Affine symmetric groups

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Semaine des jeunes de l'IDP

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   - Definition
   - Transposition, cycles and decomposition into disjoints cycles
   - Group presentation
   - $\mathfrak{S}_n$ as a reflection group
2. Affine symmetric groups
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   - Action of $\widehat{\mathfrak{S}}_n$ on the alcoves
   - Group presentation
   - Combinatorial model
   - Affine transpositions, pseudo-cycles
   - An example: affine braid groups

Permutations

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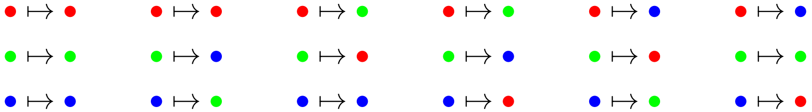
Example : permutations of a set of three colors



Symmetric groups

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When studying symmetric groups of finite sets, we can only look at the symmetric group of the sets $\llbracket 1, n \rrbracket$ for $n \in \mathbb{N}$. We call them **symmetric groups** of rank n and we denote them $\mathfrak{S}_n$ instead of $\mathfrak{S}(\llbracket 1, n \rrbracket)$.

Transpositions

A **transposition** is a permutation that fixes all but exactly two points. If $i \neq j$, we denote (i, j) the transposition that does not fix i and j. We have $(i, j) = (j, i)$, so we will always assume $i < j$ when writing a transposition like that.

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Let σ be a permutation in $\mathfrak{S}_n$. Then there exist transpositions $t_1, \ldots, t_r$ such that $\sigma = t_1 \ldots t_r$.

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A **simple transposition** is a transposition $(i, i+1)$. We denote them σ_i. If (i, j) is a transposition, then

$$(i, j) = \sigma_i \sigma_{i+1} \ldots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2} \ldots \sigma_{i+1} \sigma_i.$$

This means that simple transpositions are also a generating set of $\mathfrak{S}_n$.

Decomposition into disjoints cycles

A **cycle** is a permutation σ such that there exist distinct elements $i_1, \ldots, i_r$ such that

- $\forall 1 \leq k \leq r-1,\ \sigma(i_k) = i_{k+1}$,
- $\sigma(i_r) = i_1$
- $\forall x \in [\![1, n]\!] \setminus \{i_1, \ldots, i_r\},\ \sigma(x) = x$.

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Theorem (Decomposition into disjoints cycles)

Let $\sigma \in \mathfrak{S}_n$. There exists a unique r-tuple of non trivial cycles up to ordering and $(c_1, \ldots, c_r)$ such that:

1. *The c_i are pairwise disjoint,*
2. $\sigma = c_1 \ldots c_r$.

Group presentation

The symmetric group of rank n is isomorphic to the group of presentation

$$\left\langle s_1, \ldots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \forall 1 \leq i \leq n-1 \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \forall 1 \leq i \leq n-1 \\ s_i s_j = s_j s_i & \forall 1 \leq i, j \leq n-1,\ |i-j| > 1 \end{array} \right\rangle$$

under the isomorphism sending the simple transposition σ_i to the generator s_i.

This makes $\mathfrak{S}_n$ a Coxeter group. We say it is of type A_{n-1}.

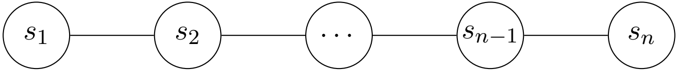

Figure: Coxeter diagram of type A_n

Reflection groups

If V is a vector space, a **reflection** of V is an element f of $\mathrm{GL}(V)$ such that

- f fixes a hyperplane pointwise,
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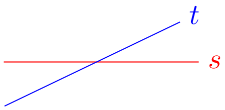

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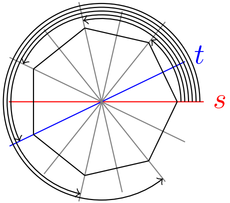

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$\mathfrak{S}_n$ as a reflection group

Let $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \; ; \; x_1 + \cdots + x_n = 0\}$. It is a subspace of $\mathbb{R}^n$ of dimension $n - 1$.

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If $1 \leq i < j \leq n$, consider the hyperplanes $H_{i,j} = \{(x_1, \ldots, x_n) \in V \; ; \; x_i - x_j = 0\}$ of V.

We denote $t_{i,j}$ the (orthogonal) reflection that fixes $H_{i,j}$.

