Geometric group theory growth and amenability

Étienne AFFALOU
École Normale Supérieure de Rennes

Supervised by Kai-Uwe BUX
Fakultät für Mathematik, Universität Bielefeld

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école normale supérieure rennes

Université de Rennes

Contents

Summary

The goal of the internship was to discover geometric group theory through the first notions of quasi-isometries, growth and amenable groups. Except for the last part about amenability, we only look at finitely generated groups and study their Cayley graphs. In the second section about growth, we are going to see how a growth type for a Cayley graph can give algebraic information about the underlying group, and vice versa. Finally, in a last part, we show that abelian groups are amenable without using the Markov–Kakutani fixed-point theorem.

Acknowledgements

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1 Quasi-isometries

1.1 Introduction, Cayley graphs

First, we introduce basic graph notations.

DEFINITION A **graph** is a pair $X = (V, E)$ where

1. V is a set, and
2. E is a set of subsets of V that contain exactly two elements.

We say that V is the set of **vertices** and E is the set of **edges**. Let $n \in \mathbb{N} \cup \{+\infty\}$.
A **path** in X of length n is a sequence of vertices $v_0, \cdots, v_n \in V$ such that $\forall j \in \{0, \cdots, n-1\}, \{v_j, v_{j+1}\} \in E$.
When $n \in \mathbb{N}$, we say that a path $v_0, \cdots, v_n$ **connects** v_0 and v_n.
The graph X is said to be **connected** when every pair of vertices can be connected by a path in X.

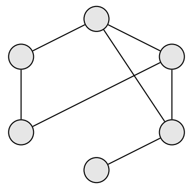


Figure 1: A connected graph with 6 vertices and 7 edges

We now endow connected graphs with a natural metric.

PROPOSITION 1.1.1 Let $X = (V, E)$ be a connected graph. Then, the map

$$\begin{array}{rccc} d & : & V \times V & \longrightarrow & \mathbb{R}_+ \\ & & (v, w) & \longmapsto & \min\{n \in \mathbb{N} \mid \text{a path of length } n \text{ connects } v \text{ and } w \text{ in } X\} \end{array}$$

is a metric on V, called metric on V associated with X.

PROOF : First, for all $v \in V$, we have $d(v, v) = 0$ since a path can be of length 0. Now if we have $v, w \in V$ such that $d(v, w) = 0$, then there is a path of length 0 connecting v to w so obviously $v = w$.
Then, for $v, w \in V$ and $n \in \mathbb{N}$, there is a path of length n from v to w if and only if there is a path of length n from w to v (by inverting the vertices) so $d(v, w) = d(w, v)$.
Finally, if $x, y, z \in V$, a path connecting x to y followed by a path connecting y to z gives a path connecting x to z, of length greater or equal to $d(x, z)$ by the very definition of d. In particular, $d(x, z) \leqslant d(x, y) + d(y, z)$.

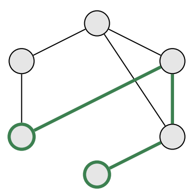


Figure 2: Green vertices are at distance 3

In order to endow groups with a metric, we define Cayley graphs.

DEFINITION Let G be a group. Let $S \subset G$ be a generating subset of G, i.e. $G = \langle S \rangle$.
The **Cayley graph** of G with respect to S is the graph $\mathrm{Cay}(G, S) = (V, E)$ where

1. the set of vertices is $V = G$, and
2. the set of edges is $E = \big\{\{g, gs\} \mid g \in G, s \in (S \cup S^{-1}) \setminus \{e\}\big\}$.

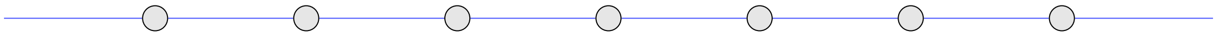


Figure 3: The graph $\mathrm{Cay}(\mathbb{Z}, \{1\})$

The Cayley graph of a group depends on the chosen generating subset (Figure 3 and 4).

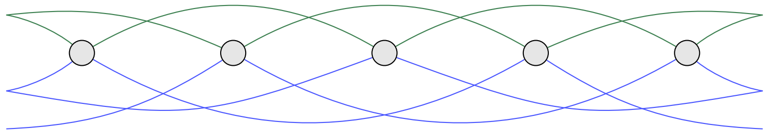


Figure 4: The graph $\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$

With this definition, G clearly acts by left translation on its Cayley graphs. Now, using proposition 1.1.1, we automatically get a metric on G for every generating subset S of G.

DEFINITION Let G be a group. Let $S \subset G$ be a generating subset of G.
The **word metric** on G with respect to S is the metric on G associated with $\mathrm{Cay}(G, S)$.
This metric depends on the generating subset S so we write d_S instead of d.
For $g \in G$, the **word length** of g with respect to S is $d_S(e, g)$.

PROPOSITION 1.1.2 Let G be a group. Let $S \subset G$ be a generating subset of G.
Then, the word metric on G with respect to S is left-invariant, i.e. $\forall h, h' \in G, \forall g \in G$,

$$d_S(gh, gh') = d_S(h, h').$$

PROOF : Let $h, h' \in G$ and let $g \in G$. Writing $n = d_S(h, h')$, $\exists s_1, \cdots, s_n \in S \cup S^{-1}$ such that

$$h = h' s_1 \cdots s_n.$$

Multiplying by g on the left gives

$$gh = (gh') s_1 \cdots s_n$$

so that $d_S(gh, gh') \leqslant n$. Now if we set $k = d_S(gh, gh')$, $\exists s'_1, \cdots, s'_k \in S \cup S^{-1}$ such that

$$gh = (gh') s'_1 \cdots s'_k.$$

We now multiply on the left by g^{-1} to get $n \leqslant k$. In the end, $k = n$.

The word metric depends on the generating subset. However, when G is finitely generated, we have a nice property on the word metrics with respect to finite generating subsets.

PROPOSITION 1.1.3 Let G be a finitely generated group. Let X, Y be finite generating subsets of G.
Then, there is a constant $C \in \mathbb{R}_+^*$ such that $\forall g, h \in G$, the following inequalities hold

$$\frac{1}{C} d_Y(g, h) \leqslant d_X(g, h) \leqslant C d_Y(g, h).$$

PROOF : We write $X = \{x_1, \cdots, x_r\}$ and $Y = \{y_1, \cdots, y_s\}$.
Let $C_1 = \max_{y \in Y \cup Y^{-1}} d_X(e, y)$. Since Y is finite, C_1 lies in $\mathbb{R}_+^*$. Now for $g, h \in G$, we have

$$g^{-1} h = y_{i_1} \cdots y_{i_n}$$

with $n = d_Y(g, h)$ and $\forall k \in \{1, \cdots, n\}, y_{i_k} \in Y \cup Y^{-1}$. Now we have

$$d_X(g, h) = d_X(g, g y_{i_1} \cdots y_{i_n}).$$

Thus, using the triangular inequality, we obtain

$$d_X(g, h) \leqslant d_X(g, g y_{i_1}) + d_X(g y_{i_1}, g y_{i_1} y_{i_2}) + \cdots + d_X(g y_{i_1} \cdots y_{i_{n-1}}, g y_{i_1} \cdots y_{i_n}).$$

Using the left-invariance of d_X gives the following inequality

$$d_X(g,h) \leqslant d_X(e, y_{i_1}) + d_X(e, y_{i_2}) + \cdots + d_X(e, y_{i_n}).$$

Now using the definition of C_1 and the fact that $n = d_Y(g,h)$, we get

$$d_X(g,h) \leqslant C_1 d_Y(g,h).$$

We set C_2 to be $\max_{x \in X \cup X^{-1}} d_Y(e,x)$. By symmetric arguments, $d_Y(g,h) \leqslant C_2 d_X(g,h)$.
But $C_2 \neq 0$ so $(1/C_2) d_Y(g,h) \leqslant d_X(g,h) \leqslant C_1 d_Y(g,h)$. To conclude, we just set $C = \max(C_1, C_2)$.

The proposition 1.1.3 motivates the definition of bilipschitz equivalence and quasi-isometry to study finitely generated groups. After defining quasi-isometries, we will try to find interesting quasi-isometry invariants.

1.2 Bilipschitz equivalences, quasi-isometries

Definition Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a map.

1. The map f is a **bilipschitz embedding** if $\exists c \in \mathbb{R}_+$ such that $\forall x, x' \in X$,

$$\frac{1}{c} d_X(x,x') \leqslant d_Y(f(x), f(x')) \leqslant c d_X(x,x').$$

2. The map f is a **bilipschitz equivalence** if it is a bilipschitz embedding, and if there is a bilipschitz embedding $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$.

3. If there is a bilipschitz equivalence $f : X \to Y$, we say that X and Y are **bilipschitz equivalent**.

Examples : One can notice that

* an isometry is a bilipchitz equivalence with $c = 1$, and
* using proposition 1.1.3, if G is a finitely generated group and if X and Y are two finite generating subsets of G, then id_G is a bilipschitz equivalence between (G, d_X) and (G, d_Y).

In order to define quasi-isometries, we introduce the following definition.

Definition Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$, $g : X \to Y$ be two maps.
The map f has **finite distance** from g if $\exists c \in \mathbb{R}_+$ such that $\forall x \in X$,

$$d_Y(f(x), g(x)) \leqslant c.$$

We now generalize isometries even further by defining quasi-isometries.

Definition Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a map.

1. The map f is a **quasi-isometric embedding** if $\exists c, b \in \mathbb{R}_+^*$ such that $\forall x, x' \in X$,

$$\frac{1}{c} d_X(x,x') - b \leqslant d_Y(f(x), f(x')) \leqslant c d_X(x,x') + b.$$

   We then say that f is a (c,b)-quasi-isometric embedding.

2. The map f is a **quasi-isometry** if it is a quasi-isometric embedding, and if there is a quasi-isometric embedding $g : Y \to X$ such that $f \circ g$ has finite distance from id_Y and $g \circ f$ has finite distance from id_X.

3. If there is a quasi-isometry $f : X \to Y$, we say that X and Y are **quasi-isometric** and we write $X \sim_{\mathrm{QI}} Y$.

Remark : Of course, a bilipschitz equivalence is a quasi-isometry.

Definition Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a map.
We say that f has quasi-dense image if $\exists c \in \mathbb{R}_+^*, \forall y \in Y, \exists x \in X$ such that

$$d_Y(f(x), y) \leqslant c.$$

In some cases, we use the following characterisation to show that two metric spaces are quasi-isometric.

Proposition 1.2.1 Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a map.
Then f is a quasi-isometry if and only if f is a quasi-isometric embedding with quasi-dense image.

Proof : First, if f is a quasi-isometry, it is a quasi-isometric embedding and there is a quasi-isometric embedding $g : Y \to X$ such that if $y \in Y$, then $d_Y(f(g(y)), y) \leqslant c$ with c being the constant appearing in the finite distance from $f \circ g$ to id_Y. Thus f has quasi-dense image.
Conversely, suppose f is a quasi-isometric embedding with dense image. Now $\exists c' \in \mathbb{R}_+^*, \forall y \in Y, \exists x_y \in X$ such that $d_Y(f(x_y), y) \leqslant c'$. The axiom of choice gives a map

$$g \ : \ \begin{array}{ccc} Y & \longrightarrow & X \\ y & \longmapsto & x_y \end{array}$$

By definition, $f \circ g$ has finite distance to id_Y. To prove that $g \circ f$ has finite distance to id_X, let $x \in X$. We have

$$d_X(g(f(x)), x) = d_X(x_{f(x)}, x).$$

But f is a quasi-isometric embedding so $\exists b, c \in \mathbb{R}_+^*$ such that

$$d_X(x_{f(x)}, x) \leqslant c d_Y(f(x_{f(x)}), f(x)) + bc \leqslant cc' + bc$$

with $cc' + bc \in \mathbb{R}_+$. This proves that $g \circ f$ has finite distance to id_X.
To conclude, we have to prove that g is a quasi-isometric embedding.
Let $y, y' \in Y$. Since f is a (c, b)-quasi-isometric embedding,

$$d_X(g(y), g(y')) = d_X(x_y, x_{y'}) \leqslant c d_Y(f(x_y), f(x_{y'})) + bc.$$

By using the triangular inequality, we get

$$c d_Y(f(x_y), f(x_{y'})) + bc \leqslant c(d_Y(f(x_y), y) + d_Y(y, y') + d_Y(y', f(x_{y'}))) + bc.$$

But $d_Y(f(x_y), y) \leqslant c'$ and $d_Y(f(x_{y'}, y') \leqslant c'$ so

$$d_X(g(y), g(y')) \leqslant c d_Y(y, y') + 2cc' + bc.$$

On the other side, we use again that f is a (c, b)-quasi-isometric embedding to write that

$$d_X(g(y), g(y')) = d_X(x_y, x_{y'}) \geqslant (1/c) d_Y(f(x_y), f(x_{y'})) - (b/c).$$

Like before, we use the triangular inequality to end up with

$$d_X(g(y), g(y')) \geqslant (1/c) d_Y(y, y') - 2(c'/c) - (b/c).$$

Setting $b' = \max(2(c'/c) + (b/c), 2cc' + bc)$, we proved that g is a (c, b')-quasi-isometric embedding.

