{\sigma}-\frac{2}{3}a_1}.\end{aligned}$$

Added to (8.1), (8.27) and $(\mathcal{P}3)_n$, we obtain for $n \in \mathbb{N}$,

$$\|(L_n\mathbf{\Pi}_n^{\perp} - \Pi_n^{\perp}L_n)\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \leqslant CC_*\varepsilon N_n^{s_0+\overline{\sigma}+\frac{2}{3}\mu_1+3-s_m}. \tag{8.36}$$

➢ *Estimate of* Q_n. We apply Taylor formula together with (8.20) leading to

$$Q_n = \int_0^1 (1-t)d_{i,\alpha}^2\mathcal{F}(U_n + t\widetilde{H}_{n+1})[\widetilde{H}_{n+1}, \widetilde{H}_{n+1}]dt.$$

Thus, (8.35) and Lemma 6.3-(iii) allow to get

$$\|Q_n\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \lesssim \varepsilon\left(1 + \|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\right)\left(\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\right)^2. \tag{8.37}$$

Combining (8.14), (8.16), (8.22) and (8.23), we find for all $s \in [s_0, S]$

$$\begin{aligned}
\|\widetilde{H}_{n+1}\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} &= \|\mathbf{\Pi}_n\mathrm{T}_n\Pi_n\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \\
&\lesssim \gamma^{-1}\left(\|\Pi_n\mathcal{F}(U_n)\|_{q,s+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}} + \|\mathfrak{I}_n\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\|\Pi_n\mathcal{F}(U_n)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_n^{2\gamma}}\right) \\
&\lesssim \gamma^{-1}\left(N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s}^{\gamma,\mathrm{O}_n^{2\gamma}} + N_n^{2\overline{\sigma}}\|\mathfrak{I}_n\|_{q,s}^{\gamma,\mathcal{O}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}}\right) \\
&\lesssim \gamma^{-1}N_n^{2\overline{\sigma}}\left(\varepsilon + \|W_n\|_{q,s}^{\gamma,\mathcal{O}}\right).
\end{aligned} \tag{8.38}$$

In the same way, according to (8.17), $(\mathcal{P}1)_n$ and $(\mathcal{P}2)_n$, we infer

$$\begin{aligned}
\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} &\lesssim \gamma^{-1}N_n^{\overline{\sigma}}\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma,\mathrm{O}_n^{2\gamma}} \\
&\lesssim C_*\varepsilon\gamma^{-1}N_n^{\overline{\sigma}}N_{n-1}^{-a_1}.
\end{aligned} \tag{8.39}$$

Choosing ε small enough and using $(\mathcal{P}1)_n$ and (8.39), we find

$$\begin{aligned}
\|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} &\leqslant C_*\varepsilon\gamma^{-1} + N_n^2\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \\
&\leqslant 1 + C\varepsilon\gamma^{-1}N_n^{\overline{\sigma}+2}N_{n-1}^{-a_1} \\
&\leqslant 1 + C\varepsilon\gamma^{-1}N_{n-1}^{3+\frac{3}{2}\overline{\sigma}-a_1}.
\end{aligned}$$

Now notice that (8.1) implies

$$a_1 \geqslant 3 + \tfrac{3}{2}\overline{\sigma}. \tag{8.40}$$

Therefore, we obtain

$$\|W_n\|_{q,s_0+2}^{\gamma,\mathcal{O}} + \|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \leqslant 2.$$

Hence, plugging this estimate and (8.39) into (8.37) and using (8.2) and (8.4), we find

$$\begin{aligned}
\|Q_n\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} &\lesssim \varepsilon\left(\|\widetilde{H}_{n+1}\|_{q,s_0+2}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\right)^2 \\
&\leqslant \varepsilon N_n^4\left(\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\right)^2 \\
&\lesssim \varepsilon C_* N_n^{2\overline{\sigma}+4}N_{n-1}^{-2a_1}.
\end{aligned}$$

By using (6.94), we deduce when $n \geqslant 1$,

$$\|Q_n\|_{q,s_0}^{\gamma,\mathcal{O}_{n+1}^{4\gamma}} \leqslant CC_*\varepsilon N_n^{2\overline{\sigma}+4-\frac{4}{3}a_1}. \tag{8.41}$$

For $n=0$, we come back to (8.38) and (8.3) to obtain for all $s\in[s_0,S]$

$$\begin{aligned}\|\widetilde{H}_1\|_{q,s}^{\gamma,\mathrm{O}_1^{4\gamma}} &\lesssim \gamma^{-1}\|\Pi_0\mathcal{F}(U_0)\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\lesssim C_*\varepsilon\gamma^{-1}.\end{aligned}\tag{8.42}$$

Finally, the inequality (8.41) becomes for $n=0$,

$$\|Q_0\|_{q,s_0}^{\gamma,\mathrm{O}_0^{4\gamma}} \lesssim C_*\varepsilon^3\gamma^{-2}.\tag{8.43}$$

➢ *Conclusion.* Inserting (8.26), (8.33), (8.36) and (8.41), into (8.21) implies for $n\in\mathbb{N}^*$,

$$\|\mathcal{F}(\widetilde{U}_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \leqslant CC_*\varepsilon\left(N_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-s_m}+N_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1}+N_n^{2\overline{\sigma}+4-\frac{4}{3}a_1}\right).$$

The parameters conditions stated in (8.1) give

$$\begin{cases} s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+2+a_1 & \leqslant & s_m\\ \overline{\sigma}+\frac{1}{3}a_1+2 & \leqslant & \mu_2\\ 2\overline{\sigma}+5 & \leqslant & \frac{1}{3}a_1.\end{cases}\tag{8.44}$$

Thus, by taking N_0 large enough, that is ε small enough, we obtain for $n\in\mathbb{N}$,

$$\begin{cases} CN_n^{s_0+2\overline{\sigma}+\frac{2}{3}\mu_1+1-s_m} & \leqslant & \frac{1}{3}N_n^{-a_1}\\ CN_n^{\overline{\sigma}+1-\mu_2-\frac{2}{3}a_1} & \leqslant & \frac{1}{3}N_n^{-a_1}\\ CN_n^{2\overline{\sigma}+4-\frac{4}{3}a_1} & \leqslant & \frac{1}{3}N_n^{-a_1},\end{cases}\tag{8.45}$$

which implies in turn that when $n\in\mathbb{N}^*$,

$$\|\mathcal{F}(\widetilde{U}_{n+1})\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}} \leqslant C_*\varepsilon N_n^{-a_1}.\tag{8.46}$$

However, when $n=0$, we plug (8.26), (8.34), (8.36) and (8.43) into (8.21) in order to get

$$\|\mathcal{F}(\widetilde{U}_1)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{4\gamma}} \leqslant CC_*\varepsilon\left(N_0^{s_0+2\overline{\sigma}+\frac{3}{2}\mu_1+1-s_m}+\varepsilon\gamma^{-2}+\varepsilon^2\gamma^{-2}\right).$$

From (8.45), one already has

$$CN_0^{s_0+2\overline{\sigma}+\frac{3}{2}\mu_1+1-s_m} \leqslant \tfrac{1}{3}N_0^{-a_1}.$$

Therefore, we need at this level to take ε small enough to ensure

$$C\left(\varepsilon\gamma^{-2}+\varepsilon^2\gamma^{-2}\right) \leqslant \tfrac{2}{3}N_0^{-a_1}.$$

This occurs since (8.2) and (8.1) imply

$$0<a<\tfrac{1}{2+a_1}.$$

Hence

$$\|\mathcal{F}(\widetilde{U}_1)\|_{q,s_0}^{\gamma,\mathrm{O}_1^{4\gamma}} \leqslant C_*\varepsilon N_0^{-a_1}.$$

This completes the proof of the estimates in $(\mathcal{P}2)_{n+1}$.

▶ **Extension and verification of** $(\mathcal{P}1)_{n+1}-(\mathcal{P}3)_{n+1}$. We shall now construct an extention of $\widetilde{H}_{n+1}$ living in the whole set of parameters and enjoying suitable decay properties. This is done by using the

C^∞ cut-off function $\chi_{n+1}:\mathcal{O}\to[0,1]$ defined by

$$\chi_{n+1}(\lambda,\omega)=\begin{cases}1 & \text{in } \mathrm{O}_{n+1}^{2\gamma}\\ 0 & \text{in } \mathcal{O}\setminus\mathrm{O}_{n+1}^{4\gamma}\end{cases}$$

and satisfying the additional growth conditions

$$\forall\alpha\in\mathbb{N}^d,\quad |\alpha|\in[\![0,q]\!],\quad \|\partial_{\lambda,\omega}^{\alpha}\chi_{n+1}\|_{L^\infty(\mathcal{O})}\lesssim\left(\gamma^{-1}N_n^{\overline{a}}\right)^{|\alpha|}. \tag{8.47}$$

Next, we shall deal with the extension H_{n+1} of $\widetilde{H}_{n+1}$ defined by

$$H_{n+1}(\lambda,\omega)\triangleq\begin{cases}\chi_{n+1}(\lambda,\omega)\widetilde{H}_{n+1}(\lambda,\omega) & \text{in} \quad \mathrm{O}_{n+1}^{4\gamma}\\ 0 & \text{in} \quad \mathcal{O}\setminus\mathrm{O}_{n+1}^{4\gamma}\end{cases} \tag{8.48}$$

and the extension U_{n+1} of $\widetilde{U}_{n+1}$ by

$$U_{n+1}\triangleq U_n+H_{n+1}. \tag{8.49}$$

We remark that

$$H_{n+1}=\widetilde{H}_{n+1}\quad\text{and}\quad\mathcal{F}(U_{n+1})=\mathcal{F}(\widetilde{U}_{n+1})\quad\text{in}\quad\mathrm{O}_{n+1}^{2\gamma}.$$

Looking at the first component of (8.49), one can write with obvious notations

$$i_{n+1}=i_n+\mathfrak{I}_{n+1}.$$

By the induction assumption $(\mathcal{P}2)_n$, (8.48) and (8.19), one has

$$\mathfrak{S}i_n(\varphi)=i_n(-\varphi)\quad\text{and}\quad\mathfrak{S}\mathfrak{I}_{n+1}(\varphi)=\mathfrak{I}_{n+1}(-\varphi).$$

Thus

$$\mathfrak{S}i_{n+1}(\varphi)=i_{n+1}(-\varphi). \tag{8.50}$$

Using Lemma A.1-(iv) together with (8.47) and the fact that $H_{n+1}=0$ in $\mathcal{O}\setminus\mathrm{O}_{n+1}^{4\gamma}$, we obtain

$$\forall s\geqslant s_0,\quad\|H_{n+1}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{q\overline{a}}\|\widetilde{H}_{n+1}\|_{q,s}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}. \tag{8.51}$$

Applying (8.51) and (8.39) we deduce that for $n\in\mathbb{N}^*$,

$$\begin{aligned}\|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}&\leqslant CN_n^{q\overline{a}}\|\widetilde{H}_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\\&\leqslant CN_n^{q\overline{a}+\overline{\sigma}}\|\widetilde{H}_{n+1}\|_{q,s_0}^{\gamma,\mathrm{O}_{n+1}^{4\gamma}}\\&\leqslant CC_*\varepsilon\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}-\frac{2}{3}a_1}.\end{aligned}$$

From (8.1), we have

$$a_2=\tfrac{2}{3}a_1-q\overline{a}-2\overline{\sigma}-1\geqslant 1. \tag{8.52}$$

Therefore, choosing ε small enough, we obtain

$$\begin{aligned}\|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}&\leqslant CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{-a_2}\\&\leqslant C_*\varepsilon\gamma^{-1}N_n^{-a_2}.\end{aligned} \tag{8.53}$$

As for the case $n=0$, we combine (8.51) and (8.42) to obtain, up to taking C_* large enough,

$$\|H_1\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{8.54}$$

We now set

$$W_{n+1} \triangleq W_n + H_{n+1}, \tag{8.55}$$

then by construction, we infer

$$U_{n+1} = U_0 + W_{n+1}.$$

Moreover, applying $(\mathcal{P}1)_n$, (8.54) and (8.53) and Lemma A.5, we infer

$$\begin{aligned}\|W_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \|H_1\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} + \sum_{k=2}^{n+1}\|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + C_*\varepsilon\gamma^{-1}\sum_{k=0}^{\infty}N_k^{-1}\\ &\leqslant \tfrac{1}{2}C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}} + CN_0^{-1}C_*\varepsilon\gamma^{-1}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_0^{q\overline{a}}.\end{aligned}$$

This completes the proof of $(\mathcal{P}1)_{n+1}$. Now gathering (8.38), (8.51) and $(\mathcal{P}3)_n$ allows to write

$$\begin{aligned}\|W_{n+1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant \|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}} + CN_n^{q\overline{a}}\|H_{n+1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{\mu_1} + CC_*\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}}\left(\varepsilon + \|W_n\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}}\right)\\ &\leqslant CC_*\varepsilon\gamma^{-1}N_n^{q\overline{a}+2\overline{\sigma}+1+\frac{2}{3}\mu_1}.\end{aligned}$$

From (8.1), we can ensure the condition

$$q\overline{a}+2\overline{\sigma}+2 = \tfrac{\mu_1}{3}, \tag{8.56}$$

in order to get

$$\begin{aligned}\|W_{n+1}\|_{q,s_m+\overline{\sigma}}^{\gamma,\mathcal{O}} &\leqslant CN_0^{-1}C_*\varepsilon\gamma^{-1}N_n^{\mu_1}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_n^{\mu_1}\end{aligned}$$

by taking ε small enough and using (8.2). This proves $(\mathcal{P}3)_{n+1}$ and the proof of Proposition 8.1 is now complete. □

Once this sequence of approximate solutions is constructed, we may obtain a non-trivial solution by passing to the limit. This is possible due the decay properties given in Proposition 8.1. Actually, we obtain the following corollary.

Corollary 8.1. *There exists $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$, the following assertions hold true. We consider the Cantor set $\mathcal{G}_\infty^\gamma$, depending on ε through γ, and defined by*

$$\mathcal{G}_\infty^\gamma \triangleq \bigcap_{n\in\mathbb{N}}\mathcal{A}_n^\gamma.$$

There exists a function

$$\begin{array}{rccl} U_\infty: & \mathcal{O} & \to & \big(\mathbb{T}^d\times\mathbb{R}^d\times H_{\mathbb{S}}^{\perp}\big)\times\mathbb{R}^d\times\mathbb{R}^{d+1}\\ & (\lambda,\omega) & \mapsto & \big(i_\infty(\lambda,\omega),\alpha_\infty(\lambda,\omega),(\lambda,\omega)\big)\end{array}$$

such that

$$\forall(\lambda,\omega)\in\mathcal{G}_\infty^\gamma,\quad \mathcal{F}(U_\infty(\lambda,\omega))=0.$$

In addition, i_∞ is reversible and $\alpha_\infty\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{R}^d)$ with

$$\alpha_\infty(\lambda,\omega)=\omega+\mathrm{r}_\varepsilon(\lambda,\omega)\quad \textit{and}\quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_0^{q\overline{a}}. \tag{8.57}$$

Moreover, there exists a q-times differentiable function $\lambda\in(\lambda_0,\lambda_1)\mapsto\omega(\lambda,\varepsilon)\in\mathbb{R}^d$ with

$$\omega(\lambda,\varepsilon)=-\omega_{\mathrm{Eq}}(\lambda)+\bar{r}_\varepsilon(\lambda),\quad \|\bar{r}_\varepsilon\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_0^{q\overline{a}} \tag{8.58}$$

and

$$\forall\lambda\in\mathcal{C}_\infty^\varepsilon,\quad \mathcal{F}\Big(U_\infty\big(\lambda,\omega(\lambda,\varepsilon)\big)\Big)=0\quad\text{and}\quad \alpha_\infty\big(\lambda,\omega(\lambda,\varepsilon)\big)=-\omega_{\mathrm{Eq}}(\lambda),$$

where the Cantor set $\mathcal{C}_\infty^\varepsilon$ is defined by

$$\mathcal{C}_\infty^\varepsilon\triangleq\Big\{\lambda\in(\lambda_0,\lambda_1)\quad\text{s.t.}\quad\big(\lambda,\omega(\lambda,\varepsilon)\big)\in\mathcal{G}_\infty^\gamma\Big\}. \tag{8.59}$$

Proof. Putting together (8.55) and (8.53), we infer

$$\|W_{n+1}-W_n\|_{q,s_0}^{\gamma,\mathcal{O}}=\|H_{n+1}\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\leqslant C_*\varepsilon\gamma^{-1}N_n^{-a_2}.$$

Thus, the telescopic series associated with the sequence $(W_n)_{n\in\mathbb{N}}$ is convergent, so the sequence itself converges. We denote its limit

$$W_\infty\triangleq\lim_{n\to\infty}W_n\triangleq(\mathfrak{I}_\infty,\alpha_\infty-\omega,0,0)$$

and

$$U_\infty\triangleq\big(i_\infty,\alpha_\infty,(\lambda,\omega)\big)=U_0+W_\infty.$$

Passing to the limit in (8.50), one obtains the reversibility property

$$\mathfrak{S}i_\infty(\varphi)=i_\infty(-\varphi).$$

By the point $(\mathcal{P}2)_n$ of Proposition 8.1, we have for small ε

$$\forall(\lambda,\omega)\in\mathcal{G}_\infty^\gamma,\quad \mathcal{F}\Big(i_\infty(\lambda,\omega),\alpha_\infty(\lambda,\omega),(\lambda,\omega),\varepsilon\Big)=0, \tag{8.60}$$

with $\mathcal{F}$ the functional defined in (6.21). We highlight that the Cantor set $\mathcal{G}_\infty^\gamma$ depends on ε through γ and (8.2). By the point $(\mathcal{P}1)_n$ of the Proposition 8.1, we have

$$\alpha_\infty(\lambda,\omega)=\omega+\mathrm{r}_\varepsilon(\lambda,\omega)\quad\text{with}\quad\|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}N_0^{q\overline{a}}.$$

We now prove the second result and check the existence of solutions to the original Hamiltonian equation. First recall that the open set $\mathcal{O}$ is defined in (6.7) by

$$\mathcal{O}=(\lambda_0,\lambda_1)\times\mathscr{U}\quad\text{with}\quad\mathscr{U}=B(0,R_0)\quad\text{for some large }R_0>0,$$

where the ball $\mathscr{U}$ is taken to contain the equilibrium frequency vector $\lambda \mapsto \omega_{\mathrm{Eq}}(\lambda)$. According to (8.57), we deduce that for any $\lambda \in (\lambda_0, \lambda_1)$, the mapping $\omega \mapsto \alpha_\infty(\lambda, \omega)$ is invertible from $\mathscr{U}$ into its image $\alpha_\infty(\lambda, \mathscr{U})$ and we have

$$\widehat{\omega} = \alpha_\infty(\lambda, \omega) = \omega + \mathrm{r}_\varepsilon(\lambda, \omega) \Leftrightarrow \omega = \alpha_\infty^{-1}(\lambda, \widehat{\omega}) = \widehat{\omega} + \widehat{\mathrm{r}}_\varepsilon(\lambda, \widehat{\omega}).$$

This gives the identity

$$\widehat{\mathrm{r}}_\varepsilon(\lambda, \widehat{\omega}) = -\mathrm{r}_\varepsilon(\lambda, \omega),$$

which implies in turn after using successive differentiation and (8.57) that $\widehat{\mathrm{r}}_\varepsilon$ satisfies the estimate

$$\|\widehat{\mathrm{r}}_\varepsilon\|_q^{\gamma, \mathcal{O}} \lesssim \varepsilon \gamma^{-1} N_0^{q\overline{a}}. \tag{8.61}$$

We now set

$$\omega(\lambda, \varepsilon) \triangleq \alpha_\infty^{-1}(\lambda, -\omega_{\mathrm{Eq}}(\lambda)) = -\omega_{\mathrm{Eq}}(\lambda) + \overline{\mathrm{r}}_\varepsilon(\lambda) \quad \text{with} \quad \overline{\mathrm{r}}_\varepsilon(\lambda) \triangleq \widehat{\mathrm{r}}_\varepsilon\big(\lambda, -\omega_{\mathrm{Eq}}(\lambda)\big).$$

As a consequence of (8.60), if we denote

$$\mathcal{C}_\infty^\varepsilon \triangleq \Big\{\lambda \in (\lambda_0, \lambda_1) \quad \text{s.t.} \quad \big(\lambda, \omega(\lambda, \varepsilon)\big) \in \mathcal{G}_\infty^\gamma\Big\},$$

then we have

$$\forall \lambda \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\Big(U_\infty\big(\lambda, \omega(\lambda, \varepsilon)\big)\Big) = 0.$$

This gives a nontrivial reversible solution for the original Hamiltonian equation provided that $\lambda \in \mathcal{C}_\infty^\varepsilon$. Since all the derivatives up to order q of ω_{Eq} are uniformly bounded on $[\lambda_0, \lambda_1]$, see Lemma 5.3-(vi), then by chain rule and (8.61), we obtain

$$\|\overline{\mathrm{r}}_\varepsilon\|_q^{\gamma, \mathcal{O}} \lesssim \varepsilon \gamma^{-1} N_0^{q\overline{a}} \quad \text{and} \quad \|\omega(\cdot, \varepsilon)\|_q^{\gamma, \mathcal{O}} \lesssim 1 + \varepsilon \gamma^{-1} N_0^{q\overline{a}} \lesssim 1. \tag{8.62}$$

This ends the proof of Corollary 8.1. ☐

8.2 Measure of the final Cantor set

The purpose of this final section is to give a lower bound of the Lebesgue measure of the Cantor set $\mathcal{C}_\infty^\varepsilon$ constructed in Corollary 8.1 via (8.59). We show that this set is massive and asymptotically when $\varepsilon \to 0$ it tends to be of full measure in (λ_0, λ_1). Note that Corollary 8.1 allows us to write the Cantor set $\mathcal{C}_\infty^\varepsilon$ in the following form

$$\mathcal{C}_\infty^\varepsilon = \bigcap_{n \in \mathbb{N}} \mathcal{C}_n^\varepsilon \quad \text{where} \quad \mathcal{C}_n^\varepsilon \triangleq \Big\{\lambda \in (\lambda_0, \lambda_1) \quad \text{s.t} \quad \big(\lambda, \omega(\lambda, \varepsilon)\big) \in \mathcal{A}_n^\gamma\Big\}. \tag{8.63}$$

The sets $\mathcal{A}_n^\gamma$ and the perturbed frequency vector $\omega(\lambda, \varepsilon)$ are respectively defined in Proposition 8.1 and in (8.57). The main result of this section reads as follows.

Proposition 8.2. *Let* q_0 *be defined as in Lemma* 5.5 *and assume that* (8.1) *and* (8.2) *hold with* $q = q_0 + 1$. *Assume the additional conditions*

$$\begin{cases} \tau_1 > d q_0 \\ \tau_2 > \tau_1 + d q_0 \\ \upsilon = \frac{1}{q_0 + 3}. \end{cases} \tag{8.64}$$

Then, there exists $C>0$ *such that*

$$\left|\mathcal{C}_{\infty}^{\varepsilon}\right| \geqslant (\lambda_1-\lambda_0) - C\varepsilon^{\frac{av}{q_0}}.$$

In particular,

$$\lim_{\varepsilon\to 0}\left|\mathcal{C}_{\infty}^{\varepsilon}\right| = \lambda_1-\lambda_0.$$

The remainder of this section is devoted to the proof of Proposition 8.2. We shall begin by giving the proof using some a priori results. These results will be proved later in Lemmata 8.1, 8.2 and 8.3. We first give a short insight about the strategy to prove Proposition 8.2. The idea is to measure the complementary set of $\mathcal{C}_{\infty}^{\varepsilon}$ in (λ_0,λ_1). To proceed with, we write

$$(\lambda_0,\lambda_1)\setminus\mathcal{C}_{\infty}^{\varepsilon} = \big((\lambda_0,\lambda_1)\setminus\mathcal{C}_{0}^{\varepsilon}\big)\sqcup\bigsqcup_{n=0}^{\infty}\big(\mathcal{C}_{n}^{\varepsilon}\setminus\mathcal{C}_{n+1}^{\varepsilon}\big). \tag{8.65}$$

The measure of each set appearing in (8.65) is estimated by using Lemma 5.6. We shall now give the proof of Proposition 8.2.

Proof. By choosing R_0 large enough, one can ensure using (8.58) that

$$\forall\lambda\in(\lambda_0,\lambda_1),\quad \omega(\lambda,\varepsilon)\in\mathscr{U}=B(0,R_0).$$

Indeed, $\mathscr{U}$ contains by construction the curve $\lambda\in(\lambda_0,\lambda_1)\mapsto\pm\omega_{\mathrm{Eq}}(\lambda)$ and by (8.58) and (8.2), one has

$$\sup_{\lambda\in(\lambda_0,\lambda_1)}|\omega(\lambda,\varepsilon)+\omega_{\mathrm{Eq}}(\lambda)|\leqslant\|\bar{\mathrm{r}}_{\varepsilon}\|_{q}^{\gamma,\mathcal{O}}\leqslant C\varepsilon\gamma^{-1}N_0^{q\overline{a}}=C\varepsilon^{1-a(1+q\overline{a})}.$$

Now, the conditions (8.1) and (8.2) imply in particular

$$0<a<\frac{1}{1+q\overline{a}}\cdot$$

Hence, by taking ε small enough, we find

$$\sup_{\lambda\in(\lambda_0,\lambda_1)}|\omega(\lambda,\varepsilon)+\omega_{\mathrm{Eq}}(\lambda)|\leqslant\|\bar{\mathrm{r}}_{\varepsilon}\|_{q}^{\gamma,\mathcal{O}}\leqslant 1.$$

As a consequence,

$$\mathcal{C}_{0}^{\varepsilon}=(\lambda_0,\lambda_1).$$

By (8.65), we can write

$$\begin{aligned}\left|(\lambda_0,\lambda_1)\setminus\mathcal{C}_{\infty}^{\varepsilon}\right| &\leqslant\sum_{n=0}^{\infty}\left|\mathcal{C}_{n}^{\varepsilon}\setminus\mathcal{C}_{n+1}^{\varepsilon}\right|\\ &\triangleq\sum_{n=0}^{\infty}\mathcal{S}_n.\end{aligned} \tag{8.66}$$

According to the notation introduced in Proposition 7.5 and Proposition 7.4 one may write

$$\begin{aligned}\mu_j^{\infty,n}(\lambda,\varepsilon) &\triangleq \mu_j^{\infty}\big(\lambda,\omega(\lambda,\varepsilon),i_n\big)\\ &=\Omega_j(\lambda)+jr^{1,n}(\lambda,\varepsilon)+r_j^{\infty,n}(\lambda,\varepsilon),\end{aligned} \tag{8.67}$$

with

$$\begin{aligned} r^{1,n}(\lambda,\varepsilon) &\triangleq c_n(\lambda,\varepsilon)-\Omega-I_1(\lambda)K_1(\lambda),\\ c_n(\lambda,\varepsilon) &\triangleq c_{i_n}(\lambda,\omega(\lambda,\varepsilon)),\\ r_j^{\infty,n}(\lambda,\varepsilon) &\triangleq r_j^{\infty}\big(\lambda,\omega(\lambda,\varepsilon),i_n\big). \end{aligned}$$

Coming back to (8.63) and using the Cantor sets introduced in Proposition 7.5, Proposition 7.6 and Proposition 7.2 one obtains by construction that for any $n\in\mathbb{N}$,

$$\mathcal{C}_n^{\varepsilon}\setminus\mathcal{C}_{n+1}^{\varepsilon}=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n),\tag{8.68}$$

with

$$\begin{aligned} \mathcal{R}_{l,j}^{(0)}(i_n) &\triangleq \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)|\leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},\\ \mathcal{R}_{l,j,j_0}(i_n) &\triangleq \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)|\leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\},\\ \mathcal{R}_{l,j}^{(1)}(i_n) &\triangleq \Big\{\lambda\in\mathcal{C}_n^{\varepsilon}\quad\text{s.t.}\quad |\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)|\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}. \end{aligned}$$

Notice that using the inclusion

$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C})$$

and the fact that $q=q_0+1$, one gets that for all $n\in\mathbb{N}$ and $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$, the curves

$$\begin{aligned} &\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon),\\ &\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon),\\ &\lambda\mapsto\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon) \end{aligned}$$

have a C^{q_0} regularity. Then, applying Lemma 5.6 combined with Lemma 8.3 gives for any $n\in\mathbb{N}$,

$$\begin{aligned} \Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big| &\lesssim \gamma^{\frac{\upsilon}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\ \Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\ \Big|\mathcal{R}_{l,j,j_0}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j-j_0\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}. \end{aligned}\tag{8.69}$$

Let us now move to the estimate of $\mathcal{S}_0$ and $\mathcal{S}_1$ defined in (8.66) that should be treated differently from the other terms. This is related to the discussion done at the beginning of the proof of Lemma 8.1 dealing with the validity of the estimate (8.74). By using Lemma 8.2, we find for all $k\in\{0,1\}$,

$$\mathcal{S}_k\lesssim\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |j|\leqslant C_0\langle l\rangle,|l|\leqslant N_k}}\Big|\mathcal{R}_{l,j}^{(0)}(i_k)\Big|+\sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |j-j_0|\leqslant C_0\langle l\rangle,|l|\leqslant N_k\\ \min(|j|,|j_0|)\leqslant c_2\gamma_1^{-\upsilon}\langle l\rangle^{\tau_1}}}\Big|\mathcal{R}_{l,j,j_0}(i_k)\Big|+\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |j|\leqslant C_0\langle l\rangle,|l|\leqslant N_k}}\Big|\mathcal{R}_{l,j}^{(1)}(i_k)\Big|.\tag{8.70}$$

Plugging (8.69) into (8.70) yields for all $k \in \{0,1\}$,

$$\begin{aligned}\mathcal{S}_k \lesssim \gamma^{\frac{1}{q_0}}\Bigg(&\sum_{|j|\leqslant C_0\langle l\rangle} |j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}} + \sum_{\substack{|j-j_0|\leqslant C_0\langle l\rangle\\ \min(|j|,|j_0|)\leqslant c_2\gamma^{-\upsilon}\langle l\rangle^{\tau_1}}} |j-j_0|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}\Bigg)\\ &+\gamma^{\frac{\upsilon}{q_0}}\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}\max_{k\in\{0,1\}}\mathcal{S}_k &\lesssim \gamma^{\frac{1}{q_0}}\Bigg(\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg)+\gamma^{\frac{\upsilon}{q_0}}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}\\ &\lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.\end{aligned}\tag{8.71}$$

Notice that the last estimate is obtained provided that we choose the parameters τ_1 and τ_2 in the following way in order to make the series convergent

$$\tau_1 > d\,q_0 \quad \text{and} \quad \tau_2 > \tau_1 + d\,q_0.\tag{8.72}$$

This condition is exactly what we required in (8.64). Concerning the estimate of $\mathcal{S}_n$ for $n \geqslant 2$ in (8.66) we may use Lemma 8.1 and Lemma 8.2, in order to get

$$\mathcal{S}_n \leqslant \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |j|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n}}\left|\mathcal{R}_{l,j}^{(0)}(i_n)\right| + \sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |j-j_0|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n\\ \min(|j|,|j_0|)\leqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}}}\left|\mathcal{R}_{l,j,j_0}(i_n)\right| + \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |j|\leqslant C_0\langle l\rangle, N_{n-1}<|l|\leqslant N_n}}\left|\mathcal{R}_{l,j}^{(1)}(i_n)\right|.$$

Remark that if $|j-j_0| \leqslant C_0\langle l\rangle$ and $\min(|j|,|j_0|) \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\max(|j|,|j_0|) = \min(|j|,|j_0|) + |j-j_0| \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1} + C_0\langle l\rangle \lesssim \gamma^{-\upsilon}\langle l\rangle^{\tau_1}.$$

Therefore, (8.69) implies

$$\mathcal{S}_n \lesssim \gamma^{\frac{1}{q_0}}\Bigg(\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{|l|>N_{n-1}}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg)+\gamma^{\frac{\upsilon}{q_0}}\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}.$$

Under the assumption, we obtain (8.72)

$$\sum_{n=2}^{\infty}\mathcal{S}_n \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.\tag{8.73}$$

Plugging (8.73) and (8.71) into (8.66) gives

$$\left|(\lambda_0,\lambda_1)\setminus\mathcal{C}_\infty^{\varepsilon}\right| \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}$$

provided that the condition (8.72) is satisfied. The condition (8.64) implies that

$$\min\left(\tfrac{\upsilon}{q_0},\tfrac{1}{q_0}-\upsilon\right)=\tfrac{\upsilon}{q_0}.$$

We then find, since $\gamma=\varepsilon^a$ according to (8.2),

$$\Big|(\lambda_0,\lambda_1)\setminus\mathcal{C}_\infty^\varepsilon\Big|\lesssim\varepsilon^{\frac{av}{q_0}}.$$

This completes the proof of Proposition 8.2. ☐

Now we are left to prove Lemma 8.1 and Lemma 8.2 used in the proof of Proposition 8.2.

Lemma 8.1. *Let $n\in\mathbb{N}\setminus\{0,1\}$ and $l\in\mathbb{Z}^d$ such that $|l|\leqslant N_{n-1}$. Then the following assertions hold true.*

(i) For $j\in\mathbb{Z}$ with $(l,j)\neq(0,0)$, we get $\mathcal{R}_{l,j}^{(0)}(i_n)=\varnothing$.

(ii) For $(j,j_0)\in(\mathbb{S}_0^c)^2$ with $(l,j)\neq(0,j_0)$, we get $\mathcal{R}_{l,j,j_0}(i_n)=\varnothing$.

(iii) For $j\in\mathbb{S}_0^c$, we get $\mathcal{R}_{l,j}^{(1)}(i_n)=\varnothing$.

(iv) For any $n\in\mathbb{N}\setminus\{0,1\}$,

$$\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\cup\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\cup\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n).$$

Proof. In all the proof, we shall use the following estimate coming from (8.11), namely, for all $n\geqslant 2$,

$$\begin{aligned}\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}&\leqslant\|U_n-U_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\&\leqslant\|H_n\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\\&\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}.\end{aligned}\tag{8.74}$$

The fact that the previous estimate is valid only for $n\geqslant 2$ is the reason why we had to treat the cases of $\mathcal{S}_0$ and $\mathcal{S}_1$ sparately in the proof of Proposition 8.2.

(i) We begin by proving that if $|l|\leqslant N_{n-1}$ and $(l,j)\neq(0,0)$, then $\mathcal{R}_{l,j}^{(0)}(i_n)\subset\mathcal{R}_{l,j}^{(0)}(i_{n-1})$. Assume for a while this inclusion and let us check how this implies that $\mathcal{R}_{l,j}^{(0)}(i_n)=\varnothing$. In view of (8.68) one obtains

$$\mathcal{R}_{l,j}^{(0)}(i_n)\subset\mathcal{R}_{l,j}^{(0)}(i_{n-1})\subset\mathcal{C}_{n-1}^\varepsilon\setminus\mathcal{C}_n^\varepsilon.$$

Now (8.68) implies in particular $\mathcal{R}_{l,j}^{(0)}(i_n)\subset\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon$ and thus we conclude

$$\mathcal{R}_{l,j}^{(0)}(i_n)\subset\big(\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon\big)\cap\big(\mathcal{C}_{n-1}^\varepsilon\setminus\mathcal{C}_n^\varepsilon\big)=\varnothing.$$

We now turn to the proof of the inclusion. Let us consider $\lambda\in\mathcal{R}_{l,j}^{(0)}(i_n)$. By construction, we get in particular that $\lambda\in\mathcal{C}_n^\varepsilon\subset\mathcal{C}_{n-1}^\varepsilon$. Moreover, by the triangle inequality, we obtain

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+jc_{n-1}(\lambda,\varepsilon)\big|&\leqslant\big|\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)\big|+|j|\big|c_n(\lambda,\varepsilon)-c_{n-1}(\lambda,\varepsilon)\big|\\&\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C|j|\|c_{i_n}-c_{i_{n-1}}\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

Therefore, combining (7.21), (8.74), (8.2) and the fact tht $\sigma_4\geqslant 2$, we infer

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+jc_{n-1}(\lambda,\varepsilon)\big|&\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon\langle j\rangle\|i_n-i_{n-1}\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}\\&\leqslant\tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon^{2-a}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

In view of the definition of γ_n in Proposition 8.1-$(\mathcal{P}2)_n$ one gets

$$\exists c_0>0,\quad \forall n\in\mathbb{N},\quad \gamma_{n+1}^{\upsilon}-\gamma_n^{\upsilon}\leqslant -c_0\,\gamma^{\upsilon}2^{-n}.$$

Now remark that (8.64), (8.1) and (8.2) imply

$$2-a-a\upsilon>1\quad\text{and}\quad a_2>\tau_1,\tag{8.75}$$

and therefore one gets $\sup\limits_{n\in\mathbb{N}}2^nN_{n-1}^{-a_2+\tau_1}<\infty$. It follows that, for ε small enough and $|l|\leqslant N_{n-1}$,

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+jc_{n-1}(\lambda,\varepsilon)\big|&\leqslant \tfrac{4\gamma_n^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\tfrac{\langle j\rangle\gamma^{\upsilon}}{2^n\langle l\rangle^{\tau_1}}\Big(-4c_0+C\varepsilon 2^nN_{n-1}^{-a_2+\tau_1}\Big)\\&\leqslant \tfrac{4\gamma_n^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.\end{aligned}$$

Consequently $\lambda\in\mathcal{R}_{l,j}^{(0)}(i_{n-1})$ and this achieves the proof.
(ii) Let $(j,j_0)\in(\mathbb{S}_0^c)^2$ and $(l,j)\neq(0,j_0)$. If $j=j_0$ then by construction $\mathcal{R}_{l,j_0,j_0}(i_n)=\mathcal{R}_{l,0}^{(0)}(i_n)$ and then the result follows from the point (i). Now let us discuss the case when $j\neq j_0$. Similarly to the point (i), in order to get the result it is enough to check that $\mathcal{R}_{l,j,j_0}(i_n)\subset\mathcal{R}_{l,j,j_0}(i_{n-1})$. Let $\lambda\in\mathcal{R}_{l,j,j_0}(i_n)$ then from the definition of this set introduced in (8.68) we deduce that $\lambda\in\mathcal{C}_n^{\varepsilon}\subset\mathcal{C}_{n-1}^{\varepsilon}$ and

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|\leqslant\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}+\varrho_{j,j_0}^n(\lambda,\varepsilon),\tag{8.76}$$

where we set

$$\varrho_{j,j_0}^n(\lambda,\varepsilon)\triangleq\big|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)-\mu_j^{\infty,n-1}(\lambda,\varepsilon)+\mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|.$$

Then coming back to (8.67), one gets

$$\begin{aligned}\varrho_{j,j_0}^n(\lambda,\varepsilon)\leqslant|j-j_0|\big|r^{1,n}(\lambda,\varepsilon)-r^{1,n-1}(\lambda,\varepsilon)\big|&+\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_j^{\infty,n-1}(\lambda,\varepsilon)\big|\\&+\big|r_{j_0}^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|.\end{aligned}\tag{8.77}$$

In view of (7.170), (8.74), (8.2) and the fact that $\sigma_4\geqslant\sigma_3$, one obtains

$$\begin{aligned}\big|r^{1,n}(\lambda,\varepsilon)-r^{1,n-1}(\lambda,\varepsilon)\big|&\lesssim\varepsilon\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_3}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon^2\gamma^{-1}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2-a}N_{n-1}^{-a_2}.\end{aligned}$$

In a similar line, using (7.241), (8.74) and (8.2) yields

$$\begin{aligned}\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_j^{\infty,n-1}(\lambda,\varepsilon)\big|&\lesssim\varepsilon\gamma^{-1}\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon^2\gamma^{-2}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Inserting the preceding two estimates into (8.77) gives

$$\varrho_{j,j_0}^n(\lambda,\varepsilon)\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\tag{8.78}$$

Putting together (8.78) and (8.76) and using $\gamma_{n+1}=\gamma_n-\varepsilon^a 2^{-n-1}$, we deduce

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big| \leqslant{}& \tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-\varepsilon^a\langle j-j_0\rangle 2^{-n}\langle l\rangle^{-\tau_2}\\ &+C\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Since $|l|\leqslant N_{n-1}$,we can write

$$-\varepsilon^a 2^{-n}\langle l\rangle^{-\tau_2}+C\varepsilon^{2(1-a)}N_{n-1}^{-a_2}\leqslant \varepsilon^a 2^{-n}\langle l\rangle^{-\tau_2}\Big(-1+C\varepsilon^{2-3a}2^n N_{n-1}^{-a_2+\tau_2}\Big).$$

Now remark that (8.1) and (8.2) yield in particular

$$a_2>\tau_2\quad\text{and}\quad a<\tfrac{2}{3}. \tag{8.79}$$

Hence, we find for ε small enough

$$\forall\, n\in\mathbb{N},\quad -1+C\varepsilon^{2-3a}2^n N_{n-1}^{-a_2+\tau_2}\leqslant 0$$

and therefore

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n-1}(\lambda,\varepsilon)\big|\leqslant \tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Consequently, $\lambda\in\mathcal{R}_{l,j,j_0}(i_{n-1})$ and the proof of the second point is now achieved.

(iii) Let $j\in\mathbb{S}_0^c$. In particular, one has $(l,j)\neq(0,0)$. We shall first prove that if $|l|\leqslant N_{n-1}$ and then $\mathcal{R}_{l,j}^{(1)}(i_n)\subset\mathcal{R}_{l,j}^{(1)}(i_{n-1})$. As in the point (i) this implies that $\mathcal{R}_{l,j}^{(1)}(i_n)=\varnothing$. Remind that the set $\mathcal{R}_{l,j}^{(1)}(i_n)$ is defined below (8.68). Consider $\lambda\in\mathcal{R}_{l,j}^{(1)}(i_n)$ then by construction $\lambda\in\mathcal{C}_n^\varepsilon\subset\mathcal{C}_{n-1}^\varepsilon$. Now by the triangle inequality we may write in view of (7.242) and (8.74) and the choice $\gamma=\varepsilon^a$

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)\big| &\leqslant \big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)\big|+|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_j^{\infty,n-1}(\lambda,\varepsilon)|\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon\gamma^{-1}|j|\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}+C\varepsilon^{2(1-a)}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Since $\gamma_{n+1}=\gamma_n-\varepsilon^a 2^{-n-1}$ and $|l|\leqslant N_{n-1}$, then

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)\big|\leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}+\tfrac{\langle j\rangle\varepsilon^a}{2^{n+1}\langle l\rangle^{\tau_1}}\Big(-1+\varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\Big).$$

Notice that (8.79) implies in particular

$$a_2>\tau_1\quad\text{and}\quad a<\tfrac{2}{3} \tag{8.80}$$

and taking ε small enough we find that

$$\forall\, n\in\mathbb{N},\quad -1+\varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\leqslant 0,$$

which implies in turn that

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n-1}(\lambda,\varepsilon)\big|\leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Consequently, $\lambda\in\mathcal{R}_{l,j}^{(1)}(i_{n-1})$ and this ends the proof of the third point.

(iv) It is an immediate consequence of (8.68) and the points (i)-(ii) and (iii) of Lemma 8.1. ☐

The next result deals with necessary conditions such that the sets in (8.68) are nonempty.

Lemma 8.2. *There exists ε_0 such that for any $\varepsilon\in[0,\varepsilon_0]$ and $n\in\mathbb{N}$ the following assertions hold true.*

(i) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}$*. If* $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$*, then* $|j|\leqslant C_0\langle l\rangle$*.*

(ii) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$*. If* $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$*, then* $|j-j_0|\leqslant C_0\langle l\rangle$*.*

(iii) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$*. If* $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$*, then* $|j|\leqslant C_0\langle l\rangle$*.*

(iv) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$*. There exists* $c_2>0$ *such that if* $\min(|j|,|j_0|)\geqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$*, then*

$$\mathcal{R}_{l,j,j_0}(i_n)\subset\mathcal{R}_{l,j-j_0}^{(0)}(i_n).$$

Proof. **(i)** Assume $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$, then we can find $\lambda\in(\lambda_0,\lambda_1)$ such that, using triangle and Cauchy-Schwarz inequalities,

$$\begin{aligned}|c_n(\lambda,\varepsilon)||j|&\leqslant 4|j|\gamma_{n+1}^{\upsilon}\langle l\rangle^{-\tau_1}+|\omega(\lambda,\varepsilon)\cdot l|\\&\leqslant 4|j|\gamma_{n+1}^{\upsilon}+C\langle l\rangle\\&\leqslant 8\varepsilon^{a\upsilon}|j|+C\langle l\rangle,\end{aligned}$$

where we have used $\gamma=\varepsilon^a$ and the fact that $(\lambda,\varepsilon)\mapsto\omega(\lambda,\varepsilon)$ is bounded. Notice that

$$c_n(\lambda,\varepsilon)=\Omega+I_1(\lambda)K_1(\lambda)+r^{1,n}(\lambda,\varepsilon)\quad\text{and}\quad\inf_{\lambda\in(\lambda_0,\lambda_1)}\big(\Omega+I_1(\lambda)K_1(\lambda)\big)>\Omega.$$

Then, from (7.17), (7.239) and Proposition 8.1 $(\mathcal{P}1)_n$, we obtain

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad\sup_{n\in\mathbb{N}}\sup_{\lambda\in(\lambda_0,\lambda_1)}|\partial_\lambda^k r^{1,n}(\lambda,\varepsilon)|&\leqslant\gamma^{-k}\sup_{n\in\mathbb{N}}\|r^{1,n}\|_q^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-k}\\&\lesssim\varepsilon^{1-ak}.\end{aligned}\tag{8.81}$$

Thus, by choosing ε small enough, we can ensure by (8.81)

$$\inf_{n\in\mathbb{N}}\inf_{\lambda\in(\lambda_0,\lambda_1)}|c_n(\lambda,\varepsilon)|\geqslant\tfrac{\Omega}{2}.$$

Hence, by taking ε small enough we find that $|j|\leqslant C_0\langle l\rangle$ for some $C_0>0$.
(ii) In the case $j=j_0$ we get by definition $\mathcal{R}_{l,j_0,j_0}(i_n)=\mathcal{R}_{l,0}^{(0)}(i_n)$, and then we use the point (i). In what follows we take $j\neq j_0$ and we assume that $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$ then there exists $\lambda\in(\lambda_0,\lambda_1)$ such that

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)|&\leqslant 2\gamma_{n+1}|j-j_0|\langle l\rangle^{-\tau_2}+|\omega(\lambda,\varepsilon)\cdot l|\\&\leqslant 2\gamma_{n+1}|j-j_0|+C\langle l\rangle\\&\leqslant 4\varepsilon^a|j-j_0|+C\langle l\rangle.\end{aligned}$$

Similarly to (8.81), we can prove

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad\sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}\sup_{\lambda\in(\lambda_0,\lambda_1)}|j||\partial_\lambda^k r_j^{\infty,n}(\lambda,\varepsilon)|&\leqslant\gamma^{-k}\sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^{\infty,n}\|_q^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1-k}\\&\lesssim\varepsilon^{1-a(1+k)}.\end{aligned}\tag{8.82}$$

By using the triangle inequality, Lemma 5.3-(v), (8.81) and (8.82) we get for $j \neq j_0$,

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)| &\geqslant |\Omega_j(\lambda)-\Omega_{j_0}(\lambda)|-|r^{1,n}(\lambda,\varepsilon)||j-j_0|-|r_j^{\infty,n}(\lambda,\varepsilon)|-|r_{j_0}^{\infty,n}(\lambda,\varepsilon)|\\ &\geqslant \big(C_0-C\varepsilon^{1-a}\big)|j-j_0|\\ &\geqslant \tfrac{C_0}{2}|j-j_0|\end{aligned}$$

provided that ε is small enough. Putting together the previous inequalities yields for ε small enough $|j-j_0|\leqslant C_0\langle l\rangle$, for some $C_0>0$.

(iii) First remark that the case $j=0$ is trivial. Now for $j\neq 0$ we assume that $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$ then there exists $\lambda\in(\lambda_0,\lambda_1)$ such that

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)| &\leqslant \gamma_{n+1}|j|\langle l\rangle^{\tau_1}+|\omega(\lambda,\varepsilon)\cdot l|\\ &\leqslant 2\varepsilon^a|j|+C\langle l\rangle.\end{aligned}$$

Using the definition (8.67) combined with the triangle inequality, Lemma 5.3-(iv), (8.81) and (8.82), we get

$$\begin{aligned}|\mu_j^{\infty,n}(\lambda,\varepsilon)| &\geqslant \Omega|j|-|j||r^{1,n}(\lambda,\varepsilon)|-|r_j^{\infty,n}(\lambda,\varepsilon)|\\ &\geqslant \Omega|j|-C\varepsilon^{1-a}|j|.\end{aligned}$$

Combining the previous two inequalities and the second condition in (8.80) implies

$$\big(\Omega-C\varepsilon^{1-a}-2\varepsilon^a\big)|j|\leqslant C\langle l\rangle.$$

Thus, by taking ε small enough we obtain $|j|\leqslant C_0\langle l\rangle$, for some $C_0>0$.

(iv) First notice that the case $j=j_0$ is trivial and follows from the definition (8.68). Let $j\neq j_0$ and $\lambda\in\mathcal{R}_{l,j,j_0}(i_n)$, then by definition

$$\big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|\leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Combining (8.67) and (5.14) with the triangle inequality we infer

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\big| &\leqslant \big|\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|\\ &\quad+|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)|+\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|.\end{aligned}$$

Thus, we find

$$\begin{aligned}\big|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\big| \leqslant &\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}+|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)|\\ &+\big|r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big|.\end{aligned}\tag{8.83}$$

Without loss of generality, we can assume that $|j_0|\geqslant|j|$ and remind that $j\neq j_0$. Then, from (5.24) and (5.22), we easily find

$$\begin{aligned}|jI_j(\lambda)K_j(\lambda)-j_0I_{j_0}(\lambda)K_{j_0}(\lambda)| &\leqslant |j|\big|I_j(\lambda)K_j(\lambda)-I_{j_0}(\lambda)K_{j_0}(\lambda)\big|+|j-j_0|\big|I_{j_0}(\lambda)K_{j_0}(\lambda)\big|\\ &\leqslant \tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.\end{aligned}$$

Applying (7.239), we find for $j \neq j_0 \in \mathbb{S}_0^c$,

$$\begin{aligned}\left|r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\right| &\leqslant C\varepsilon^{1-a}\big(|j|^{-1}+|j_0|^{-1}\big)\\ &\leqslant C\varepsilon^{1-a}\tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.\end{aligned}$$

Plugging the preceding estimates into (8.83) yields

$$\left|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\right| \leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}+C\tfrac{\langle j-j_0\rangle}{\min(|j|,|j_0|)}.$$

Therefore, if we assume $\min(|j|,|j_0|) \geqslant \frac{1}{2}C\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$ and $\tau_2>\tau_1$, then we deduce

$$\left|\omega(\lambda,\varepsilon)\cdot l+(j-j_0)c_n(\lambda,\varepsilon)\right| \leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j-j_0\rangle}{\langle l\rangle^{\tau_1}}.$$

This ends the proof of the lemma by taking $c_2=\frac{C}{2}$. ☐

We shall now establish that the perturbed frequencies $\omega(\lambda,\varepsilon)$ satisfy the Rüssmann conditions. This is done by a perturbation argument from the equilibrium linear frequencies $\omega_{\mathrm{Eq}}(\lambda)$ for which we already know by Lemma 5.5 that they satisfy the transversality conditions.

Lemma 8.3. *Let q_0, C_0 and ρ_0 as in Lemma 5.5. There exist $\varepsilon_0>0$ small enough such that for any $\varepsilon\in[0,\varepsilon_0]$ the following assertions hold true.*

(i) For all $l\in\mathbb{Z}^d\setminus\{0\}$, we have

$$\inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l\right)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(ii) For all $(l,j)\in\mathbb{Z}^{d+1}\setminus\{(0,0)\}$ such that $|j|\leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)\big)| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iii) For all $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$ such that $|j|\leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)\big)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iv) For all $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ such that $|j-j_0|\leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{\lambda\in[\lambda_0,\lambda_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

Proof. **(i)** From the triangle and Cauchy-Schwarz inequalities together with (8.62), (8.2) and Lemma 5.5-(i), we deduce

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l\right)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l\right)\right|-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\left(\mathrm{\bar{r}}_\varepsilon(\lambda)\cdot l\right)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon\gamma^{-1-q}N_0^{q\overline{a}}\langle l\rangle\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

provided that ε is small enough and

$$1-a(1+q+q\overline{a})>0. \tag{8.84}$$

Notice that the condition (8.84) is automatically satisfied by (8.2) and (8.1).

(ii) As before, using the triangle and Cauchy-Schwarz inequalities combined with (8.62), (8.81), Lemma 5.5-(ii) and the fact that $|j|\leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega(\lambda,\varepsilon)\cdot l+jc_n(\lambda,\varepsilon)\right)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega_{\mathrm{Eq}}(\lambda)\cdot l+j(\Omega+I_1(\lambda)K_1(\lambda))\big)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\left(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+jr^{1,n}(\lambda,\varepsilon)\right)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-aq}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough and with the condition (8.84).

(iii) As before, performing the triangle and Cauchy-Schwarz inequalities combined with (8.62), (8.81), (8.82), Lemma 5.5-(iii) and the fact that $|j|\leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\left(\omega_{\mathrm{Eq}}(\lambda)\cdot l+\Omega_j(\lambda)\right)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\big(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+jr^{1,n}(\lambda,\varepsilon)+r_j^{\infty,n}(\lambda,\varepsilon))\big)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough with the condition (8.84).

(iv) Arguing as in the preceding point, using (8.81), (8.82), Lemma 5.5-(iv)-(v) and the fact that $0<|j-j_0|\leqslant C_0\langle l\rangle$ (notice that the case $j=j_0$ is trivial), we have

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega(\lambda,\varepsilon)\cdot l+\mu_j^{\infty,n}(\lambda,\varepsilon)-\mu_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_\lambda^k\big(\omega_{\mathrm{Eq}}(\lambda)\cdot l+\Omega_j(\lambda)-\Omega_{j_0}(\lambda)\big)\right|\\ &\quad-\max_{k\in[\![0,q]\!]}\left|\partial_\lambda^k\big(\bar{\mathrm{r}}_\varepsilon(\lambda)\cdot l+(j-j_0)r^{1,n}(\lambda,\varepsilon)+r_j^{\infty,n}(\lambda,\varepsilon)-r_{j_0}^{\infty,n}(\lambda,\varepsilon)\big)\right|\\ &\geqslant \rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j-j_0|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough. This ends the proof of Lemma 8.3. ☐

PART II

Boundary effects on the emergence of quasi-periodic solutions for Euler equations

> L'essence des mathématiques, c'est la liberté.
>
> Georg Cantor

This part is devoted to the proof of Theorem 2.2. We also refer to Theorem 9.1 below for a more precise statement. This result is the subject of the following preprint [89] which is submitted to the journal Annals of PDE and entitled "Boundary effects on the emergence of quasi-periodic solutions for Euler equations".

Abstract

We highlight the importance of the boundary effects on the construction of quasi-periodic vortex patches solutions close to Rankine vortices and whose existence is not known in the whole space due to the resonances of the linear frequencies. Availing of the lack of invariance by radial dilation of Euler equations in the unit disc and using a Nash-Moser implicit function iterative scheme we show the existence of such structures when the radius of the Rankine vortex belongs to a suitable massive Cantor-like set with almost full Lebesgue measure.

9 Introduction

We shall now present the second result of this thesis and discuss the key ideas of its proof. We first consider a polar parametrization of a patch boundary close to the stationary solution $b\mathbb{D}$, namely

$$z(t,\theta)\triangleq R(b,t,\theta)e^{\mathrm{i}\theta},\qquad R(b,t,\theta)\triangleq\sqrt{b^2+2r(t,\theta)}.$$

The quantity of interest is the radial deformation r assumed to be of small size. We emphasize that our ansatz is slightly different from the one in the papers [87, 101] where the parametrization is written in a rotating frame with an angular velocity Ω to remedy to the degeneracy of the first frequency. This is not the case in our context due to the non-degeneracy of the first frequency according to (1.22). As explained in Lemma 10.1 and Proposition 10.1, the radial deformation solves a nonlinear and nonlocal transport PDE which admits a Hamiltonian formulation in the form

$$\partial_t r=\tfrac{1}{2}\partial_\theta\nabla E(r),\tag{9.1}$$

where E is the kinetic energy related to the stream function given by (1.4). In view of Lemma 11.1, the linearized operator at a state r close to the Rankine patch $b\mathbb{D}$ takes the form

$$\mathcal{L}_r=\partial_t+\partial_\theta\Big(V_r\cdot+\mathbf{L}_r-\mathbf{S}_r\Big),\tag{9.2}$$

where

$$V_r(b,t,\theta)\triangleq\tfrac{1}{2}\int_{\mathbb{T}}\tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)}d\eta-\tfrac{1}{R(b,t,\theta)}\int_{\mathbb{T}}\log\big(A_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta\tag{9.3}$$

$$-\tfrac{1}{R^3(b,t,\theta)}\int_{\mathbb{T}}\log\big(B_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta,\tag{9.4}$$

$\mathbf{L}_r$ is a nonlocal operator in the form

$$\mathbf{L}_r(\rho)(b,t,\theta)\triangleq\int_{\mathbb{T}}\rho(t,\eta)\log\left(A_r(b,t,\theta,\eta)\right)d\eta,\quad A_r(b,t,\theta,\eta)\triangleq\big|R(b,t,\theta)e^{\mathrm{i}\theta}-R(b,t,\eta)e^{\mathrm{i}\eta}\big|\tag{9.5}$$

and $\mathbf{S}_r$ is a smoothing nonlocal operator in the form

$$\mathbf{S}_r(\rho)(b,t,\theta)\triangleq\int_{\mathbb{T}}\rho(t,\eta)\log\left(B_r(b,t,\theta,\eta)\right)d\eta,\quad B_r(b,t,\theta,\eta)\triangleq\left|1-R(b,t,\theta)R(b,t,\eta)e^{\mathrm{i}(\eta-\theta)}\right|.\tag{9.6}$$

The operator $\mathbf{L}_r$ is of order zero and reflects the planar Euler action. Moreover, we observe two boundary effects of $\mathbb{D}$. The first one is quasi-linear in the transport part through the last term of V_r, but with a smoothing action. The second one is given by the operator $\mathbf{S}_r$ which is smoothing since it involves a smooth kernel. At the equilibrium state $r=0$, the linearized operator is a Fourier multiplier given by

$$\mathcal{L}_0=\partial_t+\tfrac{1}{2}\partial_\theta+\partial_\theta\mathcal{K}_{1,b}*\cdot-\partial_\theta\mathcal{K}_{2,b}*\cdot,$$

where

$$\mathcal{K}_{1,b}(\theta)\triangleq\tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right)\quad\text{and}\quad\mathcal{K}_{2,b}(\theta)\triangleq\log\left(\left|1-b^2e^{\mathrm{i}\theta}\right|\right).$$

Notice that the convolution with the kernel $\partial_\theta\mathcal{K}_{1,b}$ is exactly the Hilbert transform in the periodic setting. From direct computations, we may show that the kernel of $\mathcal{L}_0$ is given by the set of functions in the form

$$(t,\theta)\mapsto\sum_{j\in\mathbb{Z}^*}r_je^{\mathrm{i}(j\theta-\Omega_j(b)t)},$$

where

$$\forall j\in\mathbb{Z}^*,\quad\Omega_j(b)\triangleq\tfrac{sgn(j)}{2}\left(|j|-1+b^{2|j|}\right),\tag{9.7}$$

where we denote by sgn the sign function. Consider a finite number of Fourier modes

$$\mathbb{S}=\left\{j_1,\ldots,j_d\right\}\subset\mathbb{N}^*\quad\text{with}\quad 1\leqslant j_1<\ldots<j_d,\quad(d\in\mathbb{N}^*).$$

Then, from Proposition 11.1, we deduce that, for any $0<b_0<b_1<1$, for almost all $b\in[b_0,b_1]$, any function in the form

$$r:(t,\theta)\mapsto\sum_{j\in\mathbb{S}}r_j\cos(j\theta-\Omega_j(b)t),\quad r_j\in\mathbb{R}$$

is a quasi-periodic solution with frequency $\omega_{\mathrm{Eq}}(b)\triangleq(\Omega_j(b))_{j\in\mathbb{S}}$ of the equation $\mathcal{L}_0r=0$ which is reversible, namely $r(-t,-\theta)=r(t,\theta)$. The measure of the Cantor set in b generating these solutions is estimated using Rüssmann Lemma 5.6 requiring a lower bound on the maximal derivative of a given function up to order q_0. In our case, the value of q_0 is explicit, namely $q_0\triangleq 2j_d+2$ which is due to the polynomial structure of the $\Omega_j(b)$. The aim of this part is to prove that these structures persist at the nonlinear level, more precisely, our result reads as follows.