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Theorem

The symmetric group of order n is isomorphic to the reflection group generated by the $t_{i,j}$.

$$\mathfrak{S}_n \simeq \langle t_{i,j} \,;\, 1 \leq i < j \leq n \rangle.$$

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The isomorphism between $\mathfrak{S}_n$ and the reflection group generated by the $t_{i,j}$ is given by

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Because $\mathfrak{S}_n$ is generated by the simple reflections, the reflections $t_{i,i+1}$ also generate the reflection group above.

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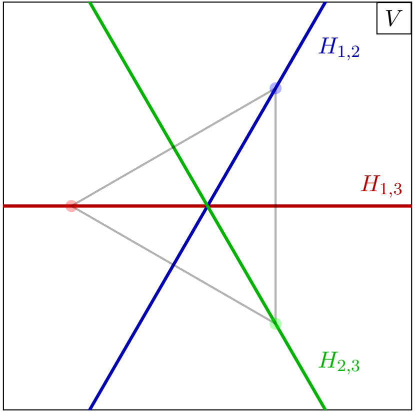

Figure: The hyperplanes $H_{i,j}$.

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Definition (Affine reflections)

An **affine reflection** is an element $f \in \mathsf{Aff}(V)$ such that $\vec{f}$ is a (linear) reflection.

Affine hyperplanes

We recall the vector space $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$ used when we studied $\mathfrak{S}_n$.

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For each $1 \leq i < j \leq n$ and $p \in \mathbb{Z}$, we define

$$H_{(i,j),p} = \{(x_1, \ldots, x_n) \in V \mid x_i - x_j = p\}.$$

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Let $t_{(i,j),p} = t_{i,j} + p(e_i - e_j)$. It is an affine reflection whose set of fixed points is $H_{(i,j),p}$.

Affine symmetric groups

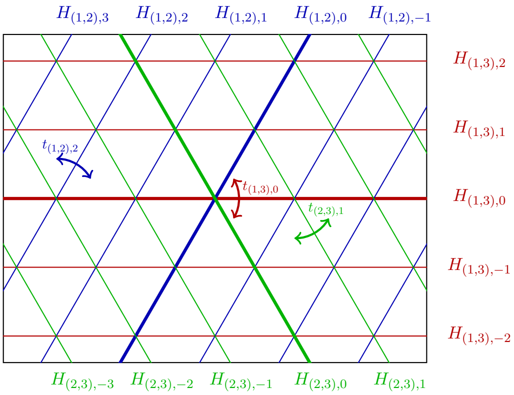

Figure: The hyperplanes $H_{(i,j),p}$ alongside some affine reflections ($n = 3$).

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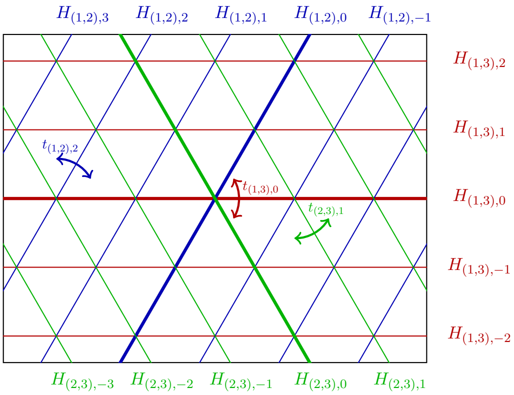

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The group generated by all the $t_{(i,j),p}$ is isomorphic to the affine symmetric group of rank n:

$$\langle t_{(i,j),p} \mid 1 \leq i < j \leq n,\ p \in \mathbb{Z} \rangle \simeq \widehat{\mathfrak{S}}_n.$$

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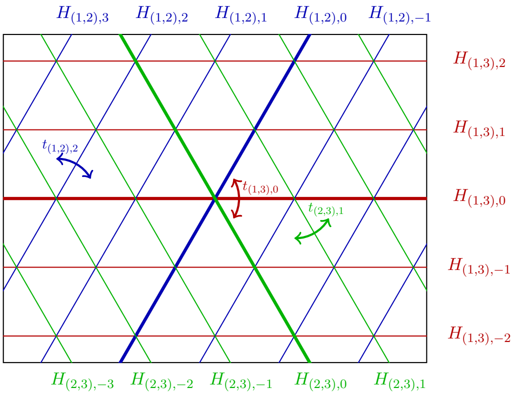

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We say that $\widehat{\mathfrak{S}}_n$ is an **affine reflection group**. It belongs to the family of affine Coxeter groups.