Remark : Two finite groups are clearly quasi-isometric, using the last proposition for example.

Proposition 1.2.2 The following propositions hold.

1. If a map is at finite distance from a quasi-isometric embedding, then it is a quasi-isometric embedding. In particular, a map at finite distance from a quasi-isometry is a quasi-isometry.
2. Let X, Y, Z be metric spaces. Let $f : X \to Y$, $f' : X \to Y$ be two maps that are at finite distance.
   - If $g : Z \to X$ is a map, then $f \circ g$ and $f' \circ g$ are at finite distance.

* If $g : Y \to Z$ is a quasi-isometric embedding, then $g \circ f$ and $g \circ f'$ are at finite distance.

3. Composing quasi-isometric embeddings provides quasi-isometric embeddings.
In particular, compositions of quasi-isometries are quasi-isometries.

Proof : To prove this proposition, we only use the basic definitions and proposition 1.2.1.

1. Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f : X \to Y$ be a (c, b)-quasi-isometric embedding and $g : X \to Y$ be a map. Suppose g has finite distance from f.
By definition, $\exists c' \in \mathbb{R}_+$ such that $\forall x \in X$, $d_Y(f(x), g(x)) \leqslant c'$. Let $x, x' \in X$. By triangular inequality,
$$d_Y(f(x), f(x')) \leqslant d_Y(f(x), g(x)) + d_Y(g(x), g(x')) + d(g(x'), f(x')) \leqslant d_Y(g(x), g(x')) + 2c'.$$
On the other hand, using the triangular inequality again gives
$$d_Y(g(x), g(x')) \leqslant d_Y(g(x), f(x)) + d_Y(f(x), f(x')) + d_Y(f(x'), g(x')) \leqslant d_Y(f(x), f(x')) + 2c'.$$
Thus,
$$d_Y(f(x), f(x')) - 2c' \leqslant d_Y(g(x), g(x')) \leqslant d_Y(f(x), f(x')) + 2c'$$
so that g is a $(c, b + 2c')$-quasi-isometric embedding.
If f is in fact a quasi-isometry, then it has quasi-dense image by proposition 1.2.1.
Then, $\exists c'' \in \mathbb{R}_+^*, \forall y \in Y, \exists x \in X$ such that $d_Y(f(x), y) \leqslant c''$. Let $y \in Y$. By definition, $\exists x \in X$ such that $d_Y(f(x), y) \leqslant c''$. Notice that g has quasi-dense image since
$$d_Y(g(x), y) \leqslant d_Y(g(x), f(x)) + d_Y(f(x), y) \leqslant c' + c''.$$
According to proposition 1.2.1, g is a quasi-isometry.

2. Let X, Y, Z be metric spaces. Let $f : X \to Y$, $f' : X \to Y$ be two maps that are at finite distance.
   * Let $g : Z \to X$ be a map. The maps f and f' are at finite distance so $\exists c' \in \mathbb{R}_+, \forall x \in X$,
   $$d_Y(f(x), f'(x)) \leqslant c'.$$
   In particular, $\forall z \in Z$, we have
   $$d_Y((f \circ g)(z), (f' \circ g)(z)) = d_Y(f(g(z)), f'(g(z))) \leqslant c'.$$
   Thus, $f \circ g$ and $f' \circ g$ are at finite distance.
   * Let $g : Y \to Z$ be a (c, b)-quasi-isometric embedding. Then, $\forall x \in X$,
   $$d_Z((g \circ f)(x), (g \circ f')(x)) = d_Z(g(f(x)), g(f'(x))) \leqslant c d_Z(f(x), f'(x)) + b \leqslant cc' + b.$$
   So $g \circ f$ and $g \circ f'$ are at finite distance.

3. Let X, Y, Z be metric spaces. Let $f : X \to Y$ be a (c, b)-quasi-isometric embedding and $g : Y \to Z$ be a (c', b')-quasi-isometric embedding. Let $x, x' \in X$. Since g is a quasi-isometric embedding,
$$\frac{1}{c'} d_Y(f(x), f(x')) - b' \leqslant d_Z(g(f(x)), g(f(x'))) \leqslant c' d_Y(f(x), f(x')) + b'.$$
But f is a (c, b)-quasi-isometric embedding so that
$$\frac{1}{cc'} d_X(x, x') - \frac{b}{c'} - b' \leqslant d_Z(g(f(x)), g(f(x'))) \leqslant cc' d_X(x, x') + bc' + b'.$$
This shows that $g \circ f$ is a quasi-isometric embedding.
If g and f are both quasi-isometries, then f has quasi-dense image by proposition 1.2.1 so $\exists c'' \in \mathbb{R}_+$ such that if $z \in Z$, $\exists y \in Y$ such that $d_Z(g(y), z) \leqslant c''$. Using proposition 1.2.1 gives a $c''' \in \mathbb{R}_+$ such that $\exists x \in X$ with $d_Y(f(x), y) \leqslant c'''$. With these notations, we have
$$d_Z(g(f(x)), z) \leqslant d_Z(g(f(x)), g(y)) + d_Z(g(y), z) \leqslant c' d_Y(f(x), y) + b' + c'' \leqslant c'c''' + b' + c''.$$
That proves that $g \circ f$ has quasi-dense image, so $g \circ f$ is a quasi-isometry.

Remark : Let X be a metric space. The set of quasi-isometries $X \to X$ is a monoid according to proposition 1.2.2. This set becomes a group when we take it modulo finite distance. The group of all quasi-isometries $X \to X$ modulo finite distance is the **quasi-isometry group** of the metric space X, written $\mathrm{QI}(X)$. Furthermore, if a metric space Y is quasi-isometric to X, then the groups $\mathrm{QI}(X)$ and $\mathrm{QI}(Y)$ are isomorphic. Indeed, if $\mathcal{F} : X \to Y$ is a quasi-isometry, then modulo finite distance, $\mathcal{F}$ is one-to-one so that the map

$$\begin{array}{cccc} \varphi & : & \mathrm{QI}(X) & \longrightarrow & \mathrm{QI}(Y) \\ & & f & \longmapsto & \mathcal{F} \circ f \circ \mathcal{F}^{-1} \end{array}$$

is a group isomorphism.

Definition Let G be a finitely generated group. Let X be a metric space.

1. The group G is **bilipschitz equivalent** to X if for some finite generating set S of G, the metric space X and the group G endowed with the word metric with respect to S are bilipschitz equivalent.
2. The group G is **quasi-isometric** to X if for some finite generating set S of G, the metric space X and the group G endowed with the word metric with respect to S are quasi-isometric and we write $G \sim_{\mathrm{QI}} X$.

Example : For $n \in \mathbb{N}^*$, we have $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{R}^n$. To prove this, we endow $\mathbb{Z}^n$ with the word metric with respect to the canonical finite generating set $S = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$.

* First, the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a $(\sqrt{n}, 0)$-quasi-isometric embedding. This comes from the fact that the biggest shortcut we make by being in $\mathbb{R}^n$ instead of $\mathbb{Z}^n$ is the path going from $(0, \cdots, 0)$ to $(1, \cdots, 1)$. In $\mathbb{R}^n$, the euclidean distance is $\sqrt{n}$ while in $\mathbb{Z}^n$, the distance is n (Figure 5).

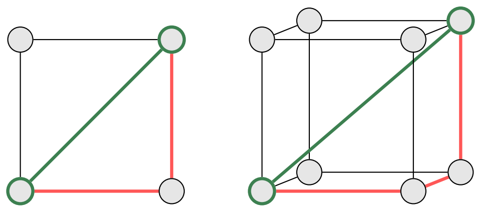


Figure 5: $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ is a $(\sqrt{n}, 0)$-quasi-isometric embedding for $n = 2$ and $n = 3$.

Indeed, for $x, x' \in \mathbb{Z}^n$, we always have $d(x, x') \leqslant d_S(x, x')$, and our point states that $d_S(x, x') \leqslant \sqrt{n} d(x, x')$.

* Now the inclusion $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ has quasi-dense image. Indeed, any element of $\mathbb{R}^n$ lies in an hypercube of side length 1 and built with integer coordinates, so the euclidean distance from this element to one of the vertices of that hypercube is at most $\sqrt{n}/2$ (Figure 6).

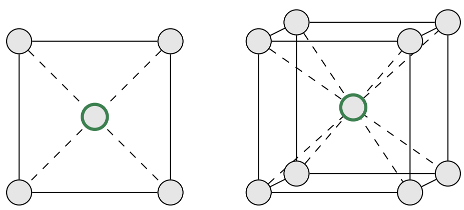


Figure 6: $\mathbb{Z}^n \hookrightarrow \mathbb{R}^n$ has quasi-dense image for $n = 2$ and $n = 3$.

Last we use proposition 1.2.1 to conclude that $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{R}^n$.

Proposition 1.2.3 Let G be a finitely generated group, H be a finite index subgroup of G. Then $H \sim_{\mathrm{QI}} G$.

Proof : To prove this, we will show that the natural inclusion $H \hookrightarrow G$ is a quasi-isometry.

* First, we show that H is finitely generated. Let $S = \{s_1, \cdots, s_m\}$ be a finite generating subset of G. Without loss of generality, we can suppose $S = S \cup S^{-1}$.
We write $G/H = \{Hg_1, \cdots, Hg_n\}$ and we set $R = \{g_1, \cdots, g_n\}$. Without loss of generality, suppose $g_1 = e$.

Now $\exists\psi : \{1,\cdots,n\}\times\{1,\cdots,m\}\to\{1,\cdots,n\}$ such that $\forall i\in\{1,\cdots,m\}, \forall j\in\{1,\cdots,n\}$, $g_j s_i \in Hg_{\psi(j,i)}$.
For $i\in\{1,\cdots,m\}$ and $j\in\{1,\cdots,n\}$, define $h_{i,j} = g_j s_i g_{\psi(j,i)}^{-1}\in H$.
If $h\in H$, then in particular, $h\in G$ so we can write

$$h = s_{\varphi(1)}\cdots s_{\varphi(r)}$$

for some $r\in\mathbb{N}$, and $\varphi:\mathbb{N}\to\{1,\cdots,m\}$. First, $s_{\varphi(1)}\in Hg_{\psi(1,\varphi(1))}$ so

$$h = h_{\varphi(1),1}g_{\psi(1,\varphi(1))}s_{\varphi(2)}\cdots s_{\varphi(r)}.$$

Now $g_{\psi(1,\varphi(1))}s_{\varphi(2)} = h_{\varphi(2),\psi(1,\varphi(1))}g_{\psi(\psi(1,\varphi(1)),\varphi(2))}$ so

$$h = h_{\varphi(1),1}h_{\varphi(2),\psi(1,\varphi(1))}g_{\psi(\psi(1,\varphi(1)),\varphi(2))}s_{\varphi(3)}\cdots s_{\varphi(r)}.$$

By repeating this process, we obtain a $k\in\{1,\cdots,n\}$ such that h is a product of elements of the form $h_{i,j}$, multiplied on the right by g_k. But then $k=1$ and $g_k=e$ since $h\in H$. Thus, any element in h can be written as a product of the form $h_{i,j}$ with finitely many $h_{i,j}$.