Theorem 9.1. *Let $0<b_0<b_1<1$, $d\in\mathbb{N}^*$ and $\mathbb{S}\subset\mathbb{N}^*$ with $|\mathbb{S}|=d$. There exists $\varepsilon_0\in(0,1)$ small enough with the following properties : For every amplitudes $\mathtt{a}=(\mathtt{a}_j)_{j\in\mathbb{S}}\in(\mathbb{R}_+^*)^d$ satisfying*

$$|\mathtt{a}|\leqslant\varepsilon_0,$$

there exists a Cantor-like set $\mathcal{C}_\infty\subset(b_0,b_1)$ with asymptotically full Lebesgue measure as $\mathtt{a}\to 0$, i.e.

$$\lim_{\mathtt{a}\to 0}|\mathcal{C}_\infty|=b_1-b_0,$$

such that for any $b\in\mathcal{C}_\infty$, the equation (9.1) admits a time quasi-periodic solution with diophantine *frequency vector* $\omega_{\mathrm{pe}}(b,\mathtt{a}) \triangleq (\omega_j(b,\mathtt{a}))_{j\in\mathbb{S}} \in \mathbb{R}^d$ *and taking the form*

$$r(t,\theta) = \sum_{j\in\mathbb{S}} \mathtt{a}_j \cos\big(j\theta + \omega_j(b,\mathtt{a})t\big) + \mathtt{p}\big(\omega_{\mathrm{pe}}(b,\mathtt{a})t,\theta\big),$$

with

$$\omega_{\mathrm{pe}}(b,\mathtt{a}) \underset{\mathtt{a}\to 0}{\longrightarrow} (-\Omega_j(b))_{j\in\mathbb{S}},$$

where $\Omega_j(b)$ *are the equilibrium frequencies defined in* (9.7) *and the perturbation* $\mathtt{p}:\mathbb{T}^{d+1}\to\mathbb{R}$ *is an even function satisfying for some large index of regularity* s *depending only on the set* $\mathbb{S}$,

$$\|\mathtt{p}\|_{H^s(\mathbb{T}^{d+1},\mathbb{R})} \underset{\mathtt{a}\to 0}{=} o(|\mathtt{a}|).$$

We shall now sketch the main steps used to prove the previous theorem. First remark that small divisors problems already appear in the proof of Proposition 11.1 to find quasi-periodic structures at the linear level from the equilibrium. We can invert the linearized operator at the equilibrium with some fixed loss of regularity. Hence, we need to use a Nash-Moser scheme to find quasi-periodic solutions for the nonlinear model. To do so, we must invert the linearized operator in a neighborhood of the equilibrium state. Since $\mathcal{L}_r$ has non constant coefficients, the task is more delicate. The basic idea consists in diagonalizing, namely to conjugate to constant coefficients operator. Actually, we may follow the procedure presented in [21], slightly modified in [87, 101], where the dynamics is decoupled into tangential and normal parts. On the tangential modes, we introduce action-angles variables (I,ϑ) allowing to reformulate the problem in terms of embedded tori. More precisely, we shall look for the zeros of the following functional

$$\mathcal{F}(i,\alpha,b,\omega,\varepsilon) \triangleq \begin{pmatrix} \omega\cdot\partial_\varphi\vartheta(\varphi) - \alpha - \varepsilon\partial_I\mathcal{P}_\varepsilon(i(\varphi)) \\ \omega\cdot\partial_\varphi I(\varphi) + \varepsilon\partial_\vartheta\mathcal{P}_\varepsilon(i(\varphi)) \\ \omega\cdot\partial_\varphi z(\varphi) - \partial_\theta\big[\mathrm{L}(b)z(\varphi) + \varepsilon\nabla_z\mathcal{P}_\varepsilon\big(i(\varphi)\big)\big] \end{pmatrix}.$$

It turns out that it is more convenient to introduce one degree of freedom through the parameter α which provides at the end of the scheme a solution for the original problem when it is fixed to $-\omega_{\mathrm{Eq}}(b)$. Given any small reversible embedded torus $i_0:\varphi\mapsto(\vartheta_0(\varphi),I_0(\varphi),z_0(\varphi))$ and any $\alpha_0\in\mathbb{R}^d$, conjugating the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ via a suitable linear diffeomorphism of the toroidal phase space $\mathbb{T}^d\times\mathbb{R}^d\times L^2_\perp$, we obtain a triangular system in the action-angle-normal variables up to error terms. To solve the triangular system, we only have to invert the linearized operator in the normal directions, which is denoted by $\widehat{\mathcal{L}}_\omega$. This is done using KAM reducibility techniques in a similar way to [7, 33, 87, 101]. According to Proposition 13.1, we can write

$$\widehat{\mathcal{L}}_\omega = \Pi_{\mathbb{S}_0}^\perp\Big(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\Big)\Pi_{\mathbb{S}_0}^\perp,$$

where $\Pi_{\mathbb{S}_0}^\perp$ is the projector in the normal directions, $\mathcal{R}$ is an integral operator and $\mathcal{L}_{\varepsilon r}$ is defined by (9.2). First, following the KAM reducibility scheme in [11, 64, 101], we can reduce the transport part and the zero order part by conjugating by a quasi-periodic symplectic invertible change of variables in the form

$$\mathscr{B}\rho(\mu,\varphi,\theta) \triangleq \Big(1 + \partial_\theta\beta(\mu,\varphi,\theta)\Big)\rho\big(\mu,\varphi,\theta+\beta(\mu,\varphi,\theta)\big).$$

More precisely, as stated in Proposition 13.2, we can find a function $V_{i_0}^\infty = V_{i_0}^\infty(b,\omega)$ and a Cantor set

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0) \triangleq \bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}} \Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad \big|\omega\cdot l + jV_{i_0}^\infty(b,\omega)\big| > \tfrac{4\gamma^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}$$

in which the following decomposition holds

$$\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_{1,b}*\cdot-\partial_\theta\mathcal{K}_{2,b}*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0,$$

where $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving Toeplitz in time integral operator enjoying good smallness properties. The operator $\mathtt{E}_n^0$ is an error term of order one associated to the time truncation of the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$. Notice that N_n is defined by

$$N_n=N_0^{\left(\frac{3}{2}\right)^n}\quad\text{with}\quad N_0\gg 1.$$

Then, we project in the normal directions by considering the operator

$$\mathscr{B}_\perp\triangleq\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.$$

Therefore, in view of Proposition 13.3, we obtain the following decomposition in $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\omega\cdot\partial_\varphi+\mathscr{D}_0+\mathscr{R}_0+\mathtt{E}_n^1\triangleq\mathscr{L}_0+\mathtt{E}_n^1,$$

where $\mathscr{D}_0\triangleq(\mathrm{i}\mu_j^0(b,\omega))_{j\in\mathbb{S}_0^c}$ is a diagonal and reversible operator and $\mathscr{R}_0=\Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ is a real and reversible Toeplitz in time remainder integral operator in $OPS^{-\infty}$ in space and satisfying nice smallness properties. The term $\mathtt{E}_n^1$ plays a similar role as the previous one $\mathtt{E}_n^0$. The next goal is to reduce the remainder term $\mathscr{R}_0$. For this aim, we implement a KAM reduction process in the Toeplitz topology as in [101, Prop. 6.5]. The result is stated in Proposition 13.4 and provides two operators Φ_∞ and $\mathscr{D}_\infty\triangleq(\mathrm{i}\mu_j^\infty(b,\omega))_{j\in\mathbb{S}_0^c}$, with $\mathscr{D}_\infty$ a diagonal and reversible operator whose spectrum is described by

$$\forall j\in\mathbb{S}_0^c,\quad\mu_j^\infty(b,\omega)=\Omega_j(b)+j\big(V_{i_0}^\infty(b,\omega)-\tfrac{1}{2}\big)+r_j^\infty(b,\omega),$$

such that in the Cantor set

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\triangleq\bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ \langle l,j-j_0\rangle\leqslant N_n\\ (l,j)\neq(0,j_0)}}\Big\{(b,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^\infty(b,\omega)-\mu_{j_0}^\infty(b,\omega)\big|>\tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}$$

the following decomposition holds

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty=\omega\cdot\partial_\varphi+\mathscr{D}_\infty+\mathtt{E}_n^2\triangleq\mathscr{L}_\infty+\mathtt{E}_n^2.$$

Now, we can invert the operator $\mathscr{L}_\infty$ when the parameters are restricted to the Cantor set

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)\triangleq\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |l|\leqslant N_n}}\Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^\infty(b,\omega)\big|>\tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\}.$$

Therefore, we are able to construct an approximate right inverse of $\widehat{\mathcal{L}}_\omega$ in the Cantor set

$$\mathcal{G}_n^\gamma(i_0)\triangleq\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0).$$

We refer to Proposition 13.5 for more details. Now we can implement a Nash-Moser scheme in a similar way to [33, 87, 101] to find a solution $(b,\omega)\mapsto(i_\infty(b,\omega),\alpha_\infty(b,\omega))$ to the equation $\mathcal{F}(i,\alpha,b,\omega,\varepsilon)=0$ provided that the parameters (b,ω) are selected among a Cantor set $\mathcal{G}_\infty^\gamma$ which is constructed as the intersection of all the Cantor sets appearing in the scheme to invert at each step the linearized operator.

To find a solution to the original problem we construct a frequency curve $b \mapsto \omega(b,\varepsilon)$ implicitly defined by solving the equation

$$\alpha_\infty(b,\omega(b,\varepsilon)) = -\omega_{\mathrm{Eq}}(b).$$

Hence, we obtain the desired result for any value of b in the Cantor set

$$\mathcal{C}_\infty^\varepsilon \triangleq \Big\{ b \in (b_0,b_1) \quad \text{s.t.} \quad (b,\omega(b,\varepsilon)) \in \mathcal{G}_\infty^\gamma \Big\}.$$

Then, it remains to check that this set is non-trivial. This is done by estimating its measure using perturbed Rüssmann conditions from the equilibrium. In Proposition 14.2, we find a lower bound for the measure of $\mathcal{C}_\infty^\varepsilon$, namely

$$|\mathcal{C}_\infty^\varepsilon| \geqslant (b_1 - b_0) - C\varepsilon^\delta \quad \text{for some } \delta = \delta(q_0,d,\tau_1,\tau_2) > 0.$$

10 Hamiltonian reformulation

In this section, we shall write down the equation governing the boundary dynamics. For that purpose, we shall consider a polar parametrization of the boundary and see that the radial deformation in there is subject to a nonlinear and nonlocal Hamiltonian equation of transport type.

10.1 Equation satisfied by the radial deformation of the patch

Given $b \in (0,1)$, consider a vortex patch $t \mapsto \mathbf{1}_{D_t}$, near the Rankine vortex $\mathbf{1}_{b\mathbb{D}}$ with a smooth boundary whose polar parametrization is given by

$$z(t,\theta) \triangleq \big(b^2 + 2r(t,\theta)\big)^{\frac{1}{2}}\, e^{\mathrm{i}\theta}, \tag{10.1}$$

where r is the radial deformation assumed to be small, namely $|r(t,\theta)| \ll 1$. In the sequel, we shall frequently use the following notations

$$\begin{aligned} R(b,t,\theta) &\triangleq \big(b^2 + 2r(t,\theta)\big)^{\frac{1}{2}}, && (10.2)\\ A_r(b,t,\theta,\eta) &\triangleq \big|R(b,t,\theta)e^{\mathrm{i}\theta} - R(b,t,\eta)e^{\mathrm{i}\eta}\big|, && (10.3)\\ B_r(b,t,\theta,\eta) &\triangleq \Big|1 - R(b,t,\theta)R(b,t,\eta)e^{\mathrm{i}(\eta-\theta)}\Big|. && (10.4) \end{aligned}$$

The equation satisfied by r is given by the following lemma.

Lemma 10.1. *For short time $T > 0$, the radial deformation r, defined through* (10.2)*, satisfies the following nonlinear and nonlocal transport PDE:*

$$\forall (t,\theta) \in [0,T] \times \mathbb{T}, \quad \partial_t r(t,\theta) + F_b[r](t,\theta) = 0, \tag{10.5}$$

where

$$\begin{aligned} F_b[r] &\triangleq -F_b^0[r] - F_b^1[r] + F_b^2[r], && (10.6)\\ F_b^0[r] &\triangleq \tfrac{1}{2}\partial_\theta r(t,\theta) \int_{\mathbb{T}} \tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)}\, d\eta, && (10.7)\\ F_b^1[r] &\triangleq \int_{\mathbb{T}} \log\big(A_r(b,t,\theta,\eta)\big)\, \partial^2_{\theta\eta}\Big(R(b,t,\theta)R(b,t,\eta)\sin(\eta-\theta)\Big) d\eta, && (10.8)\\ F_b^2[r] &\triangleq \int_{\mathbb{T}} \log\big(B_r(b,t,\theta,\eta)\big)\, \partial^2_{\theta\eta}\Big(\tfrac{R(b,t,\eta)}{R(b,t,\theta)}\sin(\eta-\theta)\Big) d\eta, && (10.9) \end{aligned}$$

where $R(b,t,\theta)$, $A_r(b,t,\theta,\eta)$ and $B_r(b,t,\theta,\eta)$ are given by (10.2)-(10.4).

Proof. We start with the vortex patch equation. Denoting $\mathbf{n}$ the outward normal vector to the boundary of the patch, the evolution equation of the boundary can be written as

$$\partial_t z(t,\theta)\cdot\mathbf{n}(t,z(t,\theta))=-\partial_\theta\boldsymbol{\Psi}(t,z(t,\theta)).$$

For a detailed proof see for instance [99, p.174]. Since $\mathbf{n}(t,z(t,\theta))=-\mathrm{i}\,\partial_\theta z(t,\theta)$ (up to a real constant of renormalization) then the complex formulation of the vortex patch equation is given by

$$\mathrm{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big)=\partial_\theta\boldsymbol{\Psi}(t,z(t,\theta)).$$

Using the parametrization (10.1), one easily checks that

$$\mathrm{Im}\Big(\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\Big)=-\partial_t r(t,\theta).$$

Thus, the vortex patch equation writes in the following way

$$\partial_t r(t,\theta)+\partial_\theta\boldsymbol{\Psi}(t,z(t,\theta))=0. \tag{10.10}$$

Now we shall compute $\partial_\theta\boldsymbol{\Psi}(t,z(t,\theta))$. Using complex notations, we have

$$\partial_\theta\boldsymbol{\Psi}(t,z(t,\theta))=\nabla\boldsymbol{\Psi}(t,z(t,\theta))\cdot\partial_\theta z(t,\theta)=2\mathrm{Re}\Big(\partial_{\overline{w}}\boldsymbol{\Psi}(t,z(t,\theta))\overline{\partial_\theta z(t,\theta)}\Big). \tag{10.11}$$

Recall, from (1.4), that the stream function $\boldsymbol{\Psi}$ writes

$$\forall w\in\mathbb{D},\quad \boldsymbol{\Psi}(t,w)=\tfrac{1}{4\pi}\int_{D_t}\log\left(|w-\xi|^2\right)dA(\xi)-\tfrac{1}{4\pi}\int_{D_t}\log\left(|\overline{\xi}w-1|^2\right)dA(\xi). \tag{10.12}$$

Let $\epsilon>0$. We set

$$f_\epsilon(\xi,\overline{\xi})\triangleq(\overline{\xi}-\overline{w})\Big[\log\left(|\xi-w|^2+\epsilon\right)-1\Big]-\left(\overline{\xi}-\tfrac{1}{w}\right)\Big[\log\left(|1-w\overline{\xi}|^2\right)-1\Big].$$

Then

$$\partial_{\overline{\xi}}f_\epsilon(\xi,\overline{\xi})=\log\left(|w-\xi|^2+\epsilon\right)-\frac{\epsilon}{|w-\xi|^2+\epsilon}-\log\left(|\overline{\xi}w-1|^2\right).$$

Using the complex version of Stokes' Theorem,

$$2\mathrm{i}\int_D\partial_{\overline{\xi}}f_\epsilon(\xi,\overline{\xi})dA(\xi)=\int_{\partial D}f_\epsilon(\xi,\overline{\xi})d\xi,$$

then passing to the limit as ϵ goes to 0, using in particular dominated convergence theorem, we obtain

$$\boldsymbol{\Psi}(t,w)=\tfrac{1}{8\mathrm{i}\pi}\int_{\partial D_t}(\overline{\xi}-\overline{w})\Big[\log\left(|\xi-w|^2\right)-1\Big]d\xi-\tfrac{1}{8\mathrm{i}\pi}\int_{\partial D_t}\left(\overline{\xi}-\tfrac{1}{w}\right)\Big[\log\left(|1-w\overline{\xi}|^2\right)-1\Big]d\xi.$$

Performing the change of variables $\xi=z(t,\eta)$, given by (10.1), and using the notation (A.3) we can write

$$\begin{aligned}\boldsymbol{\Psi}(t,w)=&\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}(\overline{z}(t,\eta)-\overline{w})\Big[\log\left(|z(t,\eta)-w|^2\right)-1\Big]\partial_\eta z(t,\eta)d\eta\\&-\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\left(\overline{z}(t,\eta)-\tfrac{1}{w}\right)\Big[\log\left(|1-w\overline{z}(t,\eta)|^2\right)-1\Big]\partial_\eta z(t,\eta)d\eta.\end{aligned}$$

It follows that

$$\begin{aligned}\partial_{\overline{w}}\mathbf{\Psi}(t,w) &= -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|z(t,\eta)-w|^2\big)\,\partial_\eta z(t,\eta)d\eta\\ &\quad-\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\big(\overline{z}(t,\eta)-\frac{1}{w}\big)\Big[\frac{\frac{1}{\overline{w}^2}}{z(t,\eta)-\frac{1}{\overline{w}}}+\frac{1}{\overline{w}}\Big]\partial_\eta z(t,\eta)d\eta.\end{aligned}\tag{10.13}$$

Direct computations lead to

$$\Big[\frac{\overline{z}(t,\eta)-\frac{1}{w}}{z(t,\eta)-\frac{1}{\overline{w}}}\Big]\partial_\eta z(t,\eta)=\partial_\eta\Big[\log\big(\big|z(t,\eta)-\frac{1}{\overline{w}}\big|^2\big)+\log\big(|w|^2\big)\Big]\Big(\overline{z}(t,\eta)-\frac{1}{w}\Big)-\partial_\eta\overline{z}(t,\eta).$$

Inserting this identity into (10.13), integrating by parts, using the morphism property of the logarithm and the periodicity imply

$$\begin{aligned}\partial_{\overline{w}}\mathbf{\Psi}(t,w) &= -\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|z(t,\eta)-w|^2\big)\,\partial_\eta z(t,\eta)d\eta\\ &\quad+\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\log\big(|1-\overline{w}z(t,\eta)|^2\big)\,\partial_\eta\overline{z}(t,\eta)\frac{1}{\overline{w}^2}d\eta\\ &\quad-\tfrac{1}{4\mathrm{i}}\int_{\mathbb{T}}\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{1}{\overline{w}}d\eta.\end{aligned}\tag{10.14}$$

As a consequence, one gets

$$\begin{aligned}2\mathrm{Re}\Big(\partial_{\overline{w}}\mathbf{\Psi}(t,z(t,\theta))\partial_\theta\overline{z}(t,\theta)\Big) &= -\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|z(t,\eta)-z(t,\theta)|^2\big)\,\mathrm{Im}\big(\partial_\eta z(t,\eta)\partial_\theta\overline{z}(t,\theta)\big)d\eta\\ &\quad+\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|1-\overline{z}(t,\theta)z(t,\eta)|^2\big)\,\mathrm{Im}\Big(\partial_\eta\overline{z}(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)^2}\Big)d\eta\\ &\quad-\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\Big(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\Big)d\eta.\end{aligned}$$

That is, by (10.11),

$$\begin{aligned}\partial_\theta\mathbf{\Psi}(t,z(t,\theta)) &= -\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|z(t,\eta)-z(t,\theta)|^2\big)\,\partial^2_{\theta\eta}\mathrm{Im}\big(z(t,\eta)\overline{z}(t,\theta)\big)d\eta\\ &\quad+\tfrac{1}{2}\int_{\mathbb{T}}\log\big(|1-\overline{z}(t,\theta)z(t,\eta)|^2\big)\,\partial^2_{\theta\eta}\mathrm{Im}\Big(\frac{z(t,\eta)}{z(t,\theta)}\Big)d\eta\\ &\quad-\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\Big(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\Big)d\eta.\end{aligned}$$

From (10.1) we immediately get

$$\begin{aligned}\mathrm{Im}\big(z(t,\eta)\overline{z}(t,\theta)\big) &= R(b,t,\theta)R(b,t,\eta)\sin(\eta-\theta),\\ \mathrm{Im}\Big(\frac{z(t,\eta)}{z(t,\theta)}\Big) &= \frac{R(b,t,\theta)}{R(b,t,\eta)}\sin(\eta-\theta),\\ \mathrm{Im}\Big(\overline{z}(t,\eta)\partial_\eta z(t,\eta)\frac{\partial_\theta\overline{z}(t,\theta)}{\overline{z}(t,\theta)}\Big) &= \frac{R^2(b,t,\eta)}{R^2(b,t,\theta)}\partial_\theta r(t,\theta)-\partial_\eta r(t,\eta).\end{aligned}$$

Combining the last four identities with (10.10) and using the notations (10.1)-(10.4) we conclude the desired result.

$\square$

We look for time quasi-periodic solutions of (10.5); that are functions in the form

$$\widehat{r}(t,\theta)=r(\omega t,\theta),$$

where $r = r(\varphi, \theta) : \mathbb{T}^{d+1} \to \mathbb{R}$, $\omega \in \mathbb{R}^d$, $d \in \mathbb{N}^*$. With this ansatz, the equation (10.5) becomes

$$\omega \cdot \partial_\varphi r(\varphi, \theta) + F_b[r](\varphi, \theta) = 0. \tag{10.15}$$

10.2 Hamiltonian structure

In this section, we show that the contour dynamics equation (10.5) has a Hamiltonian structure related to the kinetic energy

$$E(r)(t) \triangleq -\frac{1}{2\pi} \int_{D_t} \mathbf{\Psi}(t, z) dA(z), \tag{10.16}$$

which is a conserved quantity for (1.5). It is well-known that the bidimensional Euler equations admits a Hamiltonian structure and we shall see here that such structure still persists at the level of the boundary equation, which is a stronger formulation.

Proposition 10.1. *The equation* (10.5) *is a Hamiltonian equation in the form*

$$\partial_t r = \partial_\theta \nabla H(r), \quad \text{where} \quad H(r) \triangleq \tfrac{1}{2} E(r), \tag{10.17}$$

and ∇ *is the* $L^2_\theta(\mathbb{T})$*-gradient associated with the* $L^2_\theta(\mathbb{T})$ *normalized inner product*

$$\big\langle \rho_1, \rho_2 \big\rangle_{L^2(\mathbb{T})} \triangleq \int_{\mathbb{T}} \rho_1(\theta) \rho_2(\theta) d\theta.$$

Proof. In polar coordinates, the stream function, given by (10.12), at some point $w \in \mathbb{D}$ writes

$$\mathbf{\Psi}(t, w) = \int_{\mathbb{T}} \int_0^{R(b,t,\eta)} G\left(w, \ell_2 e^{\mathrm{i}\eta}\right) \ell_2 d\ell_2 d\eta \quad \text{with} \quad G(w, \xi) \triangleq \log\left(\left| \frac{w - \xi}{1 - w\overline{\xi}} \right| \right) \tag{10.18}$$

and kinetic energy E, in (10.16), reads

$$E(r)(t) = -\int_{\mathbb{T}} \int_{\mathbb{T}} \int_0^{R(b,t,\theta)} \left(\int_0^{R(b,t,\eta)} G\left(\ell_1 e^{\mathrm{i}\theta}, \ell_2 e^{\mathrm{i}\eta}\right) \ell_2 d\ell_2 \right) \ell_1 d\ell_1 d\theta d\eta.$$

Differentiating with respect to r in the direction ρ and using the symmetry of the kernel

$$G(w, \xi) = G(\xi, w)$$

yields

$$\begin{aligned} d_r E(r)[\rho](t) &= -2 \int_{\mathbb{T}} \rho(t, \theta) \left(\int_{\mathbb{T}} \int_0^{R(b,t,\eta)} G\left(R(b, t, \theta) e^{\mathrm{i}\theta}, \ell_2 e^{\mathrm{i}\eta}\right) \ell_2 d\ell_2 d\eta \right) d\theta \\ &= -2 \int_{\mathbb{T}} \rho(t, \theta) \mathbf{\Psi}\big(t, R(b, t, \theta) e^{\mathrm{i}\theta}\big) d\theta. \end{aligned}$$

Since $d_r E(r)[\rho] = \langle \nabla E, \rho \rangle_{L^2(\mathbb{T})}$ then

$$\nabla E(r)(t, \theta) = -2 \mathbf{\Psi}\big(t, R(b, t, \theta) e^{\mathrm{i}\theta}\big). \tag{10.19}$$

Finally, using (10.19) and comparing (10.17) with (10.10) we conclude the desired result. This achieves the proof of Proposition 10.1. ☐

Now, we shall present the symplectic structure associated with the Hamiltonian equation (10.17). This will be relevant later in Section 12.1 when introducing the action-angle variables. We shall also explore some symmetry property for (10.17). Observe that this latter equation implies

$$\frac{d}{dt}\int_{\mathbb{T}} r(t,\theta)d\theta = 0.$$

Therefore, we will consider the phase space with zero average in the space variable $L^2_0(\mathbb{T})$ defined in (4.23). The equation (10.17) induces on the phase space $L^2_0(\mathbb{T})$ a symplectic structure given by the symplectic 2-form $\mathcal{W}$ defined in (4.24). The Hamiltonian vector field is $X_H(r) = \partial_\theta \nabla H(r)$ is defined similarly to (4.25). We shall now look at the reversibility property of the equation (10.17). Using the change of variables $\eta \mapsto -\eta$ and parity arguments, one gets

$$F_b \circ \mathscr{S} = -\mathscr{S} \circ F_b,$$

where F_b is given by (10.6) and $\mathscr{S}$ is the involution introduced in (4.27). Then we conclude by Lemma 10.1, (10.17) and (4.28) that the Hamiltonian vector field X_H satisfies

$$X_H \circ \mathscr{S} = -\mathscr{S} \circ X_H.$$

Thus, we will look for quasi-periodic solutions satisfying the reversibility condition

$$r(-t,-\theta) = r(t,\theta).$$

11 Linearization and structure of the equilibrium frequencies

In the current section, we linearize the equation (10.5) at a given small state r close to the equilibrium. At this latter, we shall see that the linear operator is a Fourier multiplier with polynomial linear frequencies with respect to the radius of the Rankine patch $b\mathbb{D}$. At the end of this section, we also check the transversality conditions for the unperturbed frequency vector.

11.1 Linearized operator

We shall first prove that the linearized operator at a general small state r can be decomposed into the sum of a variable coefficients transport operator, a nonlocal operator of order 0 and a smoothing nonlocal operator in the variable θ. More precisely, we have the following lemma.

Lemma 11.1. *The linearized Hamiltonian equation of* (10.17) *at a state r is the time-dependent Hamiltonian system*

$$\partial_t \rho(t,\theta) = -\partial_\theta\Big(V_r(b,t,\theta)\rho(t,\theta) + \mathbf{L}_r(\rho)(b,t,\theta) - \mathbf{S}_r(\rho)(b,t,\theta)\Big),$$

where the function V_r is defined by

$$\begin{aligned} V_r(b,t,\theta) = &-\tfrac{1}{2}\int_{\mathbb{T}} \tfrac{R^2(b,t,\eta)}{R^2(b,t,\theta)}d\eta \\ &- \tfrac{1}{R(b,t,\theta)}\int_{\mathbb{T}} \log\big(A_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta \\ &- \tfrac{1}{R^3(b,t,\theta)}\int_{\mathbb{T}} \log\big(B_r(b,t,\theta,\eta)\big)\partial_\eta\big(R(b,t,\eta)\sin(\eta-\theta)\big)d\eta, \end{aligned} \tag{11.1}$$

$\mathbf{L}_r$ *is a nonlocal operator in the form*

$$\mathbf{L}_r(\rho)(b,t,\theta) = \int_{\mathbb{T}} \rho(t,\eta)\log\left(A_r(t,\theta,\eta)\right)d\eta \tag{11.2}$$

and $\mathbf{S}_r$ *is a smoothing nonlocal operator in the form*

$$\mathbf{S}_r(\rho)(b,t,\theta)=\int_{\mathbb{T}}\rho(t,\eta)\log\left(B_r(t,\theta,\eta)\right)d\eta. \tag{11.3}$$

We recall that A_r, B_r *and* R *are defined by* (10.3), (10.4) *and* (10.2), *respectively.*
Moreover, if $r(-t,-\theta)=r(t,\theta)$, *then*

$$V_r(b,-t,-\theta)=V_r(b,t,\theta). \tag{11.4}$$

Proof. In all the proof, we shall omit the dependence of our quantities with respect to the variables b and t. Notice that linearizing (10.10) amounts to compute the Gâteaux derivative of the stream function $\mathbf{\Psi}(r,z(\theta))\triangleq\mathbf{\Psi}(z(\theta))$ given by (10.12) at point r in the direction ρ (real-valued). All the computations are done at a formal level, but can be rigorously justified in a classical way in the functional context introduced in Section A. Applying the chain rule gives

$$d_r\big(\mathbf{\Psi}\big(r,z(\theta)\big)\big)[\rho]=\big(d_r\mathbf{\Psi}(r,w)[\rho]\big)_{|w=z(\theta)}+2\mathrm{Re}\left(\big(\partial_{\overline{w}}\mathbf{\Psi}(r,w)\big)_{|w=z(\theta)}d_r\overline{z}(\theta)[\rho]\right). \tag{11.5}$$

Differentiating (10.18) gives

$$d_r\mathbf{\Psi}(r,w)[\rho]=\int_{\mathbb{T}}\log\left(\left|\frac{w-R(\eta)e^{\mathrm{i}\eta}}{1-R(\eta)e^{-\mathrm{i}\eta}w}\right|\right)\rho(\eta)d\eta. \tag{11.6}$$

On the other hand, from (10.14) and the identity

$$d_rz(\theta)[\rho](\theta)=\tfrac{\rho(\theta)}{R(\theta)}e^{\mathrm{i}\theta},$$

we obtain

$$\begin{aligned}2\mathrm{Re}\left(\big(\partial_{\overline{w}}\mathbf{\Psi}(r,w)\big)_{|w=z(\theta)}d_r\overline{z}(\theta)[\rho]\right)&=-\tfrac{\rho(\theta)}{R(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\log\left(|z(\eta)-z(\theta)|^2\right)\partial_\eta\mathrm{Im}\left(z(\eta)e^{-\mathrm{i}\theta}\right)d\eta\\&\quad-\tfrac{\rho(\theta)}{R^3(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\log\left(|1-\overline{z(\theta)}z(\eta)|^2\right)\partial_\eta\mathrm{Im}\left(\overline{z}(\eta)e^{\mathrm{i}\theta}\right)d\eta\\&\quad+\tfrac{\rho(\theta)}{R^2(\theta)}\tfrac{1}{2}\int_{\mathbb{T}}\mathrm{Im}\big(\partial_\eta\overline{z}(\eta)z(\eta)\big)d\eta.\end{aligned} \tag{11.7}$$

Putting together (11.6), (11.5), (11.7) and using the identities

$$\mathrm{Im}\left(z(\eta)e^{-\mathrm{i}\theta}\right)=R(\eta)\sin(\eta-\theta),\quad\mathrm{Im}\big(\partial_\eta\overline{z}(\eta)z(\eta)\big)=-R^2(\eta),$$

we conclude the desired result. The symmetry property (11.4) is an immediate consequence of (11.1) with the change of variables $\eta\mapsto-\eta$. This achieves the proof of Lemma 11.1. ☐

The following lemma shows that the linearized operator at the equilibrium state is a Fourier multiplier. This provides an integrable Hamiltonian equation from which we shall generate, in Proposition 11.1, quasi-periodic solutions.

Lemma 11.2. *The following properties hold true.*

1. *The linearized equation of* (10.17) *at the equilibrium state* $(r=0)$ *writes*

$$\partial_t\rho=\partial_\theta\mathrm{L}(b)\rho=\partial_\theta\nabla H_{\mathrm{L}}(\rho), \tag{11.8}$$

where $\mathrm{L}(b)$ *is the self-adjoint operator on* $L^2_0(\mathbb{T})$ *defined by*

$$\mathrm{L}(b) \triangleq -\frac{1}{2} - \mathcal{K}_b * \cdot \tag{11.9}$$

with

$$\mathcal{K}_b \triangleq \mathcal{K}_{1,b} - \mathcal{K}_{2,b}, \tag{11.10}$$
$$\mathcal{K}_{1,b}(\theta) \triangleq \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right), \tag{11.11}$$
$$\mathcal{K}_{2,b}(\theta) \triangleq \log\left(|1-b^2e^{\mathrm{i}\theta}|\right). \tag{11.12}$$

It is generated by the quadratic Hamiltonian

$$H_{\mathrm{L}}(\rho) \triangleq \frac{1}{2}\langle \mathrm{L}(b)\rho, \rho\rangle_{L^2(\mathbb{T})}. \tag{11.13}$$

2. From Fourier point of view, if we write $\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*}\rho_j(t)e^{\mathrm{i}j\theta}$ *with* $\rho_{-j}(t) = \overline{\rho_j}(t)$, *then the self-adjoint operator* $\mathrm{L}(b)$ *and the Hamiltonian* H_{L} *write*

$$\mathrm{L}(b)\rho(\theta) = -\sum_{j\in\mathbb{Z}^*}\tfrac{\Omega_j(b)}{j}\rho_j e^{\mathrm{i}j\theta} \quad \textit{and} \quad H_{\mathrm{L}}(\rho) = -\sum_{j\in\mathbb{Z}^*}\tfrac{\Omega_j(b)}{2j}|\rho_j|^2, \tag{11.14}$$

where $\big(\Omega_j(b)\big)_{j\in\mathbb{Z}^*}$ *is defined by*

$$\forall j\in\mathbb{N}^*, \quad \Omega_j(b) = \frac{j-1+b^{2j}}{2} \quad \text{and} \quad \Omega_{-j}(b) = -\Omega_j(b). \tag{11.15}$$

Moreover, the reversible solutions of the equation (11.8) *take the form*

$$\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*}\rho_j\cos\left(j\theta - \Omega_j(b)t\right), \quad \rho_j\in\mathbb{R}. \tag{11.16}$$

Proof. **1.** Notice that the quantities A_r and B_r, introduced in (10.3), (10.4), can be rewritten as follows

$$\begin{aligned}A_r(b,t,\theta,\eta) &= \Big(R^2(b,t,\theta) + R^2(b,t,\eta) - 2R(b,t,\theta)R(b,t,\eta)\cos(\eta-\theta)\Big)^{\frac{1}{2}}\\ &= \Big(\big(R(b,t,\theta)-R(b,t,\eta)\big)^2 + 4R(b,t,\theta)R(b,t,\eta)\sin^2(\eta-\theta)\Big)^{\frac{1}{2}}\end{aligned} \tag{11.17}$$

and

$$B_r(b,t,\theta,\eta) = \Big(R^2(b,t,\theta)R^2(b,t,\eta) - 2R(b,t,\theta)R(b,t,\eta)\cos(\eta-\theta) + 1\Big)^{\frac{1}{2}}. \tag{11.18}$$

Taking $r=0$ in (11.17), (10.4) and (10.2) gives

$$A_0(b,t,\theta,\eta) = 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|, \quad B_0(b,t,\theta,\eta) = |1-b^2e^{\mathrm{i}(\eta-\theta)}| \quad \text{and} \quad R(b,t,\theta) = b. \tag{11.19}$$

According to (11.1), (11.2) and (11.3) we obtain, after straightforward simplifications using (11.19),

$$\begin{aligned}
V_0(b,t,\theta) &= -\tfrac{1}{2} - \tfrac{1}{2}\int_{\mathbb{T}} \log\left(4b^2\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(\eta)d\eta - \frac{1}{b^2}\int_{\mathbb{T}}\log\left(|1-b^2e^{\mathrm{i}\eta}|\right)\cos(\eta)d\eta,\\
\mathbf{L}_0(\rho)(b,t,\theta) &= \int_{\mathbb{T}}\log\left(2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\right)\rho(t,\eta)d\eta,\\
\mathbf{S}_0(\rho)(b,t,\theta) &= \int_{\mathbb{T}}\log\left(\left|1-b^2e^{\mathrm{i}(\eta-\theta)}\right|\right)\rho(t,\eta)d\eta.
\end{aligned}$$

We then see that $\mathbf{L}_0$ and $\mathbf{S}_0$ are convolution operators given by

$$\begin{aligned}
\mathbf{L}_0 &= \mathcal{K}_{1,b}*\cdot \quad \text{with} \quad \mathcal{K}_{1,b}(\theta) \triangleq \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\theta}{2}\right)\right),\\
\mathbf{S}_0 &= \mathcal{K}_{2,b}*\cdot \quad \text{with} \quad \mathcal{K}_{2,b}(\theta) \triangleq \log\left(|1-b^2e^{\mathrm{i}\theta}|\right).
\end{aligned}$$

2. To describe the operators above, it suffices to look for their actions on the Fourier basis $(e_j)_{j\in\mathbb{Z}^*}$ of $L^2_0(\mathbb{T})$. We first study the operator $\mathbf{L}_0$. Recall the following formula which can be found in [43, Lem. A.3]

$$\forall j\in\mathbb{Z}^*,\quad \int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(j\eta)d\eta = -\frac{1}{|j|}. \tag{11.20}$$

Using (11.20) together with symmetry arguments, one obtains

$$\begin{aligned}
\forall j\in\mathbb{Z}^*,\quad \mathcal{K}_{1,b}*e_j(\theta) &= \tfrac{1}{2}\int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)e^{\mathrm{i}j(\theta-\eta)}d\eta\\
&= \frac{e_j(\theta)}{2}\int_{\mathbb{T}}\log\left(\sin^2\left(\tfrac{\eta}{2}\right)\right)\cos(j\eta)d\eta \qquad (11.21)\\
&= -\frac{e_j(\theta)}{2|j|}. \qquad (11.22)
\end{aligned}$$

We now turn to the study of the operator $\mathbf{S}_0$. Using the following identity proved in [138, Lem. 3.2]

$$\forall j\in\mathbb{Z}^*,\quad \int_{\mathbb{T}}\log\left(|1-b^2e^{\mathrm{i}\eta}|\right)\cos(j\eta)d\eta = -\frac{b^{2|j|}}{2|j|}, \tag{11.23}$$

we obtain

$$\begin{aligned}
\forall j\in\mathbb{Z}^*,\quad \mathcal{K}_{2,b}*e_j(\theta) &= e_j(\theta)\int_{\mathbb{T}}\log\left(|1-b^2e^{\mathrm{i}\eta}|\right)\cos(j\eta)d\eta\\
&= -\frac{b^{2|j|}e_j(\theta)}{2|j|}. \qquad (11.24)
\end{aligned}$$

In view of the expression of V_0 and using formulae (11.20) and (11.23) we find

$$V_0(b,t,\theta) = \frac{1}{2}. \tag{11.25}$$

Notice that, the kernels $\mathcal{K}_{1,b}$ and $\mathcal{K}_{2,b}$ being even, the operator $\mathrm{L}(b)$ is self-adjoint. The identities in (11.14) follows immediately from (11.9), (11.22), (11.24) and (11.25). Then, according to (11.14), a real function ρ with Fourier representation $\rho(t,\theta) = \sum_{j\in\mathbb{Z}^*}\rho(t)e^{\mathrm{i}j\theta}$ is a solution to (11.8) if and only if

$$\forall j\in\mathbb{Z}^*,\quad \dot{\rho_j} = -\mathrm{i}\,\Omega_j(b)\rho_j,$$

where $\Omega_j(b)$ is defined by (11.15). Solving the previous ODE gives

$$\rho(t,\theta)=\sum_{j\in\mathbb{Z}^*}\rho_j(0)e^{\mathrm{i}(j\theta-\Omega_j(b)t)}.$$

Therefore, every real-valued reversible solution to (11.8) has the form (11.16). This ends the proof of Lemma 11.2. $\square$

11.2 Properties of the equilibrium frequencies

The goal of this section is to explore some important properties of the equilibrium frequencies. We shall first show some bounds on these frequencies then discuss their non-degeneracy through the transversality conditions. Such conditions are crucial in the measure estimates of the final Cantor set giving rise to quasi-periodic solutions for the linear and the nonlinear problems.

Lemma 11.3. *The following properties hold true.*

(i) For all $b\in(0,1)$, the sequence $\big(\frac{\Omega_j(b)}{j}\big)_{j\in\mathbb{N}^*}$ is strictly increasing.

(ii) For all $j\in\mathbb{Z}^*$, we have

$$\forall\, 0<b_0\leqslant b<1,\quad |\Omega_j(b)|\geqslant\frac{b_0^2}{2}|j|.$$

(iii) For all $j,j'\in\mathbb{Z}^*$, we have

$$\forall\, 0<b_0\leqslant b<1,\quad |\Omega_j(b)\pm\Omega_{j'}(b)|\geqslant\frac{b_0^2}{6}|j\pm j'|.$$

(iv) Given $0<b_0<b_1<1$ and $q_0\in\mathbb{N}$, there exists $C_0>0$ such that

$$\forall j,j'\in\mathbb{Z}^*,\qquad \max_{q\in[\![0,q_0]\!]}\sup_{b\in[b_0,b_1]}\big|\partial_b^q\big(\Omega_j(b)-\Omega_{j'}(b)\big)\big|\leqslant C_0|j-j'|.$$

Proof. **(i)** This point was proved in [86, Prop. 2].
(ii) By symmetry (11.15), it suffices to show the inequality for $j\in\mathbb{N}^*$. From (i) we have

$$\frac{\Omega_j(b)}{j}\geqslant\Omega_1(b)=\frac{b^2}{2}\geqslant\frac{b_0^2}{2}\cdot$$

(iii) In view of the symmetry (11.15), it suffices to check the property for $j,j'\in\mathbb{N}^*$. By symmetry in j,j' we may assume that $j\geqslant j'$. For $j=j'=1$ one has

$$\Omega_1(b)+\Omega_1(b)=b^2\geqslant b_0^2\cdot$$

In the case where $j\geqslant 2$ and $j'\geqslant 1$ we get

$$\Omega_j(b)+\Omega_{j'}(b)=\frac{j+j'-2}{2}+\frac{b^{2j}+b^{2j'}}{2}\geqslant(j+j')\frac{j+j'-2}{2(j+j')}\geqslant\frac{j+j'}{6}\cdot$$

Now we shall move to the difference. Using Taylor formula we obtain, for all $j > j' \geqslant 1$,

$$\begin{aligned}\Omega_j(b) - \Omega_{j'}(b) &= \frac{j-j'}{2} + \frac{b^{2j} - b^{2j'}}{2}\\ &= \frac{j-j'}{2} + \log(b)\int_{j'}^{j} b^{2x}dx\\ &\geqslant \frac{j-j'}{2}(1 + 2\log(b)b^{2j}) \geqslant \frac{j-j'}{4}.\end{aligned}$$

(iv) The case $j = j'$ is trivial, then from the symmetry (11.15) and without loss of generality we shall assume that $j > j' \geqslant 1$. First, remark that

$$\forall b \in (0,1), \quad |\Omega_j(b) \pm \Omega_{j'}(b)| \leqslant \frac{(j-1) \pm (j'-1)}{2} + \frac{b^{2j'} \pm b^{2j}}{2} \leqslant j \pm j'.$$

Now, for all $q \in \mathbb{N}^*$, one has

$$\partial_b^q\Big(\Omega_j(b) \pm \Omega_{j'}(b)\Big) = \frac{1}{2}\partial_b^q\Big(b^{2j} \pm b^{2j'}\Big).$$

Moreover, for all $q \in [\![1, q_0]\!]$ and $n \in \mathbb{N}^*$,

$$0 \leqslant \partial_b^q(b^n) \leqslant q!\binom{n}{q}b^{n-q} \leqslant \frac{n^{q_0}b_1^n}{b_0^{q_0}}.$$

Since $b_1 \in (0,1)$ then the sequence $(n^{q_0}b_1^n)_{n\in\mathbb{N}}$ is bounded. Therefore, there exists $C_0 \triangleq C_0(q_0, b_0, b_1) > 0$ such that

$$\forall n \in \mathbb{N}, \quad 0 \leqslant \partial_b^q(b^n) \leqslant C_0. \tag{11.26}$$

We deduce that for all $q \in [\![1, q_0]\!]$,

$$\Big|\partial_b^q\Big(\Omega_j(b) \pm \Omega_{j'}(b)\Big)\Big| \leqslant C_0 \leqslant C_0(j \pm j').$$

This concludes the proof of Lemma 11.3. □

Let us consider finitely many Fourier modes, called tangential sites, gathered in the tangential set $\mathbb{S}$ defined by

$$\mathbb{S} \triangleq \{j_1, \ldots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < j_2 < \ldots < j_d. \tag{11.27}$$

Now, we define the equilibrium frequency vector by

$$\omega_{\mathrm{Eq}}(b) \triangleq (\Omega_j(b))_{j\in\mathbb{S}}, \tag{11.28}$$

where $\Omega_j(b)$ is defined by (11.15). We shall now investigate the non-degeneracy and the transversality properties satisfied by ω_{Eq}. We have the following result.

Lemma 11.4. *The equilibrium frequency vector ω_{Eq} and the vector-valued function $(\omega_{\mathrm{Eq}}, 1)$ are non-degenerate on $[b_0, b_1]$ in the sense of Definition 5.1.*

Proof. ► We shall first prove that the equilibrium frequency vector ω_{Eq} is non-degenerate on $[b_0, b_1]$. Arguing by contradiction, suppose that there exists $c \triangleq (c_1, \ldots, c_d) \in \mathbb{R}^d\backslash\{0\}$ such that

$$\forall b \in [b_0, b_1], \quad \sum_{k=1}^{d} c_k\Omega_{j_k}(b) = 0. \tag{11.29}$$

Since $\Omega_j(b)$ is polynomial in b then, from (11.15), one has

$$\forall b\in\mathbb{R},\quad \sum_{k=1}^{d}c_k(j_k-1+b^{2j_k})=0. \tag{11.30}$$

Taking the limit $b\to 0$ in (11.30) gives the relation $\sum_{k=1}^{d}c_k(j_k-1)=0$, which, inserted into (11.30), implies

$$\forall b\in\mathbb{R},\quad \sum_{k=1}^{d}c_k b^{2j_k}=0.$$

Since $j_1<j_2<\ldots<j_d$, then

$$\forall k\in[\![1,d]\!],\quad c_k=0,$$

which contradicts the assumption.

▶ Next, we shall check that the function $(\omega_{\mathrm{Eq}},1)$ is non-degenerate on $[b_0,b_1]$. Suppose, by contradiction, that there exists $c\triangleq(c_1,\ldots,c_d,c_{d+1})\in\mathbb{R}^{d+1}\backslash\{0\}$ such that

$$\forall b\in[b_0,b_1],\quad c_{d+1}+\sum_{k=1}^{d}c_k\Omega_{j_k}(b)=0. \tag{11.31}$$

Since $\Omega_j(b)$ is polynomial in b then, from (11.15), one may writes

$$\forall b\in\mathbb{R},\quad c_{d+1}+\tfrac{1}{2}\sum_{k=1}^{d}c_k(j_k-1+b^{2j_k})=0. \tag{11.32}$$

Taking the limit $b\to 0$ in (11.32) yields

$$c_{d+1}+\tfrac{1}{2}\sum_{k=1}^{d}c_k(j_k-1)=0.$$

Inserting this relation into (11.32) gives

$$\forall b\in\mathbb{R},\quad \sum_{k=1}^{d}c_k b^{2j_k}=0.$$

Reasoning as in the previous point, we obtain

$$\forall k\in[\![1,d]\!],\ c_k=0$$

and then $c_{d+1}=0$, by coming back to (11.32), contradicting the assumption. □

We shall now state the transversality conditions satisfied by the unperturbed frequencies.

Lemma 11.5. [Transversality] *Let $0<b_0<b_1<1$. Set $q_0=2j_d+2$. Then, there exists $\rho_0>0$ such that the following results hold true. Recall that ω_{Eq} and Ω_j are defined in* (11.28) *and* (11.15)*, respectively.*

(i) For all $l\in\mathbb{Z}^d\setminus\{0\}$, we have

$$\inf_{b\in[b_0,b_1]}\max_{q\in[\![0,q_0]\!]}|\partial_b^q\omega_{\mathrm{Eq}}(b)\cdot l|\geqslant\rho_0\langle l\rangle.$$

(ii) For all $(l,j)\in\mathbb{Z}^d\times(\mathbb{N}^*\setminus\mathbb{S})$

$$\inf_{b\in[b_0,b_1]}\max_{q\in[\![0,q_0]\!]}\left|\partial_b^q\big(\omega_{\mathrm{Eq}}(b)\cdot l\pm\tfrac{j}{2}\big)\right|\geqslant\rho_0\langle l\rangle.$$

(iii) For all $(l,j)\in\mathbb{Z}^d\times(\mathbb{N}^*\setminus\mathbb{S})$

$$\inf_{b\in[b_0,b_1]}\max_{q\in[\![0,q_0]\!]}\left|\partial_b^q\big(\omega_{\mathrm{Eq}}(b)\cdot l\pm\Omega_j(b)\big)\right|\geqslant\rho_0\langle l\rangle.$$

(iv) For all $l\in\mathbb{Z}^d, j,j'\in\mathbb{N}^*\setminus\mathbb{S}$ *with* $(l,j)\neq(0,j')$, *we have*

$$\inf_{b\in[b_0,b_1]}\max_{q\in[\![0,q_0]\!]}\left|\partial_b^q\big(\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)\pm\Omega_{j'}(b)\big)\right|\geqslant\rho_0\langle l\rangle.$$

Proof. **(i)** Assume by contradiction that for all $\rho_0>0$, there exist $l\in\mathbb{Z}^d\setminus\{0\}$ and $b\in[b_0,b_1]$ such that

$$\max_{q\in[\![0,q_0]\!]}|\partial_b^q\omega_{\mathrm{Eq}}(b)\cdot l|<\rho_0\langle l\rangle.$$

In particular, for the choice $\rho_0=\frac{1}{m+1}$, we can construct sequences $l_m\in\mathbb{Z}^d\setminus\{0\}$ and $b_m\in[b_0,b_1]$ such that

$$\forall q\in[\![0,q_0]\!],\quad\left|\partial_b^q\omega_{\mathrm{Eq}}(b_m)\cdot\tfrac{l_m}{\langle l_m\rangle}\right|<\tfrac{1}{m+1}.\tag{11.33}$$

Since the sequences $\left(\frac{l_m}{\langle l_m\rangle}\right)_m$ and $(b_m)_m$ are bounded, then by compactness arguments and, up to an extraction, we can assume that

$$\lim_{m\to\infty}\tfrac{l_m}{\langle l_m\rangle}=\bar{c}\neq0\quad\text{and}\quad\lim_{m\to\infty}b_m=\bar{b}.$$

Therefore, denoting

$$P_0\triangleq\omega_{\mathrm{Eq}}(X)\cdot\bar{c}\in\mathbb{R}_{2j_d}[X]$$

then passing to the limit in (11.33) as $m\to\infty$ leads to

$$\forall q\in[\![0,q_0]\!],\quad P_0^{(q)}(\bar{b})=0.$$

Hence, using the particular choice of q_0, we conclude that the polynomial $(X-\bar{b})^{2j_d+3}$ divides P_0,

$$(X-\bar{b})^{2j_d+3}|P_0.$$

Since $\deg(P_0)\leqslant2j_d$, we conclude that P_0 is identically zero. This contradicts the non-degeneracy of the equilibrium frequency vector ω_{Eq} stated in Lemma 11.4.
(ii) The case $l=0,j\in\mathbb{N}^*$ is trivially satisfied. Thus, we shall consider the case $j\in\mathbb{N}$, $l\in\mathbb{Z}^d\setminus\{0\}$. By the triangle inequality combined with the boundedness of ω_{Eq} we find

$$\left|\omega_{\mathrm{Eq}}(b)\cdot l+\tfrac{j}{2}\right|\geqslant\tfrac12|j|-|\omega_{\mathrm{Eq}}(b)\cdot l|\geqslant\tfrac12|j|-C|l|\geqslant|l|$$

provided that $|j|\geqslant C_0|l|$ for some $C_0>0$. Thus, we shall restrict the proof to indices j and l with

$$|j|\leqslant C_0|l|,\quad j\in\mathbb{N},\quad l\in\mathbb{Z}^d\setminus\{0\}.\tag{11.34}$$

Arguing by contradiction as in the previous case, we may assume the existence of sequences $l_m\in\mathbb{Z}^d\setminus\{0\}$,

$j_m \in \mathbb{N}$ satisfying (11.34) and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0, q_0]\!], \quad \left|\partial_b^q \left(\omega_{\rm Eq}(b_m) \cdot \tfrac{l_m}{\langle l_m \rangle} + \tfrac{j_m}{2\langle l_m \rangle}\right)\right| < \tfrac{1}{1+m}. \tag{11.35}$$

Since the sequences $(b_m)_m$, $\left(\frac{j_m}{2\langle l_m\rangle}\right)$ and $\left(\frac{l_m}{\langle l_m\rangle}\right)$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty} b_m = \bar{b}, \quad \lim_{m\to\infty} \tfrac{j_m}{2\langle l_m\rangle} = \bar{d} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} \tfrac{l_m}{\langle l_m\rangle} = \bar{c} \neq 0.$$

Denoting

$$Q_0 \triangleq \omega_{\rm Eq}(X) \cdot \bar{c} + \bar{d} \in \mathbb{R}_{2j_d}[X]$$

and letting $m \to \infty$ in (11.35) we obtain

$$\forall q \in [\![0, q_0]\!], \quad Q_0^{(q)}(\bar{b}) = 0.$$

Consequently, using the particular choice of q_0, we get

$$(X - \bar{b})^{2j_d+3} | Q_0.$$

Since $\deg(Q_0) \leqslant 2j_d$, we conclude that Q_0 is identically zero. This contradicts Lemma 11.4.

(iii) Consider $(l,j) \in \mathbb{Z}^d \times (\mathbb{N}^* \setminus \mathbb{S})$. Then applying the triangle inequality and Lemma 11.3-(ii), yields

$$\begin{aligned}|\omega_{\rm Eq}(b) \cdot l \pm \Omega_j(b)| &\geqslant |\Omega_j(b)| - |\omega_{\rm Eq}(b) \cdot l| \\ &\geqslant \tfrac{b_0^2}{2} j - C|l| \geqslant \langle l \rangle\end{aligned}$$

provided $j \geqslant C_0\langle l\rangle$ for some $C_0 > 0$. Thus as before we shall restrict the proof to indices j and l with

$$0 \leqslant j < C_0 \langle l \rangle, \quad j \in \mathbb{N}^* \setminus \mathbb{S} \quad \text{and} \quad l \in \mathbb{Z}^d \backslash \{0\}. \tag{11.36}$$

Proceeding by contradiction, we may assume the existence of sequences $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m \in \mathbb{N} \setminus \mathbb{S}$ satisfying (11.36) and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0, q_0]\!], \quad \left|\partial_b^q \left(\omega_{\rm Eq}(b) \cdot \tfrac{l_m}{|l_m|} \pm \tfrac{\Omega_{j_m}(b)}{|l_m|}\right)_{|b=b_m}\right| < \tfrac{1}{m+1}. \tag{11.37}$$

Since the sequences $\left(\frac{l_m}{|l_m|}\right)_m$ and $(b_m)_m$ are bounded, then up to an extraction we can assume that

$$\lim_{m\to\infty} \tfrac{l_m}{|l_m|} = \bar{c} \neq 0 \quad \text{and} \quad \lim_{m\to\infty} b_m = \bar{b}.$$

Now we shall distinguish two cases.

▶ Case ❶ : $(l_m)_m$ is bounded. In this case, by (11.36) we find that $(j_m)_m$ is bounded too and thus up to an extraction we may assume $\lim_{m\to\infty} l_m = \bar{l}$ and $\lim_{m\to\infty} j_m = \bar{\jmath}$. Since $(j_m)_m$ and $(|l_m|)_m$ are sequences of integers, then they are necessary stationary. In particular, the condition (11.36) implies $\bar{l} \neq 0$ and $\bar{\jmath} \in \mathbb{N} \setminus \mathbb{S}$. Hence, denoting

$$P_{0,\bar{\jmath}} \triangleq \omega_{\rm Eq}(X) \cdot \bar{l} \pm \Omega_{\bar{\jmath}}(X) \in \mathbb{R}_{\max(2j_d, 2\bar{\jmath})}[X],$$

then taking the limit $m \to \infty$ in (11.37), yields

$$\forall q \in [\![0, q_0]\!], \quad P_{0,\bar{\jmath}}^{(q)}(\bar{b}) = 0.$$

If $\bar{\jmath} < j_d$, then in a similar way to the point (i), we find that $P_{0,\bar{\jmath}} = 0$ which contradicts Lemma 11.4, applied with $\big(\omega_{\mathrm{Eq}}, \Omega_{\bar{\jmath}}\big)$ in place of ω_{Eq}. Hence, we shall restrict the discussion to the case $\bar{\jmath} > j_d$. Since $\omega_{\mathrm{Eq}}(X) \cdot \bar{l}$ is of degree $2j_d$, then we obtain in view of our choice of q_0 that

$$\tfrac{1}{2} q! \binom{2\bar{\jmath}}{q} b^{2\bar{\jmath} - 2j_d - 1} = \partial_b^{2j_d+1} \Omega_{\bar{\jmath}}(\bar{b}) = 0.$$

This implies that $\bar{b} = 0$ which contradicts the fact that $\bar{b} \in [b_0, b_1] \subset (0,1)$.
► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m \to \infty} |l_m| = \infty$. We have two sub-cases.
• Sub-case ① : $(j_m)_m$ is bounded. In this case and up to an extraction we can assume that it converges. Then, taking the limit $m \to \infty$ in (11.37), we find

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \omega_{\mathrm{Eq}}(\bar{b}) \cdot \bar{c} = 0.$$

Therefore, we obtain a contradiction in a similar way to the point (i).
• Sub-case ② : $(j_m)_m$ is unbounded. Then up to an extraction we can assume that $\lim\limits_{m \to \infty} j_m = \infty$. We write according to (11.15)

$$\frac{\Omega_{j_m}(b)}{|l_m|} = \frac{j_m}{2|l_m|} - \frac{1}{2|l_m|} + \frac{b^{2j_m}}{2|l_m|}. \tag{11.38}$$

By (11.36), the sequence $\left(\frac{j_m}{2|l_m|}\right)_n$ is bounded, thus up to an extraction we can assume that it converges to $\bar{d}$. Moreover, since $\lim\limits_{m \to \infty} j_m = \lim\limits_{m \to \infty} |l_m| = \infty$ and $b_m \in (b_0, b_1)$, then taking the limit in (11.38), one obtains from (11.26),

$$\lim_{m \to \infty} \tfrac{\partial_b^q \Omega_{j_m}(b)_{|b=b_m}}{|l_m|} = \begin{cases} \bar{d} & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}$$

Consequently, taking the limit $m \to \infty$ in (11.37), we have

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \left(\omega_{\mathrm{Eq}}(b) \cdot \bar{c} \pm \bar{d}\right)_{|b=\bar{b}} = 0.$$

Then, in a similar way the the point (ii), we deduce that the polynomial $\omega_{\mathrm{Eq}}(X) \cdot \bar{c} + \bar{d}$ is identically zero, which is in contradiction with Lemma 11.4.
(iv) Consider $l \in \mathbb{Z}^d, j, j' \in \mathbb{N}^* \setminus \mathbb{S}$ with $(l, j) \neq (0, j')$. Then applying the triangle inequality combined with Lemma 11.3-(iii), we infer that

$$|\omega_{\mathrm{Eq}}(b) \cdot l + \Omega_j(b) \pm \Omega_{j'}(b)| \geqslant |\Omega_j(b) \pm \Omega_{j'}(b)| - |\omega_{\mathrm{Eq}}(b) \cdot l| \geqslant \tfrac{b_0^2}{6} |j \pm j'| - C|l| \geqslant \langle l \rangle$$

provided that $|j \pm j'| \geqslant c_0 \langle l \rangle$ for some $c_0 > 0$. Then it remains to check the proof for indices satisfying

$$|j \pm j'| < c_0 \langle l \rangle, \quad l \in \mathbb{Z}^d \backslash \{0\}, \quad j, j' \in \mathbb{N}^* \setminus \mathbb{S}. \tag{11.39}$$

Reasoning by contradiction as in the previous cases, we get for all $m \in \mathbb{N}$, real numbers $l_m \in \mathbb{Z}^d \setminus \{0\}$, $j_m, j'_m \in \mathbb{N}^* \setminus \mathbb{S}$ satisfying (11.39) and $b_m \in [b_0, b_1]$ such that

$$\forall q \in [\![0, q_0]\!], \quad \left| \partial_b^q \left(\omega_{\mathrm{Eq}}(b) \cdot \tfrac{l_m}{|l_m|} + \tfrac{\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)}{|l_m|} \right)_{|b=b_m} \right| < \tfrac{1}{m+1}. \tag{11.40}$$

Up to an extraction we can assume that $\lim\limits_{m \to \infty} \frac{l_m}{|l_m|} = \bar{c} \neq 0$ and $\lim\limits_{m \to \infty} b_m = \bar{b}$.
As before we shall distinguish two cases.

