The *Skorokhod* Embedding Problem for Random Walks

Course for the end of M1, under the management of Alexander Cox, University of Bath

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June-July 2014

Contents

Introduction

We will here find some solutions to the *Skorokhod* Embedding Problem for Random Walks, and prove some of their properties. This problem was first presented by *Anatoliy Volodymyrovych Skorokhod* (1930-2011), who was a Soviet (and then Ukrainian) mathematician. His scientific works are on the theory of stochastic differential equations, limit theorems of random processes, distributions in infinite-dimensional spaces, statistics of random processes and Markov processes.

Originally, this problem was presented with the Brownian Motion, and in the first section, we will show the financial reasons which make us solve this problem, and find some "optimal" solutions. But, we will here work with simple symmetric random walks, and show various results which are similar to results we could have with the Brownian Motion.

We consider a simple symmetric random walk Z_n^λ (also written Z_n^y when $\lambda = \delta_y$), running on $\mathbb{N}$, where λ is a distribution over $\mathbb{N}$, that is to say: [1]

- $\mathcal{L}\left(Z_0^\lambda\right) = \lambda$
- if $Z_n^\lambda = k \neq 0$, then $\mathbb{P}\left(Z_{n+1}^\lambda = k+1\right) = \mathbb{P}\left(Z_{n+1}^\lambda = k-1\right) = \dfrac{1}{2}$
- if $Z_n^\lambda = 0$, then $Z_{n+1}^\lambda = 0$ a.s.

We also assume that λ is integrable.

Now, we can define the *Skorokhod* Embedding Problem: given μ, we want to find a stopping time τ such that $\mathcal{L}\left(Z_\tau^\lambda\right) = \mu$.

In the second section, we will show that such a stopping time τ exists if λ and μ verify a specific condition.

Then, in the third section, we will construct two solutions to the *Skorokhod* Embedding Problem, and we will prove some of their properties.

[1] In this paper, we will always note $\mathbb{N} = \{0, 1, 2, \ldots\}$ the set of non-negative integers, $\mathbb{N}^* = \{1, 2, \ldots\}$ the set of positive integers and $\mathbb{R}^+ = [0, +\infty[$.

1 Financial Motivation

We suppose that we have an asset S_t, where $t \in [0, T]$.

The General Financial theory says that the discounted asset price $\mathrm{e}^{-rt} S_t$, where r is the interest rate is a martingale under a probability measure $\mathbb{Q}$ if the model is free of arbitrage; and $\mathbb{Q}$ satisfies: $\mathbb{P}(A) > 0 \iff \mathbb{Q}(A) > 0$.

Then, the prices of derivatives are expectations under $\mathbb{Q}$.

For example, a call option at T will pay $(S_T - K)_+$, where K is the strike price. So, today, its (arbitrage-free) price is:

$$C(K, T) = \mathrm{e}^{-rT} \mathbb{E}^{\mathbb{Q}} \left[(S_T - K)_+ \right]$$

In practice, call options[2] are "liquidly traded", so we can use them as an "input" to calibrate the model.

Now, let's take $r = 0$. We write $p(x)\mathrm{d}x$ the density of S_T under $\mathbb{Q}$. Then, we have:

$$C(K, T) = \int_K^\infty (y - K) p(y) \, \mathrm{d}y$$

$$\frac{\partial}{\partial K} [C(K, T)] = -\int_K^\infty p(y) \, \mathrm{d}y$$

$$\frac{\partial^2}{\partial^2 K} [C(K, T)] = p(K)$$

So, observing $C(K, T)$ tells us $\mathcal{L}(S_T)$ under $\mathbb{Q}$. Let's call $\mu = \mathcal{L}(S_T)$.

What we know is:

- S_0 (today's stock price);
- $\mathcal{L}(S_T)$ (under $\mathbb{Q}$);
- $(S_t)_{t \in [0,T]}$ is a $\mathbb{Q}$-martingale.

Now, we want to price "exotic" options. For example, a lookback option pays $\sup\limits_{0 \le t \le T} S_t$.

[2]Let's take an example to understand what a call option is. The 30^{th} of June, the trader A buys a call contract for 100 shares of ABC Corp from the trader B who is the call seller. The strike price for the contract is £60 per share, and the contract ends the 31^{st} of December. The current price of the share is £45, and A pays a premium up front of £15 per share, or £1,500 total. The 31^{st} of December:

- if the share values £80, then A exercises the call option by buying 100 shares of ABC Corp from B for a total of £6,000. If A decides to sell immediately those shares on the market, his profit will be £8,000 − (£1,500 + £6,000) = £500.
- if the share values £50, then A will not exercise the option (i.e., A will not buy a stock at £60 per share from B when he can buy it on the open market at £50 per share). A loses his premium, a total of £1,500. B, however, keeps the premium with no other out-of-pocket expenses, making a profit of £1,500.

So, its current price is: $\mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leq t\leq T} S_t\right]$.

Our question is: in all the models satisfying $\mathcal{L}(S_T) = \mu$ and (S_t) is a $\mathbb{Q}$-martingale, what is the model which maximises or minimises $\sup_{0\leq t\leq T} S_t$?

There is a general result which says:

"Any martingale M_t can be written as a time-change[3] of Brownian motion[4]."

Now, we have $S_t = B_{\tau_t}$ for some time-change $(\tau_t)_{t\in[0,T]}$. So we have: $\mathcal{L}(B_{\tau_T}) = \mathcal{L}(S_T) = \mu$.

And, models satisfying $\mathcal{L}(S_T) = \mu$ correspond to time changes τ_t satisfying $\mathcal{L}(B_{\tau_T}) = \mu$.

If we assume that τ_t doesn't jump, we have: $\sup_{0\leq t\leq T} S_t = \sup_{0\leq t\leq T} B_{\tau_t}$; and $\sup_{0\leq t\leq T} B_{\tau_t}$ only depends on τ_t!

Then, actually, we want to find stopping times $\tau = \tau_T$, satisfying $B_\tau \sim \mu$, and which maximise or minimise $\mathbb{E}\left[\sup_{s\leq\tau} B_s\right]$. This is called the Optimal *Skorokhod* Embedding Problem.

To make things be easier, we will work on this problem with the Brownian Motion replaced by a Simple Symmetric Random Walk.

[3] $(\tau_t)_{t\geq 0}$ is a time-change if:

- $\forall t \geq 0$, τ_t is a stopping time;
- $t \mapsto \tau_t$ is increasing.

[4] The Brownian motion B_t is the "canonical" continuous-time stochastic process. It verifies:

- $t \mapsto B_t$ is continuous (a.s.);
- $(B_t - B_s) \sim \mathcal{N}(0, t-s)$;
- for $t_0 \leq t_1 \leq \ldots \leq t_n$, $(B_{t_1} - B_{t_0}) \perp\!\!\!\perp (B_{t_2} - B_{t_1}) \perp\!\!\!\perp \ldots$

The Brownian motion can be considered as the "scaling limit of random walks".

2 Necessary Condition for the Existence of a Stopping Time

Proposition 1

1. $\left(Z_n^\lambda\right)_{n\in\mathbb{N}}$ is a martingale.
2. Let τ be an almost surely finite stopping time.
   We have:
$$\mathbb{E}\left[Z_\tau^\lambda\right] \leq \mathbb{E}\left[Z_0^\lambda\right]$$

Proof 1

For $n \in \mathbb{N}$, we note $\mathcal{F}_n = \sigma\left(Z_0^\lambda, \ldots, Z_n^\lambda\right)$.
We have:

- Z_n^λ is $\mathcal{F}_n$-measurable
- $\mathbb{E}\left[\left|Z_n^\lambda\right|\right] = \mathbb{E}\left[Z_n^\lambda\right] \leq \mathbb{E}\left[n + Z_0^\lambda\right] = n + \mathbb{E}\left[\left|Z_0^\lambda\right|\right] < \infty$
  So, for all $n \in \mathbb{N}$, $Z_n^\lambda \in \mathrm{L}^1$.
- $\mathbb{E}\left[Z_{n+1}^\lambda \middle| \mathcal{F}_n\right] = \underbrace{\mathbb{E}\left[Z_{n+1}^\lambda \mathbb{1}_{Z_n^\lambda=0} \middle| \mathcal{F}_n\right]}_{=0} + \mathbb{E}\left[Z_{n+1}^\lambda \mathbb{1}_{Z_n^\lambda>0} \middle| \mathcal{F}_n\right] = \frac{1}{2}\left(Z_n^\lambda + 1 + Z_n^\lambda - 1\right)\mathbb{1}_{Z_n^\lambda>0} = Z_n^\lambda \mathbb{1}_{Z_n^\lambda>0}$
  $= Z_n^\lambda$

So $\left(Z_n^\lambda\right)_{n\in\mathbb{N}}$ is a martingale.

For all $n \in \mathbb{N}$, $\tau \wedge n$ is a bounded stopping time.
By Optional Stopping Theorem, we have: $\mathbb{E}\left[Z_{\tau\wedge n}^\lambda\right] = \mathbb{E}\left[Z_0^\lambda\right]$.
Then, using *Fatou*'s lemma, because $Z_{\tau\wedge n}^\lambda \geq 0$:
$$\mathbb{E}\left[Z_\tau^\lambda\right] = \mathbb{E}\left[\lim_{n\to\infty} Z_{\tau\wedge n}^\lambda\right] = \mathbb{E}\left[\liminf_{n\to\infty} Z_{\tau\wedge n}^\lambda\right] \leq \liminf_{n\to\infty} \mathbb{E}\left[Z_{\tau\wedge n}^\lambda\right] = \mathbb{E}\left[Z_0^\lambda\right]$$

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Definition 1 Potential

Let τ be an almost surely finite stopping time.
We call potential of the distribution λ at time τ the function:
$$K_\tau^\lambda : \left| \begin{array}{ccc} \mathbb{N} & \to & \mathbb{R}^+ \\ y & \mapsto & \mathbb{E}\left[Z_\tau^\lambda \wedge y\right] \end{array} \right.$$
We will also note sometimes K^λ instead of K_0^λ.

Lemma 2

The knowledge of the potential K_τ^λ is equivalent to the knowledge of the distribution $\mathcal{L}\left(Z_\tau^\lambda\right)$.

Proof 2

If we know $\mathcal{L}\left(Z_\tau^\lambda\right)$, we know of course $K_\tau^\lambda(y)$ for all $y \in \mathbb{N}$.

Now, we know $\left(K_\tau^\lambda(y)\right)_{y\in\mathbb{N}}$, and we want to know $\mu := \mathcal{L}\left(Z_\tau^\lambda\right)$.

We have: $K_\tau^\lambda(y) = \mathbb{E}\left[Z_\tau^\lambda \wedge y\right] = \sum_{i=0}^{\infty}(i\wedge y)\mu(i) = \sum_{i=0}^{y-1} i\mu(i) + y\sum_{i=y}^{\infty}\mu(i) = \sum_{i=0}^{y-1} i\mu(i) + y\left(1 - \sum_{i=0}^{y-1}\mu(i)\right)$

$$= y + \sum_{i=0}^{y-1}(i-y)\mu(i)$$

Then $K_\tau^\lambda(y+1) - K_\tau^\lambda(y) = y + 1 + \sum_{i=0}^{y}(i-y-1)\mu(i) - y - \sum_{i=0}^{y-1}(i-y)\mu(i)$

$$= 1 - \mu(y) + \sum_{i=0}^{y-1}(i-y-1-i+y)\mu(i)$$

$$= 1 - \sum_{i=0}^{y}\mu(i)$$

If $y > 0$, $\left(K_\tau^\lambda(y+1) - K_\tau^\lambda(y)\right) - \left(K_\tau^\lambda(y) - K_\tau^\lambda(y-1)\right) = 1 - \sum_{i=0}^{y}\mu(i) - 1 + \sum_{i=0}^{y-1}\mu(i) = -\mu(y)$

So, $\mu(0) = 1 - K_\tau^\lambda(1) - K_\tau^\lambda(0) = 1 - K_\tau^\lambda(1)$ and for $y > 0$, $\mu(y) = -K_\tau^\lambda(y+1) + 2K_\tau^\lambda(y) - K_\tau^\lambda(y-1)$.

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We will often use the formula shown in this proof:

$$K^\lambda(y) = \sum_{i=0}^{\infty}(i\wedge y)\lambda(i) = y + \sum_{i=0}^{y-1}(i-y)\lambda(i). \tag{1}$$

Lemma 3

$\forall y \in \mathbb{N}^*$, $\left(Z_n^\lambda \wedge y + \frac{1}{2}\sum_{i=0}^{n-1}\mathbb{1}_{Z_i^\lambda = y}\right)_{n\in\mathbb{N}^*}$ is a martingale.

Proof 3

Let us write $M_n^{\lambda,y} = Z_n^\lambda \wedge y + \frac{1}{2}\sum_{i=0}^{n-1}\mathbb{1}_{Z_i^\lambda = y}$.

- $M_n^{\lambda,y}$ is $\mathcal{F}_n$-measurable
- $\mathbb{E}\left[\left|M_n^{\lambda,y}\right|\right] \le y + \frac{n}{2} < \infty$
  So, for all $n \in \mathbb{N}^*$, $M_n^{\lambda,y} \in \mathrm{L}^1$.
- $\mathbb{E}\left[M_{n+1}^{\lambda,y}\middle|\mathcal{F}_n\right] = \mathbb{E}\left[Z_{n+1}^\lambda \wedge y\middle|\mathcal{F}_n\right] + \frac{1}{2}\sum_{i=0}^{n}\mathbb{1}_{Z_i^\lambda = y}$
  And $\mathbb{E}\left[Z_{n+1}^\lambda \wedge y\middle|\mathcal{F}_n\right] = \underbrace{\mathbb{E}\left[\left(Z_{n+1}^\lambda \wedge y\right)\mathbb{1}_{Z_n^\lambda = 0}\middle|\mathcal{F}_n\right]}_{=0} + \mathbb{E}\left[\left(Z_{n+1}^\lambda \wedge y\right)\mathbb{1}_{Z_n^\lambda > 0}\middle|\mathcal{F}_n\right]$
  $$= \frac{1}{2}\left(\left(Z_n^\lambda + 1\right)\wedge y + \left(Z_n^\lambda - 1\right)\wedge y\right)\mathbb{1}_{Z_n^\lambda > 0}$$
  If $Z_n^\lambda \le y-1$, $\mathbb{E}\left[Z_{n+1}^\lambda \wedge y\middle|\mathcal{F}_n\right] = \frac{1}{2}\left(Z_n^\lambda + 1 + Z_n^\lambda - 1\right)\mathbb{1}_{Z_n^\lambda > 0} = Z_n^\lambda\mathbb{1}_{Z_n^\lambda > 0} = Z_n^\lambda$

If $Z_n^\lambda = y$, $\mathbb{E}\left[Z_{n+1}^\lambda \wedge y \middle| \mathcal{F}_n\right] = \frac{1}{2}(y+y-1)\underbrace{\mathbb{1}_{Z_n^\lambda>0}}_{=1} = y - \frac{1}{2}$

If $Z_n^\lambda \geq y+1$, $\mathbb{E}\left[Z_{n+1}^\lambda \wedge y \middle| \mathcal{F}_n\right] = \frac{1}{2}(y+y)\underbrace{\mathbb{1}_{Z_n^\lambda>0}}_{=1} = y$

So $\mathbb{E}\left[Z_{n+1}^\lambda \wedge y \middle| \mathcal{F}_n\right] = Z_n^\lambda \wedge y - \frac{1}{2}\mathbb{1}_{Z_n^\lambda=y}$

Finally, $\mathbb{E}\left[M_{n+1}^{\lambda,y} \middle| \mathcal{F}_n\right] = Z_n^\lambda \wedge y + \frac{1}{2}\sum_{i=0}^{n-1}\mathbb{1}_{Z_i^\lambda=y} = M_n^{\lambda,y}$

So $\left(M_n^{\lambda,y}\right)_{n\in\mathbb{N}^*}$ is a martingale.

