Ergodic BSDEs

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Motivation

Let us consider the SDE

$$X_t^x = x + \int_0^t \Xi(X_s^x)\,ds + \int_0^t \sigma(X_s^x)\,dW_s$$

of infinitesimal generator denoted $\mathcal{L}$. Then, Hamilton-Jacobi-Bellman PDE

$$\begin{cases} f(x, \partial_x u(t,x)\sigma(x)) - \partial_t u(t,x) + \mathcal{L}u(t,x) = 0 \\ u(0,x) = g(x) \end{cases} \tag{1}$$

is linked with the following BSDE of finite horizon

$$Y_t^{T,x} = g(X_T^x) + \int_t^T f(X_s^x, Z_s^{T,x})\,ds - \int_t^T Z_s^{T,x}\,dW_s$$

by the relation $Y_t^{T,x} = u(T-t, X_t^x)$.
Can we describe the large time behaviour of such a BSDE? Yes!

Our assumptions

- Ξ and σ are Lipschitz continuous;
- $\sigma(\mathbb{R}^d) \subset GL_d(\mathbb{R})$ and $x \mapsto \sigma(x)^{-1}$ is bounded;
- f is Lipschitz continuous with respect to $(x, z\sigma(x)^{-1})$;
- Ξ is weakly dissipative: $\langle \Xi(x), x\rangle \le \eta_1 - \eta_2 |x|^2$;
- σ has linear growth:

$$|\sigma(x)|_F^2 \le r_1 + r_2|x|^2 \quad \text{with} \quad \sqrt{r_2}\|f\|_{lip} + \frac{r_2}{2} < \eta_2.$$

An auxiliary BSDE

We consider the infinite horizon BSDE

$$Y_t^{\alpha,x} = Y_T^{\alpha,x} + \int_t^T \{f(X_s^x, Z_s^{\alpha,x}) - \alpha Y_s^{\alpha,x}\}\,ds - \int_t^T Z_s^{\alpha,x}\,dW_s$$

where $\alpha > 0$, which holds for every $0 \le t \le T < \infty$. Existence and uniqueness of the solution of this BSDE is given by [BH98]. We define a function $v^\alpha : x \mapsto Y_0^{\alpha,x}$. Our goal is now to study the limit when $\alpha \to 0$.

Key estimates

There exists a constant C, such that for every α, x and x',

$$|\alpha v^\alpha(0)| \le C$$
$$|v^\alpha(x) - v^\alpha(x')| \le C(1 + |x|^2 + |x'|^2)$$
$$|v^\alpha(x) - v^\alpha(x')| \le C(1 + |x|^2 + |x'|^2)|x - x'|.$$

Convergence results

Along a subsequence (α_n), we get:

- $\alpha_n v^{\alpha_n}(0) \to \lambda$, for a convenient real number λ;
- $v^{\alpha_n}(x) - v^{\alpha_n}(0) \to v(x)$, for a convenient function v;
- $Z^{\alpha_n,x}$ converges in $L^2_{\mathcal{P},loc}(\Omega; L^2(0,\infty;\mathbb{R}^d))$ to a process Z^x.

Taking the limit in the auxiliary leads us to

$$Y_t^x = Y_T^x + \int_t^T \{f(X_s^x, Z_s^x) - \lambda\}\,ds - \int_t^T Z_s^x\,dW_s \tag{2}$$

where $Y_t^x := v(X_t^x)$.

Uniqueness of the solution

Let (v, ζ) and $(\tilde v, \tilde\zeta)$ be couples of functions such that:

- $v, \tilde v : \mathbb{R}^d \to \mathbb{R}$ are continuous, have quadratic growth and $v(0) = \tilde v(0) = 0$;
- $\zeta, \tilde\zeta : \mathbb{R}^d \to (\mathbb{R}^d)^*$ are measurable.

If, for all $x \in \mathbb{R}^d$, the triplets $(v(X_t^x), \zeta(X_t^x), \lambda)$ and $(\tilde v(X_t^x), \tilde\zeta(X_t^x), \tilde\lambda)$ satisfy the EBSDE (2), then $\lambda = \tilde\lambda$, $v = \tilde v$ and $\zeta(X_t^x) = \tilde\zeta(X_t^x)$ $\mathbb{P}$-a.s., for a.e. $t \ge 0$ and for every $x \in \mathbb{R}^d$.

References

Philippe Briand and Ying Hu.
Stability of BSDEs with random terminal time and homogenization of semilinear elliptic PDEs.
J. Funct. Anal., 155(2):455–494, 1998.

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Ergodic BSDE with an unbounded and multiplicative underlying diffusion and application to large time behavior of viscosity solution of HJB equation.
arXiv:1801.01284, 2018.