► Case ❶ : $(l_m)_m$ is bounded. Up to an extraction we may assume that $\lim\limits_{m\to\infty} l_m = \bar{l} \neq 0$. Now according to (11.39) we have two sub-cases to discuss depending whether the sequences $(j_m)_m$ and $(j'_m)_m$ are simultaneously bounded or unbounded.

• Sub-case ① : $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case, up to an extraction we may assume that these sequences are stationary $j_m = \bar{\jmath}$ and $j'_m = \bar{\jmath}'$ with $\bar{\jmath}, \bar{\jmath}' \in \mathbb{N}^* \setminus \mathbb{S}$. Hence, denoting

$$P_{0,\bar{\jmath},\bar{\jmath}'} \triangleq \omega_{\mathrm{Eq}}(X) \cdot \bar{l} + \Omega_{\bar{\jmath}}(X) \pm \Omega_{\bar{\jmath}'}(X) \in \mathbb{R}_{\max(2j_d, 2\bar{\jmath}, 2\bar{\jmath}')}[X],$$

then, taking the limit $m \to \infty$ in (11.37), we have

$$\forall q \in [\![0, q_0]\!], \quad P^{(q)}_{0,\bar{\jmath},\bar{\jmath}'}(\bar{b}) = 0.$$

If $\max(\bar{\jmath}, \bar{\jmath}') < j_d$, then, we deduce that $P_{0,\bar{\jmath},\bar{\jmath}'} = 0$ which gives a contradiction as the previous cases, up to replacing ω_{Eq} by $\big(\omega_{\mathrm{Eq}}, \Omega_{\bar{\jmath}}, \Omega_{\bar{\jmath}'}\big)$. Therefore, we are left to study the case $\max(\bar{\jmath}, \bar{\jmath}') > j_d$. Notice that the cases $\bar{\jmath} = \bar{\jmath}'$ and $\min(\bar{\jmath}, \bar{\jmath}') > j_d$ are byproducts of point (i) and (iii). Without loss of generality, we may assume that $\bar{\jmath} > \bar{\jmath}' \geqslant j_d + 1$. In particular, since $\omega_{\mathrm{Eq}}(X) \cdot \bar{l}$ is of degree $2j_d$, then, according to our choice of q_0, we obtain

$$\begin{cases} C_1 \bar{b}^{\alpha} \pm C_2 \bar{b}^{\beta} = 0 \\ C_1 \alpha \bar{b}^{\alpha} \pm C_2 \beta \bar{b}^{\beta} = 0, \end{cases} \tag{11.41}$$

with

$$\alpha \triangleq 2\bar{\jmath} - 2j_d - 1, \quad \beta \triangleq 2\bar{\jmath}' - 2j_d - 1, \quad C_1 \triangleq q_0! \binom{2\bar{\jmath}}{q_0} \quad \text{and} \quad C_2 \triangleq q_0! \binom{2\bar{\jmath}'}{q_0}.$$

Since C_1 and C_2 are positive, we immediately get from the first equation in (11.41) that

$$C_1 \bar{b}^{\alpha} + C_2 \bar{b}^{\beta} = 0 \quad \Rightarrow \quad \bar{b} = 0.$$

This contradicts the fact that $\bar{b} \in [b_0, b_1] \subset (0, 1)$. In the case where we have the difference, the system (11.41) gives

$$\frac{C_2}{C_1} = \frac{C_2 \beta}{C_1 \alpha},$$

which implies in turn that $\alpha = \beta$, that is $\bar{\jmath} = \bar{\jmath}'$ which is excluded by hypothesis.

• Sub-case ② : $(j_m)_m$ and $(j'_m)_m$ are both unbounded and without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m = \lim\limits_{m\to\infty} j'_m = \infty$. Coming back to (11.15) we get the splitting

$$\frac{\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)}{|l_m|} = \frac{j_m \pm j'_m}{2|l_m|} + \frac{b^{2j_m} \pm b^{2j'_m}}{2|l_m|}.$$

Using once again (11.39) and up to an extraction we have $\lim\limits_{m\to\infty} \frac{j_m \pm j'_m}{|l_m|} = \bar{d}$. Thus

$$\lim_{m\to\infty} |l_m|^{-1} \partial_b^q \left(\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)\right)_{|b=b_m} = \begin{cases} \bar{d} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

By taking the limit as $m \to \infty$ in (11.40), we find

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \left(\omega_{\mathrm{Eq}}(b) \cdot \bar{c} + \bar{d}\right)_{|b=\bar{b}} = 0.$$

This leads to a contradiction as in the point (ii).

► Case ❷ : $(l_m)_m$ is unbounded. Up to an extraction we can assume that $\lim\limits_{m\to\infty} |l_m| = \infty$.
We shall distinguish three sub-cases.

• Sub-case ①. The sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. In this case and up to an extraction they will converge and then taking the limit in (11.40) yields,

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \omega_{\rm Eq}(\bar{b}) \cdot \bar{c} = 0.$$

which leads to a contradiction as before.
• Sub-case ②. The sequences $(j_m)_m$ and $(j'_m)_m$ are both unbounded. This is similar to the sub-case ② of the case ❶.
• Sub-case ③. The sequence $(j_m)_m$ is unbounded and $(j'_m)_m$ is bounded (the symmetric case is similar). Without loss of generality we can assume that $\lim\limits_{m\to\infty} j_m = \infty$ and $j'_m = \bar{\jmath}'$. By (11.39) and up to an extraction one gets $\lim\limits_{m\to\infty} \frac{j_m \pm j'_m}{|l_m|} = \bar{d}$. Once again, we have

$$\lim_{m\to\infty} |l_m|^{-1} \partial_b^q \left(\Omega_{j_m}(b) \pm \Omega_{j'_m}(b)\right)_{|b=b_m} = \begin{cases} \bar{d} & \text{if } q = 0 \\ 0 & \text{if } q \neq 0. \end{cases}$$

Hence, taking the limit in (11.40) implies

$$\forall q \in [\![0, q_0]\!], \quad \partial_b^q \left(\omega_{\rm Eq}(b) \cdot \bar{c} + \bar{d}\right)_{b=\bar{b}} = 0,$$

which also gives a contradiction as the previous cases. This completes the proof of Lemma 11.5. ☐

Notice that by selecting only a finite number of frequencies, the sum in (11.16) give rise to quasi-periodic solutions of the linearized equation (11.8), up to selecting the parameter b in a Cantor-like set of full measure. We have the following result.

Proposition 11.1. *Let* $0 < b_0 < b_1 < 1$, $d \in \mathbb{N}^*$ *and* $\mathbb{S} \subset \mathbb{N}^*$ *with* $|\mathbb{S}| = d$. *Then, there exists a Cantor-like set* $\mathcal{C} \subset [b_0, b_1]$ *satisfying* $|\mathcal{C}| = b_1 - b_0$ *and such that for all* $\lambda \in \mathcal{C}$, *every function in the form*

$$\rho(t,\theta) = \sum_{j\in\mathbb{S}} \rho_j \cos\big(j\theta - \Omega_j(b)t\big), \quad \rho_j \in \mathbb{R}^* \tag{11.42}$$

is a time quasi-periodic reversible solution to the equation (10.17) *with the vector frequency*

$$\omega_{\rm Eq}(b) = \big(\Omega_j(b)\big)_{j\in\mathbb{S}}.$$

The proof of this proposition follows in a similar way to Proposition 5.1.

12 Functional of interest and regularity aspects

The main goal of this section is to reformulate the problem in a dynamical system language more adapted to KAM techniques. More precisely, we shall write the equation (10.17) as a Hamiltonian perturbation of an integrable system, given by the linear dynamics at the equilibrium state. Then, by selecting finitely-many tangential sites and decomposing the phase space into tangential and normal subspaces we can introduce action-angle variables on the tangential part allowing to reformulate the problem in terms of embedded tori. This reduces the problem into the search for zeros of a functional $\mathcal{F}$ to which the Nash-Moser implicit function theorem will be applied. We shall also study in this section some regularity aspects for the perturbed Hamiltonian vector field appearing in $\mathcal{F}$ and needed during the Nash-Moser scheme. This approach has been intensively used before, for instance in [7, 8, 29, 28, 33].
Notice that, according to Lemmata 10.1 and 11.2, the equation (10.17), that is also (10.5), can be written

in the form

$$\partial_t r = \partial_\theta \mathrm{L}(b)(r) + X_P(r) \quad \text{with} \quad X_P(r) \triangleq \tfrac{1}{2}\partial_\theta r + \partial_\theta \mathcal{K}_b * r - F_b[r], \tag{12.1}$$

where the nonlinear functional $F_b[r]$ is introduced in (10.6) and the convolution kernel is given by (11.10). Since we shall look for small amplitude quasi-periodic solutions then it is more convenient to rescale the solution as follows $r \mapsto \varepsilon r$ with r bounded. Hence, the Hamiltonian equation (10.17) takes the form

$$\partial_t r = \partial_\theta \mathrm{L}(b)(r) + \varepsilon X_{P_\varepsilon}(r), \tag{12.2}$$

where X_{P_ε} is the Hamiltonian vector field defined by $X_{P_\varepsilon}(r) \triangleq \varepsilon^{-2} X_P(\varepsilon r)$. Notice that (12.2) is the Hamiltonian system generated by the rescaled Hamiltonian

$$\begin{aligned} \mathcal{H}_\varepsilon(r) &= \varepsilon^{-2} H(\varepsilon r) \\ &\triangleq H_{\mathrm{L}}(r) + \varepsilon P_\varepsilon(r), \end{aligned} \tag{12.3}$$

with H_{L} the quadratic Hamiltonian defined in Lemma 11.2 and $\varepsilon P_\varepsilon(r)$ containing terms of higher order more than cubic.

12.1 Reformulation with the action-angle and normal variables

Recall from (11.27) that the tangential set is defined by

$$\mathbb{S} \triangleq \{j_1, \ldots, j_d\} \subset \mathbb{N}^* \quad \text{with} \quad 1 \leqslant j_1 < j_2 < \ldots < j_d.$$

We now define the symmetrized tangential sets $\overline{\mathbb{S}}$ and $\mathbb{S}_0$ by

$$\overline{\mathbb{S}} \triangleq \mathbb{S} \cup (-\mathbb{S}) = \{\pm j,\ j \in \mathbb{S}\} \quad \text{and} \quad \mathbb{S}_0 = \overline{\mathbb{S}} \cup \{0\}. \tag{12.4}$$

Since the application $b \mapsto \omega_{\mathrm{Eq}}(b)$ is continuous then $\omega_{\mathrm{Eq}}\left([b_0, b_1]\right)$ is a compact subset of $\mathbb{R}^d$. In particular, there exists $R_0 > 0$ such that

$$\omega_{\mathrm{Eq}}\left((b_0, b_1)\right) \subset \mathscr{U} \triangleq B(0, R_0).$$

Then we define the set of parameters as

$$\mathcal{O} \triangleq (b_0, b_1) \times \mathscr{U}. \tag{12.5}$$

Then we decompose the phase space $L^2_0(\mathbb{T})$ into the following $L^2(\mathbb{T})$-orthogonal direct sum

$$L^2_0(\mathbb{T}) = L_{\overline{\mathbb{S}}} \overset{\perp}{\oplus} L^2_\perp, \quad L_{\overline{\mathbb{S}}} \triangleq \Big\{ \sum_{j \in \overline{\mathbb{S}}} r_j e_j,\ \overline{r_j} = r_{-j} \Big\}, \quad L^2_\perp \triangleq \Big\{ z = \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} z_j e_j \in L^2_0(\mathbb{T}) \Big\}, \tag{12.6}$$

where we denote $e_j(\theta) = e^{\mathrm{i} j \theta}$. The associated orthogonal projectors $\Pi_{\overline{\mathbb{S}}}, \Pi^\perp_{\mathbb{S}_0}$ are defined by

$$r = \sum_{j \in \mathbb{Z}^*} r_j e_j = v + z, \quad v \triangleq \Pi_{\overline{\mathbb{S}}} r \triangleq \sum_{j \in \overline{\mathbb{S}}} r_j e_j, \quad z \triangleq \Pi^\perp_{\mathbb{S}_0} r \triangleq \sum_{j \in \mathbb{Z} \setminus \mathbb{S}_0} r_j e_j, \tag{12.7}$$

where v and z are respectively called the tangential and normal variables. For fixed small amplitudes $(\mathtt{a}_j)_{j \in \overline{\mathbb{S}}} \in (\mathbb{R}^*_+)^d$ satisfying $\mathtt{a}_{-j} = \mathtt{a}_j$, we introduce the action-angle variables on the tangential set $L_{\overline{\mathbb{S}}}$ by making the following symplectic polar change of coordinates

$$\forall j \in \overline{\mathbb{S}}, \quad r_j = \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j}, \tag{12.8}$$

where

$$\forall j \in \overline{\mathbb{S}}, \quad I_{-j} = I_j \in \mathbb{R} \quad \text{and} \quad \vartheta_{-j} = -\vartheta_j \in \mathbb{T}. \tag{12.9}$$

Thus, any function r of the phase space L_0^2 can be represented as

$$r = A(\vartheta, I, z) \triangleq v(\vartheta, I) + z \quad \text{where} \quad v(\vartheta, I) \triangleq \sum_{j \in \overline{\mathbb{S}}} \sqrt{\mathtt{a}_j^2 + |j| I_j}\, e^{\mathrm{i}\vartheta_j} e_j\,. \tag{12.10}$$

Observe that the function $v(-\omega_{\mathrm{Eq}}(b)t, 0)$, where ω_{Eq} is defined in (11.28), corresponds to the solution of the linear system (11.8) described by (11.42). In these new coordinates, the involution $\mathscr{S}$ defined in (4.27) reads

$$\mathfrak{S} : (\vartheta, I, z) \mapsto (-\vartheta, I, \mathscr{S}z) \tag{12.11}$$

and the symplectic 2-form in (4.24) becomes, after straightforward computations using (12.8) and (12.9),

$$\mathcal{W} = \sum_{j \in \mathbb{S}} d\vartheta_j \wedge dI_j + \frac{1}{2} \sum_{j \in \mathbb{Z}\setminus\mathbb{S}_0} \frac{1}{\mathrm{i}j} dr_j \wedge dr_{-j} = \Big(\sum_{j \in \mathbb{S}} d\vartheta_j \wedge dI_j\Big) \oplus \mathcal{W}_{|L_\perp^2}, \tag{12.12}$$

where $\mathcal{W}_{|L_\perp^2}$ denotes the restriction of $\mathcal{W}$ to $L_\perp^2$. This proves that the transformation A defined in (12.10) is symplectic and in the action-angle and normal coordinates $(\vartheta, I, z) \in \mathbb{T}^d \times \mathbb{R}^d \times L_\perp^2$, the Hamiltonian system generated by $\mathcal{H}_\varepsilon$ in (12.3) transforms into the one generated by the Hamiltonian

$$H_\varepsilon = \mathcal{H}_\varepsilon \circ A. \tag{12.13}$$

Since $\mathrm{L}(b)$ in Lemma 11.2 preserves the subspaces $L_{\overline{\mathbb{S}}}$ and $L_\perp^2$ then the quadratic Hamiltonian H_{L} in (11.13) (see (11.14)) in the variables (ϑ, I, z) reads, up to an additive constant,

$$H_{\mathrm{L}} \circ A = -\sum_{j \in \mathbb{S}} \Omega_j(b) I_j + \frac{1}{2}\langle \mathrm{L}(b)\, z, z\rangle_{L^2(\mathbb{T})} = -\omega_{\mathrm{Eq}}(b) \cdot I + \frac{1}{2}\langle \mathrm{L}(b)\, z, z\rangle_{L^2(\mathbb{T})}, \tag{12.14}$$

where $\omega_{\mathrm{Eq}} \in \mathbb{R}^d$ is the unperturbed tangential frequency vector defined by (11.28). By (12.3) and (12.14), the Hamiltonian H_ε in (12.13) reads

$$H_\varepsilon = \mathcal{N} + \varepsilon \mathcal{P}_\varepsilon \quad \text{with} \quad \mathcal{N} \triangleq -\omega_{\mathrm{Eq}}(b) \cdot I + \frac{1}{2}\langle \mathrm{L}(b)\, z, z\rangle_{L^2(\mathbb{T})} \quad \text{and} \quad \mathcal{P}_\varepsilon \triangleq P_\varepsilon \circ A. \tag{12.15}$$

We look for an embedded invariant torus

$$\begin{array}{rccl} i: & \mathbb{T}^d & \rightarrow & \mathbb{R}^d \times \mathbb{R}^d \times L_\perp^2 \\ & \varphi & \mapsto & i(\varphi) \triangleq (\vartheta(\varphi), I(\varphi), z(\varphi)) \end{array} \tag{12.16}$$

of the Hamiltonian vector field

$$X_{H_\varepsilon} \triangleq (\partial_I H_\varepsilon, -\partial_\vartheta H_\varepsilon, \Pi_{\mathbb{S}_0}^\perp \partial_\theta \nabla_z H_\varepsilon) \tag{12.17}$$

filled by quasi-periodic solutions with Diophantine frequency vector ω. We point out that for the value $\varepsilon = 0$ the Hamiltonian system

$$\omega \cdot \partial_\varphi i(\varphi) = X_{H_0}(i(\varphi))$$

possesses, for any value of the parameter $b \in (b_0, b_1)$, the invariant torus

$$i_{\mathrm{flat}}(\varphi) \triangleq (\varphi, 0, 0). \tag{12.18}$$

Now we consider the family of Hamiltonians,

$$H_\varepsilon^\alpha \triangleq \mathcal{N}_\alpha + \varepsilon \mathcal{P}_\varepsilon \quad \text{where} \quad \mathcal{N}_\alpha \triangleq \alpha \cdot I + \frac{1}{2}\langle \mathrm{L}(b)\, z, z \rangle_{L^2(\mathbb{T})}, \tag{12.19}$$

which depends on the constant vector $\alpha \in \mathbb{R}^d$. For the value $\alpha = -\omega_{\mathrm{Eq}}(b)$ we have $H_\varepsilon^\alpha = H_\varepsilon$. The parameter α is introduced in order to ensure the validity of some compatibility conditions during the approximate inverse process. We look for zeros of the nonlinear operator

$$\begin{aligned} \mathcal{F}(i, \alpha, (b, \omega), \varepsilon) &\triangleq \omega \cdot \partial_\varphi i(\varphi) - X_{H_\varepsilon^\alpha}(i(\varphi)) \\ &= \begin{pmatrix} \omega \cdot \partial_\varphi \vartheta(\varphi) - \alpha - \varepsilon \partial_I \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi I(\varphi) + \varepsilon \partial_\vartheta \mathcal{P}_\varepsilon(i(\varphi)) \\ \omega \cdot \partial_\varphi z(\varphi) - \partial_\theta \big[\mathrm{L}(b) z(\varphi) + \varepsilon \nabla_z \mathcal{P}_\varepsilon\big(i(\varphi)\big)\big] \end{pmatrix}, \end{aligned} \tag{12.20}$$

where $\mathcal{P}_\varepsilon$ is defined in (12.3). For any $\alpha \in \mathbb{R}^d$, the Hamiltonian H_ε^α is invariant under the involution $\mathfrak{S}$ defined in (12.11),

$$H_\varepsilon^\alpha \circ \mathfrak{S} = H_\varepsilon^\alpha.$$

Thus, we look for reversible solutions of $\mathcal{F}(i, \alpha, (b, \omega), \varepsilon) = 0$, namely satisfying

$$\vartheta(-\varphi) = -\vartheta(\varphi), \quad I(-\varphi) = I(\varphi), \quad z(-\varphi) = (\mathscr{S} z)(\varphi). \tag{12.21}$$

We define the periodic component $\mathfrak{I}$ of the torus i by

$$\mathfrak{I}(\varphi) \triangleq i(\varphi) - (\varphi, 0, 0) = (\Theta(\varphi), I(\varphi), z(\varphi)) \quad \text{with} \quad \Theta(\varphi) = \vartheta(\varphi) - \varphi$$

and the weighted Sobolev norm of $\mathfrak{I}$ as

$$\|\mathfrak{I}\|_{q,s}^{\gamma,\mathcal{O}} \triangleq \|\Theta\|_{q,s}^{\gamma,\mathcal{O}} + \|I\|_{q,s}^{\gamma,\mathcal{O}} + \|z\|_{q,s}^{\gamma,\mathcal{O}}.$$

12.2 Regularity of the perturbed Hamiltonian vector field

This section is devoted to some regularity aspects of the Hamiltonian involved in the equation (10.17). We shall need the following lemma.

Lemma 12.1. *Let* (γ, q, s_0, s) *satisfy* (A.2). *There exists* $\varepsilon_0 \in (0, 1]$ *such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0,$$

then the operators $\partial_\theta \mathbf{L}_r$ *and* $\partial_\theta \mathbf{S}_r$*, defined in* (11.2) *and* (11.3) *write*

$$\partial_\theta \mathbf{L}_r = \partial_\theta \mathcal{K}_{1,b} * \cdot + \partial_\theta \mathbf{L}_{r,1} \quad \text{with} \quad \mathbf{L}_{r,1}(\rho)(b, \varphi, \theta) \triangleq \int_{\mathbb{T}} \rho(\varphi, \eta) \mathbb{K}_{r,1}(b, \varphi, \theta, \eta) d\eta, \tag{12.22}$$

$$\partial_\theta \mathbf{S}_r = \partial_\theta \mathcal{K}_{2,b} * \cdot + \partial_\theta \mathbf{S}_{r,1} \qquad \text{with} \quad \mathbf{S}_{r,1}(\rho)(b, \varphi, \theta) = \int_{\mathbb{T}} \rho(\varphi, \eta) \mathscr{K}_{r,1}(b, \varphi, \theta, \eta) d\eta \tag{12.23}$$

where $\mathcal{K}_{1,b}$, $\mathcal{K}_{2,b}$ *are given by* (11.11)-(11.12) *and the kernels* $\mathbb{K}_{r,1}(b, \varphi, \theta, \eta), \mathscr{K}_{r,1}(b, \varphi, \theta, \eta) \in \mathbb{R}$ *satisfy the following symmetry property: if* $r(-\varphi, -\theta) = r(\varphi, \theta)$ *then*

$$\mathbb{K}_{r,1}(b, -\varphi, -\theta, -\eta) = \mathbb{K}_{r,1}(b, \varphi, \theta, \eta), \tag{12.24}$$

$$\mathscr{K}_{r,1}(b, -\varphi, -\theta, -\eta) = \mathscr{K}_{r,1}(b, \varphi, \theta, \eta) \tag{12.25}$$

and the following estimates

$$\|\mathbb{K}_{r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}, \tag{12.26}$$

$$\|\mathscr{K}_{r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s}^{\gamma,\mathcal{O}}. \tag{12.27}$$

Moreover,

$$\|\partial_\theta \mathcal{K}_{1,b} * \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}, \tag{12.28}$$

$$\|\partial_\theta \mathcal{K}_{2,b} * \rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}}, \tag{12.29}$$

$$\|\partial_\theta \mathbf{L}_{r,1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s_0+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}, \tag{12.30}$$

$$\|\partial_\theta \mathbf{S}_{r,1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{12.31}$$

Proof. According to (11.17) we may write

$$\begin{aligned} A_r(\varphi,\theta,\eta) &= 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right|\left(\left(\frac{R(b,\varphi,\eta)-R(b,\varphi,\theta)}{2b\sin\left(\frac{\eta-\theta}{2}\right)}\right)^2 + \frac{1}{b^2}R(b,\varphi,\eta)R(b,\varphi,\theta)\right)^{\frac{1}{2}} \\ &\triangleq 2b\left|\sin\left(\tfrac{\eta-\theta}{2}\right)\right| v_{r,1}(b,\varphi,\theta,\eta). \end{aligned} \tag{12.32}$$

Notice that $v_{r,1}$ is smooth when r is smooth and small enough, and $v_{0,1}=1$. More precisely, by using Lemma A.1-(iv)-(v) combined with Lemma A.2-(ii) and the smallness condition on r, we get

$$\|v_{r,1}-1\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s+1}^{\gamma,\mathcal{O}}. \tag{12.33}$$

Using the morphism property of the logarithm, we can write

$$\begin{aligned} \log(A_r(b,\varphi,\theta,\eta)) &= \log\big(2b\big) + \tfrac{1}{2}\log\left(\sin^2\left(\tfrac{\eta-\theta}{2}\right)\right) + \log\left(v_{r,1}(b,\varphi,\theta,\eta)\right) \\ &\triangleq \log\big(2b\big) + \mathcal{K}_{1,b}(\eta-\theta) + \mathbb{K}_{r,1}(b,\varphi,\theta,\eta) \end{aligned} \tag{12.34}$$

and (12.24) immediately follows. Moreover, (11.2) and (12.34) give (12.22). Applying Lemma A.1-(v) together with (12.33) and the smallness condition on r, we obtain (12.26). Using (12.26), Lemma A.7-(ii) and the smallness property on r, we get (12.30). Similarly, from (11.18) we can link B_r^2 to B_0^2 by

$$\begin{aligned} B_r^2(b,\varphi,\theta,\eta) &= B_0^2(b,\varphi,\theta,\eta) + \Big(R^2(b,\varphi,\theta)R^2(b,\varphi,\eta)-b^4\Big) - 2\Big(R(b,\varphi,\theta)R(b,\varphi,\eta)-b^2\Big)\cos(\eta-\theta) \\ &= B_0^2(b,\varphi,\theta,\eta)\big(1+P_r(b,\varphi,\theta,\eta)\big) \end{aligned}$$

with

$$P_r(b,\varphi,\theta,\eta) \triangleq \tfrac{\left(R^2(b,\varphi,\theta)R^2(b,\varphi,\eta)-b^4\right)-2\left(R(b,\varphi,\theta)R(b,\varphi,\eta)-b^2\right)\cos(\eta-\theta)}{1+b^4-2b^2\cos(\eta-\theta)}$$

so that we can write

$$\begin{aligned} \log\big(B_r(b,\varphi,\theta,\eta)\big) &= \log\big(B_0(b,\varphi,\theta,\eta)\big) + \tfrac{1}{2}\log\big(1+P_r(b,\varphi,\theta,\eta)\big) \\ &\triangleq \mathcal{K}_{2,b}(\eta-\theta) + \mathscr{K}_{r,1}(b,\varphi,\theta,\eta) \end{aligned} \tag{12.35}$$

and (12.25) immediately follows. Moreover, (11.3) and (12.35) give (12.23). Notice that that P_r is smooth with respect to each variable and with respect to r with $P_0=0$. We conclude by Lemma A.1-(iv)-(v) and the smallness property on r that

$$\|P_r\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \|r\|_{q,s}^{\gamma,\mathcal{O}}.$$

As a consequence, composition laws in Lemma A.1 together with the smallness property on r imply (12.27). Then, using (12.27), Lemma A.7-(ii) and the smallness property on r, we get (12.31). The estimates (12.28)-(12.29) can be obtained using (11.22), (11.24) and Leibniz rule combined with the following estimate

$$\sup_{n\in\mathbb{N}}\|b\mapsto b^n\|_q^{\gamma,\mathcal{O}}\lesssim 1.$$

This ends the proof of Lemma 12.1. $\square$

We now provide tame estimates for the vector field X_P defined in (12.1).

Lemma 12.2. *Let* (γ,q,s_0,s) *satisfy* (A.2). *There exists* $\varepsilon_0\in(0,1]$ *such that if*

$$\|r\|_{q,s_0+2}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0,$$

then the vector field X_P, *defined in* (12.1) *satisfies the following estimates*

(i) $\|X_P(r)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(ii) $\|d_rX_P(r)[\rho]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+2}^{\gamma,\mathcal{O}}\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\|d_r^2X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\big(\|\rho_1\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

Proof. We first prove the estimate (iii) and the estimates (ii) and (i) then follow by Taylor formula since $d_rX_P(0)=0$ and $X_P(0)=0$. Recall from Lemma 11.1, (12.22) and (12.23) that

$$d_rX_H(r)[\rho]=-d_rF_b(r)[\rho]=-\partial_\theta\left(V_r\rho\right)-\partial_\theta\mathcal{K}_b*\rho-\partial_\theta\mathbf{L}_{r,1}\rho+\partial_\theta\mathbf{S}_{r,1}\rho.$$

According to (12.1), P is is the Hamiltonian generated by higher order more than cubic terms $H_{\geqslant 3}$. Then differentiating with respect to r the last expression we obtain

$$d_r^2X_P(r)[\rho_1,\rho_2]=-\partial_\theta\left(\big(d_rV_r[\rho_2]\big)\rho_1\right)-\partial_\theta\left(d_r\mathbf{L}_{r,1}[\rho_2]\rho_1\right)+\partial_\theta\left(d_r\mathbf{S}_{r,1}[\rho_2]\rho_1\right).\tag{12.36}$$

Recall, from (12.22) and (12.34), that

$$\mathbf{L}_{r,1}(\rho)(b,\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\log(v_{r,1}(b,\varphi,\theta,\eta))d\eta.\tag{12.37}$$

Hence by differentiation we obtain

$$d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta)=\frac{1}{2}\int_{\mathbb{T}}\rho_1(\varphi,\eta)\tfrac{\left(d_rv_{r,1}^2\right)[\rho_2](b,\varphi,\theta,\eta)}{v_{r,1}^2(b,\varphi,\theta,\eta)}d\eta.\tag{12.38}$$

Coming back to (12.32) it is obvious that the dependence in r of the functional $v_{r,1}^2$ is smooth. Straightforward calculations lead to

$$\frac{1}{2}d_rv_{r,1}^2(r)[\rho_2](b,\varphi,\theta,\eta)=\tfrac{R(b,\varphi,\theta)-R(b,\varphi,\eta)}{\sin^2\left(\frac{\eta-\theta}{2}\right)}\left(\tfrac{\rho_2(\varphi,\theta)}{R(b,\varphi,\theta)}-\tfrac{\rho_2(\varphi,\eta)}{R(b,\varphi,\eta)}\right)+\tfrac{\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)+\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)}{R(b,\varphi,\theta)R(b,\varphi,\eta)}.$$

Using (12.33) combined with the law products stated in Lemma A.1, Lemma A.2-(ii) and the smallness condition of Lemma 12.2 we find that

$$\|d_rv_{r,1}^2(r)[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{12.39}$$

According to (12.39), (12.38) and using Lemma A.1-(iv)-(v), Lemma A.7-(ii) and the smallness condition we obtain,

$$\begin{aligned}\|\partial_\theta d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim&\|d_r\mathbf{L}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ \lesssim&\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\big(\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big).\end{aligned}\tag{12.40}$$

Now we shall move to the estimate of $d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta)$. By differentiating with respect to r in (12.23) and (12.35), we obtain

$$d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1(b,\varphi,\theta)=\frac{1}{2}\int_{\mathbb{T}}\rho_1(\varphi,\eta)\frac{\big(d_rB_r^2\big)[\rho_2](b,\varphi,\theta,\eta)}{B_r^2(b,\varphi,\theta,\eta)}d\eta.$$

In view of (11.18), direct computations yield

$$\begin{aligned}\frac{1}{2}d_rB_r^2(r)[\rho_2](b,\varphi,\theta,\eta)=&\ \rho_2(\varphi,\theta)R^2(b,\varphi,\eta)+\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)\\ &-\Big(\rho_2(\varphi,\theta)\tfrac{R(b,\varphi,\eta)}{R(b,\varphi,\theta)}+\rho_2(\varphi,\eta)\tfrac{R(b,\varphi,\theta)}{R(b,\varphi,\eta)}\Big)\cos(\eta-\theta).\end{aligned}$$

Then, Lemma A.1-(*iv*)-(*v*) and the smallness condition on r imply

$$\|d_rB_r^2(r)[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|r\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

It follows from Lemma A.7-(*ii*), that

$$\begin{aligned}\|\partial_\theta d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1\|_{q,s}^{\gamma,\mathcal{O}} \lesssim&\|d_r\mathbf{S}_{r,1}(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\ \lesssim&\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\big(\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big).\end{aligned}\tag{12.41}$$

Next we shall move to the estimate of $d_rV_r[\rho_2]$. From Lemma 11.1, we can write

$$\begin{aligned}V_r=V_r^0+V_r^1+V_r^2,\quad\text{with}\quad &V_r^0(b,\varphi,\theta)\triangleq-\tfrac{1}{2}\int_{\mathbb{T}}\tfrac{R^2(b,\varphi,\eta)}{R^2(b,\varphi,\theta)}d\eta,\\ &V_r^1(b,\varphi,\theta)\triangleq-\tfrac{1}{R(b,\varphi,\theta)}\int_{\mathbb{T}}\log\big(A_r(b,\varphi,\theta,\eta)\big)\partial_\eta\big(R(b,\varphi,\eta)\sin(\eta-\theta)\big)d\eta,\\ &V_r^2(b,\varphi,\theta)\triangleq-\tfrac{1}{R^3(b,\varphi,\theta)}\int_{\mathbb{T}}\log\big(B_r(b,\varphi,\theta,\eta)\big)\partial_\eta\big(R(b,\varphi,\eta)\sin(\eta-\theta)\big)d\eta.\end{aligned}$$

Differentiating V_r^0 with respect to r in the direction ρ_2 yields

$$d_rV_r^0(r)[\rho_2](\theta)=-\int_{\mathbb{T}}\frac{\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)-\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)}{R^4(b,\varphi,\theta)}d\eta.$$

Law products in Lemma A.1 and the smallness condition in r then imply

$$\|d_rV_r^0(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|r\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.\tag{12.42}$$

Differentiating V_r^1 with respect to r in the direction ρ_2 gives

$$\begin{aligned}d_rV_r^1(r)[\rho_2](\theta)=&-\int_{\mathbb{T}}\log\big(A_r(b,\varphi,\theta,\eta)\big)\,\partial_\eta d_rf_r[\rho_2](b,\varphi,\theta,\eta)d\eta\\ &-\frac{1}{2}\int_{\mathbb{T}}\frac{\big(d_rv_{r,1}^2\big)[\rho_2](b,\varphi,\theta,\eta)}{v_{r,1}^2(b,\varphi,\theta,\eta)}\partial_\eta f_r(b,\varphi,\theta,\eta)d\eta\\ \triangleq&\ \mathcal{I}_1(\theta)+\mathcal{I}_2(\theta),\end{aligned}$$

with

$$f_r(b,\varphi,\theta,\eta)\triangleq\tfrac{R(b,\varphi,\eta)}{R(b,\varphi,\theta)}\sin(\eta-\theta).$$

Straightforward computations give

$$d_rf_r[\rho_2](b,\varphi,\theta)=\tfrac{\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)-\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)}{R^3(b,\varphi,\theta)R(b,\varphi,\eta)}\sin(\eta-\theta).$$

Then, by law products and composition laws in Lemma A.1 we immediately deduce that

$$\|\partial_\eta f_r\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim 1+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}, \tag{12.43}$$

$$\|\partial_\eta d_rf_r[\rho_2]\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\big(1+\|r\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\big(1+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\big)\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{12.44}$$

The following estimate on $\mathcal{I}_2$ can be obtained combining (12.39), (12.33) and (12.43) together with Lemma A.1-(iv)-(v) and the smallness property on r.

$$\|\mathcal{I}_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{12.45}$$

As for $\mathcal{I}_1$ we argue in a similar way to Lemma 12.1 to get

$$\|\mathcal{I}_1\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{12.46}$$

Putting together (12.45) and (12.46) yields

$$\|d_rV_r^1(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{12.47}$$

Differentiating V_r^2 with respect to r in the direction ρ_2 yields

$$\begin{aligned}d_rV_r^2(r)[\rho_2](b,\varphi,\theta)=&-\int_{\mathbb{T}}\log\big(B_r(b,\varphi,\theta,\eta)\big)\partial_\eta\Big(\tfrac{\rho_2(\varphi,\eta)R^2(b,\varphi,\theta)-3\rho_2(\varphi,\theta)R^2(b,\varphi,\eta)}{R^5(b,\varphi,\theta)R(b,\varphi,\eta)}\sin(\eta-\theta)\Big)\,d\eta\\&-\frac{1}{2}\int_{\mathbb{T}}\tfrac{\left(d_rB_r^2\right)[\rho_2](b,\varphi,\theta,\eta)}{B_r^2(b,\varphi,\theta,\eta)}\partial_\eta\Big(\tfrac{R(b,\varphi,\eta)}{R^3(b,\varphi,\theta)}\sin(\eta-\theta)\Big)d\eta.\end{aligned}$$

Arguing in a similar way as above we find

$$\|d_rV_r^2(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{12.48}$$

Putting together (12.42), (12.47) and (12.48) gives

$$\|d_rV_r(r)[\rho_2]\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_2\|_{q,s+1}^{\gamma,\mathcal{O}}+\|r\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{12.49}$$

Therefore, according to the law products in Lemma A.1, (12.49) and the smallness condition we obtain

$$\begin{aligned}\big\|\partial_\theta\big(d_rV_r(r)[\rho_2]\rho_1\big)\big\|_{q,s}^{\gamma,\mathcal{O}}&\leqslant\|d_rV_r(r)[\rho_2]\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\big\|d_rV_r(r)[\rho_2]\big\|_{q,s+1}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s_0}^{\gamma,\mathcal{O}}+\big\|d_rV_r(r)[\rho_2]\big\|_{q,s_0}^{\gamma,\mathcal{O}}\big\|\rho_1\big\|_{q,s+1}^{\gamma,\mathcal{O}}\\&\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}}+\|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Combining the latter estimate with (12.36), (12.40) and (12.41) allows to get

$$\|d_r^2 X_P(r)[\rho_1,\rho_2]\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s+2}^{\gamma,\mathcal{O}} + \|r\|_{q,s+2}^{\gamma,\mathcal{O}}\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\rho_1\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0+1}^{\gamma,\mathcal{O}}.$$

Using Sobolev embeddings we get the desired result. This concludes the proof of Lemma 12.2. ☐

Notice in particular that Lemma 12.2-(i) implies that there is no singlarity in ε for the rescaled vector field X_{P_ε} defined in (12.2). Based on the previous lemma we obtain tame estimates for the Hamiltonian vector field

$$X_{\mathcal{P}_\varepsilon} = (\partial_I \mathcal{P}_\varepsilon, -\partial_\vartheta \mathcal{P}_\varepsilon, \Pi_{\mathbb{S}}^{\perp}\partial_\theta \nabla_z \mathcal{P}_\varepsilon)$$

defined by (12.15) and (12.17). The proof can be done in a similar way to [33, Lem. 5.1].

Lemma 12.3. *Let* (γ, q, s_0, s) *satisfy* (A.2). *There exists* $\varepsilon_0 \in (0,1)$ *such that if*

$$\varepsilon \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}\|_{q,s_0+2}^{\gamma,\mathcal{O}} \leqslant 1,$$

then the perturbed Hamiltonian vector field $X_{\mathcal{P}_\varepsilon}$ *satisfies the following tame estimates,*

(i) $\|X_{\mathcal{P}_\varepsilon}(i)\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}$.

(ii) $\big\|d_i X_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath}\,]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\,\widehat{\imath}\,\|_{q,s+2}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\|\,\widehat{\imath}\,\|_{q,s_0+1}^{\gamma,\mathcal{O}}$.

(iii) $\big\|d_i^2 X_{\mathcal{P}_\varepsilon}(i)[\widehat{\imath},\widehat{\imath}\,]\big\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\,\widehat{\imath}\,\|_{q,s+2}^{\gamma,\mathcal{O}}\|\,\widehat{\imath}\,\|_{q,s_0+1}^{\gamma,\mathcal{O}} + \|\mathfrak{I}\|_{q,s+2}^{\gamma,\mathcal{O}}\big(\|\,\widehat{\imath}\,\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big)^2$.

13 Construction of an approximate right inverse

In order to apply a modified Nash-Moser scheme, we need to construct an approximate right inverse of the linearized operator associated to the functional $\mathcal{F}$, that is

$$d_{(i,\alpha)}\mathcal{F}(i_0,\alpha_0)[\widehat{\imath},\widehat{\alpha}] = \omega\cdot\partial_\varphi \widehat{\imath} - d_i X_{H_\varepsilon^{\alpha_0}}(i_0(\varphi))[\widehat{\imath}] - (\widehat{\alpha},0,0). \tag{13.1}$$

where $\mathcal{F}$ is defined in (12.20), $\alpha_0 : \mathcal{O} \to \mathbb{R}^d$ is a vector-valued function and $i_0 = (\vartheta_0, I_0, z_0)$ is an arbitrary torus close to the flat one and satisfying the reversibility condition

$$\vartheta_0(-\varphi) = -\vartheta_0(\varphi), \quad I_0(-\varphi) = I_0(\varphi) \quad \text{and} \quad z_0(-\varphi) = (\mathscr{S} z_0)(\varphi). \tag{13.2}$$

For this aim, we may follow the procedure introduced in [21] and slightly simplified in [87, Sec. 6]. The main idea consists in conjugating (13.1) by a linear diffeomorphism of the toroidal phase space $\mathbb{T}^d \times \mathbb{R}^d \times L^2_\perp$ to a triangular system in the action-angles-normal variables up to small fast decaying error terms and terms vanishing at an exact solution. Then, to solve the triangular system we are led to almost invert the linearized operator in the normal directions, given by

$$\widehat{\mathcal{L}}_\omega \triangleq \Pi_{\mathbb{S}_0}^{\perp}\Big(\omega\cdot\partial_\varphi - \partial_\theta(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) - \varepsilon\partial_\theta \mathcal{R}(\varphi)\Big)\Pi_{\mathbb{S}_0}^{\perp}, \tag{13.3}$$

where $H_\varepsilon^{\alpha_0}$ is given by (12.19),

$$\mathcal{R}(\varphi) \triangleq L_2^\top(\varphi)\partial_I\nabla_I \mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi) + L_2^\top(\varphi)\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)) + \partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi), \tag{13.4}$$

$\mathcal{P}_\varepsilon$ is defined by (12.15) and

$$L_2(\phi) \triangleq -[(\partial_\vartheta \widetilde{z}_0)(\vartheta_0(\phi))]^\top \partial_\theta^{-1}, \quad \widetilde{z}_0(\vartheta) \triangleq z_0(\vartheta_0^{-1}(\vartheta)). \tag{13.5}$$

Here, for any linear operator $A \in \mathcal{L}(\mathbb{R}^d, L^2_\perp)$ the transposed operator $A^\top : L^2_\perp \to \mathbb{R}^d$ is defined through the duality relation

$$\forall\, u \in L^2_\perp\,, \quad \forall\, v \in \mathbb{R}^d, \quad \big\langle A^\top u, v\big\rangle_{\mathbb{R}^d} = \big\langle u, Av\big\rangle_{L^2(\mathbb{T}^d)}. \tag{13.6}$$

We point out the presence of the remainder term due to the linear change of variables performed to decouple the dynamics of the action-angle components from the normal ones. For more details we refer the reader to [87, Sec. 6].

13.1 Linearized operator in the normal direction

Our main goal here is to explore the structure of the linear operator $\widehat{\mathcal{L}}_\omega$, introduced in (13.3). We have the following result. The following lemma describes the asymptotic structure of $\widehat{\mathcal{L}}_\omega$ around the equilibrium state, described in Lemma 11.1.

Proposition 13.1. *Let (γ, q, d, s_0) satisfy (A.2). Then, the operator $\widehat{\mathcal{L}}_\omega$ defined in (13.3) takes the form*

$$\widehat{\mathcal{L}}_\omega = \Pi^\perp_{\mathbb{S}_0}\Big(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R}\Big)\Pi^\perp_{\mathbb{S}_0}, \tag{13.7}$$

where

(i) the operator $\mathcal{L}_{\varepsilon r}$ is given by

$$\mathcal{L}_{\varepsilon r} \triangleq \omega\cdot\partial_\varphi + \partial_\theta\big(V_{\varepsilon r}\,\cdot\,\big) + \partial_\theta\mathbf{L}_{\varepsilon r} - \partial_\theta\mathbf{S}_{\varepsilon r}, \tag{13.8}$$

with $V_{\varepsilon r}$, $\mathbf{L}_{\varepsilon r}$ and $\mathbf{S}_{\varepsilon r}$ defined by (11.1), (11.2) and (11.3).

(ii) the function r is given by

$$r(\varphi, \cdot) = A(i_0(\varphi)), \tag{13.9}$$

satisfies the following symmetry property

$$r(-\varphi, -\theta) = r(\varphi, \theta) \tag{13.10}$$

and the following estimates

$$\|r\|^{\gamma,\mathcal{O}}_{q,s} \lesssim 1 + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s}\,, \tag{13.11}$$

$$\|\Delta_{12} r\|^{\gamma,\mathcal{O}}_{q,s} \lesssim \|\Delta_{12} i\|^{\gamma,\mathcal{O}}_{q,s} + \|\Delta_{12} i\|^{\gamma,\mathcal{O}}_{q,s_0} \max_{j\in\{1,2\}} \|\mathfrak{I}_j\|^{\gamma,\mathcal{O}}_{q,s}\,. \tag{13.12}$$

(iii) the operator $\mathcal{R}$, defined in (13.4), is an integral operator with kernel J satisfying the symmetry property

$$J(-\varphi, -\theta, -\eta) = J(\varphi, \theta, \eta) \tag{13.13}$$

and the following estimates: for all $\ell \in \mathbb{N}$,

$$\sup_{\eta\in\mathbb{T}} \|(\partial^\ell_\theta J)(*, \cdot, {\scriptscriptstyle\blacksquare}, \eta + {\scriptscriptstyle\blacksquare})\|^{\gamma,\mathcal{O}}_{q,s} \lesssim 1 + \|\mathfrak{I}_0\|^{\gamma,\mathcal{O}}_{q,s+3+\ell}, \tag{13.14}$$

$$\sup_{\eta\in\mathbb{T}} \|\Delta_{12}(\partial^\ell_\theta J)(*, \cdot, {\scriptscriptstyle\blacksquare}, \eta + {\scriptscriptstyle\blacksquare})\|^{\gamma,\mathcal{O}}_{q,s} \lesssim \|\Delta_{12} i\|^{\gamma,\mathcal{O}}_{q,s+3+\ell} + \|\Delta_{12} i\|^{\gamma,\mathcal{O}}_{q,s_0+3} \max_{j\in\{1,2\}} \|\mathfrak{I}_j\|^{\gamma,\mathcal{O}}_{q,s+3+\ell}. \tag{13.15}$$

where $*, \cdot, {\scriptscriptstyle\blacksquare}$, denote successively the variables b, φ, θ and $\mathfrak{I}_j(\varphi) = i_j(\varphi) - (\varphi, 0, 0)$.

Proof. From (12.19), (12.13), (12.10) and (12.3) we obtain

$$\begin{aligned}(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) &= \mathrm{L}(b)\Pi_{\mathbb{S}_0}^{\perp} + \varepsilon\partial_z\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))\\ &= \mathrm{L}(b)\Pi_{\mathbb{S}_0}^{\perp} + \varepsilon\Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r P_\varepsilon(A(i_0(\varphi)))\\ &= \Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r\mathcal{H}_\varepsilon(A(i_0(\varphi)))\\ &= \Pi_{\mathbb{S}_0}^{\perp}\partial_r\nabla_r H(\varepsilon A(i_0(\varphi))).\end{aligned}$$

According to the general form of the linearized operator stated in Lemma 11.1 one has

$$-\partial_\theta(\partial_z\nabla_z H_\varepsilon^{\alpha_0})(i_0(\varphi)) = \Pi_{\mathbb{S}_0}^{\perp}\big(\partial_\theta\big(V_{\varepsilon r}(b,\varphi,\boldsymbol{\cdot})\,\cdot\,\big) + \partial_\theta\mathbf{L}_{\varepsilon r} - \partial_\theta\mathbf{S}_{\varepsilon r}\big)\Pi_{\mathbb{S}_0}^{\perp}.$$

Inserting this identity into (13.3) gives (13.7). The operator $\mathcal{R}(\varphi)$ in (13.4) may be written as

$$\mathcal{R}(\varphi) = \mathcal{R}_1(\varphi) + \mathcal{R}_2(\varphi) + \mathcal{R}_3(\varphi), \quad \text{with} \quad \begin{aligned}\mathcal{R}_1(\varphi) &\triangleq L_2^{\top}(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi),\\ \mathcal{R}_2(\varphi) &\triangleq L_2^{\top}(\varphi)\partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi) &\triangleq \partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi))L_2(\varphi).\end{aligned}$$

Notice that $\mathcal{R}_1(\varphi)$, $\mathcal{R}_2(\varphi)$ and $\mathcal{R}_3(\varphi)$ have a finite-dimensional rank. In fact, from (13.5) and (13.6) one may write

$$L_2(\varphi)[\rho] = \sum_{k=1}^{d}\big\langle L_2(\varphi)[\rho], \underline{e}_k\big\rangle_{\mathbb{R}^d}\,\underline{e}_k = \sum_{k=1}^{d}\big\langle \rho, L_2^{\top}(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}\,\underline{e}_k,$$

with $(\underline{e}_k)_{k=1}^{d}$ being the canonical basis of $\mathbb{R}^d$. Hence

$$\begin{aligned}\mathcal{R}_1(\varphi)[\rho] &= \sum_{k=1}^{d}\big\langle \rho, L_2^{\top}(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_1(\varphi)[\underline{e}_k] \qquad \text{with} \quad A_1(\varphi) = L_2^{\top}(\varphi)\partial_I\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)),\\ \mathcal{R}_3(\varphi)[\rho] &= \sum_{k=1}^{d}\big\langle \rho, L_2^{\top}(\varphi)[\underline{e}_k]\big\rangle_{L^2(\mathbb{T})}A_3(\varphi)[\underline{e}_k] \qquad \text{with} \quad A_3(\varphi) = \partial_I\nabla_z\mathcal{P}_\varepsilon(i_0(\varphi)).\end{aligned}$$

Analogously, since $A_2(\varphi) \triangleq \partial_z\nabla_I\mathcal{P}_\varepsilon(i_0(\varphi)) : L^2_{\perp} \to \mathbb{R}^d$, then we may write

$$\mathcal{R}_2(\varphi)[\rho] = \sum_{k=1}^{d}\big\langle \rho, A_2^{\top}(\varphi)[\underline{e}_k]\big\rangle_{L^2\,(\mathbb{T})}L_2^{\top}(\varphi)[\underline{e}_k].$$

By setting

$$\begin{aligned}g_{k,1}(\varphi,\theta) = g_{k,3}(\varphi,\theta) = \chi_{k,2}(\varphi,\theta) &\triangleq L_2^{\top}(\varphi)[\underline{e}_k](\theta), \qquad & g_{k,2}(\varphi,\theta) &\triangleq A_2^{\top}(\varphi)[\underline{e}_k](\theta),\\ \chi_{k,1}(\varphi,\theta) &\triangleq A_1(\varphi)[\underline{e}_k](\theta), \qquad & \chi_{k,3}(\varphi,\theta) &\triangleq A_3(\varphi)[\underline{e}_k](\theta),\end{aligned}$$

we can see that the operator $\mathcal{R}$ takes the integral form

$$\mathcal{R}\rho(\varphi,\theta) = \int_{\mathbb{T}}\rho(\varphi,\eta)J(\varphi,\theta,\eta)d\eta, \quad \text{with} \quad J(\varphi,\theta,\eta) \triangleq \sum_{k'=1}^{3}\sum_{k=1}^{d} g_{k,k'}(\varphi,\eta)\chi_{k,k'}(\varphi,\theta).$$

The symmetry property (13.13) is a consequence of the definition of r and the reversibility condition (13.2) imposed on the torus i_0. The estimates (13.15), (13.14), (13.11) and (13.12) are straightforward and follow in a similar way to Proposition 7.1. ☐

13.2 Diagonalization of the linearized operator in the normal directions

This section is devoted to the reduction of the linearized operator $\widehat{\mathcal{L}}_\omega$, defined in (13.7), to constant coefficients. This procedure is done in three steps. First, we reduce the operator $\mathcal{L}_{\varepsilon r}$ introduced in (13.8) up to smoothing reminders. Then we study the action of the localization in the normal directions. Finally, we almost eliminate the remainders by using a KAM reduction procedure. We fix the following parameters.

$$\begin{aligned} s_l &\triangleq s_0+\tau_1 q+\tau_1+2, & \overline{\mu}_2 &\triangleq 4\tau_1 q+6\tau_1+3,\\ \overline{s}_l &\triangleq s_l+\tau_2 q+\tau_2, & \overline{s}_h &\triangleq \tfrac{3}{2}\overline{\mu}_2+s_l+1. \end{aligned} \tag{13.16}$$

13.2.1 Leading orders reduction

In this section, we shall straighten the transport part by using a suitable quasi-periodic symplectic change of variables and look at its conjugation action on the nonlocal terms. The reduction of the transport part is done by a KAM iterative scheme in a similar way to Proposition 7.2. The result reads as follows.

Proposition 13.2. *Let* $(\gamma,q,d,\tau_1,s_0,\overline{\mu}_2,s_l,\overline{s}_h,S)$ *satisfy* (A.2), (A.1) *and* (13.16). *Let* $\upsilon\in\left(0,\frac{1}{q+2}\right]$. *We set*

$$\sigma_1=s_0+\tau_1 q+2\tau_1+4 \quad \text{and} \quad \sigma_2=s_0+\sigma_1+3. \tag{13.17}$$

For any $(\mu_2,\mathtt{p},s_h)$ *satisfying*

$$\mu_2\geqslant\overline{\mu}_2,\quad \mathtt{p}\geqslant 0,\quad s_h\geqslant\max\left(\frac{3}{2}\mu_2+s_l+1,\overline{s}_h+\mathtt{p}\right), \tag{13.18}$$

there exists $\varepsilon_0>0$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2}\leqslant\varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_2}^{\gamma,\mathcal{O}}\leqslant 1, \tag{13.19}$$

then following assertions hold true.