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Proposition 4

Let τ be an almost surely finite stopping time, $y \in \mathbb{N}^*$.
We have: $\mathbb{E}\left[Z_\tau^\lambda \wedge y\right] \leq \mathbb{E}\left[Z_0^\lambda \wedge y\right]$.

Proof 4

For all $n \in \mathbb{N}^*$, $\tau \wedge n$ is a bounded stopping time.
By Optional Stopping Theorem, we have: $\mathbb{E}\left[M_{\tau\wedge n}^{\lambda,y}\right] = \mathbb{E}\left[M_0^{\lambda,y}\right] = \mathbb{E}\left[Z_0^\lambda \wedge y\right]$

Also: $\mathbb{E}\left[M_\tau^{\lambda,y}\right] = \mathbb{E}\left[Z_\tau^\lambda \wedge y + \frac{1}{2}\sum_{i=0}^{\tau-1}\mathbb{1}_{Z_i^\lambda=y}\right] = \mathbb{E}\left[Z_\tau^\lambda \wedge y\right] + \frac{1}{2}\sum_{i=0}^{\infty}\mathbb{P}\left(Z_i^\lambda = y,\, i \leq \tau - 1\right)$

But $\sum_{i=0}^{\infty}\mathbb{P}\left(Z_i^\lambda = y,\, i \leq \tau-1\right) \leq \sum_{i=0}^{\infty}\mathbb{P}\left(Z_i^\lambda = y\right)$

We have $y > 0$, so $\mathbb{P}\left(\forall i \in \mathbb{N}^*,\, Z_i^y \neq y\right) \geq \mathbb{P}\left(Z_y^y = 0\right) \geq \left(\frac{1}{2}\right)^y > 0$

So every $y > 0$ is transient, so $\sum_{i=0}^{\infty}\mathbb{P}\left(Z_i^\lambda = y\right) < \infty$ for each $y \in \mathbb{N}^*$.

We have $\mathbb{E}\left[Z_0^\lambda \wedge y\right] = \mathbb{E}\left[M_{\tau\wedge n}^{\lambda,y}\right] = \underbrace{\mathbb{E}\left[Z_{\tau\wedge n}^\lambda \wedge y\right]}_{\xrightarrow[n\to\infty]{}\mathbb{E}\left[Z_\tau^\lambda\wedge y\right]} + \underbrace{\frac{1}{2}\mathbb{E}\left[\sum_{i=0}^{\tau\wedge n-1}\mathbb{1}_{Z_i^\lambda=y}\right]}_{\text{bounded when } n\to\infty}$, by dominated convergence.

Finally, we have : $\mathbb{E}\left[Z_\tau^\lambda \wedge y\right] \leq \mathbb{E}\left[Z_0^\lambda \wedge y\right]$.

■

We remark that we have shown a more precise result in this proof, that we may use in the following:

$$K_\tau^\lambda(y) = K^\lambda(y) - \frac{1}{2}\mathbb{E}\left[\sum_{i=0}^{\tau-1}\mathbb{1}_{Z_i^\lambda=y}\right] \tag{2}$$

We can now derive from this a necessary condition to embed μ, starting from λ. If there is an almost surely finite stopping time τ, such that $\mu = \mathcal{L}\left(Z_\tau^\lambda\right)$, then we have, for all $y \in \mathbb{N}$, $\mathbb{E}\left[Z_\tau^\lambda \wedge y\right] \leq \mathbb{E}\left[Z_0^\lambda \wedge y\right]$, that is to say :

$$\forall y \in \mathbb{N},\, \sum_{i=0}^{\infty}(i\wedge y)\,\mu(i) \leq \sum_{i=0}^{\infty}(i\wedge y)\,\lambda(i)$$

In other words, the potential of μ needs to be under the potential of λ.

3 Resolution of the *Skorokhod* Embedding Problem

3.1 Different ways to construct a stopping time

In the next subsection, we will show that if the potential of μ is under the potential of λ, we can construct a stopping time such that $\mathcal{L}\left(Z_\tau^\lambda\right) = \mu$. But we have to know that we can construct such stopping times by many different ways. We will show several possible constructions in this subsection.
For example, we can choose $\lambda = \delta_2$ and $\mu = \frac{2}{3}\delta_1 + \frac{1}{3}\delta_3$.

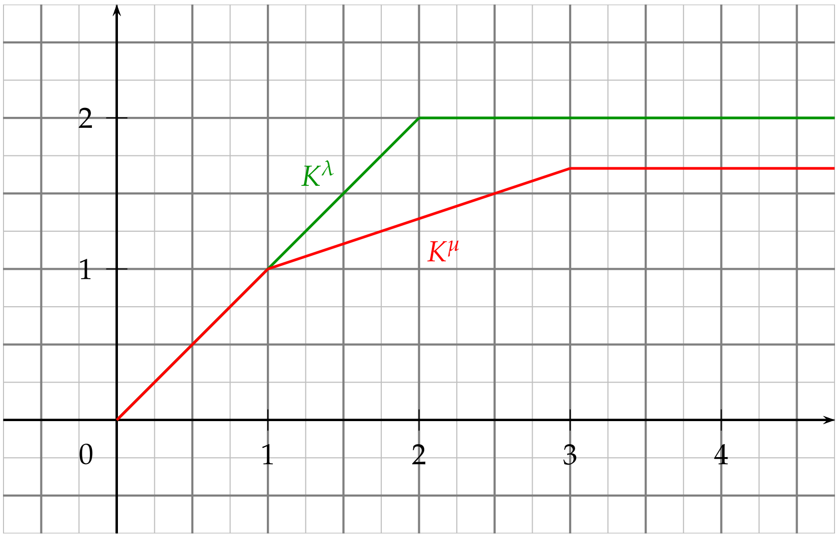

Figure 1: The distributions λ and μ verify the necessary condition.

3.1.1 Stop as "early" as possible

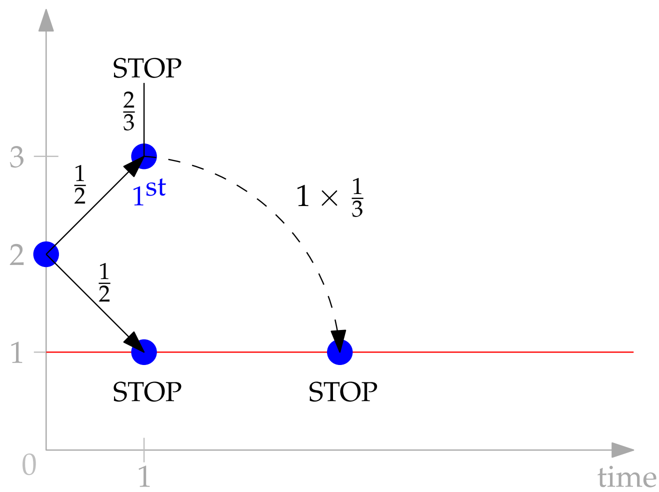

Figure 2: Behaviour of the stopped random walk. (Stop as "early" as possible)

Starting from 2, we have : $\mathbb{P}\left(Z_1^\lambda = 1\right) = \mathbb{P}\left(Z_1^\lambda = 3\right) = \frac{1}{2}$.
When we reach 1, we must stop, because the probability of getting stuck at 0 is positive.

The first time we reach 3, we stop with probability $\frac{2}{3}$.
Then, we stop when we reach 1, which is done almost surely in finite time. And we can write

$$\tau = \begin{cases} 1 & \text{if } U = 1 \\ \inf\left\{n \in \mathbb{N} \middle| Z_n^\lambda = 1\right\} & \text{if } U = 0 \end{cases},$$

where U is a *Bernoulli* random variable of parameter $\frac{2}{3}$.
This way, $\mathbb{P}\left(Z_\tau^\lambda = 3\right) = \mathbb{P}\left(Z_1^\lambda = 3, U = 1\right) = \frac{1}{2} \times \frac{2}{3} = \frac{1}{3}$.
And then, $\mathbb{P}\left(Z_\tau^\lambda = 1\right) = 1 - \mathbb{P}\left(Z_\tau^\lambda = 3\right) = 1 - \frac{1}{3} = \frac{2}{3}$.
So, we have : $Z_\tau^\lambda \sim \mu$.

3.1.2 Stop as "late" as possible

The same as before: when we reach 1, we need to stop.
When we are in 3, then, the probability that we reach 1 before 3 again is equal to $\frac{1}{4}$.
Because $\frac{1}{2} + \frac{1}{2} \times \frac{1}{4} = \frac{5}{8} < \frac{2}{3}$, we will always allow more than one visit to 3.
But $\frac{5}{8} + \frac{3}{8} \times \frac{1}{4} = \frac{23}{32} > \frac{2}{3}$, so we will need to stop sometimes at the second visit to 3.
So, we want to find p such that: $\frac{1}{2} \times \frac{3}{4} \times p + \frac{1}{2} \times \frac{3}{4} \times (1-p) \times \frac{3}{4} = \frac{1}{3}$. We find $p = \frac{5}{9}$, the probability of stopping at the second visit to 3.
We can write:

$$\tau = \inf\left\{n \in \mathbb{N} \middle| Z_n^\lambda = 1 \text{ or } \sum_{i=0}^{n} \mathbb{1}_{Z_i^\lambda = 3} = 2 + U\right\},$$

where U is a *Bernoulli* random variable of parameter $\frac{5}{9}$.
And, this way, $Z_\tau^\lambda \sim \mu$.

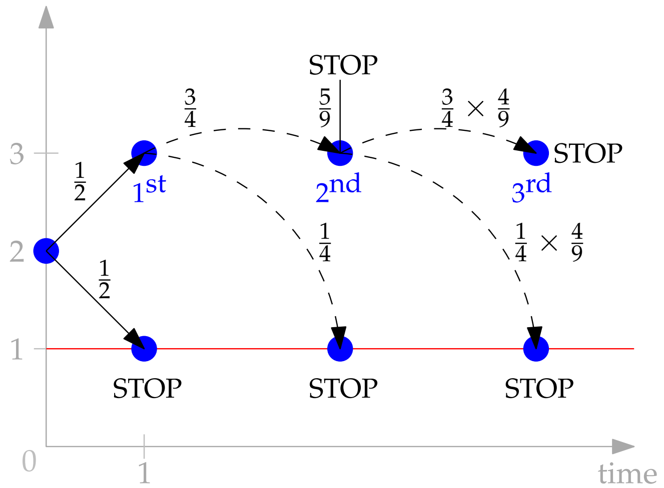

Figure 3: Behaviour of the stopped random walk. (Stop as "late" as possible)

3.1.3 Stop at the bottom or reach another point

Because Z^λ is a symmetric simple random walk, we know that, for $x < y < z \in \mathbb{N}$, we have: $\mathbb{P}_y\left(H_x < H_z\right) = \frac{z-y}{z-x}$, where H_k is the first time k is reached by the random walk.
And we can use this result: $\mathbb{P}_2\left(H_1 < H_4\right) = \frac{2}{3}$.
We have a stopping rule: we run to 1 or 4; if we hit 1, then we stop, else, we run to 3 and stop.

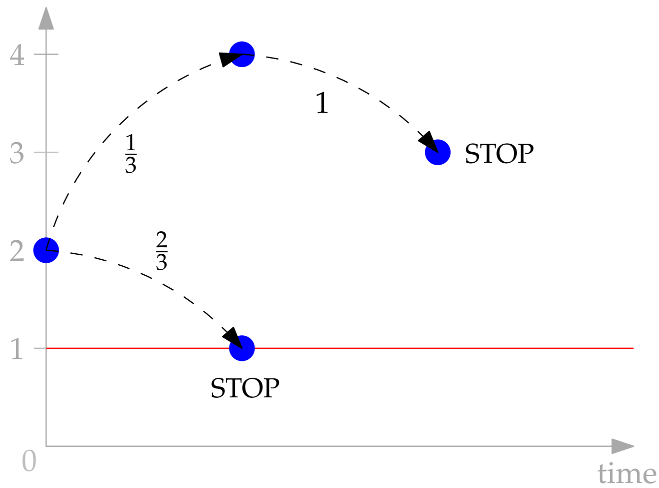

Figure 4: Behaviour of the stopped random walk. (Stop at the bottom or reach another point)

$$\tau = \inf\left\{ n \in \mathbb{N} \middle| Z_n^\lambda = 1 \text{ or } \mathbb{1}_{Z_n^\lambda = 3} \sum_{i=0}^{n} \mathbb{1}_{Z_i^\lambda = 4} = 1 \right\}$$

And again $Z_\tau^\lambda \sim \mu$.

3.2 The *Azéma-Yor*[5] solution

3.2.1 Illustration of the process

But we will choose another way of constructing a stopping time to show that the condition on the potentials is sufficient, based on the potentials themselves.
We will try to construct a stopping time, such that $Z_\tau^\lambda \sim \lambda'$, with the potential of λ' between those of λ and μ.
In the next example, we have $\lambda = \frac{1}{4}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_3 + \frac{5}{36}\delta_4 + \frac{1}{9}\delta_5$ and $\mu = \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_3$.
The process is this one:

1. First, K^λ and K^μ split at the point $(1, 1)$. We draw a line from the point $(1, 1)$, with the same slope as K^μ (in orange). This line touches K^λ at the point $\left(4, \frac{5}{2}\right)$, whose first coordinate is an integer. Our aim will be: constructing a stopping time τ' such that $Z_{\tau'}^\lambda \sim \lambda'$, with $\lambda' = \frac{1}{2}\delta_1 + \frac{7}{18}\delta_4 + \frac{1}{9}\delta_5$.

2. Then, $K^{\lambda'}$ and K^μ split at the point $\left(2, \frac{3}{2}\right)$. We do the same (in blue), but the intersection point first coordinate is not integer. Our aim will be: constructing a stopping time τ'' such that $Z_{\tau''}^{\lambda'} \sim \lambda''$, with $\lambda'' = \frac{1}{2}\delta_1 + \frac{1}{4}\delta_2 + \frac{5}{36}\delta_6 + \frac{1}{9}\delta_7$.

3. Finally, we will use the stopping time $\tau''' = \inf\left\{ n \in \mathbb{N} \middle| Z_n^{\lambda''} \leq 3 \right\}$, because it gives: $Z_{\tau'''}^{\lambda''} \sim \mu$. By this way, we have a stopping time τ (the "concatenation" of the previous stopping time constructed), such that $Z_\tau^\lambda \sim \mu$.

[5] *Jacques Azéma* (born 1939) and *Marc Yor* (1949-2014) published in 1979 an article called *"Une solution simple au problème de Skorokhod"* in which they exploit this process to solve the *Skorokhod* problem for the Brownian motion.