Transitive action of $\widehat{\mathfrak{S}}_n$ on the alcoves

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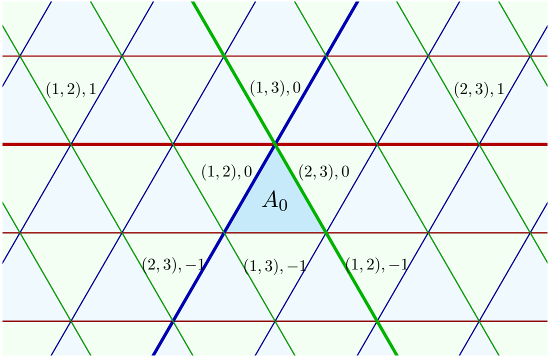

Figure: Alcoves obtained by action of the $t_{(i,j),p}$.

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The action of $\widehat{\mathfrak{S}}_n$ on the alcoves satisfies :

1. For each alcove A, there exists $\sigma \in \widehat{\mathfrak{S}}_n$ such that $\sigma(A_0) = A$ (transitive action)
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This implies that $\widehat{\mathfrak{S}}_n$ is finitely generated:

$$\widehat{\mathfrak{S}}_n \simeq \langle t_{(i,j),p} \mid 1 \leq i < j \leq n,\ p \in \mathbb{Z} \rangle \simeq \langle t_{(1,2),0},\ \ldots,\ t_{(n-1,n),0},\ t_{(1,n),-1} \rangle.$$

An example

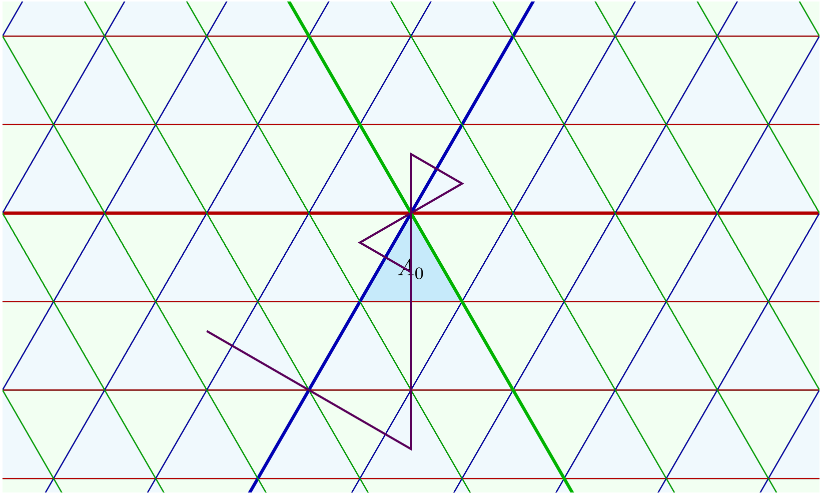

Figure: The alcove $t_{(1,2),0}t_{(1,3),-1}t_{(1,2),0}t_{(2,3),0}t_{(1,2),0}A_0$

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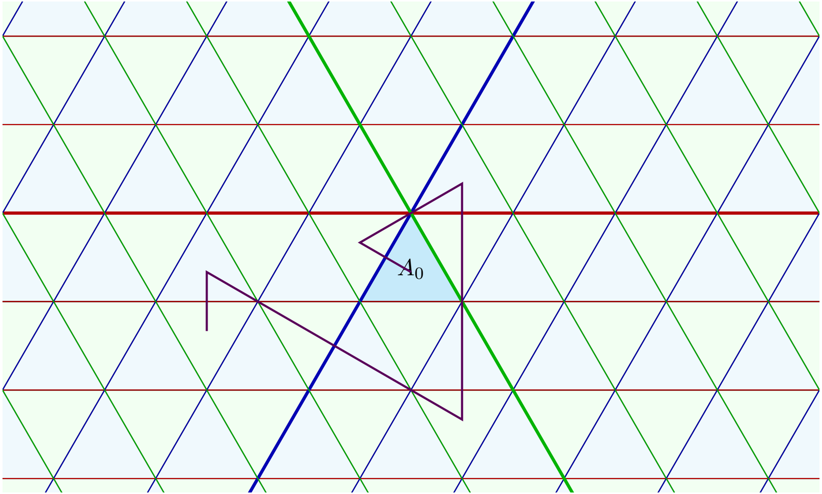

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$\widehat{\mathfrak{S}}_n$ is a Coxeter group