1. *There exist*

$$V_{i_0}^\infty\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C}) \qquad \textit{and} \qquad \beta\in W^{q,\infty,\gamma}(\mathcal{O},H^S)$$

such that with $\mathscr{B}$ *defined in* (A.12) *one gets the following results.*

   (i) *The function* $V_{i_0}^\infty$ *satisfies the estimate:*

$$\|V_{i_0}^\infty-\tfrac{1}{2}\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon. \tag{13.20}$$

   (ii) *The transformations* $\mathscr{B}^{\pm1},\mathcal{B}^{\pm1},\beta$ *and* $\widehat{\beta}$ *satisfy the following estimates: for all* $s\in[s_0,S]$,

$$\|\mathscr{B}^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|\mathcal{B}^{\pm1}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}, \tag{13.21}$$

$$\|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\beta\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right). \tag{13.22}$$

   Moreover, β *and* $\widehat{\beta}$ *satisfy the following symmetry condition:*

$$\beta(\mu,-\varphi,-\theta)=-\beta(\mu,\varphi,\theta) \quad \text{and} \quad \widehat{\beta}(\mu,-\varphi,-\theta)=-\widehat{\beta}(\mu,\varphi,\theta). \tag{13.23}$$

   (iii) *Let* $n\in\mathbb{N}$, *then in the truncated Cantor set*

$$\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ |l|\leqslant N_n}}\left\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\left|\omega\cdot l+jV_{i_0}^\infty(b,\omega)\right|>\tfrac{4\gamma^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\}, \tag{13.24}$$

we have the decomposition

$$\mathfrak{L}_{\varepsilon r}\triangleq\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}=\omega\cdot\partial_\varphi+V_{i_0}^{\infty}\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0,$$

where $\mathcal{L}_{\varepsilon r}$ *is given by* (13.8), $\mathcal{K}_b$ *is defined in Lemma 11.2 and* $\mathtt{E}_n^0=\mathtt{E}_n^0(b,\omega,i_0)$ *is a linear operator satisfying*

$$\|\mathtt{E}_n^0\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\lesssim\varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}.\tag{13.25}$$

The operator $\mathfrak{R}_{\varepsilon r}$ *is a real and reversibility preserving integral operator satisfying*

$$\forall s\in[s_0,S],\qquad\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_2}^{\gamma,\mathcal{O}}\right).\tag{13.26}$$

2. *Given two tori* i_1 *and* i_2 *both satisfying* (13.19), *we have*

$$\|\Delta_{12}V_i^{\infty}\|_q^{\gamma,\mathcal{O}}\lesssim\varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}},\tag{13.27}$$

$$\|\Delta_{12}\beta\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_1}^{\gamma,\mathcal{O}}.\tag{13.28}$$

In addition, we have

$$\max_{k\in\{0,1\}}\|\Delta_{12}(\partial_\theta^k\mathfrak{R}_{\varepsilon r})\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_2}^{\gamma,\mathcal{O}}.\tag{13.29}$$

Proof. Notice that along the proof, to simplify the notation, we shall omit the dependence with respect to the parameters b,ω kipping in mind that the functions appearing actually depend on them. We begin by setting

$$V_0=\tfrac{1}{2}\quad\text{and}\quad f_0(\varphi,\theta)\triangleq V_{\varepsilon r}(\varphi,\theta)-\tfrac{1}{2},\tag{13.30}$$

with $V_{\varepsilon r}$ defined by (11.1). According to (13.10) and (11.4), one gets

$$f_0(-\varphi,-\theta)=f_0(\varphi,\theta).\tag{13.31}$$

Notice that according to (11.1), (12.49) and Taylor formula, one has

$$\|f_0\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\left(1+\|\mathfrak{I}_0\|_{q,s+1}^{\gamma,\mathcal{O}}\right).\tag{13.32}$$

These properties allow to apply Proposition 7.2, whose proof is based on a KAM iterative scheme reduction of the perturbation term f_0 and construct β and $V_{i_0}^{\infty}$. In particular, for any $n\in\mathbb{N}$, we are able to construct a Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ in the form (13.24) in which the following reduction holds

$$\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi+\partial_\theta\big(V_{\varepsilon r}\cdot\big)\big)\mathscr{B}=\omega\cdot\partial_\varphi+V_{i_0}^{\infty}\partial_\theta+\mathtt{E}_n^0,\tag{13.33}$$

where $\mathtt{E}_n^0$ is an operator enjoying the decay property stated in (13.25). Using (13.33), (12.22), (12.23) and Lemma A.3-(i), one obtains in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ the following decomposition

$$\begin{aligned}\mathscr{B}^{-1}\mathcal{L}_{\varepsilon r}\mathscr{B}&=\mathscr{B}^{-1}\big(\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)\big)\mathscr{B}+\mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r}\mathscr{B}-\mathscr{B}^{-1}\partial_\theta\mathbf{S}_{\varepsilon r}\mathscr{B}\\&=\omega\cdot\partial_\varphi+V_{i_0}^{\infty}\partial_\theta+\mathscr{B}^{-1}\partial_\theta\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}+\mathscr{B}^{-1}\partial_\theta\mathbf{L}_{\varepsilon r,1}\mathscr{B}\\&\quad-\mathscr{B}^{-1}\partial_\theta\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathscr{B}^{-1}\partial_\theta\mathbf{S}_{\varepsilon r,1}\mathscr{B}+\mathtt{E}_n^0\\&=\omega\cdot\partial_\varphi+V_{i_0}^{\infty}\partial_\theta+\partial_\theta\mathcal{K}_{1,b}*\cdot-\partial_\theta\mathcal{K}_{2,b}*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0,\end{aligned}$$

where

$$\mathfrak{R}_{\varepsilon r} \triangleq \Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b} * \cdot\big)\mathscr{B} - \mathcal{K}_{1,b} * \cdot\Big] - \Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b} * \cdot\big)\mathscr{B} - \mathcal{K}_{2,b} * \cdot\Big] + \mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B} - \mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}. \tag{13.34}$$

Direct computations using (11.11) lead to

$$\mathcal{B}^{-1}(\mathcal{K}_{1,b} * (\mathscr{B}\rho))(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\log(\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta))d\eta,$$

where

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta) \triangleq \Big|\sin\Big(\tfrac{\eta-\theta}{2} + \widehat{h}(\varphi,\theta,\eta)\Big)\Big| \quad \text{with} \quad \widehat{h}(\varphi,\theta,\eta) \triangleq \tfrac{\widehat{\beta}(\varphi,\theta)-\widehat{\beta}(\varphi,\eta)}{2}.$$

Using elementary trigonometric identities, we can write

$$\mathscr{A}_{\widehat{\beta}}(\varphi,\theta,\eta) = \Big|\sin\Big(\tfrac{\eta-\theta}{2}\Big)\Big|\, v_{\widehat{\beta}}(\varphi,\theta,\eta) \quad \text{with} \quad v_{\widehat{\beta}}(\varphi,\theta,\eta) \triangleq \cos\big(\widehat{h}(\varphi,\theta,\eta)\big) + \frac{\sin\big(\widehat{h}(\varphi,\theta,\eta)\big)}{\tan\big(\frac{\eta-\theta}{2}\big)}\cdot$$

In view of (13.23), one finds that $v_{\widehat{\beta}}$ enjoys the following symmetry property

$$v_{\widehat{\beta}}(-\varphi,-\theta,-\eta) = v_{\widehat{\beta}}(\varphi,\theta,\eta). \tag{13.35}$$

Using the morphism property of the logarithm, one gets

$$\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b} * \rho\big)\mathscr{B} - \mathcal{K}_{1,b} * \rho\Big](\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)\mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)d\eta$$

where

$$\mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta) \triangleq \log\big(v_{\widehat{\beta}}(\varphi,\theta,\eta)\big). \tag{13.36}$$

Notice that (13.36) and (13.35) imply

$$\mathbb{K}_{\widehat{\beta},2}(-\varphi,-\theta,-\eta) = \mathbb{K}_{\widehat{\beta},2}(\varphi,\theta,\eta) \in \mathbb{R}. \tag{13.37}$$

Hence, we deduce from Lemma A.7 that $\mathcal{B}^{-1}\big(\mathcal{K}_{1,b} * \cdot\big)\mathscr{B} - \mathcal{K}_{1,b} * \cdot$ is a real and reversibility preserving Toeplitz in time operator. Writing

$$v_{\widehat{\beta}}(\varphi,\theta,\eta) = 1 + \Big(\cos\big(\widehat{h}(\varphi,\theta,\eta)\big) - 1\Big) + \frac{\sin\big(\widehat{h}(\varphi,\theta,\eta)\big)}{\tan\big(\frac{\eta-\theta}{2}\big)},$$

one finds, by Lemma A.1-(v), Lemma A.2 and (13.22),

$$\begin{aligned}\|v_{\widehat{\beta}} - 1\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} &\lesssim \|\widehat{\beta}\|_{q,s+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\Big(1 + \|\mathfrak{I}_0\|_{q,s+1+\sigma_1}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Moreover, by (13.28) and the Mean Value Theorem (applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0+2$), we find

$$\begin{aligned}\|\Delta_{12}v_{\widehat{\beta}}\|_{q,H^{\overline{s}_h+\mathtt{p}+s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}+s_0+2}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}.\end{aligned}$$

In a similar way, we deduce that

$$\|\mathbb{K}_{\widehat{\beta},2}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+1+\sigma_1}^{\gamma,\mathcal{O}}\right), \tag{13.38}$$

$$\|\Delta_{12}\mathbb{K}_{\widehat{\beta},2}\|_{q,H^{\overline{s}_h+\mathtt{p}+s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}. \tag{13.39}$$

In view of Lemma A.7 we get, from (13.38) and (13.39),

$$\max_{k\in\{0,1,2\}}\left\|\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{1,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+3+\sigma_1}^{\gamma,\mathcal{O}}\right), \tag{13.40}$$

$$\max_{k\in\{0,1\}}\left\|\Delta_{12}\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{1,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{1,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}. \tag{13.41}$$

According to (11.12), one finds

$$\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*(\mathscr{B}\rho)\big)(\varphi,\theta)-\mathcal{K}_{2,b}*\rho(\varphi,\theta)=\int_{\mathbb{T}}\rho(\varphi,\eta)\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)d\eta,$$

with

$$\begin{aligned}\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta) &\triangleq \tfrac{1}{2}\Big[\log\big(1+b^4-2b^2\cos(\eta-\theta+\widehat{h}(\varphi,\theta,\eta))\big)-\log\big(1+b^4-2b^2\cos(\eta-\theta)\big)\Big]\\ &= \tfrac{1}{2}\log\left(\tfrac{1+b^4-2b^2\cos\big(\eta-\theta+\widehat{h}(\varphi,\theta,\eta)\big)}{1+b^4-2b^2\cos(\eta-\theta)}\right).\end{aligned}$$

From (13.23), we deduce that

$$\mathscr{K}_{\widehat{\beta},2}(-\varphi,-\theta,-\eta)=\mathscr{K}_{\widehat{\beta},2}(\varphi,\theta,\eta)\in\mathbb{R}. \tag{13.42}$$

It follows from Lemma A.7 that $\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot$ is a real and reversibility preserving Toeplitz in time operator. Arguying as for (12.27) and using (13.22), we obtain

$$\begin{aligned}\|\mathscr{K}_{\widehat{\beta},2}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} &\lesssim \|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned} \tag{13.43}$$

Using Mean Value theorem, applied with $\mathtt{p}$ replaced by $\mathtt{p}+s_0+1$, one also gets by (13.28)

$$\begin{aligned}\|\Delta_{12}\mathscr{K}_{\widehat{\beta},2}\|_{q,H^{\overline{s}_h+\mathtt{p}+s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\widehat{\beta}\|_{q,\overline{s}_h+\mathtt{p}+s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+1+\sigma_1}^{\gamma,\mathcal{O}}.\end{aligned} \tag{13.44}$$

Consequently, in view of Lemma A.7, we get from (13.43)

$$\max_{k\in\{0,1,2\}}\left\|\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}\right) \tag{13.45}$$

and from (13.44)

$$\max_{k\in\{0,1\}}\left\|\Delta_{12}\partial_\theta^k\Big[\mathcal{B}^{-1}\big(\mathcal{K}_{2,b}*\cdot\big)\mathscr{B}-\mathcal{K}_{2,b}*\cdot\Big]\right\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+1+\sigma_1}^{\gamma,\mathcal{O}}. \tag{13.46}$$

Next, putting together (12.24), (13.23) and Lemma A.8, we infer that $\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}$ is a real and reversibility preserving Toeplitz in time operator. Moreover, we obtain from (A.30) in Lemma A.8, (13.22), (12.26), (13.11) and the smallness condition (13.19),

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathbb{K}_{\varepsilon r,1}\|_{q,H^{s+s_0+2}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\beta\|_{q,s+s_0+2}^{\gamma,\mathcal{O}}\|\mathbb{K}_{\varepsilon r,1}\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{13.47}$$

Applying Lemma A.1, we get from (12.32), Lemma A.2 and (13.12),

$$\begin{aligned}\|\Delta_{12}\mathbb{K}_{\varepsilon r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}&\lesssim\|\Delta_{12}v_{\varepsilon r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j\in\{1,2\}}\|\mathfrak{I}_j\|_{q,s+1}^{\gamma,\mathcal{O}}\Big).\end{aligned}$$

Added to Lemma A.8-(ii), (13.28), (13.22), (12.26) and (13.19), we infer

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathcal{B}^{-1}\mathbf{L}_{\varepsilon r,1}\mathscr{B}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+1+\sigma_1}^{\gamma,\mathcal{O}}.\tag{13.48}$$

The next task is to estimate the term $\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}$ in (13.34). Note that (12.25), (13.23) and Lemma A.8 imply that $\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}$ is a real and reversibility preserving Toeplitz in time operator. In addition, Lemma A.8 together with the estimates (12.27), (13.11) and (13.22) give

$$\begin{aligned}\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}&\lesssim\|\mathscr{K}_{\varepsilon r,1}\|_{q,H^{s+2}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}+\|\beta\|_{q,s+2}^{\gamma,\mathcal{O}}\|\mathscr{K}_{\varepsilon r,1}\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\\&\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}\right).\end{aligned}\tag{13.49}$$

Applying Lemma A.1, we get from (12.35) and (13.12),

$$\|\Delta_{12}\mathscr{K}_{\varepsilon r,1}\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\Big(\|\Delta_{12}i\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}i\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{i\in\{1,2\}}\|\mathfrak{I}_i\|_{q,s}^{\gamma,\mathcal{O}}\Big).$$

Then, combining Lemma A.8-(ii), (13.28), (13.22), (12.27) and (13.19), we get

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathcal{B}^{-1}\mathbf{S}_{\varepsilon r,1}\mathscr{B}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+1+\sigma_1}^{\gamma,\mathcal{O}}.\tag{13.50}$$

In view of (13.34), Lemma A.8 and the previous computations, we conclude that $\mathfrak{R}_{\varepsilon r}$ is a real and reversibility preserving toeplitz in time integral operator which satisfies, by (13.40), (13.45), (13.47) and (13.49),

$$\max_{k\in\{0,1,2\}}\|\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\left(1+\|\mathfrak{I}_0\|_{q,s+s_0+3+\sigma_1}^{\gamma,\mathcal{O}}\right).$$

In addition, combining (13.41), (13.46), (13.48) and (13.50) yields

$$\max_{k\in\{0,1\}}\|\Delta_{12}\partial_\theta^k\mathfrak{R}_{\varepsilon r}\|_{\text{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}}\lesssim\varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+s_0+2+\sigma_1}^{\gamma,\mathcal{O}}.$$

This ends the proof of Proposition 13.2. □

13.2.2 Projection in the normal directions

In this section, we study the effects of the localization in the normal directions for the reduction of the transport part. For that purpose, we consider the localized quasi-periodic symplectic change of coordinates defined by

$$\mathscr{B}_\perp=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp.$$

Then, the main result of this section reads as follows.

Proposition 13.3. *Let* $(\gamma, q, d, \tau_1, s_0, s_h, \overline{s}_h, \mathtt{p}, S)$ *satisfy the assumptions* (A.2), (A.1) *and* (13.18). *There exist* $\varepsilon_0 > 0$ *and* $\sigma_3 = \sigma_3(\tau_1, q, d, s_0) \geqslant \sigma_2$ *such that if*

$$\varepsilon\gamma^{-1}N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_3}^{\gamma,\mathcal{O}} \leqslant 1, \tag{13.51}$$

then the following assertions hold true.

(i) The operators $\mathscr{B}_\perp^{\pm 1}$ *satisfy the following estimate*

$$\|\mathscr{B}_\perp^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{13.52}$$

(ii) For any $n \in \mathbb{N}^*$, *in the Cantor set* $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *introduced in Proposition 13.2, we have*

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp &= \big(\omega\cdot\partial_\varphi + V_{i_0}^\infty\partial_\theta + \partial_\theta\mathcal{K}_b * \cdot\big)\Pi_{\mathbb{S}_0}^\perp + \mathscr{R}_0 + \mathtt{E}_n^1\\ &\triangleq \omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_0 + \mathscr{R}_0 + \mathtt{E}_n^1\\ &\triangleq \mathscr{L}_0 + \mathtt{E}_n^1,\end{aligned}$$

where $\mathscr{R}_0 = \Pi_{\mathbb{S}_0}^\perp\mathscr{R}_0\Pi_{\mathbb{S}_0}^\perp$ *is reversible and* $\mathscr{D}_0 = \Pi_{\mathbb{S}_0}^\perp\mathscr{D}_0\Pi_{\mathbb{S}_0}^\perp$ *is a reversible Fourier multiplier operator given by*

$$\forall (l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c, \quad \mathscr{D}_0\mathbf{e}_{l,j} = \mathrm{i}\,\mu_j^0\,\mathbf{e}_{l,j},$$

with

$$\mu_j^0(b,\omega,i_0) = \Omega_j(b) + jr^1(b,\omega,i_0) \quad \textit{and} \quad r^1(b,\omega,i_0) = V_{i_0}^\infty(b,\omega) - \tfrac{1}{2} \tag{13.53}$$

and such that

$$\|r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon \quad \text{and} \quad \|\Delta_{12}r^1\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\|\Delta_{12}i\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}}. \tag{13.54}$$

(iii) The operator $\mathtt{E}_n^1$ *satisfies the following estimate*

$$\|\mathtt{E}_n^1\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+2}^{\gamma,\mathcal{O}}. \tag{13.55}$$

(iv) The operator $\mathscr{R}_0$ *is a real and reversible Toeplitz in time operator satisfying*

$$\forall s\in[s_0,S], \quad \max_{k\in\{0,1\}}\|\partial_\theta^k\mathscr{R}_0\|_{\textnormal{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\Big(1+\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\Big) \tag{13.56}$$

and

$$\|\Delta_{12}\mathscr{R}_0\|_{\textnormal{O-d},q,\overline{s}_h+\mathtt{p}}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12}i\|_{q,\overline{s}_h+\mathtt{p}+\sigma_3}^{\gamma,\mathcal{O}}. \tag{13.57}$$

(v) Furthermore the operator $\mathscr{L}_0$ *satisfies*

$$\forall s\in[s_0,S], \quad \|\mathscr{L}_0\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s+1}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-1}\|\mathfrak{I}_0\|_{q,s+\sigma_3}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{13.58}$$

Proof. **(i)** Follows from (13.21) and Lemma A.1-(ii).
(ii) From (13.7) and the decomposition $\mathrm{Id} = \Pi_{\mathbb{S}_0} + \Pi_{\mathbb{S}_0}^\perp$ we write

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp &= \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp(\mathcal{L}_{\varepsilon r} - \varepsilon\partial_\theta\mathcal{R})\mathscr{B}_\perp\\ &= \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp - \mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathcal{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp - \varepsilon\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathcal{R}\mathscr{B}_\perp.\end{aligned}$$

According to the definitions of $\mathfrak{L}_{\varepsilon r}$ and $\mathcal{L}_{\varepsilon r}$ seen in Proposition 13.2 and in Lemma 11.1 and using (12.22), (12.23) and (11.10), one has in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$

$$\mathcal{L}_{\varepsilon r}\mathscr{B}=\mathscr{B}\mathfrak{L}_{\varepsilon r}\quad\text{and}\quad\mathcal{L}_{\varepsilon r}=\omega\cdot\partial_\varphi+\partial_\theta\left(V_{\varepsilon r}\cdot\right)+\partial_\theta\mathcal{K}_b*\cdot+\partial_\theta\mathbf{L}_{\varepsilon r,1}-\partial_\theta\mathbf{S}_{\varepsilon r,1}$$

and therefore

$$\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)+\partial_\theta\mathbf{L}_{\varepsilon r,1}-\partial_\theta\mathbf{S}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp,$$

where we have used the identities

$$\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp=\mathscr{B}_\perp^{-1}\quad\text{and}\quad[\Pi_{\mathbb{S}_0}^\perp,T]=0=[\Pi_{\mathbb{S}_0},T],$$

for any Fourier multiplier T. The structure of $\mathfrak{L}_{\varepsilon r}$ is detailed in Proposition 13.2, and from this we deduce that

$$\begin{aligned}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot+\partial_\theta\mathfrak{R}_{\varepsilon r}+\mathtt{E}_n^0\big)\Pi_{\mathbb{S}_0}^\perp\\&=\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\Pi_{\mathbb{S}_0}^\perp\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\mathscr{B}_\perp\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

It follows that

$$\begin{aligned}\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathfrak{L}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp&=\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\&=\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&\quad+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\end{aligned}$$

Consequently, in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$, one has the following reduction

$$\begin{aligned}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp=&\big(\omega\cdot\partial_\varphi+V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)\Pi_{\mathbb{S}_0}^\perp+\Pi_{\mathbb{S}_0}^\perp\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp+\mathscr{B}_\perp^{-1}\mathscr{B}\Pi_{\mathbb{S}_0}\partial_\theta\mathfrak{R}_{\varepsilon r}\Pi_{\mathbb{S}_0}^\perp\\&-\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\left(\partial_\theta\left(V_{\varepsilon r}\cdot\right)+\partial_\theta\mathbf{L}_{\varepsilon r,1}-\partial_\theta\mathbf{S}_{\varepsilon r,1}\right)\Pi_{\mathbb{S}_0}\mathscr{B}\Pi_{\mathbb{S}_0}^\perp-\varepsilon\mathscr{B}_\perp^{-1}\partial_\theta\mathcal{R}\mathscr{B}_\perp+\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp\\\triangleq&\,\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_0+\mathscr{R}_0+\mathtt{E}_n^1,\end{aligned}\tag{13.59}$$

where we set

$$\mathscr{D}_0\triangleq\big(V_{i_0}^\infty\partial_\theta+\partial_\theta\mathcal{K}_b*\cdot\big)\Pi_{\mathbb{S}_0}^\perp$$

and

$$\mathtt{E}_n^1\triangleq\mathscr{B}_\perp^{-1}\Pi_{\mathbb{S}_0}^\perp\mathscr{B}\mathtt{E}_n^0\Pi_{\mathbb{S}_0}^\perp.\tag{13.60}$$

(iii) Results from (13.60), (13.52), (13.21), (13.25) and Lemma A.1-(ii).

(iv) For the estimates (13.56) and (13.57), we refer to Lemma 7.1 and Proposition 7.4. They are based on suitable duality representations of $\mathscr{B}_\perp^{\pm1}$ linked to $\mathscr{B}^{\pm1}$.

(v) It is obtained by (12.28), (12.29), (13.20), (13.56) and Lemma A.6-(iv). ☐

13.2.3 Elimination of the remainder term

We perform here the KAM reduction of the remainder $\mathscr{R}_0$ of Proposition 13.3. This procedure allows to diagonalise the linearized operator in the normal directions, namely to conjugate it to a constant coefficients operator $\mathscr{L}_\infty$, up to fast decaying terms. We omit the proof due to its similarity with Proposition 7.5.

Proposition 13.4. *Let* $(\gamma,q,d,\tau_1,\tau_2,s_0,s_l,\overline{s}_l,\overline{s}_h,\overline{\mu}_2,S)$ *satisfy* (A.2), (A.1), (13.16). *For any* (μ_2,s_h)

satisfying

$$\mu_2 \geqslant \overline{\mu}_2 + 2\tau_2 q + 2\tau_2, \quad \text{and} \quad s_h \geqslant \frac{3}{2}\mu_2 + \overline{s}_l + 1, \tag{13.61}$$

there exist $\varepsilon_0 \in (0,1)$ *and* $\sigma_4 = \sigma_4(\tau_1, \tau_2, q, d) \geqslant \sigma_3$ *such that if*

$$\varepsilon\gamma^{-2-q} N_0^{\mu_2} \leqslant \varepsilon_0 \quad \text{and} \quad \|\mathfrak{I}_0\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}} \leqslant 1, \tag{13.62}$$

then the following assertions hold true.

(i) There exists a family of invertible linear operator $\Phi_\infty : \mathcal{O} \to \mathcal{L}\big(H^s \cap L^2_\perp\big)$ *satisfying the estimates*

$$\forall s \in [s_0, S], \qquad \|\Phi_\infty^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \|\rho\|_{q,s}^{\gamma,\mathcal{O}} + \varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma_4}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{13.63}$$

There exists a diagonal operator $\mathscr{L}_\infty = \mathscr{L}_\infty(b,\omega,i_0)$ *taking the form*

$$\mathscr{L}_\infty = \omega\cdot\partial_\varphi \Pi_{\mathbb{S}_0}^\perp + \mathscr{D}_\infty$$

where $\mathscr{D}_\infty = \mathscr{D}_\infty(b,\omega,i_0) = \Pi_{\mathbb{S}_0}^\perp \mathscr{D}_\infty \Pi_{\mathbb{S}_0}^\perp$ *is a reversible Fourier multiplier operator given by,*

$$\forall (l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c, \quad \mathscr{D}_\infty \mathbf{e}_{l,j} = \mathrm{i}\,\mu_j^\infty\,\mathbf{e}_{l,j},$$

with

$$\forall j \in \mathbb{S}_0^c, \quad \mu_j^\infty(b,\omega,i_0) = \mu_j^0(b,\omega,i_0) + r_j^\infty(b,\omega,i_0) \tag{13.64}$$

and

$$\sup_{j\in\mathbb{S}_0^c} |j|\|r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} \tag{13.65}$$

such that in the Cantor set

$$\mathscr{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0) \triangleq \bigcap_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |l|\leqslant N_n\\ (l,j)\neq(0,j_0)}} \Big\{(b,\omega)\in\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0), \big|\omega\cdot l + \mu_j^\infty(b,\omega,i_0) - \mu_{j_0}^\infty(b,\omega,i_0)\big| > \tfrac{2\gamma\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\Big\}$$

we have

$$\Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty = \mathscr{L}_\infty + \mathtt{E}_n^2,$$

and the linear operator $\mathtt{E}_n^2$ *satisfies the estimate*

$$\|\mathtt{E}_n^2\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-2}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}}. \tag{13.66}$$

Notice that the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *was introduced in Proposition 13.2, the operator* $\mathscr{L}_0$ *and the frequencies* $\big(\mu_j^0(b,\omega,i_0)\big)_{j\in\mathbb{S}_0^c}$ *were stated in Proposition 13.3.*

(ii) Given two tori i_1 *and* i_2 *both satisfying* (13.62), *then*

$$\forall j \in \mathbb{S}_0^c, \quad \|\Delta_{12} r_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}\|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} \tag{13.67}$$

$$\forall j \in \mathbb{S}_0^c, \quad \|\Delta_{12}\mu_j^\infty\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}|j|\|\Delta_{12} i\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}. \tag{13.68}$$

13.3 Construction and tame estimates for the approximate inverse

At this step, we can construct an almost approximate right inverse for $\widehat{\mathcal{L}}_\omega$ defined in (13.7). This enables to find in turn an almost approximate right inverse for the whole operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ given by (13.1).

Proposition 13.5. *Let* $(\gamma,q,d,\tau_1,s_0,s_h,\mu_2,S)$ *satisfying* (A.2), (A.1) *and* (13.61). *There exists* $\sigma\triangleq\sigma(\tau_1,\tau_2,q,d)\geqslant\sigma_4$ *such that if*

$$\varepsilon\gamma^{-2-q}N_0^{\mu_2}\leqslant\varepsilon_0\quad\text{and}\quad\|\mathfrak{I}_0\|_{q,s_h+\sigma}^{\gamma,\mathcal{O}}\leqslant 1,\tag{13.69}$$

then, the following assertions hold true.

(i) Consider the operator $\mathscr{L}_\infty$ *defined in Proposition* 13.4, *then there exists a family of linear reversible operators* $\big(\mathtt{T}_n\big)_{n\in\mathbb{N}}$ *defined in* $\mathcal{O}$ *satisfying the estimate*

$$\forall s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}$$

and such that for any $n\in\mathbb{N}$, *in the Cantor set*

$$\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)=\bigcap_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\Big\{(b,\omega)\in\mathcal{O}\quad\text{s.t.}\quad\big|\omega\cdot l+\mu_j^\infty(b,\omega,i_0)\big|>\tfrac{\gamma\langle j\rangle}{\langle l\rangle^{\tau_1}}\Big\},$$

we have

$$\mathscr{L}_\infty\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3,$$

with

$$\forall s_0\leqslant s\leqslant\overline{s}\leqslant S,\quad\|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.$$

(ii) There exists a family of linear reversible operators $\big(\mathtt{T}_{\omega,n}\big)_{n\in\mathbb{N}}$ *satisfying*

$$\forall\, s\in[s_0,S],\quad\sup_{n\in\mathbb{N}}\|\mathtt{T}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\Big)\tag{13.70}$$

and such that in the Cantor set

$$\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)\triangleq\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\tag{13.71}$$

we have

$$\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n}=\mathrm{Id}+\mathtt{E}_n,$$

where $\mathtt{E}_n$ *satisfies the following estimate*

$$\begin{aligned}\forall\, s\in[s_0,S],\quad\|\mathtt{E}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{s_0-s}\gamma^{-1}\Big(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}\Big)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}\tag{13.72}$$

Recall that $\widehat{\mathcal{L}}_\omega$, $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)$ *and* $\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ *are given by* (13.7) *and Propositions 13.2 and 13.4, respectively.*

(iii) In the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$, *we have the following splitting*

$$\widehat{\mathcal{L}}_\omega=\widehat{\mathtt{L}}_{\omega,n}+\widehat{\mathtt{R}}_n\quad\text{with}\quad\widehat{\mathtt{L}}_{\omega,n}\mathtt{T}_{\omega,n}=\mathrm{Id}\quad\text{and}\quad\widehat{\mathtt{R}}_n=\mathtt{E}_n\widehat{\mathtt{L}}_{\omega,n},$$

where $\widehat{\mathtt{L}}_{\omega,n}$ *and* $\widehat{\mathtt{R}}_n$ *are reversible operators defined in* $\mathcal{O}$ *and satisfy the following estimates*

$$\forall s\in[s_0,S],\quad \sup_{n\in\mathbb{N}}\|\widehat{\mathtt{L}}_{\omega,n}\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho\|_{q,s+1}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+1}^{\gamma,\mathcal{O}},\tag{13.73}$$

$$\begin{aligned}\forall s\in[s_0,S],\quad \|\widehat{\mathtt{R}}_n\rho\|_{q,s_0}^{\gamma,\mathcal{O}}&\lesssim N_n^{s_0-s}\gamma^{-1}\left(\|\rho\|_{q,s+\sigma}^{\gamma,\mathcal{O}}+\varepsilon\gamma^{-2}\|\mathfrak{I}_0\|_{q,s+\sigma}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}\right)\\&\quad+\varepsilon\gamma^{-3}N_0^{\mu_2}N_{n+1}^{-\mu_2}\|\rho\|_{q,s_0+\sigma}^{\gamma,\mathcal{O}}.\end{aligned}\tag{13.74}$$

Proof. **(i)** First recall from Proposition 13.4 that

$$\mathscr{L}_\infty=\omega\cdot\partial_\varphi\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty.$$

Using the projectors defined in (A.5), we can split this operator as follows

$$\begin{aligned}\mathscr{L}_\infty&=\Pi_{N_n}\omega\cdot\partial_\varphi\Pi_{N_n}\Pi_{\mathbb{S}_0}^\perp+\mathscr{D}_\infty-\Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp\\&\triangleq\mathtt{L}_n-\mathtt{R}_n,\end{aligned}\tag{13.75}$$

where

$$\mathtt{R}_n\triangleq\Pi_{N_n}^\perp\omega\cdot\partial_\varphi\Pi_{N_n}^\perp\Pi_{\mathbb{S}_0}^\perp.$$

According to the structure of $\mathscr{D}_\infty$ in Proposition 13.4, we obtain from (13.75),

$$\forall(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c,\quad \mathbf{e}_{-l,-j}\mathtt{L}_n\mathbf{e}_{l,j}=\begin{cases}\mathrm{i}\big(\omega\cdot l+\mu_j^\infty\big) & \text{if } |l|\leqslant N_n\\ \mathrm{i}\,\mu_j^\infty & \text{if } |l|>N_n.\end{cases}$$

Let us now consider the diagonal operator $\mathtt{T}_n$ defined by

$$\begin{aligned}\mathtt{T}_n\rho(b,\omega,\varphi,\theta)\triangleq&-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\tfrac{\chi\left((\omega\cdot l+\mu_j^\infty(b,\omega,i_0))\gamma^{-1}\langle l\rangle^{\tau_1}\right)}{\omega\cdot l+\mu_j^\infty(b,\omega,i_0)}\rho_{l,j}(b,\omega)\,e^{\mathrm{i}(l\cdot\varphi+j\theta)}\\&-\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|>N_n}}\tfrac{\rho_{l,j}(b,\omega)}{\mu_j^\infty(b,\omega,i_0)}\,e^{\mathrm{i}(l\cdot\varphi+j\theta)},\end{aligned}$$

where χ is the cut-off function introduced in (6.92) and $\big(\rho_{l,j}(b,\omega)\big)_{l,j}$ are the Fourier coefficients of ρ. Now recall the expansion of the perturbed eigenvalues given by Proposition 13.4, namely

$$\mu_j^\infty(b,\omega,i_0)=\Omega_j(b)+jr^1(b,\omega)+r_j^\infty(b,\omega)\quad\text{with}\quad r^1(b,\omega)=V_{i_0}^\infty(b,\omega)-\tfrac{1}{2}.$$

In view of Lemma 11.3-(iv), (13.54) and (13.65), they satisfy the following estimates

$$\forall j\in\mathbb{S}_0^c,\quad \|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}\lesssim|j|.$$

According to Lemma 11.3-(ii), (13.54), (13.65) and the smallness condition (13.69) we infer

$$|j|\lesssim\|\mu_j^\infty\|_0^{\gamma,\mathcal{O}}\leqslant\|\mu_j^\infty\|_q^{\gamma,\mathcal{O}}.$$

Computations based on Lemma A.1-(vi) give

$$\forall s\geqslant s_0,\quad \|\mathtt{T}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\gamma^{-1}\|\rho\|_{q,s+\tau_1q+\tau_1}^{\gamma,\mathcal{O}}.\tag{13.76}$$

In addition, by construction

$$\mathtt{L}_n\mathtt{T}_n = \mathrm{Id} \quad \text{in } \Lambda_{\infty,n}^{\gamma,\tau_1}(i_0) \tag{13.77}$$

since $\chi(\cdot)=1$ in this set. Gathering (13.77) and (13.75) yields

$$\begin{aligned}\forall\,(b,\omega)\in\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0),\quad \mathscr{L}_\infty\mathtt{T}_n &= \mathrm{Id}-\mathtt{R}_n\mathtt{T}_n\\ &\triangleq \mathrm{Id}+\mathtt{E}_n^3.\end{aligned} \tag{13.78}$$

Remark that by Lemma A.1-(ii),

$$\forall\, s_0\leqslant s\leqslant \overline{s}\leqslant S,\quad \|\mathtt{R}_n\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\|\rho\|_{q,\overline{s}+1}^{\gamma,\mathcal{O}}.$$

Putting this estimate with (13.76) implies

$$\forall\, s_0\leqslant s\leqslant \overline{s}\leqslant S,\quad \|\mathtt{E}_n^3\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim N_n^{s-\overline{s}}\gamma^{-1}\|\rho\|_{q,\overline{s}+1+\tau_1 q+\tau_1}^{\gamma,\mathcal{O}}. \tag{13.79}$$

(ii) We set

$$\mathtt{T}_{\omega,n}\triangleq \mathscr{B}_\perp\Phi_\infty\mathtt{T}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}, \tag{13.80}$$

where the operators $\mathscr{B}_\perp$ and Φ_∞ are defined in Propositions 13.3 and 13.4 respectively. Notice that $\mathtt{T}_{\omega,n}$ is defined in the whole range of parameters $\mathcal{O}$. Since the condition (13.69) is satisfied, then, both Propositions 13.2 and 13.4 apply and the estimate (13.70) is obtained combining (13.52), (13.63), (13.76) and (13.69). Now combining Propositions 13.3 and 13.4, we find that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathcal{O}_{\infty,n}^{\gamma,\tau_1,\tau_2}(i_0)$ the following decomposition holds

$$\begin{aligned}\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty &= \Phi_\infty^{-1}\mathscr{L}_0\Phi_\infty+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\\ &= \mathscr{L}_\infty+\mathtt{E}_n^2+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty.\end{aligned}$$

According to (13.78), one finds that in the Cantor set $\mathcal{O}_{\infty,n}^{\gamma,\tau_1}(i_0)\cap\mathscr{O}_{\infty,n}^{\gamma,\tau_2}(i_0)\cap\Lambda_{\infty,n}^{\gamma,\tau_1}(i_0)$ the following identity holds

$$\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\widehat{\mathcal{L}}_\omega\mathscr{B}_\perp\Phi_\infty\mathtt{T}_n=\mathrm{Id}+\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n,$$

which implies in turn, in view of (13.80), the following identity in $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\begin{aligned}\widehat{\mathcal{L}}_\omega\mathtt{T}_{\omega,n}&=\mathrm{Id}+\mathscr{B}_\perp\Phi_\infty\big(\mathtt{E}_n^3+\mathtt{E}_n^2\mathtt{T}_n+\Phi_\infty^{-1}\mathtt{E}_n^1\Phi_\infty\mathtt{T}_n\big)\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}\\ &\triangleq \mathrm{Id}+\mathtt{E}_n.\end{aligned} \tag{13.81}$$

Combining (13.81), (13.55), (13.66), (13.79), (13.76), (13.52), (13.63) and (13.69), we get (13.72), up to taking σ large enough.

(iii) By virtue of (13.81), one can write in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathcal{L}}_\omega=\mathtt{T}_{\omega,n}^{-1}+\mathtt{E}_n\mathtt{T}_{\omega,n}^{-1}. \tag{13.82}$$

Putting together (13.80) and (13.77), one finds in the Cantor set $\mathcal{G}_n(\gamma,\tau_1,\tau_2,i_0)$

$$\widehat{\mathtt{L}}_{\omega,n}\triangleq\mathtt{T}_{\omega,n}^{-1}=\mathscr{B}_\perp\Phi_\infty\mathtt{L}_n\Phi_\infty^{-1}\mathscr{B}_\perp^{-1}.$$

Therefore, (13.82) can be rewritten

$$\widehat{\mathcal{L}}_\omega = \widehat{\mathtt{L}}_{\omega,n} + \widehat{\mathtt{R}}_n \quad \text{with} \quad \widehat{\mathtt{R}}_n \triangleq \mathtt{E}_n \widehat{\mathtt{L}}_{\omega,n}.$$

The estimate (13.73) is obtained gathering (13.75), (13.52), (13.63) and (13.69). Finally, (13.73) together with (13.72) implies (13.74). This ends the proof of Proposition 13.5. □

The following theorem, see also Theorem 6.1, states that the linearized operator $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ in (13.1) admits an approximate right inverse on a suitable Cantor set.

Theorem 13.1. (Approximate inverse)
Let $(\gamma, q, d, \tau_1, \tau_2, s_0, s_h, \mu_2)$ *satisfy* (A.2), (A.1), (13.16) *and* (13.61). *Then there exists* $\overline{\sigma} = \overline{\sigma}(\tau_1, \tau_2, d, q) > 0$ *and a family of reversible operators* $\mathrm{T}_0 \triangleq \mathrm{T}_{0,n}(i_0)$ *such that if the smallness condition* (13.69) *holds, then for all* $g = (g_1, g_2, g_3)$, *satisfying*

$$g_1(\varphi) = g_1(\varphi), \quad g_2(-\varphi) = -g_2(\varphi) \quad \text{and} \quad g_3(-\varphi) = (\mathscr{S} g_3)(\varphi),$$

the function $\mathrm{T}_0 g$ *satisfies the following estimate*

$$\forall s \in [s_0, S], \quad \|\mathrm{T}_0 g\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1}\left(\|g\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}} + \|\mathfrak{I}_0\|_{q,s+\overline{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\right).$$

Moreover T_0 *is an almost-approximate right inverse of* $d_{i,\alpha}\mathcal{F}(i_0,\alpha_0)$ *in the Cantor set* $\mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0)$ *defined by* (13.71). *More precisely,*

$$\forall (b,\omega) \in \mathcal{G}_n(\gamma, \tau_1, \tau_2, i_0), \quad d_{i,\alpha}\mathcal{F}(i_0) \circ \mathrm{T}_0 - \mathrm{Id} = \mathcal{E}_1^{(n)} + \mathcal{E}_2^{(n)} + \mathcal{E}_3^{(n)},$$

where the operators $\mathcal{E}_1^{(n)}$, $\mathcal{E}_2^{(n)}$ *and* $\mathcal{E}_3^{(n)}$ *are defined in the whole set* $\mathcal{O}$ *with the estimates*

$$\begin{aligned}
&\|\mathcal{E}_1^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} \|\mathcal{F}(i_0,\alpha_0)\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}},\\
\forall b \geqslant 0, \quad &\|\mathcal{E}_2^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim \gamma^{-1} N_n^{-b} \big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon \|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\big),\\
\forall b \in [0,S], \quad &\|\mathcal{E}_3^{(n)} g\|_{q,s_0}^{\gamma,\mathcal{O}} \lesssim N_n^{-b} \gamma^{-2} \Big(\|g\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} + \varepsilon \gamma^{-2} \|\mathfrak{I}_0\|_{q,s_0+b+\overline{\sigma}}^{\gamma,\mathcal{O}} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}\Big)\\
&\qquad + \varepsilon \gamma^{-4} N_0^{\mu_2} N_n^{-\mu_2} \|g\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}}.
\end{aligned}$$

14 Nash-Moser iteration and measure of the final Cantor set

In this last section, we shall find a non-trivial solution $(b,\omega) \mapsto (i_\infty(b,\omega), \alpha_\infty(b,\omega))$ to the equation

$$\mathcal{F}(i, \alpha, b, \omega, \varepsilon) = 0,$$

where $\mathcal{F}$ is the functional defined in (12.20). This is done by using a Nash-Moser scheme in a similar way as the series of papers [7, 33, 87, 101]. The solutions are constructed for parameters (b,ω) belonging to the intersection of all the Cantor sets $\mathcal{G}_\infty^\gamma$ on which we are able to invert the linearized operator at the different steps. In order to find a solution to the original problem, we must rigidify the frequencies ω so that they coincide with the equilibrium frequencies. This amounts to considering a frequency curve $b \mapsto \omega(b,\varepsilon)$ implicitly defined by the equation

$$\alpha_\infty\big(b, \omega(b,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(b).$$

Considering the associated rigidified Cantor set

$$\mathcal{C}_\infty^\varepsilon = \Big\{ b \in (b_0, b_1) \quad \text{s.t.} \quad \big(b, \omega(b,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma \Big\},$$

we have a solution to the original problem provided that the measure of $\mathcal{C}_\infty^\varepsilon$ is non-zero. This will be checked, in Section 14.2, by perturbative arguments in the spirit of the previous works [7, 10, 33, 87, 101]. This proves in particular Theorem 9.1.

14.1 Nash-Moser iteration

In this section we implement the Nash-Moser scheme, which is a modified Newton method consisting in a recursive construction of approximate solutions of the equation $\mathcal{F}\big(i,\alpha,b,\omega\big) \triangleq \mathcal{F}\big(i,\alpha,b,\omega,\varepsilon\big) = 0$ where the functional $\mathcal{F}$ is defined in (12.20). At each step of this procedure, we need to construct an approximate inverse of the linearized operator at a state near the equilibrium by applying the reduction procedure developed in Section 13. This allows to get Theorem 13.1 with the suitable tame estimates associated to the final loss of regularity $\overline{\sigma}$ that could be arranged to be large enough. We point out that $\overline{\sigma}$ depends only on the shape of the Cantor set through the parameters τ_1, τ_2, d and q but it is independent of the regularity of the solutions that we want to construct. The main result of this section can be stated as follows. The proof is similar to Proposition 8.1.

Proposition 14.1. ***(Nash-Moser)***
Let $(\tau_1, \tau_2, q, d, s_0)$ satisfy (A.2) *and* (A.1). *Consider the parameters fixed by* (8.1) *and* (8.2). *There exist $C_* > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ we get for all $n \in \mathbb{N}$ the following properties.*

$(\mathcal{P}1)_n$ *There exists a q-times differentiable function*

$$\begin{array}{rccl} W_n : & \mathcal{O} & \to & E_{n-1} \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ & (b,\omega) & \mapsto & \big(\mathfrak{I}_n, \alpha_n - \omega, 0\big) \end{array}$$

satisfying

$$W_0 = 0 \quad \textit{and} \quad \textit{for} \quad n \in \mathbb{N}^*, \quad \|W_n\|_{q, s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_* \varepsilon \gamma^{-1} N_0^{q\overline{a}}.$$

By setting

$$U_0 = \Big((\varphi,0,0), \omega, (b,\omega)\Big) \quad \text{and} \quad \textit{for} \quad n \in \mathbb{N}^*, \quad U_n = U_0 + W_n \quad \text{and} \quad H_n = U_n - U_{n-1}, \tag{14.1}$$

then

$$\forall s \in [s_0, S], \ \|H_1\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \frac{1}{2} C_* \varepsilon \gamma^{-1} N_0^{q\overline{a}} \quad \textit{and} \quad \forall\, 2 \leqslant k \leqslant n, \ \|H_k\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_* \varepsilon \gamma^{-1} N_{k-1}^{-a_2}. \tag{14.2}$$

We also have for $n \geqslant 2$,

$$\|H_n\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} \leqslant C_* \varepsilon \gamma^{-1} N_{n-1}^{-a_2}. \tag{14.3}$$

$(\mathcal{P}2)_n$ *Define*

$$i_n = (\varphi, 0, 0) + \mathfrak{I}_n, \quad \gamma_n = \gamma(1 + 2^{-n}), \tag{14.4}$$

then i_n satisfies the following reversibility condition

$$\mathfrak{S} i_n(\varphi) = i_n(-\varphi), \tag{14.5}$$

where $\mathfrak{S}$ *is defined by* (12.11). *Define also*

$$\mathcal{A}_0^{\gamma} = \mathcal{O} \quad \text{and} \quad \mathcal{A}_{n+1}^{\gamma} = \mathcal{A}_n^{\gamma} \cap \mathcal{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$$

where $\mathcal{G}_n(\gamma_{n+1}, \tau_1, \tau_2, i_n)$ *is defined in Proposition* 7.6. *Consider the open sets*

$$\forall \mathtt{r} > 0, \quad \mathrm{O}_n^{\mathtt{r}} \triangleq \Big\{(b,\omega) \in \mathcal{O} \quad \text{s.t.} \quad \mathtt{dist}\big((b,\omega), \mathcal{A}_n^{2\gamma}\big) < \mathtt{r} N_n^{-\overline{a}}\Big\}$$

where $\mathtt{dist}(x,A) = \inf\limits_{y \in A} \|x-y\|$. *Then we have the following estimate*

$$\|\mathcal{F}(U_n)\|_{q,s_0}^{\gamma, \mathrm{O}_n^{2\gamma}} \leqslant C_* \varepsilon N_{n-1}^{-a_1}.$$

$(\mathcal{P}3)_n$ $\|W_n\|_{q,b_1+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_* \varepsilon \gamma^{-1} N_{n-1}^{\mu_1}$.

A non trivial reversible quasi-periodic solution of our problem is obtained as the limit of the sequence $(U_n)_{n\in\mathbb{N}}$ according to the fast convergence stated in Proposition 14.1. This is explained in the following corollary.

Corollary 14.1. *There exists* $\varepsilon_0 > 0$ *such that, for all* $\varepsilon \in (0, \varepsilon_0)$, *the following assertions hold true. We consider the Cantor set* $\mathcal{G}_{\infty}^{\gamma}$, *related to* ε *through* γ, *and defined by*

$$\mathcal{G}_{\infty}^{\gamma} \triangleq \bigcap_{n \in \mathbb{N}} \mathcal{A}_n^{\gamma}.$$

There exists a function

$$\begin{array}{rccl} U_{\infty}: & \mathcal{O} & \to & \big(\mathbb{T}^d \times \mathbb{R}^d \times L_{\perp}^2 \cap H^{s_0}\big) \times \mathbb{R}^d \times \mathbb{R}^{d+1} \\ & (b,\omega) & \mapsto & \big(i_{\infty}(b,\omega), \alpha_{\infty}(b,\omega), (b,\omega)\big) \end{array}$$

such that

$$\forall (b,\omega) \in \mathcal{G}_{\infty}^{\gamma}, \quad \mathcal{F}(U_{\infty}(b,\omega)) = 0.$$

In addition, i_{∞} *is reversible and* $\alpha_{\infty} \in W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{R}^d)$ *with*

$$\alpha_{\infty}(b,\omega) = \omega + \mathrm{r}_{\varepsilon}(b,\omega) \quad \text{and} \quad \|\mathrm{r}_{\varepsilon}\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon \gamma^{-1} N_0^{q\overline{a}}. \tag{14.6}$$

Moreover, there exists a q*-times differentiable function* $b \in (b_0, b_1) \mapsto \omega(b,\varepsilon)$ *with*

$$\omega(b,\varepsilon) = -\omega_{\mathrm{Eq}}(b) + \overline{r}_{\varepsilon}(b), \quad \|\overline{r}_{\varepsilon}\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon \gamma^{-1} N_0^{q\overline{a}}, \tag{14.7}$$

and

$$\forall b \in \mathcal{C}_{\infty}^{\varepsilon}, \quad \mathcal{F}\big(U_{\infty}(b, \omega(b,\varepsilon))\big) = 0 \quad \text{and} \quad \alpha_{\infty}\big(b, \omega(b,\varepsilon)\big) = -\omega_{\mathrm{Eq}}(b),$$

where the Cantor set $\mathcal{C}_{\infty}^{\varepsilon}$ *is defined by*

$$\mathcal{C}_{\infty}^{\varepsilon} = \Big\{ b \in (b_0, b_1) \quad \text{s.t.} \quad \big(b, \omega(b,\varepsilon)\big) \in \mathcal{G}_{\infty}^{\gamma} \Big\}. \tag{14.8}$$

Proof. In view of (14.1) and (14.2), we obtain

$$\|W_{n+1} - W_n\|_{q,s_0}^{\gamma,\mathcal{O}} = \|H_{n+1}\|_{q,s_0}^{\gamma,\mathcal{O}} \leqslant \|H_{n+1}\|_{q,s_0+\overline{\sigma}}^{\gamma,\mathcal{O}} \leqslant C_* \varepsilon \gamma^{-1} N_n^{-a_2}.$$

This implies the convergence of the sequence $(W_n)_{n\in\mathbb{N}}$. Its limit is denoted by

$$W_\infty \triangleq \lim_{n\to\infty} W_n \triangleq (\mathfrak{I}_\infty, \alpha_\infty - \omega, 0, 0)$$

and we set

$$U_\infty \triangleq \big(i_\infty, \alpha_\infty, (b,\omega)\big) = U_0 + W_\infty.$$

Taking $n \to \infty$ in (14.5) gives

$$\mathfrak{S} i_\infty(\varphi) = i_\infty(-\varphi).$$

According to Proposition 14.1-$(\mathcal{P}2)_n$, we get for small values of ε

$$\forall (b,\omega) \in \mathcal{G}_\infty^\gamma, \quad \mathcal{F}\Big(i_\infty(b,\omega), \alpha_\infty(b,\omega), (b,\omega), \varepsilon\Big) = 0, \tag{14.9}$$

where $\mathcal{F}$ is the functional defined in (12.20). We emphasize that the Cantor set $\mathcal{G}_\infty^\gamma$ depends on ε through γ fixed in (8.2). Now, from Proposition 14.1-$(\mathcal{P}1)_n$, we deduce that

$$\alpha_\infty(b,\omega) = \omega + \mathrm{r}_\varepsilon(b,\omega) \quad \text{with} \quad \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} N_0^{q\overline{a}}.$$

Next we shall prove the second result and check the existence of solutions to the original Hamiltonian equation. First recall that the open set $\mathcal{O}$ is defined in (12.5) by

$$\mathcal{O} = (b_0, b_1) \times \mathscr{U} \quad \text{with} \quad \mathscr{U} = B(0, R_0) \quad \text{for some large } R_0 > 0,$$

where the ball $\mathscr{U}$ is taken to contain the equilibrium frequency vector $b \mapsto \omega_{\mathrm{Eq}}(b)$. In view of (14.6), we obtain that for any $b \in (b_0, b_1)$, the mapping $\omega \mapsto \alpha_\infty(b,\omega)$ is invertible from $\mathscr{U}$ into its image $\alpha_\infty(b,\mathscr{U})$ and we have

$$\widehat{\omega} = \alpha_\infty(b,\omega) = \omega + \mathrm{r}_\varepsilon(b,\omega) \Leftrightarrow \omega = \alpha_\infty^{-1}(b,\widehat{\omega}) = \widehat{\omega} + \widehat{\mathrm{r}}_\varepsilon(b,\widehat{\omega}).$$

In particular,

$$\widehat{\mathrm{r}}_\varepsilon(b,\widehat{\omega}) = -\mathrm{r}_\varepsilon(b,\omega).$$

Differentiating the previous relation and using (14.6), we find

$$\|\widehat{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1} N_0^{q\overline{a}}. \tag{14.10}$$

Now, we set

$$\omega(b,\varepsilon) \triangleq \alpha_\infty^{-1}(b, -\omega_{\mathrm{Eq}}(b)) = -\omega_{\mathrm{Eq}}(b) + \overline{\mathrm{r}}_\varepsilon(b) \quad \text{with} \quad \overline{\mathrm{r}}_\varepsilon(b) \triangleq \widehat{\mathrm{r}}_\varepsilon\big(b, -\omega_{\mathrm{Eq}}(b)\big)$$

and consider the following Cantor set

$$\mathcal{C}_\infty^\varepsilon \triangleq \Big\{ b \in (b_0, b_1) \quad \text{s.t.} \quad \big(b, \omega(b,\varepsilon)\big) \in \mathcal{G}_\infty^\gamma \Big\}.$$

Then, according to (14.9), we get

$$\forall b \in \mathcal{C}_\infty^\varepsilon, \quad \mathcal{F}\Big(U_\infty\big(b, \omega(b,\varepsilon)\big)\Big) = 0.$$

This gives a nontrivial reversible solution for the original Hamiltonian equation provided that $b \in \mathcal{C}_\infty^\varepsilon$. From Lemma 11.3, we obtain that all the derivatives up to order q of ω_{Eq} are uniformly bounded on

$[b_0,b_1]$. As a consequence, the chain rule and (14.10) imply

$$\|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \lesssim \varepsilon\gamma^{-1}N_0^{q\overline{a}} \quad \text{and} \quad \|\omega(\cdot,\varepsilon)\|_q^{\gamma,\mathcal{O}} \lesssim 1+\varepsilon\gamma^{-1}N_0^{q\overline{a}} \lesssim 1. \tag{14.11}$$

This achieves the proof of Corollary 14.1. □

14.2 Measure estimates

In this last section, we check that the Cantor set $\mathcal{C}_\infty^\varepsilon$, defined in (14.8), of parameters generating non-trivial quasi-periodic solutions is non trivial. More precisely, we have the following proposition giving a lower bound measure for $\mathcal{C}_\infty^\varepsilon$.

Proposition 14.2. *Let* q_0 *be defined as in Lemma 11.5 and impose* (8.1) *and* (8.2) *with* $q=q_0+1$. *Assume the additional conditions*

$$\begin{cases} \tau_1 > dq_0 \\ \tau_2 > \tau_1 + dq_0 \\ \upsilon = \frac{1}{q_0+3}. \end{cases} \tag{14.12}$$

Then there exists $C>0$ *such that*

$$\left|\mathcal{C}_\infty^\varepsilon\right| \geqslant (b_1-b_0) - C\varepsilon^{\frac{a\upsilon}{q_0}}.$$

In particular,

$$\lim_{\varepsilon\to 0}\left|\mathcal{C}_\infty^\varepsilon\right| = b_1-b_0.$$

Remark 14.1. *The constraints listed in* (14.12) *appear naturally in the proof, see* (14.20) *and* (14.25), *for the convergence of series and for smallness conditions. Notice that these conditions agree with* (A.1) *and Proposition 13.2.*

Proof. According to Corollary 14.1, we can decompose the Cantor set $\mathcal{C}_\infty^\varepsilon$ in the following intersection

$$\mathcal{C}_\infty^\varepsilon \triangleq \bigcap_{n\in\mathbb{N}} \mathcal{C}_n^\varepsilon \quad \text{where} \quad \mathcal{C}_n^\varepsilon \triangleq \Big\{b\in(b_0,b_1) \quad \text{s.t} \quad \big(b,\omega(b,\varepsilon)\big)\in\mathcal{A}_n^\gamma\Big\}. \tag{14.13}$$

Recall that the intermediate sets $\mathcal{A}_n^\gamma$ and the perturbed frequency vector $\omega(b,\varepsilon)$ are respectively defined in Proposition 14.1 and in (14.6). Instead of measuring directly $\mathcal{C}_\infty^\varepsilon$, we rather estimate the measure of its complementary set in (b_0,b_1). Thus, we write

$$(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon = \big((b_0,b_1)\setminus\mathcal{C}_0^\varepsilon\big)\sqcup\bigsqcup_{n=0}^{\infty}\big(\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon\big). \tag{14.14}$$

Then, we have to measure all the sets appearing in the decomposition (14.14). This can be done by using Lemma 5.6 together with some trivial inclusions allowing to link the time and space Fourier modes in order to make the series converge. For more details, we refer to Lemmata 14.1, 14.2 and 14.3. From (8.2) and (14.7), one obtains

$$\sup_{b\in(b_0,b_1)}|\omega(b,\varepsilon)+\omega_{\mathrm{Eq}}(b)| \leqslant \|\mathrm{r}_\varepsilon\|_q^{\gamma,\mathcal{O}} \leqslant C\varepsilon\gamma^{-1}N_0^{q\overline{a}} = C\varepsilon^{1-a(1+q\overline{a})}.$$

Notice that the conditions (8.1) and (8.2) imply in particular

$$0<a<\frac{1}{1+q\overline{a}}.$$

Therefore, taking ε small enough yields

$$\sup_{b\in(b_0,b_1)}|\omega(b,\varepsilon)+\omega_{\mathrm{Eq}}(b)|\leqslant\|\bar{\mathrm{r}}_\varepsilon\|_q^{\gamma,\mathcal{O}}\leqslant 1.$$

Recall that $\mathscr{U}=B(0,R_0)$, then, up to taking R_0 large enough, we get

$$\forall b\in(b_0,b_1),\ \forall\varepsilon\in[0,\varepsilon_0),\quad \omega(b,\varepsilon)\in\mathscr{U}=B(0,R_0).$$

Recall that $\mathcal{A}_0^\gamma=\mathcal{O}=(b_0,b_1)\times\mathscr{U}$ then, from (14.13),

$$\mathcal{C}_0^\varepsilon=(b_0,b_1)$$

and coming back to (14.14), we find

$$\begin{aligned}\Big|(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon\Big|&\leqslant\sum_{n=0}^{\infty}\Big|\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon\Big|\\&\triangleq\sum_{n=0}^{\infty}\mathcal{S}_n.\end{aligned}\tag{14.15}$$

In accordance with the notations used in Propositions 13.3 and 13.4, we denote the perturbed frequencies associated with the reduced linearized operator at state i_n in the following way

$$\begin{aligned}\mu_j^{\infty,n}(b,\varepsilon)&\triangleq\mu_j^\infty\big(b,\omega(b,\varepsilon),i_n\big)\\&=\Omega_j(b)+jr^{1,n}(b,\varepsilon)+r_j^{\infty,n}(b,\varepsilon),\end{aligned}\tag{14.16}$$

where

$$\begin{aligned}r^{1,n}(b,\varepsilon)&\triangleq V_n^\infty(b,\varepsilon)-\tfrac{1}{2},\\V_n^\infty(b,\varepsilon)&\triangleq V_{i_n}^\infty(b,\omega(b,\varepsilon)),\\r_j^{\infty,n}(b,\varepsilon)&\triangleq r_j^\infty\big(b,\omega(b,\varepsilon),i_n\big).\end{aligned}$$

Now, according to (14.13), Propositions 13.4, 13.5 and 13.2 one can write for any $n\in\mathbb{N}$,

$$\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\|l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n),\tag{14.17}$$

where we denote

$$\begin{aligned}\mathcal{R}_{l,j}^{(0)}(i_n)&\triangleq\left\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+jV_n^\infty(b,\varepsilon)\Big|\leqslant\tfrac{4\gamma_{n+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\},\\\mathcal{R}_{l,j,j_0}(i_n)&\triangleq\left\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)\Big|\leqslant\tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}\right\},\\\mathcal{R}_{l,j}^{(1)}(i_n)&\triangleq\left\{b\in\mathcal{C}_n^\varepsilon\quad\text{s.t.}\quad\Big|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\Big|\leqslant\tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}}\right\}.\end{aligned}$$

In view of the inclusion

$$W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\hookrightarrow C^{q-1}(\mathcal{O},\mathbb{C})$$

and the fact that $q = q_0 + 1$, one obtains that for any $n \in \mathbb{N}$ the curves

$$\begin{aligned}
&b \mapsto \omega(b,\varepsilon)\cdot l + jV_n^{\infty}(b,\varepsilon), \quad (l,j) \in \mathbb{Z}^d \times \mathbb{Z}\backslash\{(0,0)\}\\
&b \mapsto \omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon), \quad (l,j,j_0) \in \mathbb{Z}^d \times (\mathbb{S}_0^c)^2\\
&b \mapsto \omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon), \quad (l,j) \in \mathbb{Z}^d \times \mathbb{S}_0^c
\end{aligned}$$

are of regularity C^{q_0}. Therefore, applying Lemma 5.6 together with Lemma 14.3 yields

$$\begin{aligned}
\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big| &\lesssim \gamma^{\frac{\upsilon}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\
\Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}},\\
\Big|\mathcal{R}_{l,j,j_0}(i_n)\Big| &\lesssim \gamma^{\frac{1}{q_0}}\langle j-j_0\rangle^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}.
\end{aligned} \tag{14.18}$$

We first estimate the measure of $\mathcal{S}_0$ and $\mathcal{S}_1$ defined in (14.15). From Lemma 14.2, we have some trivial inclusions allowing us to write for $n \in \{0,1\}$,

$$\mathcal{S}_n \lesssim \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\backslash\{(0,0)\}\\ |j|\leqslant C_0\langle l\rangle,|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big| + \sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |j-j_0|\leqslant C_0\langle l\rangle,|l|\leqslant N_n\\ \min(|j|,|j_0|)\leqslant c_2\gamma_1^{-\upsilon}\langle l\rangle^{\tau_1}}}\Big|\mathcal{R}_{l,j,j_0}(i_n)\Big| + \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |j|\leqslant C_0\langle l\rangle,|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big|. \tag{14.19}$$

Inserting (14.18) into (14.19) implies that for $n \in \{0,1\}$,

$$\begin{aligned}
\mathcal{S}_n \lesssim\ & \gamma^{\frac{1}{q_0}}\Bigg(\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}} + \sum_{\substack{|j-j_0|\leqslant C_0\langle l\rangle\\ \min(|j|,|j_0|)\leqslant c_2\gamma^{-\upsilon}\langle l\rangle^{\tau_1}}}|j-j_0|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_2+1}{q_0}}\Bigg)\\
&+\gamma^{\frac{\upsilon}{q_0}}\sum_{|j|\leqslant C_0\langle l\rangle}|j|^{\frac{1}{q_0}}\langle l\rangle^{-1-\frac{\tau_1+1}{q_0}}.
\end{aligned}$$

The first two conditions listed in (14.12) write

$$\tau_1 > d\,q_0 \quad \text{and} \quad \tau_2 > \tau_1 + d\,q_0. \tag{14.20}$$

Hence, we can make the series appearing in the following expression converge and write

$$\begin{aligned}
\max_{n\in\{0,1\}}\mathcal{S}_n &\lesssim \gamma^{\frac{1}{q_0}}\Bigg(\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}} + \gamma^{-\upsilon}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg) + \gamma^{\frac{\upsilon}{q_0}}\sum_{l\in\mathbb{Z}^d}\langle l\rangle^{-\frac{\tau_1}{q_0}}\\
&\lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.
\end{aligned} \tag{14.21}$$

Let us now move to the estimate of $\mathcal{S}_n$ for $n \geqslant 2$ defined by (14.15). Using Lemma 14.1 and Lemma 14.2, we infer

$$\mathcal{S}_n \leqslant \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\backslash\{(0,0)\}\\ |j|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(0)}(i_n)\Big| + \sum_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ |j-j_0|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n\\ \min(|j|,|j_0|)\leqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}}}\Big|\mathcal{R}_{l,j,j_0}(i_n)\Big| + \sum_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ |j|\leqslant C_0\langle l\rangle,N_{n-1}<|l|\leqslant N_n}}\Big|\mathcal{R}_{l,j}^{(1)}(i_n)\Big|.$$

Notice that if $|j-j_0| \leqslant C_0\langle l\rangle$ and $\min(|j|,|j_0|) \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\max(|j|,|j_0|) = \min(|j|,|j_0|) + |j-j_0| \leqslant \gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1} + C_0\langle l\rangle \lesssim \gamma^{-\upsilon}\langle l\rangle^{\tau_1}.$$

Hence, we deduce from (14.18) that

$$\mathcal{S}_n \lesssim \gamma^{\frac{1}{q_0}}\Bigg(\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}+\gamma^{-\upsilon}\sum_{|l|>N_{n-1}}\langle l\rangle^{\tau_1-1-\frac{\tau_2}{q_0}}\Bigg)+\gamma^{\frac{\upsilon}{q_0}}\sum_{|l|>N_{n-1}}\langle l\rangle^{-\frac{\tau_1}{q_0}}.$$

Now according to (14.20), we obtain

$$\sum_{n=2}^{\infty}\mathcal{S}_n \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}. \tag{14.22}$$

Inserting (14.22) and (14.21) into (14.15) yields

$$\Big|(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon\Big| \lesssim \gamma^{\min\left(\frac{\upsilon}{q_0},\frac{1}{q_0}-\upsilon\right)}.$$

Remark also that (14.12) implies

$$\min\left(\tfrac{\upsilon}{q_0},\tfrac{1}{q_0}-\upsilon\right)=\tfrac{\upsilon}{q_0}.$$

Consequently, using the fact that $\gamma=\varepsilon^a$ due to (8.2), we finally get

$$\Big|(b_0,b_1)\setminus\mathcal{C}_\infty^\varepsilon\Big| \lesssim \varepsilon^{\frac{a\upsilon}{q_0}}.$$

This ends the proof of Proposition 14.2. □

We shall now prove Lemmata 14.1, 14.2 and 14.3 used in the proof of Proposition 14.2.

