CATALAN combinatorics from the perspective of COXETER sortable elements of the symmetric group

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Summary

1. COXETER sortable elements of the symmetric group
   - Symmetric group
   - COXETER sortable elements
2. Non crossing partitions
   - Geometric non crossing partitions
   - Bijection with the c-sortable elements
3. Binary trees
   - Binary trees and SSA algorithm
   - Link with c-sortable elements

1. Coxeter sortable elements of the symmetric group
   - Symmetric group
   - Coxeter sortable elements

2. Non crossing partitions

3. Binary trees

Notations

Let $n \in \mathbb{N}^*$. We note $\mathfrak{S}_n$ the symmetric group of order n.

An element $\sigma \in \mathfrak{S}_n$ is represented by its one line notation : $\sigma(1)\sigma(2)\ldots\sigma(n)$.
Example : $\mathfrak{S}_3 = \{1\,2\,3,\ 2\,1\,3,\ 2\,3\,1,\ 1\,3\,2,\ 3\,1\,2,\ 3\,2\,1\}$.

System of generators : let τ_i be the simple transposition that exchanges i and $i+1$.
We can write all permutations of $\mathfrak{S}_n$ as a product of simple transposition.
Example :

$$\text{id} = 1\,2\,3 \underset{\tau_1}{\longrightarrow} 2\,1\,3 \underset{\tau_2}{\longrightarrow} 2\,3\,1 \underset{\tau_1}{\longrightarrow} 3\,2\,1 = \tau_1\tau_2\tau_1$$

$$\text{id} = 1\,2\,3 \underset{\tau_2}{\longrightarrow} 1\,3\,2 \underset{\tau_1}{\longrightarrow} 3\,1\,2 \underset{\tau_2}{\longrightarrow} 3\,2\,1 = \tau_2\tau_1\tau_2$$

COXETER elements

A COXETER element is permutation c of $\mathfrak{S}_n$ that is a great cycle of the form

$$(1, a_1, \dots, a_k, n, b_l, \dots, b_1)$$

where $k + l = n - 2$, $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_l$.

A COXETER element is the data of a partition of $\{2, \dots, n-1\}$: $L_c = \{a_1, \dots, a_k\}$ and $R_c = \{b_1, \dots, b_l\}$.

Examples : let $n = 6$, $(1, 3, 4, 6, 5, 2)$, $(1, 6, 5, 4, 3, 2)$, $(1, 2, 3, 4, 5, 6)$, $\xcancel{(1, 5, 2, 6, 3, 4)}$.

COXETER sortable elements

Let c be a COXETER element. A permutation $\sigma \in \mathfrak{S}_n$ is c-sortable [Rea05] if its one line notation avoids the following patterns :

- $ki \ldots j$ for $i < j < k$ and $j \in L_c$
- $j \ldots ki$ for $i < j < k$ and $j \in R_c$.

Example : let $n = 4$ and $c = (1, 2, 4, 3)$. We have $L_c = \{2\}$ and $R_c = \{3\}$.

1 2 3 4	1 2 4 3	1 3 2 4	1 3 4 2	1 4 2 3	1 4 3 2
2 1 3 4	2 1 4 3	2 3 1 4	2 3 4 1	2 4 1 3	2 4 3 1
3 2 1 4	3 2 4 1	3 1 2 4	3 1 4 2	3 4 2 1	3 4 1 2
4 2 3 1	4 2 1 3	4 3 2 1	4 3 1 2	4 1 2 3	4 1 3 2

We notice there are 14 c-sortable elements.

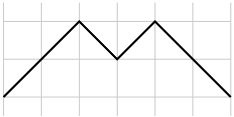
Catalan

Let $n \in \mathbb{N}^*$. For any Coxeter element c of $\mathfrak{S}_n$, there are $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ c-sortable elements.

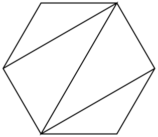
This means the c-sortable elements are in bijection with all the objects enumerated by the Catalan numbers.

Examples :

Dyck paths of length $2n$



Triangulations of $(n+2)$-gons



Well parenthesized expressions of $n+1$ factors

$$((a(bc))d)(ef)$$

1. COXETER sortable elements of the symmetric group

2. Non crossing partitions
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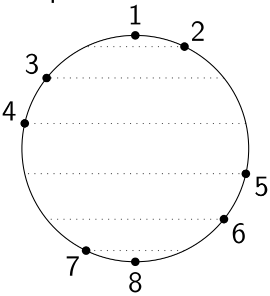
Labeling the circle

Let $n \in \mathbb{N}^*$ and place on a circle the numbers from 1 to n such that :

- 1 is at the highest point and n is at the lowest point,
- no two numbers are on the same height,
- reading from top to bottom the numbers are increasing.