* Let $S'=\{s'_1,\cdots,s'_q\}$ be a finite generating subset of H and $S=\{s_1,\cdots,s_p\}$ be a finite generating subset of G. Without loss of generality, we can suppose $S'\subset S$ (adding elements to a generating set gives a generating set). Let $h,h'\in H$. Writing $h(h')^{-1}\in H$ in terms of elements in $S'\subset S$, shows that $d_G(h,h')\leqslant d_H(h,h')$.
Now, we can write $h(h')^{-1} = s_1\cdots s_p$ with $p = d_S(h,h')$ and $\forall i\in\{1,\cdots,p\}, s_i\in S\cup S^{-1}$.
But using the notations of earlier, $s_1\in G$ so $\exists c_1\in R$ such that $s_1\in Hc_1$. We write

$$h(h')^{-1} = (s_1c_1^{-1})(c_1s_2)s_3\cdots s_p.$$

Now $c_1s_2\in G$ so $\exists c_2\in R$ such that $c_1s_2\in Hc_2$. Then,

$$h(h')^{-1} = (s_1c_1^{-1})(c_1s_2c_2^{-1})(c_2s_3)\cdots s_p.$$

By induction, we have $c_1,\cdots,c_{p-1}\in R$ such that $\forall i\in\{2,\cdots,p-1\}$, $c_{i-1}s_ic_i^{-1}\in H$ and

$$h(h')^{-1} = (s_1c_1^{-1})(c_1s_2c_2^{-1})\cdots(c_{p-2}s_{p-1}c_{p-1}^{-1})(c_{p-1}s_p)$$

with $c_{p-1}s_p\in G$. But $h(h')^{-1}\in H$ so in fact, $c_{p-1}s_p\in H$. We wrote $h(h')^{-1}$ as a product of elements in

$$\Omega = \{asb \mid a,b\in K, s\in S\}\cap H$$

which is a finite set, so if C is equal to $\max_{x\in\Omega} d_{S'}(x,e)$, we have $d_{S'}(h,h')\leqslant Cd_G(h,h')$.
The inclusion $H\hookrightarrow G$ is a quasi-isometric embedding. To conclude, according to proposition 1.2.1, we just have to show that the inclusion has quasi-dense image. Let $C' = \max_{i\in\{1,\cdots,n\}} d_S(g_i,e)$. Let $g\in G$. There is a $i\in\{1,\cdots,n\}$ such that $g\in Hg_i$. Let $h = gg_i^{-1}\in H$. By left invariance and definition of C',

$$d_S(g,h) = d_S(g,gg_i^{-1}) = d_S(e,g_i^{-1}) = d_S(g_i,e)\leqslant C'.$$

Hence, $H\hookrightarrow G$ has quasi-dense image and $H\sim_{\mathrm{QI}} G$.

Remark : In a sense, quasi-isometries only look at similarities at large scale, leaving out local details.
The main problem of geometric group theory is to classify finitely generated groups up to quasi-isometries. A first step in this direction is to find some quasi-isometry invariants.

Definition Let V be a set. A **quasi-isometry invariant** with values in V is a map

$$\begin{array}{cccc} I & : & \mathcal{G}_{fg} & \longrightarrow & V \\ & & G & \longmapsto & I(G) \end{array}$$

such that $\forall G,H\in\mathcal{G}_{f,g}$, if $G\sim_{\mathrm{QI}} H$ then $I(G)=I(H)$, where $\mathcal{G}_{fg}$ is the class of all finitely generated groups.

Remark : Let I be a quasi-isometry invariant, and G,H be two finitely generated groups. If $I(G)\neq I(H)$, then G and H are not quasi-isometric. However, if $I(G)=I(H)$, then we can not conclude anything in general.

1.3 Quasi-geodesic spaces, Švarc-Milnor lemma

Finitely generated groups with the word metric are not geodesic spaces. Indeed, the word metric clearly makes our groups discrete. This is why we relax the assumptions and define quasi-geodesic spaces.

DEFINITION Let X be a metric space. Let $(c, b) \in (\mathbb{R}_+^* \times \mathbb{R}_+)$.

1. A (c, b)-**quasi-geodesic** in X is a (c, b)-quasi-isometric embedding $\gamma : [t, t'] \to X$ with $[t, t'] \subset \mathbb{R}$. The point $\gamma(t)$ is the **start point** of γ and $\gamma(t')$ is the **end point** of γ.

2. We say that X is (c, b)-**quasi-geodesic** if there exists a (c, b)-quasi-geodesic in X with start point x and end point x' for all $x, x' \in X$.

REMARK : Quasi-geodesic spaces are a generalization of geodesic spaces.

PROPOSITION 1.3.1 Finitely generated groups endowed with the word metric are $(1, 1)$-quasi-geodesic spaces.

PROOF : The distance between two elements in a finitely generated group is defined as the length of a path in a graph the associated Cayley graph.
Since the length every edge is one, we indeed obtain a $(1, 1)$-quasi-geodesic space.

We can now state a metric formulation of the Švarc-Milnor lemma.

PROPOSITION 1.3.2 Let G be a group. Let (X, d) be a (c, b)-quasi-geodesic metric space with $c, b \in \mathbb{R}_+^*$. Suppose that G acts on X by isometries, and suppose that there is a subset B of X such that :

1. The diameter of B is finite (two elements of B are always at finite distance).

2. The G-translates of B cover all of X.

3. The set $S = \{g \in G \mid g\dot{B}' \cap B' \neq \emptyset\}$ is finite, whith $B' = \{x \in X \mid \exists y \in B, d(x, y) \leqslant 2b\}$.

Then the group G is finitely generated by S, and for all $x \in X$, the action map

$$\begin{array}{cccc} \varphi & : & G & \longrightarrow & X \\ & & g & \longmapsto & g \cdot x \end{array}$$

is a quasi-isometry with respect to the word metric on G with respect to S.

This is not the main point of this document, so we refer to section 5.4 of [Lö17] for a full proof. Instead we briefly explain how this lemma is used in practice. On the first hand, this lemma shows that if a group acts by isometries on a metric space which behaves well enough, then this group is finitely generated and has the same geometric properties as this very metric space. On the other hand, to study a metric space, we can try to find a familiar group that acts well on this metric space to use what we know about groups.

2 Growth

2.1 Growth of finitely generated groups

If G is a finitely generated group and if S is a finite generating set of G, we introduce a function $\beta_{G,S}$ that measures how fast the group is growing. This function depends on the generating set.

DEFINITION Let G be a finitely generated group. Let S be a finite generating subset of G.
The **growth function** of G with respect to S is

$$\begin{array}{rccc}\beta_{G,S} & : \mathbb{N} & \longrightarrow & \mathbb{N} \\ & r & \longmapsto & \left|B_r^{G,S}(e)\right|\end{array}$$

where $\forall r \in \mathbb{N}$, the set $B_r^{G,S}(e)$ is the closed ball of center e and radius r for the word metric on G with respect to S, that is $B_r^{G,S}(e) = \{g \in G \mid d_S(g,e) \leqslant r\}$.

EXAMPLE : We give the growth function of $\mathbb{Z}$ with respect to two finite generating sets.

* Looking at $\mathbb{Z}$ with the generating set $\{1\}$, we obviously have $\forall r \in \mathbb{N}, \beta_{\mathbb{Z},\{1\}}(r) = 2r + 1$ (Figure 7).

Figure 7: $\beta_{\mathbb{Z},\{1\}}(1) = 2 \times 1 + 1 = 3$.

* Now if we use the generating set $\{2,3\}$ for $\mathbb{Z}$, we have $\beta_{\mathbb{Z},\{2,3\}}(0) = 1$, $\beta_{\mathbb{Z},\{2,3\}}(1) = 5$ and $\forall r \in \mathbb{N}$, if $r \geqslant 2$, $\beta_{\mathbb{Z},\{2,3\}}(r) = 6r + 1$. First, $r = 1$ is detailed by Figure 8.

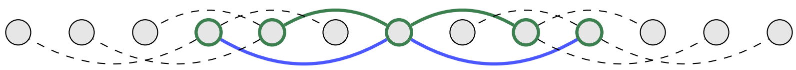


Figure 8: $\beta_{\mathbb{Z},\{2,3\}}(1) = 5$.

Then, we add 6 to the number of vertices discovered at each step (Figure 9).

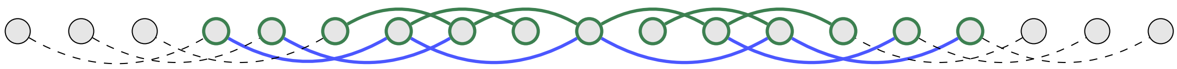


Figure 9: $\beta_{\mathbb{Z},\{2,3\}}(2) = 6 \times 2 + 1$.

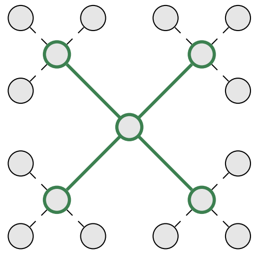
Indeed, we don't have access to more than 6 new vertices for the next step.

Notice how changing the finite generating subset did not change the fact that the growth function is affine.
We now give a more general example. In that case, the growth function increases faster.

PROPOSITION 2.1.1 Let $n \in \mathbb{N}$. Suppose $n \geqslant 2$. Let G be a free group of rank n.
Let S be a finite generating subset of G with n elements. Then, $\forall r \in \mathbb{N}$,

$$\beta_{G,S}(r) = 1 + \frac{n}{n-1}\left((2n-1)^r - 1\right).$$

PROOF : For convenience, we write $\forall r \in \mathbb{N}, u_r = \beta_{G,S}(r)$.
First, $u_0 = 1$ and since there are n elements in S, $u_1 = 1 + 2n$ (counting the inverses of the generators).
Now for $r \in \mathbb{N}$, the number of elements in the closed ball of center e and radius $r+2$ is equal to the number of elements in the closed ball of center e and radius $r+1$ added to the number of elements in the sphere of radius $r+2$. This sphere has a number of element equal to $(u_{r+1} - u_r) \times (2n-1)$ since we can discover $2n-1$ new vertices starting from an element of the sphere of radius $r+1$ (no cancellation).
See the case $n = 2$ with the following graph showing $\beta_{G,S}(1) = 1 + 4$ and $\beta_{G,S}(2) = 1 + 4 + 4 \times (2 \times 2 - 1)$.



Thus, $\forall r \in \mathbb{N}$, we have

$$u_{r+2} = u_{r+1} + (u_{r+1} - u_r) \times (2n-1) = 2nu_{r+1} - (2n-1)u_r.$$

This is a constant recursive sequence of order 2 so $\exists \lambda, \mu \in \mathbb{R}$ such that $\forall r \in \mathbb{N}$,

$$u_r = \lambda \times 1^r + \mu \times (2n-1)^r.$$

Using u_0 and u_1 gives $\mu = n/(n-1)$ and $\lambda = 1 - (n/(n-1))$.
To conclude, we just plug the values for λ and μ in our formula for u_r.

Now we give a general proposition for growth functions.

PROPOSITION 2.1.2 Let G be a finitely generated group, and S be a finite generating subset of G. Let $n = |S|$.

1. The function $\beta_{G,S}$ is **sub-multiplicative** : $\forall r, r' \in \mathbb{N}$, $\beta_{G,S}(r + r') \leqslant \beta_{G,S}(r)\beta_{G,S}(r')$.
2. If G is infinite, then $\beta_{G,S}$ is strictly increasing. In particular, $\forall r \in \mathbb{N}$, $\beta_{G,S}(r) \geqslant r$.
3. We have $\forall r \in \mathbb{N}$, $\beta_{G,S}(r) \leqslant \beta_{F(S),S}(r)$.

PROOF : We prove the three statements independently.

1. Let $r, r' \in \mathbb{N}$. Let $g \in G$. Suppose $d_S(g, e) \leqslant r + r'$. We decompose the corresponding path from e to g in :
   * a path connecting e to some $g' \in G$ of length less than r, followed by
   * a path connecting g' to g of length less than r'.