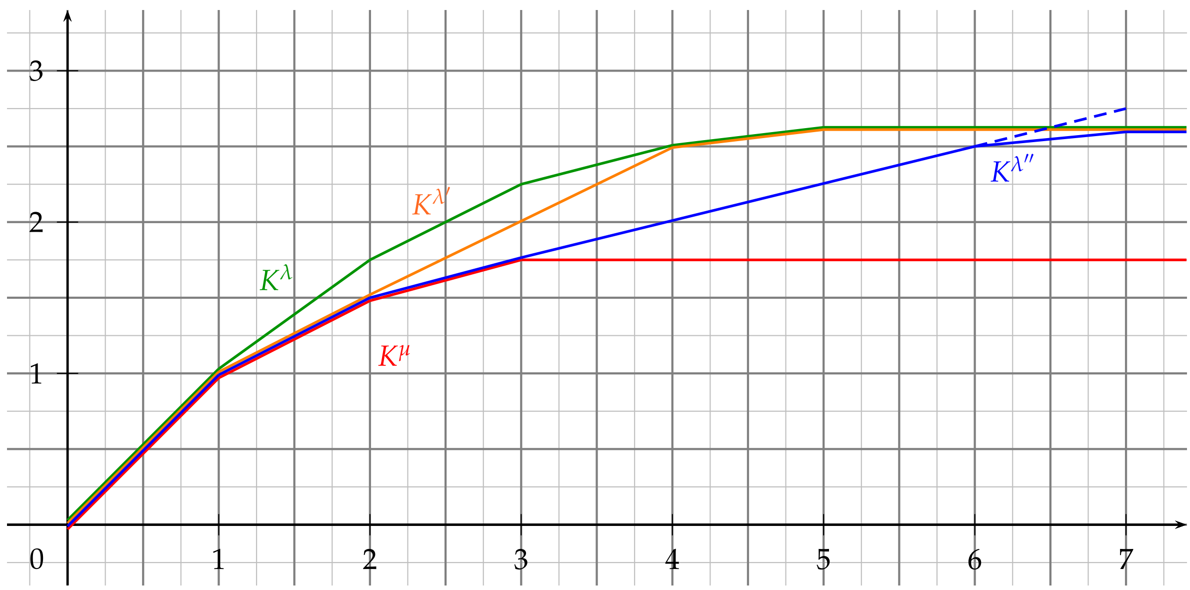

Figure 5: The potentials of the distributions λ, μ and of the intermediate distributions λ' and λ''.

3.2.2 Proof that it solves the *Skorokhod* Problem

We will now show that this method works if we have: $K^{\lambda} \geq K^{\mu}$.

Theorem 5 The *Azéma-Yor* solution for the *Skorokhod* Embedding Problem

Let λ and μ be two integrable distributions over $\mathbb{N}$.
The *Azéma-Yor* process shows that we have the equivalence:

$$\mu \text{ is embeddable starting from } \lambda \Leftrightarrow \forall y \in \mathbb{N}, \sum_{i=0}^{\infty} (i \wedge y)\, \mu(i) \leq \sum_{i=0}^{\infty} (i \wedge y)\, \lambda(i)$$

Proof 5

- First, we suppose that μ is a distribution bounded by $N \in \mathbb{N}$.
  We note $x = \inf \left\{ n \in \mathbb{N} \middle| K^{\lambda}(n) > K^{\mu}(n) \right\} = \inf \{ n \in \mathbb{N} | \lambda(n) < \mu(n) \}$. We suppose $\lambda \neq \mu$, so $x \leq N$.
  $\Delta = 1 - \sum_{i=0}^{x} \mu(i)$ is the slope of K^{μ} between x and $x+1$.

  1. If $\Delta = 0$.
     It means that $\sum_{i=0}^{x} \mu(i) = 1$.
     We write $\tau = \inf \left\{ n \in \mathbb{N} \middle| Z_n^{\lambda} \in [\![0, x]\!] \right\}$. We have $\tau < \infty$ a.s.
     $\forall k \in [\![0, x-1]\!]$, $\mathbb{P}\left(Z_{\tau}^{\lambda} = k\right) = \mathbb{P}\left(Z_0^{\lambda} = k\right) = \lambda(k) = \mu(k)$ (because $k < x$).
     And then: $\mathbb{P}\left(Z_{\tau}^{\lambda} = x\right) = 1 - \sum_{k=0}^{x-1} \mathbb{P}\left(Z_{\tau}^{\lambda} = k\right) = 1 - \sum_{k=0}^{x-1} \mu(k) = \mu(x)$.
     So, we have: $Z_{\tau}^{\lambda} \sim \mu$.

2. If $\Delta > 0$.
   We have: $\lim_{y\to+\infty} K^\mu(x) + (y-x)\Delta = +\infty$ and $\lim_{y\to+\infty} K^\lambda(y) = \mathbb{E}\left[Z_0^\lambda\right] < +\infty$ because λ is integrable.
   Because we have $K^\mu \le K^\lambda$, we have two cases.
   (a) $\exists y \in \mathbb{N},\, K^\mu(x) + (y-x)\Delta = K^\lambda(y)$.
   We define λ' by: $\lambda' = \sum_{i=0}^{x} \mu(i)\delta_i + \left(\Delta - \sum_{i=y+1}^{\infty} \lambda(i)\right)\delta_y + \sum_{i=y+1}^{\infty} \lambda(i)\delta_i$.

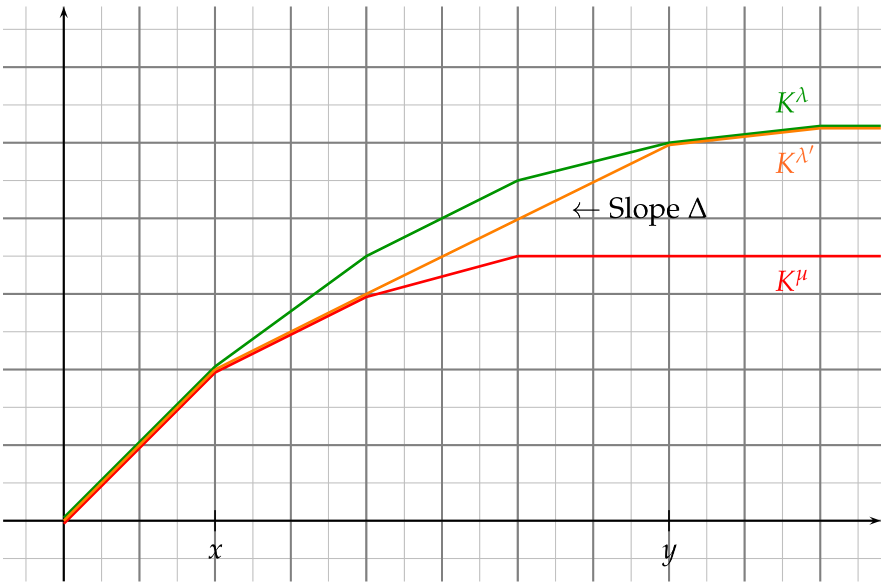

Figure 6: An illustration of the case (a).

And $\tau = \inf\left\{n \in \mathbb{N} \middle| Z_n^\lambda \in [\![0,x]\!] \cup [\![y,+\infty[\![\right\}$.

We have: $\Delta = 1 - \sum_{i=0}^{x} \mu(i) = 1 - \sum_{i=0}^{x-1} \lambda(i) - \mu(x)$

$$\ge 1 - \sum_{i=0}^{x-1} \lambda(i) - \lambda(x) = \sum_{i=x+1}^{\infty} \lambda(i) \ge \sum_{i=y+1}^{\infty} \lambda(i), \text{ so } \lambda'(y) \ge 0.$$

λ' is well defined: $\sum_{i=0}^{\infty} \lambda'(i) = \sum_{i=0}^{x} \mu(i) + \Delta - \sum_{i=y+1}^{\infty} \lambda(i) + \sum_{i=y+1}^{\infty} \lambda(i) = 1$.

If $k \le x-1$ or $k \ge y+1$, then $\mathbb{P}\left(Z_\tau^\lambda = k\right) = \mathbb{P}\left(Z_0^\lambda = k\right) = \lambda(k) = \lambda'(k)$.

If $x+1 \le k \le y-1$, then $\mathbb{P}\left(Z_\tau^\lambda = k\right) = 0 = \lambda'(k)$.

$$\mathbb{P}\left(Z_\tau^\lambda = x\right) = \sum_{k=x}^{y-1} \mathbb{P}\left(Z_\tau^\lambda = x,\, Z_0^\lambda = k\right) = \sum_{k=x}^{y-1} \mathbb{P}\left(Z_0^\lambda = k, \text{ hitting } x \text{ before } y \text{ from } k\right)$$

$$\overset{\perp\!\!\!\perp}{=} \sum_{k=x}^{y-1} \lambda(k) \frac{y-k}{y-x}$$

But we have: $K^\mu(x) + (y-x)\Delta = K^\lambda(y)$

$$\Leftrightarrow \sum_{i=0}^{\infty} (i \wedge x)\mu(i) + (y-x)\left(1 - \sum_{i=0}^{x} \mu(i)\right) = \sum_{i=0}^{\infty} (i \wedge y)\lambda(i)$$

$$\text{using (1)} \Leftrightarrow x + \sum_{i=0}^{x-1} (i-x)\mu(i) + y - x + \sum_{i=0}^{x} (x-y)\mu(i) = y + \sum_{i=0}^{y-1} (i-y)\lambda(i)$$

$$\Leftrightarrow \sum_{i=0}^{x-1} (i-y)\mu(i) + (x-y)\mu(x) = \sum_{i=0}^{x-1} (i-y)\mu(i) + \sum_{i=x}^{y-1} (i-y)\lambda(i)$$

So $K^\mu(x)+(y-x)\Delta = K^\lambda(y) \Leftrightarrow \mu(x) = \sum_{i=x}^{y-1} \frac{i-y}{x-y}\lambda(i) = \mathbb{P}\left(Z_\tau^\lambda = x\right)$.

Finally, $\mathbb{P}\left(Z_\tau^\lambda = x\right) = \mu(x) = \lambda'(x)$,

and $\mathbb{P}\left(Z_\tau^\lambda = y\right) = 1 - \sum_{i \neq y} \mathbb{P}\left(Z_\tau^\lambda = i\right) = 1 - \sum_{i\neq y} \lambda'(i) = \lambda'(y)$.

This way, we have: $Z_\tau^\lambda \sim \lambda'$.

If $k \leq x$, $K^{\lambda'}(k) = K^\mu(k)$, and if $k \geq y$, $K^{\lambda'}(k) = K^\lambda(k) \geq K^\mu(k)$.

If $x+1 \leq k \leq y-1$, $K^{\lambda'}(k) = K^{\lambda'}(x) + (k-x)\Delta = K^\mu(x) + \sum_{i=x+1}^{k} \Delta$

$$\geq K^\mu(x) + \sum_{i=x+1}^{k}\left(1 - \sum_{j=0}^{i}\mu(j)\right) = K^\mu(k).$$

And λ' is integrable because $\mathbb{E}\left[Z_0^{\lambda'}\right] = \mathbb{E}\left[Z_\tau^\lambda\right] = \mathbb{E}\left[Z_0^\lambda\right] < \infty$ by OST.

(b) $\exists y \in \mathbb{N}$, $K^\mu(x) + (y-x)\Delta < K^\lambda(y)$ and $K^\mu(x) + (y+1-x)\Delta > K^\lambda(y+1)$.

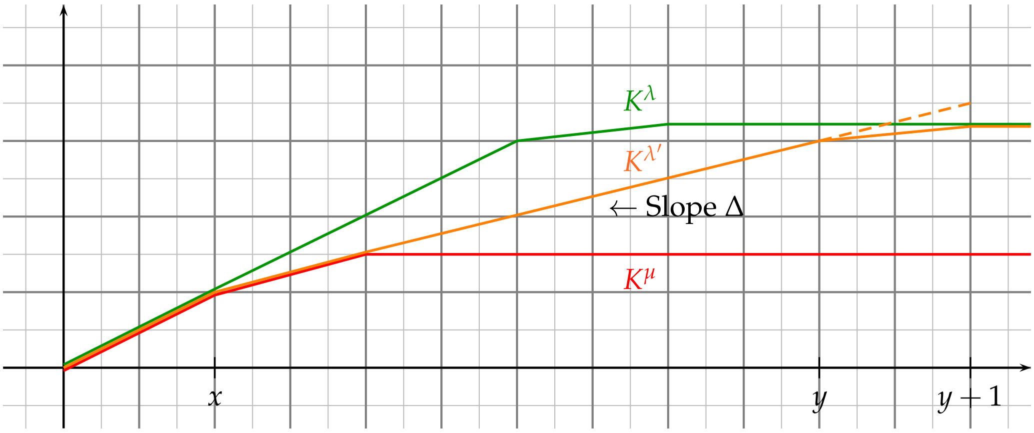

Figure 7: An illustration of the case (b).

We define λ' by:

$$\lambda'(i) = \begin{cases} \mu(i) & \text{if } i \leq x \\ 0 & \text{if } x+1 \leq i \leq y-1 \\ K^\mu(x) + (y+1-x)\Delta - K^\lambda(y+1) & \text{if } i = y \\ \Delta - \left[K^\mu(x) + (y+1-x)\Delta - K^\lambda(y+1)\right] - \sum_{k=y+2}^{\infty}\lambda(k) & \text{if } i = y+1 \\ \lambda(i) & \text{if } i \geq y+2 \end{cases}.$$

We have: $\lambda'(y+1) = K^\lambda(y+1) - [K^\mu(x) + (y-x)\Delta] - \sum_{i=y+2}^{\infty}\lambda(i)$

$$\lambda'(y+1) > K^\lambda(y+1) - K^\lambda(y) - \sum_{i=y+2}^{\infty}\lambda(i)$$

$$\lambda'(y+1) > \sum_{i=y+1}^{\infty}\lambda(i) - \sum_{i=y+2}^{\infty}\lambda(i)$$

$$\lambda'(y+1) > \lambda(y+1) \geq 0$$

Also: $\sum_{i=0}^{\infty} \lambda'(i) = \sum_{i=0}^{x} \mu(i) + K^{\mu}(x) + (y+1-x)\Delta - K^{\lambda}(y+1) + \Delta$

$$- \left[K^{\mu}(x) + (y+1-x)\Delta - K^{\lambda}(y+1)\right] - \sum_{k=y+2}^{\infty} \lambda(k) + \sum_{k=y+2}^{\infty} \lambda(k)$$

$$= \sum_{i=0}^{x} \mu(i) + \Delta = 1$$

So λ' is well defined.
If $k \leq x$, $K^{\lambda'}(k) = K^{\mu}(k)$, and if $k \geq y+1$, $K^{\lambda'}(k) = K^{\lambda}(k) \geq K^{\mu}(k)$.
If $x+1 \leq k \leq y$, then:
$K^{\lambda'}(k) = K^{\lambda'}(x) + \sum_{i=x+1}^{k} \Delta \geq K^{\mu}(x) + \sum_{i=x+1}^{k} \left(1 - \sum_{j=0}^{i} \mu(j)\right) = K^{\mu}(k)$, so $K^{\lambda'} \geq K^{\mu}$.
We have λ' is integrable, because $\mathbb{E}\left[Z_0^{\lambda'}\right] = \mathbb{E}\left[Z_\tau^{\lambda}\right] = \mathbb{E}\left[Z_0^{\lambda}\right] < \infty$ by OST.
We will construct τ as said below.