Lemma 14.1. *Let $n\in\mathbb{N}\setminus\{0,1\}$ and $l\in\mathbb{Z}^d$ such that $|l|\leqslant N_{n-1}$. Then the following assertions hold true.*

(i) For $j\in\mathbb{Z}$ with $(l,j)\neq(0,0)$, we get $\mathcal{R}_{l,j}^{(0)}(i_n)=\varnothing$.

(ii) For $(j,j_0)\in(\mathbb{S}_0^c)^2$ with $(l,j)\neq(0,j_0)$, we get $\mathcal{R}_{l,j,j_0}(i_n)=\varnothing$.

(iii) For $j\in\mathbb{S}_0^c$, we get $\mathcal{R}_{l,j}^{(1)}(i_n)=\varnothing$.

(iv) For any $n\in\mathbb{N}\setminus\{0,1\}$,

$$\mathcal{C}_n^\varepsilon\setminus\mathcal{C}_{n+1}^\varepsilon=\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(0)}(i_n)\cup\bigcup_{\substack{(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j,j_0}(i_n)\cup\bigcup_{\substack{(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c\\ N_{n-1}<|l|\leqslant N_n}}\mathcal{R}_{l,j}^{(1)}(i_n).$$

Proof. The following estimate, obtained from (14.3), turns to be very useful in the sequel. For any $n\geqslant 2$, we have

$$\begin{aligned}\|i_n-i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}} &\leqslant \|U_n-U_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant \|H_n\|_{q,s_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant C_*\varepsilon\gamma^{-1}N_{n-1}^{-a_2}.\end{aligned}\tag{14.23}$$

Since (14.23) is only true for $n\geqslant 2$, we had to estimate the measures of $\mathcal{S}_0$ and $\mathcal{S}_1$ differently in the proof of Proposition 14.2.

(i) Assume that $|l|\leqslant N_{n-1}$ and $(l,j)\neq(0,0)$. Let us prove that

$$\mathcal{R}_{l,j}^{(0)}(i_n)\subset\mathcal{R}_{l,j}^{(0)}(i_{n-1}). \tag{14.24}$$

Take $b \in \mathcal{R}_{l,j}^{(0)}(i_n)$. In view of (14.17), we have in particular that $b \in \mathcal{C}_n^\varepsilon \subset \mathcal{C}_{n-1}^\varepsilon$. In addition, the triangle inequality gives

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^\infty(b,\varepsilon)\big| &\leqslant \big|\omega(b,\varepsilon)\cdot l + jV_n^\infty(b,\varepsilon)\big| + |j|\big|V_n^\infty(b,\varepsilon) - V_{n-1}^\infty(b,\varepsilon)\big| \\ &\leqslant \tfrac{4\gamma_{n+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} + C|j|\|V_{i_n}^\infty - V_{i_{n-1}}^\infty\|_q^{\gamma,\mathcal{O}}.\end{aligned}$$

Thus, putting together (13.27), (14.23), (8.2) and the fact that $\sigma_4 \geqslant 2$, we obtain

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^\infty(b,\varepsilon)\big| &\leqslant \tfrac{4\gamma_{n+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon\langle j\rangle\|i_n - i_{n-1}\|_{q,\overline{s}_h+2}^{\gamma,\mathcal{O}} \\ &\leqslant \tfrac{4\gamma_{n+1}^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon^{2-a}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

According the definition of γ_n in Proposition 14.1-$(\mathcal{P}2)_n$, we infer

$$\exists c_0 > 0, \quad \forall n \in \mathbb{N}, \quad \gamma_{n+1}^\upsilon - \gamma_n^\upsilon \leqslant -c_0\,\gamma^\upsilon 2^{-n}.$$

Notice that (14.12), (8.1) and (8.2) give

$$2 - a - a\upsilon > 1 \quad \text{and} \quad a_2 > \tau_1, \tag{14.25}$$

which implies in turn

$$\sup_{n\in\mathbb{N}} 2^n N_{n-1}^{-a_2+\tau_1} < \infty.$$

Consequently, for ε small enough and $|l| \leqslant N_{n-1}$,

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + jV_{n-1}^\infty(b,\varepsilon)\big| &\leqslant \tfrac{4\gamma_n^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\tfrac{\langle j\rangle\gamma^\upsilon}{2^n\langle l\rangle^{\tau_1}}\Big(-4c_0 + C\varepsilon 2^n N_{n-1}^{-a_2+\tau_1}\Big) \\ &\leqslant \tfrac{4\gamma_n^\upsilon\langle j\rangle}{\langle l\rangle^{\tau_1}}.\end{aligned}$$

It follows that $b \in \mathcal{R}_{l,j}^{(0)}(i_{n-1})$ and this proves (14.24). Now, from (14.17) we deduce

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{R}_{l,j}^{(0)}(i_{n-1}) \subset \mathcal{C}_{n-1}^\varepsilon \setminus \mathcal{C}_n^\varepsilon.$$

In view of (14.24) and (14.17), we get $\mathcal{R}_{l,j}^{(0)}(i_n) \subset \mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon$ and thus we conclude

$$\mathcal{R}_{l,j}^{(0)}(i_n) \subset \big(\mathcal{C}_n^\varepsilon \setminus \mathcal{C}_{n+1}^\varepsilon\big) \cap \big(\mathcal{C}_{n-1}^\varepsilon \setminus \mathcal{C}_n^\varepsilon\big) = \varnothing.$$

This proves the first point.
(ii) Let $(j,j_0) \in (\mathbb{S}_0^c)^2$ and $(l,j) \neq (0,j_0)$. If $j = j_0$ then by construction $\mathcal{R}_{l,j_0,j_0}(i_n) = \mathcal{R}_{l,0}^{(0)}(i_n)$ and then the result is an immediate consequence of the first point. Then, we restrict the discussion to the case $j \neq j_0$. In a similar way to the point (i), we only have to check that

$$\mathcal{R}_{l,j,j_0}(i_n) \subset \mathcal{R}_{l,j,j_0}(i_{n-1}).$$

Take $b \in \mathcal{R}_{l,j,j_0}(i_n)$. Then coming back to (14.17), we deduce from the triangle inequality that $b \in \mathcal{C}_n^\varepsilon \subset \mathcal{C}_{n-1}^\varepsilon$ and

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(b,\varepsilon) - \mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}} + \varrho_{j,j_0}^n(b,\varepsilon), \tag{14.26}$$

where

$$\varrho_{j,j_0}^n(b,\varepsilon) \triangleq \big|\mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon) - \mu_j^{\infty,n-1}(b,\varepsilon) + \mu_{j_0}^{\infty,n-1}(b,\varepsilon)\big|.$$

According to (14.16), one obtains

$$\varrho^n_{j,j_0}(b,\varepsilon)\leqslant |j-j_0|\big|r^{1,n}(b,\varepsilon)-r^{1,n-1}(b,\varepsilon)\big|+\big|r^{\infty,n}_j(b,\varepsilon)-r^{\infty,n-1}_j(b,\varepsilon)\big|$$
$$+\big|r^{\infty,n}_{j_0}(b,\varepsilon)-r^{\infty,n-1}_{j_0}(b,\varepsilon)\big|. \tag{14.27}$$

From (13.54), (14.23), (8.2) and the fact that $\sigma_4\geqslant\sigma_3$, we deduce that

$$\begin{aligned}\big|r^{1,n}(b,\varepsilon)-r^{1,n-1}(b,\varepsilon)\big|&\lesssim\varepsilon\|i_n-i_{n-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma_3}\\&\lesssim\varepsilon^2\gamma^{-1}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2-a}N_{n-1}^{-a_2}.\end{aligned}$$

Similarly, (13.67), (14.23) and (8.2) imply

$$\begin{aligned}\big|r^{\infty,n}_j(b,\varepsilon)-r^{\infty,n-1}_j(b,\varepsilon)\big|&\lesssim\varepsilon\gamma^{-1}\|i_n-i_{n-1}\|^{\gamma,\mathcal{O}}_{q,\overline{s}_h+\sigma_4}\\&\lesssim\varepsilon^2\gamma^{-2}N_{n-1}^{-a_2}\\&\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Plugging the preceding two estimates into (14.27) yields

$$\varrho^n_{j,j_0}(b,\varepsilon)\lesssim\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}. \tag{14.28}$$

Gathering (14.28) and (14.26) and using $\gamma_{n+1}=\gamma_n-\varepsilon^a2^{-n-1}$, we obtain

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l+\mu^{\infty,n-1}_j(b,\varepsilon)-\mu^{\infty,n-1}_{j_0}(b,\varepsilon)\big|\leqslant{}&\tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}-\varepsilon^a\langle j-j_0\rangle2^{-n}\langle l\rangle^{-\tau_2}\\&+C\varepsilon^{2(1-a)}\langle j-j_0\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Using the fact that $|l|\leqslant N_{n-1}$, we deduce

$$-\varepsilon^a2^{-n}\langle l\rangle^{-\tau_2}+C\varepsilon^{2(1-a)}N_{n-1}^{-a_2}\leqslant\varepsilon^a2^{-n}\langle l\rangle^{-\tau_2}\Big(-1+C\varepsilon^{2-3a}2^nN_{n-1}^{-a_2+\tau_2}\Big).$$

Notice that (8.1) and (8.2) imply in particular

$$a_2>\tau_2\quad\text{and}\quad a<\tfrac{2}{3}. \tag{14.29}$$

Therefore, for ε small enough, we get

$$\forall\, n\in\mathbb{N},\quad -1+C\varepsilon^{2-3a}2^nN_{n-1}^{-a_2+\tau_2}\leqslant0,$$

which implies in turn

$$\big|\omega(b,\varepsilon)\cdot l+\mu^{\infty,n-1}_j(b,\varepsilon)-\mu^{\infty,n-1}_{j_0}(b,\varepsilon)\big|\leqslant\tfrac{2\gamma_n\langle j-j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Finally, $b\in\mathcal{R}_{l,j,j_0}(i_{n-1})$. This achieves the proof of the second point.
(iii) Let $j\in\mathbb{S}_0^c$. In particular, one has $(l,j)\neq(0,0)$. In a similar line to the first point, we shall prove that if $|l|\leqslant N_{n-1}$ and then

$$\mathcal{R}^{(1)}_{l,j}(i_n)\subset\mathcal{R}^{(1)}_{l,j}(i_{n-1}),$$

where the set $\mathcal{R}^{(1)}_{l,j}(i_n)$ is defined below (14.17). Take $b\in\mathcal{R}^{(1)}_{l,j}(i_n)$. Then, by construction, $b\in\mathcal{C}^\varepsilon_n\subset\mathcal{C}^\varepsilon_{n-1}$.

By using the triangle inequality, (13.68), (14.23) and the choice $\gamma = \varepsilon^a$, we obtain

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(b,\varepsilon)\big| &\leqslant \big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon)\big| + |\mu_j^{\infty,n}(b,\varepsilon) - \mu_j^{\infty,n-1}(b,\varepsilon)|\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon\gamma^{-1}|j|\|i_n - i_{n-1}\|_{q,\overline{s}_h+\sigma_4}^{\gamma,\mathcal{O}}\\ &\leqslant \tfrac{\gamma_{n+1}\langle j\rangle}{\langle l\rangle^{\tau_1}} + C\varepsilon^{2(1-a)}\langle j\rangle N_{n-1}^{-a_2}.\end{aligned}$$

Now recalling that $\gamma_{n+1} = \gamma_n - \varepsilon^a 2^{-n-1}$ and $|l| \leqslant N_{n-1}$, we get

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(b,\varepsilon)\big| \leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}} + \tfrac{\langle j\rangle\varepsilon^a}{2^{n+1}\langle l\rangle^{\tau_1}}\Big(-1 + \varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1}\Big).$$

As a byproduct of (14.29), we infer

$$a_2 > \tau_1 \quad \text{and} \quad a < \tfrac{2}{3}. \tag{14.30}$$

Therefore, up to taking ε small enough, we deduce

$$\forall\, n\in\mathbb{N},\quad -1 + \varepsilon^{2-3a}2^{n+1}N_{n-1}^{-a_2+\tau_1} \leqslant 0,$$

which implies in turn that

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n-1}(b,\varepsilon)\big| \leqslant \tfrac{\gamma_n\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

Finally, $b\in\mathcal{R}_{l,j}^{(1)}(i_{n-1})$ and the proof of the third point is now complete.
(iv) Follows immediately from (14.17) and the points (i)-(ii)-(iii). ☐

The following lemma provides necessary constraints on the time and space Fourier modes so that the sets in (14.17) are not void.

Lemma 14.2. *There exists ε_0 such that for any $\varepsilon\in[0,\varepsilon_0]$ and $n\in\mathbb{N}$ the following assertions hold true.*

(i) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{Z}\setminus\{(0,0)\}$. If $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(ii) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. If $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$, then $|j-j_0|\leqslant C_0\langle l\rangle$.

(iii) Let $(l,j)\in\mathbb{Z}^d\times\mathbb{S}_0^c$. If $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$, then $|j|\leqslant C_0\langle l\rangle$.

(iv) Let $(l,j,j_0)\in\mathbb{Z}^d\times(\mathbb{S}_0^c)^2$. There exists $c_2>0$ such that if $\min(|j|,|j_0|)\geqslant c_2\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$, then

$$\mathcal{R}_{l,j,j_0}(i_n)\subset\mathcal{R}_{l,j-j_0}^{(0)}(i_n).$$

Proof. **(i)** Let us assume that $\mathcal{R}_{l,j}^{(0)}(i_n)\neq\varnothing$. Then, there exists $b\in(b_0,b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l + jV_n^{\infty}(b,\varepsilon)| \leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\rangle}{\langle l\rangle^{\tau_1}}.$$

From triangle and Cauchy-Schwarz inequalities, (14.4) and (8.2), we deduce

$$\begin{aligned}|V_n^{\infty}(b,\varepsilon)||j| &\leqslant 4|j|\gamma_{n+1}^{\upsilon}\langle l\rangle^{-\tau_1} + |\omega(b,\varepsilon)\cdot l|\\ &\leqslant 4|j|\gamma_{n+1}^{\upsilon} + C\langle l\rangle\\ &\leqslant 8\varepsilon^{a\upsilon}|j| + C\langle l\rangle.\end{aligned}$$

Remark that we used the fact that $(b,\varepsilon)\mapsto\omega(b,\varepsilon)$ is bounded. Also notice that the identity

$$V_n^{\infty}(b,\varepsilon) = \tfrac{1}{2} + r^{1,n}(b,\varepsilon)$$

together with (13.20), (13.65) and Proposition 14.1-$(\mathcal{P}1)_n$ imply

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad \sup_{n\in\mathbb{N}}\sup_{b\in(b_0,b_1)}|\partial_b^k r^{1,n}(b,\varepsilon)| &\leqslant \gamma^{-k}\sup_{n\in\mathbb{N}}\|r^{1,n}\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-k}\\ &\lesssim \varepsilon^{1-ak}.\end{aligned}\tag{14.31}$$

Hence, taking ε small enough, we infer

$$\inf_{n\in\mathbb{N}}\inf_{b\in(b_0,b_1)}|V_n^{\infty}(b,\varepsilon)|\geqslant \tfrac{1}{4}.$$

Therefore, up to choosing ε small enough we can ensure $|j|\leqslant C_0\langle l\rangle$ for some $C_0>0$.
(ii) In the case $j=j_0$ we get by definition $\mathcal{R}_{l,j_0,j_0}(i_n)=\mathcal{R}_{l,0}^{(0)}(i_n)$, so this case can be treated by the first point. Then, we shall restrict the discussion to the case $j\neq j_0$. Let us assume that $\mathcal{R}_{l,j,j_0}(i_n)\neq\varnothing$. Then, there exists $b\in(b_0,b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)|\leqslant \tfrac{2\gamma_{n+1}|j-j_0|}{\langle l\rangle^{\tau_2}}.$$

By using triangle and Cauchy-Schwarz inequalities, (14.4) and (8.2), we get

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)| &\leqslant 2\gamma_{n+1}|j-j_0|\langle l\rangle^{-\tau_2}+|\omega(b,\varepsilon)\cdot l|\\ &\leqslant 2\gamma_{n+1}|j-j_0|+C\langle l\rangle\\ &\leqslant 4\varepsilon^a|j-j_0|+C\langle l\rangle.\end{aligned}$$

In a similar way to (14.31), we may obtain

$$\begin{aligned}\forall k\in[\![0,q]\!],\quad \sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}\sup_{b\in(b_0,b_1)}|j||\partial_b^k r_j^{\infty,n}(b,\varepsilon)| &\leqslant \gamma^{-k}\sup_{n\in\mathbb{N}}\sup_{j\in\mathbb{S}_0^c}|j|\|r_j^{\infty,n}\|_q^{\gamma,\mathcal{O}}\\ &\lesssim \varepsilon\gamma^{-1-k}\\ &\lesssim \varepsilon^{1-a(1+k)}.\end{aligned}\tag{14.32}$$

From the triangle inequality, Lemma 11.3-(iii), (14.31) and (14.32) we infer for $j\neq j_0$,

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)| &\geqslant |\Omega_j(b)-\Omega_{j_0}(b)|-|r^{1,n}(b,\varepsilon)||j-j_0|-|r_j^{\infty,n}(b,\varepsilon)|-|r_{j_0}^{\infty,n}(b,\varepsilon)|\\ &\geqslant \big(\tfrac{b_0^2}{6}-C\varepsilon^{1-a}\big)|j-j_0|\\ &\geqslant \tfrac{b_0^2}{12}|j-j_0|.\end{aligned}$$

Notice that the last inequality is obtained for ε sufficiently small. Gathering the previous inequalities implies that, up to choosing ε small enough, we can ensure $|j-j_0|\leqslant C_0\langle l\rangle$, for some $C_0>0$.
(iii) First notice that the case $j=0$ is obvious. Now for $j\neq 0$ we assume that $\mathcal{R}_{l,j}^{(1)}(i_n)\neq\varnothing$. Then, there exists $b\in(b_0,b_1)$ such that

$$|\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)|\leqslant \tfrac{\gamma_{n+1}|j|}{\langle l\rangle^{\tau_1}}.$$

Thus, triangle and Cauchy-Schwarz inequalities, (14.4) and (8.2) imply

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)| &\leqslant \gamma_{n+1}|j|\langle l\rangle^{-\tau_1}+|\omega(b,\varepsilon)\cdot l|\\ &\leqslant 2\varepsilon^a|j|+C\langle l\rangle.\end{aligned}$$

According to the definition (14.16) together with the triangle inequality, Lemma 11.3-(ii), (14.31) and (14.32), we obtain

$$\begin{aligned}|\mu_j^{\infty,n}(b,\varepsilon)| &\geqslant \tfrac{b_0^2}{2}|j| - |j||r^{1,n}(b,\varepsilon)| - |r_j^{\infty,n}(b,\varepsilon)|\\ &\geqslant \tfrac{b_0^2}{2}|j| - C\varepsilon^{1-a}|j|.\end{aligned}$$

Putting together the previous two inequalities and the second condition in (14.30) yields

$$\big(\tfrac{b_0^2}{2} - C\varepsilon^{1-a} - 2\varepsilon^a\big)|j| \leqslant C\langle l\rangle.$$

Finally, by choosing ε small enough we get $|j| \leqslant C_0\langle l\rangle$, for some $C_0 > 0$.

(iv) First remark that the case $j = j_0$ is obvious as a direct consequence of the definition (14.17). Let $j \neq j_0$. In view of the symmetry property $\mu_{-j}^{\infty,n} = -\mu_j^{\infty,n}$ of the perturbed eigenvalues, we can always assume that $0 < j < j_0$. Take $b \in \mathcal{R}_{l,j,j_0}(i_n)$. Then by construction

$$\big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) \pm \mu_{j_0}^{\infty,n}(b,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}}.$$

Putting together (14.16), (11.15) and the triangle inequality, we find

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + (j\pm j_0)V_n^{\infty}(b,\varepsilon)\big| \leqslant{}& \big|\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) \pm \mu_{j_0}^{\infty,n}(b,\varepsilon)\big| + \tfrac{1}{2}|b^{2j} \pm b^{2j_0}|\\ &+ \tfrac{1}{2}\big|(j-1)\pm(j_0-1) - (j\pm j_0)\big| + \big|r_j^{\infty,n}(b,\varepsilon) \pm r_{j_0}^{\infty,n}(b,\varepsilon)\big|.\end{aligned}$$

Hence, we deduce

$$\begin{aligned}\big|\omega(b,\varepsilon)\cdot l + (j\pm j_0)V_n^{\infty}(b,\varepsilon)\big| \leqslant{}& \tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}} + \tfrac{1}{2}|b^{2j} \pm b^{2j_0}|\\ &+ \tfrac{1}{2}\big|(j-1)\pm(j_0-1) - (j\pm j_0)\big| + \big|r_j^{\infty,n}(b,\varepsilon) \pm r_{j_0}^{\infty,n}(b,\varepsilon)\big|. \quad (14.33)\end{aligned}$$

Notice that

$$b^{2j} + b^{2j_0} \leqslant C\tfrac{\langle j+j_0\rangle}{j}.$$

In addition, Taylor formula implies

$$b^{2j} - b^{2j_0} \leqslant -2\ln(b)\int_j^{j_0} b^{2x}dx \leqslant \tfrac{c_1\langle j-j_0\rangle}{j},$$

where $c_1 = \sup\limits_{j\in\mathbb{N}, b\in(0,1)}\big(-2\ln(b)jb^{2j}\big) > 0$. On the other hand, one has

$$\big|(j-1)\pm(j_0-1) - (j\pm j_0)\big| = 1\pm 1 \leqslant \tfrac{\langle j+j_0\rangle}{j}.$$

Applying (13.65), we find for $j \neq j_0$,

$$\begin{aligned}\big|r_j^{\infty,n}(b,\varepsilon) \pm r_{j_0}^{\infty,n}(b,\varepsilon)\big| \leqslant{}& C\varepsilon^{1-a}\big(|j|^{-1} + |j_0|^{-1}\big)\\ \leqslant{}& C\varepsilon^{1-a}\tfrac{\langle j\pm j_0\rangle}{j}.\end{aligned}$$

Plugging the preceding estimates into (14.33) yields

$$\big|\omega(b,\varepsilon)\cdot l + (j\pm j_0)V_n^{\infty}(b,\varepsilon)\big| \leqslant \tfrac{2\gamma_{n+1}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_2}} + C\tfrac{\langle j\pm j_0\rangle}{j}.$$

Therefore, if we assume $j \geqslant \frac{1}{2}C\gamma_{n+1}^{-\upsilon}\langle l\rangle^{\tau_1}$ and $\tau_2 > \tau_1$, then we deduce

$$\left|\omega(b,\varepsilon)\cdot l + (j\pm j_0)V_n^{\infty}(b,\varepsilon)\right| \leqslant \tfrac{4\gamma_{n+1}^{\upsilon}\langle j\pm j_0\rangle}{\langle l\rangle^{\tau_1}}.$$

This achieves the proof of Lemma 14.2, taking $c_2 = \frac{C}{2}$. □

We shall now establish that the perturbed frequencies $\omega(b,\varepsilon)$ satisfy the Rüssmann conditions. This is done by a perturbation argument on the transversality conditions of the equilibrium linear frequencies $\omega_{\mathrm{Eq}}(b)$ stated in Lemma 11.5.

Lemma 14.3. *Let q_0, C_0 and ρ_0 as in Lemma 11.5. There exist $\varepsilon_0 > 0$ small enough such that for any $\varepsilon \in [0,\varepsilon_0]$ the following assertions hold true.*

(i) For all $l \in \mathbb{Z}^d \setminus \{0\}$, we have

$$\inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\left(\omega(b,\varepsilon)\cdot l\right)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(ii) For all $(l,j) \in \mathbb{Z}^{d+1}\setminus\{(0,0)\}$ such that $|j| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l + jV_n^{\infty}(b,\varepsilon)\big)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iii) For all $(l,j) \in \mathbb{Z}^d\times\mathbb{S}_0^c$ such that $|j| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon)\big)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

(iv) For all $(l,j,j_0) \in \mathbb{Z}^d\times(\mathbb{S}_0^c)^2$ such that $|j-j_0| \leqslant C_0\langle l\rangle$, we have

$$\forall n\in\mathbb{N},\qquad \inf_{b\in[b_0,b_1]}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l + \mu_j^{\infty,n}(b,\varepsilon) - \mu_{j_0}^{\infty,n}(b,\varepsilon)\big)\right| \geqslant \tfrac{\rho_0\langle l\rangle}{2}.$$

Proof. **(i)** From the triangle and Cauchy-Schwarz inequalities together with (14.11), (8.2) and Lemma 11.5-(i), we deduce

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\left(\omega(b,\varepsilon)\cdot l\right)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\left(\omega_{\mathrm{Eq}}(b)\cdot l\right)\right| - \max_{k\in[\![0,q]\!]}\left|\partial_b^k\left(\bar{\mathrm{r}}_{\varepsilon}(b)\cdot l\right)\right|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon\gamma^{-1-q}N_0^{q\overline{a}}\langle l\rangle\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

provided that ε is small enough and

$$1 - a(1+q+q\overline{a}) > 0. \tag{14.34}$$

Notice that the condition (14.34) is automatically satisfied by (8.2) and (8.1).

(ii) As before, using the triangle and Cauchy-Schwarz inequalities combined with (14.11), (14.31), Lemma 11.5-(ii) and the fact that $|j| \leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\left(\omega(b,\varepsilon)\cdot l + jV_n^{\infty}(b,\varepsilon)\right)\right| &\geqslant \max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega_{\mathrm{Eq}}(b)\cdot l + \tfrac{j}{2}\big)\right| - \max_{k\in[\![0,q]\!]}\left|\partial_b^k\left(\bar{\mathrm{r}}_{\varepsilon}(b)\cdot l + jr^{1,n}(b,\varepsilon)\right)\right|\\ &\geqslant \rho_0\langle l\rangle - C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle - C\varepsilon^{1-aq}|j|\\ &\geqslant \tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough and with the condition (14.34).

(iii) As before, using triangle and Cauchy-Schwarz inequalities combined with (14.11), (14.31), (14.32), Lemma 11.5-(iii) and the fact that $|j|\leqslant C_0\langle l\rangle$, we get

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)\big)\right|&\geqslant\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\left(\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)\right)\right|\\&\quad-\max_{k\in[\![0,q]\!]}\left|\partial_b^k\big(\overline{\mathrm{r}}_\varepsilon(b)\cdot l+jr^{1,n}(b,\varepsilon)+r_j^{\infty,n}(b,\varepsilon)\big)\big)\right|\\&\geqslant\rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j|\\&\geqslant\tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough with the condition (14.34).

(iv) Arguing as in the preceding point, using (14.31), (14.32), Lemma 11.5-(iv) and the fact that $0<|j-j_0|\leqslant C_0\langle l\rangle$ (notice that the case $j=j_0$ is trivial), we have

$$\begin{aligned}\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega(b,\varepsilon)\cdot l+\mu_j^{\infty,n}(b,\varepsilon)-\mu_{j_0}^{\infty,n}(b,\varepsilon)\big)\right|&\geqslant\max_{k\in[\![0,q_0]\!]}\left|\partial_b^k\big(\omega_{\mathrm{Eq}}(b)\cdot l+\Omega_j(b)-\Omega_{j_0}(b)\big)\right|\\&\quad-\max_{k\in[\![0,q]\!]}\left|\partial_b^k\big(\overline{\mathrm{r}}_\varepsilon(b)\cdot l+(j-j_0)r^{1,n}(b,\varepsilon)+r_j^{\infty,n}(b,\varepsilon)-r_{j_0}^{\infty,n}(b,\varepsilon)\big)\right|\\&\geqslant\rho_0\langle l\rangle-C\varepsilon^{1-a(1+q+q\overline{a})}\langle l\rangle-C\varepsilon^{1-a(1+q)}|j-j_0|\\&\geqslant\tfrac{\rho_0\langle l\rangle}{2}\end{aligned}$$

for ε small enough. This ends the proof of Lemma 14.3. ☐

PART III

Vortex rigid motion in QGSW equations

> Un Anneau pour les gouverner tous. Un Anneau pour les trouver. Un Anneau pour les amener tous et dans les ténèbres les lier.
>
> J. R. R. Tolkien

This part is devoted to the proof of Theorem 2.3. We also refer to Theorem 15.1 below for a more precise statement. This result is the subject of the following preprint [138] which is accepted for publication in the journal Asymptotic Analysis and entitled "Vortex rigid motion in quasi-geostrophic shallow-water equations".

Abstract

We prove the existence of analytic relative equilibria with holes for quasi-geostrophic shallow-water equations. More precisely, using bifurcation techniques, we establish for any $\mathbf{m}$ large enough the existence of two branches of $\mathbf{m}$-fold doubly-connected V-states bifurcating from any annulus of arbitrary size.

15 Introduction

We shall present here the last result obtained during the PhD related to the existence of relative equilibria with holes for QGSW equations. The result reads as follows.

Theorem 15.1. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N(\lambda, b) \in \mathbb{N}^*$ such that for every $\mathbf{m} \in \mathbb{N}^*$, with $\mathbf{m} \geqslant N(\lambda, b)$, there exist two curves of $\mathbf{m}$-fold doubly-connected V-states bifurcating from the annulus A_b defined in* (1.20)*, at the angular velocities*

$$\begin{aligned}\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) = &\frac{1-b^2}{2b}\Lambda_1(\lambda, b) + \frac{1}{2}\Big(\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) - \boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)\Big)\\ &\pm \frac{1}{2b}\sqrt{\Big(b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) + \boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda, b)\Big)^2 - 4b^2\Lambda_{\mathbf{m}}^2(\lambda, b)},\end{aligned}$$

where $\boldsymbol{\Omega}_{\mathbf{m}}(\lambda)$ is defined in (1.23) *and*

$$\Lambda_m(\lambda, b) \triangleq I_m(\lambda b)K_m(\lambda)$$

with I_m and K_m being the modified Bessel functions of first and second kind. In addition, the boundary of each V-state is analytic.

Before sketching the proof some remarks are in order.

Remark 15.1. *The spectrum is continuous with respect to λ and b. In particular, when we shrink $\lambda \to 0$ we find the spectrum of Euler equations detailed in* (1.21)*. However, when we shrink $b \to 0$ we obtain in part the simply connected spectrum* (1.23) *. In other words,*

$$\begin{cases} \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) \underset{\lambda \to 0}{\longrightarrow} \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(b) \\ \boldsymbol{\Omega}_{\mathbf{m}}^{+}(\lambda, b) \underset{b \to 0}{\longrightarrow} \boldsymbol{\Omega}_{\mathbf{m}}(\lambda). \end{cases}$$

These asymptotics are obtained for sufficiently large values of $\mathbf{m}$. For more details see Lemma 17.2.

Now, we intend to discuss the key steps of the proof of Theorem 15.1. Notice that for a given continuous function $f : \mathbb{T} \to \mathbb{C}$, we define its mean value by

$$⨍_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2\mathrm{i}\pi}\int_{\mathbb{T}} f(\tau)d\tau \triangleq \frac{1}{2\pi}\int_0^{2\pi} f\left(e^{\mathrm{i}\theta}\right)e^{\mathrm{i}\theta}d\theta, \tag{15.1}$$

where $d\tau$ stands for the complex integration.
First, in Section 16, we reformulate the vortex patch equation by using conformal maps. We opted for this approach to take advantage of the computations already done in [54] in this framework. Nevertheless, one could choose to formulate the problem in polar coordinates as in the previous sections. Consider an initial doubly-connected domain $D_0 = D_1 \setminus \overline{D_2}$, with D_1 and D_2 are two simply-connected domains close to the discs of radii 1 and b respectively. We introduce for $j \in \{1,2\}$ the conformal mappings $\Phi_j : \mathbb{D}^c \to D_j^c$ taking the form

$$\Phi_1(z) = z + f_1(z) = z + \sum_{n=0}^{\infty} \frac{a_n}{z^n}, \quad \Phi_2(z) = bz + f_2(z) = bz + \sum_{n=0}^{\infty} \frac{b_n}{z^n}.$$

Thus, from the contour dynamics equation, rotating doubly-connected V-states amounts to finding non-trivial zeros of the nonlinear functional $G = (G_1, G_2)$, defined for $j \in \{1,2\}$ and $w \in \mathbb{T}$ by

$$G_j(\lambda, b, \Omega, f_1, f_2)(w) \triangleq \operatorname{Im}\left\{ \Big(\Omega \Phi_j(w) + S(\lambda, \Phi_2, \Phi_j)(w) - S(\lambda, \Phi_1, \Phi_j)(w) \Big) \overline{w} \overline{\Phi_j'(w)} \right\},$$

with

$$\forall w \in \mathbb{T}, \quad S(\lambda, \Phi_i, \Phi_j)(w) \triangleq \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\left(\lambda |\Phi_j(w) - \Phi_i(\tau)| \right) d\tau.$$

For this aim, we shall implement Crandall-Rabinowitz's Theorem, starting from the elementary observation that the annulus A_b defined by (1.20) generates a trivial line of solutions for any $\Omega \in \mathbb{R}$, which will play the role of the bifurcation parameter. In the same section, we also study the regularity of G and prove that it is of class C^1 with respect to the functional spaces introduced in Section 16.2. Then, in Section 17, we compute the linearized operator at the equilibrium state and prove that it is a Fourier matrix multiplier. More precisely, for

$$\forall w \in \mathbb{T}, \quad h_1(w) = \sum_{n=0}^{\infty} a_n \overline{w}^n \quad \text{and} \quad h_2(w) = \sum_{n=0}^{\infty} b_n \overline{w}^n,$$

we have

$$d_{(f_1,f_2)} G(\lambda, b, \Omega, 0, 0)[h_1, h_2](w) = \sum_{n=0}^{\infty} (n+1) M_{n+1}(\lambda, b, \Omega) \begin{pmatrix} a_n \\ b_n \end{pmatrix} \operatorname{Im}(w^{n+1}),$$

where

$$M_n(\lambda, b, \Omega) \triangleq \begin{pmatrix} \boldsymbol{\Omega}_n(\lambda) - \Omega - b\Lambda_1(\lambda, b) & b\Lambda_n(\lambda, b) \\ -\Lambda_n(\lambda, b) & \Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_n(\lambda b) + \Omega\big] \end{pmatrix}.$$

We refer to Proposition 17.1 for more details and point out that some difficulties appear there when computing some integrals related to Bessel functions. Then, the kernel for the linearized operator $d_{(f_1,f_2)} G(\lambda, b, \Omega, 0, 0)$ is non trivial for $\Omega = \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)$, as defined in Theorem 15.1, with $\mathbf{m}$ large enough. The restriction to higher symmetry $\mathbf{m} \geqslant N(\lambda, b)$ is needed first to ensure the condition

$$\Delta_{\mathbf{m}}(\lambda, b) \triangleq \Big(b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) + \boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda, b) \Big)^2 - 4b^2 \Lambda_{\mathbf{m}}^2(\lambda, b) > 0,$$

required in the transversality condition of Crandall-Rabinowitz's Theorem and second to get the monotonicity of the sequences $\big(\boldsymbol{\Omega}_n^{\pm}(\lambda, b)\big)_{n \geqslant N(\lambda, b)}$ (to get a one-dimensional kernel), obtained from tricky asymptotic analysis on the modified Bessel functions. For more details, we refer to Proposition 18.1. The previous bifurcation occurs a priori in $C^{1+\alpha}$ regularity with $\alpha \in (0,1)$, but using an elliptic regularity argument, we prove in Lemma 18.1 the analyticity of the boundary for these V-states.

16 Functional settings

In this section, we shall reformulate the problem of finding V-states looking at the zeros of a nonlinear functional G. We also introduce the function spaces used in the analysis and study some regularity aspects for the functional G with respect to these functions spaces.

16.1 Boundary equations

In this subsection we shall obtain the system governing the patch motion. The starting point is the vortex patch equation in complex notation (1.17), which we recall here

$$\operatorname{Im}\left\{\left[\partial_t z(t,\theta)-\mathbf{v}(t,z(t,\theta))\right]\overline{\partial_\theta z(t,\theta)}\right\}=0, \tag{16.1}$$

where $\theta\mapsto z(t,\theta)$ is a parametrization of the boundary of D_t. Assuming that the patch is uniformly rotating with an angular velocity Ω, we can choose a parametrization γ in the form

$$z(t,\theta)=e^{\mathrm{i}\Omega t}z(0,\theta). \tag{16.2}$$

One readily has

$$\operatorname{Im}\left\{\partial_t z(t,\theta)\overline{\partial_\theta z(t,\theta)}\right\}=\Omega\operatorname{Re}\left\{z(0,\theta)\overline{\partial_\theta z(0,\theta)}\right\}. \tag{16.3}$$

Now, to study the second term in the equation (16.1), we use (4.4). By using (16.2), we obtain

$$\begin{aligned}\mathbf{v}(t,z(t,\theta))&=\frac{1}{2\pi}\int_{\partial D_t}K_0\left(\lambda|z(t,\theta)-\xi|\right)d\xi\\&=\frac{1}{2\pi}\int_0^1K_0\left(\lambda|e^{\mathrm{i}\Omega t}z(0,\theta)-e^{\mathrm{i}\Omega t}z(0,\eta)|\right)\partial_\eta z(t,\eta)d\eta\\&=\frac{e^{\mathrm{i}\Omega t}}{2\pi}\int_0^1K_0\left(\lambda|z(0,\theta)-z(0,\eta)|\right)\partial_\eta z(0,\eta)d\eta\\&=\frac{e^{\mathrm{i}\Omega t}}{2\pi}\int_{\partial D_0}K_0\left(\lambda|z(0,\theta)-\xi|\right)d\xi\\&=e^{\mathrm{i}\Omega t}\mathbf{v}(0,z(0,\theta)).\end{aligned}$$

Consequently using again (16.2), we get

$$\operatorname{Im}\left\{\mathbf{v}(t,z(t,\theta))\overline{\partial_\theta z(t,\theta)}\right\}=\operatorname{Im}\left\{\mathbf{v}(0,z(0,\theta))\overline{\partial_\theta z(0,\theta)}\right\}. \tag{16.4}$$

Putting together (16.3) and (16.4), the equation (16.1) can be rewritten

$$\Omega\operatorname{Re}\left\{z(0,\theta)\overline{\partial_\theta z(0,\theta)}\right\}=\operatorname{Im}\left\{\mathbf{v}(0,z(0,\theta))\overline{\partial_\theta z(0,\theta)}\right\}. \tag{16.5}$$

Let us assume that our starting domain D_0 is doubly-connected, that is

$$D_0=D_1\backslash\overline{D_2}\quad\text{with}\quad\overline{D}_2\subset D_1,$$

where D_1 and D_2 are simply-connected bounded open domains of $\mathbb{C}$. Then combining (4.4) and (16.5), one obtains for all $z \in \partial D_0 = \partial D_1 \cup \partial D_2$,

$$\begin{aligned}\Omega \mathrm{Re}\left\{z\overline{z'}\right\} &= \mathrm{Im}\left\{\frac{1}{2\pi}\int_{\partial D_0} K_0\left(\lambda|z-\xi|\right)d\xi\overline{z'}\right\}\\ &= \mathrm{Im}\left\{\left(\frac{1}{2\pi}\int_{\partial D_1} K_0\left(\lambda|z-\xi|\right)d\xi - \frac{1}{2\pi}\int_{\partial D_2} K_0\left(\lambda|z-\xi|\right)d\xi\right)\overline{z'}\right\}, \qquad (16.6)\end{aligned}$$

where z' denotes a tangent vector to the boundary ∂D_0 at the point z. The minus sign in front of the integral on ∂D_2 is here because of the orientation convention for the application of Stokes' Theorem. Following the works initiated by Burbea, see for instance [42, 54, 99, 100], we give the equation(s) to solve by using conformal mappings. For this purpose, we shall recall Riemann mapping Theorem.

Theorem 16.1 (Riemann Mapping)**.** *Let $\mathbb{D}$ denote the unit open ball and $D_0 \subset \mathbb{C}$ be a simply connected bounded domain. Then there exists a unique bi-holomorphic map called also conformal map, $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{D_0}$ taking the form*

$$\Phi(z) = az + \sum_{n=0}^{\infty}\frac{a_n}{z^n},$$

with $a > 0$ and $(a_n)_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$.

Notice that in the previous theorem, the domain is only assumed to be simply-connected and bounded. In particular, the existence of the conformal mapping does not depend on the regularity of the boundary. However, information on the regularity of the conformal mapping implies some regularity of the boundary. This is given by the following result which can be found in [144] or in [130, Thm. 3.6].

Theorem 16.2 (Kellogg-Warschawski)**.** *We keep the notations of Riemann mapping Theorem. If the conformal map $\Phi : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{D_0}$ has a continuous extension to $\mathbb{C}\backslash\mathbb{D}$ which is of class $C^{n+1+\beta}$ with $n \in \mathbb{N}$ and $\beta \in (0,1)$, then the boundary $\Phi(\mathbb{T})$ is a Jordan curve of class $C^{n+1+\beta}$.*

Assuming that D_1 and D_2 are respectively small deformations of the discs of radii 1 and b, so that the shape of D_0 is close to the annulus A_b defined in (1.20), we shall consider the parametrizations by the conformal mapping $\Phi_j : \mathbb{C}\backslash\overline{\mathbb{D}} \to \mathbb{C}\backslash\overline{D_j}$ satisfying

$$\Phi_1(z) = z + f_1(z) = z\left(1 + \sum_{n=1}^{\infty}\frac{a_n}{z^n}\right)$$

and

$$\Phi_2(z) = bz + f_2(z) = z\left(b + \sum_{n=1}^{\infty}\frac{b_n}{z^n}\right).$$

We shall now rewrite the equations by using the conformal parametrizations Φ_1 and Φ_2. First remark that for $w \in \mathbb{T}$, a tangent vector on the boundary ∂D_j at the point $z = \Phi_j(w)$ is given by

$$\overline{z'} = -\mathrm{i}\overline{w}\overline{\Phi_j'(w)}.$$

Inserting this into (16.6) and using the change of variables $\xi = \Phi_j(\tau)$ gives

$$\forall j \in \{1,2\}, \quad \forall w \in \mathbb{T}, \quad G_j(\lambda, b, \Omega, f_1, f_2)(w) = 0,$$

where

$$G_j(\lambda, b, \Omega, f_1, f_2)(w) \triangleq \mathrm{Im}\left\{\Big(\Omega\Phi_j(w) + S(\lambda, \Phi_2, \Phi_j)(w) - S(\lambda, \Phi_1, \Phi_j)(w)\Big)\overline{w}\overline{\Phi_j'(w)}\right\}, \qquad (16.7)$$

with

$$\forall (i,j)\in\{1,2\}^2,\quad \forall w\in\mathbb{T},\quad S(\lambda,\Phi_i,\Phi_j)(w)\triangleq \fint_{\mathbb{T}}\Phi_i'(\tau)K_0\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)d\tau. \tag{16.8}$$

Then, finding a non trivial uniformly rotating vortex patch for (1.6) reduces to finding zeros of the nonlinear functional

$$G\triangleq(G_1,G_2).$$

As stated in the introduction, these non trivial solutions may be obtained by bifurcation techniques from trivial solutions which are annuli. Let us recover with this formalism that indeed the annuli rotate for any angular velocity. This is given by the following result.

Lemma 16.1. *Let $b\in(0,1)$. Then the annulus A_b defined in* (1.20) *is a rotating patch for* (1.6) *for any angular velocity $\Omega\in\mathbb{R}$.*

Proof. Taking $f_1=f_2=0$ by in (16.7), we get

$$G_1(\lambda,b,\Omega,0,0)(w)=\operatorname{Im}\left\{b\overline{w}\fint_{\mathbb{T}}K_0\left(\lambda|w-b\tau|\right)d\tau-\overline{w}\fint_{\mathbb{T}}K_0\left(\lambda|w-\tau|\right)d\tau\right\}.$$

Using the changes of variables $\tau\mapsto w\tau$ and the fact that $|w|=1$, we have

$$G_1(\lambda,b,\Omega,0,0)(w)=\operatorname{Im}\left\{b\fint_{\mathbb{T}}K_0\left(\lambda|1-b\tau|\right)d\tau-\fint_{\mathbb{T}}K_0\left(\lambda|1-\tau|\right)d\tau\right\}=0.$$

Indeed for $a\in\{1,b\}$, we have by (C.3) and the change of variables $\theta\mapsto-\theta$

$$\begin{aligned}\overline{\fint_{\mathbb{T}}K_0\left(\lambda|1-a\tau|\right)d\tau}&=\overline{\frac{1}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-ae^{\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta}\\&=\frac{1}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-ae^{\mathrm{i}\theta}|\right)e^{-\mathrm{i}\theta}d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-ae^{-\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-ae^{\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta\\&=\fint_{\mathbb{T}}K_0\left(\lambda|1-a\tau|\right)d\tau.\end{aligned}\tag{16.9}$$

Similarly, we find

$$G_2(\lambda,b,\Omega,0,0)(w)=0.$$

This proves Lemma 16.1. ☐

16.2 Function spaces and regularity of the functional

We introduce here the function spaces used along this work. Throughout this part it is more convenient to think of 2π-periodic function $g:\mathbb{R}\to\mathbb{C}$ as a function of the complex variable $w=e^{\mathrm{i}\theta}$. To be more precise, let $f:\mathbb{T}\to\mathbb{R}^2$, be a continuous function, then it can be assimilated to a 2π-periodic function $g:\mathbb{R}\to\mathbb{R}^2$ via the relation

$$f(w)=g(\theta),\quad w=e^{\mathrm{i}\theta}.$$

Hence, when f is smooth enough, we get

$$f'(w)\triangleq\frac{df}{dw}=-\mathrm{i}e^{-\mathrm{i}\theta}g'(\theta).$$

Since $\frac{d}{dw}$ and $\frac{d}{d\theta}$ differ only by a smooth factor with modulus one, we shall in the sequel work with $\frac{d}{dw}$ instead of $\frac{d}{d\theta}$ which appears more suitable in the computations. In addition, if f is of class C^1 and has real Fourier coefficients, then we can easily check that

$$\left(\overline{f}\right)'(w) = -\frac{\overline{f'(w)}}{w^2}. \tag{16.10}$$

We shall now recall the definition of Hölder spaces on the unit circle.

Definition 16.1. *Let* $\alpha \in (0,1)$.

(i) We denote by $C^\alpha(\mathbb{T})$ *the space of continuous functions* f *such that*

$$\|f\|_{C^\alpha(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \sup_{\substack{(\tau,w)\in\mathbb{T}^2\\ \tau\neq w}} \frac{|f(\tau)-f(w)|}{|\tau-w|^\alpha} < \infty.$$

(ii) We denote by $C^{1+\alpha}(\mathbb{T})$ *the space of* C^1 *functions with* α*-Hölder continuous derivative*

$$\|f\|_{C^{1+\alpha}(\mathbb{T})} \triangleq \|f\|_{L^\infty(\mathbb{T})} + \left\|\frac{df}{dw}\right\|_{C^\alpha(\mathbb{T})} < \infty.$$

For $\alpha \in (0,1)$, we set

$$X^{1+\alpha} \triangleq X_1^{1+\alpha} \times X_1^{1+\alpha} \quad \text{with} \quad X_1^{1+\alpha} \triangleq \left\{ f \in C^{1+\alpha}(\mathbb{T}) \quad \text{s.t.} \quad \forall w \in \mathbb{T},\, f(w) = \sum_{n=0}^{\infty} f_n \overline{w}^n,\, f_n \in \mathbb{R} \right\}$$

and

$$Y^\alpha \triangleq Y_1^\alpha \times Y_1^\alpha \quad \text{with} \quad Y_1^\alpha \triangleq \left\{ g \in C^\alpha(\mathbb{T}) \quad \text{s.t.} \quad \forall w \in \mathbb{T},\, g(w) = \sum_{n=1}^{\infty} g_n e_n(w),\, g_n \in \mathbb{R} \right\},$$

where

$$e_n(w) \triangleq \mathrm{Im}(w^n).$$

We denote

$$B_r^{1+\alpha} \triangleq \left\{ f \in X_1^{1+\alpha} \quad \text{s.t.} \quad \|f\|_{C^{1+\alpha}(\mathbb{T})} < r \right\}.$$

We can encode the **m**-fold structure in the functional spaces by setting

$$X_{\mathbf{m}}^{1+\alpha} \triangleq X_{1,\mathbf{m}}^{1+\alpha} \times X_{1,\mathbf{m}}^{1+\alpha} \quad \text{with} \quad X_{1,\mathbf{m}}^{1+\alpha} \triangleq \left\{ f \in X_1^{1+\alpha} \quad \text{s.t.} \quad \forall w \in \mathbb{T},\, f(w) = \sum_{n=1}^{\infty} f_{\mathbf{m}n-1} \overline{w}^{\mathbf{m}n-1} \right\}$$

and

$$Y_{\mathbf{m}}^\alpha \triangleq Y_{1,\mathbf{m}}^\alpha \times Y_{1,\mathbf{m}}^\alpha \quad \text{with} \quad Y_{1,\mathbf{m}}^\alpha \triangleq \left\{ g \in Y_1^\alpha \quad \text{s.t.} \quad \forall w \in \mathbb{T},\, g(w) = \sum_{n=1}^{\infty} g_{\mathbf{m}n} e_{\mathbf{m}n}(w) \right\}.$$

The spaces $X^{1+\alpha}$ and $X_{\mathbf{m}}^{1+\alpha}$ $\big($resp. Y^α and $Y_{\mathbf{m}}^\alpha\big)$ are equipped with the strong product topology of $C^{1+\alpha}(\mathbb{T}) \times C^{1+\alpha}(\mathbb{T})$ $\big($resp. $C^\alpha(\mathbb{T}) \times C^\alpha(\mathbb{T})\big)$. We also denote

$$B_{r,\mathbf{m}}^{1+\alpha} \triangleq \left\{ f \in X_{1,\mathbf{m}}^{1+\alpha} \quad \text{s.t.} \quad \|f\|_{C^{1+\alpha}(\mathbb{T})} < r \right\} = B_r^{1+\alpha} \cap X_{1,\mathbf{m}}^{1+\alpha}.$$

We shall now investigate the regularity of the nonlinear functional G defined by (16.7). Indeed, Crandall-Rabinowitz's Theorem B.1 requires some regularity assumptions to apply and this is what we check here.

Proposition 16.1. *Let* $\lambda > 0$, $b \in (0,1)$, $\alpha \in (0,1)$ *and* $\mathbf{m} \in \mathbb{N}^*$. *There exists* $r > 0$ *such that*

(i) $G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha} \to Y^\alpha$ *is well-defined and of classe* C^1.

(ii) The restriction $G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_{r,\mathbf{m}}^{1+\alpha} \times B_{r,\mathbf{m}}^{1+\alpha} \to Y_{\mathbf{m}}^\alpha$ *is well-defined.*

(iii) The partial derivative $\partial_\Omega d_{(f_1,f_2)} G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha} \to \mathcal{L}(X^{1+\alpha}, Y^\alpha)$ *exists and is continuous.*

The following proof follows closely the lines of the proof of [85, Prop. 4.1].

Proof. **(i)** The proof proceeds in three steps. The first step is to show the well-posedness of the function $G(\lambda, b, \cdot, \cdot, \cdot)$ from $\mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha}$ to Y^α for some r small enough. Then, in the second step, we shall prove the existence and give the computation of the Gâteaux derivative of $G(\lambda, b, \cdot, \cdot, \cdot)$. Finally, in the third step, we shall prove that these Gâteaux derivatives are continuous. This will show the C^1 regularity of $G(\lambda, b, \cdot, \cdot, \cdot)$.

► **Step 1 : Show that** $G(\lambda, b, \cdot, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \times B_r^{1+\alpha} \to Y^\alpha$ **is well-defined :**
For this purpose, we split G_j into two terms, the self-induced term $\mathcal{S}_j$ and the interaction term $\mathcal{I}_j$,

$$G_j(\lambda, b, \Omega, f_1, f_2) = \mathcal{S}_j(\lambda, b, \Omega, f_j) + \mathcal{I}_j(\lambda, b, f_1, f_2), \tag{16.11}$$

where

$$\begin{aligned}\mathcal{S}_j(\lambda, b, \Omega, f_j)(w) &\triangleq \mathrm{Im}\left\{\left[\Omega \Phi_j(w) + (-1)^j S(\lambda, \Phi_j, \Phi_j)(w)\right] \overline{w}\overline{\Phi_j'(w)}\right\}, \\ \mathcal{I}_j(\lambda, b, f_1, f_2) &\triangleq (-1)^{j-1} \mathrm{Im}\left\{S(\lambda, \Phi_i, \Phi_j)(w)\overline{w}\overline{\Phi_j'(w)}\right\}.\end{aligned}$$

➢ We refer to [54, Prop. 5.7] for the study of $\mathcal{S}_j$. Only the $(-1)^j$ defers, but has no consequence. We recall here the results. There exists $r \in (0,1)$ such that for all $\alpha \in (0,1)$, we have

- $\mathcal{S}_j(\lambda, b, \cdot, \cdot) : \mathbb{R} \times B_r^{1+\alpha} \to Y_1^\alpha$ is of class C^1.
- The restriction $\mathcal{S}_j(\lambda, b, \cdot, \cdot) : \mathbb{R} \times B_{r,\mathbf{m}}^{1+\alpha} \to Y_{\mathbf{m}}^\alpha$ is well-defined.

Moreover, we have

$$\begin{aligned} d_{f_j}\mathcal{S}_j(\lambda, b, \Omega, f_j)h_j(w) &= \Omega \mathrm{Im}\left\{h_j(w)\overline{w}\overline{\Phi_j'(w)} + \Phi_j(w)\overline{w}\overline{h_j'(w)}\right\} \\ &+ (-1)^j \mathrm{Im}\left\{S(\lambda, \Phi_j, \Phi_j)(w)\overline{w}\overline{h_j'(w)} + \overline{w}\overline{\Phi_j'(w)}\left[A_1(\lambda, \Phi_j, h_j)(w) + B_1(\lambda, \Phi_j, h_j)(w)\right]\right\}, \end{aligned} \tag{16.12}$$

where

$$\begin{aligned} A_1(\lambda, \Phi_j, h_j)(w) &\triangleq \fint_{\mathbb{T}} h_j'(\tau) K_0\left(\lambda|\Phi_j(w) - \Phi_j(\tau)|\right) d\tau, \\ B_1(\lambda, \Phi_j, h_j)(w) &\triangleq \lambda \fint_{\mathbb{T}} \Phi_j'(\tau) K_0'\left(\lambda|\Phi_j(w) - \Phi_j(\tau)|\right) \frac{\mathrm{Re}\left(\left(\overline{h_j(w)} - \overline{h_j(\tau)}\right)\left(\Phi_j(w) - \Phi_j(\tau)\right)\right)}{|\Phi_j(w) - \Phi_j(\tau)|} d\tau. \end{aligned}$$

Actually, this is the most difficult part of this proof since in this case, the integrals appearing have singular kernel and the proof uses some results about singular kernels. As we shall see in the remaining of the proof, the terms concerning $\mathcal{I}_j$ are not singular.

➢ We shall first show that for $(f_1, f_2) \in B_r^{1+\alpha} \times B_r^{1+\alpha}$, we have $\mathcal{I}_j(\lambda, b, f_1, f_2) \in C^\alpha(\mathbb{T})$. According to the algebra structure of $C^{\alpha}(\mathbb{T})$, it suffices to show that for $i \neq j$, $S(\lambda, \Phi_i, \Phi_j) \in C^{\alpha}(\mathbb{T})$. For that purpose, we consider the operator $\mathcal{T}$ defined by

$$\forall w \in \mathbb{T}, \quad \mathcal{T}_{ij}\chi(w) \triangleq \fint_{\mathbb{T}} \chi(\tau) K_0\left(\lambda|\Phi_j(w) - \Phi_i(\tau)|\right) d\tau. \tag{16.13}$$

But for $w, \tau \in \mathbb{T}$, we have taking f_1 and f_2 small functions,

$$|\Phi_1(w) - \Phi_2(\tau)| \leqslant |w - b\tau| + |f_1(w)| + |f_2(\tau)| \leqslant (1+b) + \|f_1\|_{L^\infty(\mathbb{T})} + \|f_2\|_{L^\infty(\mathbb{T})} \leqslant 2(1+b) \tag{16.14}$$

and

$$|\Phi_1(w) - \Phi_2(\tau)| \geqslant |w - b\tau| - |f_1(w)| - |f_2(\tau)| \geqslant (1-b) - \|f_1\|_{L^\infty(\mathbb{T})} - \|f_2\|_{L^\infty(\mathbb{T})} \geqslant \frac{1-b}{2}. \tag{16.15}$$

Since K_0 is continuous on $\left[\frac{\lambda(1-b)}{2}, 2\lambda(1+b)\right]$, we have

$$\|\mathcal{T}_{ij}\chi\|_{L^\infty(\mathbb{T})} \lesssim \|\chi\|_{L^\infty(\mathbb{T})}.$$

Moreover, taking $w_1 \neq w_2 \in \mathbb{T}$, we have by mean value Theorem, since from (C.4) $K_0' = -K_1$ is continuous on $\left[\frac{\lambda(1-b)}{2}, 2\lambda(1+b)\right]$, and left triangle inequality

$$\begin{aligned}
|\mathcal{T}_{ij}\chi(w_1) - \mathcal{T}_{ij}\chi(w_2)| &\lesssim \int_{\mathbb{T}} |\chi(\tau)| \left|K_0\left(\lambda|\Phi_j(w_1) - \Phi_i(\tau)|\right) - K_0\left(|\lambda||\Phi_j(w_2) - \Phi_i(\tau)|\right)\right| |d\tau| \\
&\lesssim \|\chi\|_{L^\infty(\mathbb{T})} \left|\Phi_j(w_1) - \Phi_j(w_2)\right|.
\end{aligned}$$

Using that $\Phi_j \in C^{1+\alpha}(\mathbb{T}) \hookrightarrow C^{\alpha}(\mathbb{T})$, we conclude that

$$|\mathcal{T}_{ij}\chi(w_1) - \mathcal{T}_{ij}\chi(w_2)| \lesssim \|\chi\|_{L^\infty(\mathbb{T})} \|\Phi_j\|_{C^\alpha(\mathbb{T})} |w_1 - w_2|^\alpha.$$

We deduce that

$$\|\mathcal{T}_{ij}\chi\|_{C^\alpha(\mathbb{T})} \lesssim \left(1 + \|\Phi_j\|_{C^\alpha(\mathbb{T})}\right) \|\chi\|_{L^\infty(\mathbb{T})}. \tag{16.16}$$

Applying this with $\chi = \Phi_j'$, we find

$$\|S(\lambda, \Phi_i, \Phi_j)\|_{C^\alpha(\mathbb{T})} \lesssim \left(1 + \|\Phi_j\|_{C^\alpha(\mathbb{T})}\right) \|\Phi_i'\|_{L^\infty(\mathbb{T})} \lesssim \left(1 + \|\Phi_j\|_{C^{1+\alpha}(\mathbb{T})}\right) \|\Phi_i\|_{C^{1+\alpha}(\mathbb{T})} < \infty.$$

The last point to check is that the Fourier coefficients of $\mathcal{I}_j(\lambda, f_1, f_2)$ are real. According to the definition of the space $X^{1+\alpha}$, the mapping Φ_j has real coefficients. We deduce that the Fourier coefficients of Φ_j' are also real. Due to the stability of such property under conjugation and multiplication, we only have to prove that the Fourier coefficients of $S(\lambda, \Phi_i, \Phi_j)$ are real. This is checked by the following computations.