Large time behaviour

Let ξ^T be a real random variable $\mathcal{F}_T$-measurable, such that $|\xi^T| \le C(1 + |X_T^x|^2)$, and consider the solution the finite horizon BSDE

$$Y_t^{T,x} = \xi^T + \int_t^T f(X_s^x, Z_s^{T,x})\,ds - \int_t^T Z_s^{T,x}\,dW_s$$

Then, $\dfrac{Y_0^{T,x}}{T} \underset{T\to\infty}{\longrightarrow} \lambda$ uniformly in every bounded subset of $\mathbb{R}^d$. Moreover, if $\xi^T = g(X_T^x)$ with $|g(x)| \le C(1 + |x|^2)$ and $|g(x) - g(x')| \le C(1 + |x|^2 + |x'|^2)|x - x'|$, then, there exists $L \in \mathbb{R}$ such that for every x and T

$$|Y_0^{T,x} - \lambda T - Y_0^x - L| \le C(1 + |x|^2)e^{-\nu T}$$

Application to Hamilton-Jacobi-Bellman PDE

Under our assumptions, Hamilton-Jacobi-Bellman equation (1) has a unique viscosity solution u. Its large time behaviour is linked to the solution (v, λ) of the ergodic PDE

$$\mathcal{L}v(x) + f(x, \nabla v(x)\sigma(x)) = \lambda$$

and we have

$$|u(T,x) - \lambda T - v(x) - L| \le C(1 + |x|^2)e^{-\nu T}$$

Optimal ergodic problem

We consider:

- a separable metric space U;
- a bounded function $R : U \to \mathbb{R}^d$;
- a Lipschitz function $L : \mathbb{R}^d \times U \to \mathbb{R}$ w.r.t. $x \in \mathbb{R}^d$, uniformly in $a \in U$.

For any control a and horizon T, we set

$$\mathbb{P}_T^{x,a} = \mathcal{E}\left(\int_0^\cdot \sigma(X_t^x)^{-1} R(a_t)\,dW_t\right)_T \mathbb{P}$$

where $\mathcal{E}$ stands for Doléans-Dade exponential. We define the finite horizon and ergodic costs as

$$J_T(x,a) = \mathbb{E}_T^{x,a}\left[g(X_T^x) + \int_0^T L(X_t^x, a_t)\,dt\right]$$

$$J(x,a) = \limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_T^{x,a}\left[\int_0^T L(X_t^x, a_t)\,dt\right]$$

We set the Hamiltonian

$$f(x,z) = \inf_{a\in U}\{L(x,a) + z\sigma(x)^{-1}R(a)\} \tag{3}$$

For every control a, we have

$$J_T(x,a) \ge Y_0^{T,x}, \qquad J(x,a) \ge \lambda \quad \text{and} \quad \liminf_{T\to\infty} \frac{J_T(x,a)}{T} \ge \lambda$$

Moreover, if the infimum is reached for every x and z in equation (3), then we can find controls $\bar a^T$ and $\bar a$ such that

$$J_T(x, \bar a^T) = Y_0^{T,x} \quad \text{and} \quad J(x, \bar a) = \lambda$$

Finally,

$$|J_T(x, \bar a^T) - J(x, \bar a)T - Y_0^x - L| \le C(1 + |x|^2)e^{-\nu T}$$

And what to do next?

We are currently trying to extend those results to a multidimensional case, i.e. study the large time behaviour of

$$\begin{cases} f_l(x, \partial_x u_l(t,x)\sigma_l(x), \{u_m(t,x)\}_{m\neq l}) - \partial_t u_l(t,x) + \mathcal{L}_l u_l(t,x) = 0 \\ u_l(0,x) = g_l(x) \end{cases}$$

for every $1 \le l \le k$. Using randomisation techniques, we have a bridge between this PDE and the BSDE

$$Y_t^{T,x,n} = g_{N_T^n}(X_T^{x,n}) + \int_t^T f_{N_s^n}(X_s^{x,n}, Z_s^{T,x,n}, H_s^{T,x,n})\,ds - \int_t^T Z_s^{T,x,n}\,dW_s - \int_t^T H_s^{T,x,n}\,d\hat N_s$$

where $\mathcal{N}$ is a Poisson random measure over $\mathbb{R}_+ \times \{1, \dots, k\}$, $N_t^n = n + \sum_{l=1}^k l\mathcal{N}((0,t] \times \{l\})$ and

$$X_t^{x,n} = x + \int_0^t \Xi_{N_s^n}(X_s^{x,n})\,ds + \int_0^t \sigma_{N_s^n}(X_s^{x,n})\,dW_s$$

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