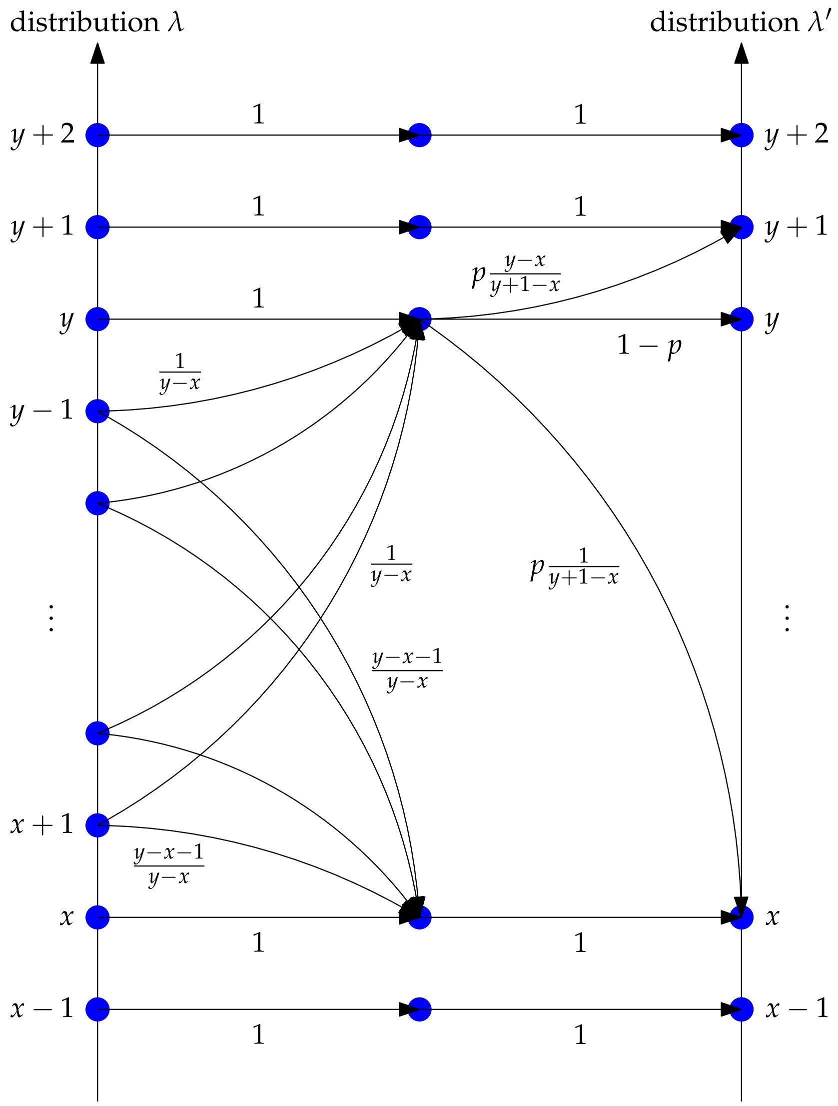

Figure 8: Illustration of what we are doing in case (b) to go from the distribution λ to the distribution λ'.

If $Z_0^{\lambda} \leq x$ or $Z_0^{\lambda} \geq y+1$, then $\tau = 0$.

If $x+1 \leq Z_0^\lambda \leq y-1$, if we reach x before y, then $\tau = \inf\{n \in \mathbb{N} | Z_n^\lambda = x\}$. The probability of this event, knowing Z_0^λ, is $\frac{y-Z_0^\lambda}{y-x}$.
Let's summarize: currently we have,

- $\sum_{i=x}^{y} \lambda(i) \dfrac{y-i}{y-x} =: \sigma_x$ of the mass which is stopped in x;
- $\sum_{i=x}^{y} \lambda(i) \dfrac{i-x}{y-x} =: \sigma_y$ of the mass which is still running in y;
- and $\lambda(y+1)$ of the mass which is stopped in $y+1$.

By the same way as before, $K^\mu(x) + (y-x)\Delta < K^\lambda(y)$ gives $\mu(x) > \sum_{i=x}^{y} \dfrac{y-i}{y-x}\lambda(i)$; and also, $K^\mu(x) + (y+1-x)\Delta > K^\lambda(y+1)$ gives $\mu(x) < \sum_{i=x}^{y} \dfrac{y+1-i}{y+1-x}\lambda(i)$.

$$
\begin{aligned}
\text{So: } \mu(x) - \sigma_x &< \sum_{i=x}^{y} \left(\frac{y+1-i}{y+1-x} - \frac{y-i}{y-x} \right) \lambda(i) \\
&< \sum_{i=x}^{y} \frac{(y-i)(y-x) + y - x - [(y-x)(y-i) + y - i]}{(y-x)(y+1-x)} \lambda(i) \\
&< \frac{1}{y+1-x} \underbrace{\sum_{i=x}^{y} \frac{i-x}{y-x} \lambda(i)}_{=\sigma_y}.
\end{aligned}
$$

Also $\mathbb{P}_y\left(H_x < H_{x+1}\right) = \dfrac{1}{y+1-x}$; we note: $p = \dfrac{\sigma_y - \lambda'(y)}{\sigma_y}$.
Then, with probability p, we run to x or $y+1$ from y, and with probability $1-p$, we stay at y.
The distribution of the mass between these three points is now this one:

- in x: $\sigma_x + \dfrac{p\sigma_y}{y+1-x}$;
- in y: $\sigma_y(1-p) = \lambda'(y)$;
- and in $y+1$: $\lambda(y+1) + p\sigma_y \dfrac{y-x}{y+1-x}$.

$$
\begin{aligned}
\text{But we have: } \sum_{i=x}^{y} \frac{i-x}{y-x}\lambda(i) &= \frac{1}{y-x}\left[\sum_{i=0}^{y} (i-x)\lambda(i) - \sum_{i=0}^{x-1} (i-x)\lambda(i) \right] \\
&= \frac{1}{y-x}\left[\sum_{i=0}^{y} (i-y)\lambda(i) + \sum_{i=0}^{y} (y-x)\lambda(i) - \left(K^\lambda(x) - x \right) \right] \\
&= \frac{1}{y-x}\left[K^\lambda(y) - y + (y-x)\sum_{i=0}^{y} \lambda(i) - K^\lambda(x) + x \right] \\
&= \frac{K^\lambda(y) - K^\lambda(x)}{y-x} - 1 + \sum_{i=0}^{y} \lambda(i)
\end{aligned}
$$

So: $\sigma_y = \dfrac{K^\lambda(y) - K^\lambda(x)}{y-x} - 1 + \sum_{i=0}^{y} \lambda(i)$ and $\sigma_x = 1 - \dfrac{K^\lambda(y) - K^\lambda(x)}{y-x} - \sum_{i=0}^{x-1} \lambda(i)$.
Let's now check that what is currently in $y+1$ is exactly $\lambda'(y+1)$.
Recall that we have the following relations:

$$\lambda'(y+1) = \Delta - \lambda'(y) - \sum_{i=y+2}^{\infty} \lambda(i),$$

$$K^\lambda(y+1) = K^\lambda(y) + \sum_{i=y+1}^{\infty} \lambda(i).$$

$$
\begin{aligned}
&\lambda(y+1) + p\sigma_y \frac{y-x}{y+1-x} \\
&\quad = \lambda(y+1) + \left(\sigma_y - \lambda'(y)\right) \frac{y-x}{y+1-x} \\
&\quad = \lambda(y+1) + \left(\frac{K^\lambda(y) - K^\lambda(x)}{y-x} - 1 + \sum_{i=0}^{y} \lambda(i) - \Delta + \lambda'(y+1) + \sum_{i=y+2}^{\infty} \lambda(i) \right) \frac{y-x}{y+1-x} \\
&\quad = \lambda(y+1) + \frac{y-x}{y+1-x}\lambda'(y+1) + \frac{y-x}{y+1-x} \left(\frac{K^\lambda(y) - K^\lambda(x)}{y-x} - \lambda(y+1) - \Delta \right) \\
&\quad = \frac{y-x}{y+1-x}\lambda'(y+1) \\
&\qquad + \frac{1}{y+1-x} \left(K^\lambda(y) - K^\lambda(x) - (y-x)\lambda(y+1) - (y-x)\Delta + (y+1-x)\lambda(y+1) \right) \\
&\quad = \frac{y-x}{y+1-x}\lambda'(y+1) + \frac{1}{y+1-x} \left(K^\lambda(y) + \lambda(y+1) - K^\lambda(x) - (y-x)\Delta \right) \\
&\quad = \frac{y-x}{y+1-x}\lambda'(y+1) \\
&\qquad + \frac{1}{y+1-x} \left(K^\lambda(y+1) - \sum_{i=y+1}^{\infty} \lambda(i) + \lambda(y+1) - K^\lambda(x) - (y-x)\Delta \right) \\
&\quad = \frac{y-x}{y+1-x}\lambda'(y+1) + \frac{1}{y+1-x} \left(K^\lambda(y+1) - K^\lambda(x) - (y-x)\Delta - \sum_{i=y+2}^{\infty} \lambda(i) \right) \\
&\quad = \frac{y-x}{y+1-x}\lambda'(y+1) + \frac{1}{y+1-x}\lambda'(y+1) \\
&\quad = \lambda'(y+1)
\end{aligned}
$$

And we have:

- for $k \leq x-1$ or $k \geq y+2$, $\mathbb{P}\left(Z^\lambda_\tau = k\right) = \mathbb{P}\left(Z^\lambda_0 = k\right) = \lambda(k) = \lambda'(k)$;
- for $x+1 \leq k \leq y-1$, $\mathbb{P}\left(Z^\lambda_\tau = k\right) = 0 = \lambda'(k)$;
- for $k = y$ or $k = y+1$, we have shown that: $\mathbb{P}\left(Z^\lambda_\tau = k\right) = \lambda'(k)$;
- so, then, we have, for $k = x$: $\mathbb{P}\left(Z^\lambda_\tau = x\right) = \lambda'(x)$.

This way, we have $Z^\lambda_\tau \sim \lambda'$.

In both cases, the point at which the two potentials split increases, and this proves that the algorithm works and ends a.s. in a finite time.

- Now, we treat the case in which μ is not bounded but only integrable.

  We write: $\mu_n = \sum_{i=0}^{n-1} \mu(i)\delta_i + \left(1 - \sum_{i=0}^{n-1} \mu(i)\right)\delta_n$.

$$
K^{\mu_n}(y) = y + \sum_{i=0}^{y} (i-y)\mu_n(i) = \begin{cases} y + \sum_{i=0}^{y} (i-y)\mu(i) = K^\mu(y) & \text{if } y \leq n-1 \\ y + \sum_{i=0}^{n-1} (i-y)\mu_n(i) + (n-y)\left(1 - \sum_{i=0}^{n-1} \mu(i)\right) & \text{if } y \geq n \end{cases}
$$

But, if $y \geq n$:

$$
\begin{aligned}
(n-y)\left(1 - \sum_{i=0}^{n-1} \mu(i)\right) &= \sum_{i=n}^{\infty} (n-y)\mu(i) \\
&= \sum_{i=n}^{y} (n-y)\mu(i) + \sum_{i=y+1}^{\infty} (n-y)\mu(i) \\
&= \sum_{i=n}^{y} (i-y)\mu(i) + \sum_{i=n}^{y} \underbrace{(n-i)}_{\leq 0}\mu(i) + \sum_{i=y+1}^{\infty} \underbrace{(n-y)}_{\leq 0}\mu(i) \\
&\leq \sum_{i=n}^{y} (i-y)\mu(i) = K^\mu(y).
\end{aligned}
$$

So, we have: $\forall y \in \mathbb{N}$, $K^{\mu_n}(y) \leq K^{\mu}(y) \leq K^{\lambda}(y)$.
And, because μ_n is bounded, we know that there exists a stopping time τ_n (given by the *Azéma-Yor* process) such that $Z^{\lambda}_{\tau_n} \sim \mu_n$.
I suppose that $Z^{\lambda}_{\tau_n} \leq n-1$.
Because $\forall z \in [0,n]$, $K^{\mu_n}(z) = K^{\mu_{n+1}}(z)$, the first steps (ie: until what we called x reaches n) of the process we used to construct τ_n and τ_{n+1} are exactly the same.
So, if we have $Z^{\lambda}_{\tau_n} \leq n-1$, then, we have $Z^{\lambda}_{\tau_{n+1}} = Z^{\lambda}_{\tau_n}$.
We write $\tau = \liminf_{n\to\infty} \tau_n$, and then, for $k \in \mathbb{N}$: $\mathbb{P}\left(Z^{\lambda}_{\tau} = k\right) = \mathbb{P}\left(Z^{\lambda}_{\tau_{k+1}} = k\right) = \mu_{k+1}(k) = \mu(k)$.

■

In the proof, it appears clearly that the *Azéma-Yor* solution is, more concretely, the strategy we called "Stop at the bottom or reach another point" (see paragraph 3.1.3, page 9).

3.2.3 Properties of the *Azéma-Yor* solution

In everything following, we will use this notation:

$$\overline{Z^{\lambda}_n} = \max\left\{Z^{\lambda}_i \middle| i \in [\![0,n]\!]\right\}$$

And we will now write τ_{AY} the stopping time given by the *Azéma-Yor* process.

Proposition 6 Bounding the maximum knowing the stopping point

Let λ and μ two integrable distributions, with $K^{\mu} \leq K^{\lambda}$.
There exists an increasing function $f : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$, such that:

$$Z^{\lambda}_{\tau_{AY}} = x \Longrightarrow f(x-1) \leq \overline{Z^{\lambda}_{\tau_{AY}}} \leq f(x)$$

Proof 6

Now, we define:

- x_1, the point where the potentials K^{λ} and K^{μ} split;
- $x_2, \ldots, x_n, \ldots$, the atoms of μ which are in $]x_1, \infty[$, such that $x_1 < x_2 < \ldots < x_n < \ldots$
  Recall that μ can have a finite number of atoms.

We write $\lambda_0 = \lambda$, $\lambda_1 = \lambda'$, $\lambda_2 = (\lambda')'$, ...; because the notation "prime" is not really good in this proof.
Also, $y_1, y_2, \ldots, y_n, \ldots \in \mathbb{R}^+$ are such that: the tangent of K^{μ} between x_k and x_{k+1} hits $K^{\lambda_{k-1}}$ at a point whose abscissa is y_k.
Finally, $\Delta_k := \sum_{i=x_{k+1}}^{\infty} \mu(i)$.
I've got:

$$y > y_k \Leftrightarrow K^{\lambda_{k-1}}(y) < K^{\mu}(x_k) + (y - x_k)\,\Delta_k,$$

because the function $y \mapsto K^{\mu}(x_k) + (y - x_k)\,\Delta_k - K^{\lambda_{k-1}}(y)$ decreases when $y \in [x_k, \infty[$ and values 0 when $y = y_k$.
This also gives me:

$$y_{k+1} > \lfloor y_k \rfloor.$$

We have: $K^\mu(x_k) + (y - x_k)\Delta_k$ is the value at y of the right-tangent to K^μ at point x_k; and: $K^\mu(x_{k+1}) + (y - x_{k+1})\Delta_{k+1}$ is the value at y of the right-tangent to K^μ at point x_{k+1}.

But for $y > x_{k+1}$, $K^\mu(x_{k+1}) + (y - x_{k+1})\Delta_{k+1} < K^\mu(x_{k+1}) + (y - x_{k+1})\Delta_k = K^\mu(x_k) + (y - x_k)\Delta_k$.
The right hand side hits $K^{\lambda_{k-1}}$ at y_k; the left hand side hits $K^{\lambda_{k-1}}$ at a point we call y'_{k+1}.
It's now obvious that we have: $y_k < y'_{k+1}$.

If $y'_{k+1} \geq \lfloor y_k \rfloor + 1$, then $y_{k+1} = y'_{k+1}$, because after $\lfloor y_k \rfloor + 1$, we have $K^{\lambda_{k-1}} = K^{\lambda_k}$.
And if $\lfloor y'_{k+1} \rfloor = \lfloor y_{k+1} \rfloor$, let's suppose that $y_{k+1} \leq \lfloor y_k \rfloor$.
It would mean that the mean of the slope of K^{λ_k} between x_{k+1} and $\lfloor y_k \rfloor$ $(\geq \Delta_k)$ is less than the mean of the slope of $K^{\lambda_{k+1}}$ between x_{k+1} and $\lfloor y_{k+1} \rfloor$ $(= \Delta_{k+1})$. And this is a contradiction, because we know $\Delta_k > \Delta_{k+1}$.

Now, we suppose that x_k is not the biggest atom of μ.

1. If $y_k \in \mathbb{N}$.
   The *Azéma-Yor* process says that if we reach y_k, we wait at y_k. But if not, we stop at x_k.
   It means that the biggest point we have reached is $\leq y_k - 1$ if we stop at x_k.
   And it's also $\geq z_0 := \min\{z \geq x_k | \lambda_{k-1}(z) > 0\}$.