By using (C.3) and the change of variables $\eta \mapsto -\eta$, one has

$$\begin{aligned}
\overline{S(\lambda,\Phi_i,\Phi_j)(w)} &= \overline{\fint_{\mathbb{T}} \Phi_i'(\tau) K_0\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right) d\tau} \\
&= \overline{\frac{1}{2\mathrm{i}\pi}\int_0^{2\pi} \Phi_i'\left(e^{\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(w)-\Phi_i\left(e^{\mathrm{i}\eta}\right))|\right) \mathrm{i}e^{\mathrm{i}\eta} d\eta} \\
&= \frac{1}{2\pi}\int_0^{2\pi} \Phi_i'\left(e^{-\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(\overline{w})-\Phi_i\left(e^{-\mathrm{i}\eta}\right)|\right) e^{-\mathrm{i}\eta} d\eta \\
&= \frac{1}{2\mathrm{i}\pi}\int_0^{2\pi} \Phi_i'\left(e^{\mathrm{i}\eta}\right) K_0\left(\lambda|\Phi_j(\overline{w})-\Phi_i\left(e^{\mathrm{i}\eta}\right)|\right) \mathrm{i}e^{\mathrm{i}\eta} d\eta \\
&= \fint_{\mathbb{T}} \Phi_i'(\tau) K_0\left(\lambda|\Phi_j(\overline{w})-\Phi_i(\tau)|\right) d\tau \\
&= S(\lambda,\Phi_i,\Phi_j)(\overline{w}).
\end{aligned}$$

▶ **Step 2 : Show the existence and compute the Gâteaux derivatives of $G(\lambda, b, \cdot, \cdot, \cdot)$:**

➢ The Gâteaux derivative of $\mathcal{I}_j$ at (f_1, f_2) in the direction $h = (h_1, h_2) \in X^{1+\alpha}$ is given by

$$\begin{aligned}
d_{(f_1,f_2)}\mathcal{I}_j(\lambda,b,f_1,f_2)h &= d_{f_1}\mathcal{I}_j(\lambda,b,f_1,f_2)h_1 + d_{f_2}\mathcal{I}_j(\lambda,b,f_1,f_2)h_2 \\
&\triangleq \lim_{t\to 0}\frac{1}{t}\left[\mathcal{I}_j(\lambda,b,f_1+th_1,f_2)-\mathcal{I}_j(\lambda,b,f_1,f_2)\right] \\
&\quad + \lim_{t\to 0}\frac{1}{t}\left[\mathcal{I}_j(\lambda,b,f_1,f_2+th_2)-\mathcal{I}_j(\lambda,b,f_1,f_2)\right]. \qquad (16.17)
\end{aligned}$$

The previous limits are understood in the sense of the strong topology of Y^α. As a consequence, we need to to prove first the pointwise existence of these limits and then we shall check that these limits exist in the strong topology of $C^\alpha(\mathbb{T})$. To be able to compute the Gâteaux dérivatives, we have to precise that since the beginning of this study we have identified $\mathbb{C}$ with $\mathbb{R}^2$. Hence $\mathbb{C}$ is naturally endowed with the Euclidean scalar product which writes for $z_1 = a_1 + \mathrm{i}b_1$ and $z_2 = a_2 + \mathrm{i}b_2$

$$\langle z_1, z_2\rangle \triangleq \mathrm{Re}(\overline{z_1}z_2) = \frac{1}{2}\left(\overline{z_1}z_2 + z_1\overline{z_2}\right) = a_1a_2 + b_1b_2.$$

By straightforward computations, we infer

$$\begin{aligned}
d_{f_j}\mathcal{I}_j(\lambda,b,f_1,f_2)h_j(w) = (-1)^{j-1}\mathrm{Im}\Big\{&\overline{w}\,\overline{h_j'(w)}S(\lambda,\Phi_i,\Phi_j)(w) \\
&+\frac{\lambda}{2}\overline{w}\,\overline{\Phi_j'(w)}\Big(\overline{h_j(w)}A(\lambda,\Phi_i,\Phi_j)(w) + h_j(w)B(\lambda,\Phi_i,\Phi_j)(w)\Big)\Big\}, \qquad (16.18)
\end{aligned}$$

where

$$A(\lambda,\Phi_i,\Phi_j)(w) \triangleq \fint_{\mathbb{T}} \Phi_i'(\tau)K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)\frac{\Phi_j(w)-\Phi_i(\tau)}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau \triangleq \fint_{\mathbb{T}} \Phi_i'(\tau)K(\lambda,w,\tau)d\tau$$

and

$$B(\lambda,\Phi_i,\Phi_j)(w) \triangleq \fint_{\mathbb{T}} \Phi_i'(\tau)K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)\frac{\Phi_j(\overline{w})-\Phi_i(\overline{\tau})}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau = \fint_{\mathbb{T}} \Phi_i'(\tau)\overline{K(\lambda,w,\tau)}d\tau.$$

Since B differs from A only with a conjugation, then, they both satisfy the same estimates in the coming analysis. For all $w \in \mathbb{T}$, we have

$$|A(\lambda,\Phi_i,\Phi_j)(w)| \lesssim \int_{\mathbb{T}} |\Phi_i'(\tau)|K_0\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)|d\tau| \lesssim \|\Phi_i'\|_{L^\infty(\mathbb{T})}.$$

So

$$\|A(\lambda,\Phi_i,\Phi_j)\|_{L^\infty(\mathbb{T})}\lesssim\|\Phi_i'\|_{L^\infty(\mathbb{T})}.$$

Let $w_1\neq w_2\in\mathbb{T}$. let $\tau\in\mathbb{T}$. Then

$$\begin{aligned}
&|K(\lambda,w_1,\tau)-K(\lambda,w_2,\tau)|\\
&=\left|K_0'\left(\lambda|\Phi_j(w_1)-\Phi_i(\tau)|\right)\frac{\Phi_j(w_1)-\Phi_i(\tau)}{|\Phi_j(w_1)-\Phi_i(\tau)|}-K_0'\left(\lambda|\Phi_j(w_2)-\Phi_i(\tau)|\right)\frac{\Phi_j(w_2)-\Phi_i(\tau)}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right|\\
&\leqslant\left|K_0'\left(\lambda|\Phi_j(w_1)-\Phi_i(\tau)|\right)-K_0'\left(\lambda|\Phi_j(w_2)-\Phi_i(\tau)|\right)\right|\\
&\quad+\left|K_0'\left(\lambda|\Phi_j(w_2)-\Phi_i(\tau)|\right)\right|\left|\frac{\Phi_j(w_1)-\Phi_i(\tau)}{|\Phi_j(w_1)-\Phi_i(\tau)|}-\frac{\Phi_j(w_2)-\Phi_i(\tau)}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right|.
\end{aligned}$$

But by right and left triangle inequalities, we get

$$\begin{aligned}
&\left|\frac{\Phi_j(w_1)-\Phi_i(\tau)}{|\Phi_j(w_1)-\Phi_i(\tau)|}-\frac{\Phi_j(w_2)-\Phi_i(\tau)}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right|\\
&=\left|\frac{\Phi_j(w_1)-\Phi_j(w_2)}{|\Phi_j(w_1)-\Phi_i(\tau)|}+(\Phi_j(w_2)-\Phi_i(\tau))\left(\frac{1}{|\Phi_j(w_1)-\Phi_i(\tau)|}-\frac{1}{|\Phi_j(w_2)-\Phi_i(\tau)|}\right)\right|\\
&\leqslant\frac{|\Phi_j(w_1)-\Phi_j(w_2)|}{|\Phi_j(w_1)-\Phi_i(\tau)|}+|\Phi_j(w_2)-\Phi_i(\tau)|\,\frac{\big||\Phi_j(w_2)-\Phi_i(\tau)|-|\Phi_j(w_1)-\Phi_i(\tau)|\big|}{|\Phi_j(w_1)-\Phi_i(\tau)|\,|\Phi_j(w_2)-\Phi_i(\tau)|}\\
&\leqslant\frac{2\,|\Phi_j(w_1)-\Phi_j(w_2)|}{|\Phi_j(w_1)-\Phi_i(\tau)|}\\
&\lesssim|\Phi_j(w_1)-\Phi_j(w_2)|\,.
\end{aligned}$$

Hence,

$$|K(\lambda,w_1,\tau)-K(\lambda,w_2,\tau)|\lesssim|\Phi_j(w_1)-\Phi_j(w_2)|\lesssim\|\Phi_j\|_{C^\alpha(\mathbb{T})}|w_1-w_2|^\alpha.$$

Thus,

$$\|A(\lambda,\Phi_i,\Phi_j)\|_{C^\alpha(\mathbb{T})}\lesssim\|\Phi_i\|_{C^{1+\alpha}(\mathbb{T})}+\|\Phi_j\|_{C^{1+\alpha}(\mathbb{T})}.$$

We conclude that,

$$\|d_{f_j}\mathcal{I}_j(\lambda,f_1,f_2)h_j\|_{C^\alpha(\mathbb{T})}\lesssim\|h_j\|_{C^{1+\alpha}(\mathbb{T})},$$

which means that $d_{f_j}\mathcal{I}_j(\lambda,b,f_1,f_2)\in\mathcal{L}(C^{1+\alpha}(\mathbb{T}),C^\alpha(\mathbb{T}))$.

➢ Concerning the other differentiation, we have

$$\begin{aligned}
d_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i(w)&=(-1)^{j-1}\mathrm{Im}\left\{\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}h_i'(\tau)K_0\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)d\tau\right.\\
&-\frac{\lambda}{2}\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}h_i(\tau)\Phi_i'(\tau)K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)\frac{\Phi_j(\overline{w})-\Phi_i(\overline{\tau})}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau\\
&\left.-\frac{\lambda}{2}\overline{w}\overline{\Phi_j'(w)}\fint_{\mathbb{T}}\overline{h_i(\tau)}\Phi_i'(\tau)K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)\frac{\Phi_j(w)-\Phi_i(\tau)}{|\Phi_j(w)-\Phi_i(\tau)|}d\tau\right\}\\
&\triangleq(-1)^{j-1}\mathrm{Im}\Big\{\overline{w}\overline{\Phi_j'(w)}\big[C(\lambda,\Phi_i,\Phi_j)(h_i)(w)+D(\lambda,\Phi_i,\Phi_j)(h_i)(w)+E(\lambda,\Phi_i,\Phi_j)(h_i)(w)\big]\Big\}.
\end{aligned}\tag{16.19}$$

Using the algebra structure of $C^\alpha(\mathbb{T})$, we obtain

$$\|d_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|C(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|D(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|E(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}.$$

From (16.16), we find

$$\|C(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i'\|_{L^\infty(\mathbb{T})}\leqslant\|h_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

In the same way as for $A(\lambda,\Phi_i,\Phi_j)$, we infer

$$\|D(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}+\|E(\lambda,\Phi_i,\Phi_j)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i\|_{L^\infty(\mathbb{T})}\leqslant\|h_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

Gathering the foregoing computations leads to

$$\|d_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)h_i\|_{C^\alpha(\mathbb{T})}\lesssim\|h_i\|_{C^{1+\alpha}(\mathbb{T})},$$

that is, $d_{f_i}\mathcal{I}_j(\lambda,b,f_1,f_2)\in\mathcal{L}(C^{1+\alpha}(\mathbb{T}),C^\alpha(\mathbb{T}))$.

➢ The last thing to check is that the convergence in (16.17) occurs in the strong topology of $C^\alpha(\mathbb{T})$. Since there are many terms involved, we shall select the more complicated one and study it. The other terms can be treated in a similar way, up to slight modifications. Let us focus on the first term of the right-hand side of (16.18). We shall prove,

$$\lim_{t\to0}S(\lambda,\Phi_i,\Phi_i+th_j)-S(\lambda,\Phi_i,\Phi_j)=0\quad\text{in}\quad C^\alpha(\mathbb{T}).$$

For more convenience, we use the following notation

$$T_{ij}(\lambda,t,w)\triangleq S(\lambda,\Phi_i,\Phi_i+th_j)(w)-S(\lambda,\Phi_i,\Phi_j)(w).$$

Consider $t>0$ such that $t\|h_j\|_{L^\infty(\mathbb{T})}<r$. According to (16.8), we get

$$\begin{aligned}T_{ij}(\lambda,t,w)&=\fint_{\mathbb{T}}\Phi_i'(\tau)\left(K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)+th_j(w)\right|\right)-K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)\right|\right)\right)d\tau\\&\triangleq\fint_{\mathbb{T}}\Phi_i'(\tau)\mathbb{K}(\lambda,t,w,\tau)d\tau.\end{aligned}$$

Applying mean value Theorem and left triangle inequality, we obtain

$$|\mathbb{K}(\lambda,t,w,\tau)|\lesssim t\|h_j\|_{L^\infty(\mathbb{T})}.$$

Consequently,

$$|T_{ij}(\lambda,t,w)|\lesssim t\|h_j\|_{L^\infty(\mathbb{T})}.$$

This implies that

$$\lim_{t\to0}\|T_{ij}(\lambda,t,\cdot)\|_{L^\infty(\mathbb{T})}=0.$$

Let us now consider $w_1\neq w_2\in\mathbb{T}$. In view of the mean value Theorem, one obtains the following estimate

$$\begin{aligned}|T_{ij}(\lambda,t,w_1)-T_{ij}(\lambda,t,w_2)|&\lesssim\int_{\mathbb{T}}|\mathbb{K}(\lambda,t,w_1,\tau)-\mathbb{K}(\lambda,t,w_2,\tau)|\,|d\tau|\\&\lesssim|w_1-w_2|\int_{\mathbb{T}}\sup_{w\in\mathbb{T}}|\partial_w\mathbb{K}(\lambda,t,w,\tau)|\,|d\tau|.\end{aligned}\tag{16.20}$$

Now remark that we can write

$$\mathbb{K}(\lambda,t,w,\tau)=\int_0^t\partial_sg(\lambda,s,w,\tau)ds\quad\text{with}\quad g(\lambda,t,w,\tau)\triangleq K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)+\tau h_j(w)\right|\right).$$

According to (16.10), one obtains

$$\partial_w g(\lambda,t,w,\tau)=\frac{\lambda}{2}K_0'\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)+th_j(w)\right|\right)$$
$$\times\frac{\left(\Phi_j'(w)+th_j'(w)\right)\left(\overline{\Phi_j(w)}-\overline{\Phi_i(\tau)}+t\overline{h_j(w)}\right)-\overline{w}^2\left(\overline{\Phi_j'(w)}+t\overline{h_j'(w)}\right)\left(\Phi_j(w)-\Phi_i(\tau)+th_j(w)\right)}{\left|\Phi_j(w)-\Phi_i(\tau)+th_j(w)\right|}.$$

After straightforward computations, we obtain for $s\in[0,t]$,

$$|\partial_s\partial_w g(\lambda,s,w,\tau)|\lesssim 1.$$

As a consequence, we infer

$$|\partial_w\mathbb{K}(\lambda,t,w,\tau)|\lesssim|t|.$$

Coming back to (16.20) and using the fact that $\alpha\in(0,1)$, we conclude

$$|T_{ij}(\lambda,t,w_1)-T_{ij}(\lambda,t,w_2)|\lesssim|t||w_1-w_2|\lesssim|t||w_1-w_2|^{\alpha}.$$

Therefore,

$$\lim_{t\to0}\|T_{ij}(t,\cdot)\|_{C^{\alpha}(\mathbb{T})}=0.$$

The second step is now achieved.

▶ Step 3 : Show that the Gâteaux derivatives of $G(\lambda,b,\cdot,\cdot,\cdot)$ are continuous :

Now we investigate for the continuity of the Gâteaux derivatives seen as operators from the neighborhood $B_r^{1+\alpha}\times B_r^{1+\alpha}$ into the Banach space $\mathcal{L}\left(X_1^{1+\alpha},Y_1^{\alpha}\right)$. Using the algebra structure of $C^{\alpha}(\mathbb{T})$, we deduce from (16.19) and (16.18) that we only have to study the continuity of the terms $S(\lambda,\Phi_i,\Phi_j)$, $A(\lambda,\Phi_i,\Phi_j)$, $B(\lambda,\Phi_i,\Phi_j)$, $C(\lambda,\Phi_i,\Phi_j)h_i$, $D(\lambda,\Phi_i,\Phi_j)h_i$ and $E(\lambda,\Phi_i,\Phi_j)h_i$. As before, we shall focus on the term $S(\lambda,\Phi_i,\Phi_j)$ for $i\neq j$ and remark that the other terms are similar. We denote

$$\Phi_1\triangleq\mathrm{Id}+f_1,\quad\Psi_1\triangleq\mathrm{Id}+g_1,\quad\Phi_2\triangleq b\mathrm{Id}+f_2,\quad\Psi_2\triangleq b\mathrm{Id}+g_2,$$

with $(f_1,f_2)\in B_r^{1+\alpha}\times B_r^{1+\alpha}$ and $(g_1,g_2)\in B_r^{1+\alpha}\times B_r^{1+\alpha}$. Let us show that

$$\|S(\lambda,\Phi_i,\Phi_j)-S(\lambda,\Psi_i,\Psi_j)\|_{C^{\alpha}(\mathbb{T})}\lesssim\|f_1-g_1\|_{C^{1+\alpha}(\mathbb{T})}+\|f_2-g_2\|_{C^{1+\alpha}(\mathbb{T})}.$$

According to (16.8), we get

$$\begin{aligned}S(\lambda,\Phi_i,\Phi_j)(w)-S(\lambda,\Psi_i,\Psi_j)(w)&=\fint_{\mathbb{T}}\left[\Phi_i'(\tau)K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)\right|\right)-\Psi_i'(\tau)K_0\left(\lambda\left|\Psi_j(w)-\Psi_i(\tau)\right|\right)\right]d\tau\\&\triangleq\fint_{\mathbb{T}}\Psi_i'(\tau)\mathbb{K}_2(\lambda,w,\tau)d\tau+\fint_{\mathbb{T}}\left(\Phi_i'(\tau)-\Psi_i'(\tau)\right)K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)\right|\right)d\tau,\end{aligned}$$

where

$$\mathbb{K}_2(\lambda,w,\tau)\triangleq K_0\left(\lambda\left|\Phi_j(w)-\Phi_i(\tau)\right|\right)-K_0\left(\lambda\left|\Psi_j(w)-\Psi_i(\tau)\right|\right).$$

We have directly

$$\left\|\fint_{\mathbb{T}}\left(\Phi_i'(\tau)-\Psi_i'(\tau)\right)K_0\left(\lambda\left|\Phi_j(\cdot)-\Phi_i(\tau)\right|\right)d\tau\right\|_{C^{\alpha}(\mathbb{T})}\lesssim\|f_i'-g_i'\|_{L^{\infty}(\mathbb{T})}\leqslant\|f_i-g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

Now set

$$L_i(\lambda,w) \triangleq \fint_{\mathbb{T}} \mathbb{K}_2(\lambda,w,\tau)\Psi_i'(\tau)d\tau,$$

By a new use of the mean value Theorem and left triangle inequality, we obtain

$$\begin{aligned}|\mathbb{K}_2(\lambda,w,\tau)| &\lesssim \big|\,|\Phi_j(w)-\Phi_i(\tau)|-|\Psi_j(w)-\Psi_i(\tau)|\,\big|\\ &\leqslant |\Phi_j(w)-\Psi_j(w)|+|\Phi_i(\tau)-\Psi_i(\tau)|\\ &\leqslant \|\Psi_j-\Phi_j\|_{L^\infty(\mathbb{T})}+\|\Psi_i-\Phi_i\|_{L^\infty(\mathbb{T})}.\end{aligned}$$

Hence, we deduce

$$\begin{aligned}\|L_i(\lambda,\cdot)\|_{L^\infty(\mathbb{T})} &\lesssim \|\Psi_i'\|_{L^\infty(\mathbb{T})}\left(\|\Psi_j-\Phi_j\|_{L^\infty(\mathbb{T})}+\|\Psi_i-\Phi_i\|_{L^\infty(\mathbb{T})}\right)\\ &\lesssim \|f_j-g_j\|_{C^{1+\alpha}(\mathbb{T})}+\|f_i-g_i\|_{C^{1+\alpha}(\mathbb{T})}.\end{aligned}$$

Take $w_1\neq w_2\in\mathbb{T}$. Applying the mean value Theorem yields

$$|L_i(\lambda,w_1)-L_i(\lambda,w_2)| \lesssim |w_1-w_2|\fint_{\mathbb{T}}\sup_{w\in\mathbb{T}}|\partial_w\mathbb{K}_2(\lambda,w,\tau)|\,|d\tau|.$$

By (16.10), we have

$$\partial_w\mathbb{K}_2(\lambda,w,\tau)=\frac{\lambda}{2}\left(\overline{\mathcal{J}(\lambda,w,\tau)}-\overline{w}^2\mathcal{J}(\lambda,w,\tau)\right),$$

where

$$\mathcal{J}(\lambda,w,\tau)\triangleq\overline{\Phi_j'(w)}(\Phi_j(w)-\Phi_i(\tau))K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)-\overline{\Psi_j'(w)}(\Psi_j(w)-\Psi_i(\tau))K_0'\left(\lambda|\Psi_j(w)-\Psi_i(\tau)|\right).$$

Notice that it can be written in the following form

$$\mathcal{J}(\lambda,w,\tau)=\mathcal{J}_1(\lambda,w,\tau)+\mathcal{J}_2(\lambda,w,\tau)+\mathcal{J}_3(\lambda,w,\tau),$$

with

$$\begin{aligned}\mathcal{J}_1(\lambda,w,\tau)&\triangleq\overline{\Phi_j'(w)}\left[(\Phi_j-\Psi_j)(w)-(\Phi_i-\Psi_i)(\tau)\right]K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right),\\ \mathcal{J}_2(\lambda,w,\tau)&\triangleq\left[\overline{\Phi_j'(w)}-\overline{\Psi_j'(w)}\right]\left[\Psi_j(w)-\Psi_i(\tau)\right]K_0'\left(\lambda|\Psi_j(w)-\Psi_i(\tau)|\right),\\ \mathcal{J}_3(\lambda,w,\tau)&\triangleq\overline{\Phi_j'(w)}\left[\Psi_j(w)-\Psi_i(\tau)\right]\left[K_0'\left(\lambda|\Phi_j(w)-\Phi_i(\tau)|\right)-K_0'\left(\lambda|\Psi_j(w)-\Psi_i(\tau)|\right)\right].\end{aligned}$$

By the same techniques as already used above, we get

$$\|\partial_w\mathbb{K}_2(\lambda,\cdot,\tau)\|_{L^\infty(\mathbb{T})}\lesssim\|f_j-g_j\|_{C^{1+\alpha}(\mathbb{T})}+\|f_i-g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

We deduce that

$$\|S(\lambda,\Phi_i,\Phi_j)-S(\lambda,\Psi_i,\Psi_j)\|_{C^\alpha(\mathbb{T})}\lesssim\|f_j-g_j\|_{C^{1+\alpha}(\mathbb{T})}+\|f_i-g_i\|_{C^{1+\alpha}(\mathbb{T})}.$$

(ii) Looking at Proposition 16.1, it is sufficient to prove the preservation of the **m**-fold symmetry. Let r be as in Proposition 16.1. Let $(f_1,f_2)\in B_{r,\mathbf{m}}^{1+\alpha}\times B_{r,\mathbf{m}}^{1+\alpha}$. Let Φ_1 and Φ_2 be the associated conformal maps

$$\Phi_1(z)=z+\sum_{n=0}^{\infty}\frac{a_n}{z^{\mathbf{m}n-1}}\qquad\text{and}\qquad\Phi_2(z)=bz+\sum_{n=0}^{\infty}\frac{b_n}{z^{\mathbf{m}n-1}}.$$

One easily obtains

$$\forall j\in\{1,2\},\quad \forall w\in\mathbb{T},\quad \Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)=e^{\frac{2i\pi}{\mathbf{m}}}\Phi_j(w)\quad\text{ and }\quad \Phi_j'\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)=\Phi_j'(w).$$

Hence, by using the change of variables $\tau\mapsto e^{\frac{2i\pi}{\mathbf{m}}}\tau$, we have for all $(i,j)\in\{1,2\}^2$ and for all $w\in\mathbb{T}$,

$$\begin{aligned}S(\lambda,\Phi_i,\Phi_j)\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)&=\fint_{\mathbb{T}}\Phi_i'(\tau)K_0\left(\lambda\left|\Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)-\Phi_i\left(\tau\right)\right|\right)d\tau\\&=e^{\frac{2i\pi}{\mathbf{m}}}\fint_{\mathbb{T}}\Phi_i'\left(e^{\frac{2i\pi}{\mathbf{m}}}\tau\right)K_0\left(\lambda\left|\Phi_j\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)-\Phi_i\left(e^{\frac{2i\pi}{\mathbf{m}}}\tau\right)\right|\right)d\tau\\&=e^{\frac{2i\pi}{\mathbf{m}}}\fint_{\mathbb{T}}\Phi_i'(\tau)K_0\left(\lambda\left|\Phi_j\left(w\right)-\Phi_i\left(\tau\right)\right|\right)d\tau\\&=e^{\frac{2i\pi}{\mathbf{m}}}S(\lambda,\Phi_i,\Phi_j)(w).\end{aligned}$$

By definition (16.7) of G_j, this immediately implies that

$$\forall j\in\{1,2\},\quad \forall w\in\mathbb{T},\quad G_j\left(\lambda,b,\Omega,f_1,f_2\right)\left(e^{\frac{2i\pi}{\mathbf{m}}}w\right)=G_j\left(\lambda,b,\Omega,f_1,f_2\right)(w).$$

So

$$G(\lambda,b,\cdot,\cdot,\cdot):\mathbb{R}\times B^{1+\alpha}_{r,\mathbf{m}}\times B^{1+\alpha}_{r,\mathbf{m}}\to Y^{\alpha}_{\mathbf{m}}.$$

(iii) Fix $j\in\{1,2\}$. By (16.11) and (16.12), we have for $f_j\in B^{1+\alpha}_r$ and $h_j\in C^{1+\alpha}(\mathbb{T})$,

$$\begin{aligned}\partial_\Omega d_{f_j}G_j(\lambda,b,\Omega,f_j)(h_j)(w)&=\partial_\Omega d_{f_j}\mathcal{S}_j(\lambda,b,\Omega,f_j)(h_j)(w)\\&=\operatorname{Im}\left\{h_j(w)\overline{w}\overline{\Phi_j'(w)}+\Phi_j(w)\overline{w}\overline{h_j'(w)}\right\}.\end{aligned}$$

As a consequence, we deduce that for $(f_j,g_j)\in(B^{1+\alpha}_r)^2$ and $h_j\in C^{1+\alpha}(\mathbb{T})$,

$$\Big\|\partial_\Omega d_{f_j}G_j(\lambda,b,\Omega,f_j)(h_j)-\partial_\Omega d_{f_j}G_j(\lambda,b,\Omega,g_j)(h_j)\Big\|_{C^\alpha(\mathbb{T})}\lesssim\|f_j-g_j\|_{C^{1+\alpha}(\mathbb{T})}\|h_j\|_{C^{1+\alpha}(\mathbb{T})}.$$

This proves the continuity of $\partial_\Omega d_{(f_1,f_2)}G(\lambda,b,\cdot,\cdot,\cdot):\mathbb{R}\times B^{1+\alpha}_r\times B^{1+\alpha}_r\to\mathcal{L}(X^{1+\alpha},Y^\alpha)$ and achieves the proof of Proposition 16.1. ☐

17 Spectral study

In this section, we study the linearized operator at the equilibrium state and look for the degeneracy conditions for its kernel.

17.1 Linearized operator

In this subsection, we compute the differential $d_{(f_1,f_2)}G(\lambda,b,\Omega,0,0)$ and show that it acts as a Fourier multiplier. More precisely, we prove the following proposition.

Proposition 17.1. *Let $\lambda>0$, $b\in(0,1)$ and $\alpha\in(0,1)$. Then for all $\Omega\in\mathbb{R}$ and for all $(h_1,h_2)\in X^{1+\alpha}$, if we write*

$$h_1(w)=\sum_{n=0}^{\infty}a_n\overline{w}^n\quad\text{and}\quad h_2(w)=\sum_{n=0}^{\infty}b_n\overline{w}^n,$$

we have for all $w \in \mathbb{T}$

$$d_{(f_1,f_2)}G(\lambda, b, \Omega, 0, 0)(h_1, h_2)(w) = \sum_{n=0}^{\infty}(n+1)M_{n+1}(\lambda, b, \Omega)\begin{pmatrix} a_n \\ b_n \end{pmatrix} e_{n+1}(w),$$

where for all $n \in \mathbb{N}^*$, the matrix $M_n(\lambda, b, \Omega)$ is defined by

$$M_n(\lambda, b, \Omega) \triangleq \begin{pmatrix} \mathbf{\Omega}_n(\lambda) - \Omega - b\Lambda_1(\lambda, b) & b\Lambda_n(\lambda, b) \\ -\Lambda_n(\lambda, b) & \Lambda_1(\lambda, b) - b\big[\mathbf{\Omega}_n(\lambda b) + \Omega\big] \end{pmatrix},$$

with $\mathbf{\Omega}_n$ defined in (1.23) *and*

$$\Lambda_n(\lambda, b) \triangleq I_n(\lambda b)K_n(\lambda).$$

Recall that the modified Bessel functions I_n and K_n are defined in Appendix C.

Proof. Since $G = (G_1, G_2)$, then for given $(h_1, h_2) \in X^{1+\alpha}$, we have

$$d_{(f_1,f_2)}G(\lambda, b, \Omega, 0, 0)(h_1, h_2) = \begin{pmatrix} d_{f_1}G_1(\lambda, b, \Omega, 0, 0)h_1 + d_{f_2}G_1(\lambda, b, \Omega, 0, 0)h_2 \\ d_{f_1}G_2(\lambda, b, \Omega, 0, 0)h_1 + d_{f_2}G_2(\lambda, b, \Omega, 0, 0)h_2 \end{pmatrix}. \tag{17.1}$$

But, with the notation introduced in the proof of Proposition 16.1, we can write

$$\begin{cases} d_{f_1}G_1(\lambda, b, \Omega, 0, 0)h_1 & = & d_{f_1}\mathcal{S}_1(\lambda, b, \Omega, 0)h_1 + d_{f_1}\mathcal{I}_1(\lambda, b, 0, 0)h_1 \\ d_{f_2}G_2(\lambda, b, \Omega, 0, 0)h_2 & = & d_{f_2}\mathcal{S}_2(\lambda, b, \Omega, 0)h_2 + d_{f_2}\mathcal{I}_2(\lambda, b, 0, 0)h_2 \\ d_{f_2}G_1(\lambda, b, \Omega, 0, 0)h_2 & = & d_{f_2}\mathcal{I}_1(\lambda, b, 0, 0)h_2 \\ d_{f_1}G_2(\lambda, b, \Omega, 0, 0)h_1 & = & d_{f_1}\mathcal{I}_2(\lambda, b, 0, 0)h_1. \end{cases} \tag{17.2}$$

We write

$$h_1(w) = \sum_{n=0}^{\infty} a_n \overline{w}^n \quad \text{and} \quad h_2(w) = \sum_{n=0}^{\infty} b_n \overline{w}^n.$$

It has already been proved in [54, Prop. 5.8] that for all $w \in \mathbb{T}$,

$$d_{f_1}\mathcal{S}_1(\lambda, b, \Omega, 0)h_1(w) = \sum_{n=0}^{\infty}(n+1)\left(\mathbf{\Omega}_{n+1}(\lambda) - \Omega\right)a_n e_{n+1}(w), \tag{17.3}$$

where

$$\mathbf{\Omega}_n(\lambda) \triangleq I_1(\lambda)K_1(\lambda) - I_n(\lambda)K_n(\lambda).$$

By a similar calculation, we get

$$d_{f_2}\mathcal{S}_2(\lambda, b, \Omega, 0)h_2(w) = -\sum_{n=0}^{\infty}(n+1)b\left(\mathbf{\Omega}_{n+1}(\lambda b) + \Omega\right)b_n e_{n+1}(w). \tag{17.4}$$

In view of (16.18), we can write

$$d_{f_1}\mathcal{I}_1(\lambda, b, 0, 0)h_1(w) = \mathcal{L}_1(h_1)(w) + \mathcal{L}_2(h_1)(w),$$

with

$$\mathcal{L}_1(h_1)(w) \triangleq \operatorname{Im}\left\{\overline{w}\overline{h_1'(w)}b \fint_{\mathbb{T}} K_0\left(\lambda|w - b\tau|\right)d\tau\right\},$$

$$\mathcal{L}_2(h_1)(w) \triangleq \operatorname{Im}\left\{\frac{\lambda b}{2}\overline{w}\fint_{\mathbb{T}} K_0'\left(\lambda|w - b\tau|\right)\frac{\overline{h_1(w)}(w - b\tau) + h_1(w)(\overline{w} - b\overline{\tau})}{|w - b\tau|}d\tau\right\}.$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we deduce

$$\overline{w} \fint_{\mathbb{T}} K_0\left(\lambda|w - b\tau|\right) d\tau = \fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau.$$

Moreover, from (16.9), we know that

$$\fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau \in \mathbb{R}.$$

So using that

$$|1 - be^{\mathrm{i}\theta}| = \left(1 - 2b\cos(\theta) + b^2\right)^{\frac{1}{2}} \qquad \text{with} \qquad b \in (0,1), \tag{17.5}$$

we obtain from (C.3),

$$\begin{aligned}\fint_{\mathbb{T}} K_0\left(\lambda|1 - b\tau|\right) d\tau &= \mathrm{Re}\left\{\frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right) e^{\mathrm{i}\theta} d\theta\right\}\\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right)\cos(\theta) d\theta.\end{aligned}$$

Now, by (C.11) and (C.3), one obtains for all $n \in \mathbb{N}^*$,

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1 - be^{\mathrm{i}\theta}|\right)\cos(n\theta) d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{m=-\infty}^{\infty} I_m(\lambda b)K_m(\lambda)\cos(m\theta)\cos(n\theta) d\theta\\ &= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} I_m(\lambda b)K_m(\lambda)\int_0^{2\pi}\cos(m\theta)\cos(n\theta) d\theta\\ &= I_n(\lambda b)K_n(\lambda).\end{aligned} \tag{17.6}$$

Notice that the inversion of symbols of summation and integration is possible due to the geometric decay at infinity given by (C.18). Then, we deduce by (16.10) that

$$\mathcal{L}_1(h_1)(w) = -\sum_{n=0}^{\infty} nbI_1(\lambda b)K_1(\lambda)a_n e_{n+1}(w).$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we infer

$$\begin{aligned}&\overline{w}\fint_{\mathbb{T}} K_0'\left(\lambda|w - b\tau|\right)\frac{\overline{h_1(w)}(w - b\tau) + h_1(w)(\overline{w} - b\overline{\tau})}{|w - b\tau|} d\tau\\ &= \fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{\overline{h_1(w)}w(1 - b\tau) + h_1(w)\overline{w}(1 - b\overline{\tau})}{|1 - b\tau|} d\tau.\end{aligned}$$

But

$$\fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{h_1(w)\overline{w}(1 - b\overline{\tau})}{|1 - b\tau|} d\tau = \sum_{n=0}^{\infty} a_n\left(\fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{(1 - b\overline{\tau})}{|1 - b\tau|} d\tau\right)\overline{w}^{n+1}$$

and

$$\fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{\overline{h_1(w)}w(1 - b\tau)}{|1 - b\tau|} d\tau = \sum_{n=0}^{\infty} a_n\left(\fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{(1 - b\tau)}{|1 - b\tau|} d\tau\right) w^{n+1}.$$

Moreover, by writting the line integral with the parametrization $\tau = e^{\mathrm{i}\theta}$ and making the change of variables $\theta \mapsto -\theta$, we get as in (16.9)

$$\fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{(1 - b\tau)}{|1 - b\tau|} d\tau \in \mathbb{R} \quad \text{and} \quad \fint_{\mathbb{T}} K_0'\left(\lambda|1 - b\tau|\right)\frac{(1 - b\overline{\tau})}{|1 - b\tau|} d\tau \in \mathbb{R}.$$

Since $\mathrm{Im}\left(\overline{w}^{n+1}\right)=-\mathrm{Im}\left(w^{n+1}\right)$, we obtain

$$\mathcal{L}_2(h_1)(w)=\sum_{n=0}^{\infty}a_n\left(\frac{\lambda b}{2}\fint_{\mathbb{T}}K_0'\left(\lambda|1-b\tau|\right)\frac{b(\overline{\tau}-\tau)}{|1-b\tau|}d\tau\right)\mathrm{Im}(w^{n+1}).$$

An integration by parts together with (17.5) and (17.6) gives

$$\begin{aligned}\frac{\lambda b}{2}\fint_{\mathbb{T}}K_0'\left(\lambda|1-b\tau|\right)\frac{b(\overline{\tau}-\tau)}{|1-b\tau|}d\tau&=\frac{\lambda b}{4\pi}\int_0^{2\pi}K_0'\left(\lambda|1-be^{\mathrm{i}\theta}|\right)\frac{b(e^{-\mathrm{i}\theta}-e^{\mathrm{i}\theta})e^{\mathrm{i}\theta}}{|1-be^{\mathrm{i}\theta}|}d\theta\\&=\frac{-b}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right)e^{\mathrm{i}\theta}d\theta\\&=\frac{-b}{2\pi}\int_0^{2\pi}K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right)\cos(\theta)d\theta\\&=-bI_1(\lambda b)K_1(\lambda).\end{aligned}$$

Therefore,

$$\mathcal{L}_2(h_1)(w)=-\sum_{n=0}^{\infty}bI_1(\lambda b)K_1(\lambda)a_ne_{n+1}(w).$$

Finally,

$$d_{f_1}\mathcal{I}_1(\lambda,b,0,0)h_1(w)=-\sum_{n=0}^{\infty}b(n+1)I_1(\lambda b)K_1(\lambda)a_ne_{n+1}(w).\tag{17.7}$$

Similar computations taking into acount the modification with b, change of signs and the fact that $|b-e^{\mathrm{i}\theta}|=|1-be^{\mathrm{i}\theta}|$ yield

$$d_{f_2}\mathcal{I}_2(\lambda,b,0,0)(h_2)(w)=\sum_{n=0}^{\infty}(n+1)I_1(\lambda b)K_1(\lambda)b_ne_{n+1}(w).\tag{17.8}$$

According to (16.19), we can write

$$d_{f_2}\mathcal{I}_1(\lambda,b,0,0)h_2(w)=\mathcal{L}_3(h_2)(w)+\mathcal{L}_4(h_2)(w),$$

with

$$\begin{aligned}\mathcal{L}_3(h_2)(w)&\triangleq\mathrm{Im}\left\{\overline{w}\fint_{\mathbb{T}}h_2'(\tau)K_0\left(\lambda|w-b\tau|\right)d\tau\right\},\\\mathcal{L}_4(h_2)(w)&\triangleq-\frac{\lambda b}{2}\mathrm{Im}\left\{\overline{w}\fint_{\mathbb{T}}K_0'\left(\lambda|w-b\tau|\right)\frac{\overline{h_2(\tau)}(w-b\tau)+h_2(\tau)(\overline{w}-b\overline{\tau})}{|w-b\tau|}d\tau\right\}.\end{aligned}$$

The change of variables $\tau\mapsto w\tau$ implies

$$\begin{aligned}\mathcal{L}_3(h_2)(w)&=\mathrm{Im}\left\{\fint_{\mathbb{T}}h_2'(w\tau)K_0\left(\lambda|1-b\tau|\right)d\tau\right\}\\&=-\sum_{n=0}^{\infty}nb_n\left(\fint_{\mathbb{T}}\overline{\tau}^{n+1}K_0\left(\lambda|1-b\tau|\right)d\tau\right)\mathrm{Im}(\overline{w}^{n+1})\\&=\sum_{n=0}^{\infty}nb_n\left(\fint_{\mathbb{T}}\overline{\tau}^{n+1}K_0\left(\lambda|1-b\tau|\right)d\tau\right)e_{n+1}(w).\end{aligned}$$

But by symmetry and (17.6)

$$\begin{aligned}\fint_{\mathbb{T}} \overline{\tau}^{n+1} K_0\left(\lambda|1-b\tau|\right) d\tau &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\mathrm{i}(n+1)\theta} K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right) e^{\mathrm{i}\theta} d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} K_0\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \cos(n\theta) d\theta \\ &= I_n(\lambda b) K_n(\lambda).\end{aligned}$$

Hence,

$$\mathcal{L}_3(h_2)(w) = \sum_{n=0}^{\infty} n I_n(\lambda b) K_n(\lambda) b_n e_{n+1}(w).$$

By using the change of variables $\tau \mapsto w\tau$ and the fact that $|w| = 1$, we have

$$\mathcal{L}_4(h_2)(w) = \frac{-\lambda b}{2} \mathrm{Im}\left\{\fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{\overline{h_2(w\tau)} w(1-b\tau) + h_2(w\tau)\overline{w}(1-b\overline{\tau})}{|1-b\tau|} d\tau\right\},$$

which also writes

$$\mathcal{L}_4(h_2)(w) = \frac{-\lambda b}{2} \sum_{n=0}^{\infty} b_n \left(\fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{(\tau^n - \overline{\tau}^n) - b(\tau^{n+1} - \overline{\tau}^{n+1})}{|1-b\tau|} d\tau\right) \mathrm{Im}(w^n).$$

We denote

$$\mathtt{I} \triangleq \frac{-\lambda b}{2} \fint_{\mathbb{T}} K_0'\left(\lambda|1-b\tau|\right) \frac{(\tau^n - \overline{\tau}^n) - b(\tau^{n+1} - \overline{\tau}^{n+1})}{|1-b\tau|} d\tau.$$

Since $\mathtt{I} \in \mathbb{R}$, we have

$$\begin{aligned}\mathtt{I} &= \frac{-\lambda b}{4\pi} \int_0^{2\pi} K_0'\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \frac{(e^{\mathrm{i}n\theta} - e^{-\mathrm{i}n\theta}) - b(e^{\mathrm{i}(n+1)\theta} - e^{-\mathrm{i}(n+1)\theta})}{|1-be^{\mathrm{i}\theta}|} e^{\mathrm{i}\theta} d\theta \\ &= \frac{\lambda b}{2\pi} \int_0^{2\pi} K_0'\left(\lambda|1-be^{\mathrm{i}\theta}|\right) \frac{\sin(\theta)}{|1-be^{\mathrm{i}\theta}|} (\sin(n\theta) - b\sin((n+1)\theta)) d\theta.\end{aligned}$$

Integrating by parts with (17.5) and using (17.6) yield

$$\begin{aligned}\mathtt{I} &= \frac{1}{2\pi}\int_0^{2\pi} K_0(\lambda|1-be^{\mathrm{i}\theta}|)\left(b(n+1)\cos((n+1)\theta) - n\cos(n\theta)\right) \\ &= b(n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) - n I_n(\lambda b) K_n(\lambda).\end{aligned}$$

Therefore,

$$d_{f_2}\mathcal{I}_1(\lambda, b, 0, 0)(h_2)(w) = \sum_{n=0}^{\infty} b(n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) b_n e_{n+1}(w). \tag{17.9}$$

Similar computations taking into acount the modification with b, change of signs and the fact that $|b - e^{\mathrm{i}\theta}| = |1 - be^{\mathrm{i}\theta}|$ imply

$$d_{f_1}\mathcal{I}_2(\lambda, b, 0, 0)(h_1)(w) = -\sum_{n=0}^{\infty} (n+1) I_{n+1}(\lambda b) K_{n+1}(\lambda) a_n e_{n+1}(w). \tag{17.10}$$

Gathering (17.1), (17.2), (17.7), (17.10), (17.3), (17.9), (17.8) and (17.4), we get the desired result. The proof of Proposition 17.1 is now complete. ☐

17.2 Asymptotic monotonicity of the eigenvalues

This subsection is devoted to the proof of Proposition 17.2 concerning the asymptotic monotonicity of the eigenvalues needed to ensure the one dimensional kernel assumption of Crandall-Rabinowitz's Theorem. But first, we have to prove their existence and this is the purpose of the following lemma.

Lemma 17.1. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N_0(\lambda,b) \in \mathbb{N}^*$ such that for all integer $n \geqslant N_0(\lambda,b)$, there exist two angular velocities*

$$\begin{aligned}\boldsymbol{\Omega}_n^{\pm}(\lambda,b) \triangleq \frac{1-b^2}{2b}\Lambda_1(\lambda,b) + \frac{1}{2}\Big(\boldsymbol{\Omega}_n(\lambda) - \boldsymbol{\Omega}_n(\lambda b)\Big)\\ \pm \frac{1}{2b}\sqrt{\Big(b\big[\boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda,b)\Big)^2 - 4b^2\Lambda_n^2(\lambda,b)}\end{aligned} \tag{17.11}$$

for which the matrix $M_n\left(\lambda, b, \boldsymbol{\Omega}_n^{\pm}(\lambda,b)\right)$ is singular.

Proof. The determinant of $M_n(\lambda,b,\Omega)$ is

$$\begin{aligned}\det\left(M_n(\lambda,b,\Omega)\right) &= \Big(\boldsymbol{\Omega}_n(\lambda) - \Omega - b\Lambda_1(\lambda,b)\Big)\Big(\Lambda_1(\lambda,b) - b\big[\boldsymbol{\Omega}_n(\lambda b) + \Omega\big]\Big) + b\Lambda_n^2(\lambda,b)\\ &= b\Omega^2 - B_n(\lambda,b)\Omega + C_n(\lambda,b),\end{aligned} \tag{17.12}$$

where

$$\begin{aligned}B_n(\lambda,b) &\triangleq (1-b^2)\Lambda_1(\lambda,b) + b\big[\boldsymbol{\Omega}_n(\lambda) - \boldsymbol{\Omega}_n(\lambda b)\big],\\ C_n(\lambda,b) &\triangleq b\left[\Big(\Lambda_1(\lambda,b) - \frac{1}{b}\boldsymbol{\Omega}_n(\lambda)\Big)\Big(b\boldsymbol{\Omega}_n(\lambda b) - \Lambda_1(\lambda,b)\Big) + \Lambda_n^2(\lambda,b)\right].\end{aligned}$$

It is a polynomial of degree two in Ω which has at most two roots. Let us compute its discriminant. After straightforward computations, we find

$$\begin{aligned}\Delta_n(\lambda,b) &\triangleq B_n^2(\lambda,b) - 4bC_n(\lambda,b)\\ &= \Big(b\big[\boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda,b)\Big)^2 - 4b^2\Lambda_n^2(\lambda,b).\end{aligned} \tag{17.13}$$

Using the asymptotic expansion of large order (C.14), we infer

$$\forall \lambda > 0, \quad \forall b \in (0,1], \quad I_n(\lambda b)K_n(\lambda) \underset{n\to\infty}{\longrightarrow} 0. \tag{17.14}$$

As a consequence,

$$\Delta_n(\lambda,b) \underset{n\to\infty}{\longrightarrow} \Delta_\infty(\lambda,b), \tag{17.15}$$

where

$$\Delta_\infty(\lambda,b) = \delta_\infty^2(\lambda,b) \quad \text{with} \quad \delta_\infty(\lambda,b) \triangleq b\big[I_1(\lambda)K_1(\lambda) + I_1(\lambda b)K_1(\lambda b)\big] - (1+b^2)I_1(\lambda b)K_1(\lambda). \tag{17.16}$$

We can rewrite $\delta_\infty(\lambda,b)$ as

$$\delta_\infty(\lambda,b) = \big[bI_1(\lambda) - I_1(\lambda b)\big]K_1(\lambda) + bI_1(\lambda b)\big[K_1(\lambda b) - bK_1(\lambda)\big].$$

According to (C.12) and (C.3), we find $K_1' < 0$ on $(0,\infty)$, which implies in turn the strict decay property of K_1 on $(0,\infty)$. Therefore, since $b \in (0,1)$, we get

$$bK_1(\lambda) < K_1(\lambda) < K_1(\lambda b).$$

Now since $b \in (0,1)$, we obtain from (C.2),

$$I_1(\lambda b) = \sum_{m=0}^{\infty} \frac{\left(\frac{\lambda b}{2}\right)^{1+2m}}{m!\Gamma(m+2)} < b \sum_{m=0}^{\infty} \frac{\left(\frac{\lambda}{2}\right)^{1+2m}}{m!\Gamma(m+2)} = bI_1(\lambda).$$

Finally,

$$\Delta_\infty(\lambda, b) > 0.$$

Thus

$$\exists N_0(\lambda, b) \in \mathbb{N}^*, \quad \forall n \in \mathbb{N}^*, \quad n \geqslant N_0(\lambda, b) \Rightarrow \Delta_n(\lambda, b) > 0. \tag{17.17}$$

Therefore, for $n \geqslant N_0(\lambda, b)$ there exist two angular velocities $\boldsymbol{\Omega}_n^-(\lambda, b)$ and $\boldsymbol{\Omega}_n^+(\lambda, b)$ for which the matrix $M_n(\lambda, b, \boldsymbol{\Omega}_n^\pm(\lambda, b))$ is singular. These angular velocities are defined by

$$\begin{aligned}\boldsymbol{\Omega}_n^\pm(\lambda, b) &\triangleq \frac{B_n(\lambda, b) \pm \sqrt{\Delta_n(\lambda, b)}}{2b}\\ &= \frac{1-b^2}{2b}\Lambda_1(\lambda, b) + \frac{1}{2}\Big(\boldsymbol{\Omega}_n(\lambda) - \boldsymbol{\Omega}_n(\lambda b)\Big)\\ &\quad \pm \frac{1}{2b}\sqrt{\Big(b\big[\boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_n(\lambda b)\big] - (1+b^2)\Lambda_1(\lambda, b)\Big)^2 - 4b^2\Lambda_n^2(\lambda, b)}.\end{aligned}$$

This ends the proof of Lemma 17.1. ☐

We shall now study the monotonicity of the eigenvalues obtained in Lemma 17.1. This is a crucial point to obtain later the one dimensional condition for the kernel of the linearized operator given by Proposition 17.1.

Proposition 17.2. *Let $\lambda > 0$ and $b \in (0,1)$. There exists $N(\lambda, b) \in \mathbb{N}^*$ with $N(\lambda, b) \geqslant N_0(\lambda, b)$ where $N_0(\lambda, b)$ is defined in Lemma 17.1 such that*

(i) The sequence $\big(\boldsymbol{\Omega}_n^+(\lambda, b)\big)_{n \geqslant N(\lambda,b)}$ is strictly increasing and converges to $\boldsymbol{\Omega}_\infty^+(\lambda, b) = I_1(\lambda)K_1(\lambda) - b\Lambda_1(\lambda, b)$.

(ii) The sequence $\big(\boldsymbol{\Omega}_n^-(\lambda, b)\big)_{n \geqslant N(\lambda,b)}$ is strictly decreasing and converges to $\boldsymbol{\Omega}_\infty^-(\lambda, b) = \frac{\Lambda_1(\lambda,b)}{b} - I_1(\lambda b)K_1(\lambda b)$.

Then, we have for all $(m, n) \in (\mathbb{N}^*)^2$ with $N(\lambda, b) \leqslant n < m$,

$$\boldsymbol{\Omega}_\infty^-(\lambda, b) < \boldsymbol{\Omega}_m^-(\lambda, b) < \boldsymbol{\Omega}_n^-(\lambda, b) < \boldsymbol{\Omega}_n^+(\lambda, b) < \boldsymbol{\Omega}_m^+(\lambda, b) < \boldsymbol{\Omega}_\infty^+(\lambda, b).$$

Proof. The convergence is an immediate consequence of (17.11), (17.15), (17.16) and (17.14). Then, we turn to the asymptotic monotonicity. For that purpose, we study the sign of the difference

$$\boldsymbol{\Omega}_{n+1}^\pm(\lambda, b) - \boldsymbol{\Omega}_n^\pm(\lambda, b) = \frac{1}{2}\Big(\big[\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_n(\lambda)\big] - \big[\boldsymbol{\Omega}_{n+1}(\lambda b) - \boldsymbol{\Omega}_n(\lambda b)\big]\Big) \pm \frac{1}{2b}\Big[\sqrt{\Delta_{n+1}(\lambda, b)} - \sqrt{\Delta_n(\lambda, b)}\Big]$$

for n large enough.

▶ We first study the difference term before the square roots. We can write

$$\begin{aligned}&\big[\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_{n+1}(\lambda b)\big] - \big[\boldsymbol{\Omega}_n(\lambda) - \boldsymbol{\Omega}_n(\lambda b)\big]\\ &= \big[\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_n(\lambda)\big] - \big[\boldsymbol{\Omega}_{n+1}(\lambda b) - \boldsymbol{\Omega}_n(\lambda b)\big]\\ &= \big[I_n(\lambda)K_n(\lambda) - I_{n+1}(\lambda)K_{n+1}(\lambda)\big] - \big[I_n(\lambda b)K_n(\lambda b) - I_{n+1}(\lambda b)K_{n+1}(\lambda b)\big]\\ &\triangleq \varphi_n(\lambda) - \varphi_n(\lambda b).\end{aligned}$$

By vitue of (C.18), we deduce

$$I_n(\lambda)K_n(\lambda) \underset{n\to\infty}{=} \frac{1}{2n} - \frac{\lambda^2}{4n^3} + o_\lambda\left(\frac{1}{n^4}\right).$$

Therefore,

$$\begin{aligned}\varphi_n(\lambda) - \varphi_n(\lambda b) &\underset{n\to\infty}{=} \lambda^2(b^2-1)\frac{(n+1)^3 - n^3}{4n^3(n+1)^3} + o_{\lambda,b}\left(\frac{1}{n^4}\right)\\ &\underset{n\to\infty}{=} \frac{3\lambda^2(b^2-1)}{4n^4} + o_{\lambda,b}\left(\frac{1}{n^4}\right).\end{aligned}$$

We conclude that

$$\frac{1}{2}\Big(\big[\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_n(\lambda)\big] - \big[\boldsymbol{\Omega}_{n+1}(\lambda b) - \boldsymbol{\Omega}_n(\lambda b)\big]\Big) \underset{n\to\infty}{=} O_{\lambda,b}\left(\frac{1}{n^4}\right). \tag{17.18}$$

► The next task is to look at the asymptotic sign of the difference $\sqrt{\Delta_{n+1}(\lambda,b)} - \sqrt{\Delta_n(\lambda,b)}$. We can write

$$\sqrt{\Delta_{n+1}(\lambda,b)} - \sqrt{\Delta_n(\lambda,b)} = \frac{\Delta_{n+1}(\lambda,b) - \Delta_n(\lambda,b)}{\sqrt{\Delta_{n+1}(\lambda,b)} + \sqrt{\Delta_n(\lambda,b)}}$$

with

$$\begin{aligned}\Delta_{n+1}(\lambda,b) - \Delta_n(\lambda,b) = b\Big(&\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_{n+1}(\lambda b) - \boldsymbol{\Omega}_n(\lambda b)\Big)\\ &\times\Big(b\big[\boldsymbol{\Omega}_{n+1}(\lambda) + \boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_{n+1}(\lambda b) + \boldsymbol{\Omega}_n(\lambda b)\big] - 2(1+b^2)\Lambda_1(\lambda,b)\Big)\\ &+ 4b^2\Big(\Lambda_n(\lambda,b) - \Lambda_{n+1}(\lambda,b)\Big)\Big(\Lambda_n(\lambda,b) + \Lambda_{n+1}(\lambda,b)\Big).\end{aligned}$$

By using (C.18), we have

$$\Lambda_n(\lambda,b) \underset{n\to\infty}{=} \frac{b^n}{2n} + \frac{\lambda^2 b^n(b^2-1)}{2n^2} + o_{\lambda,b}\left(\frac{b^n}{n^2}\right).$$

Hence, the following asymptotic expansion holds

$$\Lambda_n(\lambda,b) \pm \Lambda_{n+1}(\lambda,b) \underset{n\to\infty}{=} o_{\lambda,b}\left(\frac{1}{n^2}\right).$$

As a consequence,

$$4b^2\Big(\Lambda_n(\lambda,b) - \Lambda_{n+1}(\lambda,b)\Big)\Big(\Lambda_n(\lambda,b) + \Lambda_{n+1}(\lambda,b)\Big) \underset{n\to\infty}{=} o_{\lambda,b}\left(\frac{1}{n^2}\right). \tag{17.19}$$

In addition,

$$b\Big(\boldsymbol{\Omega}_{n+1}(\lambda) - \boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_{n+1}(\lambda b) - \boldsymbol{\Omega}_n(\lambda b)\Big) = b\Big(\varphi_n(\lambda) + \varphi_n(\lambda b)\Big) \underset{n\to\infty}{\sim} \frac{b}{n^2} \tag{17.20}$$

and

$$\begin{aligned}&b\Big[\boldsymbol{\Omega}_{n+1}(\lambda) + \boldsymbol{\Omega}_n(\lambda) + \boldsymbol{\Omega}_{n+1}(\lambda b) + \boldsymbol{\Omega}_n(\lambda b)\Big] - 2(1+b^2)\Lambda_1(\lambda,b)\\ &= 2b\Big[I_1(\lambda)K_1(\lambda) + I_1(\lambda b)K_1(\lambda b)\Big] - 2(1+b^2)I_1(\lambda b)K_1(\lambda)\\ &\qquad - b\Big[I_{n+1}(\lambda)K_{n+1}(\lambda) + I_{n+1}(\lambda b)K_{n+1}(\lambda b) + I_n(\lambda)K_n(\lambda) + I_n(\lambda b)K_n(\lambda b)\Big]\\ &\underset{n\to\infty}{\longrightarrow} 2\delta_\infty(\lambda,b),\end{aligned} \tag{17.21}$$

where $\delta_\infty(\lambda,b)$ is defined in (17.16). From (17.15), (17.16), (17.19), (17.20) and (17.21), we obtain

$$\sqrt{\Delta_{n+1}(\lambda,b)}-\sqrt{\Delta_n(\lambda,b)}\underset{n\to\infty}{\sim}\frac{b}{n^2}. \tag{17.22}$$

▶ Combining (17.18) and (17.22), we get

$$\mathbf{\Omega}_{n+1}^{\pm}(\lambda,b)-\mathbf{\Omega}_n^{\pm}(\lambda,b)\underset{n\to\infty}{\sim}\pm\frac{1}{2n^2}.$$

We conclude that there exists $N(\lambda,b)\geqslant N_0(\lambda,b)$ such that

$$\forall n\in\mathbb{N}^*,\quad n\geqslant N(\lambda,b)\Rightarrow\begin{cases}\mathbf{\Omega}_{n+1}^{+}(\lambda,b)-\mathbf{\Omega}_n^{+}(\lambda,b)>0\\ \mathbf{\Omega}_{n+1}^{-}(\lambda,b)-\mathbf{\Omega}_n^{-}(\lambda,b)<0,\end{cases}$$

i.e. the sequence $\big(\mathbf{\Omega}_n^{+}(\lambda,b)\big)_{n\geqslant N(\lambda,b)}$ $\Big($resp. $\big(\mathbf{\Omega}_n^{-}(\lambda,b)\big)_{n\geqslant N(\lambda,b)}\Big)$ is strictly increasing (resp. decreasing). This achieves the proof of Proposition 17.2. □

We shall now study both important asymptotic behaviours

$$\lambda\to0\quad\text{and}\quad b\to0.$$

The first one corresponds to the Euler case and the second one corresponds to the simply-connected case. We remark that we formally recover (at least partially) [94, Thm. B.] and [54, Thm. 5.1.] looking at these limits. More precisely, we have the following result.

Lemma 17.2. *The spectrum is continuous in the following sense.*

(i) Let $b\in(0,1)$. There exists $\widetilde{N}(b)$ such that

$$\forall n\in\mathbb{N}^*,\ n\geqslant\widetilde{N}(b)\ \Rightarrow\ \mathbf{\Omega}_n^{\pm}(\lambda,b)\underset{\lambda\to0}{\longrightarrow}\mathbf{\Omega}_n^{\pm}(b),$$

where $\mathbf{\Omega}_n^{\pm}(b)$ is defined in (1.21).

(ii) Let $\lambda>0$. There exists $\widetilde{N}(\lambda)$ such that

$$\forall n\in\mathbb{N}^*,\ n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \mathbf{\Omega}_n^{+}(\lambda,b)\underset{b\to0}{\longrightarrow}\mathbf{\Omega}_n(\lambda),$$

where $\mathbf{\Omega}_n(\lambda)$ is defined in (1.23).