   For our first step, we have at most $\beta_{G,S}(r)$ possibilities. For the second step, we have at most $|B_{r'}^{G,S}(g')|$ choices. Now notice that the number of elements in a closed ball depends only on the radius :

   $$B_{r'}^{G,S}(g') = \{g \in G \mid d_S(g, g') \leqslant r'\} = \{g'' \in G \mid d_S(g'', e) \leqslant r'\}$$

   by using the fact that $g \mapsto (g')^{-1}g$ is a bijection, and that d_S is left-invariant by proposition 1.1.2 (then write $g'g'' = g$ so that $d_S(g, g') = d_S(g'g'', g') = d_S(g'', e)$). Thus $|B_{r'}^{G,S}(g')| = \beta_{G,S}(r')$, and

   $$\beta_{G,S}(r + r') \leqslant \beta_{G,S}(r) \left| B_{r'}^{G,S}(g') \right| = \beta_{G,S}(r)\beta_{G,S}(r').$$

2. Clearly, if $r \leqslant r'$, then $B_r^{G,S}(e) \subset B_{r'}^{G,S}(e)$ so $\beta_{G,S}$ is increasing. Now if $r < r'$ and $\beta_{G,S}(r) = \beta_{G,S}(r')$, then G is finite by the definition of the growth function. This proves the second point.
3. Using the universal property of free groups, we build a homomorphism $\varphi : F(S) \to G$ defined by $\varphi|_S = \mathrm{id}_S$. Let $r \in \mathbb{N}$. By our construction of φ, we have

   $$\beta_{G,S}(r) = \left| B_r^{G,S}(e) \right| = \left| \varphi \left(B_r^{F(S),S}(e) \right) \right|.$$

   Now notice that φ is a surjective so that

   $$\beta_{G,S}(r) = \left| \varphi \left(B_r^{F(S),S}(e) \right) \right| \leqslant \left| B_r^{F(S),S}(e) \right| = \beta_{F(S),S}(r).$$

Remark : Using 2.1.2 and 2.1.1, we see that the growth function of an infinite finitely generated group growth at least *like* an affine function, and at most *like* an exponential function. Our goal is now to give a formal meaning to growing like another function.

Definition A **generalized growth function** in an increasing function $f : \mathbb{R}_+ \to \mathbb{R}_+$.
Let f, g be two generalized growth functions. We say that g **quasi-dominates** f if $\exists c, b \in \mathbb{R}_+^*, \forall r \in \mathbb{R}_+$,

$$f(r) \leqslant cg(cr + b) + b.$$

In that case, we write $f \prec g$.
The functions f and g are **quasi-equivalent** when $f \prec g$ and $g \prec f$. In that case, we write $f \sim g$.

Remark : One can check that the quasi-equivalence relation is an equivalence relation on generalized growth functions.

Proposition 2.1.3 The growth function of a group with respect to a finite generating subset can be extended into a generalized growth function.

Proof : Let G be a finitely generated group and S be a finite generating subset of G. The map

$$\begin{array}{cccc} \tilde{\beta}_{G,S} & : & \mathbb{R}_+ & \longrightarrow & \mathbb{R}_+ \\ & & r & \longmapsto & \beta_{G,S}(\lceil r \rceil) \end{array}$$

is increasing and has the same values as $\beta_{G,S}$ over $\mathbb{N}$.

This allows to extend the definition of quasi-domination and quasi-equivalence to growth functions of finitely generated groups by looking at their associated generalized growth function.
Now we hope that two quasi-isometric groups have quasi-equivalent growth functions.

Proposition 2.1.4 Let G, H be two finitely generated groups. Let S be a finite generating subset of G and T be a finite generating subset of H. Suppose that there is a quasi-isometric embedding $f : (G, d_S) \to (H, d_T)$. Then $\beta_{G,S} \prec \beta_{H,T}$. In particular, if $G \sim_{\mathrm{QI}} H$ then $\beta_{G,S} \sim \beta_{H,T}$.

Proof : First f is a quasi-isometric embedding so $\exists c, b \in \mathbb{R}_+^*$ such that $\forall g, g' \in G$,

$$\frac{1}{c} d_S(g, g') - b \leqslant d_T(f(g), f(g')) \leqslant c d_S(g, g') + b.$$

Let $r \in \mathbb{N}$. If $g \in B_r^{G,S}(e_G)$, then

$$d_T(f(g), e_H) \leqslant c d_S(g, e_G) + b \leqslant cr + b.$$

This shows that $f\left(B_r^{G,S}(e_G)\right) \subset B_{cr+b}^{H,T}(e_H)$. To conclude, let us compare $\left|f\left(B_r^{G,S}(e_G)\right)\right|$ and $\left|B_r^{G,S}(e_G)\right|$.
The function f is not surjective, but $B_r^{G,S}(e_G)$ is finite, so

$$\left|B_r^{G,S}(e_G)\right| \leqslant \max\{n \in \mathbb{N} \mid \exists x_1, \cdots, x_n \in B_r^{G,S}(e_G), f(x_1) = \cdots = f(x_n)\} \times \left|f\left(B_r^{G,S}(e_G)\right)\right|.$$

Let $g \in B_r^{G,S}(e_G)$. If $g' \in G$ is such that $f(g) = f(g')$, then $d_S(g, g') \leqslant c d_T(f(g), f(g')) + bc = bc$ so $g' \in B_{bc}^{G,S}(g)$.
But as we already saw in the proof of proposition 2.1.2, $\left|B_{bc}^{G,S}(g)\right| = \left|B_{bc}^{G,S}(e_G)\right|$. Thus,

$$\beta_{G,S}(r) \leqslant \left|B_{bc}^{G,S}(e_G)\right| \times \left|f\left(B_r^{G,S}(e_G)\right)\right| \leqslant \left|B_{bc}^{G,S}(e_G)\right| \times \left|B_{cr+b}^{H,T}(e_H)\right| = \left|B_{bc}^{G,S}(e_G)\right| \beta_{H,T}(cr + b).$$

This shows that $\beta_{G,S} \prec \beta_{H,T}$ by choosing $\max\{c, \left|B_{bc}^{G,S}(e_G)\right|\}$.

It is now natural to define the growth type of a finitely generated group.

DEFINITION Let G be a finitely generated group.

* The **growth type** of G is the quasi-equivalence class of all growth functions of G with respect to finite generating subsets of G.

* If the growth function of G with respect to some finite generating subset S is quasi-dominated by $x \mapsto x^a$ with $a \in \mathbb{R}_+$, we say that G has **polynomial growth**.

* If G has the growth type of $x \mapsto e^x$, we say that G has **exponential growth**.

* If G is neither of polynomial growth nor of exponential growth, then G is of **intermediate growth**.

EXAMPLES : We already have examples for polynomial and exponential growth.

* The additive group $\mathbb{Z}$ is of polynomial growth (see the first example of growth function).

* By proposition 2.1.1, a free group of rank $n \geqslant 2$ has growth type $x \mapsto (2n-1)^x$.

Free groups of rank at least 2 have exponential growth according to the second point of the following proposition.

PROPOSITION 2.1.5 Let $a, d \in \mathbb{R}_+$. Then

1. If $(x \mapsto x^a) \prec (x \mapsto x^d)$, then $a \leqslant d$.
2. If $a > 1$ and $d > 1$, we have $(x \mapsto a^x) \sim (x \mapsto d^x)$.

PROOF : We use real analysis basic tools to prove this.

1. Suppose $(x \mapsto x^a) \prec (x \mapsto x^d)$. By definition, $\exists c, b \in \mathbb{R}_+^*$ such that $\forall r \in \mathbb{R}_+$,
$$r^a \leqslant c(cr+b)^d + b.$$
Now if $r \in \mathbb{R}_+$, $cr + b \neq 0$ so
$$\frac{r^a - b}{(cr+b)^d} \leqslant c.$$
The function on the left is bounded, so $a - d \leqslant 0$.

2. Now suppose $a > 1$ and $d > 1$. Let $c = \ln(d)/\ln(a)$. We have $\forall r \in \mathbb{R}_+$,
$$a^{cr} = e^{cr\ln(a)} = e^{r\ln(d)} = d^r$$
so that $a^r = d^{r/c}$. This proves that $(x \mapsto a^x) \sim (x \mapsto d^x)$ since $c \in \mathbb{R}_+^*$ and $1/c \in \mathbb{R}_+^*$.

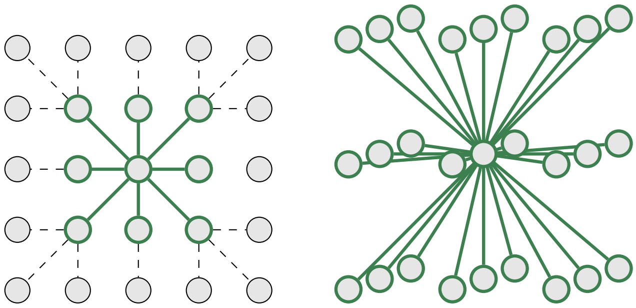
We can rephrase proposition 2.1.4 using our new vocabulary.

COROLLARY The growth type of finitely generated groups is a quasi-isometry invariant.

It is well known that a two finite-dimensional normed vector spaces are homeomorphic if and only if they have the same dimension (topological invariance of dimension). We have a similar result with quasi-isometries.

LEMMA Let $n \in \mathbb{N}$. Then $\mathbb{Z}^n$ has the growth type of $x \mapsto x^n$.

PROOF : We already covered $n = 1$, and $n = 0$ is obvious.
Now if $n \geqslant 2$, we look at $\mathbb{Z}^n$ with the set $S' = \{(x_1, \cdots, x_n) \mid \forall i \in \{1, \cdots, n\}, x_i \in \{-1, 0, 1\}\}$.
Writing $S = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$ the canonical generating subset of $\mathbb{Z}^n$, we have $\mathbb{Z}^n = \langle S' \rangle$ because $S \subset S'$. Now, if $r \in \mathbb{N}$, we have
$$\beta_{\mathbb{Z}^n, S'}(r) = (2r-1)^n.$$
Indeed, at distance r we have $2r - 1$ choices for each coordinate. See the following graphs for $n = 2$ and $n = 3$.



The group $\mathbb{Z}^n$ has the growth type of $x \mapsto x^n$ since the growth type does not depend on the finite generating subset (proposition 2.1.4 and proposition 1.1.3).

The topological invariance of dimension for quasi-isometries follows from the lemma.

PROPOSITION 2.1.6 Let $n, m \in \mathbb{N}$. Then, $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{Z}^m \Leftrightarrow n = m$. In other words, $\mathbb{R}^n \sim_{\mathrm{QI}} \mathbb{R}^m \Leftrightarrow n = m$.

PROOF : Suppose $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{Z}^m$. Using proposition 2.1.4 and the lemma, we have $(x \mapsto x^n) \sim (x \mapsto x^m)$.
Now $n = m$ by proposition 2.1.5. The second part comes from the fact that $\mathbb{Z}^n \sim_{\mathrm{QI}} \mathbb{R}^n$ (see a previous example).

REMARK : Let A be a finitely generated abelian group. The fundamental theorem of finitely generated abelian groups states that A is isomorphic to a direct product $\mathbb{Z}^n \times T$ where T is a finite abelian group. Recall that n is the **rank** of the abelian group A. We write $\mathrm{rk}_{\mathbb{Z}}(A) = n$. Now if A' is another finitely generated abelian group, combining proposition 1.2.3 and proposition 2.1.6, we have $A \sim_{\mathrm{QI}} A' \Leftrightarrow \mathrm{rk}_{\mathbb{Z}}(A) = \mathrm{rk}_{\mathbb{Z}}(A')$.
We will now study the growth of nilpotent groups.

2.2 Nilpotent groups, polynomial growth

NOTATION : If H and K are two subgroups of a group G, we set $[H, K]$ to be the subgroup of G generated by the commutators $[h, k] = hkh^{-1}k^{-1}$ for $h \in H$ and $k \in K$.

DEFINITION Let G be a group. Let $C_{(1)}(G) = G$ and $\forall n \in \mathbb{N}^*$, $C_{(n+1)}(G) = [G, C_{(n)}(G)]$.
The subgroups $(C_{(n)}(G))_{n \in \mathbb{N}^*}$ form the **lower central series** of G.
Let $k \in \mathbb{N}^*$. The group G is k-step **nilpotent** if $C_{(k+1)(G)}$ is the trivial group.
If G is nilpotent, the minimal $k \in \mathbb{N}^*$ such that $C_{(k+1)(G)}$ is the trivial group is the **nilpotency class** of G.

REMARK : Any nilpotent group is solvable. Indeed, we see by induction that the terms of the derived series appear as subsets of terms of the lower central series.

PROPOSITION 2.2.1 Let G be a group.
Then, for $n \in \mathbb{N}^*, C_{(n+1)}(G)$ is a characteristic subgroup of $C_{(n)}(G)$, and $C_{(n)}(G)/C_{(n+1)}(G)$ is abelian.