Example : $n = 8$



A *c-labeling* is a labeling such that on the left are the elements of R_c and on the right the elements of L_c

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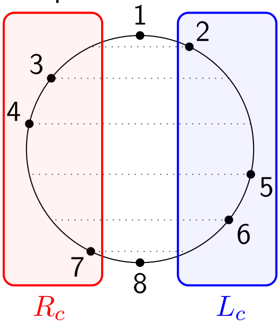


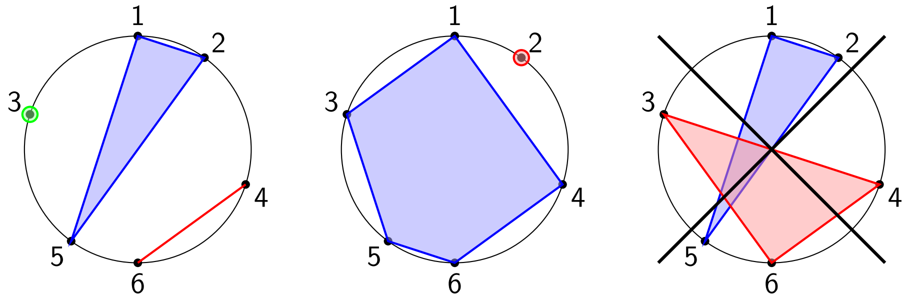
$c = (1, 2, 5, 6, 8, 7, 4, 3)$

A c-labeling is a labeling such that on the left are the elements of R_c and on the right the elements of L_c

Non crossing partitions

A c-non crossing partition is a set of non crossing polygons with vertices the marked points of a c-labeled circle. Single points and segments are considered as polygons.

Examples : $n = 6$ and $c = (1, 2, 4, 6, 5, 3)$

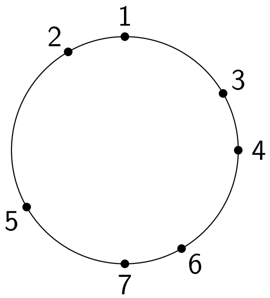


Bijection with the c-sortable elements

Theorem ([Rea05])

The set of c-sortable elements is in bijection with the set of c-non crossing partitions via an explicit map called nc_c.

Example : let $n = 7$, $c = (1, 3, 4, 6, 7, 5, 2)$ and $\sigma = 3167425$. It is a c-sortable element.

$\sigma = 3167425 \longrightarrow$ 

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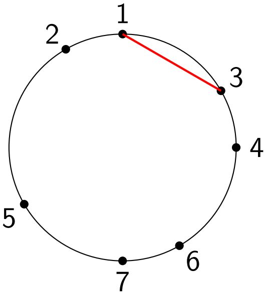

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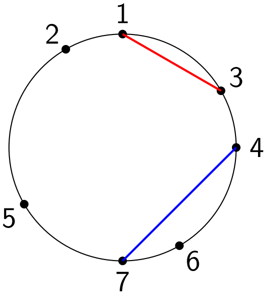

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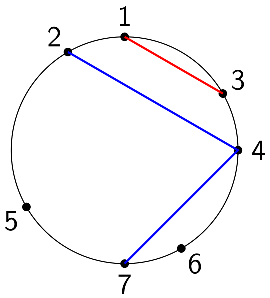

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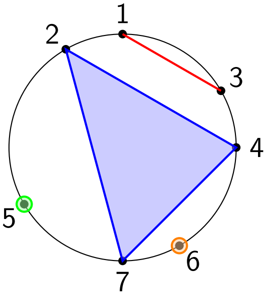
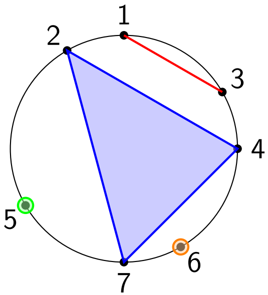


$= nc_c(\sigma)$.

Inverse map

The inverse map of nc_c can be computed by selecting the polygons of a c-non crossing partition in a specific order [Gob18].

Example : Let's use the c-non crossing partition we computed in the previous slide.