Proof. **(i)** In view of (C.13), we deduce

$$\forall\, n\in\mathbb{N}^*,\quad\forall\, b\in(0,1],\quad I_n(\lambda b)K_n(\lambda)\underset{\lambda\to0}{\longrightarrow}\frac{b^n}{2n}. \tag{17.23}$$

In what follows, we fix $b\in(0,1)$. By virtue of (17.23), the matrices M_n defined in Proposition 17.1, satisfy the following convergence

$$\forall n\in\mathbb{N}^*,\quad M_n(\lambda,b,\Omega)\underset{\lambda\to0}{\longrightarrow}M_n(b,\Omega)\triangleq\begin{pmatrix}\frac{n-1}{2n}-\frac{b^2}{2}-\Omega & \frac{b^{n+1}}{2n}\\ -\frac{b^n}{2n} & \frac{b}{2}-\frac{b(n-1)}{2n}-b\Omega\end{pmatrix}.$$

After straightforward computations, we find

$$\det\big(M_n(b,\Omega)\big)=b\Omega^2-\frac{b(1-b^2)}{2}\Omega+\frac{b}{4n^2}\left[n(1-b^2)-1+b^{2n}\right].$$

This polynomial of degree two in Ω has the discriminant

$$\Delta_n(b) \triangleq \frac{b^2}{n^2}\left[\left(\frac{n(1-b^2)}{2}-1\right)^2 - b^{2n}\right].$$

Thus, provided $\Delta_n(b) > 0$, i.e. for

$$1 + b^n - \frac{n(1-b^2)}{2} < 0, \tag{17.24}$$

we have two roots

$$\boldsymbol{\Omega}_n^{\pm}(b) \triangleq \frac{1-b^2}{4} \pm \frac{1}{2n}\sqrt{\left(\frac{n(1-b^2)}{2}-1\right)^2 - b^{2n}}.$$

Then, we recover the result found in [94, Thm. B.]. Now, observe that the sequence $n \mapsto 1 + b^n - \frac{n(1-b^2)}{2}$ is decreasing. Then there exists $\widetilde{N}(b) \in \mathbb{N}^*$ and $c_0 > 0$ such that

$$\inf_{\substack{n\in\mathbb{N}^*\\ n\geqslant \widetilde{N}(b)}} \Delta_n(b) \geqslant c_0 > 0.$$

We use the integral representation (C.10), allowing to write

$$\forall n \in \mathbb{N}^*, \quad I_n(\lambda)K_n(\lambda) - \tfrac{1}{2n} = \tfrac{1}{2}\int_0^{\infty}\Big[J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right) - 1\Big]e^{-nt}dt.$$

Now using the integral representation (C.1), we find

$$J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right) - 1 = \tfrac{1}{\pi}\int_0^{\pi}\Big[\cos\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\sin(\theta)\right) - 1\Big]d\theta.$$

The classical inequalities

$$\forall x \in \mathbb{R}, \quad |\cos(x) - 1| \leqslant \tfrac{x^2}{2} \quad \text{and} \quad \sinh(x) \leqslant \tfrac{e^x}{2}$$

provide the following estimate for $t \geqslant 0$

$$\Big|J_0\left(2\lambda\sinh\left(\tfrac{t}{2}\right)\right) - 1\Big| \leqslant \lambda^2 e^t.$$

We conclude that

$$\forall \lambda > 0, \qquad \sup_{n\in\mathbb{N}\setminus\{0,1\}}\left|I_n(\lambda)K_n(\lambda) - \tfrac{1}{2n}\right| \leqslant \lambda^2. \tag{17.25}$$

On the other hand, we set for $\varepsilon > 0$,

$$K_0^{\varepsilon}(x) = K_0(\varepsilon x) + \log\left(\tfrac{\varepsilon}{2}\right).$$

Remark that (C.7) implies

$$\lim_{\varepsilon\to 0} K_0^{\varepsilon}(x) = -\log\left(\tfrac{x}{2}\right) - \boldsymbol{\gamma}.$$

By the dominated convergence theorem, one has

$$\forall n \in \mathbb{N}^*, \quad \lim_{\varepsilon\to 0}\int_{\mathbb{T}} K_0^{\varepsilon}(|1 - be^{\mathrm{i}\theta}|)\cos(n\theta)d\eta = -\int_{\mathbb{T}}\log(|1 - be^{\mathrm{i}\theta}|)\cos(n\theta)d\theta.$$

Now one obtains from (17.6)

$$\forall n\in\mathbb{N}^*,\quad \int_{\mathbb{T}} K_0^{\varepsilon}(|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\eta=\int_{\mathbb{T}} K_0(\varepsilon|1-be^{\mathrm{i}\theta}|)\cos(n\theta)d\theta$$
$$=I_n(\varepsilon b)K_n(\varepsilon).$$

Putting together the last two equality with (17.23) yields

$$\forall n\in\mathbb{N}^*,\quad \int_{\mathbb{T}}\log\big(|1-be^{\mathrm{i}\theta}|\big)d\theta=-\tfrac{b^n}{2n}.$$

Added to (17.6), we have

$$\forall\lambda>0,\quad\forall n\in\mathbb{N}^*,\quad I_n(\lambda b)K_n(\lambda)-\tfrac{b^n}{2n}=\int_{\mathbb{T}}\Big[K_0\big(\lambda|1-be^{\mathrm{i}\theta}|\big)+\log\big(|1-be^{\mathrm{i}\theta}|\big)\Big]\cos(n\theta)d\theta.$$

Then, making appeal to the power series decompositions (C.7) and (C.2), we get

$$\forall\lambda>0,\quad\sup_{n\in\mathbb{N}^*}\big|I_n(\lambda b)K_n(\lambda)-\tfrac{b^n}{2n}\big|\lesssim\max(|\log(\lambda)|,1)\lambda^2.\tag{17.26}$$

Combining (17.13), (17.25), (17.26) and (17.23) one obtains

$$\sup_{n\in\mathbb{N}^*}\big|\Delta_n(\lambda,b)-\Delta_n(b)\big|\underset{\lambda\to0}{\longrightarrow}0.$$

Hence, there exists $\lambda_0(b)>0$ such that

$$\inf_{\lambda\in(0,\lambda_0(b)]}\ \inf_{\substack{n\in\mathbb{N}^*\\ n\geqslant\widetilde{N}(b)}}\Delta_n(\lambda,b)\geqslant\tfrac{c_0}{2}>0.$$

Therefore, we deduce from (17.11) and (17.23) that,

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(b)\ \Rightarrow\ \mathbf{\Omega}_n^{\pm}(\lambda,b)\underset{\lambda\to0}{\longrightarrow}\mathbf{\Omega}_n^{\pm}(b).$$

(ii) In what follows, we fix $\lambda>0$. By using the asymptotic (C.13), we find

$$\frac{\Lambda_1(\lambda,b)}{b}\underset{b\to0}{\longrightarrow}\frac{\lambda K_1(\lambda)}{2}\quad\text{and}\quad\forall n\in\mathbb{N}^*,\ \Lambda_n(\lambda,b)\underset{b\to0}{\sim}\tfrac{(\lambda b)^n}{2^n n!}K_n(\lambda).$$

Using the power series decomposition (C.2), the decay property of $\lambda\mapsto I_n(\lambda)K_n(\lambda)$ and the asymptotic (17.23), we get

$$\forall n\in\mathbb{N}^*,\quad\big|I_n(\lambda b)K_n(\lambda)-\tfrac{(\lambda b)^n}{2^n n!}K_n(\lambda)\big|\leqslant b^2I_n(\lambda)K_n(\lambda)\leqslant b^2.$$

Thus, we obtain from (17.13), (17.25) and (17.23)

$$\sup_{n\in\mathbb{N}^*}\Big|\Delta_n(\lambda,b)-b^2\Big[\big(\mathbf{\Omega}_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}\big)^2-\tfrac{(\lambda b)^{2n}}{2^{2n}(n!)^2}K_n^2(\lambda)\Big]\Big|\underset{b\to0}{\longrightarrow}0.\tag{17.27}$$

Notice that

$$\mathbf{\Omega}_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}\underset{n\to\infty}{\longrightarrow}I_1(\lambda)K_1(\lambda)+\tfrac{1-\lambda K_1(\lambda)}{2}.$$

Consider the function φ defined by $\forall x>0,\varphi(x)=xK_1(x)$. From (C.4), we get

$$\varphi'(x)=K_1(x)+xK_1'(x)=-xK_0(x)<0.$$

Hence φ is strictly decreasing on $(0,\infty)$. Moreover, in view of the asymptotic (C.13), we infer

$$\lim_{x\to 0}\varphi(x)=1.$$

Thus, using also (C.3), we obtain

$$\forall x>0,\quad \varphi(x)\in(0,1).$$

Therefore, we deduce that there exists $\widetilde{N}(\lambda)\in\mathbb{N}^*$ such that

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \mathbf{\Omega}_n(\lambda)+\tfrac{n-1}{2n}-\tfrac{\lambda K_1(\lambda)}{2}>0.$$

In addition, using (C.14) and up to increasing the value of $\widetilde{N}(\lambda)$ one gets

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \tfrac{(\lambda b)^{2n}}{2^{2n}(n!)^2}K_n^2(\lambda)\leqslant 1.$$

Coming back to (17.27), we infer the existence of $b_0(\lambda)\in(0,1)$ such that

$$\forall b\in(0,b_0(\lambda)),\quad \forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \Delta_n(\lambda,b)>0.$$

Thus, we get from (17.11)

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \mathbf{\Omega}_n^+(\lambda,b)\underset{b\to 0}{\longrightarrow}\mathbf{\Omega}_n(\lambda).$$

Then, we partially recover the result found in [54, Thm. 5.1.]. We also obtain, up to increasing the value of $\widetilde{N}(\lambda)$,

$$\forall n\in\mathbb{N}^*,\quad n\geqslant\widetilde{N}(\lambda)\ \Rightarrow\ \mathbf{\Omega}_n^-(\lambda,b)\underset{b\to 0}{\longrightarrow}\mathbf{\Omega}_n^-(\lambda)\triangleq\frac{\lambda nK_1(\lambda)-n+1}{2n}.$$

Unfortunately, we cannot prove bifurcation from these eigenvalues. □

18 Bifurcation from simple eigenvalues

We prove here the following result which implies the main Theorem 15.1 by a direct application of Crandall-Rabinowitz's Theorem B.1.

Proposition 18.1. *Let $\lambda>0$, $b\in(0,1)$, $\alpha\in(0,1)$ and $\mathbf{m}\in\mathbb{N}^*$ such that $\mathbf{m}\geqslant N(\lambda,b)$. Then the following assertions hold true.*

(i) There exists $r>0$ such that $G(\lambda,b,\cdot,\cdot,\cdot):\mathbb{R}\times B^{1+\alpha}_{r,\mathbf{m}}\times B^{1+\alpha}_{r,\mathbf{m}}\to Y^{\alpha}_{\mathbf{m}}$ is well-defined and of class C^1.

(ii) The kernel $\ker\Big(d_{(f_1,f_2)}G\big(\lambda,b,\mathbf{\Omega}^{\pm}_{\mathbf{m}}(\lambda,b),0,0\big)\Big)$ is one-dimensional and generated by

$$\begin{array}{rccl} v_{0,\mathbf{m}}: & \mathbb{T} & \to & \mathbb{C}^2\\ & w & \mapsto & \begin{pmatrix} b\big[\mathbf{\Omega}_{\mathbf{m}}(\lambda b)+\mathbf{\Omega}^{\pm}_{\mathbf{m}}(\lambda,b)\big]-\Lambda_1(\lambda,b)\\ -\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}\overline{w}^{\mathbf{m}-1}.\end{array}$$

(iii) The range $R\Big(d_{(f_1,f_2)}G\big(\lambda,b,\mathbf{\Omega}^{\pm}_{\mathbf{m}}(\lambda,b),0,0\big)\Big)$ is closed and of codimension one in $Y^{\alpha}_{\mathbf{m}}$.

(iv) Tranversality condition :

$$\partial_{\Omega}d_{(f_1,f_2)}G\big(\lambda,b,\mathbf{\Omega}^{\pm}_{\mathbf{m}}(\lambda,b),0,0\big)(v_{0,\mathbf{m}})\notin R\Big(d_{(f_1,f_2)}G\big(\lambda,b,\mathbf{\Omega}^{\pm}_{\mathbf{m}}(\lambda,b),0,0\big)\Big).$$

Proof. **(i)** Follows from Proposition 16.1.
(ii) Let $(h_1,h_2)\in X^{1+\alpha}_{\mathbf{m}}$. We write

$$h_1(w)=\sum_{n=1}^{\infty}a_n\overline{w}^{n\mathbf{m}-1}\qquad\text{and}\qquad h_2(w)=\sum_{n=1}^{\infty}b_n\overline{w}^{n\mathbf{m}-1}.\tag{18.1}$$

Proposition 17.1 gives

$$\forall w\in\mathbb{T},\quad d_{(f_1,f_2)}G(\lambda,b,\Omega,0,0)(h_1,h_2)(w)=\sum_{n=1}^{\infty}n\mathbf{m}M_{n\mathbf{m}}(\lambda,b,\Omega)\begin{pmatrix}a_n\\ b_n\end{pmatrix}e_{n\mathbf{m}}(w).\tag{18.2}$$

For $\Omega\in\big\{\boldsymbol{\Omega}_{\mathbf{m}}^{-}(\lambda,b),\boldsymbol{\Omega}_{\mathbf{m}}^{+}(\lambda,b)\big\}$, we have

$$\det\Big(M_{\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)=0.$$

Thus, the kernel of $d_{(f_1,f_2)}G\big(\lambda,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)$ is non trivial and it is one dimensional if and only if

$$\forall n\in\mathbb{N}^*,\quad n\geqslant 2\Rightarrow\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)\neq 0.\tag{18.3}$$

The previous condition is satisfied in view of Proposition 17.2. Hence, we have the equivalence

$$(h_1,h_2)\in\ker\Big(d_{(f_1,f_2)}G\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)\Leftrightarrow\begin{cases}\forall n\in\mathbb{N}^*,\quad n\geqslant 2\Rightarrow a_n=0=b_n\\ \begin{pmatrix}a_1\\ b_1\end{pmatrix}\in\ker\Big(M_{\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big).\end{cases}$$

Therefore, we can select as generator of $\ker\Big(d_{(f_1,f_2)}G\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)$ the following pair of functions

$$\begin{array}{rccl}v_{0,\mathbf{m}}: & \mathbb{T} & \rightarrow & \mathbb{C}^2\\ & w & \mapsto & \begin{pmatrix}b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)+\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big]-\Lambda_1(\lambda,b)\\ -\Lambda_{\mathbf{m}}(\lambda,b)\end{pmatrix}\overline{w}^{\mathbf{m}-1}.\end{array}$$

(iii) We consider the set $Z_{\mathbf{m}}$ defined by

$$Z_{\mathbf{m}}\triangleq\Bigg\{g=(g_1,g_2)\in Y^{\alpha}_{\mathbf{m}}\quad\text{s.t.}\quad\forall w\in\mathbb{T},\quad g(w)=\sum_{n=1}^{\infty}\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix}e_{n\mathbf{m}}(w),$$
$$\forall n\in\mathbb{N}^*,\quad(\mathscr{A}_n,\mathscr{B}_n)\in\mathbb{R}^2\quad\text{and}\quad\exists(a_1,b_1)\in\mathbb{R}^2,\quad M_{\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}a_1\\ b_1\end{pmatrix}=\begin{pmatrix}\mathscr{A}_1\\ \mathscr{B}_1\end{pmatrix}\Bigg\}.$$

Clearly, $Z_{\mathbf{m}}$ is a closed sub-vector space of codimension one in $Y^{\alpha}_{\mathbf{m}}$. It remains to prove that it coincides with the range of $d_{(f_1,f_2)}G\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)$. Obviously, we have the inclusion

$$R\Big(d_{(f_1,f_2)}G\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)\Big)\subset Z_{\mathbf{m}}.$$

We are left to prove the converse inclusion. Let $(g_1,g_2)\in Z_{\mathbf{m}}$. We shall prove that the equation

$$d_{(f_1,f_2)}G\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b),0,0\big)(h_1,h_2)=(g_1,g_2)$$

admits a solution $(h_1,h_2)\in X^{1+\alpha}_{\mathbf{m}}$ in the form (18.1). According to (18.2), the previous equation is equivalent to the following countable set of equations

$$\forall n\in\mathbb{N}^*,\quad n\mathbf{m}M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}a_n\\ b_n\end{pmatrix}=\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix}.$$

For $n=1$, the existence follows from the definition of $Z_{\mathbf{m}}$. Thanks to (18.3), the sequences $(a_n)_{n\geqslant 2}$ and $(b_n)_{n\geqslant 2}$ are uniquely determined by

$$\forall n\in\mathbb{N}^*,\quad n\geqslant 2\Rightarrow\begin{pmatrix}a_n\\ b_n\end{pmatrix}=\frac{1}{n\mathbf{m}}M_{n\mathbf{m}}^{-1}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\begin{pmatrix}\mathscr{A}_n\\ \mathscr{B}_n\end{pmatrix},$$

or equivalently,

$$\begin{cases}a_n &= \dfrac{\Lambda_1(\lambda,b)-b\big[\Omega_{n\mathbf{m}}(\lambda b)+\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big]}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{A}_n-\dfrac{b\Lambda_{n\mathbf{m}}(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{B}_n\\[3ex] b_n &= \dfrac{\Lambda_{n\mathbf{m}}(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{A}_n+\dfrac{\Omega_{n\mathbf{m}}(\lambda b)+\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)-b\Lambda_1(\lambda,b)}{n\mathbf{m}\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\mathscr{B}_n.\end{cases}$$

It remains to prove the regularity, that is $(h_1,h_2)\in X_{\mathbf{m}}^{1+\alpha}$. For that purpose, we show

$$w\mapsto\left(\begin{array}{c}h_1(w)-a_1\overline{w}^{\mathbf{m}-1}\\ h_2(w)-a_2\overline{w}^{\mathbf{m}-1}\end{array}\right)\in C^{1+\alpha}(\mathbb{T})\times C^{1+\alpha}(\mathbb{T}).$$

We may focus on the first component, the second one being analogous. We set

$$H_1(\lambda,b,\mathbf{m})(w)\triangleq\sum_{n=2}^{\infty}\frac{\mathscr{A}_n}{n\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}w^n,\quad H_2(w)\triangleq\sum_{n=2}^{\infty}\frac{\mathscr{B}_n}{n}w^n$$

and

$$\mathscr{G}_1(\lambda,b,\mathbf{m})(w)\triangleq\sum_{n=2}^{\infty}I_{n\mathbf{m}}(\lambda b)K_{n\mathbf{m}}(\lambda b)w^n,\quad\mathscr{G}_2(\lambda,b,\mathbf{m})(w)\triangleq\sum_{n=2}^{\infty}\frac{\Lambda_{n\mathbf{m}}(\lambda,b)}{\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}w^n.$$

If we denote $\widetilde{h}_1(w)\triangleq h_1(w)-a_1\overline{w}^{\mathbf{m}-1}$, then we can write

$$\begin{aligned}\widetilde{h}_1(w)=&C_1(\lambda,b,\mathbf{m})wH_1(\lambda,b,\mathbf{m})\left(\overline{w}^{\mathbf{m}}\right)\\ &+C_2(b,\mathbf{m})w(\mathscr{G}_1(\lambda,b,\mathbf{m})*H_1(\lambda,b,\mathbf{m}))\left(\overline{w}^{\mathbf{m}}\right)\\ &+C_2(b,\mathbf{m})w(\mathscr{G}_2(\lambda,b,\mathbf{m})*H_2)\left(\overline{w}^{\mathbf{m}}\right),\end{aligned}\tag{18.4}$$

where

$$\begin{aligned}C_1(\lambda,b,\mathbf{m})&\triangleq\frac{\Lambda_1(\lambda,b)-b\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)-bI_1(\lambda b)K_1(\lambda b)}{\mathbf{m}},\\ C_2(b,\mathbf{m})&\triangleq-\frac{b}{\mathbf{m}}.\end{aligned}$$

The convolution must be understood in the usual sense, that is

$$\forall w=e^{\mathrm{i}\theta}\in\mathbb{T},\quad f*g(w)=\fint_{\mathbb{T}}f(\tau)g(w\overline{\tau})\frac{d\tau}{\tau}=\frac{1}{2\pi}\int_0^{2\pi}f\left(e^{\mathrm{i}\eta}\right)g\left(e^{\mathrm{i}(\theta-\eta)}\right)d\eta.$$

We shall use the classical convolution law

$$L^1(\mathbb{T}) * C^{1+\alpha}(\mathbb{T}) \hookrightarrow C^{1+\alpha}(\mathbb{T}). \tag{18.5}$$

By using the decay property of the product I_nK_n and the asymptotic (C.13), we have

$$\|\mathscr{G}_1(\lambda,b,\mathbf{m})\|_{L^1(\mathbb{T})} \lesssim \|\mathscr{G}_1(\lambda,b,\mathbf{m})\|_{L^2(\mathbb{T})} = \left(\sum_{n=2}^{\infty} I_{n\mathbf{m}}^2(\lambda b)K_{n\mathbf{m}}^2(\lambda b)\right)^{\frac{1}{2}} \leqslant \frac{1}{2\mathbf{m}}\left(\sum_{n=2}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}} < \infty.$$

We also have

$$\|\mathscr{G}_2(\lambda,b,\mathbf{m})\|_{L^1(\mathbb{T})} \leqslant \|\mathscr{G}_2(\lambda,b,\mathbf{m})\|_{L^\infty(\mathbb{T})} \lesssim \sum_{n=2}^{\infty} b^{n\mathbf{m}} < \infty.$$

Hence

$$\Big(\mathscr{G}_1(\lambda,b,\mathbf{m}),\mathscr{G}_2(\lambda,b,\mathbf{m})\Big) \in \left(L^1(\mathbb{T})\right)^2. \tag{18.6}$$

We now prove that H_1 and H_2 are with regularity $C^{1+\alpha}(\mathbb{T})$.

▶ *Regularity of H_2 :*

First observe that by Cauchy-Schwarz inequality and the embedding $C^\alpha(\mathbb{T})(\hookrightarrow L^\infty(\mathbb{T})) \hookrightarrow L^2(\mathbb{T})$, we have

$$\begin{aligned}\|H_2\|_{L^\infty(\mathbb{T})} &\leqslant \sum_{n=2}^{\infty}\frac{|\mathscr{B}_n|}{n}\\ &\leqslant \left(\sum_{n=2}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\left(\sum_{n=2}^{\infty}|\mathscr{B}_n|^2\right)^{\frac{1}{2}}\\ &\lesssim \|g_2\|_{L^2(\mathbb{T})}\\ &\lesssim \|g_2\|_{C^\alpha(\mathbb{T})}.\end{aligned} \tag{18.7}$$

We now have to prove that $H_2' \in C^\alpha(\mathbb{T})$. We show that it coincides, up to slight modifications, with g_2 which is of regularity $C^\alpha(\mathbb{T})$. For that purpose, we show that we can differentiate H_2 term by term.

We denote $(S_N)_{N\geqslant 2}$ (resp. $(R_N)_{N\geqslant 2}$) the sequence of the partial sums (resp. the sequence of the remainders) of the series of functions H_2. One has

$$R_N(w) = \sum_{n=N+1}^{\infty}\frac{\mathscr{B}_n}{n}w^n.$$

Using Cauchy-Schwarz inequality, we obtain similarly to (18.7)

$$\|R_N\|_{L^\infty(\mathbb{T})} \leqslant \left(\sum_{n=N+1}^{\infty}\frac{1}{n^2}\right)^{\frac{1}{2}}\|g_2\|_{C^\alpha(\mathbb{T})} \underset{N\to\infty}{\longrightarrow} 0.$$

Hence

$$\|S_N - H_2\|_{L^\infty(\mathbb{T})} \underset{N\to\infty}{\longrightarrow} 0. \tag{18.8}$$

One has

$$S_N'(w) = \overline{w}\sum_{n=2}^{N}\mathscr{B}_n w^n \triangleq \overline{w}g_2^N(w).$$

We set

$$g_2^+(w) \triangleq \sum_{n=2}^{\infty}\mathscr{B}_n w^n.$$

By continuity of the Szegö projection defined by

$$\Pi : \sum_{n\in\mathbb{Z}} \alpha_n w^n \mapsto \sum_{n\in\mathbb{N}} \alpha_n w^n$$

from $C^\alpha(\mathbb{T})$ into itself (see [83] for more details) added to the fact that $g_2 \in C^\alpha(\mathbb{T})$, we deduce that $g_2^+ \in C^\alpha(\mathbb{T})$. Applying Bernstein Theorem of Fourier series gives that g_2^+ is the uniform limit of its Fourier series, namely

$$\|S_N' - \overline{w} g_2^+\|_{L^\infty(\mathbb{T})} \underset{N\to\infty}{\longrightarrow} 0. \tag{18.9}$$

Gathering (18.8) and (18.9), we conclude that we can differentiate H_2 term by term and get

$$H_2'(\omega) = \overline{w} g_2^+(w).$$

As a consequence,

$$H_2 \in C^{1+\alpha}(\mathbb{T}). \tag{18.10}$$

► *Regularity of* $H_1(\lambda, b, \mathbf{m})$ *:*

By using (17.12) and (C.18), we have the asymptotic expansion

$$\det\left(M_{n\mathbf{m}}(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b))\right) \underset{n\to\infty}{=} d_\infty(\lambda, b, \mathbf{m}) + \frac{\widetilde{d}_\infty(\lambda, b, \mathbf{m})}{n} + O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right), \tag{18.11}$$

with, using Proposition 17.2,

$$\begin{aligned} d_\infty(\lambda, b, \mathbf{m}) &\triangleq \left[I_1(\lambda)K_1(\lambda) - \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) - b\Lambda_1(\lambda, b)\right]\left[\Lambda_1(\lambda, b) - b\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) - bI_1(\lambda b)K_1(\lambda b)\right] \\ &= b\left[\boldsymbol{\Omega}_\infty^{+}(\lambda, b) - \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\right]\left[\boldsymbol{\Omega}_\infty^{-}(\lambda, b) - \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\right] \\ &< 0 \end{aligned}$$

and, using (17.16),

$$\begin{aligned} \widetilde{d}_\infty(\lambda, b, \mathbf{m}) &\triangleq \frac{b}{2\mathbf{m}}\left[I_1(\lambda)K_1(\lambda) - \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) - b\Lambda_1(\lambda, b)\right] - \frac{1}{2\mathbf{m}}\left[\Lambda_1(\lambda, b) - b\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) - bI_1(\lambda b)K_1(\lambda b)\right] \\ &= \frac{b\left(I_1(\lambda)K_1(\lambda) + I_1(\lambda b)K_1(\lambda b)\right) - (1+b^2)\Lambda_1(\lambda, b)}{2\mathbf{m}} \\ &= \frac{\delta_\infty(\lambda, b)}{2\mathbf{m}}. \end{aligned}$$

We denote

$$r_n(\lambda, b, \mathbf{m}) \triangleq \det\left(M_{n\mathbf{m}}(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b))\right) - d_\infty(\lambda, b, \mathbf{m}) \underset{n\to\infty}{=} \frac{\widetilde{d}_\infty(\lambda, b, \mathbf{m})}{n} + O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right). \tag{18.12}$$

We can write

$$\frac{1}{\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)} = \frac{r_n^2(\lambda, b, \mathbf{m})}{d_\infty^2(\lambda, b, \mathbf{m})\det\Big(M_{n\mathbf{m}}\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big)} - \frac{r_n(\lambda, b, \mathbf{m})}{d_\infty^2(\lambda, b, \mathbf{m})} + \frac{1}{d_\infty(\lambda, b, \mathbf{m})}.$$

Thus we can write

$$\begin{aligned}
H_1(\lambda,b,\mathbf{m})(w) &= \frac{1}{d_\infty^2(\lambda,b,\mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n r_n^2(\lambda,b,\mathbf{m})}{n\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\mathbf{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}w^n - \frac{1}{d_\infty^2(\lambda,b,\mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n r_n(\lambda,b,\mathbf{m})}{n}w^n\\
&\quad + \frac{1}{d_\infty(\lambda,b,\mathbf{m})}\sum_{n=2}^{\infty}\frac{\mathscr{A}_n}{n}w^n\\
&\triangleq \frac{1}{d_\infty^2(\lambda,b,\mathbf{m})}H_{1,1}(\lambda,b,\mathbf{m})(w) - \frac{1}{d_\infty^2(\lambda,b,\mathbf{m})}H_{1,2}(\lambda,b,\mathbf{m})(w) \qquad (18.13)\\
&\quad + \frac{1}{d_\infty(\lambda,b,\mathbf{m})}H_{1,3}(\lambda,b,\mathbf{m})(w). \qquad (18.14)
\end{aligned}$$

Now since $(\mathscr{A}_n)_{n\in\mathbb{N}^*}\in l^2(\mathbb{N}^*)\subset l^\infty(\mathbb{N}^*)$, we have

$$\left|\frac{\mathscr{A}_n r_n^2(\lambda,b,\mathbf{m})}{n\det\Big(M_{n\mathbf{m}}\big(\lambda,b,\mathbf{\Omega}_{\mathbf{m}}^{\pm}(\lambda,b)\big)\Big)}\right| \underset{n\to\infty}{=} O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right).$$

By using the link regularity/decay of Fourier coefficients, we deduce that

$$H_{1,1}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \qquad (18.15)$$

Similarly to (18.10), we can obtain

$$H_{1,3}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \qquad (18.16)$$

By the same method, we can also differentiate term by term $H_{1,2}(\lambda,b,\mathbf{m})$ and obtain

$$\forall w\in\mathbb{T},\quad \big(H_{1,2}(\lambda,b,\mathbf{m})\big)'(w)=\overline{w}\sum_{n=2}^{\infty}\mathscr{A}_n r_n(\lambda,b,\mathbf{m})w^n.$$

Notice that from (18.12), we can write

$$\forall w\in\mathbb{T},\quad w\big(H_{1,2}(\lambda,b,\mathbf{m})\big)'(w)=\widetilde{d}_\infty(\lambda,b,\mathbf{m})H_{1,3}(\lambda,b,\mathbf{m})+(\mathscr{C}*g_1^+)(w),$$

where

$$\forall w\in\mathbb{T},\quad g_1^+(w)\triangleq\sum_{n=2}^{\infty}\mathscr{A}_n w^n\quad\text{and}\quad \mathscr{C}(w)\triangleq\sum_{n=2}^{\infty}\mathscr{C}_n w^n\quad\text{with}\quad \mathscr{C}_n=O_{\lambda,b,\mathbf{m}}\left(\frac{1}{n^3}\right).$$

Using again the continuity of the Szegö projection, we have

$$g_1^+\in C^{1+\alpha}(\mathbb{T})\subset L^\infty(\mathbb{T})\subset L^1(\mathbb{T})\quad\text{and}\quad \mathscr{C}\in C^{1+\alpha}(\mathbb{T}). \qquad (18.17)$$

Using (18.16), (18.17) and (18.5), we deduce that

$$\big(H_{1,2}(\lambda,b,\mathbf{m})\big)'\in C^{1+\alpha}(\mathbb{T})\subset C^{\alpha}(\mathbb{T}).$$

Thus

$$H_{1,2}(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \qquad (18.18)$$

Gathering (18.15), (18.18) and (18.16), we conclude that

$$H_1(\lambda,b,\mathbf{m})\in C^{1+\alpha}(\mathbb{T}). \qquad (18.19)$$

Putting together (18.4), (18.19), (18.10), (18.6) and (18.5), we finally conclude

$$\widetilde{h}_1 \in C^{1+\alpha}(\mathbb{T}).$$

(iv) $\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)$ is a simple eigenvalue since $\Delta_{\mathbf{m}}(\lambda, b) > 0$. From (16.11) and (16.12), we deduce

$$\begin{cases} \partial_\Omega d_{(f_1,f_2)} G_1\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)(h_1, h_2)(w) = \operatorname{Im}\Big\{\overline{h_1'(w)} + \overline{w} h_1(w)\Big\} = -\displaystyle\sum_{n=0}^{\infty} n\mathbf{m} a_n e_{n\mathbf{m}}(w) \\ \partial_\Omega d_{(f_1,f_2)} G_2\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)(h_1, h_2)(w) = b\operatorname{Im}\Big\{\overline{h_2'(w)} + \overline{w} h_2(w)\Big\} = -\displaystyle\sum_{n=0}^{\infty} bn\mathbf{m} b_n e_{n\mathbf{m}}(w). \end{cases}$$

Thus,

$$\partial_\Omega d_{(f_1,f_2)} G\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)(v_{0,\mathbf{m}})(w) = \mathbf{m} \begin{pmatrix} \Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b) + \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big] \\ b\Lambda_{\mathbf{m}}(\lambda, b) \end{pmatrix} e_{\mathbf{m}}(w).$$

Notice that the previous expression belongs to the range of $d_{(f_1,f_2)} G\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)$ if and only if the vector

$$\begin{pmatrix} \Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b) + \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big] \\ b\Lambda_{\mathbf{m}}(\lambda, b) \end{pmatrix}$$

is a scalar multiple of one column of the matrix $M_{\mathbf{m}}\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big)$. This occurs if and only if

$$\Big(\Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b) + \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big]\Big)^2 - b^2 \Lambda_{\mathbf{m}}^2(\lambda, b) = 0. \tag{18.20}$$

Putting (18.20) together with $\det\Big(M_{\mathbf{m}}\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big)\Big) = 0$ implies

$$\Big(\Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b) + \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big]\Big)\Big((1 - b^2)\Lambda_1(\lambda, b) + b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) - \boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)\big] - 2b\boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\Big) = 0.$$

Now remark that the above equation is equivalent to

$$\Lambda_1(\lambda, b) - b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda b) + \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b)\big] = 0 \quad \text{or} \quad \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b) = \frac{1}{2b}\Big((1 - b^2)\Lambda_1(\lambda, b) + b\big[\boldsymbol{\Omega}_{\mathbf{m}}(\lambda) - \boldsymbol{\Omega}_{\mathbf{m}}(\lambda b)\big]\Big).$$

Since $b \neq 0$ and $\Lambda_{\mathbf{m}}(\lambda, b) \neq 0$, then in view of (18.20), the first equation can't be solved. Then, necessary, the second equation must be satisfied. But we notice that it corresponds to a multiple eigenvalue ($\Delta_{\mathbf{m}}(\lambda, b) = 0$), which is excluded here. Therefore, we conclude that

$$\partial_\Omega d_{(f_1,f_2)} G\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)(v_{0,\mathbf{m}}) \notin R\Big(d_{(f_1,f_2)} G\big(\lambda, b, \boldsymbol{\Omega}_{\mathbf{m}}^{\pm}(\lambda, b), 0, 0\big)\Big).$$

This ends the proof of Proposition 18.1. □

The previous proposition allows to construct, for any fixed $\lambda > 0$, $b \in (0, 1)$, $\alpha \in (0, 1)$ and $\mathbf{m} \geqslant N(\lambda, b)$ two branches of $\mathbf{m}$-fold doubly-connected V-states with regularity $C^{1+\alpha}$ bifurcating from the annulus A_b at the angular velocities $\Omega_{\mathbf{m}}^{\pm}(\lambda, b)$ for the $(QGSW)_\lambda$ equations. Actually, we have the following better result for the regularity of the boundary.

Lemma 18.1. *Let $\lambda > 0$, $b \in (0, 1)$ and $\mathbf{m} \geqslant N(\lambda, b)$. Consider a $\mathbf{m}$-fold doubly-connected V-state close to A_b for $(QGSW)_\lambda$ equations, rotating with an angular velocity Ω and associated with an initial domain $D_0 = D_1 \setminus \overline{D_2}$, where D_1 and D_2 are simply-connected domains satisfying $\overline{D_2} \subset D_1$ and parametrized by* *the following conformal mappings*

$$\Phi_1(w) = w + f_1(w), \qquad \Phi_2(w) = bw + f_2(w), \qquad f_1, f_2 \in B^{1+\alpha}_{r,\mathbf{m}}.$$

If $r > 0$ is small enough, then the boundaries ∂D_1 and ∂D_2 are analytic.

Proof. The proof is done in the spirit of [88, Sec. 5.4] by applying [113, Thm. 3.1']. We highlight that the positive number r quantifies the smallness of f_1 and f_2 in the $C^{1+\alpha}$ topology. We mention that (16.5) can also be written as follows

$$\tfrac{\Omega}{2}\partial_s\big|\gamma(0,s)\big|^2 = \partial_s\Big(\boldsymbol{\Psi}\big(0,\gamma(0,s)\big)\Big), \tag{18.21}$$

where $\boldsymbol{\Psi}$ is the velocity potential given by

$$\mathbf{v}(t,z) = \nabla^{\perp}\boldsymbol{\Psi}(t,z) = 2\mathrm{i}\partial_{\overline{z}}\boldsymbol{\Psi}(t,z), \qquad (\Delta - \lambda^2)\boldsymbol{\Psi}(t,z) = \mathbf{1}_{D_t}(z). \tag{18.22}$$

Therefore, integrating the relation (18.21), there exists for each $j \in \{1,2\}$ a constant $c_j \in \mathbb{R}$ such that

$$\forall z \in \partial D_j, \quad u_j(z) := \boldsymbol{\Psi}(0,z) - \tfrac{\Omega}{2}|z|^2 - c_j = 0.$$

Fix $j \in \{1,2\}$. By compactness of ∂D_j, there exist $M \in \mathbb{N}^*$, $(x_{k,j})_{1\leqslant k\leqslant M} \in (\partial D_j)^M$ and $\varepsilon > 0$ (small) such that we can write

$$\partial D_j \subset \bigcup_{k=1}^{M} B(x_{k,j},\varepsilon), \qquad \text{with} \qquad B(x_{k,j},\varepsilon) \cap \partial D_{3-j} = \varnothing.$$

Fix $k \in [\![1,M]\!]$ and denote

$$\Gamma_{k,j} := B(x_{k,j},\varepsilon) \cap \partial D_j, \qquad \mathcal{O}^-_{k,j} := B(x_{k,j},\varepsilon) \cap D_0, \qquad \mathcal{O}^+_{k,j} := B(x_{k,j},\varepsilon) \cap \big(\mathbb{R}^2 \setminus D_0\big).$$

Solving the Helmoltz problem (18.22) as in [101], the stream function writes

$$\boldsymbol{\Psi}(0,z) = -\frac{1}{2\pi}\int_{D_0} K_0(\lambda|z-\xi|)dA(\xi),$$

where dA denotes the planar Lebesgue measure. From (C.7)-(C.2), we can write

$$\begin{aligned}\boldsymbol{\Psi}(0,z) &= \frac{1}{2\pi}\int_{D_0}\log(|z-\xi|)dA(\xi) + \int_{D_0}\mathtt{F}(|z-\xi|)dA(\xi)\\ &:= \boldsymbol{\Psi}_1(z) + \boldsymbol{\Psi}_2(z).\end{aligned}$$

where $\mathtt{F}, \mathtt{F}'$ are bounded at 0 and $\mathtt{F}''$ is integrable at the origin. Notice that $\boldsymbol{\Psi}_1$ corresponds to the classical Euler velocity potential. Since D_0 is of regularity $C^{1+\alpha}$ then one can classically prove that

$$\boldsymbol{\Psi}_1 \in C^{1+\alpha}\big(\mathbb{R}^2, \mathbb{R}\big) \cap C^{2+\alpha}\big(\overline{D_0}, \mathbb{R}\big) \cap C^{2+\alpha}\big(\mathbb{R}^2 \setminus D_0, \mathbb{R}\big).$$

For instance, the $C^{1+\alpha}$ regularity is obtained by using [75, Exercise 4.8 (a)]. As for the $C^{2+\alpha}$ regularity, one may use in particular the "Main Lemma" in [124] applied to the Calderón-Zygmund type operator $\mathbf{1}_{D_0} \mapsto \nabla\nabla^{\perp}\boldsymbol{\Psi}_1$. The term $\boldsymbol{\Psi}_2$ being less singular, we get

$$\boldsymbol{\Psi}(0,\cdot) \in C^{1+\alpha}\big(\mathbb{R}^2, \mathbb{R}\big) \cap C^{2+\alpha}\big(\overline{D_0}, \mathbb{R}\big) \cap C^{2+\alpha}\big(\mathbb{R}^2 \setminus D_0, \mathbb{R}\big)$$

and then

$$u_j \in C^1\big(B(x_{k,j},\varepsilon)\,,\,\mathbb{R}\big) \cap C^2\big(\mathcal{O}_{k,j}^- \cup \Gamma_{k,j}\,,\,\mathbb{R}\big) \cap C^2\big(\mathcal{O}_{k,j}^+ \cup \Gamma_{k,j}\,,\,\mathbb{R}\big).$$

One can easily find from (18.22) that

$$\begin{aligned}
\forall z \in \mathcal{O}_{k,j}^+, \quad & 0 = \mathscr{F}_j(z, u_j, Du_j, D^2u_j) := (\Delta - \lambda^2)u_j(z) - \tfrac{\lambda^2}{2}\Omega|z|^2 - \lambda^2 c_j + 2\Omega,\\
\forall z \in \mathcal{O}_{k,j}^-, \quad & 0 = \mathscr{G}_j(z, u_j, Du_j, D^2u_j) := (\Delta - \lambda^2)u_j(z) - \tfrac{\lambda^2}{2}\Omega|z|^2 - \lambda^2 c_j + 2\Omega - 1.
\end{aligned}$$

Observe that the functions $\mathscr{F}_j$ and $\mathscr{G}_j$ are analytic. Thus it remains to prove that

$$\forall z \in \partial D_j, \quad \nabla u_j(z) \cdot \mathbf{n}_j(z) \neq 0, \tag{18.23}$$

where $\mathbf{n}_j$ is a normal unitary vector to ∂D_j. We can write

$$\begin{aligned}
\nabla u_j(z) \cdot \mathbf{n}_j(z) &= \nabla \mathbf{\Psi}(0,z) \cdot \mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z)\\
&= \nabla^{\perp} \mathbf{\Psi}(0,z) \cdot i\mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z)\\
&= \mathbf{v}(0,z) \cdot i\mathbf{n}_j(z) - \Omega z \cdot \mathbf{n}_j(z).
\end{aligned} \tag{18.24}$$

The normal unitary vector can be expressed as follows in terms of the conformal mapping

$$\mathbf{n}_j(z) = w\frac{\Phi_j'(w)}{|\Phi_j'(w)|} \qquad \text{if} \qquad z = \Phi_j(w), \qquad w \in \mathbb{T}.$$

On one hand, denoting $b_1 := 1$ and $b_2 := b$, we have for $z = \Phi_j(w) \in \partial D_j$,

$$\begin{aligned}
z \cdot \mathbf{n}_j(z) &= \mathrm{Re}\left\{\Phi_j(w)\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\right\}\\
&= b_j + \mathrm{Re}\left\{f_j(w)\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|} + b_j\left(\frac{b_j + \overline{f_j'(w)}}{|b_j + f_j'(w)|} - 1\right)\right\}\\
&= b_j + O(r).
\end{aligned} \tag{18.25}$$

On the other hand,

$$\begin{aligned}
\mathbf{v}(0,z) \cdot i\mathbf{n}_j(z) &= \mathrm{Re}\left\{\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\left(\fint_{\mathbb{T}} \Phi_1'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - \fint_{\mathbb{T}} \Phi_2'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\}\\
&= \mathrm{Re}\left\{\overline{w}\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_j w - \tau|\big)d\tau - b\fint_{\mathbb{T}} K_0\big(\lambda|b_j w - b\tau|\big)d\tau\right)\right\} + \mathscr{J}_1 + \mathscr{J}_2 + \mathscr{J}_3,
\end{aligned} \tag{18.26}$$

where

$$\begin{aligned}
\mathscr{J}_1 &:= \mathrm{Re}\left\{\overline{w}\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_j w - \tau|\big)d\tau - \fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau\right)\right\}\\
&\quad - \mathrm{Re}\left\{\overline{w}b\left(\fint_{\mathbb{T}} K_0\big(\lambda|b_j w - b\tau|\big)d\tau - \fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\},\\
\mathscr{J}_2 &:= \mathrm{Re}\left\{\overline{w}\left(\frac{b_j + \overline{f_j'(w)}}{|b_j + f_j'(w)|} - 1\right)\left(\fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - b\fint_{\mathbb{T}} K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\},\\
\mathscr{J}_3 &:= \mathrm{Re}\left\{\overline{w}\frac{\overline{\Phi_j'(w)}}{|\Phi_j'(w)|}\left(\fint_{\mathbb{T}} f_1'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_1(\tau)|\big)d\tau - \fint_{\mathbb{T}} f_2'(\tau)K_0\big(\lambda|\Phi_j(w) - \Phi_2(\tau)|\big)d\tau\right)\right\}.
\end{aligned}$$

We shall now prove that the terms $\mathscr{J}_1$, $\mathscr{J}_2$ and $\mathscr{J}_3$ are small. Let us start with $\mathscr{J}_3$. Recalling the notation (16.13), one has

$$|\mathscr{I}_3| \leqslant \|\mathcal{T}_{1j} f_1'\|_{L^\infty(\mathbb{T})} + \|\mathcal{T}_{2j} f_2'\|_{L^\infty(\mathbb{T})}. \tag{18.27}$$

From (16.16), we get

$$\forall (i,j) \in \{1,2\}^2, \qquad i \neq j \quad \Rightarrow \quad \|\mathcal{T}_{ij} f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i\|_{C^{1+\alpha}(\mathbb{T})} \lesssim r. \tag{18.28}$$

Now fix $i \in \{1,2\}$ and denote

$$\mathcal{K}_i(w,\tau) := K_0(\lambda|\Phi_i(w) - \Phi_i(\tau)|).$$

We mention that the triangle inequality and the mean value theorem imply that Φ_i is bi-Lipschitz, namely

$$(1-r)|w-\tau| \leqslant |\Phi_i(w) - \Phi_i(\tau)| \leqslant (1+r)|w-\tau|. \tag{18.29}$$

Recall that K_0 behaves like a logarithm at 0 and using (C.7) we can write

$$K_0'(z) = -\frac{1}{z} + \mathtt{G}(z), \qquad \mathtt{G} \text{ bounded at } 0. \tag{18.30}$$

Therefore, for any $\delta \in (0,1)$, we have

$$|\mathcal{K}_i(w,\tau)| \lesssim \frac{1}{|w-\tau|^\delta} \quad \text{and} \quad |\partial_w \mathcal{K}_i(w,\tau)| \lesssim \frac{1}{|w-\tau|^{1+\delta}}.$$

Thus, applying [83, Lem. 1], we infer

$$\|\mathcal{T}_{ii} f_i'\|_{L^\infty(\mathbb{T})} \lesssim \|f_i'\|_{L^\infty(\mathbb{T})} \lesssim r. \tag{18.31}$$

Putting together (18.27), (18.28) and (18.31), we deduce

$$|\mathscr{I}_3| \lesssim r. \tag{18.32}$$

From the previous computations, one also obtains

$$|\mathscr{I}_2| \lesssim r. \tag{18.33}$$

As for $\mathscr{I}_1$, we may use Taylor formula to write

$$\begin{aligned} & K_0\big(\lambda|\Phi_j(w) - \Phi_i(\tau)|\big) - K_0\big(\lambda|b_j w - b_i \tau|\big) \\ & \quad = \lambda\big(|\Phi_j(w) - \Phi_i(\tau)| - |b_j w - b_i\tau|\big) \int_0^1 K_0'\Big(\lambda|b_j w - b_i \tau| + \lambda t\big(|\Phi_j(w) - \Phi_i(\tau)| - |b_j w - b_i \tau|\big)\Big) dt. \end{aligned}$$

The triangular inequality and the mean value theorem imply

$$\Big||\Phi_j(w) - \Phi_i(\tau)| - |b_j w - b_i \tau|\Big| \leqslant |f_j(w) - f_i(\tau)| \leqslant \begin{cases} 2r & \text{if } i \neq j, \\ r|w-\tau| & \text{if } i = j. \end{cases}$$

Hence using (18.30), (18.29), (16.14) and (16.15), we deduce

$$|\mathscr{I}_1| \lesssim r. \tag{18.34}$$

Moreover, according to the computations carried out in Proposition 17.1 (see also [101, Lem. 3.2]), we have

$$\overline{w}\fint_{\mathbb{T}} K_0(\lambda|z-\tau|)d\tau = I_1(\lambda)K_1(\lambda), \qquad \overline{w}\fint_{\mathbb{T}} K_0(\lambda|w-b\tau|)d\tau = \overline{w}\fint_{\mathbb{T}} K_0(\lambda|bw-\tau|)d\tau = \Lambda_1(\lambda,b). \tag{18.35}$$

Therefore, in view of (18.26), (18.32), (18.33), (18.34), (18.35) and Proposition 17.2, we infer

$$\begin{aligned}\forall z\in\partial D_1,\quad \mathbf{v}(0,z)\cdot\mathbf{in}_1(z) &= I_1(\lambda)K_1(\lambda)-b\Lambda_1(\lambda,b)+O(r)\\ &= \Omega_\infty^+(\lambda,b)+O(r)\end{aligned} \tag{18.36}$$

and

$$\begin{aligned}\forall z\in\partial D_2,\quad \mathbf{v}(0,z)\cdot\mathbf{in}_2(z) &= \Lambda_1(\lambda,b)-bI_1(\lambda b)K_1(\lambda b)+O(r)\\ &= b\Omega_\infty^-(\lambda,b)+O(r).\end{aligned} \tag{18.37}$$

Combining (18.24), (18.25), (18.36) and (18.37), we deduce by triangular inequality

$$\forall z\in\partial D_1,\quad |\nabla u_1(z)\cdot n_1(z)|\geqslant|\Omega_\infty^+(\lambda,b)-\Omega|-Cr$$

and

$$\forall z\in\partial D_2,\quad |\nabla u_2(z)\cdot n_2(z)|\geqslant b|\Omega_\infty^-(\lambda,b)-\Omega|-Cr.$$

The Crandall-Rabinowitz Theorem implies that Ω is close to $\Omega_{\mathbf{m}}^\pm(\lambda,b)$. Hence, according to Proposition 17.2, we can say

$$\Omega_\infty^\pm(\lambda,b)-\Omega\neq 0.$$

Thus, up to take r sufficiently small, we get (18.23). Consequently, $\Gamma_{k,j}$ is analytic from which we deduce by reconstruction that ∂D_j is also analytic. $\square$

APPENDICES

A Functional setting and technical lemmata

In this appendix, we set up the general topological framework for both the functions and the operators classes that are used in Parts I and II. We also provide some classical results on the law products, composition rule, Toeplitz operators, etc... First we begin by presenting some notations.

Notations. Along this document we shall make use of the following parameters and sets.

- We denote by
$$\mathbb{N} \triangleq \{0,1,\cdots\}, \qquad \mathbb{Z} \triangleq \{\cdots,-1,0,1,\cdots\}$$
the set of natural numbers and the set of integers, respectively, and we set
$$\mathbb{N}^* \triangleq \mathbb{N}\backslash\{0\}, \qquad \mathbb{Z}^* \triangleq \mathbb{Z}\backslash\{0\}.$$
The set of real (resp. complex) numbers is denoted $\mathbb{R}$ (resp. $\mathbb{C}$). We also use the following notation
$$\mathbb{R}_+^* \triangleq (0,\infty), \qquad \mathbb{R}_+ \triangleq \mathbb{R}_+^* \cup \{0\}.$$
- The integer d is the number of excited frequencies that will generate the quasi-periodic solutions. This is the dimension of the space where lies the frequency vector $\omega \in \mathbb{R}^d$, that will be a perturbation of the equilibrium frequency vectors.
- The integer q is the index of regularity of our functions/operators with respect to the parameters λ or b and ω. It is chosen as
$$q \triangleq q_0 + 1,$$
with q_0 being the non-degeneracy index provided by Lemmata 5.5 or 11.5.
- The real parameters γ, τ_1 and τ_2 satisfy
$$0 < \gamma < 1, \quad \tau_2 > \tau_1 > d \tag{A.1}$$
and are linked to different Diophantine conditions, see for instance Propositions 7.2 and 7.5. The choice of τ_1 and τ_2 will be finally fixed in (8.64). We point out that the parameter γ appears in the weighted Sobolev spaces and will be fixed in Proposition 8.1 with respect to the rescaling parameter ε giving the smallness condition of the solutions around the equilibrium.
- The real number s is the Sobolev index regularity of the functions in the variables φ and θ. The index s will vary between s_0 and S,
$$S \geqslant s \geqslant s_0 > \tfrac{d+1}{2} + q + 2, \tag{A.2}$$
where S is a fixed large number.

- For a given continuous complex function $f:\mathbb{T}^n\to\mathbb{C}$, $n\geqslant 1$, $\mathbb{T}\triangleq\mathbb{R}/2\pi\mathbb{Z}$, we denote by
$$\int_{\mathbb{T}^n} f(x)dx\triangleq\frac{1}{(2\pi)^n}\int_{[0,2\pi]^n} f(x)dx. \tag{A.3}$$
Notice that $\mathbb{T}$ will also be considered as the unit circle, namely, the boundary of the unit disc $\mathbb{D}$.

- We denote by $(\mathbf{e}_{l,j})_{(l,j)\in\mathbb{Z}^d\times\mathbb{Z}}$ the Hilbert basis of the $L^2(\mathbb{T}^{d+1},\mathbb{C})$,
$$\mathbf{e}_{l,j}(\varphi,\theta)\triangleq e^{\mathrm{i}(l\cdot\varphi+j\theta)},$$
and we endow this space with the Hermitian inner product
$$\big\langle\rho_1,\rho_2\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})}\triangleq\int_{\mathbb{T}^{d+1}}\rho_1(\varphi,\theta)\overline{\rho_2(\varphi,\theta)}d\varphi d\theta. \tag{A.4}$$

A.1 Function spaces

We shall introduce the function spaces that will be frequently used along the document. They are given by weighted Sobolev spaces with respect to the parameter γ in (A.1). Given $\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})$, we may decompose it in Fourier expansion as
$$\rho=\sum_{(l,j)\in\mathbb{Z}^{d+1}}\rho_{l,j}\,\mathbf{e}_{l,j}\quad\text{ where }\quad\rho_{l,j}\triangleq\big\langle\rho,\mathbf{e}_{l,j}\big\rangle_{L^2(\mathbb{T}^{d+1},\mathbb{C})}.$$
Next, we introduce for $s\in\mathbb{R}$ the complex Sobolev space $H^s(\mathbb{T}^{d+1},\mathbb{C})$ by
$$H^s(\mathbb{T}^{d+1},\mathbb{C})\triangleq\Big\{\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\|\rho\|_{H^s}^2\triangleq\sum_{(l,j)\in\mathbb{Z}^{d+1}}\langle l,j\rangle^{2s}|\rho_{l,j}|^2<\infty\Big\},$$
where $\langle l,j\rangle\triangleq\max(1,|l|,|j|)$ with $|\cdot|$ denoting either the ℓ^1 norm in $\mathbb{R}^d$ or the absolute value in $\mathbb{R}$. The real Sobolev spaces can be viewed as closed sub-spaces of the preceding one,
$$\begin{aligned}H^s\triangleq H^s(\mathbb{T}^{d+1},\mathbb{R})&\triangleq\Big\{\rho\in H^s(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\,\rho(\varphi,\theta)=\overline{\rho(\varphi,\theta)}\Big\}\\&=\Big\{\rho\in H^s(\mathbb{T}^{d+1},\mathbb{C})\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\,\rho_{-l,-j}=\overline{\rho_{l,j}}\Big\}.\end{aligned}$$
We shall also make use of the following subspaces of H^s taking into account of some particular symmetries on odd and even functions,
$$\begin{aligned}H^s_{\textnormal{even}}&\triangleq\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\,\rho(-\varphi,-\theta)=\rho(\varphi,\theta)\Big\}\\&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\,\rho_{-l,-j}=\rho_{l,j}\Big\}\end{aligned}$$
and
$$\begin{aligned}H^s_{\textnormal{odd}}&\triangleq\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(\varphi,\theta)\in\mathbb{T}^{d+1},\,\rho(-\varphi,-\theta)=-\rho(\varphi,\theta)\Big\}\\&=\Big\{\rho\in H^s\quad\text{s.t.}\quad\forall\,(l,j)\in\mathbb{Z}^{d+1},\,\rho_{-l,-j}=-\rho_{l,j}\Big\}.\end{aligned}$$

For $N \in \mathbb{N}^*$, we define the cut-off frequency projectors on $H^s(\mathbb{T}^{d+1}, \mathbb{C})$ as follows

$$\Pi_N \rho \triangleq \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ \langle l,j\rangle \leqslant N}} \rho_{l,j}\mathbf{e}_{l,j} \quad \text{and} \quad \Pi_N^{\perp} = \mathrm{Id} - \Pi_N. \tag{A.5}$$

We shall also make use of the following mixed weighted Sobolev spaces.

$$\begin{aligned} W^{q,\infty,\gamma}(\mathcal{O}, H^s) &\triangleq \Big\{\rho : \mathcal{O} \to H^s \quad \text{s.t.} \quad \|\rho\|_{q,s}^{\gamma,\mathcal{O}} < \infty\Big\},\\ W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) &\triangleq \Big\{\rho : \mathcal{O} \to \mathbb{C} \quad \text{s.t.} \quad \|\rho\|_{q}^{\gamma,\mathcal{O}} < \infty\Big\}, \end{aligned}$$

where $\mu \in \mathcal{O} \mapsto \rho(\mu) \in H^s$ and

$$\begin{aligned} \|\rho\|_{q,s}^{\gamma,\mathcal{O}} &\triangleq \sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\ |\alpha|\leqslant q}} \gamma^{|\alpha|} \sup_{\mu\in\mathcal{O}} \|\partial_\mu^\alpha \rho(\mu,\cdot)\|_{H^{s-|\alpha|}},\\ \|\rho\|_{q}^{\gamma,\mathcal{O}} &\triangleq \sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\ |\alpha|\leqslant q}} \gamma^{|\alpha|} \sup_{\mu\in\mathcal{O}} |\partial_\mu^\alpha \rho(\mu)|. \end{aligned} \tag{A.6}$$

Note that a function $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$ can be written in the form

$$\rho(\mu, \varphi, \theta) = \sum_{(l,j)\in\mathbb{Z}^{d+1}} \rho_{l,j}(\mu)\mathbf{e}_{l,j}(\varphi, \theta).$$

Remark A.1.

- *From Sobolev embeddings, we obtain*

$$W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}) \hookrightarrow C^{q-1}(\mathcal{O}, \mathbb{C}).$$

- *The spaces* $\big(W^{q,\infty,\gamma}(\mathcal{O}, H^s), \|\cdot\|_{q,s}^{\gamma,\mathcal{O}}\big)$ *and* $\big(W^{q,\infty,\gamma}(\mathcal{O}, \mathbb{C}), \|\cdot\|_{q}^{\gamma,\mathcal{O}}\big)$ *are complete.*
- *For needs related to the use of the kernels of integral operators, we will have to duplicate the variable* θ*. Thus we may define the weighted Sobolev space* $W^{q,\infty,\gamma}(\mathcal{O}, H^s_{\varphi,\theta,\eta})$ *similarly as above and denote the corresponding norm by* $\|\cdot\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}$*.*

In the next lemma we collect some useful classical results dealing with various operations in weighted Sobolev spaces. The proofs are very close to those in [29, 28, 33], so we omit them.

Lemma A.1. *Let* (γ, q, d, s_0, s) *satisfying* (A.2)*, then the following assertions hold true.*

(i) Space translation invariance: Let $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$*, then for all* $\eta \in \mathbb{T}$*, the function* $(\varphi, \theta) \mapsto \rho(\varphi, \eta + \theta)$ *belongs to* $W^{q,\infty,\gamma}(\mathcal{O}, H^s)$*, and satisfies*

$$\|\rho(\cdot, \eta + \cdot)\|_{q,s}^{\gamma,\mathcal{O}} = \|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

(ii) Projectors properties: Let $\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$*, then for all* $N \in \mathbb{N}^*$ *and for all* $t \in \mathbb{R}_+^*$*,*

$$\|\Pi_N \rho\|_{q,s+t}^{\gamma,\mathcal{O}} \leqslant N^t \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \quad \text{and} \quad \|\Pi_N^{\perp} \rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant N^{-t} \|\rho\|_{q,s+t}^{\gamma,\mathcal{O}},$$

where the projectors are defined in (A.5)*.*

(iii) Interpolation inequality: Let $q < s_1 \leqslant s_3 \leqslant s_2$ *and* $\overline{\theta} \in [0, 1]$*, with* $s_3 = \overline{\theta} s_1 + (1 - \overline{\theta}) s_2$*.*

If $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_2})$*, then* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},H^{s_3})$ *and*

$$\|\rho\|_{q,s_3}^{\gamma,\mathcal{O}}\lesssim\left(\|\rho\|_{q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}}\left(\|\rho\|_{q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iv) Product laws:

(a) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q,s_0}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}+\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

(b) Let $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$ *and*

$$\|\rho_1\rho_2\|_{q}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q}^{\gamma,\mathcal{O}}.$$

(c) Let $(\rho_1,\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})\times W^{q,\infty,\gamma}(\mathcal{O},H^s)$*. Then* $\rho_1\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *and*

$$\|\rho_1\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\rho_1\|_{q}^{\gamma,\mathcal{O}}\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(v) Composition law: Let $f\in C^\infty(\mathcal{O}\times\mathbb{R},\mathbb{R})$ *and* $\rho_1,\rho_2\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *such that*

$$\|\rho_1\|_{q,s}^{\gamma,\mathcal{O}},\|\rho_2\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C_0$$

for an arbitrary constant $C_0>0$ *and define the pointwise composition*

$$\forall(\mu,\varphi,\theta)\in\mathcal{O}\times\mathbb{T}^{d+1},\quad f(\rho)(\mu,\varphi,\theta)\triangleq f(\mu,\rho(\mu,\varphi,\theta)).$$

Then $f(\rho_1)-f(\rho_2)\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$ *with*

$$\|f(\rho_1)-f(\rho_2)\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C(s,d,q,f,C_0)\|\rho_1-\rho_2\|_{q,s}^{\gamma,\mathcal{O}}.$$

(vi) Composition law 2: Let $f\in C^\infty(\mathbb{R},\mathbb{R})$ *with bounded derivatives. Let* $\rho\in W^{q,\infty,\gamma}(\mathcal{O},\mathbb{C})$*. Then*

$$\|f(\rho)-f(0)\|_{q}^{\gamma,\mathcal{O}}\leqslant C(q,d,f)\|\rho\|_{q}^{\gamma,\mathcal{O}}\left(1+\|\rho\|_{L^\infty(\mathcal{O})}^{q-1}\right).$$

This estimate is also true for $\gamma=1$*, corresponding to the classical Sobolev space* $W^{q,\infty}(\mathcal{O},\mathbb{C})$*.*

The following technical lemma turns out to be very useful in the study of the linearized operators.