PROOF : For $n = 1$, we have to show that $D(G) = [G, G]$ is a characteristic subgroup of G and that the quotient $G/D(G)$ is abelian. This very well known fact is true because commutators are stable under automorphisms and commutators generates $D(G)$. The quotient group is called the abelianization of G and it is easy to check that it is abelian.
Now let $n \in \mathbb{N}^*$. Suppose $C_{(n+1)}(G)$ is a characteristic subgroup of $C_{(n)}(G)$ and $C_{(n)}(G)/C_{(n+1)}(G)$ is abelian.
We are going to show : $C_{(n+2)}(G)$ is a characteristic subgroup of $C_{(n+1)}(G)$, and $C_{(n+1)}(G)/C_{(n+2)}(G)$ is abelian.
Let φ be an automorphism of $C_{(n+1)}(G)$. Generators of $C_{(n+2)}(G)$ are $[g, x]$ with $g \in G$ and $x \in C_{(n+1)}(G)$.
But $\varphi([g, x]) = \varphi(gxg^{-1}x^{-1}) = \varphi(g)\varphi(x)\varphi(g)^{-1}\varphi(x)^{-1} = [\varphi(g), \varphi(x)]$ with $\varphi(g) \in G$ and $\varphi(x) \in C_{(n+1)}(G)$ by definition of φ. Thus $\varphi([g, x]) \in C_{(n+2)}(G)$ and $C_{(n+2)}(G)$ is a characteristic subgroup of $C_{(n+1)}(G)$.
Let $a, b \in C_{(n+1)}(G)$. To show that $(aC_{(n+2)}(G))(bC_{(n+2)}(G)) = (bC_{(n+2)}(G))(aC_{(n+2)}(G))$ we just have to prove one inclusion because the statement is symmetric. Let $g, h \in G$ and $x, y \in C_{(n+1)}(G)$. We start by noticing that

$$a[g, x]b[h, y] = bb^{-1}a[g, x]ba^{-1}a[h, y] = b(b^{-1}a)[g, x](b^{-1}a)^{-1}a[h, y]$$

so that

$$a[g,x]b[h,y] = b[(b^{-1}a)g(b^{-1}a)^{-1},(b^{-1}a)x(b^{-1}a)^{-1}]a[h,y]$$

with $(b^{-1}a)g(b^{-1}a)^{-1} \in G$ and $(b^{-1}a)x(b^{-1}a)^{-1} \in C_{(n+1)}(G)$. Thus if X, Y are commutator generators of $C_{(n+2)}(G)$, then $aXbY = bX'aY$ with X' a commutator generator of $C_{(n+2)}(G)$. In fact, Y can even be an element of $C_{(n+2)}(G)$ as it does not interfere with $aXb = bX'a$.
This last equality can be used to generalize to products of commutators, so that the quotient is abelian.

Now that we know that the quotients of two consecutive groups in the lower central series of G are abelian, we can define the homogeneous dimension of a finitely generated nilpotent group.

Definition Let G be a finitely generated nilpotent group of nilpotency class k.
The **homogeneous dimension** of G is the integer

$$d(G) = \sum_{i=1}^{k} i \operatorname{rk}_{\mathbb{Z}}(C_{(i)}(G)/C_{(i+1)}(G)).$$

Note that $d(G)$ is an integer because of proposition 2.2.1. The sum is finite by definition of nilpotency.

Theorem 2.2.1 Let G be a finitely generated nilpotent group. Then G has the growth type of $x \mapsto x^{d(G)}$.

This theorem requires multiple lemmas that are not detailed here. Instead we only outline the idea of the proof (see paragraph 14.2 of [DK18] for details).

Idea We work by induction on the nilpotency class of G.
If G has nilpotency class 1, then $[G,G]$ is trivial so that G is abelian. As a finitely generated abelian group, G has the same rank as $\mathbb{Z}^{\operatorname{rk}_{\mathbb{Z}}(G)}$. This shows that $G \sim_{\mathrm{QI}} \mathbb{Z}^{\operatorname{rk}_{\mathbb{Z}}(G)}$ with $\operatorname{rk}_{\mathbb{Z}}(G) = d(G)$ so that G has the growth rate of $x \mapsto x^{d(G)}$.
Now if $k \geqslant 2$ is an integer, we suppose that for all finitely generated nilpotent group of nilpotency class less than $k-1$ our statement holds. Then we take G a finitely generated nilpotent group of nilpotency class k.
We then set H to be $C_{(k)}(G)$ and work with the homogeneous dimension of G/H. Using the induction hypothesis on G/H and lemmas allows us to conclude.

In fact, a finitely generated group that has polynomial growth is virtually nilpotent (has a nilpotent subgroup of finite index). The following theorem is known as Gromov's polynomial growth theorem.

Theorem 2.2.2 Let G be a finitely generated group.
Then G has polynomial growth if and only if G is virtually nilpotent.

The book [Lö17] contains references for a proof of this theorem. We can easily deduce the following proposition.

Proposition 2.2.2 A finitely generated group that has polynomial growth has the growth type of $x \mapsto x^n$ for some $n \in \mathbb{N}$.

Proof : A finitely generated group with polynomial growth is virtually nilpotent. Such group G has the growth type $x \mapsto x^{d(G)}$ with $d(G) \in \mathbb{N}$.

2.3 Cactus groups

The main goal of this part is to study the growth of Cactus groups. First, we recall the definition of cactus groups by generators and relations.

Definition Let $n \in \mathbb{N}$. Suppose $n \geqslant 2$. The **cactus group** J_n is the group finitely generated by

$$s_{p,q} \text{ for } (p,q) \in \mathbb{N}^2 \text{ such that } 1 \leqslant p < q \leqslant n$$

with the following relations $\forall (p,q,m,r) \in \mathbb{N}^4$ such that $1 \leqslant p < q \leqslant n$ et $1 \leqslant m < r \leqslant n$,

* $s_{p,q}^2 = 1$.
* if $[p,q] \cap [m,r] = \emptyset$, then $s_{p,q}s_{m,r} = s_{m,r}s_{p,q}$.
* if $[m,r] \subset [p,q]$, then $s_{p,q}s_{m,r} = s_{p+q-r,p+q-m}s_{p,q}$.

REMARK : The cactus group J_2 is isomorphic to the cyclic group of order 2, and J_2 is the only finite cactus groups. Indeed if $n \geqslant 3$, then $s_{1,2}s_{2,3}$ has infinite order since there is no relation to simplify a power of this element.

IDEA We are going to study the geometry of the Cayley graph of J_3 and J_4. As stated in proposition 1.1.3, the chosen generating subset does not change the large scale geometric properties of the group, so we would like to reduce the number of generators to have a clear picture of how the group looks.
In order to remove useless generators and relations from our groups, we use **Tietze transformations**. These transformations work as follows. In a presentation of a group, we can

* **remove a generator** if one of the relations expresses this generator as a word in the other generators. As a result, we have to replace any occurrence of this generator by the word in the other generators.
* **remove a relation** if this relation can be obtained using the other relations.

We will now apply these two rules to the definition of J_3 and J_4.

PROPOSITION 2.3.1 There is an isomorphism between J_3 and $\langle s_{1,2}, s_{1,3} \mid s_{1,2}^2 = s_{1,3}^2 = 1\rangle$.

PROOF : The definition of J_3 gives the following relations :

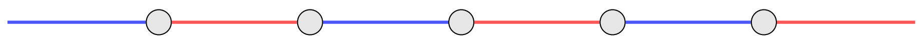
$$s_{1,2}^2 = s_{1,3}^2 = s_{2,3}^2 = 1 \qquad s_{1,3}s_{1,2} = s_{2,3}s_{1,3} \qquad s_{1,3}s_{2,3} = s_{1,2}s_{1,3}.$$

But if $s_{1,3}s_{1,2} = s_{2,3}s_{1,3}$, $s_{1,3}^2s_{1,2}s_{1,3} = s_{1,3}s_{2,3}s_{1,3}^2$ and using the fact that $s_{1,3}^2 = 1$ gives $s_{1,2}s_{1,3} = s_{1,3}s_{2,3}$ so we can get rid of the relation $s_{1,3}s_{2,3} = s_{1,2}s_{1,3}$. Now $s_{2,3} = s_{1,3}s_{1,2}s_{1,3}$ using the relations $s_{1,3}s_{1,2} = s_{2,3}s_{1,3}$ and $s_{1,3}^2 = 1$. Thus, we can remove this relation and the generator $s_{2,3}$. At this point, we have the following relations

$$s_{1,2}^2 = s_{1,3}^2 = (s_{1,3}s_{1,2}s_{1,3})^2 = 1.$$

But the relation $(s_{1,3}s_{1,2}s_{1,3})^2 = 1$ is obtained using the other two relations $s_{1,2}^2 = s_{1,3}^2 = 1$.
Thus we are allowed to only keep $s_{1,2}, s_{1,3}$ and the relations $s_{1,2}^2 = s_{1,3}^2 = 1$.

COROLLARY The Cayley graph of J_3 with the presentation of proposition 2.3.1 is the following.



The blue color corresponds to $s_{1,2}$ and the red color corresponds to $s_{1,3}$.
Thus, we have a quasi-isometry $J_3 \sim_{\mathrm{QI}} \mathbb{Z}$. In particular, J_3 has linear growth according to proposition 2.1.4.

We saw that J_2 is finite and that J_3 is quasi-isometric to $\mathbb{Z}$. Our next stop is J_4. First, we apply Tietze transformations.

PROPOSITION 2.3.2 There is an isomorphism between J_4 and the following group :

$$\langle s_{1,2}, s_{1,3}, s_{1,4} \mid s_{1,2}^2 = s_{1,3}^2 = s_{1,4}^2 = (s_{1,2}s_{1,4})^4 = (s_{1,4}s_{1,3}s_{1,2}s_{1,3})^2 = 1\rangle.$$

PROOF : As we saw with the group J_3, the relations of the type $s_{1,3}s_{1,2} = s_{2,3}s_{1,3}$ and $s_{1,3}s_{2,3} = s_{1,2}s_{1,3}$ are in fact the same, so we only pick one of them in our starting relations. We list the remaining relations.

* $s_{1,2}^2 = s_{1,3}^2 = s_{1,4}^2 = s_{2,3}^2 = s_{2,4}^2 = s_{3,4}^2 = 1$ and $s_{1,2}s_{3,4} = s_{3,4}s_{1,2}$.
* $s_{1,4}s_{1,2} = s_{3,4}s_{1,4}$, $s_{1,4}s_{1,3} = s_{2,4}s_{1,4}$ and $s_{1,4}s_{2,3} = s_{2,3}s_{1,4}$.
* $s_{1,3}s_{1,2} = s_{2,3}s_{1,3}$ and $s_{2,4}s_{2,3} = s_{3,4}s_{2,4}$.

Now $s_{1,4}s_{1,2} = s_{3,4}s_{1,4}$ gives $s_{3,4} = s_{1,4}s_{1,2}s_{1,4}$ so we can remove the relation $s_{1,4}s_{1,2} = s_{3,4}s_{1,4}$, the generator $s_{3,4}$ and the relation $s_{3,4}^2 = 1$ becomes trivial as with the case of J_3. We make the exact same work to write $s_{2,4} = s_{1,4}s_{1,3}s_{1,4}$ and $s_{2,3} = s_{1,3}s_{1,2}s_{1,3}$. We are left with the generators $s_{1,2}, s_{1,3}, s_{1,4}$ and the relations

$$s_{1,2}^2 = s_{1,3}^2 = s_{1,4}^2 = 1 \qquad s_{1,2}s_{3,4} = s_{3,4}s_{1,2} \qquad s_{2,4}s_{2,3} = s_{3,4}s_{2,4}$$

where we just have to replace $s_{3,4}, s_{2,3}$ and $s_{2,4}$ with words in $s_{1,2}, s_{1,3}, s_{1,4}$. Doing so gives us the result.

We now draw the Cayley graph of J_4, with respect to this generating subset. We use blue for $s_{1,2}$, green for $s_{1,4}$ and red for $s_{1,3}$. Just drawing the relevant relations using our colors gives these octagons (Figure 10).

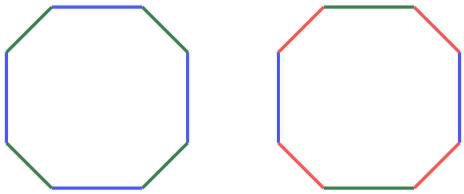


Figure 10: $(s_{1,2}s_{1,4})^4 = 1$ and $(s_{1,4}s_{1,3}s_{1,2}s_{1,3})^2 = 1$.