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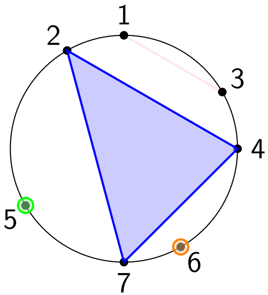

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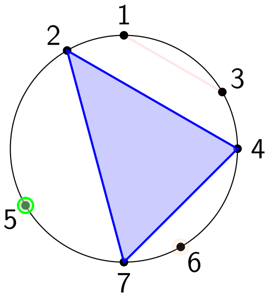

$\longrightarrow \{3,1\}, \{6\},$

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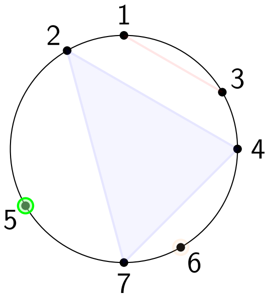

$\longrightarrow \{3,1\}, \{6\}, \{7,4,2\},$

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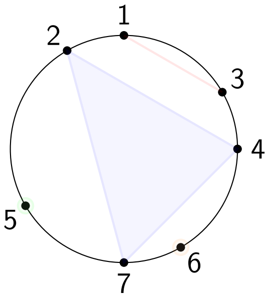

$\longrightarrow \{3,1\},\{6\},\{7,4,2\},\{5\}$

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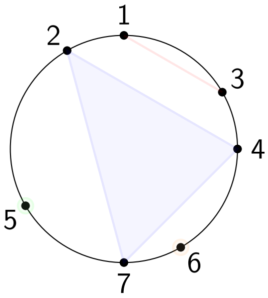

$\longrightarrow \{3,1\},\{6\},\{7,4,2\},\{5\} \longrightarrow 3167425 = \sigma$

1. Coxeter sortable elements of the symmetric group

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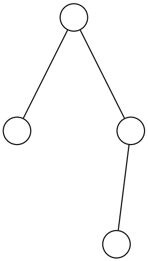
3. Binary trees

- Binary trees and SSA algorithm
- Link with c-sortable elements

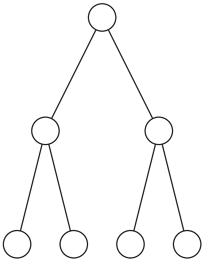
Binary trees

A binary tree is either an empty tree or a node with exactly one left child and one right child. The size of a binary tree is the number of nodes in the tree.

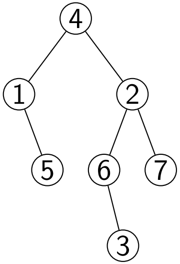
Examples :



A binary tree.



A complete binary tree.

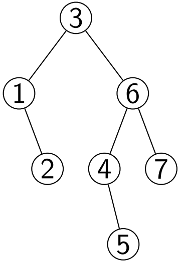


A labeled binary tree.

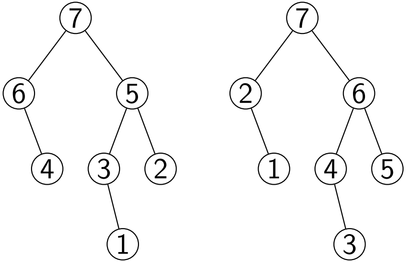
Descending trees, binary search trees

A binary search tree is a labeled binary tree such that the label of each node is larger than the labels of its left child and smaller than the labels of its right child.

Examples :



A binary search tree



Two descending trees of the same shape

SSA algorithm

Theorem (SSA algorithm [HNT04])

There is an explicit bijection

$$\mathfrak{S}_n \simeq \left\{ (T, Q) \;\middle|\; \begin{array}{l} T \text{ is a binary search tree of size } n \text{ and} \\ Q \text{ is a descending tree of the same shape as } T \end{array} \right\}$$

Example : Let $n = 7$ and $\sigma = 2154763$.

$T(\sigma) =$ 　　　　$Q(\sigma) =$

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Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 7$)

$T(\sigma) =$ 3 $\qquad Q(\sigma) =$ 7

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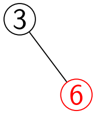
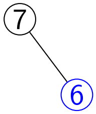
Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 6$)

$T(\sigma) =$ 

$Q(\sigma) =$ 

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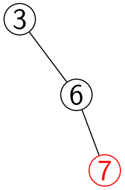
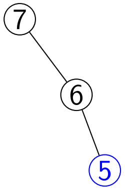
Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 5$)

$T(\sigma) =$



$Q(\sigma) =$



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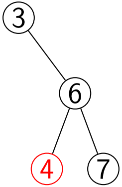
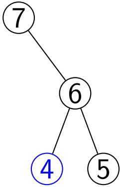
Example : Let $n = 7$ and $\sigma = 2154763$. (position $= 4$)

$T(\sigma) =$



$Q(\sigma) =$



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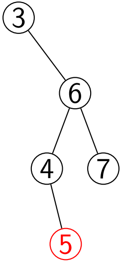
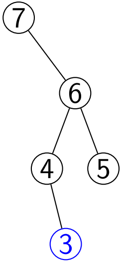