   If $k > 1$, then we have: $K^{\lambda_{k-1}}(x_k) = K^\mu(x_k) = K^\mu(x_{k-1}) + (x_k - x_{k-1})\Delta_{k-1} < K^{\lambda_{k-2}}(x_k)$ (because there's no μ-atom between x_{k-1} and x_k).
   We have $K^\mu(x_{k-1}) + (\lfloor y_{k-1} \rfloor + 1 - x_{k-1})\Delta_{k-1} > K^{\lambda_{k-2}}(\lfloor y_{k-1} \rfloor + 1)$, because $y_{k-1} < \lfloor y_{k-1} \rfloor + 1$.
   So $x_k < \lfloor y_{k-1} \rfloor + 1$, that is to say: $x_k \leq \lfloor y_{k-1} \rfloor$, and $z_0 = \lfloor y_{k-1} \rfloor$, because λ_{k-1} has no atom between x_{k-1} and $\lfloor y_{k-1} \rfloor$ (the potential is a non-broken line between those points).

   Finally, if we stop in x_k, it means that my maximum is between : $\left| \begin{array}{ccll} x_1 & \text{and} & y_1 - 1 & \text{if } k = 1 \\ \lfloor y_{k-1} \rfloor & \text{and} & y_k - 1 & \text{if } k > 1 \end{array} \right.$.

2. If $y_k \notin \mathbb{N}$.
   The *Azéma-Yor* process now says that if we reach $\lfloor y_k \rfloor$, we can: reach and stop at x_k, or wait at $\lfloor y_k \rfloor$, or even reach and wait at $\lfloor y_k \rfloor + 1$.
   But it's quite the same: if we stop at x_k, the biggest point we can have reached is $\lfloor y_k \rfloor$.

   What we did in the previous case remains the same and:
   if we stop in x_k, it means that my maximum is between : $\left| \begin{array}{ccll} x_1 & \text{and} & \lfloor y_1 \rfloor & \text{if } k = 1 \\ \lfloor y_{k-1} \rfloor & \text{and} & \lfloor y_k \rfloor & \text{if } k > 1 \end{array} \right.$.

So, we can define f by pieces: $\left| \begin{array}{rcll} f(j) & = & j + \frac{1}{2} & \text{if } j < x_1 \\ f(x_k) & = & y_k - \frac{1}{2} & \text{if } x_k \text{ is a } \mu\text{-atom of case 1.} \\ f(x_k) & = & \lfloor y_k \rfloor & \text{if } x_k \text{ is a } \mu\text{-atom of case 2.} \\ f(j) & = & f(x_k) & \text{if } j \in \rrbracket x_k, x_{k+1} \llbracket \\ f(j) & = & \infty & \text{if } j \geq x_k \text{ and } x_k \text{ is the biggest atom of } \mu \end{array} \right.$.
And one can check that f is an increasing function, such that: if we stop at x_k, my maximum is between $f(x_k - 1)$ and $f(x_k)$.

■

Theorem 7 Maximisation of the probability of reaching a point

Let λ and μ be two integrable distributions, with $K^\mu \leq K^\lambda$.
τ_{AY} maximises the quantity $\mathbb{P}\left(\overline{Z^\lambda_\tau} \geq x\right)$, where $x \in \mathbb{N}$, over all the almost surely finite stopping times τ such that $Z^\lambda_\tau \sim \mu$.

Proof 7

We will first find a family of bounds of this probability when τ is a general stopping time such that $Z^\lambda_\tau \sim \mu$, and then, we will show that $\mathbb{P}\left(\overline{Z^\lambda_{\tau_{AY}}} \geq x\right)$ is equal to one of these bounds.

- We take $z \in \mathbb{N}$, $z < x$.
  We can show that we have:
  $$\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} \leq \frac{\left(Z^\lambda_\tau - z\right)_+}{x - z} + \frac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} \tag{3}$$
  If $\overline{Z^\lambda_\tau} \geq x$, then, $\dfrac{\left(Z^\lambda_\tau - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} = \begin{cases} \frac{Z^\lambda_\tau - z + x - Z^\lambda_\tau}{x - z} = 1 = \mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} & \text{if } z \leq Z^\lambda_\tau \\ \frac{x - Z^\lambda_\tau}{x - z} \geq 1 = \mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} & \text{if } z > Z^\lambda_\tau \end{cases}.$
  And if $\overline{Z^\lambda_\tau} < x$, then, $\dfrac{\left(Z^\lambda_\tau - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x} = \dfrac{\left(Z^\lambda_\tau - z\right)_+}{x - z} \geq 0 = \mathbb{1}_{\overline{Z^\lambda_\tau} \geq x}$.

  Now, we have also:
  $$\forall n \in \mathbb{N},\ \frac{\left(Z^\lambda_{\tau\wedge n} - z\right)_+}{x - z} + \frac{x - Z^\lambda_{\tau\wedge n}}{x - z}\mathbb{1}_{\overline{Z^\lambda_{\tau\wedge n}} \geq x} \leq \frac{x}{x - z}$$
  If $Z^\lambda_{\tau\wedge n} \leq z$ and $\overline{Z^\lambda_{\tau\wedge n}} \geq x$, then $\dfrac{\left(Z^\lambda_{\tau\wedge n} - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_{\tau\wedge n}}{x - z}\mathbb{1}_{\overline{Z^\lambda_{\tau\wedge n}} \geq x} = \dfrac{x - Z^\lambda_{\tau\wedge n}}{x - z} \leq \dfrac{x}{x - z}$.
  If $Z^\lambda_{\tau\wedge n} \leq z$ and $\overline{Z^\lambda_{\tau\wedge n}} < x$, then $\dfrac{\left(Z^\lambda_{\tau\wedge n} - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_{\tau\wedge n}}{x - z}\mathbb{1}_{\overline{Z^\lambda_{\tau\wedge n}} \geq x} = 0 \leq \dfrac{x}{x - z}$.
  If $Z^\lambda_{\tau\wedge n} > z$ and $\overline{Z^\lambda_{\tau\wedge n}} \geq x$ then $\dfrac{\left(Z^\lambda_{\tau\wedge n} - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_{\tau\wedge n}}{x - z}\mathbb{1}_{\overline{Z^\lambda_{\tau\wedge n}} \geq x} = \dfrac{x - z}{x - z} \leq \dfrac{x}{x - z}$.
  And if $Z^\lambda_{\tau\wedge n} > z$ and $\overline{Z^\lambda_{\tau\wedge n}} < x$, then $\dfrac{\left(Z^\lambda_{\tau\wedge n} - z\right)_+}{x - z} + \dfrac{x - Z^\lambda_{\tau\wedge n}}{x - z}\mathbb{1}_{\overline{Z^\lambda_{\tau\wedge n}} \geq x} = \dfrac{Z^\lambda_{\tau\wedge n} - z}{x - z} \leq 1 \leq \dfrac{x}{x - z}$.
  So, by bounded convergence, we can take the expectancy in (3) and get:
  $$\mathbb{P}\left(\overline{Z^\lambda_\tau} \geq x\right) \leq \mathbb{E}\left[\frac{\left(Z^\lambda_\tau - z\right)_+}{x - z}\right] + \mathbb{E}\left[\frac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x}\right].$$
  The first part of the right hand side is all right, because it is independent from τ: $\mathcal{L}\left(Z^\lambda_\tau\right) = \mu$.
  Let's work on the second part!
  $\mathbb{E}\left[\dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x}\right] = \displaystyle\sum_{y=0}^{\infty} \mathbb{E}\left[\dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x}\mathbb{1}_{Z^\lambda_0 = y}\right]$
  We now have two cases:
  - If $y \geq x$, then $\overline{Z^\lambda_\tau} \geq x$, and $\mathbb{E}\left[\dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{\overline{Z^\lambda_\tau} \geq x}\mathbb{1}_{Z^\lambda_0 = y}\right] = \mathbb{E}\left[\dfrac{x - Z^\lambda_\tau}{x - z}\mathbb{1}_{Z^\lambda_0 = y}\right]$.
  - If $y < x$, then $\left(\dfrac{x - Z^\lambda_{n\wedge H_x}}{x - z}\mathbb{1}_{Z^\lambda_0 = y}\right)$ is a martingale, where $H_x = \inf\left\{i \in \mathbb{N} \middle| Z^\lambda_i = x\right\}$ can be equal to ∞.
    $\dfrac{x - Z^\lambda_{n\wedge H_x}}{x - z}\mathbb{1}_{Z^\lambda_0 = y}$ is $Z^\lambda_{n\wedge H_x}$-measurable and integrable (because Z^λ_n is a martingale).
    And $\mathbb{E}\left[\dfrac{x - Z^\lambda_{(n+1)\wedge H_x}}{x - z}\mathbb{1}_{Z^\lambda_0 = y}\middle| Z^\lambda_{0\wedge H_x}, \ldots, Z^\lambda_{n\wedge H_x}\right]$
    $$= \begin{cases} \mathbb{E}\left[\frac{x - Z^\lambda_{n+1}}{x - z}\mathbb{1}_{Z^\lambda_0 = y}\middle| Z^\lambda_0, \ldots, Z^\lambda_n\right] = \frac{x - Z^\lambda_{n\wedge H_x}}{x - z}\mathbb{1}_{Z^\lambda_0 = y} & \text{if } n + 1 \leq H_x \\ 0 = \frac{x - Z^\lambda_{n\wedge H_x}}{x - z}\mathbb{1}_{Z^\lambda_0 = y} & \text{if } n \geq H_x \end{cases}.$$

Also, this martingale is bounded by 0 and $\frac{x}{x-z}$, this way we can apply the Optional Stopping Theorem: $\mathbb{E}\left[\frac{x-Z^\lambda_{\tau\wedge H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\right] = \mathbb{E}\left[\frac{x-Z^\lambda_{0\wedge H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\right] = \frac{x-y}{x-z}\lambda(y)$.

$$\begin{aligned}
\text{But } \mathbb{E}\left[\frac{x-Z^\lambda_{\tau\wedge H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\right] &= \mathbb{E}\left[\frac{x-Z^\lambda_{\tau\wedge H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\right] + \mathbb{E}\left[\frac{x-Z^\lambda_{\tau\wedge H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}<x}\right] \\
&= \underbrace{\mathbb{E}\left[\frac{x-Z^\lambda_{H_x}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\right]}_{=0} + \mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}<x}\right] \\
&= \mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}<x}\right]
\end{aligned}$$

So: $\mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\right] = \mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\right] - \frac{x-y}{x-z}\lambda(y)$.

$$\begin{aligned}
\text{Finally, } \mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\right] &= \sum_{y=0}^{\infty}\mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\mathbb{1}_{Z_0^\lambda=y}\right] \\
&= \sum_{y=0}^{\infty}\mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\mathbb{1}_{Z_0^\lambda=y}\right] - \sum_{y=0}^{x-1}\frac{x-y}{x-z}\lambda(y) \\
&= \mathbb{E}\left[\frac{x-Z^\lambda_{\tau}}{x-z}\right] - \sum_{y=0}^{x-1}\frac{x-y}{x-z}\lambda(y), \text{ independent from } \tau.
\end{aligned}$$

This way, we have shown that for all $z < x$, we have:

$$\mathbb{P}\left(\overline{Z^\lambda_\tau}\geq x\right) \leq \mathbb{E}\left[\frac{(Z^\lambda_\tau - z)_+}{x-z}\right] + \mathbb{E}\left[\frac{x-Z^\lambda_\tau}{x-z}\mathbb{1}_{\overline{Z^\lambda_\tau}\geq x}\right],$$

and the right hand side is totally independent from τ!

- Now, our goal is to show that we have, for one $z_0 < x$:

$$\mathbb{1}_{\overline{Z^\lambda_{\tau_{AY}}}\geq x} = \frac{(Z^\lambda_{\tau_{AY}} - z_0)_+}{x-z_0} + \frac{x-Z^\lambda_{\tau_{AY}}}{x-z_0}\mathbb{1}_{\overline{Z^\lambda_{\tau_{AY}}}\geq x}. \tag{4}$$

Then, we will do the same as before: take the expectancy, to have:

$$\mathbb{P}\left(\overline{Z^\lambda_{\tau_{AY}}}\geq x\right) = \mathbb{E}\left[\frac{(Z^\lambda_{\tau_{AY}} - z_0)_+}{x-z_0}\right] + \mathbb{E}\left[\frac{x-Z^\lambda_{\tau_{AY}}}{x-z_0}\mathbb{1}_{\overline{Z^\lambda_{\tau_{AY}}}\geq x}\right].$$

And because in the right hand side we will be able to replace τ_{AY} by any other τ verifying $Z^\lambda_\tau \sim \mu$, and because of what we showed before, we will have:

$$\text{for all almost surely finite stopping time } \tau \text{ such that } Z^\lambda_\tau \sim \mu,\ \mathbb{P}\left(\overline{Z^\lambda_{\tau_{AY}}}\geq x\right) \geq \mathbb{P}\left(\overline{Z^\lambda_\tau}\geq x\right)$$

$$\begin{aligned}
\text{We have: } (4) &\Leftrightarrow \frac{Z^\lambda_{\tau_{AY}} - z_0}{x-z_0}\mathbb{1}_{\overline{Z^\lambda_{\tau_{AY}}}\geq x} = \frac{(Z_{\tau_{AY}} - z_0)_+}{x-z_0} \\
&\Leftrightarrow \begin{cases} \text{if } \overline{Z^\lambda_{\tau_{AY}}}\geq x: & Z^\lambda_{\tau_{AY}} - z_0 = \left(Z^\lambda_{\tau_{AY}} - z_0\right)_+ \Leftrightarrow Z^\lambda_{\tau_{AY}} - z_0 \geq 0 \\ \text{if } \overline{Z^\lambda_{\tau_{AY}}} < x: & 0 = \left(Z^\lambda_{\tau_{AY}} - z_0\right)_+ \Leftrightarrow Z^\lambda_{\tau_{AY}} - z_0 \leq 0 \end{cases}.
\end{aligned}$$

We know that there exists f increasing, such that:

$$Z^\lambda_{\tau_{AY}} = z \Longrightarrow f(z-1) \leq \overline{Z^\lambda_{\tau_{AY}}} \leq f(z).$$

So:

$$Z^\lambda_{\tau_{AY}} \geq z \Longrightarrow f(z-1) \leq \overline{Z^\lambda_{\tau_{AY}}} \text{ and } \overline{Z^\lambda_{\tau_{AY}}} < f(z-1) \Longrightarrow Z^\lambda_{\tau_{AY}} < z;$$

$$Z^{\lambda}_{\tau_{AY}} \leq z \Longrightarrow \overline{Z^{\lambda}_{\tau_{AY}}} \leq f(z) \text{ and } \overline{Z^{\lambda}_{\tau_{AY}}} > f(z) \Longrightarrow Z^{\lambda}_{\tau_{AY}} > z.$$

We choose:

$$z_0 = \min \{z \in \mathbb{N} | f(z) \geq x\}.$$

This way, we have:

$$\overline{Z^{\lambda}_{\tau_{AY}}} < x \Longrightarrow \overline{Z^{\lambda}_{\tau_{AY}}} < f(z_0) \Longrightarrow Z^{\lambda}_{\tau_{AY}} < z_0 + 1 \Longrightarrow Z^{\lambda}_{\tau_{AY}} \leq z_0 \Longrightarrow Z^{\lambda}_{\tau_{AY}} - z_0 \leq 0;$$

$$\overline{Z^{\lambda}_{\tau_{AY}}} \geq x \Longrightarrow \overline{Z^{\lambda}_{\tau_{AY}}} > f(z_0 - 1) \Longrightarrow Z^{\lambda}_{\tau_{AY}} > z_0 - 1 \Longrightarrow Z^{\lambda}_{\tau_{AY}} \geq z_0 \Longrightarrow Z^{\lambda}_{\tau_{AY}} - z_0 \geq 0.$$

And this way we prove that (4) is true for at least this $z_0 = \min \{z \in \mathbb{N} | f(z) \geq x\}$, and the theorem is proved.