That means that we must find these octagons in the Cayley graph of J_4. Now we start from our three generators of order two and we grow the corresponding graph until we reach depth 3 (Figure 11).

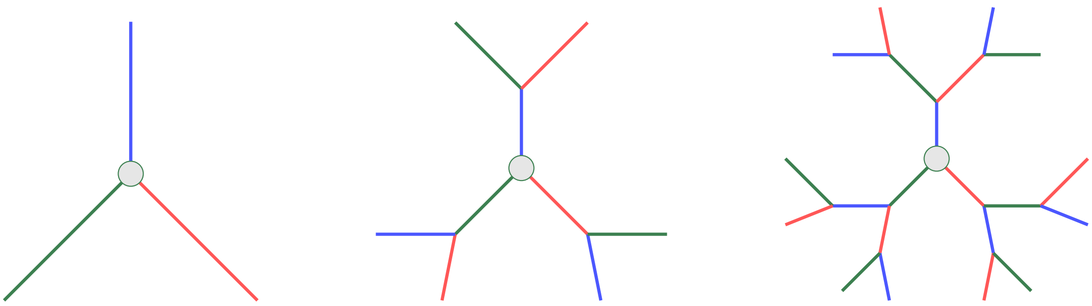


Figure 11: Cayley graph of J_4 up to depth 1, 2 and 3.

For depth 4 and deeper, things change as relations must form. Indeed, going from depth 3 to depth 4, we are going to complete 3 octagons, one coming from the relation $(s_{1,2}s_{1,4})^4 = 1$ and two coming from the other non-trivial relation $(s_{1,4}s_{1,3}s_{1,2}s_{1,3})^2 = 1$. At depth 5, we get the following graph (Figure 12).

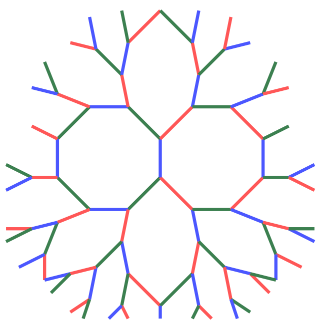


Figure 12: Cayley graph of J_4.

The colors don't really matter here, what we can see just by looking at the shape of this graph and understanding how it grows, is that it looks like an octagonal tiling of the hyperbolic plane. For now we will try to find the exact growth function β of J_4 with respect to the generating subset $\{s_{1,2}, s_{1,3}, s_{1,4}\}$. First of all, we can try to count by hand and find a pattern.

r	0	1	2	3	4	5
$\beta(r)$	1	4	10	22	43	79

Using version 10.6 of SageMath (see [The25]), we can speed up the counting process. We get the following table.

r	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\beta(r)$	1	4	10	22	43	79	142	250	436	757	1309	2260	3898	6718	11575	19939

Here is the code used to find these values. It just builds every word of length i and checks if we already saw it.

```
J4 = groups.misc.Cactus(4) # Cactus groups are already implemented in SageMath.
gens = J4.group_generators()
generators, words, k = [gens[(1, i)] for i in range(2, 5)], [J4.one()], 0
for i in range(16): # We list words of length i in the list words.
    new_words, m = words[:], len(words)
    for j in range(k,m): # We start at k because we only want words of length i-1.
        for gen in generators:
            new_word = words[j]*gen # With our choice of k, words[j] has length i-1.
            if not new_word in new_words:
                new_words.append(new_word)
    k = len(words)-1
    print("(", i,",", len(words), ")") # This is where we read the values of beta(r).
    words = new_words
```

Now can we find a pattern in the table ? Let $(s_n)_{n\in\mathbb{N}^*}$ be the sequence defined by $\forall n \in \mathbb{N}^*$, $s_n = \beta(n) - \beta(n-1)$. The integer s_n represents the number of elements in the sphere of radius n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
s_n	3	6	12	21	36	63	108	186	321	552	951	1638	2820	4857	8364

Remark : It seems like $\forall n \in \mathbb{N}^*$, we have $s_{n+4} = s_{n+3} + s_{n+2} + s_{n+1} - s_n$. For example,

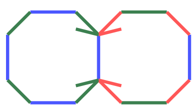
* $s_4 + s_3 + s_2 - s_1 = 21 + 12 + 6 - 3 = 36 = s_5$,
* $s_5 + s_4 + s_3 - s_2 = 36 + 21 + 12 - 6 = 63 = s_6$ and
* $s_6 + s_5 + s_4 - s_3 = 63 + 36 + 21 - 12 = 108 = s_7$.

We now prove all of our previous statements about this Cayley graph.

Lemma The Cayley graph of J_4 with respect to the generating subset $\{s_{1,2}, s_{1,3}, s_{1,4}\}$ can be seen as an octagonal tiling of the hyperbolic plane.

Proof : First, we state that an edge spans exactly two octagons. We have three cases.

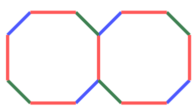
* If the edge is blue, then we can glue two different relator discs on it. If we try to add one more, by the symmetries of our relators, it will be one that is already glued.



Trying to glue an additional octagon.

Indeed, if we glue a new tilted relator on our blue central edge, it will span the same octagons.

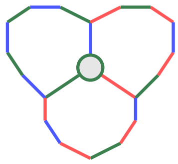
* If the edge is green, we use the exact same argument as for the blue one (we can exchange green and blue edges in our relators).
* If the edge is red, we can glue two different octagons to it, and not one more seeing our relators (a red edge in our relators has a blue edge on one side and a green edge on the other side).



As before, we can only glue a relator that spans one of the octagons already here.

Thus, a red edge also has exactly two octagons around it.

Now considering a vertex, it is connected to three other vertices by three different colors. These three vertices have two octagons around them for at most six octagons in total. In fact, we only get three octagons because we know two consecutive sides for our octagons which decides the relator we are going to glue.

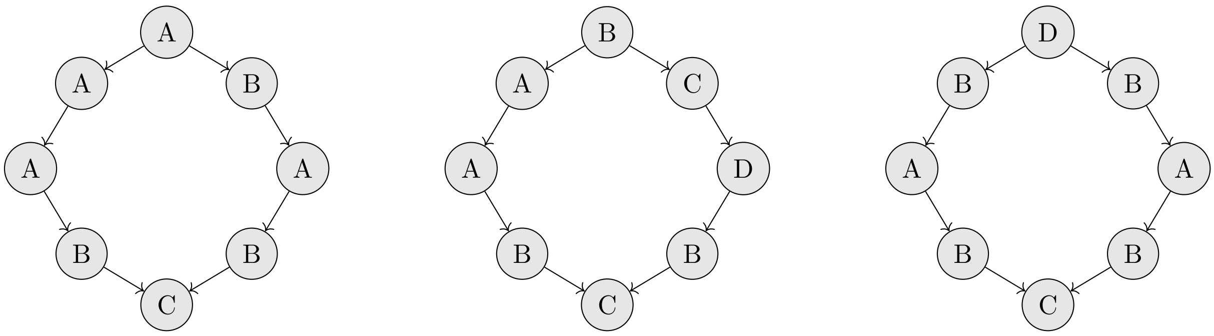


Around a vertex are three octagons.

We get the same result if we swap blue and green to get the other configuration for the colors around a vertex. This explains why our graph can be seen as an octagonal tiling of the hyperbolic plane.

We are now going to label the vertices of our Cayley graph.

PROPOSITION 2.3.3 There exists a vertex labelling with letters **A**, **B**, **C** and **D** such that any octagon that does not contain the identity element or one of the three generators is one of (or an axial reflection of) the following octagon types

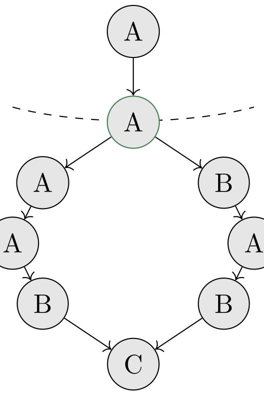


with arrows meaning increasing distance. We name the three types 1, 2 and 4 based on the starting letter of the octagon in the alphabet.

PROOF : We are going to prove the following statement by induction on n.
For all $n \in \mathbb{N}$ such that $n \geqslant 2$, there is a vertex labelling of the ball of radius $n+3$ and center 1 such that all octagons that intersect the ball of radius n and center 1 in an arc of at least three vertices are of said type.
The right vertex labelling for the case $n = 2$ is six type 1 octagons.
Now let $n \in \mathbb{N}$ with $n \geqslant 2$ and suppose that there is a vertex labelling of the ball of radius $n+3$ and center 1 such that all octagons that intersect the ball of radius n and center 1 in an arc of at least three vertices are of said type. How can we extend this labelling to the next step ? We extend the labelling by looking at every case.

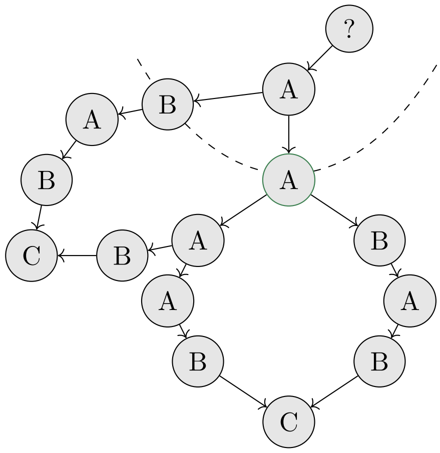
1. If we are at distance n on a vertex labelled **A**, then by induction hypothesis, there is only one vertex above (closer to identity) and two vertices below. Above this **A**, there is a **A** or a **B** by induction hypothesis (previous octagons are of type 1, 2 or 4).

   - If we have a **A** at distance n and a **A** at distance $n-1$ just above, then we label as follows.



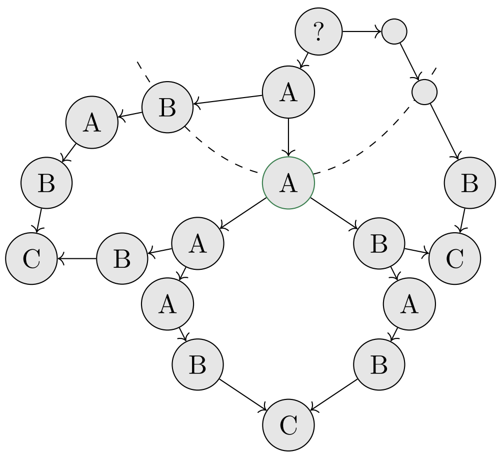
The dashed line represents the radius n ball.

We have to check if this labelling is compatible with what was the previous labelling. On the top left, we have a **A**, **A**, **A** suite so there must be a type 1 octagon that starts at the first **A**. The setup must be as follows (notice how our choice for putting **A** on the left and **B** on the right does not matter).

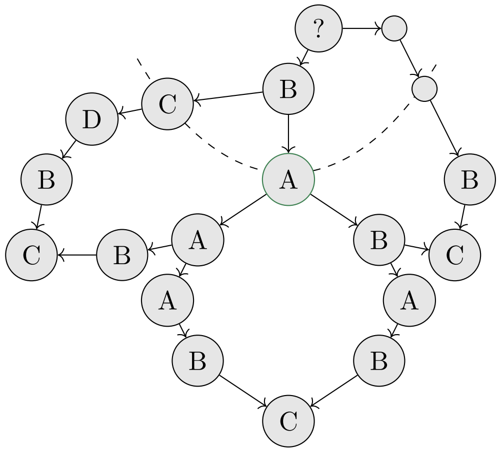


This situation is forced.

Now the octagon on the right must end with **B C** on both sides but there is no additional constraints. In the end we obtain something like below. Notice how the next two octagons are going to be labelled using our three types.



- The second case is when there is a **B** above the **A**. On one side, the **B**, **A**, **A** forces a type 2 octagon, and on the other side a **B**, **A**, **B** situation gives a **C** next.