$Q(\sigma) =$



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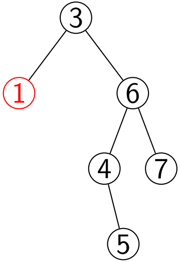
Example : Let $n = 7$ and $\sigma = 2\textcolor{red}{1}54763$. (position $= \textcolor{blue}{2}$)

$T(\sigma) =$



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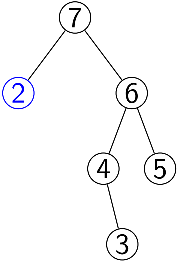

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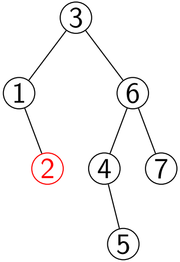
$T(\sigma) =$



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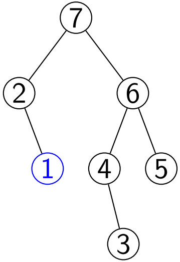

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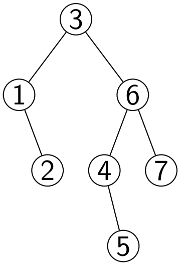

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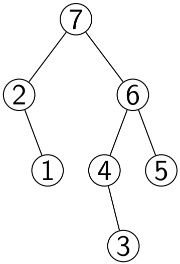

Sylvester congruence

SSA algorithm : $\sigma \in \mathfrak{S}_n$ is encoded by $(T(\sigma), Q(\sigma))$.

What happens if we forget $Q(\sigma)$? Can we describe all $\sigma' \in \mathfrak{S}_n$ s.t. $T(\sigma) = T(\sigma')$?

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What happens if we forget $Q(\sigma)$? Can we describe all $\sigma' \in \mathfrak{S}_n$ s.t. $T(\sigma) = T(\sigma')$?

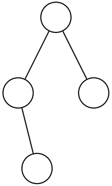
Yes! $T(\sigma) = T(\sigma')$ iff σ' can be obtained from σ by a series of transformations of the form $ki \dots j \leftrightarrow ik \dots j$ with $i < j < k \longrightarrow$ Sylvester congruence $\sigma \equiv \sigma'$.

Example, $\sigma' = 5421763$ has the same binary search tree than $\sigma = 2154763$

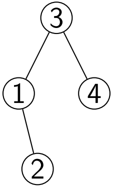
$$2154763 \rightarrow 2514763 \rightarrow 2541763 \rightarrow 5241763 \rightarrow 5421763$$

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :



$\leftrightarrow$



$$\leftrightarrow \left\{ \begin{matrix} 2143 \\ 2413 \\ 4213 \end{matrix} \right\} \leftrightarrow 2143$$

A binary tree

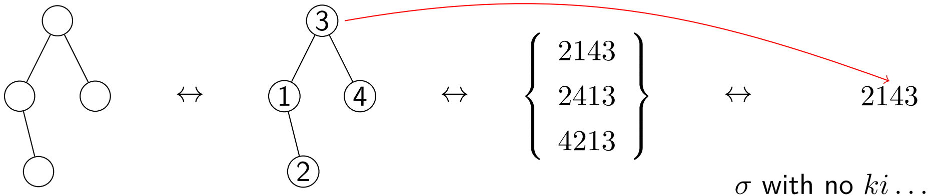
A binary search tree

A sylvester class

σ with no $ki \ldots j$ pattern
$=$ a c-sortable element

Link with c-sortable elements

If $c = (1, 2, 3, \ldots, n-1, n)$, then we have the following one to one maps :



A binary tree — A binary search tree — A sylvester class — σ with no $ki\ldots j$ pattern $=$ a c-sortable element

The map from binary trees to c-sortable elements can be directly obtained with a postfix reading of the associated binary search tree.

Thank you for your attention!

Further questions

- What about other Coxeter elements? $\longrightarrow$ another tree structure : Cambrian trees [CP17].

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- What about other finite Coxeter groups? $\longrightarrow$ W-Catalan numbers [Rei97].

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- What about other Coxeter elements? $\longrightarrow$ another tree structure : Cambrian trees [CP17].
- What about other finite Coxeter groups? $\longrightarrow$ W-Catalan numbers [Rei97].
- What about infinite Coxeter groups? $\longrightarrow$ It's complicated... Non crossing partitions are well defined (in a more algebraic way) as well as Reading bijection, but it never is surjective. In the affine type, there's a hope to generalize the c-sortable elements to a larger family that becomes in bijection with the non-crossing partitions $\longrightarrow$ my thesis.

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