■

3.3 The *Root*'s barrier[6] solution

3.3.1 Illustration of the process

This construction is the strategy we called "Stop as late as possible" (see paragraph 3.1.2, page 9).

Let's have a look on what happens to the potential when we work like this. In this example, we have $\lambda = \frac{1}{4}\delta_2 + \frac{1}{2}\delta_3 + \frac{1}{4}\delta_4$ and $\mu = \frac{1}{6}\delta_1 + \frac{7}{9}\delta_3 + \frac{1}{18}\delta_6$.

What we do in this example is this:

1. We see that the two potentials (red and green) are equal in 1, an atom of μ. It means that we will always stop everything when we will reach the point 1.

2. First, we will see what would happen if we stopped nothing at time 0 in points 3 and 6. We get the dashed orange potential (partly above the blue one), of the distribution: $\frac{1}{8}\delta_1 + \frac{1}{4}\delta_2 + \frac{1}{4}\delta_3 + \frac{1}{4}\delta_4 + \frac{1}{8}\delta_5$. The dashed part goes below the potential of μ. So we needed to stop some mass at time 0 at the point 3. The choice we make is to stop at 3 at time 0 with probability $\frac{2}{3}$ (the proof will explain this choice), and always after time 0. We obtain this distribution, whose potential is in blue:
$\frac{1}{8}\delta_1 + \frac{1}{3} \times \frac{1}{4}\delta_2 + \left(\frac{1}{4} + \frac{2}{3} \times \frac{1}{2}\right)\delta_3 + \frac{1}{3} \times \frac{1}{4}\delta_4 + \frac{1}{8}\delta_5 = \frac{1}{8}\delta_1 + \frac{1}{12}\delta_2 + \frac{7}{12}\delta_3 + \frac{1}{12}\delta_4 + \frac{1}{8}\delta_5$.

3. We are at time 1 now. And we always have to stop when we reach 1 or 3.

4. Then, we carry on. We obtain the yellow potential at time 2, assuming that we stop nothing at 6. The matching distribution is:
$\left(\frac{1}{8} + \frac{1}{24}\right)\delta_1 + \left(\frac{1}{24} + \frac{7}{12} + \frac{1}{24}\right)\delta_3 + \frac{1}{16}\delta_4 + \frac{1}{24}\delta_5 + \frac{1}{16}\delta_6 = \frac{1}{6}\delta_1 + \frac{2}{3}\delta_3 + \frac{1}{16}\delta_4 + \frac{1}{24}\delta_5 + \frac{1}{16}\delta_6$.

5. If we carry on, we will find a time n at which we will have to stop at 6 too. And the potentials we will draw will move towards the potential of μ.

[6] *D.H. Root* published in 1969 *"The existence of certain stopping times on Brownian motion"* in which he presents this class of solutions.

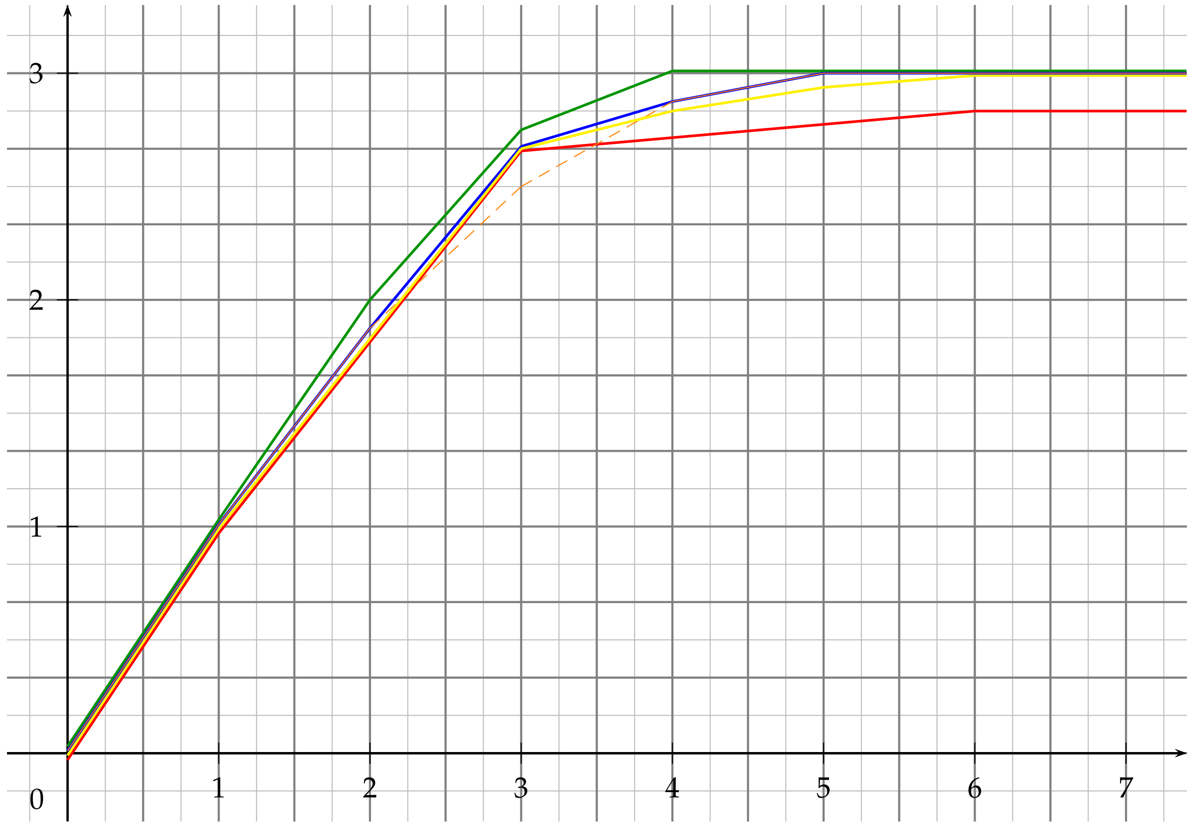

Figure 9: The potentials of the distributions λ, μ and of the intermediate ones.

3.3.2 Proof that it solves the *Skorokhod* Problem

Theorem 8 The *Root*'s barrier solution for the *Skorokhod* Embedding Problem

Let λ and μ be two integrable distributions over $\mathbb{N}$.
The *Root*'s barrier process works if we have:

$$\forall y \in \mathbb{N}, \sum_{i=0}^{\infty} (i \wedge y)\, \mu(i) \leq \sum_{i=0}^{\infty} (i \wedge y)\, \lambda(i).$$

Proof 8

We write $\lambda_n(x)$ the mass which is in x at step n, and $p_{x,n}$ the probability of stopping at x at step n, knowing that we are currently in x.
We get the following equalities:

$$\forall n \in \mathbb{N}, \lambda_{n+1}(x) = \begin{cases} \frac{1-p_{x-1,n}}{2}\lambda_n(x-1) + p_{x,n}\lambda_n(x) + \frac{1-p_{x+1,n}}{2}\lambda_n(x+1) & \text{if } x \geq 1 \\ p_{0,n}\lambda_n(0) + \frac{1-p_{1,n}}{2}\lambda_n(1) & \text{if } x = 0 \end{cases}.$$

(We can see that the case $x = 0$ is the same as the first one if we say $\lambda_n(-1) = 0$.)

In the following, we will use the fact that we know: $p_{0,n} = 1$ for all $n \in \mathbb{N}$, which means that we always stop when we reach 0.

We have:

$$
\begin{aligned}
K^{\lambda_{n+1}}(x) &= x + \sum_{i=0}^{x-1} (i-x)\lambda_{n+1}(i) \\
&= x + \sum_{i=0}^{x-1} (i-x)\frac{1-p_{i-1,n}}{2}\lambda_n(i-1) + \sum_{i=0}^{x-1} (i-x)p_{i,n}\lambda_n(i) + \sum_{i=0}^{x-1} (i-x)\frac{1-p_{i+1,n}}{2}\lambda_n(i+1) \\
&= x + \sum_{i=0}^{x-2} (i+1-x)\frac{1-p_{i,n}}{2}\lambda_n(i) + \sum_{i=0}^{x-1} (i-x)p_{i,n}\lambda_n(i) + \sum_{i=0}^{x} (i-1-x)\frac{1-p_{i,n}}{2}\lambda_n(i) \\
&= x + \sum_{i=0}^{x-2} \frac{1-p_{i,n}}{2}\lambda_n(i) + \sum_{i=0}^{x-2} (i-x)\frac{1-p_{i,n}}{2}\lambda_n(i) + \sum_{i=0}^{x-1} (i-x)p_{i,n}\lambda_n(i) - \sum_{i=0}^{x} \frac{1-p_{i,n}}{2}\lambda_n(i) \\
&\quad + \sum_{i=0}^{x} (i-x)\frac{1-p_{i,n}}{2}\lambda_n(i) \\
&= x - \frac{1-p_{x-1,n}}{2}\lambda_n(x-1) - \frac{1-p_{x,n}}{2}\lambda_n(x) - (x-1-x)\frac{1-p_{x-1,n}}{2}\lambda_n(x-1) + \sum_{i=0}^{x-1} (i-x)\frac{1-p_{i,n}}{2}\lambda_n(i) \\
&\quad + \sum_{i=0}^{x-1} (i-x)p_{i,n}\lambda_n(i) + (x-x)\frac{1-p_{x,n}}{2}\lambda_n(x) + \sum_{i=0}^{x-1} (i-x)\frac{1-p_{i,n}}{2}\lambda_n(i) \\
&= x - \frac{1-p_{x,n}}{2}\lambda_n(x) + \sum_{i=0}^{x-1} (i-x)\lambda_n(i) \\
&= K^{\lambda_n}(x) - \frac{1-p_{x,n}}{2}\lambda_n(x).
\end{aligned}
$$

The stopping time we are constructing implies that the sequence $(p_{x,n})$ follow the rules:

$$
\begin{cases}
\text{if } p_{x,n} > 0, \text{ then } p_{x,n+1} = 1 \\
\text{if } \mu(x) = 0 \text{ and } x \neq 0, \text{ then } \forall n \in \mathbb{N},\ p_{x,n} = 0 \\
p_{0,0} = 1
\end{cases}
$$

If $K^{\lambda_{n-1}}(x) - \frac{1}{2}\lambda_{n-1}(x) \leq K^{\mu}(x)$, then, we write $p_{x,n-1} = \dfrac{K^{\mu}(x) - \left(K^{\lambda_{n-1}}(x) - \frac{1}{2}\lambda_{n-1}(x)\right)}{\frac{1}{2}\lambda_{n-1}(x)}$ and $p_{x,n} = 1$.

If we reach this case, then, we will have: $\forall m \geq n$, $K^{\lambda_m}(x) = K^{\mu}(x)$.

If $K^{\lambda_{n-1}}(x) - \dfrac{1}{2}\lambda_{n-1}(x) > K^{\mu}(x)$, we write $p_{x,n-1} = 0$.

This way, we always have $K^{\lambda_n}(x) \geq K^{\mu}(x)$.

Now, what we need to show is that:

$$\mu(x) > 0 \Longrightarrow \exists n \in \mathbb{N},\ K^{\lambda_n}(x) \leq K^{\mu}(x).$$

We will show this by contradiction: let x be an atom of μ such that $\forall n \in \mathbb{N}$, $K^{\lambda_n}(x) > K^{\mu}(x)$.

We define those numbers:

$$y_0 := \max\left\{ y \in [\![0, x-1]\!] \middle| y = 0 \text{ or } \left(\mu(y) > 0 \text{ and } \exists n \in \mathbb{N},\ K^{\lambda_n}(y) = K^{\mu}(y)\right)\right\}$$

$$y_1 := \min\left\{ y \in [\![x+1, \infty[\![\middle| \mu(y) > 0 \text{ and } \exists n \in \mathbb{N},\ K^{\lambda_n}(y) = K^{\mu}(y)\right\} \text{ (it may not be defined!)}$$

1. We suppose that y_1 exists.
   We write: $n_0 = \min\left\{n \in \mathbb{N} \middle| K^{\lambda_n}(y_0) = K^{\mu}(y_0) \text{ and } K^{\lambda_n}(y_1) = K^{\mu}(y_1)\right\}$.
   Now, I've got:

   $$\forall n \geq n_0,\ p_{y_0,n} = p_{y_1,n} = 1$$

   $$\forall n \in \mathbb{N},\ \forall z \in [\![y_0+1, y_1-1]\!],\ p_{z,n} = 0$$

   The second fact is because of the definition of y_0 and y_1.
   We will never stop between $y_0 + 1$ and x, and never between x and $y_1 - 1$.

Because we know that if we are between y_0 and y_1, I'm sure that we will hit one of them and then stop, we have:

$$\forall z \in [\![y_0+1, y_1-1]\!],\ \lambda_n(z) \underset{n\to\infty}{\longrightarrow} 0.$$

Also, we recall that we have $\forall n \geq n_0$, $K^{\lambda_n}(y_0) = K^{\mu}(y_0)$ and $K^{\lambda_n}(y_1) = K^{\mu}(y_1)$.
So, between y_0 and y_1, the potential of λ_n has for limit the straight line between the points $(y_0, K^{\mu}(y_0))$ and $(y_1, K^{\mu}(y_1))$.
That is to say:

$$\forall z \in [\![y_0+1, y_1-1]\!],\ K^{\lambda_n}(z) \underset{n\to\infty}{\longrightarrow} K^{\mu}(y_0) + \frac{K^{\mu}(y_1) - K^{\mu}(y_0)}{y_1 - y_0}(z - y_0).$$

But, because of its decreasing slope, we know that K^{μ} is concave.
So K^{μ} is over its chords, and we finally have:

$$K^{\mu}(x) = K^{\mu}(y_0) + \frac{K^{\mu}(y_1) - K^{\mu}(y_0)}{y_1 - y_0}(x - y_0) \Longrightarrow \mu(x) = 0.$$

So, it means that: $K^{\mu}(x) > \lim\limits_{n\to\infty} K^{\lambda_n}(x)$. Which gives us our contradiction.

2. We suppose that y_1 doesn't exist in this 2^{nd} case.
We write now $n_0 := \min\left\{n \in \mathbb{N} \middle| K^{\mu}(y_0) = K^{\lambda_n}(y_0)\right\}$.
And we've got now:

$$\forall n \geq n_0,\ p_{y_0,n} = 1$$
$$\forall z \geq y_0 + 1,\ \forall n \in \mathbb{N},\ p_{z,n} = 0$$

It means that we will stop at y_0 after step n_0, and we never stop strictly over y_0.
If we are over y_0, we are then sure that we will hit y_0 in a finite time.
So, $\forall z \geq y_0$, $\lambda(z) \underset{n\to\infty}{\longrightarrow} 0$ and $\forall n \geq n_0$, $K^{\mu}(y_0) = K^{\lambda_n}(y_0)$.
And it means $\forall z > y_0$, $K^{\lambda_n}(z) \underset{n\to\infty}{\longrightarrow} K^{\mu}(y_0)$.
But $K^{\mu}(x) \geq K^{\mu}(y_0) + \mu(x)(x - y_0) > K^{\mu}(y_0) = \lim\limits_{n\to\infty} K^{\lambda_n}(x)$.
And we finally have a contradiction.