The group with the following presentation

$$\left\langle s_0,\, s_1,\, \ldots,\, s_n \;\middle|\; \begin{array}{ll} s_i^2 = 1 & \forall 0 \le i \le n \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} & \forall 0 \le i \le n \quad (s_{n+1} = s_0) \\ s_i s_j = s_j s_i & \forall 0 \le i, j \le n,\ |i-j| \text{ mod } n > 1 \end{array} \right\rangle$$

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is isomorphic to $\widehat{\mathfrak{S}}_{n+1}$ via the isomorphism

$$\begin{array}{lll} s_0 & \mapsto & t_{(1,n),-1} \\ s_i & \mapsto & t_{(i,i+1),0} \quad \forall 1 \leq i \leq n. \end{array}$$

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$\widehat{\mathfrak{S}}_n$ is a Coxeter group of type $\widetilde{A}_{n-1}$.

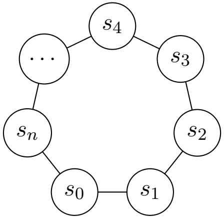

Figure: Coxeter diagram of type $\widetilde{A}_n$

Periodic permutations of $\mathbb{Z}$

Combinatorial definition:

$$\widehat{\mathfrak{S}}_n = \left\{ \sigma \in \mathfrak{S}(\mathbb{Z}) \;\middle|\; \begin{array}{l} \forall x \in \mathbb{Z},\ \sigma(x+n) = \sigma(x) + n \\ \sigma(1) + \cdots + \sigma(n) = 1 + \cdots + n = n(n+1)/2 \end{array} \right\}.$$

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Example, $n = 4$:

$$\begin{pmatrix} \dots & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ \dots & -2 & -11 & 0 & -1 & 2 & -7 & 4 & 3 & 6 & -3 & 8 & \dots \end{pmatrix}.$$

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Window notation:

$$[4,\ 3,\ 6,\ -3]\,.$$

Affine transpositions

If $x \not\equiv y[n]$, we note (x, y) the affine permutation which exchanges $x + pn$ and $y + pn$ for all $p \in \mathbb{Z}$. For any $k \in \mathbb{Z}$, the equality $(x, y) = (x + kn, y + kn)$ holds.

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The set of all affine transpositions is

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The set of all affine transpositions is

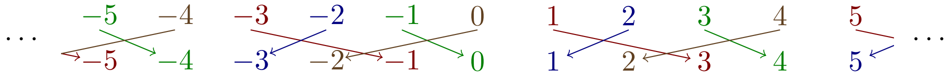
$$T = \{(x, y) \mid x, y \in \mathbb{Z},\ x \not\equiv y[n]\} = \{(i, j)_p \mid 1 \leq i < j \leq n,\ p \in \mathbb{Z}\}.$$

Bijection between $\widehat{\mathfrak{S}}_n$ and the geometric definition:

$$\forall 1 \leq i < j < n,\ \forall p \in \mathbb{Z},\ t_{(i,j),p} \leftrightarrow (i, j)_p.$$

Cycles

Example: $[3,\ 1,\ 4,\ 2]$.



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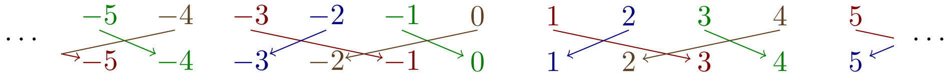

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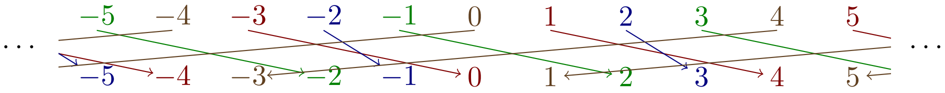
Definition (Cycle)

A **cycle** $(i_1, \dots, i_r)$ where each i_k is in a distinct class modulo n is the affine permutation sending $i_k + an$ to $i_{k+1} + an$ if $i < r$ and $i_r + an$ to $i_1 + an$ for all $a \in \mathbb{Z}$, and fixing all the other points.

Note that an affine transposition is a cycle of length 2.

Pseudo-cycles

Example: $[4,\ 3,\ 6,\ -3]$.



This is a product of pseudo-cycles: $[4,\ 3,\ 6,\ -3] = (1,4)_{[-1]}(2,3)_{[1]}$.