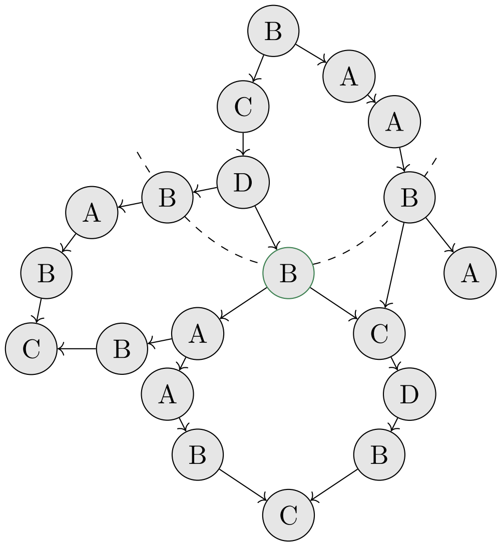


This situation is forced in this case.

Again, we can can choose the next two octagons in our list of octagon types.

2. Next is the case where we are at distance n on a vertex labelled **B**. Again there are two sub-cases.

   - If the vertex above is a **D**, then the **D**, **B**, **A** suite forces a type 4 octagon one one side. The other side is on a **D**, **B**, **C** situation that only happens at the end of type 2 octagons. In the end, the labelling looks like this.

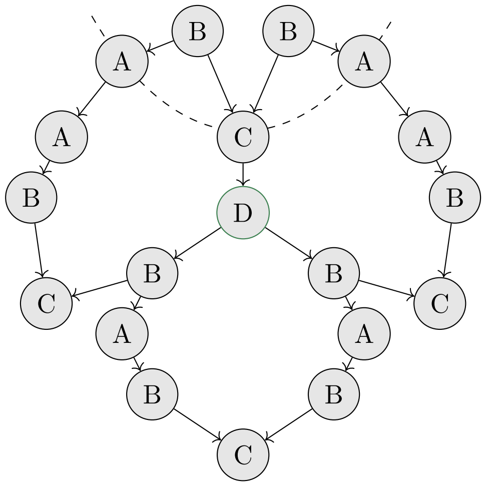


This situation arises when we have a **D** above a **B**.

   On the left we can pick a type 1 octagon, and on the right, the choice is already made for a type 2.

   - The second sub-case is when there is a **A** before the **B**. On one side we obtain a type 1 octagon, while the other sides ends in a type 2 or 4 due to the **A**, **B**, **C** chain. The result we obtain works just like the previous sub-case.

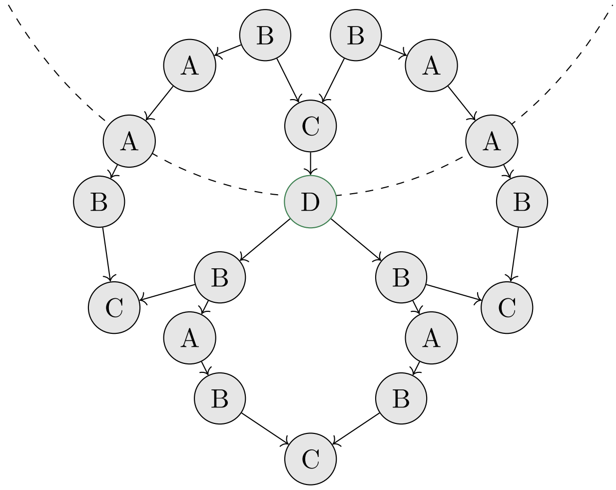
3. Our next case is when we are at a **C** vertex at distance n. By induction hypothesis, there is no starting octagon from a **C** type vertex. Furthermore, above a **C** type vertex, there could only be a **B** type vertex and after a **C** we can only have a **D**. Thus we have a **B**, **C**, **D** on each side so by induction hypothesis, they must both be type 2 octagons.



This is forced for a **C** type at distance n.

We can extend this using our octagon types.

4. Our last case is the easiest. If a **D** type is at distance n, then there must be a **C** just above, and two **B** below. We conclude by just looking at our previous case graph (we can fill with two type 2 octagons).

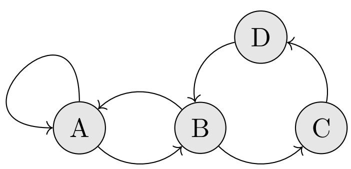


Forced situation when looking at a **D** type vertex.

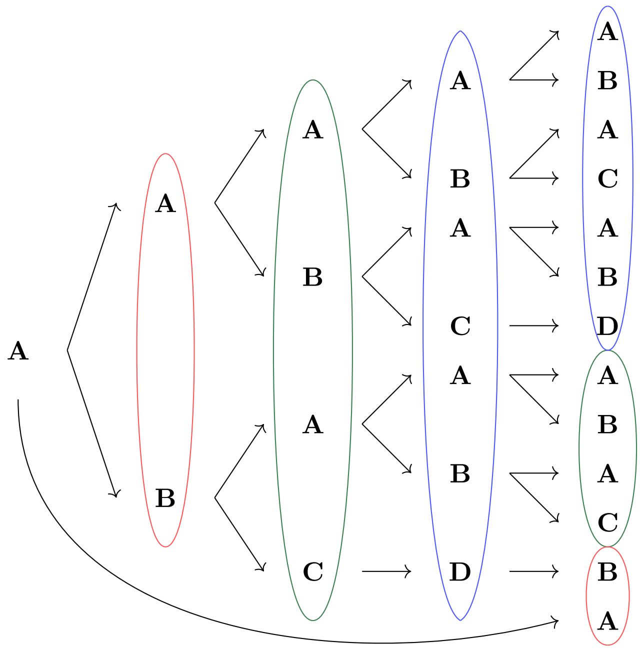
This concludes our proof by induction as every case has been covered.

Proposition 2.3.4 For $n \in \mathbb{N}^*$, $s_{n+4} = s_{n+3} + s_{n+2} + s_{n+1} - s_n$.

Proof : Using proposition 2.3.3, we obtain rules for what vertex type is next according to the vertex type we are currently. This is summarized in the following graph.



For example, after a vertex of type **A** comes a vertex of type **A** and a vertex of type **B**. Since we started our induction with **A** types, let us look at how a type **A** vertex grows (what goes next ?).



At the fifth step, we see that we have the fourth step (in blue), the third step (in green) and the second step (in red) except we don't have the whole second step as we don't have the first step. This goes to show that the announced recurrence formula holds.

We have everything we need to get the exact formula for the growth function of J_4 with our reduced set of generators.

PROPOSITION 2.3.5 Let $P = X^4 - X^3 - X^2 - X + 1$ in $\mathbb{C}[X]$.
The polynomial P has two real roots a and b and two conjugate complex roots c and $\overline{c}$. The complex roots and one of the real roots (say b) have modulus strictly smaller than 1. With these notations,

$$a = \frac{\sqrt{13} + 1 + \sqrt{2(\sqrt{13} - 1)}}{4}$$

and $\beta(r)$ grows like a^r.

PROOF : First, using proposition 2.3.4, $\exists \lambda_a, \lambda_b, \lambda_c, \lambda_{\overline{c}} \in \mathbb{C}$ such that $\forall n \in \mathbb{N}^*$, we have

$$s_n = \lambda_a a^n + \lambda_b b^n + \lambda_c c^n + \lambda_{\overline{c}} \overline{c}^n.$$

The numbers $\lambda_a, \lambda_b, \lambda_c, \lambda_{\overline{c}}$ are found easily using the first four values of our sequence and inverting a four by four matrix (numerically). With the definition of $\beta(r)$, if $r \in \mathbb{N}^*$, we have

$$\beta(r) - \beta(0) = \sum_{k=1}^{r} s_k = \lambda_a \sum_{k=1}^{r} a^k + \lambda_b \sum_{k=1}^{r} b^k + \lambda_c \sum_{k=1}^{r} c^k + \lambda_{\overline{c}} \sum_{k=1}^{r} \overline{c}^k$$

so that

$$\beta(r) = 1 + \lambda_a \frac{a^{r+1} - a}{a - 1} + \lambda_b \frac{b^{r+1} - b}{b - 1} + \lambda_c \frac{c^{r+1} - c}{c - 1} + \lambda_{\overline{c}} \frac{\overline{c}^{r+1} - \overline{c}}{\overline{c} - 1}.$$

If we only keep the dominant term, we find that $\beta(r)$ grows like λa^r with $\lambda = \lambda_a/(1 - (1/a))$.

Now to check our result, we can plot the exact points we found using SageMath (see [The25]) and the curve $r \mapsto \lambda a^r$ with $\lambda \simeq 5.766$ and $a \simeq 1.7221$ (Figure 13). The value for λ is obtained by using the exact value of λ_a and the exact value of a (the exact values were found using a symbolic solving package that solved the polynomial equation explicitly and inverted the right matrix).

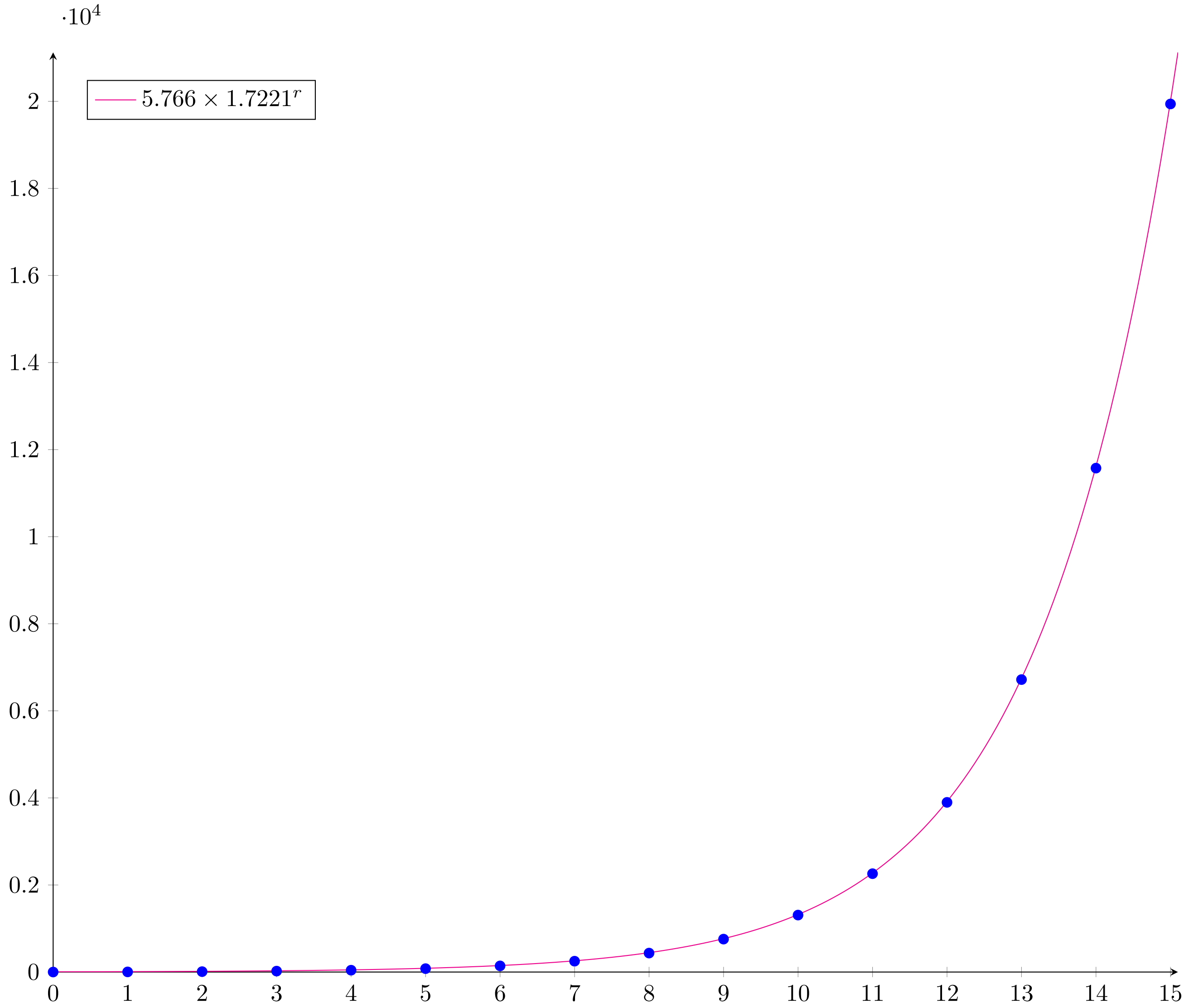


Figure 13: The blue dots are the exact values of $\beta(r)$ for $r \in \{0, \cdots, 15\}$.