So we have: $\forall x$ atom of μ, $\exists n_0 \in \mathbb{N}$, $\forall n \geq n_0$, $K^{\lambda_n}(x) = K^{\mu}(x)$.
Then, because we never stop at points which are not atoms of μ, K^{λ_n} tends to a straight line between two consecutive atoms of μ.
It's the same as K^{μ}!
So, we finally have: $\forall x \in \mathbb{N}$, $K^{\lambda_n}(x) \underset{n\to\infty}{\longrightarrow} K^{\mu}(x)$.
But λ_n is the law of $Z^{\lambda}_{\tau\wedge n}$, so the limit Z^{λ}_{τ} has law μ because the limit of K^{λ_n} is K^{μ}.

■

In the beginning of the previous proof, we have proved a relation which true for any stopping time τ:

$$\forall y \in \mathbb{N},\ K^{\lambda}_{\tau\wedge(n+1)}(y) = K^{\lambda}_{\tau\wedge n}(y) - \frac{1 - \mathbb{P}\left(\tau = n \middle| Z^{\lambda}_{\tau\wedge n} = y\right)}{2}\mathbb{P}\left(Z^{\lambda}_{\tau\wedge n} = y\right) \tag{5}$$

3.3.3 Properties of the *Root*'s barrier solution

Now, we will write τ_R the stopping time given by this process.
We also write:

$$\Delta^2 K^{\lambda}(y) = K^{\lambda}(y+1) - 2K^{\lambda}(y) + K^{\lambda}(y-1) = -\lambda(y).$$

Let's have a look to what we also showed in the proof.
At step n, we have two cases:

- If $\mathbb{P}\left(\text{Stopping at } y \text{ at step } n \middle| Z_n^\lambda = y\right) = 0$,
then $K_{\tau_R \wedge (n+1)}^\lambda(y) = K_{\tau_R \wedge n}^\lambda(y) + \frac{1}{2}\Delta^2 K_{\tau_R \wedge n}^\lambda(y) \geq K^\mu(y)$.

- Else we have $K_{\tau_R \wedge (n+1)}^\lambda(y) = K^\mu(y) \geq K_{\tau \wedge n}^\lambda(y) + \frac{1}{2}\Delta^2 K_{\tau \wedge n}^\lambda(y)$.

In other words,

- If $K^\mu(y) < K_{\tau_R \wedge (n+1)}^\lambda(y)$,
then $K_{\tau_R \wedge n}^\lambda(y) + \frac{1}{2}\Delta^2 K_{\tau_R \wedge n}^\lambda(y) = K_{\tau_R \wedge (n+1)}^\lambda(y)$.

- Else $K^\mu(y) = K_{\tau_R \wedge (n+1)}^\lambda(y)$ and $K_{\tau_R \wedge n}^\lambda(y) + \frac{1}{2}\Delta^2 K_{\tau_R \wedge n}^\lambda(y) \leq K^\mu(y) = K_{\tau_R \wedge (n+1)}^\lambda(y)$.

It means that $(K_n)_{n \in \mathbb{N}} := \left(K_{\tau_R \wedge n}^\lambda\right)_{n \in \mathbb{N}}$ solves all these equations:

$$\forall n \in \mathbb{N}, \forall y \in \mathbb{N}, K_{n+1}(y) = \max\left\{K^\mu(y), K_n(y) + \frac{1}{2}\Delta^2 K_n(y)\right\} \tag{6}$$

Definition 2 Super-solutions to (6)

Let $\left(\widetilde{K_n}\right)_{n \in \mathbb{N}}$ be a sequence of functions over $\mathbb{N}$.
We say that $\left(\widetilde{K_n}\right)_{n \in \mathbb{N}}$ is a super-solution to (6), if and only if:

$$\widetilde{K_0} = K^\lambda \text{ and } \forall n \in \mathbb{N}, \forall y \in \mathbb{N}, \max\left\{K^\mu(y), \widetilde{K_n}(y) + \frac{1}{2}\Delta^2 \widetilde{K_n}(y)\right\} \leq \widetilde{K_{n+1}}(y).$$

Proposition 9 Link between the *Skorokhod* Embedding Problem and (6)

Let τ be any solution to the *Skorokhod* Embedding Problem, with starting distribution λ and target distribution μ', with $K^{\mu'} \geq K^\mu$.

The sequence $\left(K_{\tau \wedge n}^\lambda\right)_{n \in \mathbb{N}}$ is a super-solution to (6).

Proof 9

Obviously, we have: $K_{\tau \wedge 0}^\lambda = K^\lambda$, and $\forall n \in \mathbb{N}$, $K_{\tau \wedge n}^\lambda \geq K^{\mu'} \geq K^\mu$.
Then, using (5), we have, for all $y \in \mathbb{N}$ and $n \in \mathbb{N}$:

$$\begin{aligned}
K_{\tau \wedge (n+1)}^\lambda(y) &= K_{\tau \wedge n}^\lambda(y) - \frac{1 - \mathbb{P}\left(\tau = n \middle| Z_{\tau \wedge n}^\lambda = y\right)}{2} \mathbb{P}\left(Z_{\tau \wedge n}^\lambda = y\right) \\
&= K_{\tau \wedge n}^\lambda(y) - \frac{\mathbb{P}\left(\tau \neq n \middle| Z_{\tau \wedge n}^\lambda = y\right)}{2} \mathbb{P}\left(Z_{\tau \wedge n}^\lambda = y\right) \\
&= K_{\tau \wedge n}^\lambda(y) - \frac{1}{2}\mathbb{P}\left(\tau \neq n \text{ and } Z_{\tau \wedge n}^\lambda = y\right)
\end{aligned}$$

But, $\mathbb{P}\left(\tau \neq n \text{ and } Z_{\tau \wedge n}^\lambda = y\right) \leq \mathbb{P}\left(Z_{\tau \wedge n}^\lambda = y\right) = -\Delta^2 K_{\tau \wedge n}^\lambda(y)$.
So, $K_{\tau \wedge (n+1)}^\lambda(y) = K_{\tau \wedge n}^\lambda(y) - \frac{1}{2}\mathbb{P}\left(\tau \neq n \text{ and } Z_{\tau \wedge n}^\lambda = y\right) \geq K_{\tau \wedge n}^\lambda(y) + \frac{1}{2}\Delta^2 K_{\tau \wedge n}^\lambda(y)$.

■

Lemma 10

If $(K_n)_{n\in\mathbb{N}}$ is a solution to (6) and $\left(\widetilde{K_n}\right)_{n\in\mathbb{N}}$ is a super-solution to (6),
Then, we have: $\forall n \in \mathbb{N}$, $\forall y \in \mathbb{N}$, $K_n(y) \le \widetilde{K_n}(y)$.

Proof 10

This can be proved by induction.

- If $n = 0$:
  We have $\forall y \in \mathbb{N}$, $K_0(y) = \widetilde{K_0}(y)$, because we know $K_0 = K^\lambda = \widetilde{K_0}$.
- If $n > 0$, I suppose $\forall y \in \mathbb{N}$, $K_{n-1}(y) \le \widetilde{K_{n-1}}(y)$.
  Let's take $y \in \mathbb{N}$.
  If $K_n(y) = K^\mu(y)$, then $\widetilde{K_n}(y) \ge K^\mu(y) = K_n(y)$.
  Else, $$\begin{aligned} K_n(y) = K_{n-1}(y) + \frac{1}{2}\Delta^2 K_{n-1}(y) &= K_{n-1}(y) + \frac{1}{2}\left(K_{n-1}(y+1) - 2K_{n-1}(y) + K_{n-1}(y-1)\right) \\ &= \frac{1}{2}K_{n-1}(y-1) + \frac{1}{2}K_{n-1}(y+1) \\ &\le \frac{1}{2}\widetilde{K_{n-1}}(y-1) + \frac{1}{2}\widetilde{K_{n-1}}(y+1) \text{ (by induction)} \\ &\le \widetilde{K_{n-1}}(y) + \frac{1}{2}\Delta^2\widetilde{K_{n-1}}(y) \\ &\le \widetilde{K_n}(y) \end{aligned}$$

■

Proposition 11 τ_R **maximises the expectancy of** $\tau \wedge n$

Let τ be any solution to the *Skorokhod* Embedding Problem, with starting distribution λ and target distribution μ', with $K^{\mu'} \ge K^\mu$.

We have:
$$\forall n \in \mathbb{N},\ \mathbb{E}\left[\tau_R \wedge n\right] \ge \mathbb{E}\left[\tau \wedge n\right]$$

Proof 11

Thanks to Proposition 9 and Lemma 10 (see pages 25 and 26), we have:

$$\forall n \in \mathbb{N},\ \forall y \in \mathbb{N},\ K^\lambda_{\tau_R\wedge n}(y) \le K^\lambda_{\tau\wedge n}(y).$$

Also, using (2) (see page 7), we have:

$$\forall n \in \mathbb{N},\ \forall y \in \mathbb{N},\ K^\lambda_{\tau\wedge n}(y) = K^\lambda(y) - \frac{1}{2}\mathbb{E}\left[\sum_{i=0}^{\tau\wedge n-1} \mathbb{1}_{Z_i^\lambda = y}\right].$$

We have, finally: $\sum_{y=0}^{\infty} \underbrace{\left(K^{\lambda}_{\tau_R \wedge n}(y) - K^{\lambda}_{\tau \wedge n}(y)\right)}_{\leq 0} = \frac{1}{2} \sum_{y=0}^{\infty} \mathbb{E}\left[\sum_{i=0}^{\tau \wedge n-1} \mathbb{1}_{Z_i^{\lambda}=y}\right] - \mathbb{E}\left[\sum_{i=0}^{\tau_R \wedge n-1} \mathbb{1}_{Z_i^{\lambda}=y}\right]$

$$= \frac{1}{2}\mathbb{E}\left[\sum_{i=0}^{\tau \wedge n-1} \sum_{y=0}^{\infty} \mathbb{1}_{Z_i^{\lambda}=y} - \sum_{i=0}^{\tau_R \wedge n-1} \sum_{y=0}^{\infty} \mathbb{1}_{Z_i^{\lambda}=y}\right]$$

$$= \frac{1}{2}\mathbb{E}\left[\tau \wedge n - \tau_R \wedge n\right]$$

So, it means that, for all $n \in \mathbb{N}$, we have: $\mathbb{E}[\tau \wedge n] - \mathbb{E}[\tau_R \wedge n] \leq 0$.

$\blacksquare$

Theorem 12 τ_R **maximises the expectancy of** $f(\tau)$ **for every concave increasing** f

Let τ be any solution to the *Skorokhod* Embedding Problem, with starting distribution λ and target distribution μ', with $K^{\mu'} \geq K^{\mu}$.
Let $f : \mathbb{N} \to \mathbb{R}$ be a concave and increasing function.
We have:

$$\mathbb{E}\left[f(\tau_R)\right] \geq \mathbb{E}\left[f(\tau)\right]$$

Proof 12

If $\mathbb{E}[f(\tau_R)] = \infty$, the result is obvious.

We now suppose that $\mathbb{E}[f(\tau_R)] < \infty$.
For $k \geq 1$, we define:

$$\Delta^2 f(k) = f(k+1) - 2f(k) + f(k-1) = (f(k+1) - f(k)) - (f(k) - f(k-1)).$$

Because f is concave, we have: $\Delta^2 f(k) \leq 0$, for all $k \geq 1$.

Let's suppose that f is bounded.
We will prove this formula:

$$\forall n \in \mathbb{N},\ f(n) = f(0) - \sum_{k=1}^{\infty} (k \wedge n)\Delta^2 f(k) \tag{7}$$

We have, for $n \in \mathbb{N}$, and $N \geq n$:

$$\sum_{k=1}^{N} (k \wedge n)\Delta^2 f(k)$$

$$= \sum_{k=1}^{n} k\Delta^2 f(k) + n \sum_{k=n+1}^{N} \Delta^2 f(k)$$

$$= \sum_{k=1}^{n} [kf(k+1) - 2kf(k) + kf(k-1)] + n \sum_{k=n+1}^{N} [f(k+1) - 2f(k) + f(k-1)]$$

$$= \sum_{k=2}^{n+1} (k-1)f(k) + \sum_{k=1}^{n} -2kf(k) + \sum_{k=0}^{n-1} (k+1)f(k) + n\left[\sum_{k=n+2}^{N+1} f(k) + \sum_{k=n+1}^{N} -2f(k) + \sum_{k=n}^{N-1} f(k)\right]$$

$$= \sum_{k=2}^{n} \underbrace{(k-1-2k+k+1)}_{=0} f(k) + nf(n+1) - 2f(1) - (n+1)f(n) + 2f(1) + 1f(0)$$

$$+ n\left[\sum_{k=n+2}^{N-1} 0f(k) + f(N+1) + f(N) - 2f(N) - 2f(n+1) + f(n+1) + f(n)\right]$$

$$= nf(n+1) - (n+1)f(n) + f(0) + nf(N+1) - nf(N) - nf(n+1) + nf(n)$$

So finally:

$$\sum_{k=1}^{N} (k \wedge n)\Delta^2 f(k) = f(0) - f(n) + n[f(N+1) - f(N)]$$

But because f is concave bounded, we have: $f(N+1) - f(N) \underset{N \to \infty}{\longrightarrow} 0$, and:

$$\sum_{k=1}^{\infty} (k \wedge n)\Delta^2 f(k) = f(0) - f(n).$$

This way, we prove (7).

Then, using Proposition 11 (see page 26), we have, supposing f bounded, and knowing $\Delta^2 f(k) \leq 0$:

$$\mathbb{E}[f(\tau)] = f(0) - \sum_{k=1}^{\infty} \mathbb{E}[k \wedge \tau]\Delta^2 f(k) \leq f(0) - \sum_{k=1}^{\infty} \mathbb{E}\left[k \wedge \tau_R\right] \Delta^2 f(k) = \mathbb{E}\left[f\left(\tau_R\right)\right]$$

Then, if f is not bounded, for all $N \in \mathbb{N}$, we have: $f_N := f \wedge N$ is bounded, concave and increasing.
So, for all $N \in \mathbb{N}$, we have: $\mathbb{E}\left[f_N(\tau)\right] \leq \mathbb{E}\left[f_N\left(\tau_R\right)\right]$.
Also, we have, for all $N \in \mathbb{N}$: $f(0) \wedge 0 \leq f_N(\tau) \leq f_{N+1}(\tau)$ and the same with τ_R.
So, we can use the monotone convergence for both sides, and we have: $\mathbb{E}[f(\tau)] \leq \mathbb{E}\left[f\left(\tau_R\right)\right]$.

■

Proposition 13 Expectancy of solutions to the *Skorokhod* Embedding Problem

Let τ be any solution to the *Skorokhod* Embedding Problem, with starting distribution λ and target distribution μ, still both integrable.

1. If λ and μ have different means, that is to say, if $\sum_{y=0}^{\infty} y\lambda(y) > \sum_{y=0}^{\infty} y\mu(y)$,

   Then $\mathbb{E}[\tau] = \infty$.

2. If λ and μ have the same mean, that is to say, if $\sum_{y=0}^{\infty} y\lambda(y) = \sum_{y=0}^{\infty} y\mu(y)$,

   And if λ has a finite 2nd moment: $\sum_{y=0}^{\infty} y^2\lambda(y) < \infty$,

   Then $\mathbb{E}[\tau] = \sum_{y=0}^{\infty} y^2(\mu(y) - \lambda(y))$ (this quantity might be infinite).