Pseudo-cycles

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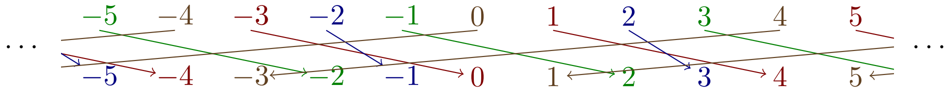

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Definition (Pseudo-cycle)

A **pseudo-cycle** $(i_1, \ldots, i_r)_{[p]}$ where each i_k is in a distinct class modulo n and $p \in \mathbb{Z}$ is the bijection of $\mathbb{Z}$ sending $i_k + an$ to $i_{k+1} + an$ if $i < r$ and $i_r + an$ to $i_1 + an + pn$ for all $a \in \mathbb{Z}$, and fixing all the other points.

Pseudo-cycles

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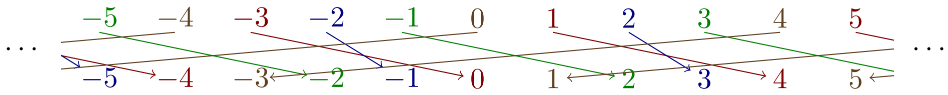

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Remarks:

1. If $p = 0$, we obtain a cycle.
2. If $p \neq 0$, a pseudo cycle is not an affine permutation.

Decomposition into disjoint pseudo-cycles

Theorem (Decomposition into disjoint pseudo-cycles)

Let $\sigma \in \widehat{\mathfrak{S}}_n$. There exists a unique r-tuple of pseudo-cycles up to ordering $(c_1, \ldots, c_r)$ such that:

1. *The sum of the indices of the c_i vanishes,*
2. *The c_i are pairwise disjoint,*
3. $\sigma = c_1 \ldots c_r$.

This is a decomposition of an element of $\widehat{\mathfrak{S}}_n$ as a product of elements of $\mathfrak{S}(\mathbb{Z})$ (in fact, of $\overline{\mathfrak{S}}_n$, the extended affine symmetric group of rank n).

Braid groups

The symmetric group of order n is linked to the braid group on n strands denoted B_n.

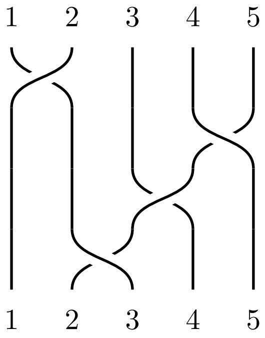

Figure: A braid on 5 strands. It induces the permutation $2\,5\,1\,3\,4$.

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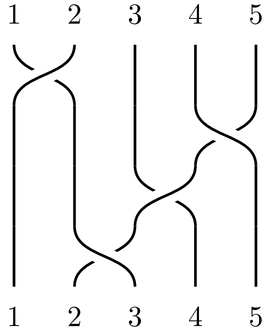

Figure: A braid on 5 strands. It induces the permutation $2\,5\,1\,3\,4$.

Similarly, the affine symmetric group of rank n appears in braids on n strands on a cylindrical annulus.

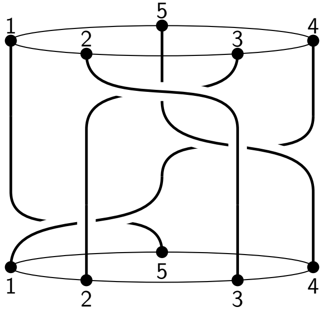

Figure: An affine braid on 5 strands. It induces the affine permutation $[-1, 3, 2, 5, 6]$.

Thank you for your attention!

Some other generalizations of the symmetric group

The affine symmetric group is not the only generalization of the symmetric group. Here is a list of some other ones:

- Extended affine symmetric group $\overline{\mathfrak{S}}_n$. It has the same combinatorial description as $\widehat{\mathfrak{S}}_n$ but without the condition on the sum of the n-th first values. It is not a reflection group. It is isomorphic to the semi-direct product $\mathfrak{S}_n \ltimes \mathbb{Z}^n$. It has countably infinite many elements. It is finitely generated.
- Infinite symmetric group $\mathfrak{S}_\infty$. It is the group consisting of all permutations of $\mathbb{Z}$ that fixes all but finitely many elements. It is the direct limit $\varinjlim \mathfrak{S}_n$. It is not a reflection group. It has countably infinite many elements. It is not finitely generated.
- Symmetric group over $\mathbb{Z}$ $\mathfrak{S}(\mathbb{Z})$. It is the set of all permutations of the set $\mathbb{Z}$. It is not a reflection group. It has uncountably infinite many elements. It is not finitely generated.