PROPOSITION 2.3.6 Let G be a finitely generated group. Let H be a finitely generated subgroup of G. Let T be a finite generating subset of H and S be a finite generating subset of G. Then, $\beta_{H,T} \prec \beta_{G,S}$.

PROOF : If we set S' to be $S \cup T$, then S' is a finite generating subset of G.
Now with S', it is clear that $\beta_{H,T} \prec \beta_{G,S'}$ since a ball of radius r in H with respect to T is always in the ball of radius r in G with respect to S' (there might be a shortcut using elements in $S' \setminus T = S$.
But G with the word metric with respect to S' is quasi-isometric to G with the word metric with respect to S, so using proposition 2.1.4, we can conclude.

Using proposition 2.3.6, we deduce the following theorem on the growth of cactus groups.

THEOREM 2.3.1 Let $n \in \mathbb{N}$. Suppose $n \geqslant 4$. Then the cactus group J_n has exponential growth.

PROOF : We write β_n the growth function of J_n with respect to the canonical generators.
By proposition 2.3.5, the group J_4 has exponential growth with base $a \simeq 1.7221$ so that $a > 1$.
Now J_n has a subgroup isomorphic to J_4, by taking only generators of J_4. By proposition 2.3.6, $(r \mapsto a^r) \prec \beta_n$.
Now we combine 2.1.2 and 2.1.1 to obtain $\beta_n \prec (r \mapsto m^r)$ with $m = n(n-1)-1$ (there are $n(n-1)/2$ generators).
But $n \geqslant 4$ so $m = n(n-1) - 1 \geqslant 4 \times 3 - 1 = 11$ so $m > 1$. So by proposition 2.1.5, $(r \mapsto m^r) \sim (r \mapsto a^r)$.

This proves that $\beta_n \sim (r \mapsto a^r) \sim (r \mapsto m^r)$ and that β_n has exponential growth.

Now, we try to do the same with J_5. First we use Tietze transformations as we did with J_4. This is a long process that we will not detail, but the calculations were checked using GAP (see [Gap]). The result is that J_5 is generated by four elements $s_{1,2}, s_{1,3}, s_{1,4}, s_{1,5}$ of order 2 with six relations (does not include the fact that the generators are of order 2). If we keep blue for $s_{1,2}$, red for $s_{1,3}$, green for $s_{1,4}$ and add black for $s_{1,5}$, the six relations are all shaped like the following octagons (Figure 14).

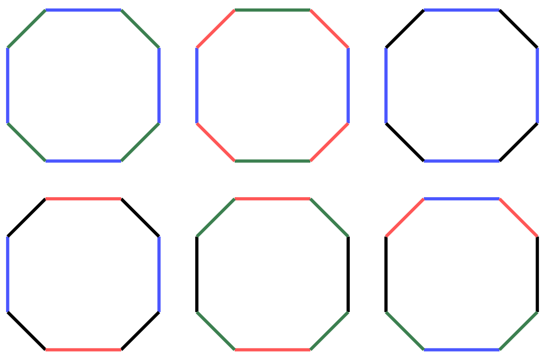


Figure 14: The six non-trivial relations required to generate J_5 with only four generators.

It is very difficult to say how the whole Cayley graph of J_5 with this generating subset looks like.

3 Amenability

Notation : If X is a set, we write $\ell^\infty(X, \mathbb{R})$ the set of all bounded functions $X \to \mathbb{R}$.

3.1 Invariant means

Our first definition of amenability involves invariant means on a specific real vector space.

Definition Let G be a group.
A G-**invariant mean** on $\ell^\infty(G, \mathbb{R})$ is a $\mathbb{R}$-linear map $m : \ell^\infty(G, \mathbb{R}) \to \mathbb{R}$ such that

1. $m(x \mapsto 1) = 1$.
2. $\forall f \in \ell^\infty(G, \mathbb{R})$, if $\forall x \in G,\ f(x) \geqslant 0$, then $m(f) \geqslant 0$.
3. $\forall g \in G, \forall f \in \ell^\infty(G, \mathbb{R})$, we have $m(g \cdot f) = m(f)$ where $g \cdot f : x \mapsto f(g^{-1}x)$.

If there exists a G-invariant mean on $\ell^\infty(G, \mathbb{R})$, we say that G is **amenable**.

Example : Let G be a finite group. We introduce the following well defined map

$$\begin{array}{rccc} m & : & \ell^\infty(G, \mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & f & \longmapsto & \frac{1}{|G|} \sum\limits_{h \in G} f(h) \end{array} .$$

Of course, if $f \in \ell^\infty(G, \mathbb{R})$, then $m(f) \in \mathbb{R}$ because G is finite and f is bounded.
Clearly, the map m is $\mathbb{R}$-linear, $m(x \mapsto 1) = 1$, and if $f : G \to \mathbb{R}$ is a positive bounded function, then $m(f) \geqslant 0$.
Now let $g \in G$ and let $f \in \ell^\infty(G, \mathbb{R})$. Now we have

$$m(g \cdot f) = \frac{1}{|G|} \sum_{h \in G} (g \cdot f)(h) = \frac{1}{|G|} \sum_{h \in G} f(g^{-1}h) = \frac{1}{|G|} \sum_{h' \in G} f(h') = m(f)$$

since the map $h' \mapsto g^{-1}h$ is a bijection. Thus G is amenable. We just proved the following proposition.

Proposition 3.1.1 Finite groups are amenable.

There is another definition for amenability.

Proposition 3.1.2 Let G be a group. Then G is amenable if and only if $\exists \mu : \mathcal{P}(G) \to [0,1]$ such that

1. $\mu(G) = 1$
2. $\forall S, T \in \mathcal{P}(G)$, if $S \cap T = \emptyset$ then $\mu(S \cup T) = \mu(S) + \mu(T)$.
3. $\forall S \in \mathcal{P}(G), \forall g \in G, \mu(gS) = \mu(S)$.

We use the term **measure** for μ as an abuse of notation (here μ is not necessarily sigma-additive).

Proof : Suppose that such a map μ exists. We can define a map

$$\begin{array}{cccc} m & : & \ell^\infty(G,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & f & \longmapsto & \int_G f d\mu \end{array}$$

where the integral is defined just as the Lebesgue integral. It is easy to see that m is a G-invariant mean on $\ell^\infty(G,\mathbb{R})$, using basic propositions that come from the definition of our integral.
Conversely, if G is amenable with mean m, then we introduce

$$\begin{array}{cccc} \mu & : & \mathcal{P}(G) & \longrightarrow & [0,1] \\ & & A & \longmapsto & m(\mathbf{1}_A) \end{array}$$

with $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$. The map μ is as wanted in the proposition.

To conclude this document, we prove the following theorem.

Theorem 3.1.1 Abelian groups are amenable.

First, we need the following extension proposition.

Proposition 3.1.3 If we have a short exact sequence of groups

$$1 \to N \to G \to Q \to 1$$

such that N and Q are amenable, then G is also amenable.

Proof : Without loss of generality, we can suppose that $N \subset G$ and that $Q = G/N$.
We write m_N and m_Q the corresponding invariant means. The map

$$\begin{array}{cccc} m & : & \ell^\infty(G,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ & & f & \longmapsto & m_Q(gN \mapsto m_N(n \mapsto f(gn))) \end{array}$$

is well defined, linear, and we have

1. $m(x \mapsto 1) = m_Q(gN \mapsto m_N(n \mapsto 1)) = m_Q(gN \mapsto 1) = 1$.
2. If $f \in \ell^\infty(G,\mathbb{R})$ is positive, $m(f) \geqslant 0$ because $gN \mapsto m_N(n \mapsto f(gn))$ and $n \mapsto f(gn)$ are positive.
3. Let $g \in G$. Let $f \in \ell^\infty(G,\mathbb{R})$. We have

$$m(g \cdot f) = m_Q(hN \mapsto m_N(n \mapsto (g \cdot f)(hn))) = m_Q(hN \mapsto m_N(n \mapsto f(hn)))$$

   so that $m(g \cdot f) = m(f)$.

This shows that G is amenable.

Remark : In particular, finite direct products of amenable groups are amenable.

3.2 Følner sequence

In this section, we are going to prove that the additive group $\mathbb{Z}$ is amenable. First we define Følner sequences.

Definition Let G be a group. Let X be a countable set. Suppose that G acts on X.
A **Følner sequence** for X is a sequence of finite subsets (F_i) of X such that $\forall g \in G$,

$$\frac{|gF_i \triangle F_i|}{|F_i|} \to 0 \text{ as } i \to +\infty.$$

Recall that if A and B are two sets, then $A \triangle B = (A \cup B) \setminus (A \cap B)$.

Example : A Følner sequence for $\mathbb{Z}$ is $(B_i^{G,\{1\}}(0))$ (the balls of radius i and center 0). Indeed, the fixed translates overlap in a bigger and bigger area as we take greater radiuses for the balls.

Proposition 3.2.1 Let G be a countable group that has a Følner sequence. Then G is amenable.

To prove this, we are going to use filters. For a full course on this topic, see chapter 1 of [Bou07].

Proof : We are going to use the equivalent definition from proposition 3.1.2.
Let (F_i) be a Følner sequence for G. Let $\mathcal{U}$ be a ultrafilter on $\mathbb{N}$ that contains the intervals $[n, +\infty[$ for all $n \in \mathbb{N}$.
We define the map using the ultralimit

$$\begin{array}{rccc} \mu & : & \mathcal{P}(G) & \longrightarrow & [0,1] \\ & & A & \longmapsto & \mathcal{U} - \lim \frac{|A \cap F_i|}{|F_i|} \end{array}.$$

Of course, μ is well defined, and $\mu(G) = 1$ since the ultralimit is equal to the limit when it exists.
Let $S, T \in \mathcal{P}(G)$ such that $S \cap T = \emptyset$. We have

$$|(S \cup T) \cap F_i| = |S \cap F_i| + |T \cap F_i|$$

since S and T are disjoint. It follows that $\mu(S \cup T) = \mu(S) + \mu(T)$ because the ultralimit is compatible with addition. Now let $S \in \mathcal{P}(G)$ and let $g \in G$. Notice that we have the following

$$\left| \frac{|gS \cap F_i|}{|F_i|} - \frac{|S \cap F_i|}{|F_i|} \right| = \left| \frac{|S \cap g^{-1}F_i|}{|F_i|} - \frac{|S \cap F_i|}{|F_i|} \right| = \left| \frac{|S \cap g^{-1}F_i| - |S \cap F_i|}{|F_i|} \right|.$$

But $|S \cap g^{-1}F_i| - |S \cap F_i| \leqslant |S \cap (g^{-1}F_i \triangle F_i)|$ so by the definition of Følner sequences,

$$\left| \frac{|gS \cap F_i|}{|F_i|} - \frac{|S \cap F_i|}{|F_i|} \right| \to 0$$

which proves that $\mu(gS) = \mu(S)$. By proposition 3.1.2, G is amenable.

Using this proposition and the last example, we obtain the following proposition.

Proposition 3.2.2 Seen as an additive group, $\mathbb{Z}$ is amenable.

Proof : This group has a Følner sequence so we just use proposition 3.2.1 to conclude.

We are getting closer to our theorem about abelian groups.

Proposition 3.2.3 Finitely generated abelian groups are amenable.

Proof : Let G be a finitely generated abelian group. The fundamental theorem of finitely generated abelian groups states that G is of the form $\mathbb{Z}^n \times T$ for some $n \in \mathbb{N}$ and some finite abelian group T.
If $n = 0$, then we just use proposition 3.1.1 to conclude that $\mathbb{Z}^n$ is amenable.
Else, we prove by induction that $\mathbb{Z}^n$ is amenable using propositions 3.2.2 and 3.1.3.
Since T is finite, it is amenable by proposition 3.1.1, so that G is amenable by proposition 3.1.3.

To conclude the proof of our theorem on amenability of abelian groups, we use the following proposition.

Proposition 3.2.4 Let G be a group. Suppose that every finitely generated subgroup of G is amenable. Then G is amenable.

This comes from the fact that finitely generated subgroups of G form an ascending directed system of subgroups of G that cover all of G. See [Lö17] for more details on this proof.
Now every finitely generated subgroup of an abelian group is a finitely generated abelian group so is amenable. Thus, an abelian group is amenable itself.

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