Proof 13

1. We know, thanks to Proposition 1 (see page 5), that $\left(Z_n^\lambda\right)$ is a martingale.
   Because we know that it is a martingale with bounded differences, we can apply the Optional Stopping Theorem with any τ integrable.
   This way we have:

$$\mathbb{E}[\tau] < \infty \Longrightarrow \mathbb{E}\left[Z_\tau^\lambda\right] = \mathbb{E}\left[Z_0^\lambda\right] = \sum_{y=0}^{\infty} y\lambda(y)$$

So, by taking the contrapositive:

$$\mathbb{E}\left[Z_\tau^\lambda\right] \neq \sum_{y=0}^{\infty} y\lambda(y) \Longrightarrow \mathbb{E}[\tau] = \infty$$

Because we know $\mathbb{E}\left[Z_\tau^\lambda\right] = \sum_{y=0}^{\infty} y\mu(y) \neq \sum_{y=0}^{\infty} y\lambda(y)$, we get $\mathbb{E}[\tau] = \infty$.

2. We write $\mathcal{F}_n = \sigma\left(Z_0^\lambda, \ldots, Z_n^\lambda\right)$.
We have:

- For each $n \in \mathbb{N}$, $\left(Z_n^\lambda\right)^2 - \sum_{i=0}^{n-1} \mathbb{1}_{Z_i^\lambda \neq 0}$ is $\mathcal{F}_n$-measurable.
- For each $n \in \mathbb{N}$,

$$\mathbb{E}\left[\left(Z_n^\lambda\right)^2 - \sum_{i=0}^{n-1} \mathbb{1}_{Z_i^\lambda \neq 0}\right] \leq \mathbb{E}\left[\left(Z_0^\lambda + n\right)^2\right] = n^2 + 2n\mathbb{E}\left[Z_0^\lambda\right] + \mathbb{E}\left[\left(Z_0^\lambda\right)^2\right] < \infty,$$

because λ has a finite moment of order 2.
- Finally, we have:
$\mathbb{E}\left[\left(Z_{n+1}^\lambda\right)^2 \middle| \mathcal{F}_n\right] = \begin{cases} 0 & \text{if } Z_n^\lambda = 0 \\ \frac{1}{2}\left(Z_n^\lambda - 1\right)^2 + \frac{1}{2}\left(Z_n^\lambda + 1\right)^2 = \left(Z_n^\lambda\right)^2 + 1 & \text{if } Z_n^\lambda \neq 0 \end{cases}$
So, $\mathbb{E}\left[\left(Z_{n+1}^\lambda\right)^2 \middle| \mathcal{F}_n\right] = \left(Z_n^\lambda\right)^2 + \mathbb{1}_{Z_n^\lambda \neq 0}$.
And: $\mathbb{E}\left[\left(Z_{n+1}^\lambda\right)^2 - \sum_{i=0}^{n} \mathbb{1}_{Z_i^\lambda \neq 0} \middle| \mathcal{F}_n\right] = \left(Z_n^\lambda\right)^2 - \sum_{i=0}^{n-1} \mathbb{1}_{Z_i^\lambda \neq 0}$.

This way, we prove that $\left(\left(Z_n^\lambda\right)^2 - \sum_{i=0}^{n-1} \mathbb{1}_{Z_i^\lambda \neq 0}\right)_{n \in \mathbb{N}}$ is a martingale.

(a) If τ is bounded, we can use the Optional Stopping Theorem, and we have:

$$\mathbb{E}\left[\left(Z_\tau^\lambda\right)^2 - \sum_{i=0}^{\tau-1} \mathbb{1}_{Z_i^\lambda \neq 0}\right] = \mathbb{E}\left[\left(Z_0^\lambda\right)^2\right] = \sum_{y=0}^{\infty} y^2\lambda(y)$$

We can see that if we write $\tau \wedge H_0$ instead of τ, we still get an embedding of μ. So, we do not change anything if we suppose that we always stop the first time we reach 0.
So $\mathbb{E}\left[\sum_{i=0}^{\tau-1} \mathbb{1}_{Z_i^\lambda \neq 0}\right] = \mathbb{E}[\tau]$.
Finally, we get: $\mathbb{E}[\tau] = \mathbb{E}\left[\left(Z_\tau^\lambda\right)^2\right] - \sum_{y=0}^{\infty} y^2\lambda(y)$.
So, if τ is bounded, we get:

$$\mathbb{E}[\tau] = \sum_{y=0}^{\infty} y^2(\mu(y) - \lambda(y)).$$

(b) If τ is unbounded, then, for all $n \in \mathbb{N}$, we have: $\tau \wedge n$ is a bounded stopping time.
Let's write $\mu_n = \mathcal{L}\left(Z_{\tau \wedge n}^\lambda\right)$.
Because $\tau \wedge n$ is integrable, thanks to what we did in the beginning of this proof, we know that λ and μ_n have the same mean m.
So, we can apply the case 2. (a), and we get:

$$\mathbb{E}[\tau \wedge n] = \sum_{y=0}^{\infty} y^2\left(\mu_n(y) - \lambda(y)\right)$$

i. If $\sum_{y=0}^{\infty} y^2\mu(y) < \infty$.

We have, using (1) (see page 6):

$$
\begin{aligned}
K^{\lambda}(y) &= y + \sum_{i=0}^{y-1} (i-y)\lambda(i) \\
&= y + \sum_{i=0}^{\infty} i\lambda(i) - y\sum_{i=0}^{\infty}\lambda(i) - \sum_{i=y}^{\infty}(i-y)\lambda(i) \\
&= m + \sum_{i=y}^{\infty}(y-i)\lambda(i)
\end{aligned}
$$

Let's note ν any distribution with a finite 2[nd] moment and mean m:

$$
\begin{aligned}
\sum_{y=0}^{\infty}\left[K^{\nu}(y) - K^{\lambda}(y)\right] &= \sum_{y=0}^{\infty}\sum_{i=y}^{\infty}(y-i)\left(\nu(i)-\lambda(i)\right) \\
&= \sum_{i=0}^{\infty}\sum_{y=0}^{i}(y-i)\left(\nu(i)-\lambda(i)\right) \text{ (we can use \textit{Fubini}: see below)} \\
&= \sum_{i=0}^{\infty}\left(-\nu(i)+\lambda(i)\right)\frac{(i+1)i}{2} \\
&= \frac{-1}{2}\left[\sum_{i=0}^{\infty} i^2\left(\nu(i)-\lambda(i)\right) + \sum_{i=0}^{\infty} i\nu(i) - \sum_{i=0}^{\infty} i\lambda(i)\right] \\
&= \frac{-1}{2}\sum_{i=0}^{\infty} i^2\left(\nu(i)-\lambda(i)\right)
\end{aligned}
$$

One equality uses *Fubini*; so we have to show:

$$
\sum_{i=0}^{\infty}\sum_{y=0}^{\infty}\mathbb{1}_{y\geq i}|y-i||\nu(i)-\lambda(i)| = \sum_{i=0}^{\infty}\frac{i^2+i}{2}|\nu(i)-\lambda(i)| \leq \sum_{i=0}^{\infty}\frac{i^2+i}{2}\nu(i) + \sum_{i=0}^{\infty}\frac{i^2+i}{2}\lambda(i) < \infty
$$

And it's true, because λ and ν have finite 2[nd] moments.
We know that $\mu_n = \mathcal{L}\left(Z^{\lambda}_{\tau\wedge n}\right)$ has mean m.

$$
\mathbb{E}\left[\left(Z^{\lambda}_{\tau\wedge n}\right)^2\right] \leq \mathbb{E}\left[\left(Z^{\lambda}_0 + (\tau\wedge n)\right)^2\right] \leq \mathbb{E}\left[\left(Z^{\lambda}_0 + n\right)^2\right] \leq \mathbb{E}\left[\left(Z^{\lambda}_0\right)^2\right] + 2n\mathbb{E}\left[Z^{\lambda}_0\right] + n^2 < \infty
$$

So, we can use the previous equality with $\nu = \mu_n$.
Finally,

$$
\mathbb{E}[\tau\wedge n] = \sum_{y=0}^{\infty} y^2\left(\mu_n(y)-\lambda(y)\right) = -2\sum_{y=0}^{\infty}\left[K^{\mu_n}(y) - K^{\lambda}(y)\right].
$$

On the left, I can use monotone convergence because $(\tau\wedge n)_{n\in\mathbb{N}}$ is increasing:

$$
\lim_{n\to\infty}\mathbb{E}[\tau\wedge n] = \mathbb{E}[\tau].
$$

On the right, because $K^{\mu_n}(y) = K^{\lambda}_{\tau\wedge n}(y)$ is decreasing with n (because $\tau\wedge n$ is increasing) and $\mu_n \underset{n\to\infty}{\longrightarrow} \mu$, we can do the same and get:

$$
\lim_{n\to\infty}\sum_{y=0}^{\infty}\left[K^{\mu_n}(y) - K^{\lambda}(y)\right] = \sum_{y=0}^{\infty}\left[K^{\mu}(y) - K^{\lambda}(y)\right].
$$

So, we have, because λ and μ have the same mean m, and μ has a finite 2[nd] moment:

$$
\mathbb{E}[\tau] = -2\sum_{y=0}^{\infty}\left[K^{\mu}(y) - K^{\lambda}(y)\right] = \sum_{y=0}^{\infty} y^2(\mu(y)-\lambda(y)).
$$

ii. If $\sum_{y=0}^{\infty} y^2\mu(y) = \infty$.

We still have:

$$\mathbb{E}[\tau \wedge n] = \sum_{y=0}^{\infty} y^2 \left(\mu_n(y) - \lambda(y)\right)$$

Because $\left(Z^{\lambda}_{\tau\wedge n}\right)^2$ is non-negative, we can use *Fatou*'s lemma and get:

$$\sum_{y=0}^{\infty} y^2\mu(y) = \mathbb{E}\left[\liminf_{n\to\infty}\left(Z^{\lambda}_{\tau\wedge n}\right)^2\right] \leq \liminf_{n\to\infty}\mathbb{E}\left[\left(Z^{\lambda}_{\tau\wedge n}\right)^2\right] = \lim_{n\to\infty}\sum_{y=0}^{\infty} y^2\mu_n(y)$$

So, using monotone convergence for $\mathbb{E}[\tau \wedge n]$:

$$\mathbb{E}[\tau] = \lim_{n\to\infty}\mathbb{E}[\tau\wedge n] = \lim_{n\to\infty}\sum_{y=0}^{\infty} y^2\mu_n(y) \geq \infty.$$

Finally,

$$\mathbb{E}[\tau] = \infty = \sum_{y=0}^{\infty} y^2(\mu(y) - \lambda(y)).$$

■

Theorem 14 τ_R **minimises** $\mathbb{V}\mathbf{ar}(\tau)$

We suppose that λ and μ are two integrable distributions, with the same mean, and we suppose that λ has a finite 2nd moment.

Let τ be any solution to the *Skorokhod* Embedding Problem, with starting distribution λ and target distribution μ; we suppose $\mathbb{E}\left[\tau^3\right] < \infty$.

We have: $\mathbb{V}\mathrm{ar}(\tau)$ is well defined and

$$\mathbb{V}\mathrm{ar}\left(\tau_R\right) \leq \mathbb{V}\mathrm{ar}(\tau)$$

Proof 14

We define, for $N \in \mathbb{N}$:

$$f_N(x) = 2N(x\wedge N) - (x\wedge N)^2 = \begin{cases} 2Nx - N^2 & \text{if } x \leq N \\ N^2 & \text{if } x \geq N \end{cases}$$

This way, f_N is continuously differentiable, and:

$$f_N'(x) = \begin{cases} 2N - 2x & \text{if } x < N \\ 0 & \text{if } x > N \end{cases}$$

So, f_N' is decreasing and non-negative; and finally f_N is increasing and concave.

We can use Theorem 12 (see page 27), and we get:

$$\begin{aligned}\mathbb{E}\left[f_N\left(\tau_R\right)\right] \geq \mathbb{E}\left[f_N(\tau)\right] &\iff 2N\,\mathbb{E}\left[\tau_R\wedge N\right] - \mathbb{E}\left[\left(\tau_R\wedge N\right)^2\right] \geq 2N\,\mathbb{E}[\tau\wedge N] - \mathbb{E}\left[(\tau\wedge N)^2\right] \\ &\iff 2N\left(\mathbb{E}\left[\tau_R\wedge N\right] - \mathbb{E}[\tau\wedge N]\right) \geq \mathbb{E}\left[\left(\tau_R\wedge N\right)^2\right] - \mathbb{E}\left[(\tau\wedge N)^2\right]\end{aligned}$$

But we have: $\mathbb{E}[\tau] - \mathbb{E}[\tau \wedge N] = \mathbb{E}\left[0\, \mathbb{1}_{\tau<N} + (\tau - N)\mathbb{1}_{\tau \geq N}\right] = \mathbb{E}\left[(\tau - N)_+\right]$.
So: $2N\left(\mathbb{E}\left[\tau_R\right] - \mathbb{E}\left[(\tau_R - N)_+\right] - \mathbb{E}[\tau] + \mathbb{E}\left[(\tau - N)_+\right]\right) \geq \mathbb{E}\left[(\tau_R \wedge N)^2\right] - \mathbb{E}\left[(\tau \wedge N)^2\right]$.

And thanks to Proposition 13 (see page 28), we have: $\mathbb{E}[\tau] = \mathbb{E}\left[\tau_R\right]$.
Finally, we have:

$$2N\left(\mathbb{E}\left[(\tau - N)_+\right] - \mathbb{E}\left[(\tau_R - N)_+\right]\right) \geq \mathbb{E}\left[(\tau_R \wedge N)^2\right] - \mathbb{E}\left[(\tau \wedge N)^2\right] \tag{8}$$

We suppose $\mathbb{E}\left[\tau^3\right] < \infty$.

We have: $\mathbb{E}\left[\tau^3\right] = 3\mathbb{E}\left[\int_0^\tau s^2\, \mathrm{d}s\right] = 3\mathbb{E}\left[\int_0^\infty s^2 \mathbb{1}_{\tau \geq s}\, \mathrm{d}s\right] = 3\int_0^\infty \mathbb{E}\left[s^2 \mathbb{1}_{\tau \geq s}\right]\, \mathrm{d}s = 3\int_0^\infty s^2 \mathbb{P}\left(\tau \geq s\right)\, \mathrm{d}s$

So, because $s^2\mathbb{P}\left(\tau \geq s\right) \geq 0$ and $\mathbb{E}\left[\tau^3\right] < \infty$, we know that: $s^2\mathbb{P}\left(\tau \geq s\right) \underset{s\to\infty}{\longrightarrow} 0$.
In other words:

$$\forall \varepsilon > 0,\ \exists s_\varepsilon \in \mathbb{R},\ \forall s \geq s_\varepsilon,\ \mathbb{P}\left(\tau \geq s\right) \leq \frac{\varepsilon}{s^2}$$

Finally, we have: $\dfrac{\partial}{\partial s}\mathbb{E}\left[(\tau - s)_+\right] = \mathbb{E}\left[\mathbb{1}_{\tau \geq s}\right] = \mathbb{P}\left(\tau \geq s\right)$ and for all $N \geq s_\varepsilon$:

$$N\mathbb{E}\left[(\tau - N)_+\right] = N\int_N^\infty \mathbb{P}(\tau \geq r)\, \mathrm{d}r \leq N\int_N^\infty \frac{\varepsilon}{r^2}\, \mathrm{d}r = N\left[-\frac{\varepsilon}{r}\right]_N^\infty = N\left(0 + \frac{\varepsilon}{N}\right) = \varepsilon$$

If we do $N \to \infty$ in the equality (8), we get (using monotone convergence on the right-hand side):

$$0 \geq \mathbb{E}\left[\tau_R^2\right] - \mathbb{E}\left[\tau^2\right]$$

But thanks to Proposition 13, we have:

$$\mathbb{V}\mathrm{ar}(\tau) \geq \mathbb{V}\mathrm{ar}\left(\tau_R\right)$$

■

Thanks

I would like to thank Alexander Cox for his availability, and for all his help before and during all my course.

Also, thanks to Caroline, Terry and Oliver who welcomed me during my course, and who also made me enjoy my trip to England.

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