Lemma A.2. *Let* (γ,q,d,s_0,s) *satifying* (A.2) *and* $f\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*.*
We consider the function $g:\mathcal{O}\times\mathbb{T}^d_\varphi\times\mathbb{T}_\theta\times\mathbb{T}_\eta\to\mathbb{C}$ *defined by*

$$g(\mu,\varphi,\theta,\eta)\triangleq\begin{cases}\frac{f(\mu,\varphi,\eta)-f(\mu,\varphi,\theta)}{\sin\left(\frac{\eta-\theta}{2}\right)} & \text{if } \theta\neq\eta\\ 2\partial_\theta f(\mu,\varphi,\theta) & \text{if } \theta=\eta.\end{cases}$$

Then

(i) $\forall k\in\mathbb{N},\quad\sup_{\eta\in\mathbb{T}}\|(\partial_\theta^k g)(*,\cdot,\blacksquare,\eta+\blacksquare)\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|\partial_\theta f\|_{q,s+k}^{\gamma,\mathcal{O}}\lesssim\|f\|_{q,s+k+1}^{\gamma,\mathcal{O}}.$

(ii) $\|g\|_{q,H^s_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\lesssim\|f\|_{q,s+1}^{\gamma,\mathcal{O}}.$

Proof. **(i)** Since the differentiation with respect to μ can be transported from g to f, then it is enough to check the result for $q=0$ and therefore we shall remove the dependence in μ. We start with expanding f into its Fourier series,

$$f(\varphi,\theta)=\sum_{(l,j)\in\mathbb{Z}^{d+1}}f_{l,j}\mathbf{e}_{l,j}(\varphi,\theta).$$

Thus, one can write

$$\begin{aligned}g(\varphi,\theta,\eta)&=\sum_{(l,j)\in\mathbb{Z}^{d+1}}f_{l,j}\frac{e^{\mathrm{i}j\eta}-e^{\mathrm{i}j\theta}}{\sin\left(\frac{\eta-\theta}{2}\right)}e^{\mathrm{i}l\cdot\varphi}\\&=2\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ j\neq0}}f_{l,j}e^{\mathrm{i}j\frac{\theta+\eta}{2}}\frac{\sin\left(j\frac{\eta-\theta}{2}\right)}{\sin\left(\frac{\eta-\theta}{2}\right)}e^{\mathrm{i}l\cdot\varphi}.\end{aligned}$$

We shall introduce the Chebychev polynomials of second kind $(U_n)_{n\in\mathbb{N}}$. They are defined for all $n\in\mathbb{N}$ by the following relation

$$\forall\theta\in\mathbb{R},\quad\sin(\theta)U_n(\cos(\theta))=\sin((n+1)\theta).$$

Using these polynomials, we obtain a new formulation for g, namely

$$g(\varphi,\theta,\eta)=2\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ j\neq0}}\frac{jf_{l,j}}{|j|}e^{\mathrm{i}j\frac{\theta+\eta}{2}}U_{|j|-1}\left(\cos\left(\frac{\theta-\eta}{2}\right)\right)e^{\mathrm{i}l\cdot\varphi}.$$

Differentiating in θ yields by Leibniz rule

$$\partial_\theta^k g(\varphi,\theta,\eta)=2\mathrm{i}\sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ j\neq0}}\sum_{m=0}^{k}\binom{k}{m}\frac{j^{k+1-m}f_{l,j}\mathrm{i}^{k-m}}{|j|2^{k-m}}e^{\mathrm{i}j\frac{\theta+\eta}{2}}\partial_\theta^m\left(U_{|j|-1}\left(\cos\left(\frac{\theta-\eta}{2}\right)\right)\right).\tag{A.7}$$

For all $j\in\mathbb{N}^*$, we consider the function f_j defined by

$$f_j(\theta)=U_{j-1}(\cos(\theta))=\frac{\sin(j\theta)}{\sin(\theta)}.$$

Notice that f_j is even and 2π-periodic. Thus, we restrict its study to the interval $[0,\pi]$. Also remark that

$$f_j(\pi-\theta)=(-1)^jf_j(\theta).$$

Hence, we restrict the study to the interval $[0,\frac{\pi}{2}]$. We first consider the function f_j on the interval $[\frac{\pi}{6},\frac{\pi}{2}]$. There, the function f_j writes as the quotient of two smooth functions with non vanishing denominator. Therefore, differentiating in θ leads to

$$\forall k\in\mathbb{N},\qquad\sup_{\theta\in[\frac{\pi}{6},\frac{\pi}{2}]}\left|\partial_\theta^kf_j(\theta)\right|\lesssim|j|^k.$$

Now we look at the behaviour close to 0 by looking at the function f_j restricted to $[0,\frac{\pi}{4}]$. Using Taylor Formula, we can write

$$\begin{aligned}f_j(\theta)&=\tfrac{\sin(j\theta)}{\theta}\times\tfrac{\theta}{\sin(\theta)}\\&=j\int_0^1\cos(tj\theta)dt\times\tfrac{\theta}{\sin(\theta)}.\end{aligned}$$

The function $\theta \mapsto \frac{\theta}{\sin(\theta)}$ being smooth on $[0, \frac{\pi}{4}]$, then differentiating in θ leads to

$$\forall k \in \mathbb{N}, \qquad \sup_{\theta \in [0, \frac{\pi}{4}]} \left|\partial_\theta^k f_j(\theta)\right| \lesssim |j|^{k+1}.$$

Combining the previous estimates, one gets

$$\forall j \in \mathbb{N}^*, \quad \forall k \in \mathbb{N}, \quad \sup_{\theta \in \mathbb{R}} \left|\partial_\theta^k \Big(U_{j-1}\left(\cos\left(\theta\right)\right)\Big)\right| \lesssim |j|^{k+1}. \tag{A.8}$$

Gathering (A.7) and (A.8), we deduce that

$$(\partial_\theta^k g)(\varphi, \theta, \eta + \theta) = \sum_{\substack{(l,j) \in \mathbb{Z}^{d+1} \\ j \neq 0}} c_{l,j,k}(\eta) \mathbf{e}_{l,j}(\varphi, \theta),$$

with

$$\sup_{\eta \in \mathbb{T}} |c_{l,j,k}(\eta)| \lesssim |j|^{k+1} |f_{l,j}|.$$

Therefore,

$$\begin{aligned}
\sup_{\eta \in \mathbb{T}} \|(\partial_\theta^k g)(\cdot, \bullet, \eta + \bullet)\|^2_{H^s_{\varphi,\theta}} &= \sum_{\substack{(l,j) \in \mathbb{Z}^{d+1} \\ j \neq 0}} \langle l, j\rangle^{2s} \sup_{\eta \in \mathbb{T}} |c_{l,j,k}(\eta)|^2 \\
&\lesssim \sum_{(l,j) \in \mathbb{Z}^{d+1}} \langle l, j\rangle^{2s} |j|^{2k+2} |f_{l,j}|^2 \\
&\lesssim \|\partial_\theta f\|^2_{H^{s+k}}.
\end{aligned}$$

This concludes the proof of Lemma A.2.

(ii) It suffices to prove the case $q = 0$. Recall the following classical norm estimate

$$\|g\|_{H^s_{\varphi,\theta,\eta}} \lesssim \|g\|_{H^s_{\varphi,\theta} L^2_\eta} + \|g\|_{L^2_\theta H^s_{\varphi,\eta}}. \tag{A.9}$$

By the translation invariance property

$$\begin{aligned}
\|g\|^2_{L^2_\theta H^s_{\varphi,\eta}} &= \int_0^{2\pi} \|g(\cdot, \theta + \bullet, \bullet)\|^2_{H^s_{\varphi,\eta}} d\theta \\
&\lesssim \sup_{\theta \in \mathbb{T}} \|g(*, \cdot, \theta + \bullet, \bullet)\|_{H^s_{\varphi,\eta}}.
\end{aligned}$$

Using the first point and the symmetry g in (η, θ) we obtain

$$\|g\|_{L^\infty_\theta H^s_{\varphi,\eta}} \lesssim \|f\|_{s+1}.$$

Introducing the Bessel potential J^s defined in Fourier by

$$\forall j \in \mathbb{Z}^d, \quad (J^s u)_j = \max(1, |j|)^s u_j, \tag{A.10}$$

a use of Fubini's Theorem implies

$$\|g\|_{H^s_{\varphi,\theta} L^2_\eta} = \|J^s_{\varphi,\theta}\, g\|_{L^2_\varphi L^2_\theta L^2_\eta} = \|J^s_{\varphi,\theta}\, g\|_{L^2_\eta L^2_\varphi L^2_\theta} = \|g\|_{L^2_\eta H^s_{\varphi,\theta}}.$$

Since g is symmetric in the variables θ and η, we get

$$\|g\|_{L^2_\eta H^s_{\varphi,\theta}} = \|g\|_{L^2_\theta H^s_{\varphi,\eta}}.$$

Combining the foregoing estimates leads to

$$\|g\|_{H^s_{\varphi,\theta,\eta}} \lesssim \|f\|_{s+1}.$$

This ends the proof of Lemma A.2. □

We now turn to the presentation of quasi-periodic symplectic change of variables needed for the reduction of the transport part of the linearized operators in the construction of the approximate inverses in the normal directions. Let $\beta : \mathcal{O} \times \mathbb{T}^{d+1} \to \mathbb{T}$ be a smooth function such that $\sup\limits_{\mu\in\mathcal{O}} \|\beta(\mu,\cdot,\centerdot)\|_{\mathrm{Lip}} < 1$ then the map

$$(\varphi,\theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \beta(\mu,\varphi,\theta)) \in \mathbb{T}^{d+1}$$

is a diffeomorphism with inverse having the form

$$(\varphi,\theta) \in \mathbb{T}^{d+1} \mapsto (\varphi, \theta + \widehat{\beta}(\mu,\varphi,\theta)) \in \mathbb{T}^{d+1}.$$

Moreover, one has the relation

$$y = \theta + \beta(\mu,\varphi,\theta) \Longleftrightarrow \theta = y + \widehat{\beta}(\mu,\varphi,y). \tag{A.11}$$

Define the operators

$$\mathscr{B} \triangleq (1 + \partial_\theta\beta)\mathcal{B}, \tag{A.12}$$

with

$$\mathcal{B}\rho(\mu,\varphi,\theta) \triangleq \rho\big(\mu,\varphi,\theta + \beta(\mu,\varphi,\theta)\big). \tag{A.13}$$

By straightforward computations we obtain

$$\mathscr{B}^{-1}\rho(\mu,\varphi,y) = \Big(1 + \partial_y\widehat{\beta}(\mu,\varphi,y)\Big)\rho\big(\mu,\varphi,y + \widehat{\beta}(\mu,\varphi,y)\big) \tag{A.14}$$

and

$$\mathcal{B}^{-1}\rho(\mu,\varphi,y) = \rho\big(\mu,\varphi,y + \widehat{\beta}(\mu,\varphi,y)\big).$$

The following lemma gives some elementary algebraic properties for $\mathcal{B}^{\pm1}$ and $\mathscr{B}^{\pm1}$.

Lemma A.3. *The following assertions hold true.*

(i) The action of $\mathscr{B}^{-1}$ on the derivative is given by

$$\mathscr{B}^{-1}\partial_\theta = \partial_\theta\mathcal{B}^{-1}.$$

(ii) The conjugation of the transport operator by $\mathscr{B}$ keeps the same structure

$$\mathscr{B}^{-1}\Big(\omega\cdot\partial_\varphi + \partial_\theta\big(V(\varphi,\theta)\,\cdot\,\big)\Big)\mathscr{B} = \omega\cdot\partial_\varphi + \partial_y\big(\mathscr{V}(\varphi,y)\,\cdot\,\big),$$

with

$$\mathscr{V}(\varphi,y) \triangleq \mathcal{B}^{-1}\Big(\omega\cdot\partial_\varphi\beta(\varphi,\theta) + V(\varphi,\theta)\big(1 + \partial_\theta\beta(\varphi,\theta)\big)\Big).$$

(iii) Denote by $\mathscr{B}^\star$ *the* $L^2_\theta(\mathbb{T})$*-adjoint of* $\mathscr{B}$*, then*

$$\mathscr{B}^\star = \mathcal{B}^{-1} \quad \textit{and} \quad \mathcal{B}^\star = \mathscr{B}^{-1}.$$

Now we shall state the following result proved in [64] for $q = 1$ and which can be obtained by induction for a general $q \in \mathbb{N}^*$ up to slight modifications. We also refer to [28, (A.2)].

Lemma A.4. *Let* (q, d, γ, s_0) *as in* (A.2)*. Let* $\beta \in W^{q,\infty,\gamma}\big(\mathcal{O}, H^\infty(\mathbb{T}^{d+1})\big)$ *such that*

$$\|\beta\|_{q,2s_0}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0, \tag{A.15}$$

with ε_0 *small enough. Then the following assertions hold true.*

(i) The linear operators $\mathcal{B}, \mathscr{B} : W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big) \to W^{q,\infty,\gamma}\big(\mathcal{O}, H^s(\mathbb{T}^{d+1})\big)$ *are continuous and invertible, with*

$$\forall s \geqslant s_0, \quad \|\mathcal{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}}\left(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\right) + C\|\beta\|_{q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \tag{A.16}$$

and

$$\forall s \geqslant s_0, \quad \|\mathscr{B}^{\pm 1}\rho\|_{q,s}^{\gamma,\mathcal{O}} \leqslant \|\rho\|_{q,s}^{\gamma,\mathcal{O}}\left(1 + C\|\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\right) + C\|\beta\|_{q,s+1}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}. \tag{A.17}$$

(ii) The functions β *and* $\widehat{\beta}$ *are linked through*

$$\forall s \geqslant s_0, \quad \|\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\|\beta\|_{q,s}^{\gamma,\mathcal{O}}. \tag{A.18}$$

(iii) Let $\beta_1, \beta_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^\infty(\mathbb{T}^{d+1}))$ *satisfying* (A.15)*. If we denote*

$$\Delta_{12}\beta \triangleq \beta_1 - \beta_2 \quad \text{and} \quad \Delta_{12}\widehat{\beta} \triangleq \widehat{\beta}_1 - \widehat{\beta}_2,$$

then they are linked through

$$\forall s \geqslant s_0, \quad \|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \leqslant C\left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}} + \|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j\in\{1,2\}}\|\beta_j\|_{q,s+1}^{\gamma,\mathcal{O}}\right). \tag{A.19}$$

Proof. **(i)-(ii)** For (A.16) and (A.18), we refer to [28, (A.2)] and [64, Lem. A.3.]. The estimate (A.17) is obtained from (A.16) and law product in Lemma A.1.
(iii) One has by Taylor Formula

$$\begin{aligned}\Delta_{12}\widehat{\beta}(y) &= \widehat{\beta}_1(y) - \widehat{\beta}_2(y)\\ &= \beta_2(y + \widehat{\beta}_2(y)) - \beta_1(y + \widehat{\beta}_1(y))\\ &= -\Delta_{12}\beta(y + \widehat{\beta}_2(y)) - \Delta_{12}\widehat{\beta}(y)\int_0^1 \partial_\theta\beta_1(y + \widehat{\beta}_1(y) - t\Delta_{12}\widehat{\beta}(y))dt.\end{aligned}$$

Hence

$$\Delta_{12}\widehat{\beta}(y) = \tfrac{-\mathcal{B}_2^{-1}\Delta_{12}\beta(y)}{1+\mathscr{I}(y)} \quad \text{with} \quad \mathscr{I}(y) \triangleq \int_0^1 \partial_\theta\beta_1(y + \widehat{\beta}_1(y) - t\Delta_{12}\widehat{\beta}(y))dt.$$

By composition estimate in Lemma A.1, one has

$$\left\|\tfrac{1}{1+\mathscr{I}}\right\|_{q,s}^{\gamma,\mathcal{O}} \lesssim 1 + \|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}}.$$

Thus, applying the law product in Lemma A.1 implies

$$\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} \lesssim \big(1+\|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}}\big)\,\|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}} + \big(1+\|\mathscr{I}\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\,\|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}.$$

Using (A.16), (A.18) and (A.15) yields

$$\begin{aligned}\|\mathcal{B}_2^{-1}\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}\left(1+\|\widehat{\beta}_2\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\|\widehat{\beta}_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\lesssim \|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\end{aligned}$$

and

$$\begin{aligned}\|\mathscr{I}\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}\left(1+\|\widehat{\beta}_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}\right)+\left(\|\widehat{\beta}_1\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\right)\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\\ &\lesssim \|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}.\end{aligned}$$

Putting together the foregoing estimates gives

$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}} &\leqslant C\left(1+\|\beta_1\|_{q,s+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\right)\big(1+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\,\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\quad+C\left(1+\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}+\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\right)\big(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s}^{\gamma,\mathcal{O}}\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\big).\end{aligned}\tag{A.20}$$

From the triangle inequality, (A.18) and (A.15), one has

$$\begin{aligned}\|\Delta_{12}\widehat{\beta}\|_{q,s_0}^{\gamma,\mathcal{O}} &\leqslant \|\widehat{\beta}_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\widehat{\beta}_2\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant \|\beta_1\|_{q,s_0}^{\gamma,\mathcal{O}}+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\\ &\leqslant 2\varepsilon_0.\end{aligned}$$

From Sobolev embeddings we infer that

$$\max_{j\in\{1,2\}}\|\beta_j\|_{q,s_0+1}^{\gamma,\mathcal{O}}\leqslant\max_{j\in\{1,2\}}\|\beta_j\|_{q,2s_0}^{\gamma,\mathcal{O}}\leqslant\varepsilon_0.$$

Thus, by choosing ε_0 small enough, we can ensure

$$C\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\|\beta_1\|_{q,s_0+1}^{\gamma,\mathcal{O}}\big(1+\|\beta_2\|_{q,s_0}^{\gamma,\mathcal{O}}\big)\,\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\leqslant\frac{1}{2}\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}.$$

Inserting this term into the left hand side in (A.20) and using Sobolev embeddings, we find

$$\|\Delta_{12}\widehat{\beta}\|_{q,s}^{\gamma,\mathcal{O}}\leqslant C\left(\|\Delta_{12}\beta\|_{q,s}^{\gamma,\mathcal{O}}+\|\Delta_{12}\beta\|_{q,s_0}^{\gamma,\mathcal{O}}\max_{j\in\{1,2\}}\|\beta_j\|_{q,s+1}^{\gamma,\mathcal{O}}\right).$$

This ends the proof of Lemma A.4. ☐

We shall also prove here the following result which is frequently used in the reduction procedure for the linearized operators.

Lemma A.5. *Let $N_0\geqslant 2$. Consider the sequence $(N_m)_{m\in\mathbb{N}}$ defined by* (6.94). *Then for all $\alpha>0$, we have*

$$\sum_{k=m}^{\infty}N_k^{-\alpha}\underset{m\to\infty}{\sim}N_m^{-\alpha}.$$

Proof. We consider the positive decaying function

$$t \in \mathbb{R}_+^* \mapsto N_0^{-\alpha\left(\frac{3}{2}\right)^t} = \exp\left(-\alpha \ln(N_0) e^{t \ln\left(\frac{3}{2}\right)}\right),$$

and apply to it a series-integral comparison, namely

$$\sum_{k=m+1}^{\infty} N_k^{-\alpha} \leqslant \int_m^{\infty} \exp\left(-\alpha \ln(N_0) e^{t \ln\left(\frac{3}{2}\right)}\right) dt = \int_0^{\infty} \exp\left(-\alpha \ln(N_0) e^{u \ln\left(\frac{3}{2}\right)} e^{m \ln\left(\frac{3}{2}\right)}\right) du.$$

Now remark that

$$N_m^{\alpha} \exp\left(-\alpha \ln(N_0) e^{u \ln\left(\frac{3}{2}\right)} e^{m \ln\left(\frac{3}{2}\right)}\right) = \exp\left(\alpha \ln(N_0)\left(1 - e^{u \ln\left(\frac{3}{2}\right)}\right) e^{m \ln\left(\frac{3}{2}\right)}\right).$$

Since

$$\forall u \in \mathbb{R}_+^*, \quad 1 - e^{u \ln\left(\frac{3}{2}\right)} < 0,$$

then we deduce that

$$\forall u \in \mathbb{R}_+^*, \quad N_m^{\alpha} \exp\left(-\alpha \ln(N_0) e^{u \ln\left(\frac{3}{2}\right)} e^{m \ln\left(\frac{3}{2}\right)}\right) \underset{m \to \infty}{\longrightarrow} 0$$

and

$$\forall u \in \mathbb{R}_+^*, \forall m \in \mathbb{N}, \quad 0 \leqslant N_m^{\alpha} \exp\left(-\alpha \ln(N_0) e^{u \ln\left(\frac{3}{2}\right)} e^{m \ln\left(\frac{3}{2}\right)}\right) \leqslant N_0^{\alpha} \exp\left(-\alpha \ln(N_0) e^{u \ln\left(\frac{3}{2}\right)}\right) \in L^1(\mathbb{R}_+).$$

Applying dominated convergence theorem, we obtain

$$\sum_{k=m+1}^{\infty} N_k^{-\alpha} \underset{m \to \infty}{=} o\left(N_m^{-\alpha}\right).$$

As a consequence

$$\sum_{k=m}^{\infty} N_k^{-\alpha} = N_m^{-\alpha} + \sum_{k=m+1}^{\infty} N_k^{-\alpha} \underset{m \to \infty}{\sim} N_m^{-\alpha}.$$

□

A.2 Operators

We shall focus in this section on some useful norms related to suitable operators class. These notions were used before in [7, 29, 28, 33]. We consider a smooth family of bounded operators on Sobolev spaces $H^s(\mathbb{T}^{d+1}, \mathbb{C})$, that is a smooth map $T : \mu = (\lambda, \omega) \in \mathcal{O} \mapsto T(\mu) \in \mathcal{L}(H^s(\mathbb{T}^{d+1}, \mathbb{C}))$ of linear continuous operators on Sobolev space $H^s(\mathbb{T}^{d+1}, \mathbb{C})$, with $\mathcal{O}$ being an open bounded set of $\mathbb{R}^{d+1}$. Then we find it convenient to encode $T(\mu)$ in terms of the infinite dimensional matrix $\left(T_{l_0,j_0}^{l,j}(\mu)\right)_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2 \\ (j,j_0)\in\mathbb{Z}^2}}$ with

$$T(\mu)\mathbf{e}_{l_0,j_0} = \sum_{(l,j)\in\mathbb{Z}^{d+1}} T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j} \qquad \text{where} \qquad T_{l_0,j_0}^{l,j}(\mu) \triangleq \left\langle T(\mu)\mathbf{e}_{l_0,j_0}, \mathbf{e}_{l,j}\right\rangle_{L^2(\mathbb{T}^{d+1})}. \tag{A.21}$$

Next, we need to fix a notation that we are implicitly using along the document. For a given family of multi-parameter operators $T(\mu)$, it acts on $W^{q,\infty,\gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C}))$ in the following sense,

$$\rho \in W^{q,\infty,\gamma}(\mathcal{O}, H^s(\mathbb{T}^{d+1}, \mathbb{C})), \qquad (T\rho)(\mu, \varphi, \theta) \triangleq T(\mu)\rho(\mu, \varphi, \theta).$$

A.2.1 Toeplitz in time operators

In this short section we shall introduce a suitable class of Toeplitz operators.

Definition A.1. *We say that an operator $T(\mu)$ is Toeplitz in time (actually in the variable φ) if its Fourier coefficients defined by* (A.21)*, satisfy*

$$\forall\, l, l_0, j, j_0 \in \mathbb{Z}, \quad T_{l_0,j_0}^{l,j}(\mu) = T_{0,j_0}^{l-l_0,j}(\mu).$$

Or equivalently

$$T_{l_0,j_0}^{l,j}(\mu) = T_{j_0}^{j}(\mu, l-l_0) \quad \text{with} \quad T_{j_0}^{j}(\mu,l) \triangleq T_{0,j_0}^{l,j}(\mu).$$

The action of a Toeplitz operator $T(\mu)$ on a function $\rho = \displaystyle\sum_{(l_0,j_0)\in\mathbb{Z}^{d+1}} \rho_{l_0,j_0}\mathbf{e}_{l_0,j_0}$ is then given by

$$T(\mu)\rho = \sum_{\substack{(l,l_0)\in(\mathbb{Z}^d)^2\\ (j,j_0)\in\mathbb{Z}^2}} T_{j_0}^{j}(\mu, l-l_0)\rho_{l_0,j_0}\mathbf{e}_{l,j}. \tag{A.22}$$

We encounter several operators acting only on the variable θ and that can be considered as φ-dependent operators $T(\mu,\varphi)$ taking the form

$$T(\mu,\varphi)\rho(\varphi,\theta) = \int_{\mathbb{T}} K(\mu,\varphi,\theta,\eta)\rho(\varphi,\eta)d\eta.$$

One can easily check that those operators are Toeplitz and therefore they satisfy (A.22).
For $q\in\mathbb{N}$ and $s\in\mathbb{R}$, we can equip Toeplitz operators with the off-diagonal norm given by,

$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \triangleq \sum_{\substack{\alpha\in\mathbb{N}^{d+1}\\ |\alpha|\leqslant q}} \gamma^{|\alpha|} \sup_{\mu\in\mathcal{O}} \|\partial_\mu^\alpha(T)(\mu)\|_{\text{O-d},s-|\alpha|}, \tag{A.23}$$

where

$$\|T\|_{\text{O-d},s}^2 \triangleq \sum_{(l,m)\in\mathbb{Z}^{d+1}} \langle l,m\rangle^{2s} \sup_{j-k=m} |T_j^k(l)|^2. \tag{A.24}$$

We mention that the off-diagonal norm (A.24) has first been introduced in [25, Def. 3.2]. This norm is of important use during the KAM reduction of the remainder. The cut-off projectors $(P_N)_{N\in\mathbb{N}^*}$ are defined as follows:

$$\left(P_N T(\mu)\right)\mathbf{e}_{l_0,j_0} = \sum_{\substack{(l,j)\in\mathbb{Z}^{d+1}\\ |l-l_0|,|j-j_0|\leqslant N}} T_{l_0,j_0}^{l,j}(\mu)\mathbf{e}_{l,j} \quad \text{and} \quad P_N^\perp T = T - P_N T. \tag{A.25}$$

In the next lemma we shall gather classical results whose proofs are very close to those in [33] concerning pseudo-differential operators. We recall that the weighted norms on functions that will be used below are defined in (A.6).

Lemma A.6. *Let (γ,q,d,s_0,s) satisfying* (A.2)*. Let T, T_1 and T_2 be Toeplitz in time operators.*

(i) Projectors properties : Let $N\in\mathbb{N}^*$. Let $t\in\mathbb{R}_+$. Then

$$\|P_N T\rho\|_{\text{O-d},q,s+t}^{\gamma,\mathcal{O}} \leqslant N^t \|T\rho\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \quad \textit{and} \quad \|P_N^\perp T\rho\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \leqslant N^{-t}\|T\rho\|_{\text{O-d},q,s+t}^{\gamma,\mathcal{O}}.$$

(ii) Interpolation inequality : Let $q<s_1\leqslant s_3\leqslant s_2$, $\overline{\theta}\in[0,1]$ with $s_3=\overline{\theta}s_1+(1-\overline{\theta})s_2$. Then

$$\|T\|_{\text{O-d},q,s_3}^{\gamma,\mathcal{O}} \lesssim \left(\|T\|_{\text{O-d},q,s_1}^{\gamma,\mathcal{O}}\right)^{\overline{\theta}} \left(\|T\|_{\text{O-d},q,s_2}^{\gamma,\mathcal{O}}\right)^{1-\overline{\theta}}.$$

(iii) Composition law :

$$\|T_1T_2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\lesssim\|T_1\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|T_2\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}+\|T_1\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|T_2\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}.$$

(iv) Link between operators and off-diagonal norms :

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|T\|_{\text{O-d},q,s_0}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}+\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s_0}^{\gamma,\mathcal{O}}.$$

In particular

$$\|T\rho\|_{q,s}^{\gamma,\mathcal{O}}\lesssim\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}}\|\rho\|_{q,s}^{\gamma,\mathcal{O}}.$$

A.2.2 Reversible and reversibility preserving operators

In this section we intend to collect some definitions and properties related to different reversibility notions for operators and give practical characterizations. We shall also come back to Toeplitz operators defined before in Section A.2.1 and discuss two important examples frequently encountered in this document and given by multiplications and integral operators. First, we give the following definitions following [10, Def. 2.2].

Definition A.2. *Introduce the following involution*

$$(\mathscr{S}_2\rho)(\varphi,\theta)\triangleq\rho(-\varphi,-\theta).\tag{A.26}$$

We say that an operator $T(\mu)$ is

- *real if for all $\rho\in L^2(\mathbb{T}^{d+1},\mathbb{C})$, we have*

$$\overline{\rho}=\rho\quad\Longrightarrow\quad\overline{T\rho}=T\rho.$$

- *reversible if*

$$T(\mu)\circ\mathscr{S}_2=-\mathscr{S}_2\circ T(\mu).$$

- *reversibility preserving if*

$$T(\mu)\circ\mathscr{S}_2=\mathscr{S}_2\circ T(\mu).$$

We now detail the following characterizations needed at several places in this document and the proofs are quite easy and follow from Fourier expansion. One can find a similar result in [10, Lem. 2.6].

Proposition A.1. *Let T be an operator. Then T is*

- *real if and only if*

$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=\overline{T_{l_0,j_0}^{l,j}}.$$

- *reversible if and only if*

$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=-T_{l_0,j_0}^{l,j}.$$

- *reversibility-preserving if and only if*

$$\forall(l,l_0,j,j_0)\in(\mathbb{Z}^d)^2\times\mathbb{Z}^2,\quad T_{-l_0,-j_0}^{-l,-j}=T_{l_0,j_0}^{l,j}.$$

In what follows, we shall focus on two particular cases of operators which will be of constant use throughout this document. Namely, multiplication and integral operators.

Definition A.3. *Let T be an operator as in Section A.2. We say that*

- *T is a multiplication operator if there exists a function $M : (\mu, \varphi, \theta) \mapsto M(\mu, \varphi, \theta)$ such that*
$$(T\rho)(\mu, \varphi, \theta) = M(\mu, \varphi, \theta)\rho(\mu, \varphi, \theta).$$
- *T is an integral operator if there exists a function (called the kernel) $K : (\mu, \varphi, \theta, \eta) \mapsto K(\mu, \varphi, \theta, \eta)$ such that*
$$(T\rho)(\mu, \varphi, \theta) = \int_{\mathbb{T}} \rho(\mu, \varphi, \eta)K(\mu, \varphi, \theta, \eta)d\eta.$$

We intend to prove the following lemma.

Lemma A.7. *Let (γ, q, d, s_0, s) satisfy* (A.2)*, then the following assertions hold true.*

(i) Let T be a multiplication operator by a real-valued function M, then the following holds true.

- *If $M(\mu, -\varphi, -\theta) = M(\mu, \varphi, \theta)$, then T is real and reversibility preserving Toeplitz in time and space operator.*
- *If $M(\mu, -\varphi, -\theta) = -M(\mu, \varphi, \theta)$, then T is real and reversible Toeplitz in time and space operator.*

Moreover,
$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \|M\|_{q,s+s_0}^{\gamma,\mathcal{O}}.$$

(ii) Let T be an integral operator with a real-valued kernel K.

- *If $K(\mu, -\varphi, -\theta, -\eta) = K(\mu, \varphi, \theta, \eta)$, then T is a real and reversibility preserving Toeplitz in time operator.*
- *If $K(\mu, -\varphi, -\theta, -\eta) = -K(\mu, \varphi, \theta, \eta)$, then T is a real and reversible Toeplitz in time operator.*

In addition,
$$\|T\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim \int_{\mathbb{T}} \|K(*, \cdot, \bullet, \eta + \bullet)\|_{q,s+s_0}^{\gamma,\mathcal{O}} d\eta \lesssim \|K\|_{q,H^{s+s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}$$

and
$$\begin{aligned}\|T\rho\|_{q,s}^{\gamma,\mathcal{O}} &\lesssim \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \int_{\mathbb{T}} \|K(*, \cdot, \bullet, \eta + \bullet)\|_{q,s}^{\gamma,\mathcal{O}} d\eta + \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \int_{\mathbb{T}} \|K(*, \cdot, \bullet, \eta + \bullet)\|_{q,s_0}^{\gamma,\mathcal{O}} d\eta \\ &\lesssim \|\rho\|_{q,s_0}^{\gamma,\mathcal{O}} \|K\|_{q,H^{s}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} + \|\rho\|_{q,s}^{\gamma,\mathcal{O}} \|K\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}}\end{aligned}$$

where the notation $*, \cdot, \bullet$ denote μ, φ, θ, respectively.

Proof. We point out that the proofs will be implemented for the particular case $q = 0$ and the general case can be done similarly by differentiating with respect to μ and using Leibniz rule.
(i) Since M is a real-valued function, then we get by the definition
$$\begin{aligned}\overline{T_{-l_0,-j_0}^{-l,-j}} &= \int_{\mathbb{T}^{d+1}} M(\varphi, \theta)\overline{\mathbf{e}_{-l_0,-j_0}(\varphi, \theta)\mathbf{e}_{l,j}(\varphi, \theta)}d\varphi d\theta \\ &= \int_{\mathbb{T}^{d+1}} M(\varphi, \theta)\mathbf{e}_{l_0,j_0}(\varphi, \theta)\mathbf{e}_{-l,-j}(\varphi, \theta)d\varphi d\theta = T_{l_0,j_0}^{l,j}.\end{aligned}$$

This shows in view of Proposition A.1 that the operator T is a real. It remains to check the reversibility preserving property. We write from the definition

$$\begin{aligned}T(\mathscr{S}_2\rho)(\varphi,\theta) &= M(\varphi,\theta)\rho(-\varphi,-\theta)\\ &= M(-\varphi,-\theta)\rho(-\varphi,-\theta)\\ &= \mathscr{S}_2\left(T\rho\right)(\varphi,\theta).\end{aligned}$$

This gives the desired result. As to the reversible Toepliz structure, it can be checked in a similar way. To achieve the proof of the first point it remains to establish the suitable estimate. Using a duality argument $H^{s+s_0}-H^{-s-s_0}$, we may write,

$$|T_j^{j'}(l)| = \left|\int_{\mathbb{T}^{d+1}} M(\varphi,\theta)\mathbf{e}_{l,j-j'}(\varphi,\theta)d\varphi d\theta\right| \lesssim \langle l, j-j'\rangle^{-s-s_0}\|M\|_{H^{s+s_0}}.$$

It follows that

$$\begin{aligned}\|T\|^2_{\text{O-d},s} &= \sum_{(l,m)\in\mathbb{Z}^{d+1}}\langle l,m\rangle^{2s}\sup_{j-j'=m}|T_j^{j'}(l)|^2\\ &\lesssim \|M\|^2_{H^{s+s_0}}\sum_{(l,m)\in\mathbb{Z}^{d+1}}\langle l,m\rangle^{2s}\langle l,m\rangle^{-2s-2s_0}\\ &\lesssim \|M\|^2_{H^{s+s_0}}.\end{aligned}$$

Therefore we find

$$\|T\|_{\text{O-d},s}\lesssim\|M\|_{H^{s+s_0}}.$$

(ii) By assumption, K is real and thus

$$\begin{aligned}\overline{T^{-l,-j}_{-l_0,-j_0}} &= \int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)\overline{\mathbf{e}_{-l_0,-j_0}(\varphi,\eta)\mathbf{e}_{l,j}(\varphi,\theta)}d\varphi d\theta\\ &= \int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)\mathbf{e}_{l_0,j_0}(\varphi,\eta)\mathbf{e}_{-l,-j}(\varphi,\theta)d\varphi d\theta d\eta = T^{l,j}_{l_0,j_0}.\end{aligned}$$

This implies, according to Proposition A.1, that T is a real operator. Now we shall check the reversibility preserving. The reversibility can be checked in a similar way. By the change of variables $\eta\mapsto-\eta$, we may write,

$$\begin{aligned}T(\mathscr{S}_2\rho)(\varphi,\theta) &= \int_{\mathbb{T}} K(\varphi,\theta,\eta)\rho(-\varphi,-\eta)d\eta\\ &= \int_{\mathbb{T}} K(-\varphi,-\theta,-\eta)\rho(-\varphi,-\eta)d\eta\\ &= \int_{\mathbb{T}} K(-\varphi,-\theta,\eta)\rho(-\varphi,\eta)d\eta = \mathscr{S}_2\left(T\rho\right)(\varphi,\theta).\end{aligned}$$

From Fubini's theorem and the duality $H^{s+s_0}_{\varphi,\theta}-H^{-s-s_0}_{\varphi,\theta}$, we infer,

$$\begin{aligned}|T_j^{j'}(l)| &= \left|\int_{\mathbb{T}^{d+2}} K(\varphi,\theta,\eta)e^{\mathrm{i}(l\cdot\varphi+j\theta-j'\eta)}d\varphi d\theta d\eta\right|\\ &= \left|\int_{\mathbb{T}^{d+1}} e^{\mathrm{i}(l\cdot\varphi+(j-j')\theta)}\left(\int_{\mathbb{T}} K(\varphi,\theta,\eta+\theta)e^{-\mathrm{i}j'\eta}d\eta\right)d\varphi d\theta\right|\\ &\lesssim \langle l,j-j'\rangle^{-s-s_0}\int_{\mathbb{T}}\|K(*,\cdot,\bullet,\eta+\bullet)\|_{H^{s+s_0}_{\varphi,\theta}}d\eta.\end{aligned}$$

Hence, we deduce that

$$\|T\|_{\text{O-d},s} \lesssim \int_{\mathbb{T}} \|K(*,\cdot,{\scriptstyle\blacksquare},\eta+{\scriptstyle\blacksquare})\|_{H^{s+s_0}_{\varphi,\theta}} d\eta.$$

The last estimate in Lemma A.7 can be obtained from the expression

$$(T\rho)(\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\theta+\eta)K(\varphi,\theta,\theta+\eta)d\eta,$$

combined with the law products and the translation invariance in Lemma A.1-(i)-(iv).
This concludes the proof of Lemma A.7. $\square$

In the following lemma we shall study the action of a change of variables as in (A.13) on an integral operator. More precisely, we shall need two partial change of coordinates $\mathcal{B}^1$ and $\mathcal{B}^2$ acting respectively on the variables θ and η and defined through

$$\begin{aligned}(\mathcal{B}^1\rho)(\mu,\varphi,\theta,\eta) &\triangleq \rho\big(\mu,\varphi,\theta+\beta_1(\mu,\varphi,\theta),\eta\big), \\ (\mathcal{B}^2\rho)(\mu,\varphi,\theta,\eta) &\triangleq \rho\big(\mu,\varphi,\theta,\eta+\beta_2(\mu,\varphi,\eta)\big),\end{aligned} \tag{A.27}$$

with β_1,β_2 two smooth functions satisfying (A.15). A similar result is proved in [33, Lem. 2.34] for pseudo-differential integral operators, so we omit the proof here. We also include the difference estimate which is useful to study the stability of the Cantor sets in Section 8.2. The proof of the difference estimate is standard and we shall also skip it here.

Lemma A.8. *Let* (γ,q,d,s_0,s) *satisfy* (A.2). *Given* $r\in W^{q,\infty,\gamma}(\mathcal{O},H^s)$*, we consider a* C^∞ *function in the form*

$$K:(\mu,\varphi,\theta,\eta)\mapsto K(\mu,\varphi,\theta,\eta).$$

We consider the integral operator associated to K*, namely*

$$T\rho(\mu,\varphi,\theta) = \int_{\mathbb{T}} \rho(\varphi,\eta)K(\mu,\varphi,\theta,\eta)d\eta.$$

Then the following assertions hold true.

(i) Let $\mathcal{B}^1$ *and* $\mathcal{B}^2$ *as in* (A.27) *associated to* β_1 *and* β_2*, respectively and enjoying the smallness condition* (A.15)*. Then,*

$$\|\mathcal{B}^1\mathcal{B}^2K\|^{\gamma,\mathcal{O}}_{q,H^s_{\varphi,\theta,\eta}} \lesssim \|K\|^{\gamma,\mathcal{O}}_{q,H^s_{\varphi,\theta,\eta}} + \Big(\max_{i\in\{1,2\}}\|\beta_i\|^{\gamma,\mathcal{O}}_{q,s}\Big)\|K\|^{\gamma,\mathcal{O}}_{q,H^{s_0}_{\varphi,\theta,\eta}}. \tag{A.28}$$

Now, assume that $\beta_1=\beta_2=\beta$ *satisfies the following symmetry conditions*

$$\beta(\mu,-\varphi,-\theta) = -\beta(\mu,\varphi,\theta). \tag{A.29}$$

Consider $\mathscr{B}$, $\mathcal{B}$ *be quasi-periodic changes of variables as in* (A.12)-(A.13)*, then*

- *if* $K(\mu,-\varphi,-\theta,-\eta)=K(\mu,\varphi,\theta,\eta)$*, then* $\mathcal{B}^{-1}T\mathscr{B}$ *is a real and reversibility preserving Toeplitz in time integral operator.*
- *if* $K(\mu,-\varphi,-\theta,-\eta)=-K(\mu,\varphi,\theta,\eta)$*, then* $\mathcal{B}^{-1}T\mathscr{B}$ *is a real and reversible Toeplitz in time inegral operator.*

In this case, for any $k\in\mathbb{N}$*,*

$$\|\partial_\theta^k\mathcal{B}^{-1}T\mathscr{B}\|^{\gamma,\mathcal{O}}_{\text{O-d},q,s} \lesssim \|K\|^{\gamma,\mathcal{O}}_{q,H^{s+s_0+k}_{\varphi,\theta,\eta}} + \|\beta\|^{\gamma,\mathcal{O}}_{q,s+s_0+k}\|K\|^{\gamma,\mathcal{O}}_{q,H^{s_0}_{\varphi,\theta,\eta}}. \tag{A.30}$$

(ii) Introduce $\mathcal{B}_r$ a quasi-periodic change of variables as in (A.13) *associated to β_r (linked to r) Consider r_1, $r_2 \in W^{q,\infty,\gamma}(\mathcal{O}, H^s)$. Denote*

$$\Delta_{12} r \triangleq r_1 - r_2, \quad \Delta_{12} f_r \triangleq f_{r_1} - f_{r_2}$$

for any quantity f_r depending on r and assume that there exist $\varepsilon_0 > 0$ small enough such that

$$\forall i \in \{1,2\}, \quad \|\beta_{r_i}\|_{q,2s_0}^{\gamma,\mathcal{O}} + \|K_{r_i}\|_{q,H^{s_0+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \leqslant \varepsilon_0. \tag{A.31}$$

Then, for any $k \in \mathbb{N}$, the following estimate holds

$$\begin{aligned}\|\Delta_{12}\partial_\theta^k \mathcal{B}_r^{-1} T_r \mathscr{B}_r\|_{\text{O-d},q,s}^{\gamma,\mathcal{O}} \lesssim{}& \|\Delta_{12} K_r\|_{q,H^{s+s_0+k}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} + \|\Delta_{12}\beta_r\|_{q,s+s_0+k}^{\gamma,\mathcal{O}} \\ &+ \Big(\max_{i\in\{1,2\}} \|\beta_{r_i}\|_{q,s+s_0+k}^{\gamma,\mathcal{O}}\Big) \|\Delta_{12} K_r\|_{q,H^{s_0}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} \\ &+ \Big(\max_{i\in\{1,2\}} \|K_{r_i}\|_{q,H^{s+s_0+k+1}_{\varphi,\theta,\eta}}^{\gamma,\mathcal{O}} + \max_{i\in\{1,2\}} \|\beta_{r_i}\|_{q,s+s_0+k+1}^{\gamma,\mathcal{O}}\Big) \|\Delta_{12}\beta_r\|_{q,s_0}^{\gamma,\mathcal{O}}.\end{aligned} \tag{A.32}$$

B Crandall-Rabinowitz's Theorem

Now, we recall the classical Crandall-Rabinowitz's Theorem. This result was first proved in [50] and it is one of the most common theorems appearing in the bifurcation theory. A convenient reference in the subject is [112]. We briefly explain the core of local bifurcation theory.

Consider a function $F : \mathbb{R} \times X \to Y$ with X and Y two Banach spaces. Assume that for all Ω in a non-empty interval I we have $F(\Omega, 0) = 0$. This provides a line of solutions

$$\big\{(\Omega, 0), \quad \Omega \in I\big\}.$$

Now take some $(\Omega_0, 0)$ with $\Omega_0 \in I$. The implicit function Theorem explains that if $DF(\Omega_0, 0)$ is invertible, then the line $\{(\Omega, 0), |\Omega - \Omega_0| \leqslant \varepsilon\}$ is the only curve of solutions close to $(\Omega_0, 0)$, i.e. for ε small enough. (Local) bifurcation theory is the study of situations where this is not true, that is, close to $(\Omega_0, 0)$ there exists (at least) another line of solutions. In this case, we say that $(\Omega_0, 0)$ is a bifurcation point. Crandall-Rabinowitz's Theorem gives sufficient conditions to construct a bifurcation curve and states as follows.

Theorem B.1 (Crandall-Rabinowitz)**.** *Let X and Y be two banach spaces. Let V be a neighborhood of* 0 *in X and let*

$$\begin{array}{rccc} F: & \mathbb{R} \times V & \to & Y \\ & (\Omega, x) & \mapsto & F(\Omega, x) \end{array}$$

be a function of classe C^1 with the following properties

(i) (Trivial solution) $\forall\, \Omega \in \mathbb{R}$, $F(\Omega, 0) = 0$.

(ii) (Regularity) $\partial_\Omega F$, $d_x F$ and $\partial_\Omega d_x F$ exist and are continuous.

(iii) (Fredholm property) $\ker\left(d_x F(0,0)\right) = \langle x_0 \rangle$ *and* $Y/R\left(d_x F(0,0)\right)$ *are one dimensional and* $R\left(d_x F(0,0)\right)$ *is closed in Y.*

(iv) (Transversality assumption) $\partial_\Omega d_x F(0,0)(x_0) \notin R\left(d_x F(0,0)\right)$.

If χ is any complement of $\ker(d_x F(0,0))$ *in X, then there exist a neighborhood U of $(0,0)$, an interval $(-a,a)$ $(a>0)$ and continuous functions*

$$\psi : (-a,a) \to \mathbb{R} \quad \text{and} \quad \phi : (-a,a) \to \chi$$

such that $\psi(0)=0$, $\phi(0)=0$ and

$$\Big\{(\Omega,x)\in U \quad \text{s.t.} \quad F(\Omega,x)=0\Big\} = \Big\{\big(\psi(s), sx_0 + s\phi(s)\big) \quad \text{s.t.} \quad |s|<a\Big\} \cup \Big\{(\Omega,0)\in U\Big\}.$$

C Modified Bessel functions

In this appendix we shall collect some properties about Bessel and modified Bessel functions that were used in the preceding sections. We refer to [145] for an almost exhaustive presentation of these special functions.

We define first the Bessel functions of order $\nu \in \mathbb{C}$ by

$$J_\nu(z) \triangleq \sum_{m=0}^{\infty} \frac{(-1)^m \left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)|<\pi.$$

Notice that when $\nu \in \mathbb{N}$ we have the following integral representation, see [121, p. 115].

$$J_\nu(x) = \frac{1}{\pi}\int_0^\pi \cos\big(x\sin\theta - \nu\theta\big)d\theta. \tag{C.1}$$

We shall also introduce the Bessel functions of imaginary argument also called modified Bessel functions of first and second kind

$$I_\nu(z) \triangleq \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{\nu+2m}}{m!\Gamma(\nu+m+1)}, \quad |\arg(z)|<\pi \tag{C.2}$$

and

$$K_\nu(z) \triangleq \frac{\pi}{2}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin(\nu\pi)}, \quad \nu\in\mathbb{C}\setminus\mathbb{Z}, \quad |\arg(z)|<\pi.$$

For $j\in\mathbb{Z}$, we define $K_j(z) = \lim\limits_{\nu\to j} K_\nu(z)$. We give now useful properties of modified Bessel functions.

▶ **Symmetry and positivity properties** (see [1, p. 375]) **:**

$$\forall j\in\mathbb{N}, \quad \forall\lambda\in\mathbb{R}_+^*, \quad I_{-j}(\lambda)=I_j(\lambda)\in\mathbb{R}_+^* \quad \text{and} \quad K_{-j}(\lambda)=K_j(\lambda)\in\mathbb{R}_+^*. \tag{C.3}$$

▶ **Derivatives and anti-derivatives** (see [1, p. 376]) **:**

If we set $\mathcal{Z}_\nu(z)=I_\nu(z)$ or $e^{\mathrm{i}\nu\pi}K_\nu(z)$, then for all $\nu\in\mathbb{R}$, we have

$$\mathcal{Z}_\nu'(z) = \mathcal{Z}_{\nu-1}(z) - \frac{\nu}{z}\mathcal{Z}_\nu(z) = \mathcal{Z}_{\nu+1}(z) + \frac{\nu}{z}\mathcal{Z}_\nu(z). \tag{C.4}$$

and

$$\frac{d}{dz}\left(z^{\nu+1}\mathcal{Z}_{\nu+1}(z)\right) = z^{\nu+1}\mathcal{Z}_\nu(z). \tag{C.5}$$

▶ **Power series extension for** K_j (see [1, p. 375]) **:**

$$\begin{aligned} K_j(z) =& \frac{1}{2}\left(\frac{z}{2}\right)^{-j}\sum_{k=0}^{j-1}\frac{(j-k-1)!}{k!}\left(\frac{-z}{4}\right)^k + (-1)^{j+1}\ln\left(\frac{z}{2}\right)I_j(z) \\ &+ \frac{1}{2}\left(\frac{-z}{2}\right)^j\sum_{k=0}^{\infty}\left(\psi(k+1)+\psi(j+k+1)\right)\frac{\left(\frac{z^2}{4}\right)^k}{k!(j+k)!}, \end{aligned} \tag{C.6}$$

where

$$\psi(1) \triangleq -\gamma \text{ (Euler's constant)} \quad \text{and} \quad \forall m \in \mathbb{N}^*,\ \psi(m+1) \triangleq \sum_{k=1}^{m}\frac{1}{k} - \gamma.$$

In particular

$$K_0(z) = -\log\left(\frac{z}{2}\right)I_0(z) + \sum_{m=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2m}}{(m!)^2}\psi(m+1), \tag{C.7}$$

so K_0 behaves like a logarithm at 0.

▶ **Integral representation for** K_ν (see [121, p. 133]) **:**
For all $a, b > 0$ for any $\nu, \mu \in \mathbb{C}$ satisfying $-1 < \mathrm{Re}(\nu) < 2\mathrm{Re}(\mu) + \frac{3}{2}$ one has

$$\int_0^\infty \frac{x^{\nu+1}J_\nu(bx)}{(x^2+a^2)^{\mu+1}}dx = \frac{a^{\nu-\mu}b^\mu}{2^\mu\Gamma(\mu+1)}K_{\nu-\mu}(ab). \tag{C.8}$$

▶ **Nicholson's integral representation** (see [145, p. 441]) **:**
Let $j \in \mathbb{N}$ then

$$(I_jK_j)(z) = \tfrac{2(-1)^j}{\pi}\int_0^{\frac{\pi}{2}} K_0(2z\cos(\tau))\cos(2j\tau)d\tau. \tag{C.9}$$

Another similar representation can be found in [121, p. 140]

$$(I_jK_j)(\lambda) = \frac{1}{2}\int_0^\infty J_0\big(2\lambda\sinh(t/2)\big)e^{-jt}dt. \tag{C.10}$$

▶ **Holomorphic property of the product** I_jK_j **:**
Let $j \in \mathbb{N}$. Then the function $z \mapsto (I_jK_j)(z)$ is holomorphic on the half plane $\mathrm{Re}(z) > 0$.

▶ **Decay property for the product** $I_\nu K_\nu$ (see [13] and [54]) **:**
The application $(\lambda, \nu) \mapsto I_\nu(\lambda)K_\nu(\lambda)$ is strictly decreasing in each variable $(\lambda, \nu) \in (\mathbb{R}_+^*)^2$.

▶ **Beltrami's summation formula** (see [145, p. 361]) **:** Let $0 < b < a$. Then

$$\forall\theta \in \mathbb{R}, \quad K_0\left(\sqrt{a^2+b^2-2ab\cos(\theta)}\right) = \sum_{m=-\infty}^{\infty} I_m(b)K_m(a)\cos(m\theta). \tag{C.11}$$

▶ **Ratio bounds** (see [14]) **:**
For all $n \in \mathbb{N}$, for all $\lambda \in \mathbb{R}_+^*$, we have

$$\begin{cases} \dfrac{\lambda I_n'(\lambda)}{I_n(\lambda)} < \sqrt{\lambda^2+n^2} \\[2ex] \dfrac{\lambda K_n'(\lambda)}{K_n(\lambda)} < -\sqrt{\lambda^2+n^2} \end{cases} \tag{C.12}$$

▶ **Asymptotic expansion of small argument** (see [1, p. 375]) **:**

$$\forall n \in \mathbb{N}^*, \quad I_n(\lambda) \underset{\lambda \to 0}{\sim} \frac{\left(\frac{1}{2}\lambda\right)^n}{\Gamma(n+1)} \quad \text{and} \quad K_n(\lambda) \underset{\lambda \to 0}{\sim} \frac{\Gamma(n)}{2\left(\frac{1}{2}\lambda\right)^n}. \tag{C.13}$$

▶ **Asymptotic expansion of high order** (see [1, p. 377]) **:**

$$\forall \lambda > 0, \quad I_\nu(\lambda) \underset{\nu \to \infty}{\sim} \frac{1}{\sqrt{2\pi\nu}} \left(\frac{e\lambda}{2\nu}\right)^\nu \quad \text{and} \quad K_\nu(\lambda) \underset{\nu \to \infty}{\sim} \sqrt{\frac{\pi}{2\nu}} \left(\frac{e\lambda}{2\nu}\right)^{-\nu}. \tag{C.14}$$

▶ **Asymptotic expansion of large argument for the product** $I_j K_j$ (see [1, p. 378]) **:**

$$\forall N \in \mathbb{N}^*, \quad I_j(\lambda) K_j(\lambda) \underset{\lambda \to \infty}{\sim} \frac{1}{2\lambda} \left(1 + \sum_{m=1}^{N} \frac{\alpha_{j,m}}{(2\lambda)^{2m}}\right), \tag{C.15}$$

with

$$\alpha_{j,m} \triangleq (-1)^m \frac{(2m)!}{4^m \left(m!\right)^2} P_m(\mu_j), \quad P_m(X) \triangleq \prod_{\ell=1}^{m} \left(X - (2\ell-1)^2\right), \quad \mu_j \triangleq 4j^2. \tag{C.16}$$

In particular,

$$I_j(\lambda) K_j(\lambda) \underset{\lambda \to \infty}{\longrightarrow} 0. \tag{C.17}$$

▶ **Asymptotic expansion of high order for the product** $I_j K_j$ (see [102]) **:**

$$\forall \lambda > 0, \quad \forall b \in (0,1], \quad I_j(\lambda b) K_j(\lambda) \underset{n \to \infty}{\sim} \frac{b^j}{2j} \left(\sum_{m=0}^{\infty} \frac{b_m(\lambda b)}{j^m}\right) \left(\sum_{m=0}^{\infty} (-1)^m \frac{b_m(\lambda)}{j^m}\right), \tag{C.18}$$

where for each $m \in \mathbb{N}$, $b_m(\lambda)$ is a polynomial of degree m in λ^2 defined by

$$b_0(\lambda) \triangleq 1 \quad \text{and} \quad \forall m \in \mathbb{N}^*, \ b_m(\lambda) \triangleq \sum_{k=1}^{m} (-1)^{m-k} \frac{S(m,k)}{k!} \left(\frac{\lambda^2}{4}\right)^k$$

and the $S(m,k)$ are Stirling numbers of second kind defined recursively by

$$\forall (m,k) \in (\mathbb{N}^*)^2, \quad S(m,k) = S(m-1,k-1) + kS(m-1,k),$$

with

$$S(0,0) = 1, \quad \forall m \in \mathbb{N}^*, \ S(m,1) = 1 \quad \text{and} \quad S(m,0) = 0 \quad \text{and if } m < k \text{ then } S(m,k) = 0.$$

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UNIVERSITÉ DE
RENNES 1

Titre : Structures quasi-périodiques pour des modèles de transport non-linéaires issus de la mécanique des fluides

Mot clés : Poches de tourbillon, Théorie KAM, Schéma de Nash-Moser, Solutions quasi-périodiques

Résumé : Nous étudions l'existence de poches de tourbillon quasi-périodiques en temps pour les équations d'Euler et les équations quasi-geostrophic shallow-water (QGSW) qui sont deux modèles de transport non-linéaires et non-locaux bidimensionnels. Les poches sont des solutions faibles de la classe de Yudovich décrites par l'évolution de domaines planaires dont l'étude repose sur la dynamique de leur bord. Tout domaine initial radial fournit une solution stationnaire et il est naturel de se demander si l'on peut trouver, proche de ses points d'équilibre, des solutions périodiques ou quasi-périodiques. Le premier cas a été largement étudié par le passé via des techniques de bifurcation, et nous apportons ici un résultat dans cette lignée pour le cas des poches doublement-connexe en rotation uniforme pour les équations QGSW. Le second cas est moins évident et constitue le noyau dur de cette thèse. En utilisant les théories de KAM et de Nash-Moser, nous montrons que quitte à choisir un paramètre dans un ensemble admissible de type Cantor et de mesure presque pleine, il est possible de générer des poches quasi-périodiques proches des tourbillons de Rankine ; solutions stationnaires associées aux disques. Pour les équations QGSW, le rayon de Rossby joue le rôle de ce paramètre qui apparaît naturellement dans les équations. Pour les équations d'Euler dans le disque unité, la non-invariance par dilatation du modèle permet de créer un paramètre géométrique : le rayon des tourbillons de Rankine.

Title: Quasi-periodic structures for nonlinear tranport fluid models

Keywords: Vortex patches, KAM theory, Nash-Moser scheme, Quasi-periodic solutions

Abstract: We study the existence of time quasi-periodic vortex patches for Euler and quasi-geostrophic shallow-water (QGSW) equations which are bidimensional nonlinear and nonlocal transport-type fluid models. Vortex patches are weak solutions in the Yudovich class described by the evolution of planar domains whose study relies on their boundary dynamics. Any radial initial domain provides a stationary solution and it is natural to ask whether we can find, close to these equilibrium points, periodic or quasi-periodic solutions. The first case has been widely studied in the past by using bifurcation theory, and here we give a result in this direction concerning the existence of doubly-connected uniformly rotating patches for QGSW equations. The second in less obvious and is the core of this thesis. By using KAM and Nash-Moser theories, we show that up to select a parameter among an admissible massive Cantor-like set, it is possible to construct quasi-periodic vortex patch solutions close to Rankine vortices ; stationary solutions associated with discs. For the QGSW equations, the Rossby radius plays the role of this parameter appearing naturally in the equations. For Euler equations set in the unit disc, the non-invariance by radial dilation allows to create a geometrical parameter : the radius of the Rankine vortices.